ABSTRACT

Reflections on General Relativity from Perspectives of Black Hole Physics and Hořava-Lifshitz Gravity

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The physical logic of the Theory of General Relativity is summarized employing differential geometry of surfaces in three dimensional Euclidean space as a visual assistance for understanding. The applications of general relativity in fields as cosmology and black hole physics is summarized. Of the unsolved problems such as cosmology of very early universe, dark matter and dark energy, black hole singularity and quantization of gravity, some are more fundamental, while some are more about the specific application of general relativity to particular situations. As an application of general relativity, the Hawking temperature and tunneling rate for the Fermion tunnelling process was obtained for a regular black hole which has no singularity inside the body. Hořava-Lifshitz theory, proposed by some physicists as an attempt to fix the quantization problem of general relativity, is employed in the study of a holography problem.

Reflections on General Relativity from Perspectives of Black Hole Physics and Hořava-Lifshitz Gravity

by

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I suspect that truly I am that very person who had struggled the most in the entire human history in getting a Ph.D. in physics. After the completion of my B.S. in physics in 2003, I have been investing my entire last 12 years (2003-2015) to the pursuit of a doctorate in physics. During the entire 12 years, I did not suspend fighting for my final target even for a single day. Coming to the end of this unforgettable journey of life, I congratulate myself for the amazing perseverance and determination of myself. I owe my thanks to many people who have helped along my way so that no matter how far the journey I have to travel, I can get to the end.

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To All Who Have Been Caring and Supportive to My Life!

CHAPTER ONE

Introduction

The Theory of Relativity and Quantum Mechanics are the two cornerstone successes of the scientific revolution at the beginning of the 20th century. Quantum Mechanics revealed the untraceable behavior of individual microscopic physical objects and introduced wave function to describe the spacetime distribution of physical quantities of the physical object from a statistical perspective. Quantum Mechanics has thus became the theoretical foundation of material science and elementary particle physics, both whose prosperous developments glorified the 20th century.

Theory of Relativity opened another window of human understanding for the mystery of the physical world. The special theory of relativity pointed out for the first time the unity of space and time and their dependence on the motion of the subjective object. The objectivity of spacetime thus became limited and this fundamentally changed the view of human being towards the large scale physical universe.

This chapter provides a short summary of the basic ideas, the successes, problems and new developments of the theory of relativity, not through spicy argument and rigorous proof, but within an enjoyable volume of discourse. It also collects some of the basic formula needed in the following work of this dissertation.

1.1 Introduction to General Relativity

It is Newton's first law of mechanics which states that a mass point under no influence of external forces, would stay in a static state or maintain its motion of constant velocity. And thus such a frame of reference which satisfies Newton's first law of mechanics (or say, the frame of reference that allows the mass point inside it, if under no influence of external forces, to stay in a static state or to maintain its motion of constant velocity) is called an inertial frame of reference. But whether which among infinitely many possible actual frames of reference in the real physical world is the fundamental inertial frame of reference becomes a untouchable question, and this fundamental frame would only later be found not to exist.

Newtonian mechanics is valid for any pair of mutually inertia frames of reference connected by Galilean transformation. But Maxwell's theory of electromagnetism could not be maintained between an inertia pair of frames of reference if they are connected by Galilean transformation. The finding of the constancy of the speed of light in the electromagnetic theory requires the Lorentz transformation between pair of mutually inertia frames of reference. And thus was found the theory of special relativity. A Lorentz transformation, which is a coordinate transformation between two mutually inertia frames of reference, or two observers, guarantees that these two mutually inertia observers are to see the same laws of physics—they may observe different values of the same set of physical quantities, but this set of physical quantities satisfy the same group of physical equations.

The relativity between an inertia pair of frames connected by Galilean transformation appearing in the Newtonian description of the physical world is empirically sufficient for the majority of physical phenomena at macroscopic scales and in the low range of speed with respect to the speed of light. An inertia pair of frames involved in motion of high speed relative to speed of light respects nonetheless special relativity which is mathematically expressed as Lorentz transformation. And special relativity contains Newtonian/Galilean relativity as a low speed limit.

Special relativity allows mutually inertial observers to agree on the same laws of physics. But because of the variety of the motion of various local regions, the majority of the vast various local physical systems and observers are non-inertial relative to each other. It remains a question how would the multitude of mutually non-inertial observers or physical systems communicate what they each observe and agree on the same laws of physics. For a unique group of physical laws to be valid for every observer in the universe, it requires the theory of general relativity which builds a general coordinate transformation between any two mutually accelerating frames. To have the same physics anywhere, any time, for anyone in the universe, the physical laws have to satisfy the general covariance of coordinate transformation. And because locally the physical processes under gravity would be equivalent to those in an oppositely accelerating frame, the theory of general relativity, which builds ways for mutually non-inertial frames to communicate, would be a theory of gravity.

The location of physical events in spacetime is described by spacetime coordinates. Different observers do not even agree on the value of the coordinates of the same physical events. Neither do the different observers have to use the same type of coordinate systems. But those different observers want to be able to communicate their physics with each other – there needs to be a coordinate transformation existing between them. It is the mathematical quantity of tensors which keeps its format under the transformation of coordinates and the tensorial equation which keeps its format under transformation of coordinates that are to be employed as the suitable mathematical language for a physical theory of general relativity. The general covariance of physics is mathematically achieved by tensors and tensorial equations. As being hinted already by the effects of time dilation and length contraction in special relativity, the coordinate transformation between mutually accelerating observers in general relativity is expected to induce even more complicated spacetime effects. And the two different observers could possibly even see different spacetime effects for every different point in the whole spacetime region they agree to have both seen. Thus the metric tensor, a mathematical quantity capable of recording the spacetime effects at every local spacetime point in domain sobserbable for both observers enters into and plays the key role in the theory of general relativity.

1.2 Introduction to Differential Geometry

The suitable mathematical apparatus for the theory of general relativity is the Riemann geometry and Pseudo-Riemannia geometry in $n-$dimensional curved space, which was first proposed by Bernhard Riemann (1826-1866) in 1854 and has been developed into a powerful but highly abstract branch of modern mathematics. While the differential geometry of curves and surfaces in 3-dimensional Euclidean space, whose early development owes greatly to Leonhard Euler (1707-1783) and Johann Gauss (1777-1855) laid the theoretical foundation for Riemann geometry. A review of differential geometry of curves and surfaces in 3-dimensional Euclidean space, which will be provided in the following in a very accessible way, would serves as a very helpful visual assistance for the understanding of general relativity. This review is a concise summary of several classical textbooks $[1,2]$ on a subject around one to two hundred years old.

The same group of physical quantities can be described by two different coordinate systems, corresponding to two different observers, u^α and $\bar{u}^\alpha$ $(\alpha = 1, ..., n)$, with a coordinate transformation between them,

$$u^\alpha = u^\alpha(\bar{u}^\beta), \quad \bar{u}^\alpha = \bar{u}^\alpha(u(\beta)) \tag{2.1}$$

The infinitesimal change of coordinates being

$$du^\alpha = \frac{\partial u^\alpha}{\partial \bar{u}^\beta} d\bar{u}^\beta, \quad d\bar{u}^\alpha = \frac{\partial \bar{u}^\alpha}{\partial u^\beta} du^\beta \tag{2.2}$$

Obviously,

$$du^\alpha = \frac{\partial u^\alpha}{\partial \bar{u}^\beta} d\bar{u}^\beta = \frac{\partial u^\alpha}{\partial \bar{u}^\beta}\frac{\partial \bar{u}^\beta}{\partial u^\sigma} du^\sigma = \delta^\alpha_\sigma du^\sigma = du^\alpha$$

Thus

$$\frac{\partial u^\alpha}{\partial \bar{u}^\beta}\frac{\partial \bar{u}^\beta}{\partial u^\sigma} = \delta^\alpha_\sigma \tag{2.3}$$

the Kroneck delta relation is satisfied between the forward and inverse transformation matrices if the transformation is reversible.

Narrowing down to a 3-dimensional perspective: the parametric representation of a curve is given as $\mathbf{x} = (x_1(t), x_2(t), x_3(t))$, in 3-dimensional Euclidean space. The infinitesimal arc length vector connecting two neighboring points along the curve,

$$d\mathbf{x} = \mathbf{x}(t + dt) - \mathbf{x}(t) = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, ...\right) dt = \left(\frac{dx_i}{dt}\right) dt \tag{2.4}$$

As the interval between the two neighboring point gets to zero, $dt \to 0$, the length $|d\mathbf{x}| \to 0$, and the direction of the infinitesimal arc length vector $d\mathbf{x}$ tends to the tangent to the curve during that small interval. The infinitesimal arc length element is the scalar product:

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} = \sum_{i=1}^{3} \left(\frac{dx_i}{dt}\right)^2 \cdot dt^2 \tag{2.5}$$

If the representation parameter t is taken as the arc length s itself,

$$ds^2 = \frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{x}}{ds} ds^2 \tag{2.6}$$

$$\Rightarrow \mathbf{t} \equiv \frac{d\mathbf{x}}{ds}, \quad |\mathbf{t}| \equiv \left|\frac{d\mathbf{x}}{ds}\right| = 1 \tag{2.7}$$

Because the infinitesimal arc length vector $d\mathbf{x}$ is along the tangent direction to the curve at the corresponding point. The division of vector $d\mathbf{x}$ to the numerical arc length parameter ds is still a vector to the same direction as $d\mathbf{x}$ itself. The division by numerical infinitesimal length ds only scales the length of the vector, not to change the direction. The division of those two infinitesimal quantities becomes a finite quantity. Thus $\mathbf{t}$ should be the unit tangent vector to the curve at the corresponding point.

Similarly, for a surface in 3-dimensional Euclidean space in the parametric form,

$$\mathbf{x} = (x_1(u^1, u^2), x_2(u^1, u^2), x_3(u^1, u^2)) \tag{2.8}$$

$$\mathbf{x}_1 \equiv \frac{\partial \mathbf{x}}{\partial u^1}, \quad \mathbf{x}_2 \equiv \frac{\partial \mathbf{x}}{\partial u^2} \tag{2.9}$$

are the two tangent vectors to the surface along coordinates u^1, u^2 on the surface. These two tangent vectors make up a plane which is tangent to the surface at the point being considered and is thus called the tangent plane to the surface at the relevant point. These two tangent vectors do not align with each other otherwise it would mean the coordinates u^1, u^2 do not make a suitable coordinate system to describe the surface. The two tangent vectors are also usually not unit length because it is generally not easy to keep both of the coordinates u^1, u^2 the same as the arc length s. The order of differentiation for smooth surface will be interchangeable,

$$\mathbf{x}_{\alpha\beta} \equiv \frac{\partial^2 \mathbf{x}}{\partial u^\alpha u^\beta} = \frac{\partial^2 \mathbf{x}}{\partial u^\beta u^\alpha} \equiv \mathbf{x}_{\beta\alpha} \tag{2.10}$$

The tangent vector for a curve $\mathbf{x} = \mathbf{x}(u^1(t), u^2(t))$ lying on a surface would be,

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{x}}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \mathbf{x}}{\partial u^2}\frac{du^2}{dt} = \mathbf{x}_1 {u^1}' + \mathbf{x}_2 {u^2}' \tag{2.11}$$

where prime denotes derivative with respect to the curve parameter t.

The infinitesimal arc length element along this curve is

$$\begin{aligned} ds^2 &= \left(\frac{d\mathbf{x}}{dt}\right)^2 dt^2 \\ &= (\mathbf{x}_1 du^1 + \mathbf{x}_2 du^2)^2 \\ &= \mathbf{x}_1 \cdot \mathbf{x}_1 (du^1)^2 + 2(\mathbf{x}_1 \cdot \mathbf{x}_2) du^1 du^2 + \mathbf{x}_2 \cdot \mathbf{x}_2 (du^2)^2 \end{aligned} \tag{2.12}$$

Thus comes the definition of the Riemann metric tensor,

$$g_{\alpha\beta} \equiv \mathbf{x}_\alpha \cdot \mathbf{x}_\beta = \mathbf{x}_\beta \cdot \mathbf{x}_\alpha = g_{\beta\alpha}, (\alpha, \beta = 1, 2) \tag{2.13}$$

Thus the arc length squared can be denoted as,

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta \tag{2.14}$$

Repeated indices mean summation over all the allowable values of the index following Einstein summation convention. This equation is called the first fundamental form of differential geometry. It is the foundation of the intrinsic geometry of $n-$dimensional curved space, the Riemann geometry.

For a given arc length with fixed value, but described by different coordinate systems u^α, $\bar{u}^\mu$, in between a reversible transformation exists,

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta = g_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^\mu} \frac{\partial u^\beta}{\partial \bar{u}^\nu} d\bar{u}^\mu d\bar{u}^\nu = \bar{g}_{\mu\nu} d\bar{u}^\mu d\bar{u}^\nu = d\bar{s}^2 \tag{2.15}$$

Thus the transformation of the metric tensor induced by a coordinate transformation,

$$\begin{aligned} \bar{g}_{\mu\nu} &= g_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^\mu} \frac{\partial u^\beta}{\partial \bar{u}^\nu} \\ g_{\alpha\beta} &= \bar{g}_{\mu\nu} \frac{\partial \bar{u}^\mu}{\partial u^\alpha} \frac{\partial \bar{u}^\nu}{\partial u^\beta} \end{aligned} \tag{2.16}$$

Here comes the definition of tensor. Tensors are mathematical objects which take specific value in each specific coordinate system, change the value along with the coordinate transformation, but keep the format along with the coordinate transformation. For example, the metric tensor keeps its format along with coordinate transformation. This is why it is called a tensor. Metric tensor is an example of a covariant tensor of the second rank. To elaborate, metric tensor in different coordinate systems do not need to have the same values, for each of their componets,

$$g_{\alpha\beta} \leftrightarrow \bar{g}_{\mu\nu}$$

but the tensorial equation keeps the format throughout,

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta \leftrightarrow d\bar{s}^2 = \bar{g}_{\mu\nu} d\bar{u}^\mu d\bar{u}^\nu$$

Here we also have $ds^2 = d\bar{s}^2$ as a quantity kept constant under the coordinate transformation. It has no free index and is called a scalar. According to general covariance, physics laws should be expressed in tensor form for them to be kept invariant for different observers anywhere, anytime in the universe.

Briefly mentioning that, since

$$du^\alpha = \frac{\partial u^\alpha}{\partial \bar{u}^\beta} d\bar{u}^\beta \tag{2.17}$$

the infinitesimal coordinate elements du^α, and $d\bar{u}^\beta$ satisfy the definition of contravariant tensor of the first rank (also called contravariant vectors or merely vectors). The same form can be followed by an general vector which has finitely valued components,

$$a^\alpha = \frac{\partial u^\alpha}{\partial \bar{u}^\beta} \bar{a}^\beta \tag{2.18}$$

If there is a scalar function ϕ in the space, who has a constant value at fixed points in space regardless of how the coordinate systems being built, then

$$\frac{\partial \phi}{\partial u^\alpha} = \frac{\partial \phi}{\partial \bar{u}^\mu} \frac{\partial \bar{u}^\mu}{\partial u^\alpha} \tag{2.19}$$

The gradient $\frac{\partial \phi}{\partial u^\alpha}$ of a scalar function ϕ is called a covariant tensor of the first rank. Similarly, a general covariant tensor of first rank has the following form,

$$a_\alpha = \frac{\partial \bar{u}^\beta}{\partial u^\alpha} \bar{a}_\beta \tag{2.20}$$

The method of building tensors of higher ranks is similar,

$$R^{\mu\nu\ldots}_{\alpha\beta\ldots} \equiv \bar{R}^{\bar{\mu}\bar{\nu}\ldots}_{\bar{\alpha}\bar{\beta}\ldots} \left(\frac{\partial \bar{u}^{\bar{\alpha}}}{\partial u^\alpha} \frac{\partial \bar{u}^{\bar{\beta}}}{\partial u^\beta} \ldots \right) \left(\frac{\partial u^\mu}{\partial \bar{u}^\mu} \frac{\partial u^\nu}{\partial \bar{u}^\nu} \ldots \right) \tag{2.21}$$

Note that the transformation matrices of contravariant and covariant vectors are inverse to each other.

Also can be defined is the contravariant inverse metric tensor $g^{\alpha\beta}$ to the original covariant metric tensor $g_{\beta\sigma}$ such that

$$g^{\alpha\beta} g_{\beta\sigma} \equiv \delta^\alpha_\sigma \tag{2.22}$$

The metric and inverse metric tensors are used to switch covariant and contravariant tensors into each other,

$$a_\tau \equiv g_{\tau\alpha} a^\alpha \quad = \quad \left(\bar{g}_{\kappa\sigma} \frac{\partial \bar{u}^\kappa}{\partial u^\tau} \frac{\partial \bar{u}^\sigma}{\partial u^\alpha} \right) \left(\frac{\partial u^\alpha}{\partial \bar{u}^\beta} \bar{a}^\beta \right)$$

$$\begin{aligned} &= \bar{g}_{\kappa\sigma}\frac{\partial\bar{u}^\kappa}{\partial u^\tau}\delta^\sigma_\beta\bar{a}^\beta = \bar{g}_{\kappa\beta}\bar{a}^\beta\frac{\partial\bar{u}^\kappa}{\partial u^\tau} \\ &\equiv \bar{a}_\kappa\frac{\partial\bar{u}^\kappa}{\partial u^\tau} \end{aligned} \tag{2.23}$$

This much about tensors is essential for the understanding of differential geometry.

In the following, we will discuss about the geometry of curves and surfaces in 3-dimensional Euclidean space.

