ABSTRACT

Chaotic Properties of Set-valued Dynamical Systems

Tim Tennant, Ph.D.

Advisor: Brian E. Raines, D.Phil

In this thesis, many classical results of topological dynamics are adapted to the set-valued case. In particular, focus is given to the notions of topological entropy and the specification property. These properties are defined in the context of set-valued functions, and examples are given both of classical theorems that extend naturally to this setting and of theorems which have no clear analogue. Particular attention is paid to a result which states that if a dynamical system has the specification property, there exist invariant non-atomic measures with full support.

Chaotic Properties of Set-valued Dynamical Systems

by

Tim Tennant, B.S.

A Dissertation

Approved by the Department of Mathematics

Lance L. Littlejohn, Ph.D., Chairperson

Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

Approved by the Dissertation Committee

Brian E. Raines, D.Phil, Chairperson

David Ryden, Ph.D.

Will Brian, Ph.D.

Jonathan Harrison, Ph.D.

Alexander Pruss, Ph.D.

Accepted by the Graduate School
May 2016

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

The author wishes to acknowledge many fruitful conversations with James Kelly, Jonathan Meddaugh and Brian Raines.

Also, I want to say thank you to Brook Randall, who started a math club when I was a kid. I did not fall in love with math until I fell in love with doing math with the friends I made there. I want to say thank you to Stephen Rodi, who took a scared sheltered kid and made that kid feel welcome in an Austin Community College classroom. It happened to be the first classroom that kid had seen in over a decade, and that first semester of Calculus I was what convinced him to keep taking more and more math classes. I want to say thank you to Brian Raines, who was willing to take on more graduate students than he knew what to do with. Be it at a blackboard or over a game of Rex, his mix of patience and a sharp wit was exactly what I needed to get through this. Last of all, I need to thank my first math teacher for her endless support. Thanks, Mom.

To you, dear reader.

CHAPTER ONE

Introduction

The term 'dynamical system' is widely used in different areas of mathematics and physics, and, before we discuss my results, a brief history of the field and how it has evolved to become the area of mathematics that I am studying is in order. In section 1.1, we discuss the origin of the field and give the first major theorem of dynamical systems, Poincaré's Recurrence Theorem. In section 1.2, we introduce Birkhoff's Ergodic Theorem and explain some of its significance. From there we show in section 1.3 how dynamical systems give fascinating results about the topological structure of inverse limit spaces with particular attention paid to the ideas of indecomposability and chainability. In section 1.4, we introduce various definitions of chaos that will appear throughout this thesis and give some well-known results. Finally, in section 1.5, we begin our discussion on set-valued dynamical systems and their inverse limits. In this section, we give examples and recent results showing that inverse limits of set-valued dynamical systems are fundamentally different from the single-valued case.

The focus of Chapter Two is the extension of the notion of Topological Entropy to set-valued dynamical systems. Similarly, the focus of Chapter Three is to extend the notion of the Specification Property to set-valued dynamical systems. Chapter Four continues directly from Chapter Three and focuses on showing the existence of invariant measures which are non-atomic and have full support.

1.1 Dynamical Systems as Solutions of Differential Equations

Dynamical systems were introduced by Henri Poincaré as he studied celestial motion, [46, 47]. A good reference for this history was written by Phillip Holmes in 1990, [22]. Poincaré defined a dynamical system as a manifold X and a collection of functions $(\phi_t)_{t\geq 0}$ such that for each $t \geq 0$, $\phi_t : X \to X$ is a continuous function. This construction is used to model change with respect to time when time is considered as a continuous variable. For the case where time is considered in discrete steps, the collection of functions $\{\phi^n : n \in \mathbb{N}\}$ is used. In either context, ϕ_t was the solution of an ordinary differential equation. From this work, one of the first theorems dealing with dynamical motion was proven, Poincaré's Recurrence Theorem.

Definition 1.1. Let (X, Σ, μ) define a probability measure space. Let $f : X \to X$ be a continuous function. We say f is *invariant* with respect to μ if $\mu(A) = \mu(f^{-1}(A))$, for all $A \in \Sigma$.

Theorem 1.2 (Poincaré). *Let (X, Σ, μ) be a probability measure space, and let $f : X \to X$ be a continuous function which is invariant with respect to μ. Then, for any $A \in \Sigma$ and μ-almost-every point $x \in A$, the set*

$$A_x = \{n \in \mathbb{N} : f^n(x) \in A\}$$

is infinite.

This theorem formalizes the idea that under iteration, points that are moved by a function will return close to where they started infinitely often. Later, we will define the term *topological transitivity*, which extends this idea.

1.2 Dynamical Systems and Ergodic Theory

George Birkhoff studied the works of Poincaré in detail, and combined ideas of dynamical systems and measure theory. This resulted in the Birkhoff Ergodic Theorem in 1931, one of the first major theorems in ergodic theory.

Definition 1.3. Let (X, Σ, μ) be a probability measure space, and let $f : X \to X$ be a function invariant with respect to μ. We say f is *ergodic* if, for any $A \in \Sigma$, $f^{-1}(A) = A$ implies that either $\mu(A) = 1$ or $\mu(A) = 0$.

Theorem 1.4 (Birkhoff). *Let (X, Σ, μ) be a probability measure space, let $f : X \to X$ be an ergodic endomorphism invariant with respect to μ, and let $g : X \to \mathbb{R}$ be a measurable function. Then, for μ almost-every $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} g \circ f^i(x) = \int g d\mu.$$

The left hand side of the equation is called the time average of f. To see why, consider the case that the function g is a characteristic function χ_A for some $A \in \Sigma$. The left hand expression then becomes the percentage of iterates of x which reside in A. In this case, the right hand side of the expression is the measure of A, which is called the space average. More colloquially, the ergodic theorem can be stated as 'the space average and time average are almost everywhere the same'.

1.3 Continuum Theory and Inverse Limits

The study of dynamical systems led to the study of the underlying invariant subspaces in their own right, and there have been many interesting results linking topological spaces to dynamical properties. One commonly studied category of spaces in dynamical systems are *inverse limit spaces*, introduced by Solomon Lefschetz in 1942, [35].

Definition 1.5. Let Λ be a directed set, and, for each $\lambda \in \Lambda$, let X_λ be a topological space. For each α,$\beta \in \Lambda$ with $\alpha < \beta$, let $f_\alpha^\beta : X_\beta \to X_\alpha$ be a continuous function satisfying the following properties:

- $f_\alpha^\alpha : X_\alpha \to X_\alpha$ is the identity map for all $\alpha \in \Lambda$.
- If $\alpha, \beta, \gamma \in \Lambda$ such that $\alpha \leq \beta$ and $\beta \leq \gamma$, then $f_\alpha^\gamma = f_\alpha^\beta \circ f_\beta^\gamma$.

These function are called *bonding maps* of the *inverse sequence* $\{X_\alpha, f_\alpha^\beta, \Lambda\}$. The *inverse limit* is the subset of the product space $\prod_{\lambda\in\Lambda} X_\lambda$ defined below.

$$\varprojlim(X_\alpha, f_\alpha^\beta, \Lambda) := \{x \in \prod_{\lambda\in\Lambda} X_\lambda : f_\alpha^\beta(x_\beta) = x_\alpha, \text{ for all } \alpha \in \Lambda \text{ and for all } \beta \geq \alpha\}.$$

Note that this space is a subset of the product space and therefore inherits the product topology. For the majority of this thesis, the directed set Λ will be the natural numbers $\mathbb{N}$, and for most examples the spaces X_α will all be the unit interval $[0,1]$. In this case, the definition reduces to

$$\varprojlim([0,1], f_n, \mathbb{N}) = \{(x_j)_{j=0}^{\infty} : x_i = f_{i+1}(x_{i+1}), \text{ for all } i \geq 0\}.$$

When studying the dynamics of a function $f : X \to X$, it is often helpful to examine the inverse limit generated by f, $\varprojlim(X, f)$. When there is no ambiguity with respect to X, we write this set as $\varprojlim(f)$. The following basic properties of inverse limits can be found in [20, 35].

Theorem 1.6. *Consider the inverse system* $\{X_\alpha, f_\alpha^\beta, \Lambda\}$.

(1) If X_α is a compact space for all $\alpha \in \Lambda$, then $\varprojlim(X_\alpha, f_\alpha^\beta)$ is a compact space.

(2) If X_α is a connected space for all $\alpha \in \Lambda$, then $\varprojlim(X_\alpha, f_\alpha^\beta)$ is a connected space.

(3) If X_α is a Hausdorff space for all $\alpha \in \Lambda$, then $\varprojlim(X_\alpha, f_\alpha^\beta)$ is a Hausdorff space.

In particular, if the indexing set Λ is countable, then if X_α is a *continuum* for each $\alpha \in \Lambda$, then the inverse limit, $\varprojlim(X_\alpha, f_\alpha^\beta)$, is a continuum.

Definition 1.7. A topological space X is a *continuum* if X is compact, connected, and metrizable.

One interesting question that people have studied in continuum theory is necessary conditions for an inverse limit to be *chainable*.

Definition 1.8. Let X be a metric space, and let $\epsilon > 0$. We call a finite collection of open sets $\mathcal{C} = \{C_i : 1 \leq i \leq n\}$ a *chain* if $C_i \cap C_j \neq \varnothing$ if and only if $|i - j| \leq 1$. For each $1 \le i \le n$, the set C_i is called a *link* of $\mathcal{C}$. We call $\mathcal{C}$ an ϵ-*chain* if $\mathcal{C}$ is a chain and $\text{diam}(C_i) < \epsilon$ for all $1 \le i \le n$. We say X is *chainable* if X can be covered by an ϵ-chain for all $\epsilon > 0$.

In the literature, chainable continua have also been called 'snake-like' and 'arc-like'. As an example, an arc is a chainable continuum. An example of a continuum that is not chainable is a simple graph with four vertices, 3 of which have degree 1 and the fourth vertex having degree 3. Jolly and Rogers showed the existence of four interval maps such that if a continuum was chainable then it was the inverse limit of a sequence of those four maps, [27]. Ingram and Cook improved this result by showing the existence of two bonding maps of the interval which could generate any chainable continuum, [17]. Conversely, the inverse limit of any sequence of interval maps was shown to be a chainable continuum by Isbell in 1959, [25]. One example of a chainable continuum that has been widely studied is the *pseudo-arc*. The property of chainability was used by R.H. Bing when studying the pseudo-arc, and by O.H. Hamilton to create fixed point theorems for chainable continua, [10, 11, 21]. The pseudo-arc was shown by Edwin Moise to be homeomorphic to each of it's non-trivial subcontinua in 1948, [42]. The following description of the pseudo-arc is due to Bing, [9].

Definition 1.9. Let $\mathcal{C} = \{C_i : 1 \le i \le n\}$ be a chain. We say the set

$$\mathcal{C}^\star = \bigcup_{C \in \mathcal{C}_i} C.$$

is the *sum* of $\mathcal{C}$.

Definition 1.10. Let $\mathcal{C} = \{C_i : 1 \le i \le n\}$ and $\mathcal{D} = \{D_i : 1 \le i \le m\}$ be chains such that each link of $\mathcal{C}$ is contained in some link of $\mathcal{D}$. We say $\mathcal{C}$ is *crooked* in $\mathcal{D}$ if for each two links D_j and D_k with $k - j > 2$, there exists natural numbers $1 \le a < b < c < d \le n$ such that $C_a \subset D_j$, $C_b \subset D_{k-1}$, $C_c \subset D_{j+1}$, and $C_d \subset D_k$.

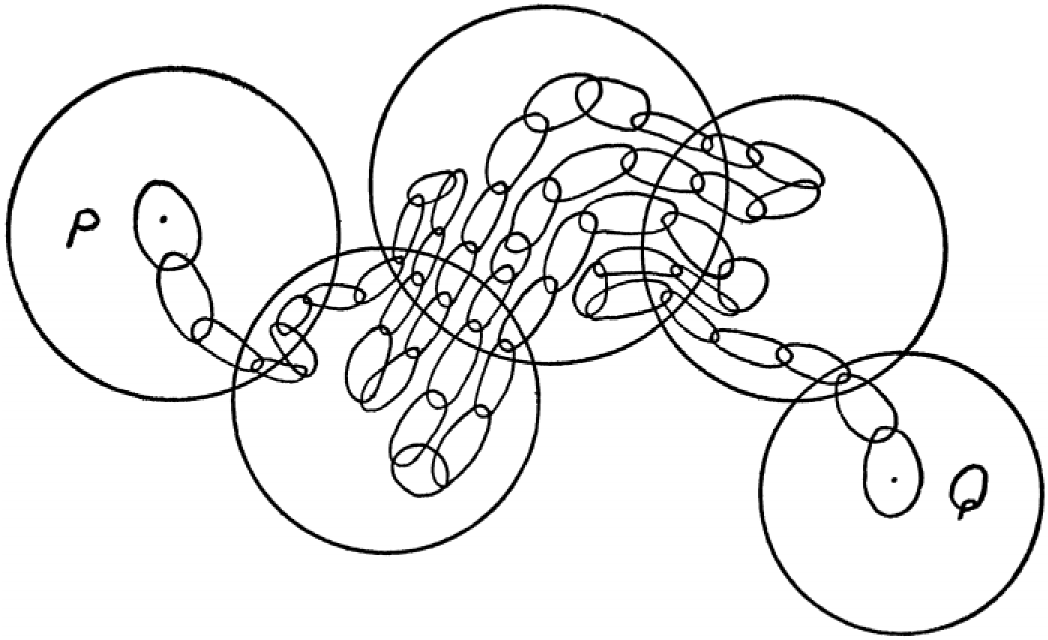

Figure 1.1. A crooked chain, [9].

Definition 1.11. Let p and q be points in a compact metric X. Let $\{\mathcal{C}_i = \{C_j^i : 1 \le j \le n_i\}_{i=1}^{\infty}$ be a sequence of chains such that

- $p \in C_1^i$ and $q \in C_{n_i}^i$ for all $i \ge 1$.
- $\mathcal{C}_{i+1}$ is crooked in $\mathcal{C}_i$ for all $i \ge 1$.
- $\operatorname{diam}(\mathcal{C}_i) < 1/i$ for all $i \ge 1$.
- The closure of each link of $\mathcal{C}_{i+1}$ is contained in a link of $\mathcal{C}_i$.

Then a *pseudo-arc* is the intersection

$$M = \bigcap_{i=1}^{\infty} \mathcal{C}_i^{\star}.$$

George Henderson showed in 1964 that the pseudo-arc can be constructed as an inverse limit which has only one bonding function. This raised the question, can every chainable continuum be constructed as an inverse limit using only one bonding function. William Mahavier answered this question when he showed the existence of a chainable continuum that could not be the inverse limit of a sequence with a single bonding map, [36]. As an example, a family of interval maps that are widely studied are the tent maps, defined below for $1 \le \alpha \le 2$.

$$f_\alpha(x) = \begin{cases} \alpha x & 0 \le x \le 1/2 \\ \alpha - \alpha x & 1/2 < x \le 1 \end{cases}$$

The function f_2 is commonly called the *full tent map*, and it is shown below.

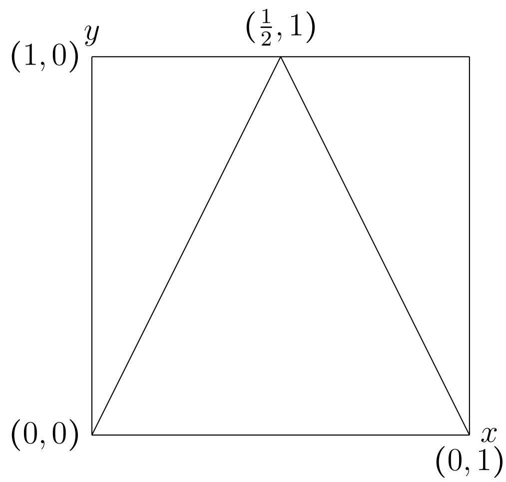

Figure 1.2. The full tent map.

The inverse limit using the single bonding map f_2, $\varprojlim([0,1], f_2)$, defines a chainable continuum called the Knaster continuum. This is one of the first examples of an *indecomposable* continuum.

Definition 1.12. Let X be a continuum. We say X is *decomposable* if there exist proper subcontinua U and V such that $X = U \cup V$. If X is not decomposable, we say X is *indecomposable*.

The first indecomposable continuum was produced by Luitzen Egbertus Jan Brouwer in 1910, [16]. Perhaps the most commonly studied indecomposable continuum is the pseudo-arc, constructed by Knaster in 1922, [34].

A long-standing open problem involving the family of tent maps was Ingram's Conjecture, which states that for distinct α and β in $[1/2, 1]$, the inverse limits $\varprojlim(f_\alpha)$ and $\varprojlim(f_\beta)$ are not homeomorphic. This was a widely studied problem, with partial results given first by Marcy Barge and Beverly Diamond in 1995, and further work done later by Lois Kailhofer, Louis Block, Slagjana Jakimovic, James Keesling, Brian Raines, and Sonja Štimac, [12, 13, 28, 29]. The conjecture was later shown to be true in 2012 by Marcy Barge, Henk Bruin, and Sonja Štimac, [5].

1.4 Topological Chaos

Results like Ingram's Conjecture which rely on dynamical properties of functions to give results about the topological structure of spaces are quite interesting to me. Often, such dynamical properties are called 'chaotic'. In this section we introduce definitions of chaos that recur throughout this thesis, as well as mention classical results involving these definitions. One important result that involves multiple ideas we have mentioned so far was given by Marcy Barge and Joe Martin, [7, 8].

Theorem 1.13 (Barge and Martin). *Let X be a compact metric space, and let $f : X \to X$ be a continuous function. If the dynamical system (X, f) has positive topological entropy, then the inverse limit $\varprojlim(X, f)$ contains an indecomposable subcontinuum.*

Topological entropy is a measure of the complexity of the dynamics of a function, and a function which has positive topological entropy is sometimes referred to as 'chaotic'. It was first introduced by Adler, Konheim, and McAndrew in 1965, [1]. In 1970, Bowen presented an equivalent definition in the context of metric spaces, [15].

The study of topological entropy includes a variety of topics including sufficient conditions for a function to have positive or infinite topological entropy, the relationship between the topological entropy of a function and the structure of its inverse limit space, and what types of spaces admit positive entropy homeomorphisms, [6, 41, 43, 58].

As topological entropy is one of the primary ideas that is extended in this thesis, we give a thorough introduction and definition below. The following definitions can be found in [57].

Definition 1.14. Let (X,d) be a metric space, and let $f : X \to X$ be continuous. Define a new metric $d_n : X \to \mathbb{R}^+$ by $d_n(x,y) = \max\{d(f^i(x), f^i(y)) : 0 \leq i \leq n-1\}$. We denote the open ball of radius ϵ with respect to this metric by $B_n(x,\epsilon)$.

Definition 1.15. We say a set A is (n,ϵ)-*separated* if for all $x, y \in A$ with $x \neq y$, we have that $d_n(x,y) \geq \epsilon$. The maximum cardinality of an (n,ϵ)-separated set is denoted $s(n,\epsilon)$.

Definition 1.16. We say a set A is (n,ϵ)-*spanning* if for all $x \in X$, there exists $y \in A$ such that $x \in B_n(y,\epsilon)$. The minimum cardinality of an (n,ϵ)-spanning set is denoted $r(n,\epsilon)$.

Definition 1.17. Let X be a metric space, and let $f : X \to X$ be continuous. The *topological entropy* of f is

$$h(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln(s(n,\epsilon)).$$

Equivalently, the topological entropy is

$$h(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln(r(n,\epsilon)).$$

Another definition of chaos central to this thesis is the *specification property*, introduced by Rufus Bowen, [15]. It bears mentioning that while Bowen introduced the idea behind the specification property, it was named by Karl Sigmund in [53]. Although he titled his definition as the specification property, it is now known as the *weak specification property*. We give his definition and the current definition below.

Definition 1.18. Let X be a compact metric space, and let $f : X \to X$ be continuous. We say the dynamical system (X, f) has the *(weak) specification property* if for every $\epsilon > 0$ there is an integer $M(\epsilon)$ such that for any choice of points $x_1, x_2 \in X$ and natural numbers $a_1 \le b_1 < a_2 \le b_2$, with $a_2 - b_1 > M(\epsilon)$, and any natural $p > b_2 - a_1 + M(\epsilon)$, there exists a periodic point $x \in X$ with period p such that

$$d(f^i(x), f^i(x_1)) < \epsilon \quad \text{for } a_1 \le i \le b_1,$$

$$d(f^i(x), f^i(x_2)) < \epsilon \quad \text{for } a_2 \le i \le b_2.$$

Definition 1.19. Let X be a compact metric space, and let $f : X \to X$ be continuous. We say the dynamical system (X, f) has the *specification property* if for every $\epsilon > 0$ and $n \in \mathbb{N}$ there is a natural number $M(\epsilon)$ such that for any choice of points $x_1, \ldots x_n \in X$ and natural numbers $a_1 \le b_1 < a_2 \le b_2 < \ldots < a_n \le b_n$, with $a_{i+1} - b_i > M(\epsilon)$, and any natural $p > b_n - a_1 + M(\epsilon)$, there exists a periodic point $x \in X$ with period p such that

$$d(f^i(x), f^i(x_j)) < \epsilon \quad \text{for } 1 \le j \le n, \text{ and } a_j \le i \le b_j,$$

The specification property is extremely strong, and its presence implies the presence of many other definitions of chaos. In particular it implies *Devaney chaos*, which is a commonly used notion introduced by Robert Devaney, [19].

Definition 1.20. A dynamical system (X, f) is said to be *Devaney Chaotic* if

- The set of periodic points

  $$\text{Per}(X) = \{x \in X : \text{ there exists } n \in \mathbb{N} \text{ with } f^n(x) = x.\}$$

  is dense in X.

- For any pair of open sets U and V, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \ne \varnothing$. A function satisfying this property is called *topologically transitive*.

- There exists some $\epsilon > 0$ such that for any two points $x \neq y$ in X, there exists $N \in \mathbb{N}$ such that $d(f^N(x), f^N(y)) > \epsilon$. This property is called Sensitive Dependence on Initial Conditions (SDIC).

The third condition is commonly not listed, as it was shown to be superfluous in the case where X is an infinite metric space, [4]. Now with these various definitions of chaos, an important matter to consider is the relative strength of these conditions. Some classical results that we extend to the set-valued case are shown below.

Theorem 1.21. *Let (X, f) be a dynamical system, with X being a compact metric space. If (X, f) has the specification property, then the topological entropy $h(f) > 0$, and (X, f) is Devaney chaotic.*

These definitions are well-studied, and there are many more definitions of chaos. Sylvie Ruette compiled a list of known implications for the case of $X = [0, 1]$, shown below in Figure 1.4 [50].

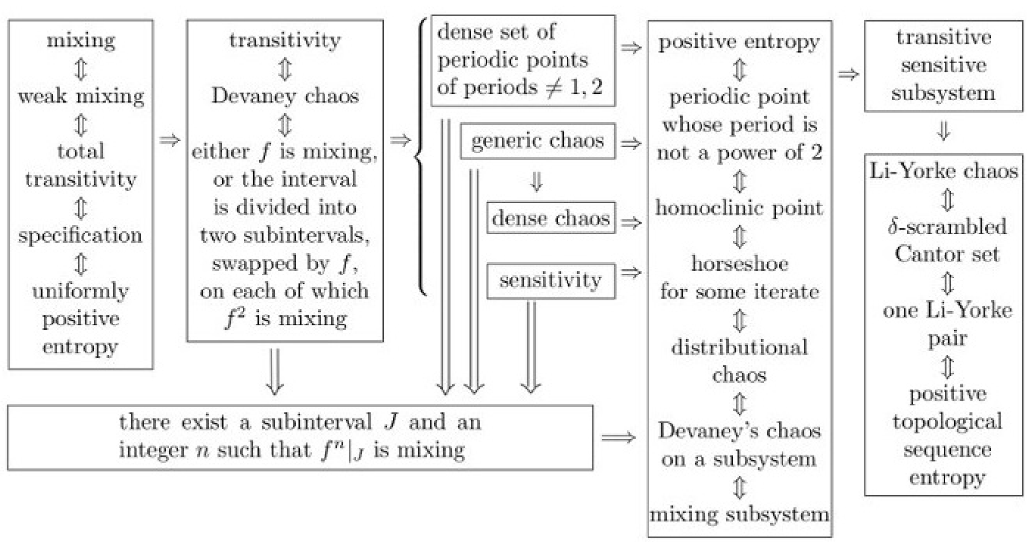

Figure 1.3. Chaotic implications of interval maps.

1.5 Inverse Limits of Set-Valued Dynamical Systems

So far, we have only considered dynamical systems (X, f) where X is a compact metric space, and $f : X \to X$ is continuous. Most of this thesis involves set-valued dynamical systems, in which the image of a point is a closed set. In this case, we replace the requirement of continuity with *upper-semi-continuity*, defined below.

Definition 1.22. Let X be a compact metric space, and let 2^X denote the collection of nonempty closed subsets of X. A function $F : X \to 2^X$ is *upper-semi-continuous*,(usc.), at $x \in X$ if for every open set $U \subset X$ which contains the set $F(x)$, there exists an open set $V \subset X$ containing x such that $F(t) \subset U$ for all $t \in V$. F is said to be usc. if it is usc. at every point.

Interest in the study of inverse limits in which the bonding maps are set-valued was renewed in 2004 when William Mahavier and Tom Ingram began publishing many interesting results and posing many questions [24, 37]. They first showed that many long-standing results have no clear analog in this new setting. For example, two results we mentioned earlier are that for a single-valued dynamical system, the factor spaces being connected implies that the inverse limit is connected, and that if the factor space is always the interval $[0, 1]$, then the inverse limit will be chainable. For a set-valued dynamical system, this is no longer the case, as there are examples of inverse limits constructed with a single bonding function $F : [0, 1] \to 2^{[0,1]}$ in which the inverse limit is not connected. After these results were published, many researchers began working on various problems in this area. We give some definitions and show some recent results below.

Van Nall gives results not attempting to find complicated structures in the inverse limit, but instead under what conditions relatively simple inverse limits exist, [44, 45].

Theorem 1.23. *Let* $([0,1], F)$ *be a surjective dynamical system. If* $\varprojlim(F)$ *is a finite graph, then* $\varprojlim(F)$ *is an arc.*

Scott Varagona has done work on determining when set-valued inverse limits are indecomposable [55, 56].

Theorem 1.24. *Suppose* $F : [0,1] \to 2^{[0,1]}$ *is usc. and there is some non-empty closed nowhere dense set* $A \subset [0,1]$ *with the following properties.*

(1) $F(a) = [0,1]$ *for all* $a \in A$*.*

(2) $F|_{[0,1]-A}$ *is an open continuous single-valued function.*

(3) For each $a \in A$*,* $y \in [0,1]$ *and* $\epsilon > 0$*:*

  (a) If there exists some $b \in [0,1]$ *with* $b > a$*, then there exists some* $x_1 \in [0,1] - A$ *such that* $x_1 \in (a, a+\epsilon)$ *and* $F(x_1) = y$*.*

  (b) If there exists some $b \in [0,1]$ *with* $b < a$*, then there exists some* $x_2 \in [0,1] - A$ *such that* $x_2 \in (a-\epsilon, a)$ *and* $F(x_2) = y$*.*

Then $\varprojlim(F)$ *is an indecomposable continuum.*

My research began with studying these set-valued inverse limits, but transitioned soon after to studying definitions of chaos for set-valued systems. We begin with topological entropy, and continue with the specification property.

CHAPTER TWO

Topological Entropy

2.1 Introduction

In 2004, Mahavier began the study of inverse limits of upper semi-continuous, set-valued functions, [37]. In recent years, there has been significant research in this area, primarily focusing on the continuum theoretic properties of these inverse limits. Many of the fundamental results concerning inverse limits of set-valued functions can be found in [23].

In this thesis, we focus on the dynamics of upper semi-continuous, set-valued functions. In this chapter, we provide a generalization of Bowen's definition of topological entropy which may be applied to set-valued functions, and we demonstrate that some well-known results extend naturally to the more general setting while others do not.

In Section 2.2 we give some background definitions and present a definition for topological entropy of a set-valued function. We then begin our discussion of the topic by exploring some properties of topological entropy which generalize naturally to set-valued functions. We then show, in Section 2.3 that the topological entropy of a set-valued function is equal to the topological entropy of the shift map on its orbit spaces. (The orbit spaces are analogous to inverse limit spaces and are defined in Section 2.2.) We also show that there is no loss of generality in assuming that the set-valued functions are surjective. In Section 2.4 we extend the notions of topological conjugacy and semi-conjugacy to set-valued functions and show that the results concerning these properties also generalize naturally to set-valued functions.

Next, we discuss some of the ways in which results concerning topological entropy of set-valued functions differ from the results in the traditional setting. In

Section 2.5, we demonstrate the relationship between the topological entropy of a set-valued function and that of its iterates. Finally, we present sufficient conditions for a set-valued function to have positive topological entropy in Section 2.6 and sufficient conditions for infinite topological entropy in Section 2.7.

2.2 Preliminary Definitions

Given a compact metric space X, we denote by 2^X the set of all non-empty compact subsets of X.

If X and Y are compact metric spaces, a function $F : X \to 2^Y$ is said to be *upper semi-continuous* at a point $x \in X$ if, for every open set $V \subseteq Y$ containing $F(x)$, there exists an open set $U \subseteq X$ containing x such that $F(t) \subseteq V$ for all $t \in U$. F is said to be *upper semi-continuous* if it is upper semi-continuous at each point of X.

The *graph* of a function $F : X \to 2^Y$ is defined to be the set

$$\Gamma(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

Ingram and Mahavier show, in [24], that if X and Y are compact Hausdorff spaces, then $F : X \to 2^Y$ is upper semi-continuous if, and only if, $\Gamma(F)$ is closed in $X \times Y$. If $f : X \to Y$, we may think of f as a set-valued function by defining a function $\tilde{f} : X \to 2^Y$ by $\tilde{f}(x) = \{f(x)\}$. In this case, $\tilde{f}$ is upper semi-continuous if and only if f is continuous. For increased distinction, we will refer to an upper semi-continuous function $F : X \to 2^Y$ as a *set-valued function* and a continuous function $f : X \to Y$ as a *mapping*.

If X, Y, and Z are compact metric spaces, $F : X \to 2^Y$ and $G : Y \to 2^Z$, we define $G \circ F : X \to 2^Z$ by

$$G \circ F(x) = \bigcup_{y \in F(x)} G(y).$$

If F and G are upper semi-continuous, then $G \circ F$ is as well.

In this thesis, we will be focusing on the setting where X is a compact metric space and $F : X \to 2^X$ is upper semi-continuous. In this case, the pair (X, F) is called a *topological dynamical system*. We define F^0 to be the identity on X, and for each $n \in \mathbb{N}$, we let $F^n = F \circ F^{n-1}$.

We begin the process of defining topological entropy for set-valued functions by defining multiple types of orbits for the system (X, F). A *forward orbit* for the system is a sequence $(x_0, x_1, x_2, \ldots)$ in X such that for each $i \geq 0$, $x_{i+1} \in F(x_i)$. A *backward orbit* is a sequence $(\ldots, x_{-2}, x_{-1}, x_0)$ in X such that for each $i \leq -1$, $x_{i+1} \in F(x_i)$. A *full orbit* is a sequence $(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$ in X such that for each $i \in \mathbb{Z}$, $x_{i+1} \in F(x_i)$.

Finally, we will also consider finite orbits. Given a natural number n, an *n-orbit* for the system (X, F) is a finite sequence $(x_0, \ldots, x_{n-1})$ in X such that for each $i = 0, \ldots, n-2$, $x_{i+1} \in F(x_i)$.

When it is clear which type of orbit we are considering, we will denote an orbit by $\mathbf{x}$. A full orbit $\mathbf{x}$ is called *periodic* if there exists $m \in \mathbb{N}$ such that $x_i = x_{i+m}$ for all $i \in \mathbb{Z}$. If $\mathbf{x}$ is periodic, the *period* of $\mathbf{x}$ is the smallest number $m \in \mathbb{N}$ for which $x_i = x_{i+m}$ for all $i \in \mathbb{Z}$.

Definition 2.1. Given a set $A \subseteq X$, and $n \in \mathbb{N}$, we define the following orbit spaces:

$$\begin{aligned}
\mathrm{Orb}_n(A, F) &= \{n\text{-orbits } (x_0, \ldots, x_{n-1}) : x_0 \in A\} \\
\overrightarrow{Orb}(A, F) &= \{\text{forward orbits } (x_0, x_1, \ldots) : x_0 \in A\} \\
\overleftarrow{Orb}(A, F) &= \{\text{backward orbits } (\ldots, x_{-1}, x_0) : x_0 \in A\} \\
\mathrm{Orb}(A, F) &= \{\text{full orbits } (\ldots, x_{-1}, x_0, x_1, \ldots) : x_0 \in A\}
\end{aligned}$$

Each of these is given the subspace topology inherited as a subset of the respective product space. Let d be the metric on X, and suppose that the diameter of X is equal to 1. For each $n \in \mathbb{N}$, we define a metric D on $\prod_{i=1}^{n} X$ by

$$D(\mathbf{x}, \mathbf{y}) = \max_{0 \leq i \leq n-1} d(x_i, y_i).$$

If $\mathbb{A} \in \{\mathbb{Z}, \mathbb{Z}_{\geq 0}, \mathbb{Z}_{\leq 0}\}$ then we define a metric ρ on $\prod_{i\in\mathbb{A}} X$ by

$$\rho(\mathbf{x}, \mathbf{y}) = \sup_{i\in\mathbb{A}} \frac{d(x_i, y_i)}{|i| + 1}.$$

Also, for any set $L \subseteq \mathbb{A}$, we define the *projection map* $\pi_L : \prod_{i\in\mathbb{A}} X \to \prod_{i\in L} X$ by $\pi_L(\mathbf{x}) = (x_i)_{i\in L}$.

In the past decade, there has been a significant amount of research concerning the inverse limits of upper semi-continuous set-valued functions. As it is typically defined, the inverse limit of the system (X, F) indexed by $\mathbb{Z}_{\geq 0}$ is equal to $\overleftarrow{Orb}(X, F)$, and the inverse limit of the system indexed by $\mathbb{Z}$ is equal to $\mathrm{Orb}(X, F)$. Also, $\overrightarrow{Orb}(X, F)$ would be equal to the inverse limit of the system (X, F^{-1}) where $F^{-1} : X \to 2^X$ is defined by $x \in F^{-1}(y)$ if, and only if, $y \in F(x)$. (Note that for F^{-1} to be well-defined, it is assumed that F is surjective, in the sense that for all $y \in X$, there exists $x \in X$ such that $y \in F(x)$.)

In the case where f is a mapping, there is less need for this distinction between the various orbit spaces. In that case, $\overleftarrow{Orb}(X, f)$ is homeomorphic to $\mathrm{Orb}(X, f)$, and, for each $n \in \mathbb{N}$, $\mathrm{Orb}_n(X, f)$ is homeomorphic to X.

We now begin our definition of topological entropy. For ease of reading, we repeat some definitions given on pages 6-7 before generalizing to set-valued functions.

Definition 2.2. Let X be a compact metric space. A set $S \subseteq X$ is called ϵ-*separated* if for each $x, y \in S$, $x \neq y$, $d(x, y) \geq \epsilon$. Let $f : X \to X$ be a mapping, and let $n \in \mathbb{N}$. We say $S \subseteq X$ is (n, ϵ)-*separated* if for $x, y \in S$ with $x \neq y$, we have that

$$\max_{0\leq i\leq n-1} d\left(f^i(x), f^i(y)\right) \geq \epsilon.$$

We denote by $s_{n,\epsilon}(f)$ the largest cardinality of an (n, ϵ)-separated set with respect to f. When there is no ambiguity, we shall use $s_{n,\epsilon}$.

Definition 2.3. Given $\epsilon > 0$, the ϵ-*entropy* of f is defined to be

$$h(f, \epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log s_{n,\epsilon},$$

and the *topological entropy* of f is defined to be

$$h(f) = \lim_{\epsilon\to 0} h(f,\epsilon).$$

To adapt this definition to the context of set-valued functions, we work in $\mathrm{Orb}_n(X,F)$ with the metric defined above, to preserve the idea of "separated" meaning separated in at least one coordinate.

Definition 2.4. Let (X,F) be a topological dynamical system, and let $n \in \mathbb{N}$ and $\epsilon > 0$. An (n,ϵ)-*separated* set for F is an ϵ-separated subset of $\mathrm{Orb}_n(X,F)$. We denote by $s_{n,\epsilon}(F)$, the largest cardinality of an (n,ϵ)-separated set with respect to F. When no ambiguity shall arise, we simply write $s_{n,\epsilon}$.

Definition 2.5. Given $\epsilon > 0$, the ϵ-*entropy* of F is defined to be

$$h(F,\epsilon) = \limsup_{n\to\infty} \frac{1}{n} \log s_{n,\epsilon},$$

and the *topological entropy* of F is defined to be

$$h(F) = \lim_{\epsilon\to 0} h(F,\epsilon).$$

Just as in the case of a mapping on X, we may give an equivalent definition using spanning sets rather than separated sets.

Definition 2.6. Let X be a compact metric space. A set $S \subseteq X$ is called ϵ-*spanning* if for each $y \in X$, there exists $x \in S$ with $d(x,y) < \epsilon$.

Let (X,F) be a topological dynamical system, and let $n \in \mathbb{N}$ and $\epsilon > 0$. An (n,ϵ)-*spanning* set for F is an ϵ-spanning subset of $\mathrm{Orb}_n(X,F)$. We denote by $r_{n,\epsilon}(F)$, the smallest cardinality of an (n,ϵ)-spanning set with respect to F. We define the *topological entropy* of F to be

$$h(F) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r_{n,\epsilon}(F).$$

The following results show that these two definitions are equivalent.

Lemma 2.7. $r_n(\epsilon) \leq s_n(\epsilon) \leq r_n(\frac{\epsilon}{2})$.