First, define the principal normal vector for a curve: a curve has at a point a unit tangent vector $\mathbf{t}$, such that $\mathbf{t}\cdot\mathbf{t} = 1$, which after being differentiated produces the orthogonality between the vectors,

$$\mathbf{t}\cdot\frac{d\mathbf{t}}{ds} = \mathbf{t}\cdot\dot{\mathbf{t}} = 0$$

Define the unit principle normal vector to a curve at some point on the curve as,

$$\mathbf{p}(s) = \frac{\dot{\mathbf{t}}(s)}{|\dot{\mathbf{t}}(s)|} = \frac{\dot{\mathbf{t}}(s)}{\kappa(s)} = \frac{\ddot{\mathbf{x}}(s)}{\kappa(s)} \tag{2.24}$$

where the dot on top of quantities denotes differentiation with respect to the arc length s. κ is the curvature of the curve which describes how sharply the curve is bending at the corresponding point and arises the definition of curvature radius as the inverse of the curvature,

$$\kappa(s) = |\dot{\mathbf{t}}(s)| = \sqrt{\ddot{\mathbf{x}}\cdot\ddot{\mathbf{x}}} = \frac{1}{\rho} \tag{2.25}$$

It can be shown that the orthogonal vectors (the unit tangent vector $\mathbf{t} = \dot{\mathbf{x}}$, and the unit principal normal vector $\mathbf{p} = \frac{1}{\kappa}\ddot{\mathbf{x}}$) make up a plane to which the curve is locally lying in. Taylor expanding a point on the curve $Q = \mathbf{x}(s+ds)$ near point $P = \mathbf{x}(s)$,

$$\mathbf{x}(s+ds) = \mathbf{x}(s) + \dot{\mathbf{x}}ds + \frac{1}{2}\ddot{\mathbf{x}}ds^2 + \cdots$$

$$\begin{aligned} Arc - Length - Vector[QP] &= \mathbf{x}(s+ds) - \mathbf{x}(s) \\ &= \dot{\mathbf{x}}ds + \frac{1}{2}\ddot{\mathbf{x}}ds^2 + \cdots \end{aligned}$$

$$= \mathbf{t}ds + \frac{1}{2}\kappa\mathbf{p}ds^2 + \cdots \tag{2.26}$$

The local arc length vector $[QP]$ is expanded by the unit tangent vector $\mathbf{t}$ and the unit principal normal vector $\mathbf{p}$. So the curve is locally lying in a plane expanded by $\mathbf{t}$, $\mathbf{p}$ called the osculating plane to the curve. The curve grows in the direction of $\mathbf{t}$ and bends to the direction of $\mathbf{p}$ with the curvature value κ.

Not only are there two tangent vectors to 2-dimensional surface at any point on it, the surface has at any point also a normal vector, which can be scaled into a unit normal vector. The unit normal vector, together with the two tangent vectors can be made to form a right-handed basis vector set for the 3-dimensional Euclidean space,

$$\mathbf{n} \equiv \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{\sqrt{g}} \tag{2.27}$$

where g is the determinant of the metric tensor, and the definition of the components of the metric tensor as the products between tangent vectors is used,

$$|\mathbf{x}_1 \times \mathbf{x}_2|^2 = |\mathbf{x}_1|^2|\mathbf{x}_2|^2 \sin^2\theta = |\mathbf{x}_1|^2|\mathbf{x}_2|^2(1-\cos^2\theta) = g_{11}g_{22}-(\mathbf{x}_1\cdot\mathbf{x}_2)^2 = g_{11}g_{22}-g_{12}^2 = g$$

The second fundamental form of differential geometry will now be defined and its geometric meaning will be given. Figure 1.1 [7] gives the coordinate system u^1,u^2 of a 2-dimensional surface S and the unit normal vector to the surface at a point P. A plane containing the normal vector $\mathbf{n}$ can cut the surface S at any angle which can be characterized by the ratio between u^2 and u^1. The curves of intersection, or cutting, are called the normal sections.

In Figure 1.1, the normal vector $\mathbf{n}$ to the surface at point P and the principal normal vector to the curve of intersection coincide with each other, $\mathbf{p} = \mathbf{n}$. Then, $\ddot{\mathbf{x}} = \kappa_n\mathbf{p} = \kappa_n\mathbf{n}$. In the sense that $\mathbf{p} = \mathbf{n}$, κ_n is said to be positive value at point P. We also have the following relations,

$$\mathbf{x}_\alpha = \frac{\partial\mathbf{x}}{\partial u^\alpha}, \quad \mathbf{x}_\alpha \cdot \mathbf{n} = 0, \quad \mathbf{x}_{\alpha\beta} = \mathbf{x}_{\beta\alpha} = \frac{\partial^2\mathbf{x}}{\partial u^\alpha u^\beta}$$

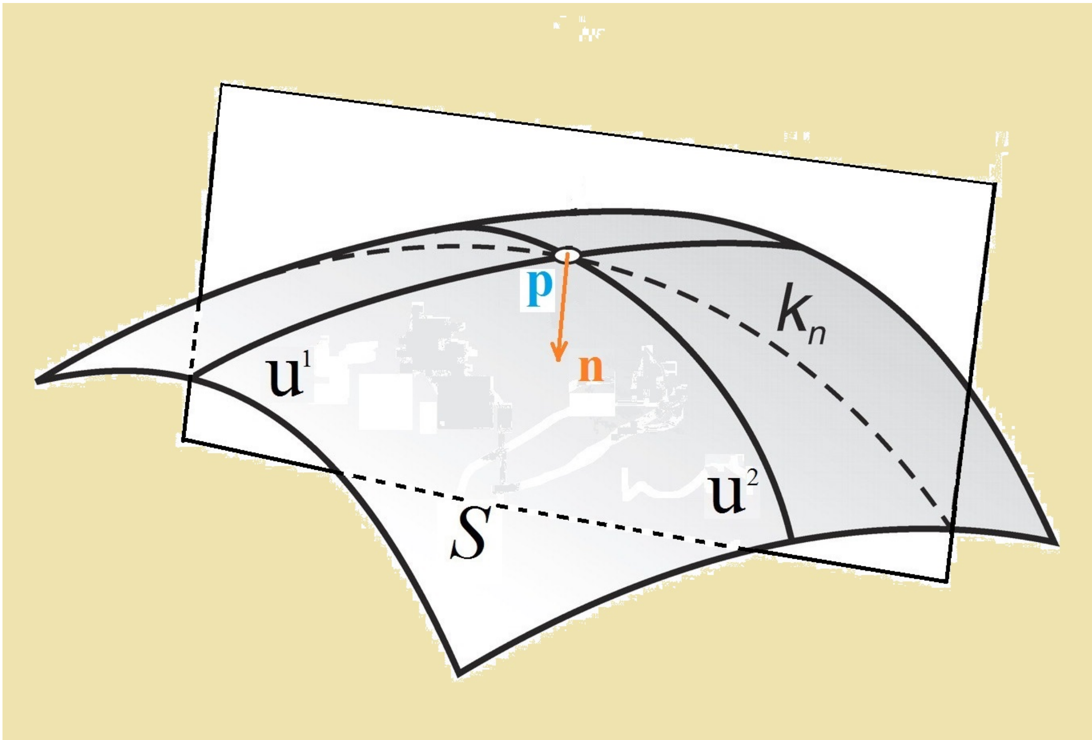

Figure 1.1. Normal section

$$\Rightarrow \dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial u^\alpha}\frac{du^\alpha}{ds} = \mathbf{x}_\alpha \dot{u}^\alpha, \quad \ddot{\mathbf{x}} = \mathbf{x}_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta + \mathbf{x}_\alpha \ddot{u}^\alpha$$

and the defining equation is obtained for $b_{\alpha\beta}$,

$$\begin{aligned}\ddot{\mathbf{x}}\cdot\mathbf{n} &= (\mathbf{x}_{\alpha\beta}\cdot\mathbf{n})\,\dot{u}^\alpha\dot{u}^\beta \\ &\equiv (b_{\alpha\beta})\,\dot{u}^\alpha\dot{u}^\beta \\ &= \kappa_n(\mathbf{p}\cdot\mathbf{n}) \\ &= \kappa_n \end{aligned} \tag{2.28}$$

Differentiating $\mathbf{x}_\alpha\cdot\mathbf{n} = 0$,

$$\mathbf{x}_{\alpha\beta}\cdot\mathbf{n} + \mathbf{x}_\alpha\cdot\mathbf{n}_\beta = 0, \quad \mathbf{n}_\beta \equiv \frac{\partial \mathbf{n}}{\partial u^\beta}$$

$$\Rightarrow b_{\alpha\beta} = \mathbf{x}_{\alpha\beta}\cdot\mathbf{n} = -\mathbf{x}_\alpha\cdot\mathbf{n}_\beta$$

Summarizing the above results:

$$ds^2 = g_{\alpha\beta}du^\alpha du^\beta = d\mathbf{x}\cdot d\mathbf{x}$$

is the first fundamental form of differential geometry.

$$b_{\alpha\beta}du^\alpha du^\beta = -\mathbf{x}_\alpha \cdot \mathbf{n}_\beta du^\alpha du^\beta = -d\mathbf{x} \cdot d\mathbf{n} \tag{2.29}$$

is called the second fundamental form of differential geometry.

$$\kappa_n = b_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} = \frac{b_{\alpha\beta}du^\alpha du^\beta}{ds^2} = \frac{b_{\alpha\beta}du^\alpha du^\beta}{g_{\rho\sigma}du^\rho du^\sigma} \tag{2.30}$$

The curvature for a normal section (called the normal curvature) equals to the ratio between the second and the first fundamental forms.

At a fixed point P on the surface, the values of $b_{\alpha\beta}$ and $g_{\alpha\beta}$ depend only on the choice of the coordinate system and the intrinsic geometry of the surface itself. However, the value of κ_n not only depend on the intrinsic geometry of the surface in the local region, but also depends on the direction of the cutting plane. This direction can be characterized by the ratio between du^2 and du^1 in the neighborhood of point P and can vary between negative infinity to positive infinity, as a real number.

The meaning of the second form $b_{\alpha\beta}du^\alpha du^\beta$ is shown below:

Let $Q(\mathbf{x} = \mathbf{x}(u^1+du^1, u^2+du^2))$ be a point close to the point $P(\mathbf{x} = \mathbf{x}(u^1, u^2))$. The distance from point Q to the tangent plane of the surface at point P is $D(Q)$

$$\begin{aligned} D(Q) &= \mathbf{n} \cdot \left[\mathbf{x}(u^1 + du^1, u^2 + du^2) - \mathbf{x}(u^1, u^2)\right] + \cdots \\ &= \mathbf{n} \cdot \left[\mathbf{x}(u^1, u^2) + \mathbf{x}_\alpha du^\alpha + \frac{1}{2!}\mathbf{x}_{\alpha\beta} du^\alpha du^\beta - \mathbf{x}(u^1, u^2)\right] + \cdots \\ &= \frac{1}{2}(\mathbf{n} \cdot \mathbf{x}_{\alpha\beta}) du^\alpha du^\beta \\ &= \frac{1}{2} b_{\alpha\beta} du^\alpha du^\beta \end{aligned} \tag{2.31}$$

Here again use orthogonality $\mathbf{x}_\alpha \cdot \mathbf{n} = 0$ between tangent vectors $\mathbf{x}_\alpha$ and the normal vector $\mathbf{n}$. We conclude that the second fundamental form of differential geometry is double the distance of a point from the tangent plane to a neighboring point.

To study the property of κ_n, expanding out the indices in the κ_n equation,

$$\kappa_n = \frac{b_{\alpha\beta}du^\alpha du^\beta}{g_{\rho\sigma}du^\rho du^\sigma} = \frac{b_{11}(du^1)^2 + 2b_{12}du^1 du^2 + b_{22}(du^2)^2}{g_{11}(du^1)^2 + 2g_{12}du^1 du^2 + g_{22}(du^2)^2}$$

Dividing both numerator and denominator by du^1 (and denoting $t = \frac{du^2}{du^1}$),

$$\kappa_n = \frac{b_{11} + 2b_{12}t + b_{22}t^2}{g_{11} + 2g_{12}t + g_{22}t^2} \tag{2.32}$$

We then conclude that: given the geometry of the surface at a point, a normal section has a certain value of curvature knowing its direction $t = \frac{du^2}{du^1}$ in the tangent plane. To see in what directions curvature κ_n can possibly reach some extreme values, differentiate κ_n with respect to t, noting that the denominator is always positive,

$$\frac{\partial \kappa_n}{\partial t} = 0 \Rightarrow (b_{12}+b_{22}t)(g_{11}+2g_{12}t+g_{22}t^2)-(g_{12}+g_{22}t)(b_{11}+2b_{12}t+b_{22}t^2) = 0 \tag{2.33}$$

which simplifies to a quadratic equation,

$$(g_{12}b_{22} - g_{22}b_{12})t^2 + (g_{11}b_{22} - g_{22}b_{11})t + (g_{11}b_{12} - g_{12}b_{11}) \equiv at^2 + bt + c = 0 \tag{2.34}$$

This equation has two real distinctive roots because its discriminant is positive,

$$\begin{aligned} \triangle &= b^2 - 4ac = (g_{11}b_{22} - g_{22}b_{11})^2 - 4(g_{12}b_{22} - g_{22}b_{12})(g_{11}b_{12} - g_{12}b_{11}) \\ &= 4\frac{g_{11}g_{22} - g_{12}^2}{g_{11}^2}(g_{11}b_{12} - g_{12}b_{11})^2 + \left[g_{11}b_{22} - g_{22}b_{11} - \frac{2g_{12}}{g_{11}}(g_{11}b_{12} - g_{12}b_{11})\right]^2 \\ &> 0 \end{aligned} \tag{2.35}$$

We conclude that along two directions (called principal directions), the maximum and minimum of the curvature of the normal sections to the surface is reached respectively. The corresponding curvatures are called the principal curvatures. It can be proved that the two principal directions are perpendicular to each other. In the tangent plane, the two principal directions can be denoted by $\frac{du^2}{du^1} = \frac{t_1}{1}$ and $\frac{du^2}{du^1} = \frac{t_2}{1}$. Thus these two direction vectors can be denoted as $\mathbf{t}_1 = 1\mathbf{x}_1 + t_1\mathbf{x}_2$, and $\mathbf{t}_2 = 1\mathbf{x}_1 + t_2\mathbf{x}_2$. The scalar product between these two direction vectors is,

$$\begin{aligned} \mathbf{t}_1 \cdot \mathbf{t}_2 &= (1\mathbf{x}_1 + t_1\mathbf{x}_2) \cdot (1\mathbf{x}_1 + t_2\mathbf{x}_2) \\ &= g_{11}1 \cdot 1 + g_{12}(t_1 + t_2) + g_{22}t_1 \cdot t_2 \end{aligned} \tag{2.36}$$

After substituting $t_1 + t_2 = -\frac{b}{a}$, $t_1 \cdot t_2 = \frac{c}{a}$ from above quadratic equation,

$$\begin{aligned}
\mathbf{t}_1 \cdot \mathbf{t}_2 &= g_{11} - g_{12}\frac{b}{a} + g_{22}\frac{c}{a} \\
&= \frac{1}{a}(g_{11}a - g_{12}b + g_{22}c) \\
&= \frac{1}{a}\left[g_{11}(g_{12}b_{22} - g_{22}b_{12}) - g_{12}(g_{11}b_{22} - g_{22}b_{11}) + g_{22}(g_{11}b_{12} - g_{12}b_{11})\right] \\
&= 0 \qquad (2.37)
\end{aligned}$$

Finally we can get the equation for the normal curvature:

$$\kappa_n = \frac{b_{11} + 2b_{12}t + b_{22}t^2}{g_{11} + 2g_{12}t + g_{22}t^2} = \frac{(b_{11} + b_{12}t) + (b_{12} + b_{22}t)t}{(g_{11} + g_{12}t) + (g_{12} + g_{22}t)t}$$

$$\Rightarrow \kappa_n = \frac{b_{11} + b_{12}t}{g_{11} + g_{12}t} = \frac{b_{12} + b_{22}t}{g_{12} + g_{22}t}$$

Cancelling t produces a quadratic equation for κ_n,

$$(g_{11}g_{22} - g_{12}^2)\kappa_n^2 - (g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22})\kappa_n + (b_{11}b_{22} - b_{12}^2) = 0 \qquad (2.38)$$

where $g_{11}g_{22} - g_{12}^2 = g$, $b_{11}b_{22} - b_{12}^2 = b$ are the determinants for metrices $g_{\alpha\beta}$, $b_{\alpha\beta}$.

$$\kappa_n^2 - \left[\left(\frac{g_{22}}{g}b_{11}\right) + 2\left(-\frac{g_{12}}{g}\right)b_{12} + \left(\frac{g_{11}}{g}\right)b_{22}\right]\kappa_n + \frac{b}{g} = 0$$

Using the direct computation of the components of inverse metric $g^{\alpha\beta}$ in terms of the components covariant metric $g_{\alpha\beta}$, this equation has a compact form,

$$\kappa_n^2 - g^{\alpha\beta}b_{\alpha\beta}\kappa_n + \frac{b}{g} = 0 \qquad (2.39)$$

Thus the Gaussian curvature is defined as,

$$K = \kappa_1\kappa_2 = \frac{b}{g} \qquad (2.40)$$

The mean curvature is,

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}g^{\alpha\beta}b_{\alpha\beta} = \frac{1}{2}b_\alpha^\alpha \qquad (2.41)$$

The sign of κ_n is determined by the numerator of its expression since the denominator, the length squared, is always positive, $ds^2 > 0$. It is always possible to make a coordinate transformation to make the second fundamental form (a quadratic form) diagonal, such that

$$b_{\alpha\beta}du^\alpha du^\beta = b_{11}(du^1)^2 + 2b_{12}du^1du^2 + b_{22}(du^2)^2 \to b_{11}(du^1)^2 + b_{22}(du^2)^2$$

The expressions for the invariant determinant of matrix $b_{\alpha\beta}$ and κ_n simplify,

$$b = b_{11}b_{22} - b_{12}^2 = b_{11}b_{22} \tag{2.42}$$

$$\kappa_n = \frac{b_{\alpha\beta}du^\alpha du^\beta}{g_{\rho\sigma}du^\rho du^\sigma} = \frac{b_{11}(du^1)^2 + b_{22}(du^2)^2}{ds^2}, \quad ds^2 > 0 \tag{2.43}$$

The classification of geometric shape of surfaces in the local region:

Case 1: If $b = det(b_{\alpha\beta}) = b_{11} \cdot b_{22} > 0$, b_{11}, b_{22} have the same sign, then the curvature κ_n of every normal section in all directions $\frac{du^2}{du^1}$ would also have the same sign. That is, all normal sections bend to the same direction relative to the fixed unit normal vector $\mathbf{n}$ to surface S at point P, and the surface is completely lying on the same side of the local tangent plane to the surface. The maximum and minimum values of κ_n, called the principal curvatures, is reached along the coordinate lines u^1, u^2 respectively. The Gaussian curvature is positive $K = \kappa_1\kappa_2 > 0$. P is called an elliptic point of the surface, Figure 1.2 [8]. Note that every point on the ellipsoid (including the sphere) is an elliptic point.