Theorem 2.8. *Let* (X, F) *be a dynamical system.* $F\colon X \to 2^X$, *the definitions of topological entropy using* (n,ϵ)*-separated sets and* (n,ϵ)*-spanning sets are equivalent, and so we have that*

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s_{n,\epsilon}(F) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r_{n,\epsilon}(F).$$

Thus, either notion may be used to define the topological entropy of F.

2.3 Topological Entropy of the Shift Map on an Orbit Space

In [15], Bowen shows that the entropy of a mapping on X is equal to the entropy of the shift map on the inverse limit space. In this section, we establish analogous results by showing that the entropy of F is equal to the entropy of the shift maps on any of the orbit spaces defined in Definition 2.1.

Theorem 2.9. *Let* (X, F) *be a topological dynamical system. If* $\sigma : \overrightarrow{Orb}(X,F) \to \overrightarrow{Orb}(X,F)$ *is the shift map defined by*

$$\sigma\left(x_0, x_1, x_2, \ldots\right) = \left(x_1, x_2, x_3, \ldots\right),$$

then $h(\sigma) = h(F)$.

Proof. Let $n \in \mathbb{N}$ and $\epsilon > 0$. We will show that $s_{n,\epsilon}(F) \leq s_{n,\epsilon}(\sigma)$. Let $S \subseteq \mathrm{Orb}_n(X,F)$ be an (n,ϵ)-separated set for F of maximal cardinality. Each n-orbit $(x_0,\ldots,x_{n-1}) \in S$ may be extended to an infinite forward orbit in $\overrightarrow{Orb}(X,F)$. Let $T \subseteq \overrightarrow{Orb}(X,F)$ be the set of all such forward orbits.

Claim: T is an (n,ϵ)-separated set for σ as defined in Definition 2.2.

To see this, let $\mathbf{x}, \mathbf{y} \in T$. Then $(x_0,\ldots,x_{n-1})$ and $(y_0,\ldots,y_{n-1})$ are in S, so $d(x_j, y_j) \geq \epsilon$ for some $0 \leq j \leq n-1$. Thus,

$$\rho\left(\sigma^j(\mathbf{x}), \sigma^j(\mathbf{y})\right) = \sup_{i\geq 0}\frac{d\left(x_{i+j}, y_{i+j}\right)}{i+1} \geq d\left(x_j, y_j\right) \geq \epsilon.$$

Thus we have that $s_{n,\epsilon}(F) \leq s_{n,\epsilon}(\sigma)$ for all $n \in \mathbb{N}$ and $\epsilon > 0$. If follows that $h(F) \leq h(\sigma)$.

Next, fix $\epsilon > 0$, and choose $k \in \mathbb{N}$ with $1/k < \epsilon$. We show that for each $n \in \mathbb{N}$, $s_{n,\epsilon}(\sigma) \leq s_{n+k,\epsilon}(F)$. Let $S \subseteq \overrightarrow{Orb}(X,F)$ be an (n,ϵ)-separated set for σ of maximal cardinality (as defined in Definition 2.2). Then, for each $\mathbf{x}, \mathbf{y} \in S$, there exists $j = 0, \ldots, n-1$ such that $\rho(\sigma^j(\mathbf{x}), \sigma^j(\mathbf{y})) \geq \epsilon$. Thus, there exists $i \in \mathbb{N}$ such that

$$\epsilon \leq \frac{d\left(x_{i+j}, y_{i+j}\right)}{i+1} \leq d\left(x_{i+j}, y_{i+j}\right).$$

Since $1/k < \epsilon$, it follows that $i+1 < k$. Thus we have that $i < k$ and $j \leq n-1$, so $i+j < n+k-1$.

Therefore, if $T = \{(x_0, \ldots, x_{n+k-1}) : \mathbf{x} \in S\}$, then T is an $(n+k,\epsilon)$-separated set for F. Moreover,

$$s_{n,\epsilon}(\sigma) = |S| = |T| \leq s_{n+k,\epsilon}(F),$$

and it follows that $h(\sigma) \leq h(F)$. □

In order to establish similar results for the shift maps on $\overleftarrow{Orb}(X,F)$ and $\mathrm{Orb}(X,F)$, we must first establish that there is no loss of generality in assuming that F is surjective. Bowen established this fact for mappings in [15].

Definition 2.10. Let X be a compact metric space, and $f : X \to X$ be a mapping. A point $x \in X$ is called *non-wandering* if for every open set $U \subseteq X$ containing x, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \varnothing$.

Theorem 2.11 (Bowen). *Let X be a compact metric space, and $f : X \to X$ be a mapping. If Ω is the set of non-wandering points then $h(f) = h(f|_\Omega)$.*

Note that if $C = \bigcap_{n\in\mathbb{N}} f^n(X)$, then C contains all the non-wandering points, so it follows from Theorem 2.11 that the entropy of f is equal to the entropy of $f|_C$. We show in the following lemma that the same holds for upper semi-continuous set-valued functions.

Lemma 2.12. *Let (X, F) be a topological dynamical system, and let $C = \bigcap_{n \in \mathbb{N}} F^n(X)$. Then $h(F) = h(F|_C)$.*

Proof. First, note that $F(C) = C$. Also, since $C = \bigcap_{n \in \mathbb{N}} F^n(X)$, it follows that

$$\overrightarrow{Orb}\,(C, F|_C) = \bigcap_{n \in \mathbb{N}} \sigma^n \left(\overrightarrow{Orb}(X, F) \right).$$

Let $\widetilde{C} = \overrightarrow{Orb}(C, F|_C)$. Since σ is a mapping, we have from Theorem 2.11 that $h(\sigma) = h(\sigma|_{\widetilde{C}})$. Then, by Theorem 2.9, we have that $h(F) = h(\sigma)$, and $h(F|_C) = h(\sigma|_{\widetilde{C}})$. The result follows. □

Theorem 2.13.

(1) *If $\sigma : \overleftarrow{Orb}(X, F) \to \overleftarrow{Orb}(X, F)$ is the shift map defined by*

$$\sigma(\ldots, x_{-2}, x_{-1}, x_0) = (\ldots, x_{-3}, x_{-2}, x_{-1})$$

then $h(\sigma) = h(F)$.

(2) *If $\sigma : \mathrm{Orb}(X, F) \to \mathrm{Orb}(X, F)$ is the shift map defined by $\sigma(\mathbf{x}) = \mathbf{y}$ where for each $i \in \mathbb{Z}$, $y_i = x_{i+1}$, then $h(\sigma) = h(F)$.*

Proof. For either shift map, σ, the same argument as in the proof of Theorem 2.9 may be used to show that $h(\sigma) \le h(F)$. Then by Lemma 2.12, we may suppose without loss of generality that F is surjective. Thus, each n-orbit for F may be extended to an infinite backward (or full) orbit, so the argument used in Theorem 2.9 may be used to show that $h(F) \le h(\sigma)$. □

Corollary 2.14. *Let (X, F) be a topological dynamical system with F surjective. Then $h(F) = h(F^{-1})$.*

Theorem 2.9 and Theorem 2.13 are significant for multiple reasons. First, all of the shift maps considered are mappings, and the shift on $\mathrm{Orb}(X, F)$ is a homeomorphism. Thus, the large volume of research on the topic of topological entropy of mappings and homeomorphisms may be applied to study the entropy of set-valued functions.

Second, there are multiple ways in which topological entropy may be defined which, in the context of mappings, are all equivalent. Theorem 2.9 and Theorem 2.13 show that any definition of topological entropy for set-valued functions which generalizes one of the definitions for topological entropy of mappings is equivalent to Definition 2.5 so long as a theorem such as Theorem 2.9 or Theorem 2.13 holds for that definition.

2.4 Topological Conjugacy and Semi-conjugacy

Another concept regarding topological entropy which generalizes to the context of set-valued functions is the notion of topological conjugacy and semi-conjugacy.

Definition 2.15. Let (X,F) and (Y,G) be topological dynamical systems. We say that G is *topologically semi-conjugate* to F if there exists a continuous surjection $\varphi : X \to Y$ such that for all $x \in X$,

$$G \circ \varphi(x) \subseteq \varphi \circ F(x).$$

The surjection φ is called a *topological semi-conjugacy* from (X,F) to (Y,G).

We say that F and G are *topologically conjugate* if there exists a homeomorphism $\varphi : X \to Y$ such that $G \circ \varphi = \varphi \circ F$. The homeomorphism φ is called a *topological conjugacy* between (X,F) and (Y,G).

The following theorems generalize well-known results regarding the topological entropy of topologically conjugate or semi-conjugate mappings (see [57, Theorem 7.2])

Theorem 2.16. *Let (X,F) and (Y,G) be topological dynamical systems. If G is topologically semi-conjugate to F, then $h(G) \leq h(F)$.*

Proof. Let $\varphi : X \to Y$ be a topological semi-conjugacy from (X,F) to (Y,G). Let $\epsilon > 0$, and choose $\delta > 0$ so that if $a, b \in X$ with $d(a,b) < \delta$, then $d(\varphi(a), \varphi(b)) < \epsilon/2$. For each $n \in \mathbb{N}$, define $\Phi_n : \mathrm{Orb}_n(X,F) \to Y^n$ by

$$\Phi_n\left(x_0, \ldots, x_{n-1}\right) = \left(\varphi\left(x_0\right), \ldots, \varphi\left(x_{n-1}\right)\right).$$

We show that for each $n \in \mathbb{N}$, $\mathrm{Orb}_n(Y,G) \subseteq \Phi_n[\mathrm{Orb}_n(X,F)]$. Let $\mathbf{y} \in \mathrm{Orb}_n(Y,G)$. Choose any $x_0 \in \varphi^{-1}(y_0)$. Now suppose that for each $0 \le i \le n-2$ and each $0 \le j \le i$, $x_j \in \varphi^{-1}(y_j)$ has been chosen such that $(x_0, x_1, \ldots, x_i) \in \mathrm{Orb}_{i+1}(X,F)$. Since

$$y_{i+1} \in G\left(y_i\right) = G \circ \varphi\left(x_i\right) \subseteq \varphi \circ F\left(x_i\right),$$

there exists $x_{i+1} \in F(x_i)$ such that $\varphi(x_{i+1}) = y_{i+1}$. In this manner, we construct an n-orbit $\mathbf{x} \in \mathrm{Orb}_n(X,F)$ such that $\Phi_n(\mathbf{x}) = \mathbf{y}$.

Fix $n \in \mathbb{N}$, and let S be an (n,δ)-spanning set for F of minimum cardinality. Let $T = \Phi_n(S)$. Then T $\epsilon/2$-spans $\mathrm{Orb}_n(Y,G)$. To see this, let $\mathbf{y} \in \mathrm{Orb}_n(Y,G)$, and choose $\mathbf{x} \in \Phi_n^{-1}(\mathbf{y})$. Since S is an (n,δ)-spanning set, there exists $\mathbf{s} \in S$ such that $D(\mathbf{s}, \mathbf{x}) < \delta$. Then $\Phi_n(\mathbf{s}) \in T$, and it follows from the choice of δ that $D(\Phi_n(\mathbf{s}), \mathbf{y}) < \epsilon/2$.

Since T is not necessarily a subset of $\mathrm{Orb}_n(Y,G)$, it may not satisfy the definition of an $(n,\epsilon/2)$-spanning set for G. However, we may use T to construct an (n,ϵ)-spanning set for G. For each $\mathbf{t} \in T$, if the D-ball centered at $\mathbf{t}$ of radius $\epsilon/2$ intersects $\mathrm{Orb}_n(Y,G)$, then choose any $\mathbf{t}'$ in that intersection. Let T' be the collection of all such points $\mathbf{t}'$, and note that $|T'| \le |T|$. It follows from the triangle inequality that T' is an (n,ϵ)-spanning set for G.

Therefore, for all $n \in \mathbb{N}$,

$$r_{n,\delta}(F) = |S| \ge |T'| \ge r_{n,\epsilon}(G).$$

It follows that $h(F) \ge h(G)$. □

If two systems are topologically conjugate, then, in particular, each is topologically semi-conjugate to the other. Hence, the following theorem follows immediately from Theorem 2.16.

Theorem 2.17. *If (X, F) and (Y, G) are topologically conjugate dynamical systems, then $h(F) = h(G)$.*

2.5 Topological Entropy of Iterates of a Set-valued Function

One result concerning topological entropy of mappings which does not always hold in the context of upper semi-continuous set-valued functions is the relationship of the entropy of a function to the entropy of its iterates. In the setting of mappings on compact metric spaces, the following result is known, (see [57, Theorem 7.10] for a proof).

Theorem 2.18. *Let X be a compact metric space, and let $f : X \to X$ be continuous. Then for all $k \in \mathbb{N}$, $h(f^k) = kh(f)$.*

This need not hold in general for upper semi-continuous set-valued functions. However, we show in Theorem 2.21 that for any topological dynamical system (X, F) and any $k \in \mathbb{N}$, $h(F) \leq h(F^k) \leq kh(F)$. We begin with the following lemma.

Lemma 2.19. *Let (X, F) be a topological dynamical system, $n \in \mathbb{N}$, $\epsilon > 0$, and S an (n, ϵ)-separated set for F. Let $k, m \in \mathbb{N}$, such that $(m-1)k < n \leq mk$, and let $L = n - (m-1)k$.*

For each $i = 0, \ldots, L-1$, let

$$A_i = \{i, i+k, i+2k, \ldots, i+(m-1)k\},$$

and for each $i = L, \ldots, k-1$, let

$$A_i = \{i, i+k, i+2k, \ldots, i+(m-2)k\}.$$

If, for each $i = 0, \ldots, k-1$, S_i is chosen to be an $\epsilon/2$-separated subset of $\pi_{A_i}(S)$ of maximal cardinality, then

$$|S| \leq \prod_{i=0}^{k-1} |S_i|$$

Proof. Define $T \subseteq X^n$ to be the set

$$T = \bigcap_{i=0}^{k-1} \pi_{A_i}^{-1}(S_i).$$

Then $|T| = \prod_{i=0}^{k-1} |S_i|$. Now, T is not necessarily a subset of S (or even of $\mathrm{Orb}_n(X, F)$) nor is S necessarily a subset of T. However, we will show that $|S| \leq |T|$ by demonstrating that $|S \setminus T| \leq |T \setminus S|$.

Suppose $\mathbf{x} \in S \setminus T$. For each $j = 0, \ldots, k-1$, consider the point $\pi_{A_j}(\mathbf{x})$, and define

$$T_j(\mathbf{x}) = \left\{ \mathbf{y} \in S_j : D\left(\mathbf{y}, \pi_{A_j}(\mathbf{x})\right) < \frac{\epsilon}{2} \right\}.$$

Since $\mathbf{x}$ is not in T, there is some $0 \leq j \leq k-1$ such that $\pi_{A_j}(\mathbf{x}) \notin S_j$, and hence $\pi_{A_j}(\mathbf{x}) \notin T_j(\mathbf{x})$. However, since S_j is the largest $\epsilon/2$-separated subset of $\pi_{A_j}(S)$, it follows that $T_j(\mathbf{x}) \neq \varnothing$ for each $0 \leq j \leq k-1$.

Now define

$$T(\mathbf{x}) = \bigcap_{i=0}^{k-1} \pi_{A_i}^{-1}\left[T_i(\mathbf{x})\right].$$

Then for each $\mathbf{z} \in T(\mathbf{x})$, $D(\mathbf{x}, \mathbf{z}) < \epsilon/2$. Hence, since $\mathbf{x} \in S$, and S is ϵ-separated, $\mathbf{z} \notin S$. Since this holds for all $\mathbf{z} \in T(\mathbf{x})$, we have that $T(\mathbf{x}) \cap S = \varnothing$. Moreover, since for each $0 \leq j \leq k-1$, $|T_j(\mathbf{x})| \geq 1$, it follows that $|T(\mathbf{x})| \geq 1$. Hence, for each point $\mathbf{x} \in S \setminus T$, there is at least one point $\mathbf{z} \in T(\mathbf{x}) \setminus S \subseteq T \setminus S$.

Finally, if $\mathbf{x}, \mathbf{y} \in S \setminus T$, then $T(\mathbf{x}) \cap T(\mathbf{y}) = \varnothing$. This is because if there were a sequence $\mathbf{z}$ in $T(\mathbf{x}) \cap T(\mathbf{y})$, then $D(\mathbf{x}, \mathbf{y}) \leq D(\mathbf{x}, \mathbf{z}) + D(\mathbf{y}, \mathbf{z}) < \epsilon$ which would contradict S being ϵ-separated. Therefore, we have that $|T \setminus S| \geq |S \setminus T|$, and the result follows.

□

Lemma 2.20. *Let (X, F) be a topological dynamical system, and let $k \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ and $\epsilon > 0$, if $m \in \mathbb{N}$ is chosen such that $(m-1)k < n \leq mk$, then*

$$s_{n,\epsilon}(F) \leq \left[s_{m,\epsilon/2}\left(F^k\right)\right]^k.$$

Proof. Let $n \in \mathbb{N}$ and $\epsilon > 0$, and fix $m \in \mathbb{N}$ such that $(m-1)k < n \leq mk$. Let S be an (n, ϵ)-separated set for F of maximal cardinality, and let $L = n - (m-1)k$. For each $i = 0, \ldots, L-1$, let

$$A_i = \{i, i+k, i+2k, \ldots, i+(m-1)k\},$$

and for each $i = L, \ldots, k-1$, let

$$A_i = \{i, i+k, i+2k, \ldots, i+(m-2)k\}.$$

For each $i = 1, \ldots, k-1$ choose S_i to be an $\epsilon/2$-separated subset of $\pi_{A_i}(S)$ of maximal cardinality. By Lemma 2.19,

$$|S| \leq \prod_{i=0}^{k-1} |S_i|.$$

Moreover, for $i = 0, \ldots, L-1$, S_i is an $(m, \epsilon/2)$-separated set for F^k, and for $i = L, \ldots, k-1$, S_i is an $(m-1, \epsilon/2)$-separated set for F^k. In either case, we have that $|S_i| \leq s_{m,\epsilon/2}(F^k)$. Therefore

$$s_{n,\epsilon}(F) = |S| \leq \prod_{i=0}^{k-1} |S_i| \leq \left[s_{m,\epsilon/2}\left(F^k\right)\right]^k.$$

□

Theorem 2.21. *Let (X, F) be a topological dynamical system, and let $k \in \mathbb{N}$. Then*

$$h(F) \leq h\left(F^k\right) \leq kh(F).$$

Proof. To show that $h(F^k) \leq kh(F)$, let $n \in \mathbb{N}$, and let S be an (n, ϵ)-separated set for F^k of maximal cardinality. For each $(x_0, \ldots, x_{n-1}) \in S$, choose $(y_0, \ldots, y_{nk-1}) \in \mathrm{Orb}_{nk}(F, X)$ such that for each $i = 1, \ldots, n-1$, $y_{ik} = x_i$, and let $\widetilde{S}$ be the set of all such nk-orbits for F.

Then $\widetilde{S}$ is an (nk, ϵ)-separated set for F with the same cardinality as S but not necessarily of maximal cardinality. It follows that

$$s_{n,\epsilon}\left(F^k\right) \le s_{nk,\epsilon}(F)$$

and hence

$$\limsup_{n\to\infty} \frac{1}{n} \log s_{n,\epsilon}\left(F^k\right) \le k \limsup_{n\to\infty} \frac{1}{nk} \log s_{nk,\epsilon}(F).$$

Therefore $h(F^k) \le kh(F)$.

To show the other inequality, note that from Lemma 2.20, if $n \in \mathbb{N}$, and $m \in \mathbb{N}$ is chosen so that $(m-1)k < n \le mk$, then

$$s_{n,\epsilon}(F) \le \left[s_{m,\epsilon/2}\left(F^k\right)\right]^k.$$

In this construction, $m \to \infty$ as $n \to \infty$, so

$$\limsup_{n\to\infty} \frac{1}{n} \log s_{n,\epsilon}(F) \le \limsup_{m\to\infty} \frac{1}{n} \log \left[s_{m,\epsilon/2}\left(F^k\right)\right]^k = \limsup_{m\to\infty} \frac{\alpha}{m} \log s_{m,\epsilon/2}\left(F^k\right)$$

where $\alpha = mk/n$.

It follows from the inequality, $(m-1)k < n \le mk$ that $\alpha \to 1$ as $n \to \infty$. Hence, we have that $h(F) \le h(F^k)$.

□

Corollary 2.22. *Let (X, F) be a topological dynamical system, and let $k \in \mathbb{N}$. Then the following hold.*

(1) $h(F) = 0$ if, and only if, $h(F^k) = 0$.

(2) $h(F) = \infty$ if, and only if, $h(F^k) = \infty$.

(3) $0 < h(F) < \infty$ if, and only if, $0 < h(F^k) < \infty$.

The inequality $h(F) \le h(F^k) \le kh(F)$ is most interesting when the entropy of F is positive and finite. From Theorem 2.18, we have that for any mapping f,

$h(f^k) = kh(f)$ for all $k \in \mathbb{N}$. Next, we give an example of two set-valued functions on the two element set $\{0,1\}$: one where $h(F^2) = h(F)$, and one where $h(F) < h(F^2) < 2h(F)$.

Example 2.23. Let $X = \{0,1\}$.

(1) Let $F : X \to 2^X$ be defined by $F(0) = \{1\}$, and $F(1) = \{0,1\}$. Then $h(F) = \log\varphi$, where $\varphi = (1+\sqrt{5})/2$, and $h(F^2) = \log 2$.

(2) Let $G : X \to 2^X$ be defined by $G(0) = G(1) = \{0,1\}$. Then for all $k \in \mathbb{N}$, $h(G^k) = h(G) = \log 2$.

Proof. Note that if $0 < \epsilon < 1$, then for all $n \in \mathbb{N}$, the entire space of n-orbits is an (n,ϵ)-separated set (for F and G respectively).

For F, the sequence $(s_{n,\epsilon})_{n=1}^{\infty}$ is the Fibonacci sequence beginning with $(2,3)$. Thus, $s_{n,\epsilon} \approx 5^{-1/2}\varphi^{n+2}$, and we have that $h(F) = \log\varphi$.

Now $F^2(0) = F^2(1) = \{0,1\}$, so $\mathrm{Orb}(X,F^2) = \{0,1\}^{\mathbb{Z}}$, and the entropy of the shift on this space is known to be $\log 2$. Thus $h(F^2) = \log 2$ which is strictly between $h(F)$ and $2h(F)$.

Note that $G = F^2$, so we have that $h(G) = \log 2$. Also, for any $k \in \mathbb{N}$, $G^k = G$, so, in particular, $h(G^k) = h(G)$. □

In this example, we had that $G^k = G$ for all $k \in \mathbb{N}$. This is not necessary, however, for their entropies to be equal. In the following example we present a function $F : [0,1] \to 2^{[0,1]}$ for which $F^2 \neq F$ but $h(F^2) = h(F)$. (The inverse limits of F and F^2 are discussed in [24, Example 4].)

Example 2.24. Let $I = [0,1]$, and let $F : I \to 2^I$ be defined by

$$F(x) = \begin{cases} \left\{x + \frac{1}{2}, \frac{1}{2} - x\right\} & x \leq \frac{1}{2} \\ \left\{x - \frac{1}{2}, \frac{3}{2} - x\right\} & x \geq \frac{1}{2} \end{cases}$$

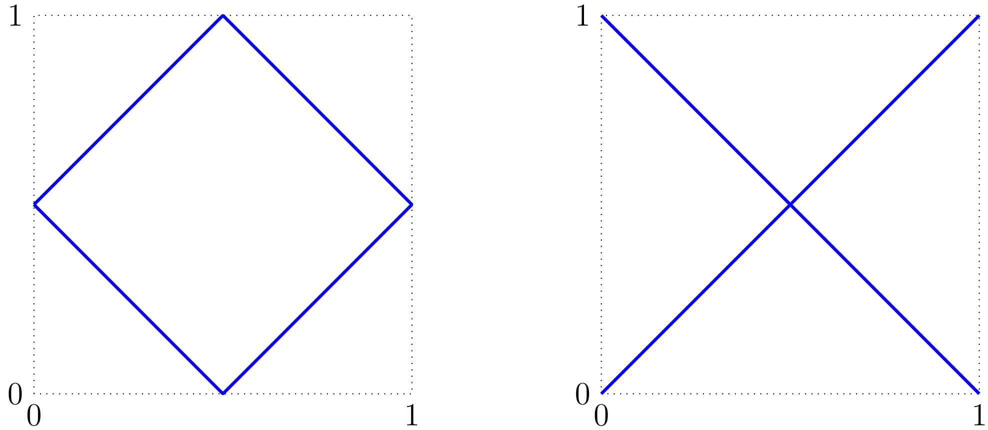

Figure 2.1. Set-valued function F (left) and F^2 (right) from Example 2.24

Then, $F^2 \neq F$, but $h(F^2) = h(F) = \log 2$. (The graphs of F and F^2 are pictured in Figure 2.1.)

Proof. For each $0 < \epsilon < 1/4$, let A_ϵ be the largest ϵ-separated subset of the set

$$\left[0 + \frac{\epsilon}{2}, \frac{1}{2} - \frac{\epsilon}{2}\right] \cup \left[\frac{1}{2} + \frac{\epsilon}{2}, 1 - \frac{\epsilon}{2}\right].$$

Note that the cardinality of A_ϵ is no more than three less than the largest cardinality for an ϵ-separated subset of I.

Moreover, for each $a \in A_\epsilon$, $F(a)$ contains exactly two points, and those points are at least ϵ apart from each other. It follows that for each $n \in \mathbb{N}$,

$$|A_\epsilon| 2^n \leq s_{n,\epsilon}(F) \leq (|A_\epsilon| + 3)\, 2^n,$$

and thus, $h(F) = \log 2$.

A similar argument shows that that $h(F^2) = \log 2$. □

2.6 Positive Topological Entropy

Each of the examples from Section 2.5 illustrates functions with positive topological entropy, where the positive entropy may be witnessed on any compact subset.

An interesting question is to determine "minimal" conditions for a set-valued function to have positive entropy. In this section, we establish conditions which are sufficient for a set-valued function to have positive entropy, and we demonstrate that set-valued functions satisfying these conditions may exhibit seemingly minimal chaotic behavior.

We also discuss the relationship between periodicity and positive topological entropy. A mapping on $[0,1]$ has positive topological entropy if, and only if, it has a periodic point whose period is not a power of 2. We demonstrate that this equivalence does not hold for set-valued functions on the interval.

We begin with sufficient conditions for a set-valued function to have positive topological entropy.

Proposition 2.25. *Let (X,F) be a topological dynamical system. Let $a,b \in X$, with $a \neq b$. If $\{a,b\} \subseteq F(a)$ and $\{a,b\} \subseteq F(b)$, then $h(F) \geq \log 2$.*

Proof. For each $n \in \mathbb{N}$ and each $0 < \epsilon < d(a,b)$, the set $\{a,b\}^n \subseteq \mathrm{Orb}_n(X,F)$ is an (n,ϵ)-separated set. Thus, $s_{n,\epsilon} \geq 2^n$. It follows that $h(F) \geq \log 2$. □

Under the assumptions of Proposition 2.25, a has two distinct periodic orbits, $(a,a,a,\ldots)$ and $(a,b,a,b,\ldots)$. The next theorem generalizes Proposition 2.25 by focusing on this property.

In this theorem, given two finite sequences $\mathbf{u} = (u_i)_{i=0}^n$ and $\mathbf{v} = (v_i)_{i=0}^n$, we define $\mathbf{uv}$ to be the sequence $(u_0,\ldots,u_n,v_0,\ldots,v_n)$. We also define a *finite word of length* m from $\{\mathbf{u},\mathbf{v}\}$ to be a sequence of the form $\mathbf{a_1 a_2 \ldots a_m}$ where for each $1 \leq j \leq m$, $\mathbf{a_j} \in \{\mathbf{u},\mathbf{v}\}$.

Theorem 2.26. *Let (X,F) be a topological dynamical system. Suppose there exists a point $p \in X$ and two distinct periodic orbits $\mathbf{a}$ and $\mathbf{b}$ such that $a_0 = b_0 = p$. Then $h(F) > 0$.*

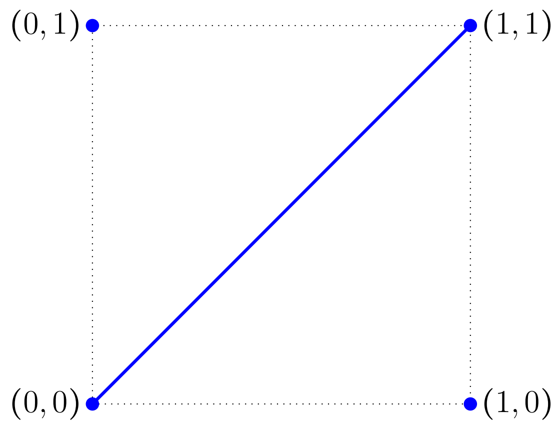

Figure 2.2. Set-valued function from Example 2.27.

Proof. Let m be the period of $\mathbf{a}$, let k be the period of $\mathbf{b}$, and let l be the least common multiple of m and k. Let $\mathbf{u} = (a_0, \ldots, a_{l-1})$, and let $\mathbf{v} = (b_0, \ldots, b_{l-1})$. Note that $p \in F(a_{l-1})$ and $p \in F(b_{l-1})$, so any finite word from $\{\mathbf{u}, \mathbf{v}\}$ is a finite orbit for F. Also, since $\mathbf{a}$ and $\mathbf{b}$ are not equal, neither are $\mathbf{u}$ and $\mathbf{v}$, so there exists $0 \le j \le l-1$ such that $u_j \neq v_j$.

For each $n \in \mathbb{N}$, let S_n be the set of all finite words of length n from $\{\mathbf{u}, \mathbf{v}\}$. Then S_n consists of nl-orbits. Moreover, if $0 < \epsilon < d(u_j, v_j)$, then S_n is an (nl, ϵ)-separated set, and $|S_n| = 2^n$. It follows that $s_{nl,\epsilon} \ge 2^n$, and hence, $h(F) \ge (\log 2)/l > 0$. □

Example 2.27. Let $I = [0,1]$, and let $F : I \to 2^I$ be defined by $F(x) = \{x\}$ for $0 < x < 1$, and $F(0) = F(1) = \{0,1\}$ (pictured in Figure 2.2). Then, according to Proposition 2.25, $h(F) > 0$, and, in fact, $h(F) = \log 2$.

What makes Example 2.27 interesting is that the positive entropy is witnessed on the nowhere dense set $\{0,1\}$. Our next two results illustrate that such a thing with continuous set-valued functions.

The following proposition can be found within the proof of a theorem due to Jaquette [26]. We state the result in a slightly different way than how it appears in [26], so we include a proof.

In Proposition 2.28 and Theorem 2.30, we will use the following notation:

If (X, F) is a topological dynamical system, and $Z \subseteq X$, then for each $n \in \mathbb{N}$ and $\epsilon > 0$, we define $s_{n,\epsilon}(Z, F)$ to be the largest cardinality of an ϵ-separated subset of $\mathrm{Orb}_n(Z, F) = \{\mathbf{x} \in \mathrm{Orb}_n(X, F) : x_0 \in Z\}$.

Proposition 2.28. *Let X be a compact metric space, and let $f : X \to X$ be continuous. If Z is a dense subset of X, then*

$$h(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_{n,\epsilon}(Z, f).$$

Proof. By definition,

$$h(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_{n,\epsilon}(X, f),$$

so it suffices to show that for each $n \in \mathbb{N}$ and $\epsilon > 0$,

$$s_{n,\epsilon}(Z, f) \leq s_{n,\epsilon}(X, f) \leq s_{n,\epsilon/2}(Z, f).$$

Since $Z \subseteq X$, it follows that $s_{n,\epsilon}(Z, f) \leq s_{n,\epsilon}(X, f)$. It remains to show the other inequality.

Recall that $\mathrm{Orb}_n(X, f)$ has the metric D defined by $D(\mathbf{x}, \mathbf{y}) = \max\{d(x_i, y_i) : 0 \leq i \leq n-1\}$ for $\mathbf{x}, \mathbf{y} \in \mathrm{Orb}_n(X, f)$. Since f is continuous, the projection map $\pi_0 : \mathrm{Orb}_n(X, f) \to X$ is a homeomorphism. Thus, since Z is dense in X, it follows that $\mathrm{Orb}_n(Z, f)$ is dense in $\mathrm{Orb}_n(X, f)$.

Let $n \in \mathbb{N}$ and $\epsilon > 0$, and let $S \subseteq \mathrm{Orb}_n(X, f)$ be an (n, ϵ)-separated set of maximal cardinality for f. Since $\mathrm{Orb}_n(Z, f)$ is dense in $\mathrm{Orb}_n(X, f)$, for each $\mathbf{x} \in S$, we may choose $\widetilde{\mathbf{x}} \in \mathrm{Orb}_n(Z, f)$ such that $D(\mathbf{x}, \widetilde{\mathbf{x}}) < \epsilon/4$. Let $\widetilde{S} = \{\widetilde{\mathbf{x}} : \mathbf{x} \in S\}$.

Then, for each $\mathbf{x}, \mathbf{y} \in S$ with $\mathbf{x} \neq \mathbf{y}$, we have that

$$\begin{aligned} D(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) &\geq D(\mathbf{x}, \mathbf{y}) - D(\mathbf{x}, \widetilde{\mathbf{x}}) - D(\mathbf{y}, \widetilde{\mathbf{y}}) \\ &> \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{4} \end{aligned}$$

$$= \frac{\epsilon}{2}.$$

It follows that $|S| = |\widetilde{S}|$ and that $\widetilde{S}$ is an $(n, \epsilon/2)$-separated set for f. Moreover, since $\widetilde{S} \subseteq \mathrm{Orb}_n(Z, f)$, we have that

$$s_{n,\epsilon}(X, f) = |S| = |\widetilde{S}| \leq s_{n,\epsilon/2}(Z, f),$$

and the result follows. □

Example 2.27 illustrates that this result does not hold in general for upper semi-continuous set-valued functions. However, we show in Theorem 2.30 that it does hold for set-valued functions which are continuous with respect to the Hausdorff metric which we define now.

Definition 2.29. Let X be a compact metric space with metric d. Given a point $x \in X$ and $\epsilon > 0$, let $B(x, \epsilon)$ represent the ball of radius ϵ centered at x. We define the *Hausdorff metric*, $\mathcal{H}_d$, on 2^X as follows: if $C, D \in 2^X$,

$$\mathcal{H}_d(C, D) = \inf\left\{\epsilon > 0 : D \subseteq \bigcup_{c \in C} B(c, \epsilon), \text{ and } C \subseteq \bigcup_{d \in D} B(d, \epsilon)\right\}$$

Theorem 2.30. *Let (X, F) be a topological dynamical system such that $F : X \to 2^X$ is continuous with respect to the Hausdorff metric on 2^X. If Z is a dense subset of X, then*

$$h(F) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_{n,\epsilon}(Z, F).$$

Proof. From Theorem 2.9, we have that the entropy of F is equal to the entropy of the shift map σ on $\overrightarrow{Orb}(X, f)$. Thus, since σ is a mapping, in light of Proposition 2.28, it suffices to show that $\overrightarrow{Orb}(Z, f)$ is dense in $\overrightarrow{Orb}(X, f)$.

Recall that $\prod_{i=0}^{\infty} X$ has the metric ρ defined for $\mathbf{x}, \mathbf{y} \in \overrightarrow{Orb}(X, F)$ by

$$\rho(\mathbf{x}, \mathbf{y}) = \sup_{i \geq 0} \frac{d(x_i, y_i)}{i + 1}.$$

Define $\widehat{F} : X \to 2^{\Pi X}$ by $\widehat{F}(x) = \overrightarrow{Orb}(x, F)$. Then, $\widehat{F}$ is continuous with respect to the Hausdorff metric $\mathcal{H}_\rho$ on $2^{\Pi X}$. Thus, for any $\mathbf{x} \in \overrightarrow{Orb}(X, F)$ and $\epsilon > 0$, we may choose $\delta > 0$ to witness the continuity of $\widehat{F}$ at x_0. Since Z is dense in X, there exists $t \in Z$ such that $d(x_0, t) < \delta$. Then

$$\mathcal{H}_\rho \left[\overrightarrow{Orb}(x_0, F), \overrightarrow{Orb}(t, F) \right] < \epsilon,$$

so there exists $\mathbf{y} \in \overrightarrow{Orb}(t, F) \subseteq \overrightarrow{Orb}(Z, F)$ such that $\rho(\mathbf{x}, \mathbf{y}) < \epsilon$. □

For mappings on the interval $[0,1]$ we have the following two results concerning periodicity.

Theorem 2.31 (Šarkovs′kiĭ [51]). *Define the relation* $\prec$ *on* $\mathbb{N}$ *by*

$$3 \prec 5 \prec 7 \prec \cdots \prec 3 \cdot 2 \prec 5 \cdot 2 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

If $f : [0,1] \to [0,1]$ *is continuous, and has a periodic point of period* $n \in \mathbb{N}$*, then it has a periodic point of period* m*, for all* $n \prec m$*.*

We also have the following result which relates periodicity to positive topological entropy. A proof may be found in [30, Section 15.3]

Theorem 2.32. *Let* $f : [0,1] \to [0,1]$ *be continuous. Then* $h(f) = 0$ *if, and only if, the period of every periodic point is a power of* 2.

The following example illustrates that neither of these results necessarily hold for set-valued functions on the interval.

Example 2.33. Let $F : [0,1] \to 2^{[0,1]}$ be defined by $F(x) = \{0\}$ for all $x \neq 1/3, 2/3, 1$, $F(1/3) = \{0, 2/3\}$, $F(2/3) = \{0, 1\}$, and $F(1) = \{0, 1/3\}$. Then F has three periodic orbits of period three and a fixed point but no other periodic orbits. Moreover, $h(F) = 0$.