Case 2: If $b = b_{11} \cdot b_{22} = 0$, one of b_{11}, b_{22} and one of κ_1, κ_2 is zero. Gaussian curvature $K = \kappa_1\kappa_2 = 0$. P is called a parabolic point of the surface, Figure 1.3 [9]. We do not consider the case when both b_{11}, b_{22} are zero for the surface to be a flat plane.

Case 3: If $b = b_{11} \cdot b_{22} < 0$, b_{11}, b_{22} are opposite signs. Point P is called a hyperbolic point (or saddle point), Figure 1.4 [10], where one normal section curves down, while the orthogonal normal section bends up, so that the Gaussian curvature has a negative value $K = \kappa_1\kappa_2 < 0$. The tangent plane at point P cuts the surface

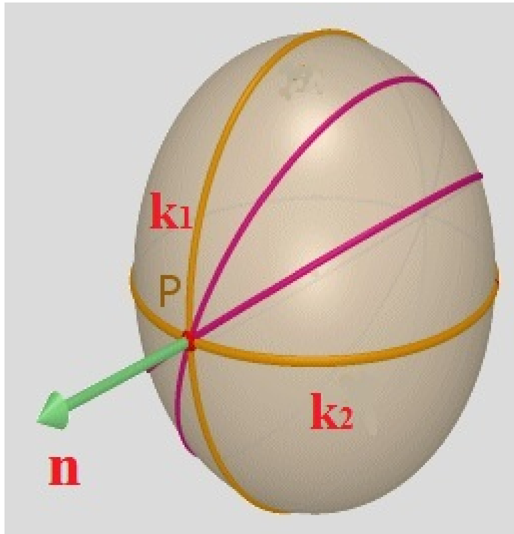

Figure 1.2. Elliptic point

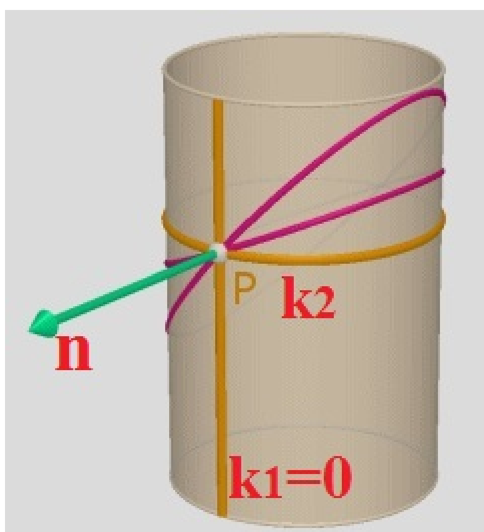

Figure 1.3. Parabolic point

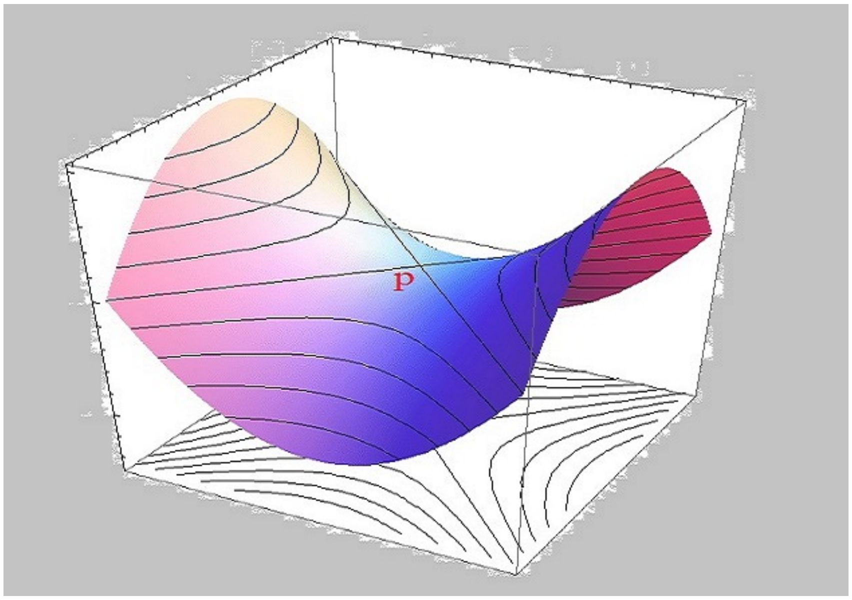

Figure 1.4. Hyperbolic point

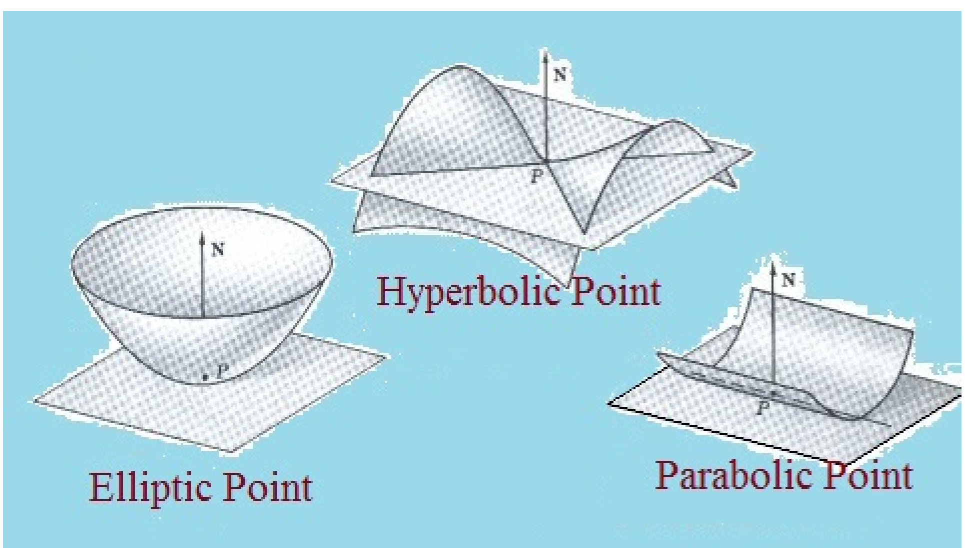

Figure 1.5. Elliptic point, hyperbolic point and parabolic point.

along two different directions, seen in both Figure 1.4 and Figure 1.5 [11]

$$\frac{du^2}{du^1} = \pm\sqrt{\left|\frac{b_{11}}{b_{22}}\right|}$$

The above important results on the classification of geometric shapes of 2-dimensional surfaces in 3-dimensional Euclidean space provides the fundamental preparation for the abstract insights into Riemann geometry of n-dimensional spaces.

Now we can introduce the important concepts of Christoffel symbols and Riemann curvature tensor. We know that $\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial u^1}$, $\mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial u^2}$ are the tangent vectors along coordinate curves u^1 and u^2 to a 2-dimensional surface at a point on the surface, and that $\mathbf{n}$ is the unit normal vector to the surface at that point. These three vectors form a complete vector basis for the 3-dimensional Euclidean space inside which the surface is embedding.

$\mathbf{x} = \mathbf{x}(u^1, u^2)$ is the position vector of any point of the surface in the 3-dimensional Euclidean space. Any order of differentiation of the position vector with respective to the surface parameters u^1, u^2 is of course still a vector in the 3-dimensional Euclidean space. As emphasized above, the tangent vectors $\mathbf{x}_\alpha$ and the unit normal vector $\mathbf{n}$ form a complete vector basis for the 3-dimensional Euclidean space, so the vector $\mathbf{x}_{\alpha\beta} = \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta}$ should have an expression in terms of these three basis vectors, with the following undetermined format, $\Gamma^\gamma_{\alpha\beta}$, and $a_{\alpha\beta}$ being expansion components,

$$\mathbf{x}_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}\mathbf{x}_\gamma + a_{\alpha\beta}\mathbf{n}$$

Doting $\mathbf{n}$ on both sides and knowing $\mathbf{x}_\gamma \cdot \mathbf{n} = 0$,

$$\Rightarrow a_{\alpha\beta} = \mathbf{n} \cdot \mathbf{x}_{\alpha\beta} = b_{\alpha\beta} \tag{2.44}$$

Doting $\mathbf{x}_\lambda$ on both sides, defining the Christoffel symbols,

$$\Gamma_{\alpha\beta\lambda} \equiv \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}_\lambda = \Gamma^\gamma_{\alpha\beta}\mathbf{x}_\gamma \cdot \mathbf{x}_\lambda = \Gamma^\gamma_{\alpha\beta} \cdot g_{\gamma\lambda} \tag{2.45}$$

Doting $g^{\lambda\kappa}$ on both sides and using $g_{\gamma\lambda}g^{\lambda\kappa} = \delta^{\kappa}_{\gamma}$,

$$\Gamma_{\alpha\beta\lambda} \cdot g^{\lambda\kappa} = \Gamma^{\kappa}_{\alpha\beta} \equiv \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}^{\kappa} = \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}_{\lambda} g^{\lambda\kappa} \tag{2.46}$$

$\Gamma_{\alpha\beta\lambda}$ is called the Christoffel symbols of the first kind. $\Gamma^{\kappa}_{\alpha\beta}$ is called the Christoffel symbols of the second kind.

Because $\mathbf{x}_{\alpha\beta} = \mathbf{x}_{\beta\alpha}$, the first two indices of Christoffel symbols are symmetric, i.e., $\Gamma_{\alpha\beta\lambda} = \Gamma_{\beta\alpha\lambda}$, $\Gamma^{\kappa}_{\alpha\beta} = \Gamma^{\kappa}_{\beta\alpha}$.

Differentiating $g_{\alpha\lambda} = \mathbf{x}_{\alpha} \cdot \mathbf{x}_{\lambda}$, and rotating the indices to get,

$$\begin{aligned} \frac{g_{\alpha\lambda}}{\partial u^{\beta}} &= \mathbf{x}_{\alpha\lambda} \cdot \mathbf{x}_{\beta} + \mathbf{x}_{\alpha} \cdot \mathbf{x}_{\lambda\beta} = \Gamma_{\alpha\beta\lambda} + \Gamma_{\lambda\beta\alpha} \\ \frac{g_{\lambda\beta}}{\partial u^{\alpha}} &= \Gamma_{\lambda\alpha\beta} + \Gamma_{\beta\alpha\lambda} \\ \frac{g_{\beta\alpha}}{\partial u^{\lambda}} &= \Gamma_{\beta\lambda\alpha} + \Gamma_{\alpha\lambda\beta} \end{aligned} \tag{2.47}$$

Summing up the first two equations, subtracting the third one, and noting the first two indices being symmetric, we arrive at:

$$\Gamma_{\alpha\beta\lambda} = \frac{1}{2}\left[\frac{\partial g_{\lambda\beta}}{\partial u^{\alpha}} + \frac{\partial g_{\lambda\alpha}}{\partial u^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\lambda}}\right] \tag{2.48}$$

Christoffel symbols are thus expressed in terms of the first derivatives of the components of the metric tensor. This definition which we get in 3-dimensional Euclidean space special case is actually correct in n-dimensional Riemann geometry. Note that Christoffel symbols are not a tensor.

Finally we get the following formula, called the formula of Gauss:

$$\mathbf{x}_{\alpha\beta} = \frac{\partial^2 \mathbf{x}}{\partial u^{\alpha}\partial u^{\beta}} = \Gamma^{\lambda}_{\alpha\beta}\mathbf{x}_{\lambda} + b_{\alpha\beta}\mathbf{n} \tag{2.49}$$

On the other hand, since

$$\frac{d}{du^{\alpha}}(\mathbf{n} \cdot \mathbf{n} = 1) \Rightarrow \mathbf{n} \cdot \mathbf{n}_{\alpha} = 0 \tag{2.50}$$

Thus $\mathbf{n}_{\alpha}$ must be a vector in the tangent plane, formally denoted as $\mathbf{n}_{\alpha} = c^{\gamma}_{\alpha}\mathbf{x}_{\gamma}$,

$$\frac{d}{du^{\alpha}}(\mathbf{n} \cdot \mathbf{x}_{\sigma} = 0) \Rightarrow -\mathbf{x}_{\alpha\sigma} \cdot \mathbf{n} = -b_{\alpha\sigma} = \mathbf{n}_{\alpha} \cdot \mathbf{x}_{\sigma} = c^{\gamma}_{\alpha}\mathbf{x}_{\gamma} \cdot \mathbf{x}_{\sigma} = c^{\gamma}_{\alpha} g_{\gamma\sigma} \tag{2.51}$$

The formula of Weingarten comes as following,

$$\mathbf{n}_\alpha \equiv \frac{\partial \mathbf{n}}{\partial u^\alpha} = -b_{\alpha\sigma} g^{\sigma\beta} \mathbf{x}_\beta = -b_\alpha^\beta \mathbf{x}_\beta \tag{2.52}$$

Both formula of Weingarten and formula of Gauss are the differential equations of space vectors either tangent or normal to a surface at every point on the surface in the 3-dimensional Euclidean space. For those differential equations to be integrable to reproduce these vectors as smooth solutions of those equations (which again would imply the existence of the smooth or differentiable surface itself), it requires the interchangeable order of differentiation as the integrability conditions:

$$\mathbf{x}_{\alpha\beta\lambda} \equiv \frac{\mathbf{x}_{\alpha\beta}}{\partial u^\lambda} = \frac{\mathbf{x}_{\alpha\lambda}}{\partial u^\beta} \equiv \mathbf{x}_{\alpha\lambda\beta} \tag{2.53}$$

Differentiating formulae of Gauss and substituting formulae of Weingarten,

$$\begin{aligned} \mathbf{x}_{\alpha\beta\lambda} &= \left[\frac{\partial \Gamma^\kappa_{\alpha\beta}}{\partial u^\lambda} + \Gamma^\kappa_{\alpha\beta}\Gamma^\sigma_{\kappa\lambda} - b_{\alpha\beta} b^\sigma_\lambda\right] \mathbf{x}_\sigma + \left[\Gamma^\rho_{\alpha\beta} b_{\rho\lambda} + \frac{\partial b_{\alpha\beta}}{\partial u^\lambda}\right] \mathbf{n} \\ \mathbf{x}_{\alpha\lambda\beta} &= \left[\frac{\partial \Gamma^\sigma_{\alpha\lambda}}{\partial u^\beta} + \Gamma^\kappa_{\alpha\lambda}\Gamma^\sigma_{\kappa\beta} - b_{\alpha\lambda} b^\sigma_\beta\right] \mathbf{x}_\sigma + \left[\Gamma^\rho_{\alpha\lambda} b_{\rho\beta} + \frac{\partial b_{\alpha\lambda}}{\partial u^\beta}\right] \mathbf{n} \end{aligned} \tag{2.54}$$

The basis vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{n}$ are all independent of each other. Equating the above two equations and comparing the coefficients of the basis vectors, we get the formulae of Mainardi-Godazzi,

$$\Gamma^\rho_{\alpha\beta} b_{\rho\lambda} - \Gamma^\rho_{\alpha\lambda} b_{\rho\beta} + \frac{\partial b_{\alpha\beta}}{\partial u^\lambda} - \frac{\partial b_{\alpha\lambda}}{\partial u^\beta} = 0 \tag{2.55}$$

and the definition of Riemann curvature tensor:

$$R^\sigma_{\alpha\lambda\beta} = b_{\alpha\beta} b^\sigma_\lambda - b_{\alpha\lambda} b^\sigma_\beta = \frac{\partial \Gamma^\sigma_{\alpha\beta}}{\partial u^\lambda} - \frac{\partial \Gamma^\sigma_{\alpha\lambda}}{\partial u^\beta} + \Gamma^\kappa_{\alpha\beta}\Gamma^\sigma_{\alpha\kappa} - \Gamma^\kappa_{\alpha\lambda}\Gamma^\sigma_{\kappa\beta} \tag{2.56}$$

Riemann curvature is a fundamental characterization of the curving property of surfaces. The familiar Gaussian curvature is one component of Riemann curvature.

The concept of geodesic will be explained in the following: if a plane cuts a surface S at point P at a right angle, so that the plane contains the local normal vector to the surface, the curve of intersection is called a normal section whose curvature at the point is called the normal curvature κ_n. But if the plane is not cutting at a right angle, the curve of intersection would have a larger curvature κ than the corresponding normal curvature. Treating curvatures as vectors, then that part of projection of κ along the normal vector is just the familiar normal curvature κ_n, and the part of the projection of κ in the tangent plane is called the geodesic curvature κ_g. A curve on the surface is called a geodesic if each point on the curve has a zero value geodesic curvature. Then we know that a curve on a surface is a geodesic for the surface if each point of the curve is locally the intersection of the surface with some plane locally normal to the surface. Or every small part of a geodesic to a surface is locally a part of a normal section to the surface. A geodesic can be shown as the arc of minimum length whose equation can be obtained by the method of variation. We briefly show this here.