2.7 Infinite Topological Entropy and the Structure of Orbit Spaces

Finally, we explore the concept of infinite topological entropy and its relationship to the structure of the orbit spaces. We begin by presenting sufficient conditions for a set-valued function to have infinite topological entropy. We then consider set-valued functions on $[0,1]$ for which the image and inverse image of a point is connected. We present in Example 2.37 such a function whose entropy is zero, yet whose forward orbit space contains a Hilbert cube (a countable product of non-degenerate closed intervals).

Theorem 2.34. *Let (X,F) be a topological dynamical system. If there exists an infinite set $A \subseteq X$ such that for all $a \in A$, $F(a) \supseteq A$, then $h(F) = \infty$.*

Proof. For each $\epsilon > 0$, choose A_ϵ to be an ϵ-separated subset of A of maximum cardinality, and let $\alpha(\epsilon) = |A_\epsilon|$. Since for each $a \in A$, $A \subseteq F(a)$, we have that for each $n \in \mathbb{N}$, $A^n \subseteq \mathrm{Orb}_n(X,F)$. In particular, A_ϵ^n is a subset of $\mathrm{Orb}_n(X,F)$ and is ϵ-separated. Therefore, $s_{n,\epsilon} \geq [\alpha(\epsilon)]^n$ which implies that $h(F,\epsilon) \geq \log \alpha(\epsilon)$.

Since A is an infinite set, $\alpha(\epsilon) \to \infty$ as $\epsilon \to 0$, so $h(F) = \infty$. □

Corollary 2.35. *Let (X,F) be a topological dynamical system. If there exists an infinite set $A \subseteq X$ and a $k \in \mathbb{N}$ such that for all $a \in A$, $F^k(a) \supseteq A$, then $h(F) = \infty$.*

Proof. By Theorem 2.34, we have that $h(F^k) = \infty$, so from Corollary 2.22, it follows that $h(F) = \infty$. □

For a set-valued function satisfying the hypotheses of either Theorem 2.34 or Corollary 2.35, its forward orbit space would contain a copy of $A^{\mathbb{N}}$. It is crucial however that this is a countable product of one infinite set. We demonstrate in Example 2.37 that an orbit space may contain a countable product of infinite sets while the set-valued function has zero entropy.

Before Example 2.37 we define what is meant by a monotone set-valued function.

Definition 2.36. A function $F : X \to 2^X$ is called *monotone* if for each $x \in X$, $F(x)$ and $F^{-1}(x)$ are each connected.

Recall that a compact, connected, metric space is called a *continuum*, and that a continuum in which every proper subcontinuum is nowhere dense is called *indecomposable*.

Barge and Diamond prove in [6] that if f is a piece-wise monotone mapping on a finite graph G, then $h(f) > 0$ if and only if $\mathrm{Orb}(G, f)$ contains an indecomposable subcontinuum. Example 2.37 demonstrates that this does not hold in general for set-valued functions.

Example 2.37. Let $F : [0,1] \to 2^{[0,1]}$ be the monotone function defined for each $x \in [0,1]$ by $F(x) = [0, x]$. Then $\overrightarrow{Orb}([0,1], F)$ contains copies of the Hilbert cube, and $h(F) = 0$.

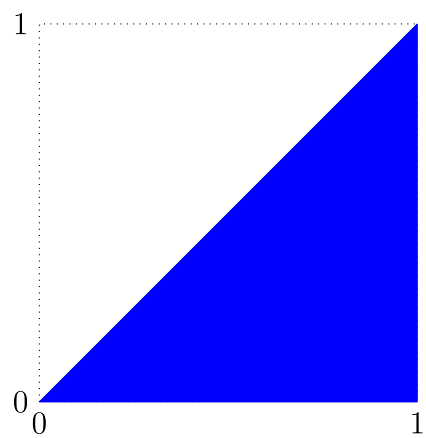

Figure 2.3. Set-valued Function from Example 2.37

Proof. First, note that, in particular, $\overrightarrow{Orb}([0,1], F)$ contains the Hilbert cube

$$\prod_{i=1}^{\infty}\left[\frac{1}{2^i}, \frac{1}{2^{i-1}}\right].$$

To show that $h(F) = 0$, we show that $h(\sigma) = 0$ where σ is the shift map on $\overrightarrow{Orb}([0,1], F)$. First, we claim that the set of non-wandering points is equal to the set of constant sequences (i.e. fixed points for σ). To see this, let $\mathbf{x} \in \overrightarrow{Orb}([0,1],F)$, and suppose that $\mathbf{x}$ is not fixed by σ. Then there exists some $j \in \mathbb{N}$, such that $x_{j+1} \neq x_j$. From the definition of F, it follows that $x_{j+1} < x_j$, and, for all $i > j$, $x_i \leq x_{j+1} < x_j$.

Fix disjoint intervals I_1 and I_2 such that $x_j \in I_1$ and $x_{j+1} \in I_2$, and let $U = \pi_j^{-1}(I_1) \cap \pi_{j+1}^{-1}(I_2)$. Then $\sigma^i(U)$ is disjoint from U for all $i \in \mathbb{N}$. Hence, the only non-wandering points are the fixed points, so σ restricted the non-wandering points is the identity. Thus, by Theorem 2.11, $h(\sigma) = 0$. □

CHAPTER THREE

Specification Property

3.1 Introduction

In this chapter we continue our exploration of the dynamics of set-valued functions and their inverse limit spaces. For ease of reading, we list some definitions from previous chapters here. For the duration of this chapter, X will denote a compact metric space and 2^X will denote the hyperspace of nonempty closed subsets of X. Let $F : X \to 2^X$ be an upper semi-continuous function. We call F a *set-valued function.* The *forward orbit space induced by* F is the space

$$\overrightarrow{Orb}(X,F) = \{(x_j)_{j=0}^{\infty} \in X^{\mathbb{N}} : x_{i+1} \in F(x_i)\}$$

considered as a subspace of the Tychonoff product $X^{\mathbb{N}}$. This space is similar to the *inverse limit* of (X,F), which is the space of backward orbits.

$$\varprojlim(X,F) = \overleftarrow{Orb}(X,F) = \{(x_j)_{j=0}^{\infty} \in X^{\mathbb{N}} : x_i \in F(x_{i+1}, \text{ for all } i \in \mathbb{N}\}$$

Associated with these orbit spaces is a *shift map*

$$\sigma(x_0, x_1, \dots) = (x_1, x_2 \dots).$$

This is a continuous well-defined function which mimics the dynamics of F on X. Of course, the trade-off for well-definedness is the inherently more complicated topology of the domain space, $\varprojlim F$. These mappings and their inverse limits have arisen in several applications of topological dynamics to economics, [32, 49].

In section 2 of this chapter, we extend the definition of the specification property from the usual single-valued function setting to the setting of set-valued mappings and recall the definition of topological entropy. In section 3, we prove that, as in the single-valued case, specification implies topological mixing and positive entropy for usc. set-valued functions. We also extend the notion of shadowing to the set-valued case. We show that if X is compact and connected and F has shadowing and a dense set of periodic points then it also has a slightly weaker version of the specification property (as is known in the single-valued setting [2].) We then give a few results on the dynamics induced on σ, specifically, we show that if F has the specification property then so does σ. We end the chapter with a few results regarding invariant measures for set-valued mappings, which is the focus of chapter four. This is closely related to results from [31].

Recall that the sequence $(x, f(x), f^2(x), \ldots)$ is the *orbit* of x under f. If there exists $n \in \mathbb{N}$ such that $f^n(x) = x$, we call x *periodic*, with period n.

We begin with a few simple extensions of definitions from the single-valued case. Notice that since F is a set-valued mapping, orbits of F are no longer uniquely determined by their initial condition.

Definition 3.1. An *orbit* of a point $x \in X$ is a sequence $(x_i)_{i=0}^{\infty}$ such that $x_{i+1} \in F(x_i)$ and $x_0 = x$.

Definition 3.2. Let $x \in X$. Let $(x_i)_{i=0}^{\infty}$ be an orbit of x. The orbit is said to be *periodic* if there is some $n \in \mathbb{N}$ such that $(x_i)_{i=0}^{\infty} = (x_j)_{j=n}^{\infty}$. The smallest such n is called the *period* of the orbit.

Note that it is not necessarily the case that if there is some $j \in \mathbb{N}$ such that $x_0 = x_j$, then the orbit $(x_j)_{j=0}^{\infty}$ is periodic.

Definition 3.3. The *fibre* of a point x, denoted $\overrightarrow{Orb}(x)$, is the collection of all orbits of x. In chapter 4, particular detail will be paid to these sets.

3.2 The Specification Property

Definition 3.4. We say that the dyamical system (X, F) has the *specification property* if, for any $\epsilon > 0$, there exists $M \in \mathbb{N}$ dependent only on ϵ such that, for any $x^1, \ldots, x^n \in$

X, any $a_1 \leq b_1 < \ldots < a_n \leq b_n$ with $a_{i+1} - b_i > M$, and any orbits $(x^i_j)_{j=0}^{\infty}$, and for any $P > M + b_n - a_1$, there exists a point z that has an orbit $(z_j)_{j=0}^{\infty}$ such that $d(z_j, x^i_j) < \epsilon$ for $i \in \{1, \ldots, n\}$ and $a_i \leq j \leq b_i$, and $z_P = z$.

Note that this definition requires F to be surjective, in the sense that for any $y \in X$, there exists $x \in X$ such that $y \in F(x)$.

Definition 3.5. We say that the dynamical system (X, F) is *topologically mixing* if, for any non-empty open U and V in X, there is an $M \in \mathbb{N}$ such that for any $m > M$ there is an $x \in U$ with an orbit $(x_j)_{j=0}^{\infty}$ with $x_m \in V$.

The notion of topological entropy has been studied extensively since it was introduced in 1965. Positive entropy is a strong indicator of topological chaos. We recall the definition from Chapter Two.

Definition 3.6. Let (X, F) be a dynamical system. The *topological entropy* of F is

$$h(F) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log(r_n(\epsilon)),$$

where $r_n(\epsilon)$ is the minimal cardinality of an (n, ϵ) spanning set.

Recall also that it is possible to define topological entropy for a set-valued function using $s_n(\epsilon)$ instead of $r_n(\epsilon)$. In that case the topological entropy of F is given by

$$h(F) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log(s_n(\epsilon)).$$

3.3 Results for Set-Valued Dynamical Systems.

We begin by considering the implications of the specification property for a set-valued mapping, F.

Theorem 3.7. *Let (X, F) be a dynamical system. If $F : X \to 2^X$ is an usc. set-valued mapping that has the specification property, then F has topological mixing.*

Proof. Let U and V be non-empty open sets. Let $x \in U$, $y \in V$. Let $\epsilon > 0$ be chosen such that $B_\epsilon(x) \subset U$ and $B_\epsilon(y) \subset V$. Let $M \in \mathbb{N}$ witness the specification property for this ϵ. Let $y^1 \in X$ be chosen such that there is an orbit $(y_j^1)_0^\infty$ with $y_{M+1}^1 = y$. Then for any orbit $(x_i)_0^\infty$ of x there is a point z^1 such that for $a_1 = b_1 = 0$, and $a_2 = b_2 = M + 1$, there is a point z^1 that has an orbit $(z_i^1)_0^\infty$, such that

$$d(z^1, x) < \epsilon \quad \text{and} \quad d(z_{M+1}, y_{M+1}^1) < \epsilon.$$

Thus $z^1 \in U$, and $z_{M+1}^1 \in V$. Now let $y^m \in X$ such that there is an orbit $(y_j^m)_{j=0}^\infty$ with $y_{M+m}^m = y$. Then for $a_1 = b_1 = 0$, and $a_2 = b_2 = M + m$, there is a point z^m with an orbit $(z_j^2)_0^\infty$ such that

$$d(z^2, x) < \epsilon \quad \text{and} \quad d(z_{M+m}^2, y_{M+m}^m) < \epsilon.$$

Then $z^m \in U$, and $z_{M+m}^m \in V$. □

Lemma 3.8. *For $\epsilon_1 < \epsilon_2$, $s_n(\epsilon_1) \leq s_n(\epsilon_2)$.*

Theorem 3.9. *Let (X, F) be a dynamical system. If $F: X \to 2^X$ is an usc. set-valued mapping that has the specification property, then F has positive entropy.*

Proof. Let $x, y \in X$. Let $\epsilon > 0$ such that $d(x, y) > 3\epsilon$. Let M witness the specification property for this ϵ. Let $z^1, \ldots, z^n$ be chosen such that $z^i \in \{x, y\}$ for some fixed n. Let $(z_j^i)_{j=0}^\infty$ be an orbit of each z^i. Let $a_i = b_i = (i-1)M$. Then there is some z that has an orbit $(z_j)_{j=0}^\infty$ such that $d(z_{(i-1)M}, z_{(i-1)M}^i) < \epsilon$. Let $\hat{z}^1, \ldots, \hat{z}^n$ be a different such choice of x's and y's. Then for orbits $(\hat{z}_j^i)_{j=0}^\infty$ there is some $\hat{z}$ (possibly equal to z) that has an orbit $(\hat{z}_j)_{j=0}^\infty$ that follows the $\hat{z}^i$s. Now, there is some $i \in \{1, \ldots, n\}$ such that $\hat{z}^i \neq z^i$. Then $d(z_{(i-1)M}, \hat{z}_{(i-1)M}) > \epsilon$, for that fixed i. Thus there are 2^n many (nM, ϵ)-separated orbits. Then for a fixed ϵ,

$$\limsup_{n\to\infty} \frac{1}{n} \log(s_n(\epsilon)) \geq \limsup_{n\to\infty} \frac{1}{nM} \log 2^n = \frac{1}{M} \log 2.$$

Thus by Lemma 3.9, F has positive entropy. □

Whenever a chaotic property such as specification is introduced, a natural question is to determine sufficient conditions for it to appear. The following theorem gives a sufficient condition to get a slightly weaker property than specification. First, we introduce the notion of shadowing.

Definition 3.10. A sequence $(x_i)_{i=0}^{\infty}$ is called a *δ-pseudo-orbit* if $d(F(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{N}$.

Definition 3.11. Let $F\colon X \to 2^X$ be a set-valued map. We say F has *shadowing* if, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $(x_i)_{i=0}^{\infty}$, there exists a point $z \in X$ with an orbit $(z_i)_0^{\infty}$ such that $d(z_i, x_i) < \epsilon$ for all $i \in \mathbb{N}$.

Theorem 3.12. *Let (X, F) be a dynamical system. If X is a continuum, and F has shadowing and a dense set of points that each have at least one periodic orbit, Then for any $\epsilon > 0$, there is an $M \in \mathbb{N}$ such that for any $x^1, \ldots, x^n \in X$, any $a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_n \leq b_n$, any orbits $(x_j^i)_{j=0}^{\infty}$, there is a point $z \in X$ with a orbit $(z_j)_{j=0}^{\infty}$ such that $d(z_j, x_j^i) < \epsilon$ for $1 \leq i \leq n$, $a_i \leq j \leq b_i$.*

Proof. Let $\epsilon > 0$. Let δ witness shadowing for this ϵ. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a finite cover of X with diameter of A_i less than $\frac{\delta}{2}$, and centered at points $q^i \in A_i$ such that q^i has at least one periodic orbit, and denote the length of that orbit by p_i. Let M_i be the collection of sums of i many elements from the list of $\{p_1, \ldots, p_m\}$, i.e.,

$$M_i = \{p_{j_1} + p_{j_2} + \ldots + p_{j_i} : p_{j_k} \in \{p_1, \ldots, p_n\},\ \text{for } 1 \leq k \leq i\}.$$

Let $M_0 = \prod_1^m M_i$, and $M = 2M_0$. This will be the M witnessing specification for the given ϵ.

Let $x^1, \ldots, x^n$ be points of X. Let $a_1 \leq b_1 < a_2 \leq \ldots \leq b_n$, with $a_{i+1} - b_n > M$. Let $(x_j^i)_{j=0}^{\infty}$ be orbits of these points.

We build a pseudo-orbit that follows the orbits of the points x^i, and then invoke shadowing. Let r_i and c_i' be natural numbers such that $a_{i+1} - b_i = c_i' M_0 + r_i$, with $0 \le r_i < M_0$, for $1 \le i \le n$. Note that as $a_{i+1} - b_i > M$, c_i' is at least 2.

It is important to note that for any two points u, v in X, there is a finite sequence $s_1, \dots, s_r$ with $r \le m$, where m is the cardinality of $\mathcal{A}$, given above, such that

$$d(s_i, s_{i+1}) < \delta, \quad d(u, s_1) < \delta, \quad d(s_r, v) < \delta,$$

and $s_i \in \{q_1, \dots, q_m\}$ for each $1 \le i \le r$. So for each pair $x^i_{b_i + r_i}$, $x^{i+1}_{a_{i+1}}$, choose a finite sequence $s^i_1, \dots, s^i_{\eta_i}$ such that $d(s^i_j, s^i_{j+1}) < \delta,$, $d(x^i_{b_i+r_i}, s^i_1) < \delta$, and $d(s^i_{\eta_i}, x^{i+1}_{a_{i+1}}) < \delta$. Then each s^i_j has a periodic orbit of length p^i_j, with $p^i_j = p_n$, for some $1 \le n \le m$. Let $(s^i_{k,j})^\infty_{j=0}$ be that periodic orbit, and so we have $s^i_{k,p^i_k} = s^i_{k,0} = s^i_k$, for each $1 \le i \le n$, and $1 \le j \le \eta_i$.

Let $c_i = \dfrac{c_i' M_0}{\sum_{k=0}^{\eta_i} p^i_k}$. Now we are ready to define our pseudo-orbit. Let $(w_k)^\infty_{k=0}$ be defined such that

$$w_k = \begin{cases} x^1_k & 0 \le k < b_1 + r_1 \\ s^1_{1,k} & b_1 + r_1 \le k < b_1 + r_1 + c_1 p^1_1 \\ s^1_{2,k} & b_1 + r_1 + c_1 p^1_1 \le k < b_1 + r_1 + c_1(p^1_1 + p^1_2) \\ \vdots & \vdots \\ s^1_{\eta_1,k} & b_1 + r_1 + c_1 \sum_{j=0}^{\eta_1 - 1} p^1_j \le k < b_1 + r_1 + c_1 \sum_{j=0}^{\eta_1} p^1_j \\ x^2_k & a_2 \le k < b_2 + r_2 \\ s^2_{1,k} & b_2 + r_2 \le k < b_2 + r_2 + c_2 p^2_1 \\ \vdots & \vdots \\ x^n_k & a_n \le k < b_n + r_n \\ \vdots & \vdots \\ s^n_{\eta_n,k} & b_n + r_n + c_n \sum_{j=0}^{\eta_n - 1} p^n_j \le k < b_n + r_n + c_n \sum_{j=0}^{\eta_n} p^n_j \end{cases}$$

By shadowing, there is a point $z \in X$ with an orbit $(z_j)^\infty_{j=0}$ that ϵ-shadows the pseudo-orbit $(w_j)_{j=0}^{\infty}$, and clearly $(z_j)_{j=0}^{\infty}$ is the orbit such that $d(z_j, x_j^i) < \epsilon$, for $1 \le i \le n$, $a_i \le j \le b_i$. $\square$

3.4 Inverse Limits

There have been many publications on inverse limits of set-valued relations, but the associated dynamics of the bonding map and how the two relate have not been studied in depth. Here we study some dynamical properties that arise in the inverse limit setting.