For a section of curve C: $\mathbf{x}(s) = \mathbf{x}(u^1(s), u^2(s))$, $s_1 < s < s_2$, the arc length is the integration of the unit tangent vector along its arc length parameter,

$$s = \int_{s_1}^{s_2} ds = \int_{s_1}^{s_2} \sqrt{g_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}}ds = \int_{s_1}^{s_2} \sqrt{g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta}ds \equiv \int_{s_1}^{s_2} L(u^\alpha, \dot{u}^\beta, s)ds \tag{2.57}$$

It is a well-known result form classical mechanics that the variation of the shape of the curve produces the Euler-Lagrangian equation,

$$\frac{\partial L}{\partial u^\mu} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{u}^\mu}\right) = 0 \tag{2.58}$$

Computing both terms directly,

$$\frac{\partial L}{\partial u^\mu} = \frac{1}{2\sqrt{g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta}}\frac{\partial g_{\alpha\beta}}{\partial u^\mu}\dot{u}^\alpha\dot{u}^\beta = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial u^\mu}\dot{u}^\alpha\dot{u}^\beta \tag{2.59}$$

$$\begin{aligned} \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{u}^\mu}\right) &= \frac{d}{ds}\left[\frac{1}{2\sqrt{g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta}}\left(g_{\mu\alpha}\dot{u}^\alpha + g_{\mu\beta}\dot{u}^\beta\right)\right] \\ &= g_{\mu\alpha}\ddot{u}^\alpha + \frac{1}{2}\left[\frac{\partial g_{\mu\alpha}}{\partial u^\beta} + \frac{\partial g_{\mu\beta}}{\partial u^\alpha}\right]\dot{u}^\alpha\dot{u}^\beta \end{aligned} \tag{2.60}$$

Subtracting and using the definition of Chiristoffel symbol, we get,

$$\begin{aligned}
g_{\mu\alpha}\ddot{u}^{\alpha} &+ \frac{1}{2}\left[\frac{\partial g_{\mu\alpha}}{\partial u^{\beta}} + \frac{\partial g_{\mu\beta}}{\partial u^{\alpha}}\right]\dot{u}^{\alpha}\dot{u}^{\beta} - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial u^{\mu}}\dot{u}^{\alpha}\dot{u}^{\beta} \\
&= g_{\mu\alpha}\ddot{u}^{\alpha} + \frac{1}{2}\left[\frac{\partial g_{\mu\alpha}}{\partial u^{\beta}} + \frac{\partial g_{\mu\beta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\mu}}\right]\dot{u}^{\alpha}\dot{u}^{\beta} \\
&= g_{\mu\alpha}\ddot{u}^{\alpha} + \Gamma_{\alpha\beta\mu}\dot{u}^{\alpha}\dot{u}^{\beta} \\
&= 0 \qquad (2.61)
\end{aligned}$$

Raising the index, we get the so called geodesic equation,

$$\ddot{u}^{\tau} + \Gamma^{\tau}_{\alpha\beta}\dot{u}^{\alpha}\dot{u}^{\beta} = 0 \Longleftrightarrow \frac{d^2u^{\tau}}{ds^2} + \Gamma^{\tau}_{\alpha\beta}\frac{du^{\alpha}}{ds}\frac{du^{\beta}}{ds} = 0 \qquad (2.62)$$

This is a differential equation of the second order for a vector position function with only one variable parameter $u^{\tau}(s)$. In principle, if given two initial conditions, for example, the positions of the two end points of the geodesic, or the position of only one end and the direction of the geodesic at this point, the trajectory of geodesic can be obtained as a solution to this equation.

To be logically strict, it takes a few steps to prove that the curves satisfying geodesic equation are really geodesics, that is, with zero geodesic curvature, see Figure 1.6 [13].

Given that the curve of intersection by a non-normal plane with a curved surface has the curvature κ, the corresponding normal curvature would be $\kappa_n = \kappa\cos\theta$, and the geodesic curvature is the projection onto the tangent plane, $\kappa_g = \kappa\sin\theta$. Expressed in vectorial form,

$$\mathbf{k} = \mathbf{k_n} + \mathbf{k_g} = \kappa_n\mathbf{n} + \kappa_g\tilde{\mathbf{n}} \qquad (2.63)$$

Doting $\tilde{n}$ on the above equation,

$$\tilde{\mathbf{n}}\cdot\mathbf{k} = \tilde{\mathbf{n}}\cdot\kappa\mathbf{p} = \tilde{\mathbf{n}}\cdot\dot{\mathbf{t}} = (\mathbf{n}\times\mathbf{t})\cdot\dot{\mathbf{t}} = \kappa_g$$

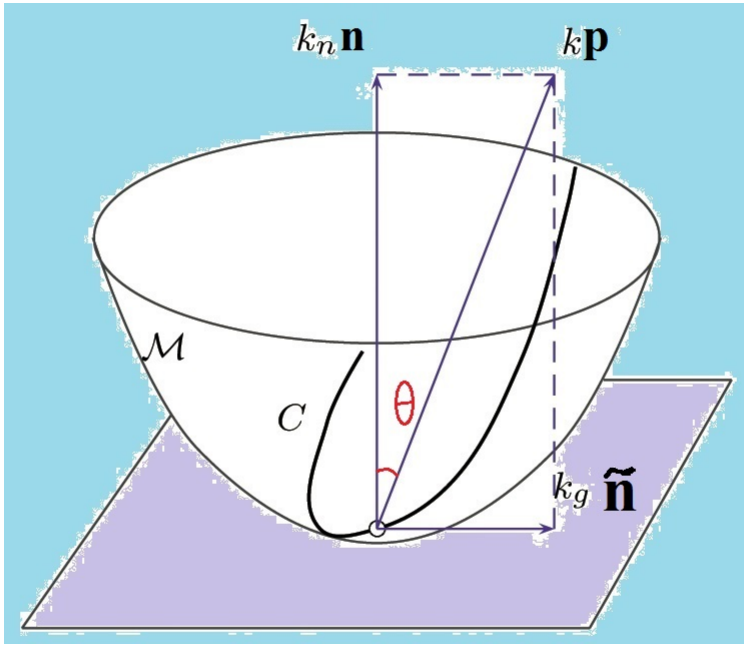

Figure 1.6. Geodesic Curvature

Switching the order of vector product,

$$\kappa_g = (\mathbf{n} \times \mathbf{t}) \cdot \dot{\mathbf{t}} = (\mathbf{t} \times \dot{\mathbf{t}}) \cdot \mathbf{n} = (\dot{\mathbf{x}} \times \ddot{\mathbf{x}}) \cdot \mathbf{n} \tag{2.64}$$

A geodesic is defined as a curve which satisfies the zero geodesic curvature requirement,

$$\kappa_g \equiv 0 \tag{2.65}$$

Using the parametric expression of a curve on a 2-dimensional surface,

$$\dot{\mathbf{x}} = \mathbf{x}_\alpha \dot{u}^\alpha$$

$$\ddot{\mathbf{x}} = \mathbf{x}_{\alpha\beta}\dot{u}^\alpha \dot{u}^\beta + \mathbf{x}_\tau \ddot{u}^\tau$$

Using the formula of Gauss,

$$\begin{aligned}
\kappa_g &= (\dot{\mathbf{x}} \times \ddot{\mathbf{x}}) \cdot \mathbf{n} = \left[\mathbf{x}_\lambda \dot{u}^\lambda \times (\mathbf{x}_{\alpha\beta}\dot{u}^\alpha \dot{u}^\beta + \mathbf{x}_\tau \ddot{u}^\tau)\right] \cdot \mathbf{n} \\
&= \left\{\mathbf{x}_\lambda \dot{u}^\lambda \times [(\Gamma^\tau_{\alpha\beta}\mathbf{x}_\tau + b_{\alpha\beta}\mathbf{n})\dot{u}^\alpha \dot{u}^\beta + \mathbf{x}_\tau \ddot{u}^\tau]\right\} \cdot \mathbf{n} \\
&= \left\{\mathbf{x}_\lambda \dot{u}^\lambda \times [(\Gamma^\tau_{\alpha\beta}\dot{u}^\alpha \dot{u}^\beta + \ddot{u}^\tau)\mathbf{x}_\tau + b_{\alpha\beta}\mathbf{n}\dot{u}^\alpha \dot{u}^\beta]\right\} \cdot \mathbf{n}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{\mathbf{x}_\lambda \dot{u}^\lambda \times [(\Gamma^\tau_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta + \ddot{u}^\tau)\mathbf{x}_\tau]\right\} \cdot \mathbf{n} \\
&= (\Gamma^\tau_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta + \ddot{u}^\tau)\dot{u}^\lambda(\mathbf{x}_\lambda \times \mathbf{x}_\tau) \cdot \mathbf{n} \\
&= 0 \qquad (2.66)
\end{aligned}
$$

Note that by the definition of the unit normal vector $\mathbf{n}$ in terms of the cross product of the tangent vectors $\Rightarrow \mathbf{x}_1, \mathbf{x}_2$, $\mathbf{x}_\lambda \times \mathbf{x}_\tau = \pm\sqrt{g}\mathbf{n}$. Also note that $\dot{u}^\lambda \neq 0$. Then the geodesic condition $\kappa_g = 0$ is produced exactly if the same geodesic equation as we get in the above holds true,

$$
\Gamma^\tau_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta + \ddot{u}^\tau = 0 \qquad (2.67)
$$

We summarize: a geodesic of a surface is the curve on the surface which has zero geodesic curvature, being a normal section to the surface in every local region, and being the curve of shortest/extreme length between two neighboring points on the surface.

We have almost exhausted the mathematical manipulations needed to construct the physical theory in a curved spacetime with the introduction of covariant differentiation. Ordinary differentiation is not a tensorial operation under coordinate transformations. It needs to be generalized into covariant differentiation for the construction of the theory of general relativity.

The direct ordinary differentiation of a vector

$$
\frac{\partial \bar{a}^\alpha}{\partial \bar{u}^\rho} = \frac{\partial}{\partial \bar{u}^\rho}\left(a^\beta \frac{\partial \bar{u}^\alpha}{\partial u^\beta}\right) = \frac{\partial a^\beta}{\partial u^\sigma}\frac{\partial \bar{u}^\alpha}{\partial u^\beta}\frac{\partial u^\sigma}{\partial \bar{u}^\rho} + a^\kappa \frac{\partial^2 \bar{u}^\alpha}{\partial u^\kappa \partial u^\sigma}\frac{\partial u^\sigma}{\partial \bar{u}^\rho} \qquad (2.68)
$$

is not a tensor, due to the second part, which we want to eliminate. It is done in the following way: differentiating the position vector and using the formula of Gauss,

$$
\mathbf{x}_\alpha = \frac{\partial \mathbf{x}}{\partial u_\alpha} = \frac{\partial \mathbf{x}}{\partial \bar{u}^\lambda}\frac{\partial \bar{u}^\lambda}{\partial u^\alpha} = \mathbf{x}_{\bar{\lambda}}\frac{\partial \bar{u}^\lambda}{\partial u^\alpha}
$$

$$
\begin{aligned}
\mathbf{x}_{\alpha\beta} &= \mathbf{x}_{\bar{\lambda}\bar{\epsilon}}\frac{\partial \bar{u}^\lambda}{\partial u^\alpha}\frac{\partial \bar{u}^\epsilon}{\partial u^\beta} + \mathbf{x}_{\bar{\sigma}}\frac{\partial^2 \bar{u}^\sigma}{\partial u^\alpha \partial u^\beta} \\
&= \left[\bar{\Gamma}^\sigma_{\lambda\epsilon}\mathbf{x}_{\bar{\sigma}} + \bar{b}_{\lambda\epsilon}\bar{\mathbf{n}}\right]\frac{\partial \bar{u}^\lambda}{\partial u^\alpha}\frac{\partial \bar{u}^\epsilon}{\partial u^\beta} + \mathbf{x}_{\bar{\sigma}}\frac{\partial^2 \bar{u}^\sigma}{\partial u^\alpha \partial u^\beta}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\bar{\Gamma}^{\sigma}_{\lambda\epsilon}\frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}}\frac{\partial \bar{u}^{\epsilon}}{\partial u^{\beta}} + \frac{\partial^2 \bar{u}^{\sigma}}{\partial u^{\alpha}\partial u^{\beta}}\right]\mathbf{x}_{\bar{\sigma}} + \bar{b}_{\lambda\epsilon}\frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}}\frac{\partial \bar{u}^{\epsilon}}{\partial u^{\beta}}\bar{\mathbf{n}} \\
&= \Gamma^{\gamma}_{\alpha\beta}\mathbf{x}_{\gamma} + b_{\alpha\beta}\mathbf{n} \\
&= \Gamma^{\gamma}_{\alpha\beta}\frac{\partial \bar{u}^{\sigma}}{\partial u^{\gamma}}\mathbf{x}_{\bar{\sigma}} + b_{\alpha\beta}\mathbf{n} \qquad (2.69)
\end{aligned}
$$

Note that $\mathbf{n} = \bar{\mathbf{n}}$, because the unit normal vector is an intrinsic quantity and is determined by the local shape of the surface, not by the choice of coordinate system. But the tangent vectors to the coordinate curves depend on different choice of coordinate systems. So comparing the coefficients, we obtain,

$$
b_{\alpha\beta}\mathbf{n} = \bar{b}_{\lambda\epsilon}\frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}}\frac{\partial \bar{u}^{\epsilon}}{\partial u^{\beta}} \qquad (2.70)
$$

$$
\Gamma^{\gamma}_{\alpha\beta}\frac{\partial \bar{u}^{\sigma}}{\partial u^{\gamma}} = \bar{\Gamma}^{\sigma}_{\lambda\epsilon}\frac{\partial \bar{u}^{\lambda}}{\partial u^{\alpha}}\frac{\partial \bar{u}^{\sigma}}{\partial u^{\beta}} + \frac{\partial^2 \bar{u}^{\sigma}}{\partial u^{\alpha}\partial u^{\beta}} \qquad (2.71)
$$

We want to combine the above equation into the formula of ordinary differentiation to cancel the unwanted part,

$$
\begin{aligned}
\frac{\partial^2 \bar{u}^{\alpha}}{\partial u^{\kappa}\partial u^{\sigma}} &= \Gamma^{\beta}_{\kappa\lambda}\frac{\partial \bar{u}^{\alpha}}{\partial u^{\beta}} - \bar{\Gamma}^{\alpha}_{\tau\epsilon}\frac{\partial \bar{u}^{\tau}}{\partial u^{\kappa}}\frac{\partial \bar{u}^{\epsilon}}{\partial u^{\sigma}} \\
\frac{\partial \bar{a}^{\alpha}}{\partial \bar{u}^{\rho}} &= \frac{\partial a^{\beta}}{\partial u^{\sigma}}\frac{\partial \bar{u}^{\alpha}}{\partial u^{\beta}}\frac{\partial u^{\sigma}}{\partial \bar{u}^{\rho}} + a^{\kappa}\left(\Gamma^{\beta}_{\kappa\lambda}\frac{\partial \bar{u}^{\alpha}}{\partial u^{\beta}} - \bar{\Gamma}^{\alpha}_{\tau\epsilon}\frac{\partial \bar{u}^{\tau}}{\partial u^{\kappa}}\frac{\partial \bar{u}^{\epsilon}}{\partial u^{\sigma}}\right)\frac{\partial u^{\sigma}}{\partial \bar{u}^{\rho}} \\
\Rightarrow \frac{\partial \bar{a}^{\alpha}}{\partial \bar{u}^{\rho}} &= \left(\frac{\partial a^{\beta}}{\partial u^{\sigma}} + a^{\kappa}\Gamma^{\beta}_{\kappa\sigma}\right)\frac{\partial \bar{u}^{\alpha}}{\partial u^{\beta}}\frac{\partial u^{\sigma}}{\partial \bar{u}^{\rho}} - \bar{a}^{\tau}\bar{\Gamma}^{\alpha}_{\tau\rho}
\end{aligned}
$$

where we have used $a^{\kappa}\frac{\partial \bar{u}^{\tau}}{\partial u^{\kappa}} = \bar{a}^{\tau}$ and $\frac{\partial \bar{u}^{\epsilon}}{\partial u^{\sigma}}\frac{\partial u^{\sigma}}{\partial \bar{u}^{\rho}} = \delta^{\epsilon}_{\rho}$.

Finally, define the covariant differentiation,

$$
\frac{\partial \bar{a}^{\alpha}}{\partial \bar{u}^{\rho}} + \bar{a}^{\tau}\bar{\Gamma}^{\alpha}_{\tau\rho} = \left(\frac{\partial a^{\beta}}{\partial u^{\sigma}} + a^{\kappa}\Gamma^{\beta}_{\kappa\sigma}\right)\frac{\partial u^{\alpha}}{\partial \bar{u}^{\beta}}\frac{\partial u^{\sigma}}{\partial \bar{u}^{\rho}} \qquad (2.72)
$$

which obeys the tensor transformation law. This is an extension of the ordinary differentiation, and is regarded as covariant differentiation. We use short hand notation,

$$
\frac{D\bar{a}^{\alpha}}{D\bar{u}^{\rho}} = \bar{a}^{\alpha}_{;\rho} = \frac{\partial \bar{a}^{\alpha}}{\partial \bar{u}^{\rho}} + \bar{a}^{\tau}\bar{\Gamma}^{\alpha}_{\tau\rho}
$$

$$\frac{Da^\beta}{Du^\sigma} = a^\beta_{;\sigma} \quad = \quad \frac{\partial a^\beta}{\partial u^\sigma} + a^\kappa \Gamma^\beta_{\kappa\sigma} \tag{2.73}$$

Similarly, the covariant differentiation for a covariant vector is,

$$\begin{aligned} \bar{a}_{\alpha;\rho} &= \frac{\partial \bar{a}^\alpha}{\partial \bar{u}^\rho} - \bar{a}^\tau \bar{\Gamma}^\alpha_{\tau\rho} \\ \bar{a}_{\beta;\sigma} &= \frac{\partial \bar{a}^\beta}{\partial u^\sigma} - a^\kappa \Gamma^\beta_{\kappa\sigma} \end{aligned} \tag{2.74}$$

Covariant derivatives in flat space reduce to ordinary derivatives.