Definition 3.13. Let (X, F) be a dynamical system. Recall the forgetful shift map $\sigma: \varprojlim F \to \varprojlim F$, defined for $x = (x_1, x_2, \ldots)$ as $\sigma(x) = (x_2, x_3, \ldots)$.

Theorem 3.14. *Let (X, F) be a dynamical system. Let F have the specification property on X. Then the dynamical system $(\varprojlim(X, F), \sigma)$ has the specification property.*

Proof. Let $\epsilon > 0$. Let $k \in \mathbb{N}$ such that

$$\sum_{i=k}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2}.$$

Let M' witness specification for $\frac{\epsilon}{2}$. Let $M = M' + k$. Let $x^1, x^2, \ldots, x^n \in \varprojlim F$. Let $1 = a_1 \le b_1 < a_2 \le b_2 \le \ldots \le b_n$ with $a_{i+1} - b_i > M$. Note that $x^i = (x_1^i, x_2^i, \ldots)$, with $x_j^i \in F(x_{j+i}^i)$. Consider, for each i, the finite piece of the orbit $(x_j^i)_{j=a_i}^{b_i+k}$.
We construct a point in the inverse limit that that traces our orbits.

Let $\alpha_1 = 0$, $\beta_1 = b_n + k - a_n$, and for $2 \le i \le n$, $\alpha_i = b_n - b_{n-(i-1)}$, and $\beta_i = b_n + k - a_{n-(i-1)}$. Note that $\alpha_{i+1} - \beta_i > M'$. Let $y_1 = x_{b_n+k}^n$, $y_2 = x_{b_{n-1}+k}^{n-1}, \ldots$ such that $y_i = x_{b_n+k}^{n-(i-1)}$ for each $1 \le i \le n$. So by the specification property, there exists a point z' in X, that has a periodic orbit of length $D > M + b_n - a_1$, denoted $(z_j')_{j=0}^{\infty}$ such that

$$|(z_j') - (x_{b_n+k-j}^{n-(i-1)})| < \frac{\epsilon}{2}, \quad \text{for} \quad 1 \le i \le n, \;\; \alpha_i \le j \le \beta_i.$$

Let z be the point in $\varprojlim F$ whose first coordinate is z'_{b_n+k}, second coordinate is z'_{b_n+k-1}, in general whose ith coordinate is $z'_{b_n+k-(i-1)}$, where we trace out the known periodic orbit of z' for negative indices. We show that z witnesses specification for $x^1, \ldots, x^n$, and the a_i's, b_i's. To see this, let $a_i \le a \le b_i$.

$$\begin{aligned} d(\sigma^a(z), \sigma^a(x^i)) &= \sum_{j=1}^{k-1} \frac{|\pi_j(\sigma^a(z)) - \pi_j(\sigma^a(x^i))|}{2^j} + \sum_{j=k}^{\infty} \frac{|\pi_j(\sigma^a(z)) - \pi_j(\sigma^a(x^i))|}{2^j} \\ &\le \sum_{j=1}^{k-1} \frac{|(z'_{b_n+k-a-j}) - (x^i_{a+j})|}{2^j} + \sum_{j=k}^{\infty} \frac{1}{2^j} \\ &\le \sum_{j=1}^{k-1} \frac{|(z'_{b_n+k-a-j}) - (x^i_{a+j})|}{2^j} + \frac{\epsilon}{2}. \end{aligned}$$

Now, note that as $a_i \le a \le b_i$, we have

$$b_n + k - b_i - (k-1) \le b_n + k - a - j \le b_n + k - a_i - 1$$

$$\alpha_{n-(i-1)} \le b_n - b_i + 1 \le b_n + k - a - i \le b_n + k - a_i - 1 \le \beta_{n-(i-1)}.$$

Then $|(z'_{b_2+k-a-j}) - (x^i_{a+j})| < \frac{\epsilon}{2}$, for each j. Thus our sum becomes

$$\le \sum_{1}^{k-1} \frac{\frac{\epsilon}{2}}{2^j} + \frac{\epsilon}{2} < \epsilon.$$

As $\sigma^D(z) = z$, we have that $\varprojlim F$ has specification via σ. □

For the next theorem, we need to define the inverse of a set-valued map on X.

Definition 3.15. Let $F\colon X \to 2^X$ be a set-valued map. Then $F^{-1}\colon X \to 2^X$ is the set-valued map on X such that $F^{-1}(x) = \{y \in X : x \in F(y)\}$.

Lemma 3.16. *Let (X, F) be a dynamical system. $\overrightarrow{Orb}(X, F)$ and $\varprojlim(X, F)$ are equal as sets.*

This tells us that chaotic properties of F will be reflected in the structure of the inverse limit of F^{-1}, and vice-versa. The following theorem connects the ideas of the specification property and inverse limits.

Theorem 3.17. *Let (X, F) be a dynamical system. F has specification if, and only if, F^{-1} does as well.*

Proof. Suppose F has specification. Let $\epsilon > 0$. Let M witness specification for F and this ϵ. Let $x^1, \ldots, x^n \in X$. Let $a_1 \le b_i < \ldots \le b_n$ with $a_{i+1} - b_i > M$. Let $P > M + b_n - a_1$. Let $(x^i_j)_{j=0}^{\infty}$ be an orbit of x^i via F^{-1}, for each $1 \le i \le n$. Now consider the orbit segments $(x^i_j)_{j=a_i}^{b_i}$. We have that $x^i_{j+1} \in F^{-1}(x^i_j)$, and so $x^i_j \in F(x^i_{j+1})$. Let

$$(y^i_j)_{j=a_i}^{b_i} = (x_j^{n-(i-1)})_{j=b_{n-(i-1)}}^{a_{n-(i-1)}}.$$

By specification for F, there is some z with an orbit $(z_j)_{j=0}^{\infty}$ such that

$$d(z_j, y^i_j) < \epsilon, \quad 1 \le i \le n, \ \ a_i \le j \le b_i.$$

The result follows from the proof of the previous theorem. The converse follows similarly. □

This gives a way to determine a class of relations that have the specification property. To be more precise, the inverses of continuous single-valued functions that have specification will have specification. As an example, the inverse of the tent map will have specification, see Figure 3.1.

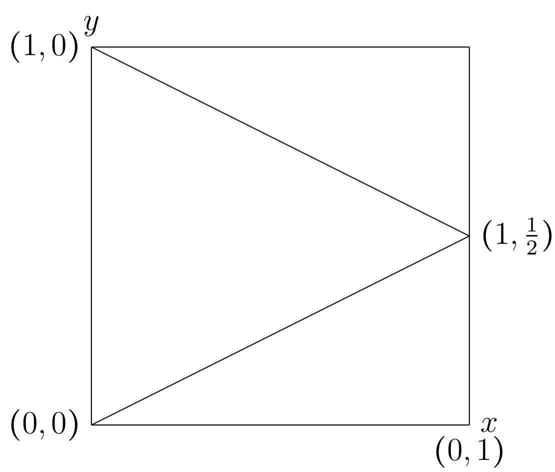

Figure 3.1.

Definition 3.18. Let (X, F) be a dynamical system. Let $\gamma\colon \varprojlim F \to 2^{\varprojlim F}$ be a set-valued function on $\varprojlim F$ such that $\gamma((x_0, x_1, \ldots)) = \{(x_{-1}, x_0, x_1, \ldots) : x_{-1} \in F(x_0)\}$.

Note that this set-valued function γ is similar to the forgetful shift map $\sigma\colon \varprojlim F \to \varprojlim F$.

Corollary 3.19. *Let F be a set-valued map on X, a compact metric that has specification. Then γ has specification.*

Proof. Identify with the shift map σ a set-valued map $\sigma\colon \varprojlim F \to 2^{\varprojlim F}$ such that $\sigma((x_0, x_1, \ldots)) = \{(x_1, x_2, \ldots)\}$. Then σ and γ are inverses as defined above, and so the result holds by Theorem 9. □

Lemma 3.20. *Let $F\colon X \to 2^X$ be a set-valued map. If F has mixing, then F^{-1} has mixing.*

Proof. Let U and V be non-empty open sets, we wish to show that there exists M such that for $m > M$, there exists a point in U that has an orbit $(x_j)_{j=0}^{\infty}$ under F^{-1} such that $x_m \in V$. As F has mixing, let $N \in \mathbb{N}$ such that for all $n > N$, there exists $y^n \in V$ such that y^n has an orbit $(y_j^n)_{j=0}^{\infty}$ under F with $y_n^n \in U$. Then letting y_n^n be our choice for a point in U, we see that under F^{-1}, this point has an orbit whose nth iterate lands in V. □

A well-known theorem of A.M. Blokh states that for single-valued functions on the interval. topological mixing is equivalent to the specification property. The following corollary extends this result.

Corollary 3.21. *Let $([0,1], f)$ be a single-valued dynamical system on the unit interval. If $F : [0,1] \to 2^{[0,1]}$ be defined as $F(x) = f^{-1}(x)$, then F has mixing if, and only if F has specification.*

Proof. It is sufficient to note Theorem 3.17, Lemma 3.20, and the exceptional theorem from Blokh [14], which states that mixing and the specification property are equivalent on single-valued interval maps. □

3.5 Measures on a Set-valued Dynamical System.

The specification property yields many good results in the measure spaces of single-valued functions, in particular the fact that in the space of invariant measures, there is a dense G_δ set of non-atomic measures with full support [18]. Here we give some preliminary results towards the existence of invariant measures with full support which are non-atomic, and we continue this topic in chapter 4. We begin with the theorem of Denker, Grillenberger, and Sigmund.

Theorem 3.22 (Denker et al). *Let (X, f) be a dynamical system, and denote the space of invariant measures by $\mathcal{M}(X)$. If F has the specification property, then the set of non-atomic invariant measures with full support forms a countable intersection of dense open sets.*

To begin adapting this to the set-valued case, we introduce the notion of an invariant measure on a set-valued map. Aubin, Frankowska and Lasota [3] gave the following notion of an invariant measure, which Akin and Miller [40] showed to be equivalent to many other notions of an invariant measure.

Definition 3.23. Let $F: X \to 2^X$ be a set-valued map. Let $P(X)$ be the space of Borel probability measures on X. Then a measure $\mu \in P(X)$ is said to be *invariant* if

$$\mu(B) \le \mu(F^{-1}(B)),$$

for all Borel sets B of X.

To see a trivial example of an F-invariant measure, suppose $F: X \to 2^X$ has a point x with a periodic orbit $(x_j)_{j=0}^{\infty}$, with period n. Then define the periodic measure δ_x by

$$\delta_x(B) = \frac{|\{j \in \{0, 1, \dots, n-1\} : x_j \in B\}|}{n},$$

for any Borel set B. To see that δ_x is invariant, let B be a Borel set in X, and suppose that there are k many distinct elements of $\{x_j : j \in \{0, 1, \dots, n-1\}\}$ in B. Then $F^{-1}(B)$ will have at least k many distinct elements of $\{x_j : j \in \{0, 1, \dots, n-1\}\}$, and so the measure of $F^{-1}(B)$ will be at least the measure of B.

Theorem 3.24. *Let $F: X \to 2^X$ be a set-valued map. The set of F-invariant measures with support X is either empty or a dense G_δ set in the space of F-invariant measures on X.*

Proof. Let μ be a F-invariant measure with support X. Let U be an open nonempty subset of X, and so $\mu(U) > 0$. Denote $D(U) = \{\mu | \mu \text{ is } F\text{-invariant}, \mu(U) = 0\}$. Then $D(U)$ is a closed collection of measures. Also, $D(U)$ has no interior. To see this, let $\nu \in D(U)$, $\epsilon > 0$ and consider the measure $\nu_\epsilon = (1-\epsilon)\nu + \epsilon\mu$. $\nu_\epsilon(U) = \epsilon\mu(U) > 0$, and so ν_ϵ is not in $D(U)$. Letting ϵ go to zero, $\nu_\epsilon \to \nu$, and so for each point of $D(U)$, there is a sequence of measures outside of $D(U)$ approaching that point. Thus $D(U)$ is nowhere dense.

Now as X is a compact metric, it is second countable, so let $\{U_i\}_1^\infty$ be a countable basis for X. Then $\bigcup_{i=1}^{\infty} D(U_i)$, being a countable collection of nowhere dense sets, is of first category. Thus the complement of this union is a dense G_δ set. As any measure in the complement of this union must have full support, the proof is complete. □

We now construct an invariant measure with full support, which gives us the following theorem.

Theorem 3.25. *Let $F: X \to 2^X$ be a set-valued map. If X has a dense set of points with periodic orbits (note that F having specification gives this), then a dense G_δ set of invariant measures have full support.*

Proof. It suffices to build an invariant measure with full support. Let $\{x^i : i \in \mathbb{N}\}$ be a dense set of points with periodic orbits $(x^i_j)_{j=0}^{\infty}$, and $x^i_{j_i} = x_0$. Consider the measure

$$\mu = \sum_{i=1}^{\infty} \frac{1}{2^i} \delta_{x^i},$$

with δ_x as defined above. To see that μ is a measure, let $\{U_n\}_{n=1}^{\infty}$ be a countable collection of disjoint measurable sets. For each $n \in \mathbb{N}$, let

$$k^i_n = |\{j \in \{0, 1, \dots, j_i - 1\} : x^i_j \in U_n\}|.$$

Then we have that

$$\mu(U_n) = \sum_{i=1}^{\infty} \frac{1}{2^i} \delta_{x^i}(U_n) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{k^i_n}{j_i}, \text{ and so } \sum_{n=1}^{\infty} \mu(U_n) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{k^i_n}{j_i}.$$

Now we consider $\mu(\cup_{n=1}^{\infty} U_n)$.

$$\begin{aligned}
\mu(\cup_{n=1}^{\infty} U_n) &= \sum_{i=1}^{\infty} \frac{1}{2^i} \delta_{x^i}(\cup_{n=1}^{\infty} U_n) \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\{j \in \{0, 1, \dots, j_i - 1\} : x^i_j \in \cup_{n=1}^{\infty} U_n\}|}{j_i} \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sum_{n=1}^{\infty} |\{j \in \{0, 1, \dots, j_i - 1\} : x^i_j \in U_n\}|}{j_i} \\
&= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2^i} \frac{k^i_n}{j_i.} \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{k^i_n}{j_i.} \\
&= \sum_{n=1}^{\infty} \mu(U_n).
\end{aligned}$$

We now show that μ is invariant. Let B be a borel set, let $m_i = |\{j \in \{0, 1, \dots, j_i - 1\} : x^i_j \in B\}|$, and let $n_i = |\{j \in \{0, 1, \dots, j_i - 1\} : x^i_j \in F^{-1}(B)\}|$. Then

$$\mu(B) = \sum_{i=1}^{\infty} \frac{m_i}{2^i j_i}.$$

$$\mu(F^{-1}(B)) = \sum_{i=1}^{\infty} \frac{n_i}{2^i j_i}.$$

For μ to be invariant, it suffices to show that $n_i \geq m_i$. To see this, let $j \in \{0, \dots, j_i - 1\}$ such that $x^i_j \in B$. Then as $x^i_{j-1} \in F^{-1}(B)$, we see that $n_i \geq m_i$. □

CHAPTER FOUR

Inducing Invariant Measures on Multi-valued Dynamical Systems

4.1 Introduction

In the context of dynamical systems of compact metric spaces, say $f : X \to X$, with f continuous, there are many results regarding the existence of measures on X that are invariant with respect to f. In particular, Denker, Grillenberger, and Sigmund show that in the presence of the specification property, the space of invariant measures on (X, f) has a dense G_δ set of non-atomic measures with full support that are invariant with respect to f, [18]. In the context of set-valued functions, relatively little is known about the existence of invariant measures. Kennedy, Raines, and Stockman ask under what conditions an upper-semi-continuous(usc) set-valued function admits an invariant measure, [31]. There are many other results on constructing invariant measures for inverse limit spaces, [52], [54], and these have primarily been motivated by economic models that lend themselves to an analysis via inverse limit spaces, see [33], [39], and [38]. In this chapter, we first include the above-mentioned theorem of Denker, et al, along with necessary background definitions. We also give a definition of a measure with respect to a set-valued function as well as a notion of invariance due to Aubin, Frankowska, and Lasota, [3]. In Section 3 we note that in the presence of the specification property, the results of Denker, et al, show the existence of non-atomic invariant measures with full support in the orbit space of set-valued dynamical systems. We use these measures to induce measures on the underlying space X using projection maps. This chapter focuses on these induced measures, and we first show that they inherit the properties of invariance and full support. We give an example showing that it is not true in general that these induced measures will be non-atomic, and in Section 4 we discuss our main results, which link the dynamical property of multiperiodicity to systems in which measures induced from orbit spaces will be atomic. Multiperiodicity is a condition which can only be found in set-valued dynamical systems, and states that there exists points with distinct periodic orbits. From the existence of such a point, our first main result shows a construction of a non-atomic measure in the orbit space of (X, F) which induces an atomic measure on X.

Theorem 4.25. Let X be a compact metric space, and let $F : X \to 2^X$ be usc. If there exists a point $x \in X$ which is multiperiodic, then there exists an invariant measure on $\mathrm{Orb}^+(X)$ which is non-atomic, but the associated induced measure on X is atomic.

Our second main result is in the converse direction. We call a measure on the orbit space fibre-atomic if the induced measure on X is atomic. Starting with a fibre-atomic measure, we give conditions on the dynamical system which show the existence of a multiperiodic point.

Theorem 4.26. Let $F : X \to 2^X$ be usc and suppose that for all $x \in X$, $|F^{-1}(x)| < \infty$. Let μ be a fibre-atomic, non-atomic invariant measure on $\mathrm{Orb}^+(X)$. If X has only finitely many fibre-atoms, then there exists a multiperiodic point in X.

4.2 Background

Let X be a compact metric space, and let 2^X be the hyperspace of nonempty closed subsets of X.

Definition 4.1. Let $F : X \to 2^X$. We say F is *upper-semi-continuous* at a point $x \in X$ if, for any open set V containing $F(x)$, there exists an open set $U \subset X$ such that $F(U) \subset V$. We say F is *upper-semi-continuous* (usc) if it is upper-semi-continuous at every point.

Ingram showed in [23] that F is usc if and only if the graph of F is a closed subset of $X \times X$.

Definition 4.2. The *inverse limit space induced by* F is the space

$$\varprojlim F = \{(x_0, x_1, \dots) \in X^{\mathbb{N}} : x_{i-1} \in F(x_i)\}$$

considered as a subspace of the Tychonoff product space $X^{\mathbb{N}} := \prod_{n=0}^{\infty} X$.

Associated with the inverse limit are a number of continuous functions. First we define the projection maps.

Definition 4.3. The *n-th projection map* is a function $\pi_n : \varprojlim F \to X$ defined as $\pi_n((x_j)_{j=0}^{\infty}) = x_n$.

Associated with the inverse limit is a natural shift map, sometimes called the forgetful shift.

Definition 4.4. The *shift map* is a map $\sigma : \varprojlim F \to \varprojlim F$ defined for each $(x_j)_{j=0}^{\infty} \in \varprojlim F$ as

$$\sigma(x_0, x_1, \dots) = (x_1, x_2 \dots).$$

This is a continuous well-defined single-valued function which mimics the dynamics of F on X. In the context of set-valued dynamical systems, it is sometimes more natural to work in the space of forward orbits, defined below.

Definition 4.5. Let X be a compact metric space, and $F : X \to 2^X$ be usc. A sequence $(x_j)_{j=0}^{\infty}$ is said to be an *orbit* of $x \in X$ if $x_0 = x$, and $x_{i+1} \in F(x_i)$ for each $i \in \mathbb{N}$.

Definition 4.6. Let X be a compact metric space, and $F : X \to 2^X$ be usc. The *forward orbit space induced by* F is the space

$$\mathrm{Orb}^{+}(F) = \{(x_0, x_1, \dots) \in X^{\mathbb{N}} x_{i+1} \in F(x_i)\},$$

This space also has a natural shift map, which we denote using σ. While the results in this chapter are given in the context of $\mathrm{Orb}^+(F)$, all of them are applicable to $\varprojlim F$, with minor changes. Some examples of these changes are given at the end of the chapter.

Before we introduce chaotic properties of set-valued functions, one aspect which differentiates set-valued dynamical systems from single-valued is the notion of periodicity.

Definition 4.7. Let X be a compact metric space and let $F : X \to 2^X$ be usc. Then $x \in X$ has a *periodic orbit* $(x_j)_{j=0}^{\infty}$ if $(x_j)_{j=0}^{\infty} \in \mathrm{Orb}(F)$ and there exists $n \in \mathbb{N}$ with $\sigma^n((x_j)_{j=0}^{\infty}) = (x_j)_{j=0}^{\infty}$.

Note that as F is set-valued, there may be points in X with many distinct periodic orbits. For completeness we now give some basic definitions from the theory of single valued dynamical systems and measure theory.

Definition 4.8. Let X be a compact metric space. A measure μ on X has *full support* if the measure of any open set is strictly positive.

Definition 4.9. Let X be a compact metric space, and let μ be a measure on X. A point $x \in X$ is an *atom* with respect to μ if $\mu(\{x\}) > 0$. If μ has at least one atom, μ is called *atomic*. If μ has no atoms, μ is called *non-atomic*.

Definition 4.10. Let X be a compact metric space, let μ be a measure on X, and let $f : X \to X$. μ is *invariant* with respect to f if $\mu(A) = \mu(f^{-1}(A))$, for all measurable sets A.

Definition 4.11. Let X be a compact metric, and $f : X \to X$. $\mathfrak{M}(X)$ is the space of measures invariant with respect to f on X.

Definition 4.12. Let X be a compact metric, and $f : X \to X$. $\mathfrak{N}(X)$ is the subset of $\mathfrak{M}(X)$ consisting of non-atomic measures with full support.

Definition 4.13. Let X be a compact metric space, and $f : X \to X$ a continuous map. Then f is said to have the *specification property* if, for any $\epsilon > 0$, there exists an $M \in \mathbb{N}$ with the following properties: for any collection of points $x_1, \ldots, x_n \in X$, and any natural numbers $a_1 \le b_1 < a_2 \le b_2 < \ldots < a_n \le b_n$ which satisfy $a_i - b_{i-1} > M$, for each $2 \le i \le n$, and any $P \in \mathbb{N}$ with $P > M + b_n - a_1$, there exists a periodic point $y \in X$ such that

$$d(f^j(y), f^j(x_i)) < \epsilon, \quad \text{for } a_i \le j \le b_i, \quad \text{for } 1 \le i \le n;$$

$$f^P(y) = y.$$

The first result is due to Denker, et al, who give a sufficient condition for the existence of non-atomic invariant measures with full support.

Theorem 4.14 (Proposition 21.10, Proposition 21.12, [18].). *Let X be a compact metric space. If $f : X \to X$ has specification, then $\mathfrak{N}(X)$ is a dense G_δ subset of $\mathfrak{M}(X)$.*

Before we extend this result to set-valued dynamical systems, we give some definitions of above-mentioned concepts in the context of set-valued functions.

Definition 4.15. Let $F : X \to 2^X$ be usc, $F^{-1} : X \to 2^X$ is an usc function defined as follows:

$$F^{-1}(x) = \{y \in X : x \in F(y)\}.$$

Aubin, et al first introduced the notion of μ being invariant with respect to a set-valued map F in [3] as

$$\mu(F(A)) \le \mu(F^{-1}(A)),$$

for all measurable sets A. Akin and Miller later took multiple definitions of invariance in the set-valued context, including the notion of Aubin, et al, and showed that they were all equivalent in [40]. For the remainder of the thesis, we will use the definition of invariant given by Aubin et al. As a note on why this definition is required, consider the following example. For a single-valued dynamical system (X, f), a trivially invariant measure is the measure that gives full weight to a point fixed under f. The natural analog to this for a set-valued dynamical system (X, F) is to let $x \in X$ such that $x \in F(x)$, and define the measure μ to be

$$\mu(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

However, this measure may not be invariant in the traditional sense, as it is possible that there exists $y \in F(x)$ such that $x \neq y$. Then $\mu(\{y\}) = 0$, but $\mu(F^{-1}(y)) = 1$.

Another definition which we extend to usc set-valued functions is the specification property.

Definition 4.16. Let $F : X \to 2^X$ be usc. F is said to have the *specification property* if, for all $\epsilon > 0$, there exists $M \in \mathbb{N}$ with the following properties: for any collection of points $x^1, \ldots x^n \in X$, any orbits of these points $(x^i_j)_{j=0}^{\infty}$, $1 \leq i \leq n$, any collection of natural numbers $a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_n \leq b_n$ with $a_{j+1} - b_j > M$, and for any $P \in \mathbb{N}$ with $P > b_n - a_1 + M$, we have the following, there exists a point $z \in X$ with a periodic orbit $(z_j)_{j=0}^{\infty}$ satisfying

$$d(z_j, x^i_j) < \epsilon, \text{ for } a_i \leq j \leq b_i, \text{ for } 1 \leq i \leq n;$$

$$z_{P+k} = z_k, \text{ for } k \in \mathbb{N}.$$

We have the following theorem which gives us the ability to apply Theorem 1 on orbit spaces. In [48], it was shown that if (X, F) has specification, then $(\varprojlim X, \sigma)$ has specification. A similar proof gives the following result.

Theorem 4.17. (X, F) *has specification if and only if* $(Orb^+(X), \sigma)$ *has specification.*

Since σ is a single-valued map and orbit spaces are compact metric spaces, it is immediate that Theorem 1 applies. Therefore $\mathfrak{N}(\mathrm{Orb}(F))$ is a dense G_δ of $\mathfrak{M}(\mathrm{Orb}(F))$. We give a method of using measures from $\mathfrak{N}(\mathrm{Orb}(F))$ to induce measures on X. We show these are invariant with respect to F and have full support. Unfortunately, these measures may no longer be non-atomic.

Theorem 4.18. *Let X be a compact metric space, and $F : X \to 2^X$ be usc. If μ is a measure on $Orb^+(X)$, then the function $\hat{\mu}(A) := (\mu(\pi_0^{-1}(A)))$ is a measure on X.*

Proof. Recall that for $\hat{\mu}$ to be a measure, the following must hold:

- $\hat{\mu}(\varnothing) = 0$.
- $\hat{\mu}(\cup_{j=0}^{\infty} A_j) = \sum_{j=0}^{\infty} \hat{\mu}(A_j)$, for any mutually disjoint collection of measurable sets.

That the first condition holds is clear. To see that the second condition holds, let $\{A_j\}_{j=0}^{\infty}$ be a collection of mutually disjoint measurable sets. Then

$$\begin{aligned}
\hat{\mu}\left(\bigcup_{j=0}^{\infty} A_j\right) &= \mu\left(\pi_0^{-1}\left(\bigcup_{j=0}^{\infty} A_j\right)\right) \\
&= \mu\left(\bigcup_{j=0}^{\infty} \pi_0^{-1}(A_j)\right) \\
&= \sum_{j=0}^{\infty} \mu(\pi_0^{-1}(A_j)) \\
&= \sum_{j=0}^{\infty} \hat{\mu}(A_j).
\end{aligned}$$

□

Theorem 4.19. *Let X be a compact metric space, and $F : X \to 2^X$ be usc. If μ has full support on $Orb^+(X)$, then $\hat{\mu}$ has full support on X.*

Proof. Let A be an open subset of X. Then $\hat{\mu}(A) = \mu(\pi_0^{-1}(A)) > 0$, as μ has full support. □

Theorem 4.20. *Let X be a compact metric space, and $F : X \to 2^X$ be usc. If μ is invariant on $Orb^+(X)$, then $\hat{\mu}$ is invariant on X.*

Proof. Let A be a measurable subset of X. Consider $\hat{\mu}(A)$ and $\hat{\mu}(F^{-1}(A))$.

$$\begin{aligned}
\hat{\mu}(A) &= \mu(\{(x_j)_{j=0}^{\infty} \in \mathrm{Orb}^+(X) : x_0 \in A\}) \\
&= \mu(\{(x_j)_{j=0}^{\infty} \in \mathrm{Orb}^+(X) : x_1 \in A\}) \text{ (by the invariance of } \mu) \\
&\le \mu(\{(x_j)_{j=0}^{\infty} \in \mathrm{Orb}^+(X) : F(x_0) \cap A \neq \varnothing\}) \\
&= \mu(\pi_0^{-1}(F^{-1}(A))) \\
&= \hat{\mu}(F^{-1}(A)).
\end{aligned}$$

□

Such measures on X will not always inherit the property of being non-atomic. Consider the function whose graph is shown in Figure 1.

$$F(x) = \begin{cases} \frac{1}{2} & x \in [0,1] \\ [0,1] & x = \frac{1}{2}. \end{cases}$$

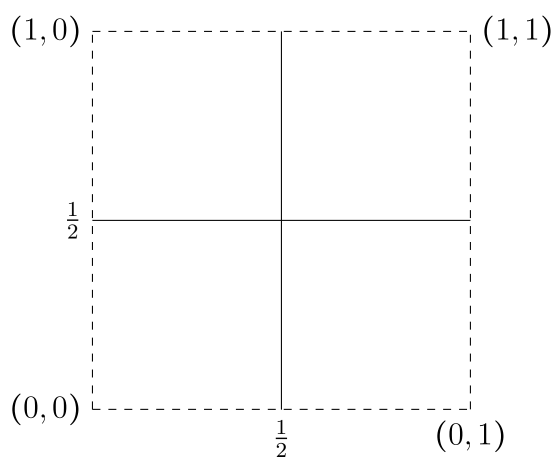

Figure 4.1.

This function has specification, and as such there there is a dense G_δ set of non-atomic measures with full support on the orbit spaces. However, none of these measures induce a non-atomic measure on X.

To see that this function has specification, first observe that for any orbit $(x_j)_{j=0}^{\infty} \in \mathrm{Orb}(F)$, $x_i \neq 1/2$ implies that $x_{i+1} = 1/2$. Similarly, $x_{i+1} \neq 1/2$ implies that $x_i = 1/2$. Let $\epsilon > 0$, and choose $M = 2$. Let $x^1, \ldots, x^n \in [0,1]$, $(x_j^i)_{j=0}^{\infty}$ be orbits of x^i, for $1 \leq i \leq n$, $a_1 \leq b_1 < a_2 \leq \ldots < a_n \leq b_n$ be natural numbers such that $a_i - b_{i-1} > M$, and let $P > M + b_n - a_1$. Then choose $z \in [0,1]$ such that the orbit $(z_j)_{j=0}^{\infty}$ may be defined as follows.

$$z_j = \begin{cases} \frac{1}{2} & j = 1, 2, \ldots, a_1 - 1, \\ \\ x_j^1 & j = a_1, a_1 + 1, \ldots, b_1, \ldots, a_2 - 2 \\ \\ \frac{1}{2} & j = a_2 - 1, \\ \vdots & \vdots \\ x_j^n & j = a_n, a_n + 1, \ldots, b_n, \ldots, P - 1 \\ \\ z_{j-P} & j \geq P. \end{cases}$$

Then $d(z_j, x_j^i) = 0 < \epsilon$ for $a_i \leq j \leq b_i$, $1 \leq i \leq n$, and the orbit of z is periodic. Thus F has specification on $[0,1]$.

By Theorem 2, σ has specification on $\mathrm{Orb}^+(X)$. By Theorem 1, there exists a dense G_δ of non-atomic invariant measures on $\mathrm{Orb}^+(X)$ with full support. For any such measure μ, the induced measure $\hat{\mu}$ will be atomic. To see this, consider

$$\hat{\mu}\left(\frac{1}{2}\right) = \mu\left(\{(x_j)_{j=0}^{\infty} \in \mathrm{Orb}^+(X) : x_0 = 1/2\}\right).$$

We show that $\pi_0^{-1}(1/2)$ has positive measure. Notice that by our above observation, this set contains the open set $\pi_1^{-1}([0,1]-\{1/2\})$. As μ has full support, $\hat{\mu}(\{1/2\}) > 0$.

4.4 Atoms

In this section, our goal is to determine conditions under which our process of inducing invariant measures will preserve the property of being non-atomic.

Definition 4.21. Let X be a compact metric space, and let $F : X \to 2^X$ be usc. We call a measure μ on $\mathrm{Orb}(F)$ *fibre-atomic* if μ is non-atomic, but the induced measure $\hat{\mu}$ is atomic. We call the points of X witnessing the atomicity of $\hat{\mu}$ the *fibre-atoms* of μ.

Lemma 4.22. *Let X be a compact metric space. Let $F : X \to 2^X$ be usc. If there exists $x \in X$ and $A \subset F(x)$ such that $\{x\} \times A$ is open in the graph of F, then for any measure μ on $Orb^+(X)$ which has full support, the induced measure $\hat{\mu}$ will be atomic.*

Proof. Note that $\pi_0^{-1}(x)$ contains the open set $\pi_0^{-1}(x) \cap \pi_1^{-1}(A)$ in $\mathrm{Orb}^+(X)$. Then as μ has full support,

$$\hat{\mu}(x) = \mu(\pi_0^{-1}(x)) > 0.$$

□

This result, while elementary, is enlightening because using this result one can tell visually based on the graph whether or not it is possible for the induced measure to be non-atomic.

To find an alternative criterion for the induced measure to be atomic, we introduce the notion of multiperiodicity.

Definition 4.23. Let X be a compact metric space, and let $F : X \to 2^X$ be usc. A point $x \in X$ is *multiperiodic* if it has at least two distinct periodic orbits.

Lemma 4.24. *Let X be a compact metric space, and let $F : X \to 2^X$ be usc. If there exists a point $x \in X$ which is multiperiodic,then there is a subset of $\pi_0^{-1}(x)$ homeomorphic to $\{0,1\}^{\mathbb{N}}$.*

Proof. We construct the homeomorphism directly. Let x be multiperiodic. Then there exist finite distinct sequences $A = \{x_0, x_1, \ldots, x_{p-1}\}$ and $B = \{y_0, y_1, \ldots, y_{q-1}\}$ such that $x_0 = y_0 = x$, $x_i \in F(x_{i-1})$ for $1 \le i \le p-1$, $y_i \in F(y_{i-1})$ for $1 \le i \le q-1$, and $x \in F(x_{p-1})$, $x \in F(y_{q-1})$. Further, choose x, A, and B in such a way that $y_{q-1} \ne x_{p-1}$. Note that although A and B are distinct, it is not necessarily the case that $x_i \ne y_i$ for all i. To make these sequences the same length, let $U = \{u_0, u_1, \ldots, u_{pq-1}\}$, with $u_{i+kp} = x_i$ for $0 \le i < p$, $0 \le k < q$. Similarly, let $V = \{v_0, v_1, \ldots, v_{pq-1}\}$, with $v_{i+kq} = y_i$ for $0 \le i < q$, $0 \le k < p$.

We define a map $\gamma : \{U, V\}^{\mathbb{N}} \to \mathrm{Orb}^+(X)$. Let $e := (e_0, e_1, \ldots) \in \{U, V\}^{\mathbb{N}}$, i.e. $e_i \in \{U, V\}$ for each $i \in \mathbb{N}$. The map γ will be the natural one, identifying each U with the string $(u_0, \ldots u_{pq-1})$ and each V with the string $(v_0, \ldots, v_{pq-1})$, and then concatenating. More formally, for each $0 \le j \le pq-1$, let $\alpha_j : \{U, V\} \to X$ by $\alpha_j(U) = u_j$, $\alpha_j(V) = v_j$. Then define $\gamma(e) = (z_j)_{j=0}^{\infty}$, where

$$z_j = \alpha_{j \bmod pq}(e_{\lfloor \frac{j}{pq} \rfloor}).$$

To see that γ is continuous, let $\epsilon > 0$. Let $\delta = \epsilon$. Let $n \in \mathbb{N}$ be the least number such that $1/(2^n) < \delta$. Let $a, b \in \{U, V\}^{\mathbb{N}}$ such that $d(a, b) < \delta$. Then $a_i = b_i$ for all $i \le n$. Then $\gamma(a)_i = \gamma(b)_i$ for $i \le pqn$. Thus $d(\gamma(a), \gamma(b)) < \epsilon$. Clearly $\{0,1\}^{\mathbb{N}}$ is homeomorphic to $\{U, V\}^{\mathbb{N}}$ and the proof is complete. □

Theorem 4.25. *Let X be a compact metric space, and let $F : X \to 2^X$ be usc. If there exists a point $x \in X$ which is multiperiodic, then there exists an invariant measure on $\mathrm{Orb}^+(X)$ which is non-atomic, but the associated induced measure on X is atomic.*

Proof. Let U and V be given as in the proof of Lemma 7. Let $Y = \{U, V\}^{\mathbb{N}}$. We use the function $\gamma : Y \to \text{Orb}(F)$ constructed in the proof of Lemma 7 to induce a measure on $\text{Orb}^+(X)$.

Let μ be the product measure on Y, sometimes called the uniform measure. To obtain a measure on $\text{Orb}(F)$, we define a map $\gamma : Y \to \text{Orb}^+(X)$ and define a measure on $\text{Orb}(F)$ in terms of μ. Let $e := (e_0, e_1, \ldots) \in Y$, i.e. $e_i \in \{U, V\}$ for each $i \in \mathbb{N}$. The map γ will be the natural one, identifying each U with the string $(u_0, \ldots u_{pq-1})$ and each V with the string $(v_0, \ldots, v_{pq-1})$, and then concatenating. More formally, for each $0 \leq j \leq pq - 1$, let $\alpha_j : \{U, V\} \to X$ by $\alpha_j(U) = u_j$, $\alpha_j(V) = v_j$. Then define $\gamma(e) = (z_j)_{j=0}^{\infty}$, where

$$z_j = \alpha_{j \bmod pq}(e_{\lfloor \frac{j}{pq} \rfloor}).$$

It is tempting to define the measure of a set A in $\text{Orb}(F)$ as $\mu(\gamma^{-1}(A))$. To see that this will not be invariant with respect to σ, consider the following example. Let $A = \pi_0^{-1}(x)$. Notice that $\gamma^{-1}(A) = Y$, and so $\mu(\gamma^{-1}(A)) = 1$. However, as $pq - 1 > 0$, $\gamma^{-1}(\sigma^{-1}(A)) = \varnothing$, and therefore has measure 0.