Covariant derivative is shown to be useful in the definition of parallel transport. The concept of parallelism in flat space needs to be generalized in the context of curved spaces. Parallelism: if two vectors located at two neighboring points on a surface are making the same angle with the geodesic passing through these two points, these two vectors are said to be parallel to each other. It means the direction cosine of these vectors with the geodesic tangent vector is constant along the arc length parameter s of the geodesic. The variation of the direction cosine between the vector and the geodesic tangent vector with respect to the arc length parameter s of the geodesic produces the so called equation of parallel tansport,

$$\frac{Da^\alpha}{Ds} \equiv \frac{Da^\alpha}{Du^\sigma}\frac{du^\sigma}{ds} = \left(\frac{\partial a^\alpha}{\partial u^\sigma} + a^\kappa \Gamma^\alpha_{\kappa\sigma}\right)\frac{du^\sigma}{ds} = \frac{da^\alpha}{ds} + a^\kappa \Gamma^\alpha_{\kappa\sigma}\frac{du^\sigma}{ds} = 0 \tag{2.75}$$

If at a point with arc length parameter $s = s_0$, the components of a vector is given as $a^\alpha(s_0)$ in terms of the local coordinate tangent vectors $\mathbf{x}_\alpha(s_0)$, then the components of all the vectors at various positions of s, who are equal length as the original vector and are parallel to the original vector, in terms of the coordinate tangent vectors at each new location s, can be solved out from this equation.

With so much preparation, now the meaning of Einstein equation can be appreciated:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.76}$$

Here $R_{\mu\nu} = g_{\alpha\beta} R_{\alpha\mu\beta\nu} = R^{\beta}_{\ \mu\beta\nu}$, and $R = g^{\mu\nu} R_{\mu\nu}$. The left hand of Einstein equation are the geometric quantities characterizing the local geometric shape of the space, and the right hand side is the energy-momentum tensor $T_{\mu\nu}$ which describes the distribution and the motion of the mass and energy. So we summarize that the Einstein equation predicts that the spacetime geometry is not flat, but curved, and how it curves depends on the local distribution of mass and energy. Of course the motion of the mass and energy would also be confined to the curved spacetime.

This concludes our brief introduction to the logic and the mathematical basics of the theory of general relativity.

1.3 Applications and Problems of General Relativity

The theory of general relativity has become the theoretical foundation of modern cosmology and black hole physics. Black hole physics is the topic of the next chapter. On the other hand, quantization of gravity is believed to be necessary in the physical processes involving microscopic and high energy scales. But the first attempt to quantize the theory of general relativity has failed because the quantized theory of general relativity is not renormalizable. Hořava-Lifshitz theory is an theoretical proposal which at the price of breaking the Lorentz invariance (which is proved true for all typical gravitational and particle theory), brings in higher order spatial derivative terms into the theory. The appearance of these higher order spatial derivative terms makes Hořava-Lifshitz theory a power counting renormalizable theory of gravity. This would be the topic of another chapter.

We will make a short review of cosmology here and point out some of the questions remaining to be answered.

We have a hugely vast universe. Stars look like moving day by day if being observed on earth. But when we rise up to the perspective of the sun, the relative positions of the stars in the Milky Way galaxy are very constant. The evolution period of the solar system about the center of Milky Way is determined to be about 230 million years – a very very long time compared with the history of human civilization. Then outside of our homeland Milky Way galaxy, the Hubble Space Telescope has located up to now about 100 billion galaxies in our universe. The universe is huge. It looks almost uniform, almost isotropic. It is almost static and stable. Not exactly static, everything is actually in motion. And the radial motion looks like a bit more significant than the angular motion.

It is reasonable that the universe is a 3-dimensional Euclidean space plus an independent flow of time in the eyes of Newton. And even if special relativity is taken into account, the Minkowski spacetime is still a good average of the geometry of the spacetime. Of course in the small scale, the spacetime geometry in the huge vacuum and near the massive stars are not the same. But in the large scale, the universe is almost flat and uniform.

It was a great discovery in history that Edwin Hubble found that the universe is expanding. All of the galaxies in every direction are leaving us. And they are leaving at the speed roughly proportional to the distance from us – the Hubble's law. The only explanation to this phenomenon is that the universe as a whole is expanding.

The following Freedman-Robertson-Walker metric is one suitable to describe a universe which is almost unifrom, isotrpic and expanding in the large scale,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) \tag{3.77}$$

where $a(t)$ is the scale factor for the spatial extent of the universe. $a(t)$ is a function of time and allows the universe to undergo the process of expansion or contraction. K is a characteristic quantity to be specified by more accurate cosmological observations whose value would categorize the geometry of the entire universe to be an flat spacetime if $K = 0$, closed spacetime $K = 1$, or open spacetime if $K = -1$. The concept of open, closed or flat universe is just a higher dimensional generalization of the spaces of elliptic, hyperbolic or parabolic points we have just studied in the previous section of differential geometry in 3-dimensional Euclidean space.

Two major types of material contents are there in the universe: the particle-like matter which comprises the shining stars we see in the night sky, and the radiation which was discovered in the 1960s as Cosmic Microwave Background radiation (CMB).

For particle-like, or dust-like material, which has mass density but no pressure, the decrease of the mass density along with the scale of the universe would be,

$$\rho_{matter} \propto a^{-3} \tag{3.78}$$

While the energy density of the radiation, or the photon gas drops faster,

$$\rho_{radiation} \propto a^{-4} \tag{3.79}$$

Here we know that, along with the expansion of the universe, the mass density of photon gas drops quicker than the particle-like or solid matter. Then we deduce that, even though today's universe has relatively more solid matter than photon gas, but if we consider the reversed process, when the universe was younger and younger, the relative mass density of the photon gas to that of the solid matter should be the higher and higher the earlier in the history of the universe. Then there should be a time early in the history of the universe when the mass density of the photon gas was equal to, and when even earlier, be higher than that of the solid matter. Then the history of the universe should be roughly be divided into two major stages, the earlier stage when the radiation was the dominant energy content of the universe, and the later stage when the solid matter is the dominant energy/matter content.

When the universe was younger, the scale of the universe was smaller, the radiation was more squeezed, and the temperature of the radiation was higher. When early in the universe, when the temperature and pressure of the photon gas was high, the photon gas could be called the photon soup. The particle-like matter can not exist in the hot photon soup – they were melt in the hot photon soup. So the earlier in the universe, we did not have particle-like matter. All matter existed in the elementary particle form. Only when the universe expands, and cools down, the elementary particles coalesce into the particle-like matter we see everywhere in the universe today. This process is called Big Bang Nuclear Synthesis. Thermodynamic calculations show that it was happening roughly in the first few minutes of the life of the universe. It is one of the greatest successes of cosmology.

But mainly for two reasons we believe there should be dark matter in the universe. The universe can not cool down so quickly as we had if there were not dark matter. And we did see the direct evidence of dark matter in most of the galaxies by studying the spiraling speed of the galaxies. Most of the galaxies should have a smearing dark matter distribution so that the spiraling speed of the outer stars are not decreasing so quickly with the radius. But whether the dark matter content is more likely black holes or supersymmetric particles, it is the present topic of research.

We also think we should have something called dark energy in the universe. Because it was recently discovered by the supernova luminosity measurements that the universe at present day is expanding faster than we expected. It could be possible according to cosmology theory built on the theory of general relativity only if we had some fancy type of energy, the dark energy which is not attracting the universe but further extending the universe.

A stage called inflation is supposed to be happening in the very early part of the radiation period of the universe. We see almost uniform temperature in the CMB in all directions of the universe. But we think there should not be enough time for distant parts of the universe to communicate in the process of the cosmological evolution so that they can all pick up a common temperature. We speculate that a small region of physically connected early universe was suddenly inflated into large scale so that they are not able to communicate anymore even by the speed of light. But they all bring the same information at the time when they lost contact. It is like the situation that people from the same family suddenly lost contact from each other due to some disaster, later we the intermediate have the chance to meet all of them one by one and spot the same genetics in all the family members and recognize them as from the same family. But how was the inflation ended and reheat the universe and switch back to the normal expansion process, it is still a problem under study.

To understand what was happening even before that, in the high energy, small scale, we need a quantum cosmology theory.

We summarize, dark matter, dark energy, reheating or end of inflation and quantum cosmology, these are the four major problems of cosmological study of today.

Then we end a very brief chapter on the basic ideas of general relativity, differential geometry and cosmology.

CHAPTER TWO

Black Hole Physics

2.1 Black Holes and Black Hole Thermodynamics

After the theory of general relativity was proposed by Einstein in 1915, the first solution ever obtained for Einstein equation of gravitational field was a black hole solution. It is the Schwarzchild static black hole solution [3–6],

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad (2.1)$$

Schwarzschild black hole is static. While taking the limit of $r \Rightarrow \infty$, the Schwarzschild solution reduces to the Minkowski metric in spherical polar coordinates,

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad (2.2)$$

When r is small, it is not exactly same meaning as radius because the spacetime is strongly curved in the region when r is small. But in the large distance limit, it reduces to the normal radius as seen by a distant observer of the black hole. The Schwarzchild black hole has a horizon and surface of infinite red shift both coincide at the location $r = 2GM$. While at $r = 0$ it has an intrinsic singularity. The point of singularity is the position where the mass of the black hole is located at.

Later in total four types of black hole solutions have been found. Mass M, charge Q and angular momentum J are the only three properties a black hole can carry. All the four black holes have mass of course. They differ only in whether they are charged and rotating:

1) Schwarzchild black hole: has mass, but no net charge and no angular momentum;

2) Reissner-Norstrom black hole: has mass and net charge, but no angular momentum;

3) Kerr black hole: has mass and angular momentum;

4) Kerr-Newman black hole: has mass, net charge and angular momentum.

Take a quick look at the Reissner-Norstrom black hole:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{\epsilon^2}{r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{\epsilon^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2.3)$$

where $\epsilon = G(Q^2 + P^2)$, Q is the total electric charge of the black hole, and P is the total magnetic charge of the black hole. When $\epsilon \to 0$, the solution also reduces to the Schwarzschild solution.

The charged black hole solutions have their theoretical interests. But the possibility for the existence of charged stars and charged black hole in the real universe is very low. While angular momentum is a universal phenomenon in the universe. Thus of more practical interest is the Kerr black hole solution.

The Kerr black hole written in Boyer-Lindquist coordinates is the following,

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 + \frac{2GMar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) \\ &\quad + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2 \end{aligned} \quad (2.4)$$

where

$$\Delta(r) = r^2 - 2GMr + a^2 \quad (2.5)$$

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta \quad (2.6)$$

and the angular momentum per unit mass,

$$a = \frac{J}{M} \quad (2.7)$$

The value of the Riemann invariant $R^{abcd}R_{abcd}$ shows us that the Kerr metric has only one intrinsic singularity and that is when $\rho^2 = 0$. Since

$$\rho^2 = r^2 + a^2 cos^2\theta = 0 \quad (2.8)$$

it follows that $r = \cos\theta = 0$. $\cos\theta = 0$ specifies the equator plane. But $r = 0$ is not the origin point but rather a disk. Together $\cos\theta = 0$ and $r = 0$ gives the shape of the singularity of Kerr black hole as a horizontal rotating massive thin ring,

$$x^2 + y^2 = a^2, z = 0 \quad (2.9)$$

The Kerr black hole is physically interpreted as the spacetime geometry induced by a rotating massive thin disk spinning in the center of the black hole.

For the Schwarzchild black hole, the horizon and the surface of infinite gravitational red shift coincide with each other. But these two types of surfaces are separate for the Kerr black hole. The surfaces of infinite red shift in the Kerr solution are again given by the vanishing of the time component of metric tensor,

$$g_{00} = \frac{r^2 - 2Mr + a^2 cos^2\theta}{\rho^2} = 0$$

which gives the positions of the surfaces of infinite red shift (in natural unit $G = 1$),

$$r = r_{s\pm} = M \pm \sqrt{M^2 - a^2 \cos^2\theta} \tag{2.10}$$

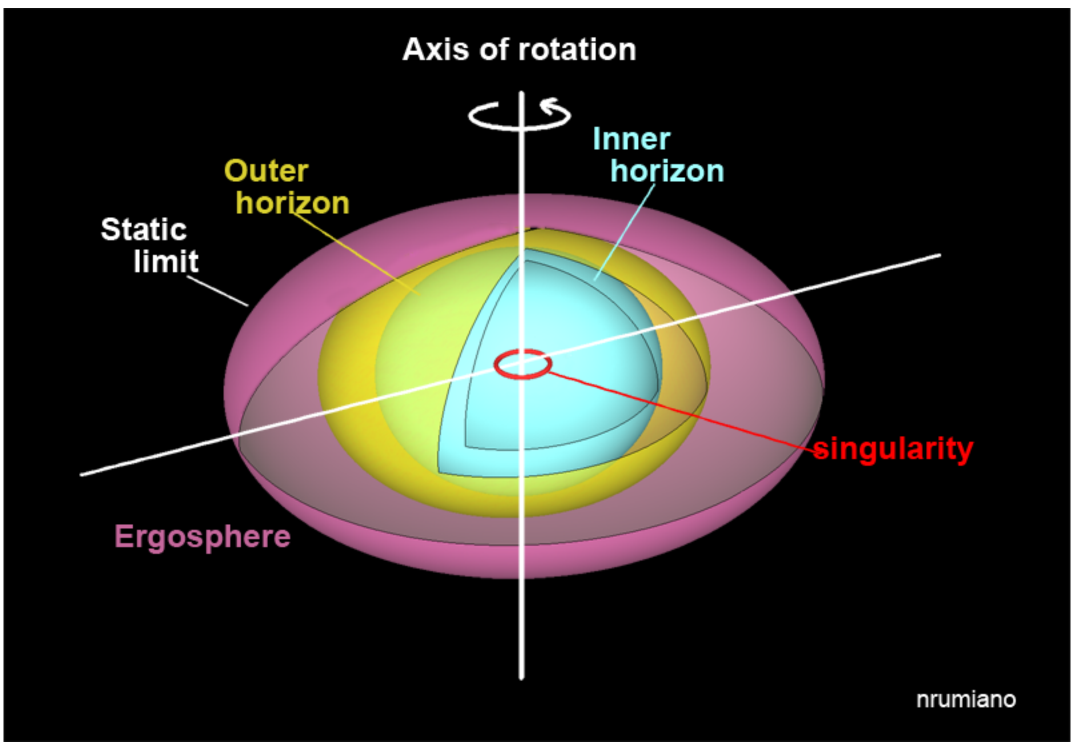

Figure 2.1. Shape of Event Horizon and Ergosphere of Kerr Black Hole.

In the Schwarzschild limit $a \Rightarrow 0$, the radius of the outer surface r_{s+} reduces to $r = 2M$ and the radius of the inner surface r_{s-} shrinks to $r = 0$. We are concerned with the case $a^2 < M^2$, when the spin is small compared with the mass.

While the position of the event horizon is determined by,

$$\Delta = r^2 - 2Mr + a^2 = 0 \tag{2.11}$$

which results in two null event horizons (assuming $a^2 < M^2$)

$$r = r_{\pm} = M \pm \sqrt{(M^2 - a^2)} \tag{2.12}$$

Similarly, in the limit $a \Rightarrow 0$, the two event horizons reduce to $r = 2M$ and $r = 0$. It again confirms that the surface of infinite red shift and the event horizons coincide for the Schwarzschild black hole. The outer event horizon $r = r_+$ lies entirely within the outer surface of infinite red shift. The region in between these two surfaces is called the ergoshpere, see Figure 2.1 [12].

Black hole thermodynamics is another interesting topic.

People did not think of black hole as a thermodynamic system before 1970s. It was mainly by the work Jacob Bekenstein and Stephen Hawking that the black hole thermodynamic was built.

Assuming a closed universe system composed of a black hole and a thermodynamic system. The thermodynamic system carries some amount of entropy which would never decrease with time according to the second law of thermodynamics. But due to the gravitational attraction of the black hole, the thermodynamic system falls into the black hole and after passing through the black hole horizon, the outside universe is left with nothing thermodynamic. If the black hole is not a thermodynamic system itself, the total entropy of the closed universe system composed of the black hole and the thermodynamic system would have decreased after the falling of the thermodynamic system into the black hole. So it was concluded that black holes need to be a thermodynamic system. They have both entropy and temperature.

The microscopic interpretation of entropy implies that black hole must have some microscopic structure inside because if all of the black hole mass, charge and angular momentum are accumulated to the single point of singularity or to the thin disk of singularity, it is not imaginable how can those singularities provide the mechanism for the existence of the entropy for the black hole.

The temperature of black hole can be defined by the following thought experiment. Suppose the energy of a box of radiation with temperature T and mass m somewhere far away from a black hole is being converted into the mechanical work by a heat engine in the process when the box is being gradually delivered approaching the black hole horizon. The size of the box is determined by the de Broglie wavelength of the radiation,

$$d \approx \frac{\hbar c}{k_B T}$$

The radiation is then released into the black horizon and one working cycle of the heat engine is completed. Then the work intractable in the process is,

$$\begin{aligned} W &= mc^2 + mg\frac{d}{2} = mc^2 - m\frac{c^4}{4GM}\frac{d}{2} \\ &= mc^2\left(1 - \frac{\hbar c^3}{8GMk_BT}\right) \end{aligned} \tag{2.13}$$

Compare with the definition of the efficiency of the heat engine,

$$\eta = \frac{W}{mc^2} = 1 - \frac{T_H}{T} = 1 - \frac{\hbar c^3}{8GMk_BT} \tag{2.14}$$

Thus the black hole temperature, or Hawking temperature T_H is identifies as,

$$T_H = \frac{\hbar c^3}{8GMk_B} \tag{2.15}$$

This result from simplified analysis is very close to the exact result proved by Hawking in 1975 [25],

$$T_H = \frac{\hbar c^3}{8\pi GMk_B} = \frac{\kappa}{4\pi k_B} \tag{2.16}$$

where κ is the the surface gravity on the horizon of the black hole.

From here the Bekeknstein entropy S_B for a Schwarzschild black hole can be derived. Using the first law of thermodynamics $dE = T_H dS$, and $E = Mc^2$,

$$dE = c^2 dM = T_H dS = \frac{\hbar c^3}{8\pi GMk_B} dS_B \tag{2.17}$$

$$\Rightarrow \frac{1}{k_B} dS_B = \frac{4\pi G}{\hbar c} dM^2$$

Assuming a black hole is growing from zero mass to the final mass gradually, so we can do the integral of above equation,

$$\frac{S_B}{k_B} = \frac{4\pi G}{\hbar c} M^2 = \frac{\pi}{G\hbar c}(2GM)^2 = \frac{4\pi G}{\hbar c} M^2 \tag{2.18}$$

Use $r_H = \frac{2GM}{c^2}$ for the radius of event horizon of the Schwarzchild black hole,

$$\frac{S_B}{k_B} = \frac{\pi c^3}{G\hbar}(r_H)^2 = \frac{c^3 A_H}{4G\hbar} = \frac{A_H}{4l_P^2} \tag{2.19}$$

The definition of the Planck length $l_p = \sqrt{\frac{G\hbar}{c^3}}$ is used in the last equal sign. A_H is the surface of the event horizon for the Schwarzchild black hole. Thus comes the famous result that the black hole entropy is proportional to the surface of the black hole horizon. The proof of the non-decreasing total surface of processes involving one or more black hole by Hawking eventually established the second law for thermodynamics for the black hole,

$$dM = TdA + \Omega dJ + \Phi dQ$$

$\frac{dA}{dt} \geqslant 0$, both the black hole horizon surface and entropy of closed thermodynamic systems are non-decreasing functions of time.

2.2 Introduction to Regular Black Holes

Penrose and Hawking [14–17] have proved in the classical theory of general relativity the so called singularity theorem which argues that under quite general assumptions, the spacetime evolution will inevitably lead to some singularity. Physical laws lose their predictability at those singularities. Attempts have been made to solve the problem of singularity.

The hope comes from the expectation that under situations where quantum effects become strong, the behavior of gravity could possibly greatly deviate from that predicted by the classical theory of general relativity. Many attempts have been made to construct black hole solutions without singularity inside. They are called regular black holes.

James Bardeen was the first to proposed a regular black hole solution in 1968 [18]. Bardeen argued that the metric components of the ReissnerNordstrom black hole can be replaced by the form

$$g_{tt} = g^{rr} = 1 - \frac{2mr^2}{(r2 + Q2)^{3/2}}$$

which has the same spacetime geometry outside the black hole but has no singularity inside, thus was called a regular black hole. Several similar black hole solutions were proposed [19–24]. Some of those results propose regular black hole solutions but without specify the energy-momentum sources. Some are using the nonliear electromagnetic field as the energy-momentum source. The nature of the these regular black holes in the region far from the position $r = 0$ is the same as ordinary black hole, but the property is greatly different near the position $r = 0$. The value of all of the geometric quantities such as curvatures are finite in the whole range of the coordinate parameters. For example, one charged regular black hole solution is given as [24],

$$\begin{aligned} ds^2 = & - \left(1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2}\right) dt^2 \\ & + \left(1 - \frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \end{aligned} \tag{2.20}$$

and is sourced by the non-linear electromagnetic field,

$$E = qr^4 \left(\frac{r^2 - 5q^2}{(r^2+q^2)^4} + \frac{15}{2}\frac{m}{(r^2+q^2)^{7/2}}\right) \tag{2.21}$$

This black hole behaves like normal Reissner-Nordstrm black hole in the large r limit,

$$-g_{tt} = 1 - \frac{2m}{r} + \frac{q^2}{r^2} + O\left(\frac{1}{r^3}\right)$$

$$E = \frac{q}{r^2} + O\left(\frac{1}{r^3}\right)$$

The geometric characteristic quantities of this black hole have been shown to be well-behaved through out the range of parameters. This is one of the typical ways of constructing regular black hole solutions.

The other typical way of regular black hole solution construction relies on the understanding that there could exist a fundamental minimal length scale arising from some quantum mechanism.

Due to the existence of a minimal length scale θ in the noncommutative geometry inspired black hole theory, the mass distribution inside the black hole is not singular, but smearing. Given the mass distribution,

$$\rho(r) = \frac{M}{4\pi\theta^{3/2}} \exp\left(-\frac{r^2}{4\theta}\right) \tag{2.22}$$

The Schwarzschild-like black hole solution from the noncommutative geometry inspired black hole theory [26] reads,

$$\begin{aligned} ds^2 = & -\left(1 - \frac{2MG}{\sqrt{\pi}r}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)\right) dt^2 \\ & + \left(1 - \frac{2MG}{\sqrt{\pi}r}\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right)\right)^{-1} dr^2 + r^2 d\Omega^2 \end{aligned} \tag{2.23}$$

where γ is the lower incomplete Gamma function, with the definition,

$$\gamma\left(\frac{3}{2}, \frac{r^2}{4\theta}\right) \equiv \int_0^{\frac{r^2}{4\theta}} t^{\frac{1}{2}} e^{-t} dt \tag{2.24}$$

For the traditional black hole, because the Hawking temperature of the black hole is inversely proportional to the black hole mass, the temperature rises up when the black hole mass reduces due to the Hawking radiation. Thus, it predicts that the eventual black hole evaporation would produce infinitely high temperature. It is another unnatural phenomenon for the black hole. But for the above noncommutative black hole, under the condition $r > \sqrt{\theta}\ l_P$, the quantum fluctuations of the noncommutative spacetime manifold cure the curvature singularity for the black hole and the resulting black after a temperature peak, will come to a slowdown of the Hawking emission before the final evaporation. Both of the singularity and infinite temperature problems seem to be solved at a single stroke.

Many extensions of above solution are possible. One possible extension is to rotating the black hole up. A solution of Kerr-like black hole with smearing mass distribution based on the noncommutative black hole theory is presented in the paper "On a general class of regular rotating black holes based on a smeared mass distribution," by A. Larranaga, A. Cardenas-Avendano, D. A. Torres, Physics Letters B, 743 (2015) 492-502. But unfortunately this paper was retracted based on parts on the publication were regarded as plagiarism on one of P. Nicolini's publication, "The Final Stage of Gravitationally Collapsed Thick Matter Layers," 812084, Advances in High Energy Physics. But the authors Larranaga-Cardenas-Avendano-Torres deserve their credit on extending the regular black hole to the rotating case [27, 28]. The following is a summary of their work:

The gravitational source being taken as a smearing mass layer,

$$\rho(r) = Ar^n \exp\left(-\frac{r^2}{l^2}\right) \tag{2.25}$$

here $l^2 = 4\theta$ as the convention in P.Nicolini's publication. A is a normalization constant for the mass. Mass distribution is a function of r,

$$m(r) = \int_0^r 4\pi\bar{r}^2\rho(\bar{r})d\bar{r} = 2\pi A l^{n+3}\gamma\left(\frac{3+n}{2}, \frac{r^2}{l^2}\right) \tag{2.26}$$

The Schwarzchild-like black solution is written as,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + rd\Omega^2 \tag{2.27}$$

where

$$f(r) = 1 - \frac{M}{4\pi r\Gamma\left(\frac{n+3}{2}\right)}\gamma\left(\frac{3+n}{2}, \frac{r^2}{l^2}\right) \tag{2.28}$$

Using the conventional method called Newman-Janis algorithm, which is a method of complex coordinate transformation capable of transforming non-rotating black hole solutions into rotating black hole solutions. A change of coordinates to the outgoing Eddington-Finkelstein coordinates followed by a complex coordinate transformation produces the followiing Kerr-like black hole with a mearing mass distribution,

$$
\begin{aligned}
ds^2 = & - \frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 - 2a\sin^2\theta \left(1 - \frac{\Delta - a^2\sin^2\theta}{\Sigma}\right) dt d\phi \\
& + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left[\Sigma + a^2\sin^2\theta \left(2 - \frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)\right] \sin^2\theta d\phi^2
\end{aligned} \tag{2.29}
$$

where

$$
\begin{aligned}
\Sigma &= r^2 + a^2\cos^2\theta \\
\Delta &= r^2 - 2m(r)r + a^2
\end{aligned} \tag{2.30}
$$

The position of the horizons of this black hole is given by $\Delta(r_H) = 0$,

$$
r_H^2 - \frac{M r_H}{4\pi\Gamma\left(\frac{n+3}{2}\right)} \gamma\left(\frac{3+n}{2}, \frac{r_H^2}{l^2}\right) + a^2 = 0 \tag{2.31}
$$

We will study the fermion tunneling process for this regular black hole.

2.3 Hawking Radiation and Fermion Tunneling

Black holes were believed to be completely black before Hawking argued that there should be black body radiation emitted out from the event horizon of black holes [25]. After Hawking, people believe black holes are radiating probably as a black body radiator. We will study the fermion tunneling process for the above regular black hole in the following. The method we are using here was proposed, developed and summarized in those research papers [29–33].

The motion of fermions is described by Dirac equation in curved spacetime,

$$
\gamma^\mu D_\mu \psi + \frac{m}{\hbar}\psi = 0 \tag{2.32}
$$

$$
D_\mu = \partial_\mu + \frac{i}{2}\Gamma_\mu^{\alpha\beta}\Pi_{\alpha\beta} + \frac{iqA_\mu}{\hbar} \tag{2.33}
$$

q, m and A_μ are the charge, mass of the fermion and electric potential in the background respectively. The fermion wave function is given,

$$\Psi = \psi(t,r,x^\mu)\, e^{\frac{i}{\hbar}S(t,r,x^\mu)} \tag{2.34}$$

Inserting the wave function into Dirac equation, dividing by the exponential terms and multiplying by $\hbar$. In the semiclassical approximation, keeping terms only to the leading order in $\hbar$,

$$\gamma^\mu\left((\partial_\mu\psi)e^{\frac{i}{\hbar}S} + \psi e^{\frac{i}{\hbar}S}\frac{i}{\hbar}\partial_\mu S + \frac{i}{2}\Gamma^{\alpha\beta}_{\mu}\Pi_{\alpha\beta}\psi e^{\frac{i}{\hbar}S} + \frac{iqA_\mu}{\hbar}\psi e^{\frac{i}{\hbar}S}\right) + \frac{m}{\hbar}\psi e^{\frac{i}{\hbar}S} = 0$$

$$\gamma^\mu\left((\partial_\mu\psi) + \psi\frac{i}{\hbar}\partial_\mu S + \frac{i}{2}\Gamma^{\alpha\beta}_{\mu}\Pi_{\alpha\beta}\psi + \frac{iqA_\mu}{\hbar}\psi\right) + \frac{m}{\hbar}\psi = 0$$

$$\Rightarrow i\gamma^\mu\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\psi + m\psi = 0 \tag{2.35}$$

Multiplying both sides of this equation by the matrix $-i\gamma^\nu\left(\frac{\partial S}{\partial x^\nu} + qA_\nu\right)$, and noticing that $-i\gamma^\nu\left(\frac{\partial S}{\partial x^\nu} + qA_\nu\right)\psi = m\psi$,

$$\gamma^\nu\left(\frac{\partial S}{\partial x^\nu} + qA_\nu\right)\gamma^\mu\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\psi + m^2\psi = 0 \tag{2.36}$$

Exchanging the index $\mu \longleftrightarrow \nu$,

$$\gamma^\mu\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\gamma^\nu\left(\frac{\partial S}{\partial x^\nu} + qA_\nu\right)\psi + m^2\psi = 0$$

Adding up, and using the relationship $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$,

$$\{\gamma^\mu,\gamma^\nu\}\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\left(\frac{\partial S}{\partial x^\nu} + qA_\nu\right)\psi + 2m^2\psi = 0$$

$$2\Big[g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\left(\frac{\partial S}{\partial x^\nu} + qA_\nu\right) + m^2\Big]\psi = 0 \tag{2.37}$$

The phase part is separated from the wave function. For the wave function to have non-trivial solution, we require the vanishing of the phase part of the above equation. then we get the Hamilton-Jacobi equation,

$$g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\left(\frac{\partial S}{\partial x^\nu} + qA_\nu\right) + m^2 = 0. \tag{2.38}$$

We solve this equation outside of the regular black with smearing mass distribution given in the previous section. This black is Kerr-like from outside. But there is no singularity inside the black hole because of the smearing mass distribution. Metric of this regular Kerr-like black hole in Boyer-Lindquist coordinates is given,

$$\begin{aligned} ds^2 = & -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - 2a\sin^2\theta\left(1-\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dtd\phi + \frac{\Sigma}{\Delta}dr^2 \\ & + \Sigma d\theta^2 + \left[\Sigma + a^2\sin^2\theta\left(2-\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)\right]\sin^2\theta d\phi^2 \end{aligned} \tag{2.39}$$

where

$$\begin{aligned} \Sigma &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 - 2m(r)r + a^2 \end{aligned} \tag{2.40}$$

Slight simplification,

$$\begin{aligned} ds^2 &= -\frac{\Delta - a^2\,sin^2\theta}{\Sigma}dt^2 - 2a\,sin^2\theta\,\frac{r^2+a^2-\Delta}{\Sigma}\,dt\,d\phi \\ &+\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{(r^2+a^2)^2 - \Delta a^2 sin^2\theta}{\Sigma}sin^2\theta\,d\phi^2 \end{aligned} \tag{2.41}$$

If given in the matrix form, it is,

$$g_{\mu\nu} = \begin{bmatrix} -\frac{\Delta - a^2\sin^2\theta}{\Sigma} & 0 & 0 & -\frac{a\sin^2\theta\left(r^2+a^2-\Delta\right)}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ -\frac{a\sin^2\theta\left(r^2+a^2-\Delta\right)}{\Sigma} & 0 & 0 & \frac{((r^2+a^2)^2 - a^2\Delta\sin^2\theta)\sin^2\theta}{\Sigma} \end{bmatrix}$$

The inverse metric can be computed as,

$$g^{\mu\nu} = \begin{bmatrix} -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Delta\Sigma} & 0 & 0 & -\frac{a\left(r^2+a^2-\Delta\right)}{\Delta\Sigma} \\ 0 & \frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ -\frac{a\left(r^2+a^2-\Delta\right)}{\Delta\Sigma} & 0 & 0 & \frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma\sin^2\theta} \end{bmatrix}$$

Now we are ready to expand the Hamilton-Jacobi equation out in the spacetime metic. We also choose a suitable form for the undermined solution of Hamilton-Jacobi equation. Considering the spherical symmetry of Kerr-like black hole event horizon, separating the variables for the action S as,

$$S = -\omega t + j\phi + R(r) + P(\theta) \tag{2.42}$$

where ω and j are energy and angular momentum for the particle.

The black hole is uncharged, assuming any electric potential being aroused by any other causes is negligible, $A_\mu = 0$. Then the Hamilton-Jacobi equation simplifies,

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + m^2 = 0. \tag{2.43}$$

Plugging in the inverse metric, with the undetermined expression for the action S,

$$\Rightarrow g^{tt}\,\omega^2 + 2g^{t\phi}\,\omega j + g^{rr} \left(\frac{\partial R}{\partial r}\right)^2 + g^{\theta\theta} \left(\frac{\partial P}{\partial \theta}\right)^2 + g^{\phi\phi}\, j^2 + m^2 = 0 \tag{2.44}$$

Completely expanding all the terms,

$$\begin{aligned} &-\frac{\omega^2 (r^2+a^2)^2}{\Delta\Sigma} + \frac{\omega^2 a^2 \sin^2\theta}{\Sigma} + 2\omega j a \frac{r^2+a^2}{\Sigma\Delta} - 2\omega j a \frac{1}{\Sigma} \\ &+ j^2 \frac{1}{\Sigma \sin^2\theta} - j^2 a^2 \frac{1}{\Sigma\Delta} + \left(\frac{dR}{dr}\right)^2 \frac{\Delta}{\Sigma} + \frac{1}{\Sigma}\left(\frac{dP}{d\theta}\right)^2 + m^2 = 0 \end{aligned} \tag{2.45}$$

Combine some terms into complete squares,

$$\frac{1}{\Sigma}\left(a\omega \sin\theta - \frac{j}{\sin\theta}\right)^2 - \frac{[\omega(r^2+a^2) - ja]^2}{\Sigma\Delta} + \frac{\Delta}{\Sigma}\left(\frac{dR}{dr}\right)^2 + \frac{1}{\Sigma}\left(\frac{dP}{d\theta}\right)^2 + m^2 = 0$$

Solving for $\frac{dR}{dr}$, multiplying Σ, dividing by Δ,

$$\left(\frac{dR}{dr}\right)^2 = \frac{[\omega(r^2+a^2) - ja]^2}{\Delta^2} - \frac{1}{\Delta}\left(a\omega\sin\theta - \frac{j}{\sin\theta}\right)^2 - \frac{1}{\Delta}\left(\frac{dP}{d\theta}\right)^2 - \frac{m^2\Sigma}{\Delta} \tag{2.46}$$

We are caring about the radial part of wave transmission factor $R(r)$ because it determines the radial transmission of the fermions. Only the radially transmitted part of the wave can propagate far enough to reach the distant observer. We will find $R(r)$ by integrating across black hole's outer event horizon r_H, thus $\Delta \to 0$, throwing away lower order terms,

$$\Rightarrow \left(\frac{dR}{dr}\right)^2 = \frac{[\omega(r^2 + a^2) - ja]^2}{\Delta^2} \tag{2.47}$$

$$\frac{dR}{dr} = \pm\frac{\omega(r^2 + a^2) - ja}{\Delta} \tag{2.48}$$

Defining

$$f(r) = \frac{\Delta}{r^2 + a^2}, \qquad \omega_0 = \frac{ja}{{r_H}^2 + a^2} \tag{2.49}$$

$$\frac{dR}{dr} = \pm\frac{\omega - \omega_0}{f(r)} \tag{2.50}$$

Integrating along r. But Hawking radiation comes from just across the outer event horizon of the black hole, for which $r = r_+ \pm \epsilon$, where ϵ is infinitesimal. We care about only the integration coming from $r = r_H$. Then by residual theorem,

$$R_\pm = \pm(\omega - \omega_0) \int_{r_H - \epsilon}^{r_H + \epsilon} \frac{1}{f(r)} dr = \pm i\pi \frac{(\omega - \omega_0)}{f'} \tag{2.51}$$

R_+ and R_- corresponds to incoming and outgoing wave function respectively.

$$ImS = ImR = ImR_+ - ImR_-$$

The tunneling rate for the fermion is:

$$\Gamma = \exp(-2ImS) = \exp\left(-4\pi\frac{(\omega - \omega_0)}{f'}\right) = \exp\left(-\frac{(\omega - \omega_0)}{T_H}\right) \tag{2.52}$$

$$T_H = \frac{f'(r_0)}{4\pi} \tag{2.53}$$

where $\Delta = r^2 - 2m(r) + a^2$, and $m(r) = 2\pi A\gamma(\frac{n+3}{2}, \frac{r^2}{l^2})$.

$$f'(r_H) = \frac{1}{4\pi}\frac{d}{dr}\left(\frac{r^2 - 2m(r) + a^2}{r^2 + a^2}\right)\Big|_{r=r_H} \tag{2.54}$$

Note that the position of the horizon is the solution of the following equation,

$$r_H^2 - \frac{Mr_H}{4\pi\Gamma\left(\frac{n+3}{2}\right)}\gamma\left(\frac{3+n}{2}, \frac{r_H^2}{l^2}\right) + a^2 = 0 \tag{2.55}$$

The numerical value of fermion tunneling rate and Hawking temperature can be found by computer method. Because of the mass distribution, both these values are modified from the traditional Kerr black hole where mass is located at the singularity, rather than distribution inside the black hole, $m(r) \to M = constant$.