Returning to the proof, to build an invariant measure we first define a sequence of maps $\gamma_i : Y \to \text{Orb}^+(X)$ by

$$\gamma_i(e) = \sigma^i(\gamma(e)), \quad 0 \leq i \leq pq - 1.$$

The uniform measure is known to be invariant with respect to the shift on Y, which we denote by $\rho : Y \to Y$ such that $\rho(e_0, e_1, \ldots) = (e_1, e_2, \ldots)$. The dynamics of ρ on Y mimic the dynamics of σ^{pq} on $\gamma(Y)$. We use the maps $\{\gamma_i\}_0^{pq-1}$ to 'fill in' the measure of each shift. Now we are ready to define our measure ν on $\text{Orb}^+(X)$. For $C \subset \text{Orb}^+(X)$, define

$$\nu(C) = \frac{1}{pq} \sum_{i=0}^{pq-1} \mu(\gamma_i^{-1}(C)).$$

We show that this is a non-atomic invariant measure on $\text{Orb}^+(X)$. To see that ν is non-atomic, let $(y_j)_{j=0}^{\infty} \in \text{Orb}^+(X)$. For $0 \leq i \leq pq - 1$, consider the set $\gamma_i^{-1}((y_j)_{j=0}^{\infty})$, and note that $|\gamma_i^{-1}((y_j)_{j=0}^{\infty})| \leq 1$. Then as μ is non-atomic, it follows that ν is non-atomic. To see that ν is a measure, observe that $\nu(\varnothing) = 0$, so it remains to show that the measure of a disjoint union of sets is the sum of their measures. Let $\{A_n\}_1^{\infty}$ be a collection of disjoint subsets of $\mathrm{Orb}(F)$.

$$\begin{aligned}
\nu\left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) &= \frac{1}{pq}\sum_{i=0}^{pq-1} \mu\left(\gamma_i^{-1}(\dot{\bigcup_{n=1}^{\infty}} A_n)\right) \\
&= \frac{1}{pq}\sum_{i=0}^{pq-1} \mu\left(\gamma^{-1}(\sigma^{-i}(\dot{\bigcup_{n=1}^{\infty}} A_n))\right) \\
&= \frac{1}{pq}\sum_{i=0}^{pq-1} \mu\left(\dot{\bigcup_{n=1}^{\infty}} \gamma^{-1}(\sigma^{-i}(A_n))\right) \\
&= \frac{1}{pq}\sum_{i=1}^{pq-1}\sum_{n=1}^{\infty} \mu(\gamma^{-1}(\sigma^{-i}(A_n))) \\
&= \sum_{n=1}^{\infty}\frac{1}{pq}\sum_{i=1}^{pq-1} \mu(\gamma^{-1}(\sigma^{-i}(A_n))) \\
&= \sum_{n=1}^{\infty}\nu(A_n)
\end{aligned}$$

We now have that ν is a non-atomic measure on $(\mathrm{Orb}(F)$. To see that ν is invariant with respect to σ, let $C \subset \mathrm{Orb}^+(X)$.

$$\begin{aligned}
\nu(\sigma^{-1}(C)) &= \frac{1}{pq}\sum_{i=0}^{pq-1} \mu(\gamma_i^{-1}(\sigma^{-1}(C))) \\
&= \frac{1}{pq}\sum_{i=0}^{pq-1} \mu(\gamma^{-1}(\sigma^{-i-1}(C))).
\end{aligned}$$

Now consider the difference $\nu(C) - \nu(\sigma^{-1}(C))$. Expanded, this becomes

$$\frac{1}{pq}\left[\mu(\gamma^{-1}(C)) + \ldots + \mu(\gamma^{-1}(\sigma^{-pq-1}(C))) - (\mu(\gamma^{-1}(\sigma^{-1}(C))) + \ldots + \mu(\gamma^{-1}(\sigma^{-pq}(C)))\right].$$

Note that almost all terms in the summations cancel, leaving us with

$$\nu(C) - \nu(\sigma^{-1}(C)) = \frac{1}{pq}\mu(\gamma^{-1}(C)) - \frac{1}{pq}\mu(\gamma^{-1}(\sigma^{-pq}(C))).$$

For ν to be invariant, this difference must be 0. To see that this will hold, note that

$$\gamma^{-1}(\sigma^{-pq}(C)) = \rho^{-1}(\gamma^{-1}(C)),$$

and therefore by the invariance of μ with respect to ρ we have that

$$\mu(\gamma^{-1}(C)) = \mu(\rho^{-1}(\gamma^{-1}(C))).$$

Therefore ν is invariant with respect to σ.

To show that ν is fibre-atomic, recall our multiperiodic point x and consider $\nu(\pi_0^{-1}(x))$.

$$\begin{aligned}\nu(\pi_0^{-1}(x)) &= \frac{1}{pq}\sum_{i=0}^{pq-1}\mu(\gamma_i^{-1}(\pi_0^{-1}(x))) \\ &\geq \frac{1}{pq}\mu(\gamma^{-1}(\pi_0^{-1}(x))) \\ &= \frac{1}{pq}.\end{aligned}$$

□

We now consider the converse direction.

Theorem 4.26. *Let $F : X \to 2^X$ be usc and suppose that for all $x \in X$, $|F^{-1}(x)| < \infty$. Let μ be a fibre-atomic, non-atomic invariant measure on $Orb^+(X)$. If X has only finitely many fibre-atoms, then there exists a multiperiodic point in X.*

Proof. Let $x_1, \ldots, x_n$ be the fibre-atoms of μ. Notice that as $|F^{-1}(x)| < \infty$ for all x and that μ is invariant, it is clear that a fibre-atom must appear in the pre-image of any fibre-atom. Relabel a subset of fibre-atoms in the following way:

$$\begin{aligned} &y_1 = x_1. \\ &y_2 \in F^{-1}(y_1), \;\; y_2 \neq y_1. \\ &\;\;\vdots \\ &y_m \in F^{-1}(y_{m-1}), \;\; y_m \notin \{y_1, \ldots, y_{m-1}\}. \end{aligned}$$

To be clear, each $y_j \in \{x_1, \ldots, x_n\}$ and $1 \leq m \leq n$. Further, choose m to be as large as possible. Thus for any $x_i \in F^{-1}(y_m)$, $x_i = y_j$ for some $j < m$.

Now, for each $1 \le j \le m$, let

$$A_j = \{y_i : y_i \in \bigcup_{k=1}^{\infty} F^{-k}(y_j)\}.$$

Note that for any distinct $y_i, y_j \in A_m$, there is a path from y_i to y_j. More precisely, as $y_i \in A_m$, there is some $n \in \mathbb{N}$ such that $y_m \in F^n(y_i)$, and $y_j \in F^{m-j}(y_m)$. Hence, there is a point $(a_j)_{j=0}^{\infty}$ in $\mathrm{Orb}(F)$ such that $a_0 = y_i$, and $a_{n+m-j} = y_j$. Also note that $A_1 \supset A_2 \supset \ldots \supset A_m$. We consider the following cases.

(1) $|A_m| = 1$.

Let $\{y_i\} = A_m$. Then we have that $A_m = A_i$. As y_i is a fibre-atom, $\mu(\pi_0^{-1}(y_i)) = \delta > 0$. As μ is invariant, we have that

$$\mu(\sigma^{-1}(\pi_0^{-1}(y_i))) = \delta.$$

As $|A_i| = 1$, this implies that

$$\mu((\pi_0^{-1}(y_i)) \cap (\pi_1^{-1}(y_i))) = \delta.$$

By induction, sets of the form

$$K_n = \bigcap_{j=0}^{n} \pi_j^{-1}(y_i)$$

all have measure δ. Thus by continuity of μ from above, the singleton $\mathbf{y} = (y_i, y_i, \ldots)$ has measure δ, contradicting the fact that μ is non-atomic.

(2) $|A_m| \ge 2$.

Here, there are again two cases to consider. Either the elements of A_m form a simple periodic orbit (i.e., the elements of A_m form a single-valued cycle), or they do not. More formally, either $|F^{-1}(y_i) \cap Y| = 1$ for all $y_i \in A_m$ or it does not.

(2a) Suppose that $|F^{-1}(y_i) \cap Y| = 1$ for all $y_i \in A_m$. Label the elements of A_m as $a_1, \ldots, a_k$, such that $a_i \in F^{-1}(a_{i-1})$, for $1 < i \leq k$, and $a_1 \in F^{-1}(a_k)$. We construct a similar contradiction as in the case of $|A_m| = 1$. Let $\mu(\pi_0^{-1}(a_1)) = \delta > 0$. Then

$$\mu(\sigma^{-1}(\pi_0^{-1}(a_1))) = \delta.$$

As F is finite-to-one and the elements of A_m form a cycle, this means that

$$\mu(\pi_0^{-1}(a_2) \cap \pi_1^{-1}(a_1)) = \delta.$$

Similarly,

$$\mu(\bigcap_{i=0}^{k-1} \pi_i^{-1}(a_{i+1})) = \delta.$$

Then the singleton $\mathbf{y} = (a_1, a_2, \ldots a_k, a_1, a_2, \ldots, a_k, \ldots)$ will have measure δ, by continuity of μ from above. This contradicts μ being non-atomic.

(2b) This leaves the case that the elements of A_m do not form a single orbit. Then there exists $y_i \in A_m$ such that $F(y_i) \supset \{y_j, y_l\}$ for some $1 \leq j \neq l \leq m$. Now, by our note above, we know that any element of A_m eventually maps to any other. Then there exist $b_1, b_2, \ldots, b_r \in A_m$ such that $b_1 = y_j$, $b_r = y_i$, and $b_{n+1} \in F(b_n)$, for $1 \leq n \leq r-1$ and $c_1, c_2 \ldots c_s$ such that $c_1 = y_l$, $c_s = y_i$, and $c_{n+1} \in F(c_n)$, for $1 \leq n \leq s-1$. Then y_i has two distinct periodic orbits, $\mathbf{x} = (y_i, b_1, \ldots b_{r-1}, y_i, \ldots)$ and $\mathbf{y} = (y_i, c_1, \ldots c_{s-1}, y_i, \ldots)$.

□

Although our results are all in the context of $\text{Orb}^+(X)$, similar results of course exist for $\varprojlim F$, with minor adjustments. For example, in Theorems 4.25 and 4.26 we need to change the finite-to-one condition so that $|F(x)| < \infty$, for all $x \in X$.

BIBLIOGRAPHY

[1] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.*, 114(2):pp. 309–319, 1965.

[2] Tatsuya Arai and Naotsugu Chinen. P-chaos implies distributional chaos and chaos in the sense of Devaney with positive topological entropy. *Topology Appl.*, 154(7):1254–1262, 2007.

[3] Jean-Pierre Aubin, Hélène Frankowska, and Andrzej Lasota. Poincaré's recurrence theorem for set-valued dynamical systems. *Ann. Polon. Math.*, 54(1):85–91, 1991.

[4] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey. On Devaney's definition of chaos. *Amer. Math. Monthly*, 99(4):332–334, 1992.

[5] Marcy Barge, Henk Bruin, and Sonja Štimac. The Ingram conjecture. *Geom. Topol.*, 16(4):2481–2516, 2012.

[6] Marcy Barge and Beverly Diamond. The dynamics of continuous maps of finite graphs through inverse limits. *Trans. Amer. Math. Soc.*, 344(2):773–790, 1994.

[7] Marcy Barge and Joe Martin. Chaos, periodicity, and snakelike continua. *Trans. Amer. Math. Soc.*, 289(1):355–365, 1985.

[8] Marcy Barge and Joe Martin. The construction of global attractors. *Proc. Amer. Math. Soc.*, 110(2):523–525, 1990.

[9] R. H. Bing. A homogeneous indecomposable plane continuum. *Duke Math. J.*, 15:729–742, 1948.

[10] R. H. Bing. Concerning hereditarily indecomposable continua. *Pacific J. Math.*, 1:43–51, 1951.

[11] R. H. Bing. Snake-like continua. *Duke Math. J.*, 18:653–663, 1951.

[12] Louis Block, Slagjana Jakimovik, Lois Kailhofer, and James Keesling. On the classification of inverse limits of tent maps. *Fund. Math.*, 187(2):171–192, 2005.

[13] Louis Block, James Keesling, Brian Raines, and Sonja Štimac. Homeomorphisms of unimodal inverse limit spaces with a non-recurrent critical point. *Topology Appl.*, 156(15):2417–2425, 2009.

[14] A. M. Blokh. Decomposition of dynamical systems on an interval. *Uspekhi Mat. Nauk*, 38(5(233)):179–180, 1983.

[15] Rufus Bowen. Topological entropy and axiom A. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 23–41. Amer. Math. Soc., Providence, R.I., 1970.

[16] L. E. J. Brouwer. Zur Analysis Situs. *Math. Ann.*, 68(3):422–434, 1910.

[17] H. Cook and W. T. Ingram. Obtaining AR-like continua as inverse limits with only two bonding maps. *Glasnik Mat. Ser. III*, 4 (24):309–312, 1969.

[18] Manfred Denker, Christian Grillenberger, and Karl Sigmund. *Ergodic theory on compact spaces*. Lecture Notes in Mathematics, Vol. 527. Springer-Verlag, Berlin-New York, 1976.

[19] Robert Devaney. *An introduction to chaotic dynamical systems*. Addison - Wesley, 1989.

[20] Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology*. Princeton University Press, Princeton, New Jersey, 1952.

[21] O. H. Hamilton. A fixed point theorem for pseudo-arcs and certain other metric continua. *Proc. Amer. Math. Soc.*, 2:173–174, 1951.

[22] Phillip. Holmes. Poincare, celestial mechanics, dynamical-systems theory and "chaos". *Physics Reports*, 1990.

[23] W. T. Ingram. *An introduction to inverse limits with set-valued functions*. Springer Briefs in Mathematics. Springer, New York, 2012.

[24] W. T. Ingram and William S. Mahavier. Inverse limits of upper semi-continuous set valued functions. *Houston J. Math.*, 32(1):119–130, 2006.

[25] J. R. Isbell. Embeddings of inverse limits. *Ann. of Math. (2)*, 70:73–84, 1959.

[26] Jonathan Jaquette. Existence of topological entropy preserving subsystems weakly embeddable in symbolic dynamical systems.

[27] R. F. Jolly and J. T. Rogers, Jr. Inverse limit spaces defined by only finitely many distinct bonding maps. *Fund. Math.*, 68:117–120, 1970.

[28] Lois Kailhofer. A partial classification of inverse limit spaces of tent maps with periodic critical points. *Topology Appl.*, 123(2):235–265, 2002.

[29] Lois Kailhofer. A classification of inverse limit spaces of tent maps with periodic critical points. *Fund. Math.*, 177(2):95–120, 2003.

[30] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995.

[31] Judy Kennedy, Brian E. Raines, and David R. Stockman. Basins of measures on inverse limit spaces for the induced homeomorphism. *Ergodic Theory Dynam. Systems*, 30(4):1119–1130, 2010.

[32] Judy Kennedy, David R. Stockman, and James A. Yorke. Inverse limits and an implicitly defined difference equation from economics. *Topology Appl.*, 154(13):2533–2552, 2007.

[33] Judy Kennedy, David R. Stockman, and James A. Yorke. The inverse limits approach to chaos. *J. Math. Econom.*, 44(5-6):423–444, 2008.

[34] B Knaster. *Fundam. Math.*, 1922.

[35] Solomon Lefschetz. *Algebraic Topology.* 1942.

[36] William S. Mahavier. A chainable continuum not homeomorphic to an inverse limit on [0, 1] with only one bonding map. *Proc. Amer. Math. Soc.*, 18:284–286, 1967.

[37] William S. Mahavier. Inverse limits with subsets of $[0,1] \times [0,1]$. *Topology Appl.*, 141(1-3):225–231, 2004.

[38] Alfredo Medio and Brian E. Raines. Inverse limit spaces arising from problems in economics. *Topology Appl.*, 153(18):3437–3449, 2006.

[39] Alfredo Medio and Brian E. Raines. Backward dynamics in economics. The inverse limit approach. *J. Econom. Dynam. Control*, 31(5):1633–1671, 2007.

[40] Walter Miller and Ethan Akin. Invariant measures for set-valued dynamical systems. *Trans. Amer. Math. Soc.*, 351(3):1203–1225, 1999.

[41] Michał Misiurewicz. Horseshoes for mappings of the interval. *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, 27(2):167–169, 1979.

[42] Edwin E. Moise. An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua. *Trans. Amer. Math. Soc.*, 63:581–594, 1948.

[43] Christopher Mouron. Positive entropy homeomorphisms of chainable continua and indecomposable subcontinua. *Proc. Am. Math. Soc.*, 139(8):2783–2791, 2011.

[44] Van Nall. Finite graphs that are inverse limits with a set valued function of $[0,1]$. *Topology and its applications*, 158:1226–1233, 2011.

[45] Van Nall. the only finite graph that is aan inverse limit with a set valued funciton on $[0,1]$ is an arc. *Topology and its applications*, 159:733–736, 2012.

[46] Henri Poincare. New methods of celestial mechanics. 1892-1899.

[47] Henri Poincare. Lectures on celestial mechanics. 1905-1910.

[48] Brian Raines and Tim Tennant. The specification propery on a set-valued map and its inverse limit. *Houston Journal of Mathematics, to appear.*

[49] Brian E. Raines and David R. Stockman. Fixed points imply chaos for a class of differential inclusions that arise in economic models. *Trans. Amer. Math. Soc.*, 364(5):2479–2492, 2012.

[50] Sylvie Ruette. Chaos for continuous interval maps. http://www.math.u-psud.fr/ ruette/abstracts/abstract-chaos-int.html.

[51] O. M. Šarkovs′kiĭ. Co-existence of cycles of a continuous mapping of the line into itself. *Ukrain. Mat. Ž.*, 16:61–71, 1964.

[52] Casey Sherman. A Lebesgue-like measure for inverse limit spaces of piecewise strictly monotone maps of an interval. *Topology Appl.*, 159(8):2062–2070, 2012.

[53] Karl Sigmund. On dynamical systems with the specification property. *Trans. Amer. Math. Soc.*, 190:285–299, 1974.

[54] David R. Stockman. Uniform measures on inverse limit spaces. *Appl. Anal.*, 88(2):293–299, 2009.

[55] Scott Varagona. Inverse limits with upper semi-continuous bonding functions and indecomposability. *Houston Journal of Mathematics*, pages 1017–1034, 2011.

[56] Scott Varagona. *Simple Techniques for Detecting Decomposability or Indecomposability of Generalized Inverse Limits*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–Auburn University.

[57] Peter Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[58] Xiangdong Ye. The dynamics of homeomorphisms of hereditarily decomposable chainable continua. *Topology and its Applications*, 64(1):85–93, 1995.