Here we end our discussion of black hole radiation.

CHAPTER THREE

Hořava-Lifshitz Gravity

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This chapter is a published research article co-authored by the author of this dissertation. Dr. A. Wang and Dr. G. Cleaver are two Baylor physics professors. They are the major designers of the philosophy of the research project. The rest four are the specific research performers. Dr. J. Yang and Dr. M. Tian were two visiting scholars to Baylor physics department from Chinese universities. X. Wang and Y. Deng were two Baylor physics Ph.D. students. They grew from greenhands of research into skilled researchers of theoretical physics through this research experience. All four visiting scholars and graduate students cross-check each others' work during the research process. They are approximately equal contributors to this research project. Y. Deng is the author of this dissertation.

3.1 Introduction to Hořava-Lifshitz Gravity

It was discovered that the straightford quantization of the theory of general relativity is not renormalizable. A suitable quantum gravity theory has been in sought for the last three to four decades. Hořava-Lifshitz theory was proposed as power-counting renormalizable gravity theory. The theory breaks the Lorentz invariance at high energy and small distance scales. But it may not bring in practical trouble to physical experiments typically being carried out at low energy and long distance scales. Theories which break the Lorentz invariance, the so called non-relativistic theories have attracted some interest due to its value both condensed matter physics and in renormalizable gravity theory.

Non-relativistic gauge/gravity correspondence may provide valuable tools to study strongly coupling systems encountered in condensed matter physics [35]. If such correspondence indeed exists, instead of directly studying those strongly coupling systems, one can study the corresponding weakly coupling systems of gravity, which are much easier to handle, and often well within our abilities.

The non-relativistic quantum field theories (NQFT) are usually assumed to possess either the Schrödinger [36] or the Lifshitz [37] symmetry. In the Lifshitz symmetry, the algebra consists of operators as the rotations M_{ij}, spatial translations P_i, time translations H, and dilatations D. These generators satisfy the standard commutation relations for M_{ij}, P_k and H, while with D the relations read,

$$[D, M_{ij}] = 0, \quad [D, P_i] = iP_i, \quad [D, H] = izH, \tag{3.1}$$

where z denotes the Lifshitz dynamical exponent, and determines the relative scaling between the time and spatial coordinates [38],

$$x^i \to \ell x^i, \quad t \to \ell^z t \tag{3.2}$$

This algebra is often called the Lifshitz algebra, as it generalizes the symmetry of Lifshitz fixed points [35].

The gauge/gravity duality requires that the space-time on the gravitational theory side must possess the same symmetry as the gauge theory. The symmetry of a space-time is characterized by its Killing vectors ζ_μ [39], satisfying the Killing equations,

$$\zeta_{\mu;\nu} + \zeta_{\nu;\mu} = 0 \tag{3.3}$$

where a semicolon ";" denotes the covariant derivative with respect to the spapcetime metric $g_{\mu\nu}$. It was found that for the Lifshitz spacetime [37],

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 d\mathbf{x}^2 \tag{3.4}$$

where $d\mathbf{x}^2 \equiv \sum_{i=1}^{d} dx^i dx^i$, its Killing vectors are to be $\zeta^\mu \partial_\mu \equiv (M, P, H, D)$

$$
\begin{aligned}
&M_{ij} = -i\left(x_i \partial_j - x_j \partial_i\right), \quad P_i = -i\partial_i \\
&H = -i\partial_t, \quad D = -i\left(zt\partial_t + x^i\partial_i - r\partial_r\right)
\end{aligned}
\tag{3.5}
$$

where $x_i \equiv \delta_{ij}x^j$. These vectors satisfy precisely the same Lifshitz algebra in the above. Thus the Lifsitz spacetime shall correspond to a non-relativistic quantum field theory which lives on the boundary $r = \infty$ of the Lifshitz spacetime.

Note that the metric is invariant under the rescaling (3.2), provided that r is scaling as $r \to \ell^{-1}r$. Clearly, this is non-relativistic for $z \neq 1$, and to produce such a space-time in Einstein's theory of general relativity (GR), matter fields must be present, in order to create such a particular direction. In [37], this was realized by two p-form gauge fields with $p = 1, 2$, and was also generalized to other cases [40].

The so called Hořava-Lifshitz theory, the attempt as a viable theory of quantum gravity, recently proposed by Hořava [41] based on the anisotropic scaling (3.2), has several remarkable features [42]. The HL theory is based on the perspective that Lorentz symmetry should appear as an emergent symmetry at long distances, but can be fundamentally absent at short ones [43]. In the UV regime, the system exhibits a strong anisotropic scaling between space and time, given by Eq.(3.2). To have the theory be power-counting renormalizable, the Lifshitz dynamical exponent z must be no less than D in the $(D + 1)$-dimensional spacetime [41, 44]. At long distances, high-order curvature corrections become negligible, and the lowest order terms take over, whereby the Lorentz invariance is expected to be "accidentally restored."

Since in the HL gravity the anisotropic scaling (3.2) is built in. It is natural to expect that the HL gravity provides a minimal holographic dual for non-relativistic Lifshitz-type field theories. Indeed, recently it was showed that the Lifshitz spacetime (3.4) is a vacuum solution of the HL gravity in (2+1) dimensions, and that the full structure of the $z = 2$ anisotropic Weyl anomaly can be reproduced in dual field theories [45], while its minimal relativistic gravity counterpart yields only one of two independent central charges in the anomaly. This speculation has been further confirmed by the existence of other types of the Lifshitz spacetimes, including Lifshitz solitons [46, 47].

Since spatial high-order operators are necessarily appear in the HL gravity in order to be power-counting renormalizable, it provides an ideal place to study such effects. In the framework of GR, this was studied in [48], and found that these effects only shift the values of z. Here, we will first show that this is true also in the HL gravity. Then, we study the effects on a scalar field and the corresponding two-point correlation functions. We find that, while in the infrared the asymptotic behavior of a (probe) scalar field near the boundary is similar to that studied in [37], it gets dramatically modified in the UV limit, because of the presence of the high-order operators in this regime. Then, according to the gauge/gravity duality, this in turn affects the two-point correlation functions. This is expected, as in the UV the high-order operators will dominate, and the asymptotic behavior of the scalar field will be determined by these high-order operators.

A brief introduction to the non-projectable HL gravity in (2+1)-dimensional spacetimes will be given in the following. The stability and ghost-free conditions in terms of the independently coupling constants of the theory will be discussed. The Lifshitz space-time (3.4) is not only a solution of the HL gravity in the IR limit, but also a solution of the full theory. The only difference is that the Lifshitz dynamical exponent z is shifted. A scalar field propagating on the Lifshitz background (3.4) will be studied. To compare our results with the ones obtained in [37], we set $z = 2$. We will calculate the two-point correlation functions, and find their main properties in the IR as well as in the UV limit.

Because of the anisotropic scaling (3.2), the gauge symmetry of the theory is broken down to the foliation-preserving diffeomorphism, Diff(M, $\mathcal{F}$),

$$\delta t = -f(t), \quad \delta x^i = -\zeta^i(t, \mathbf{x}) \tag{3.6}$$

for which the lapse function N, shift vector N^i, and 3-spatial metric g_{ij} transform as

$$\begin{aligned} \delta N &= \zeta^k \nabla_k N + \dot{N} f + N \dot{f} \\ \delta N_i &= N_k \nabla_i \zeta^k + \zeta^k \nabla_k N_i + g_{ik} \dot{\zeta}^k + \dot{N}_i f + N_i \dot{f} \\ \delta g_{ij} &= \nabla_i \zeta_j + \nabla_j \zeta_i + f \dot{g}_{ij} \end{aligned} \tag{3.7}$$

where $\dot{f} \equiv df/dt$, ∇_i denotes the covariant derivative with respect to g_{ij}, $N_i = g_{ik} N^k$, and $\delta g_{ij} \equiv \tilde{g}_{ij}\left(t, x^k\right) - g_{ij}\left(t, x^k\right)$, etc.

Due to the Diff(M, $\mathcal{F}$) diffeomorphisms (3.6), one more degree of freedom appears in the gravitational sector - a spin-0 graviton. Using the gauge freedom (3.6), without loss of the generality, one can always set

$$N^i = 0 \tag{3.8}$$

for which the remaining gauge freedom is

$$t = \hat{f}(t'), \quad x^i = \hat{\zeta}^i(x') \tag{3.9}$$

In the rest of this section, we shall leave the gauge choice open, and in particular not restrict ourselves to the gauge (3.8).

The Riemann and Ricci tensors R_{ijkl} and R_{ij} of the 2D leaves t = constant are uniquely determined by the 2D Ricci scalar R via the relations [49],

$$\begin{aligned} R_{ijkl} &= \frac{1}{2}\left(g_{ik} g_{jl} - g_{il} g_{jk}\right) R \\ R_{ij} &= \frac{1}{2} g_{ij} R, \ (i, j = 1, 2) \end{aligned} \tag{3.10}$$

The general action of the HL theory without the projectability condition in (2+1)-dimensional spacetimes is given by [46]

$$S = \zeta^2 \int dt d^2 x N \sqrt{g} \Big(\mathcal{L}_K - \mathcal{L}_V + \zeta^{-2} \mathcal{L}_M \Big) \tag{3.11}$$

where $g = \det(g_{ij})$, $\zeta^2 = 1/(16\pi G)$, and

$$\begin{aligned} \mathcal{L}_K &= K_{ij}K^{ij} - \lambda K^2 \\ \mathcal{L}_V &= \gamma_0 \zeta^2 + \beta a_i a^i + \gamma_1 R + \frac{1}{\zeta^2}\Big[\gamma_2 R^2 + \beta_1 \left(a_i a^i\right)^2 + \beta_2 \left(a^i{}_i\right)^2 \\ &\quad + \beta_3 a_i a^i a^j{}_j + \beta_4 a^{ij} a_{ij} + \beta_5 a^i a_i R + \beta_6 a^i{}_i R\Big] \end{aligned} \tag{3.12}$$

with $\Delta \equiv g^{ij}\nabla_i \nabla_j$, and

$$\begin{aligned} K_{ij} &= \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right) \\ a_i &= \frac{N_{,i}}{N}, \quad a_{ij} = \nabla_i a_j \end{aligned} \tag{3.13}$$

$\mathcal{L}_M$ is the Lagrangian of matter fields. Then, the corresponding field equations and conservation laws are given explicitly in [46].

The Stability and Ghost-free Conditions are to be discussed in the following.

It is easy to show that the Minkowski space-time

$$\left(\bar{N}, \bar{N}^i, \bar{g}_{ij}\right) = (1, 0, \delta_{ij}) \tag{3.14}$$

is a solution of the above HL gravity with $\gamma_0 = 0$. Then, its linear perturbations are given by

$$\begin{aligned} \delta N &= n, \quad \delta N_i = \partial_i B - S_i, \\ \delta g_{ij} &= -2\psi \delta_{ij} + \left(\partial_i \partial_j - \delta_{ij}\partial^2\right) E + 2F_{(i,j)} \end{aligned} \tag{3.15}$$

where $F_{(i,j)} \equiv (F_{i,j} + F_{j,i})/2$, and

$$\partial^i S_i = \partial^i F_i = 0 \tag{3.16}$$

It is interesting to note that in the decompositions (3.15) no tensor mode appears in δg_{ij}. This is closely related to the fact that in (2+1)-dimensional spacetimes, spin-2 massless gravitons do not exist.

Then, the infinitesimal gauge transformations (3.4) can be written as

$$f = \epsilon(t), \quad \zeta^i = \partial^i\zeta + \eta^i, \quad (\partial_i\eta^i = 0) \tag{3.17}$$

under which the quantities defined in Eq.(3.15) transfer as,

$$\begin{aligned} \tilde{n} &= n + \dot{\epsilon}, \quad \tilde{B} = B + \dot{\zeta} \\ \tilde{E} &= E + \zeta, \quad \tilde{\psi} = \psi - \frac{1}{2}\partial^2\zeta \\ \tilde{S}_i &= S_i + \dot{\eta}_i, \quad \tilde{F}_i = F_i + \eta_i \end{aligned} \tag{3.18}$$

Thus, from the above we can construct three scalar and one vector gauge-invariants,

$$\begin{aligned} \Psi &\equiv \psi + \frac{1}{2}\partial^2 E, \quad \Phi \equiv B - \dot{E}, \\ \Sigma &\equiv \partial^2 n, \quad \Phi_i \equiv S_i - \dot{F}_i \end{aligned} \tag{3.19}$$

Using the above gauge freedom, without loss of the generality, we can always set

$$E = 0, \quad F_i = 0 \tag{3.20}$$

which will uniquely fix the gauge freedom represented by ζ and η_i, while leave $\epsilon(t)$ unspecified.

To further study the above linear perturbations, let us consider the scalar and vector perturbations separately. For the scalar perturbations, under the gauge (3.20), the remaining scalars are n, B and ψ, with which it can be shown that the gravitational sector of the action to the second-order takes the form,

$$\begin{aligned} S_g^{(2)} &= \zeta^2 \int dt d^2x N\sqrt{g}\Big\{2(1-2\lambda)\dot{\psi}^2 + 2(1+2\lambda)\dot{\psi}\partial^2 B + (1-\lambda)(\partial^2 B)^2 + \beta n\partial^2 n \\ &\quad -2\gamma_1 n\partial^2\psi - \frac{1}{\zeta^2}\big[4\gamma_2(\partial^2\psi)^2 + (\beta_2+\beta_4)(\partial^2 n)^2 + 2\beta_6(\partial^2 n)(\partial^2\psi)\big]\Big\} \end{aligned} \tag{3.21}$$

Its variations with respect to ψ, B *and* n yield, respectively,

$$\ddot{\psi} + \frac{1}{2}\partial^2 \dot{B} + \frac{\gamma_1}{2(1-2\lambda)}\partial^2 n + \frac{4\gamma_2 \partial^4 \psi + \beta_6 \partial^4 n}{2\zeta^2(1-2\lambda)} = 0 \tag{3.22}$$

$$(1-2\lambda)\dot{\psi} + (1-\lambda)\partial^2 B = 0 \tag{3.23}$$

$$\beta n - \gamma_1 \psi - \frac{\beta_2 + \beta_4}{\zeta^2}\partial^2 n - \frac{\beta_6}{\zeta^2}\partial^2 \psi = 0 \tag{3.24}$$

From Eq.(3.23) we can find B in terms of ψ, and then substituting it into (3.21) we obtain,

$$\begin{aligned} S_g^{(2)} &= \zeta^2 \int dt d^2 x N \sqrt{g} \Bigg\{ \frac{1-2\lambda}{1-\lambda}\dot{\psi}^2 + \beta n \partial^2 n - 2\gamma_1 n \partial^2 \psi \\ &\quad - \frac{1}{\zeta^2}\Big[4\gamma_2 (\partial^2 \psi)^2 + (\beta_2 + \beta_4)(\partial^2 n)^2 + 2\beta_6 (\partial^2 n)(\partial^2 \psi)\Big]\Bigg\} \end{aligned} \tag{3.25}$$

Then, the ghost-free condition require

$$\frac{1-2\lambda}{1-\lambda} \geq 0 \tag{3.26}$$

that is,

$$(i)\ \lambda > 1 \quad \text{or} \quad (ii)\ \lambda \leq \frac{1}{2} \tag{3.27}$$

From Eqs.(3.22)-(3.24), on the other hand, we can get a master equation for ψ, which in momentum space can be written in the form

$$\ddot{\psi}_k + \omega_k^2 \psi_k = 0 \tag{3.28}$$

where

$$\begin{aligned} \omega_k^2 &= \frac{1-\lambda}{1-2\lambda}\left(\frac{4\gamma_2 k^4}{\zeta^2} + \left(\frac{\beta_6 k^4}{\zeta^2} - \gamma_1 k^2\right)\frac{\gamma_1 - \frac{\beta_6 k^2}{\zeta^2}}{\beta + \frac{(\beta_2+\beta_4)k^2}{\zeta^2}}\right) \\ &= \begin{cases} -\frac{1-\lambda}{1-2\lambda}\frac{\gamma_1^2 k^2}{\beta}, & k^2/\zeta \ll 1 \\ \frac{1-\lambda}{1-2\lambda}\left(4\gamma_2 - \frac{\beta_6^2}{\beta_2+\beta_4}\right)\frac{k^4}{\zeta^2}, & k^2/\zeta \gg 1 \end{cases} \end{aligned} \tag{3.29}$$

Thus, to have the mode be stable in the infrared (IR), we must require

$$\beta < 0 \tag{3.30}$$

while its stability condition in the ultraviolet (UV) requires

$$\gamma_2 \geq \frac{\beta_6^2}{4(\beta_2 + \beta_4)} \tag{3.31}$$

In the intermediate range, by properly choosing the free parameters the mode can be always stable, and such requirement does not impose any save constraints. So, in the following we do not consider it any further, and simply assume that it is always satisfied. It should be noted that the conditions (3.27), (3.30) and (3.31) are valid only for the cases $\lambda \neq 1$, for which Eq.(3.30) tells that β must be strictly negative, and in particular cannot be zero.

When $\lambda = 1$, from Eq.(3.23) we find that

$$\dot{\psi} = 0 \tag{3.32}$$

that is, ψ does not represent a propagative mode, and we can always set it to zero by properly choosing the boundary conditions. Then, Eqs.(3.22) and (3.24) reduce to,

$$\dot{B} - \gamma_1 n - \frac{\beta_6}{\zeta^2}\partial^2 n = 0 \tag{3.33}$$

$$\frac{\beta_2 + \beta_4}{\zeta^2}\partial^2 n - \beta n = 0 \tag{3.34}$$

From the last equation, we can see that n does not represent a propagative mode either, and can be set to zero by properly choosing the boundary conditions. Then, Eq.(3.33) yields $\dot{B} = 0$, that is, B is also not a propagative mode.

Therefore, in the case $\lambda = 1$ there is no gravitational propagative mode, similar to the relativistic case [49]. As a result, *all the free parameters in this case are free, as long as the stability and ghost-free conditions are concerned.*

As a corollary, we find that the HL theory with $\beta = 0$ is viable only when $\lambda = 1$. Otherwise, the corresponding scalar mode will become unstable, as one can see clearly from Eq.(3.29).

For the vector perturbations, under the gauge (3.20), the remaining vector is S_i, with which it can be shown that the gravitational sector of the action to the second-order takes the form,

$$S_g^{(2)} = -\frac{\zeta^2}{2}\int dtd^2xN\sqrt{g}\big(S^i\partial^2S_i\big) \tag{3.35}$$

from which we find that,

$$\partial^2 S^i = 0 \tag{3.36}$$

That is, there is no propagative vector mode in the HL gravity, even the Lorentz symmetry is violated.

In summary, the above analysis shows: *(i) In the case $\lambda \neq 1$, only spin-0 gravitons exist in the (2+1)-dimensional non-projectable HL gravity.* Their stability and ghost-free conditions require the independent coupling constants must satisfy the conditions of Eqs.(3.27), (3.30) and (3.31). *(ii) In the case $\lambda = 1$, the gravitational sector of the HL gravity has no free propagation mode, similar to its relativistic counterpart.* Then, all the free parameters in this case are free, as long as the stability and ghost-free conditions are concerned.

We will consider the detailed balance condition to simplify the analysis.

To reduce the number of the coupling constants, Hořava imposed the detailed balance condition [41]. The main idea is to introduce a superpotential W on the leaves $t =$ Constant,

$$W = \int d^2x\sqrt{g}\mathcal{L}_W\left(R_{ij}, a_k, \nabla_l\right) \tag{3.37}$$

so that the potential part of the action is given by

$$\hat{\mathcal{L}}_V^{(DB)} = E_{ij}G^{ijkl}E_{kl}, \quad E_{ij} \equiv \frac{1}{\sqrt{g}}\frac{\delta W}{\delta g^{ij}} \tag{3.38}$$

where G^{ijkl} denotes the generalized de Witt metric on the space of metrics, and is given by

$$G^{ijkl} \equiv \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda g^{ij}g^{kl} \tag{3.39}$$

Power-counting renormalizibility requires that the dimension of $\mathcal{L}_W$ must be greater or equal to $2d$, that is, $[\mathcal{L}_W] \geq 2d$. Taking the lowest dimension, one can see that in (2+1)-dimensional space-times, $\mathcal{L}_W$ in general can be cast in the form,

$$\mathcal{L}_W = w\left(R + \mu a_i a^i - 2\Lambda_W\right) \tag{3.40}$$

where w, μ and Λ_W are three coupling constants. Plugging the above into Eq.(3.38) and taking Eq.(3.6) into account, we find that

$$\begin{aligned} E_{ij} &= w\left[\mu\left(a_i a_j - \frac{1}{2}g_{ij}a_k a^k\right) + \Lambda_W g_{ij}\right] \\ \hat{\mathcal{L}}_V^{(DB)} &= \frac{w^2}{2}\left[\mu^2\left(a_i a^i\right)^2 + 4\left(1-2\lambda\right)\Lambda_W^2\right] \end{aligned} \tag{3.41}$$

To have a healthy IR limit, the detailed balance condition is frequently allowed to be broken softly [41,50,51] by adding all the low dimensional relevant terms, $R,\ a_i a^i,\ \Lambda$, into $\hat{\mathcal{L}}_V^{(DB)}$, so that the potential is finally given by

$$\mathcal{L}_V^{(DB)} = 2\Lambda + \beta a_i a^i + \gamma_1 R + \frac{\beta_1}{\zeta^2}\left(a_i a^i\right)^2 \tag{3.42}$$

where $\beta_1 \equiv w^2\mu^2/2$ and $\Lambda \equiv \gamma_0\zeta^2/2$. Comparing it with $\mathcal{L}_V$ given by Eq.(3.12), one can see that this is equivalent to set $\gamma_2 = 0 = \beta_n$ $(2 \leq n \leq 6)$.

3.3 Lifshitz Spacetimes in (2+1)-dimensions

In this section we are going to study static vacuum spacetimes with the ADM variables given by

$$\begin{aligned} N &= r^z f(r), \quad N^i = 0 \\ g_{ij} &= \text{diag}\left(\frac{g^2(r)}{r^2} r^2\right) \end{aligned} \tag{3.43}$$

in the coordinates (t, r, x), where z is the dynamical Lifshitz exponent. Then, we find that

$$R_{ij} = \frac{rg' - g}{r^2 g}\delta_i^r\delta_j^r + \frac{r^2\left(rg' - g\right)}{g^3}\delta_i^\theta\delta_j^\theta$$

$$a_i = \frac{(zf + rf')}{rf}\delta_i^r, \quad K_{ij} = 0 \tag{3.44}$$

From Eq.(3.44) we find that

$$\begin{aligned}
R &= \frac{2\left(rg' - g\right)}{g^3} \\
\Delta R &= \frac{2r}{g^7}\left[15r^2 g'^3 - rgg'\left(21g' + 10rg''\right) + g^2\left(6g' + r\left(6g'' + rg^{(3)}\right)\right)\right] \\
a_{ij} &= \frac{f\left(g\left(f' + rf''\right) - rf'g'\right) - rgf'^2 - zf^2 g'}{rf^2 g}\delta_i^r \delta_j^r + \left(\frac{r^2\left(zf + rf'\right)}{fg^2}\right)\delta_i^\theta \delta_j^\theta \\
a^{ij}a_{ij} &= \frac{1}{f^4 g^6}\left\{f^2 g^2\left(zf + rf'\right)^2 + r^2\left[rgf'^2 + zf^2 g' - f\left(g\left(f' + rf''\right) - rf'g'\right)\right]^2\right\} \\
a_i a^i &= \frac{\left(zf + rf'\right)^2}{f^2 g^2} \\
a^i{}_i &= \frac{zf^2\left(g - rg'\right) - r^2 gf'^2 + rf\left(f'\left(2g - rg'\right) + rgf''\right)}{f^2 g^3}
\end{aligned} \tag{3.45}$$

and applying these results to produces,

$$\begin{aligned}
L_V = {} & \zeta^2 g_0 + \frac{1}{g^2}\left\{\beta z^2 - 2\gamma_1 + \beta r\frac{f'}{f}\left(2z + r\frac{f'}{f}\right)\right\} + 2\gamma_1 r\frac{g'}{g^3} \\
& + \frac{1}{\zeta^2 g^4}\left\{2\left(2\gamma_2 - \gamma_4 z^2\right) + z^2\left(\beta_1 z^2 + \beta_2 + \beta_3 z + \beta_4\right)\right. \\
& + \frac{1}{f}\left(2rz\left(-2\gamma_4 + 2\beta_1 z^2 + 2\beta_2 + 2\beta_3 z + \beta_4\right)f' + r^2 z\left(2\beta_2 + \beta_3 z\right)f''\right) \\
& + \frac{1}{f^2}\left(r^2\left(-2\gamma_4 + 6\beta_1 z^2 + 2\left(2 - z\right)\beta_2 + z\left(5 - z\right)\beta_3 + 2\beta_4\right)\left(f'\right)^2\right. \\
& \left. + 2r^3\left(2\beta_2 + \beta_3 z + \beta_4\right)f''f' + r^4\left(\beta_2 + \beta_4\right)\left(f''\right)^2\right) \\
& + \frac{1}{f}\left(2rz\left(-2\gamma_4 + 2\beta_1 z^2 + 2\beta_2 + 2\beta_3 z + \beta_4\right)f' + r^2 z\left(2\beta_2 + \beta_3 z\right)f''\right) \\
& + \frac{1}{f^3}\left(2r^3\left(2\beta_1 z - 2\beta_2 - \left(z - 1\right)\beta_3 - \beta_4\right)\left(f'\right)^3\right. \\
& \left.\left. - r^4\left(2\beta_2 - \beta_3 + 2\beta_4\right)f''\left(f'\right)^2\right) + r^4\frac{\left(f'\right)^4}{f^4}\left(\beta_1 + \beta_2 - \beta_3 + \beta_4\right)\right\} \\
& + \frac{1}{\zeta^2 g^5}\left\{\left(2r\left(-4\gamma_2 + 6\beta_6 + z^2\gamma_4\right) - rz^2\left(2\beta_2 + z\beta_3\right)\right)g' + 2r^2\beta_6\left(6g'' + rg^{(3)}\right)\right. \\
& + \frac{g'}{f}\left(r^2 z\left(4\gamma_4 - 6\beta_2 - 3\beta_3 r - 2\beta_4\right)f' - 2r^3 z\left(\beta_2 + \beta_4\right)f''\right) \\
& + \frac{g'}{f^2}\left(r^3\left(2\gamma_4 + 2\left(z - 2\right)\beta_2 - 3z\beta_3 + 2\left(z - 1\right)\beta_4\right)\left(f'\right)^2\right.
\end{aligned}$$

$$
-2r^4\left(\beta_2+\beta_4\right)f''f'\Big)+r^4\frac{\left(f'\right)^3}{f^3}\left(2\beta_2-\beta_3+2\beta_4\right)g'\Bigg\}
$$

$$
+\frac{1}{\zeta^2 g^6}\left\{r^2\left(2\left(2\gamma_2-21\beta_6\right)+z^2\left(\beta_2+\beta_4\right)\right)\left(g'\right)^2\right.
$$

$$
\left.-20\beta_6 r^3 g''g'+r^3\frac{f'}{f}\left(g'\right)^2\left(\beta_2+\beta_4\right)\left(2z+r\frac{f'}{f}\right)\right\}+30\beta_6 r^3\frac{(g')^3}{\zeta^2 g^7} \tag{3.46}
$$

Inserting the above into the general action (3.11), for the vacuum case $\mathcal{L}_M = 0$, we obtain

$$
S_g = -V_x\zeta^2\int dt dr r^z f g\mathcal{L}_V\left(f^{(n)}, g^{(m)}, r\right) \tag{3.47}
$$

where $V_x \equiv \int dx,\ I^{(n)} \equiv d^n I(r)/dr^n$, and $\mathcal{L}_V$ is given by Eq.(3.46). Then, it can be shown that in the present case there are only two independent equations, which can be cast in the forms,

$$
\sum_{n=0}^{3}(-1)^n\frac{d^n}{dr^n}\left(\frac{\delta\mathcal{L}_g}{\delta f^{(n)}}\right) = 0 \tag{3.48}
$$

$$
\sum_{n=0}^{3}(-1)^n\frac{d^n}{dr^n}\left(\frac{\delta\mathcal{L}_g}{\delta g^{(n)}}\right) = 0 \tag{3.49}
$$

where $\mathcal{L}_g \equiv r^z f g\mathcal{L}_V$.

In terms of f, g and their derivatives, these two equations are given in the following long expression.

$$
\begin{aligned}
0 \;=\; & -r^z\zeta^2\gamma_0 g+\frac{r^z}{g}\left\{2\gamma_1+r\beta\left[2(z+2)+\left(2z+4-r\frac{f'}{f}\right)\frac{f'}{f}+2rf''\right]\right\}\\
& -\frac{r^z}{g^2}\left\{2\gamma_1+2\beta r\left(z+r\frac{f'}{f}\right)\right\}g'\\
& +\frac{r^z}{\zeta^2 g^3}\left\{-4\gamma_2-2z(z+2)\gamma_4+z^3(3z+4)\beta_1-z^2(2z+3)\beta_2\right.\\
& -z^2(z^2-2)\beta_3+z(z+2)\beta_4\\
& +\frac{2r}{f}\left(\left[-2(z+2)\gamma_4+6z^2(z+3)\beta_1-2(2z^2+5z+2)\beta_2\right.\right.\\
& \left.+2z(z^2-2z+1)\beta_3-(z^2+3z+2)\beta_4\right]f'\\
& +r\left[-2\gamma_4+6\beta_1 z^2-(z^2+11z+14)\beta_2-2z(z-1)\beta_3-(z^2+8z+13)\beta_4\right]f''\\
& -2r^2(4+z)\left[\beta_2+\beta_4\right]f^{(3)}-r^3\left[\beta_2+\beta_4\right]f^{(4)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{r^2}{f^2}\left(2\left[\gamma_4+3z(z+6)\beta_1+(z^2+6z+4)\beta_2-(z^2+6z-3)\beta_3\right.\right.\\
&\left.+(z^2+6z+8)\beta_4\right](f')^2\\
&+4r\left[12\beta_1 z+4z(z+5)\beta_2-2(2z-1)\beta_3+(2z+13)\beta_4\right]f''f'\\
&\left.+4r^2\left[\beta_2+\beta_4\right]f^{(3)}f'+3r^2\left[\beta_2+\beta_4\right](f'')^2\right)\\
&-\frac{4r^3}{f^3}\left(\left[(3z-4)\beta_1+(z+2)\beta_2-(z-2)\beta_3+(z+3)\beta_4\right]f'\right.\\
&\left.-r\left[3\beta_1-2\beta_2-\beta_3-2\beta_4\right]f''\right)(f')^2-\frac{3r^4}{f^4}\left(3\beta_1-\beta_2-\beta_3-\beta_4\right)(f')^4\\
&+\frac{r^z}{\zeta^2 g^4}\left\{\left(2r\left[4\gamma_2-6\beta_6-6\beta_1 z^3+z(z^2+9z+6)\beta_2\right.\right.\right.\\
&\left.-z^2(3-2z)\beta_3+z(z^2+4z+1)\beta_4\right]g'\\
&\left.+2r^2\left[-6\beta_6+2\gamma_4 z+2z(z+3)\beta_2+z(2z+5)\beta_4\right]g''+2r^3\left[-\beta_6+z(\beta_2+\beta_4)\right]g^{(3)}\right)\\
&+\frac{1}{f}\left(\left(2r^2\left[2(6+z)\gamma_4-18\beta_1 z^2+3(z^2+9z+8)\beta_2\right.\right.\right.\\
&\left.+6z(z-1)\beta_3+(3z^2+17z+18)\beta_4\right]f'\\
&\left.+2r^3\left[2\gamma_4+3(10+3z)\beta_2+(29+9z)\beta_4\right]f''+12r^4\left[\beta_2+\beta_4\right]f^{(3)}\right)g'\\
&+2r^3\left(\left[2\gamma_4+2(2z+5)\beta_2+(4z+9)\beta_4\right]g''+2r\left[\beta_2+\beta_4\right]g^{(3)}\right)f'\\
&\left.+8r^4\left[\beta_2+\beta_4\right]f''g''\right)+\frac{1}{f^2}\left(\left(-2r^3\left[\gamma_4+18\beta_1 z+6(2+z)\beta_2\right.\right.\right.\\
&\left.+3(1-2z)\beta_3+2(8+3z)\beta_4\right]f'\\
&\left.\left.-18r^4\left[\beta_2+\beta_4\right]f''\right)f'g'-4r^4\left[\beta_2+\beta_4\right](f')^2g''\right)\\
&\left.-\frac{4r^4}{f^4}\left(3\beta_1-2\beta_2-\beta_3-2\beta_4\right)(f')^3g'\right\}\\
&+\frac{r^z}{\zeta^2 g^5}\left\{\left(-r^2\left[4\gamma_2-42\beta_6+16\gamma_4 z+z(42+15z)\beta_2+z(34+15z)\beta_4\right]g'\right.\right.\\
&\left.+20r^3\left[\beta_6+z(\beta_2+\beta_4)\right]g''\right)g'+\frac{2}{f}\left(\left(-r^3\left[8\gamma_4+(36+15z)\beta_2+(32+15z)\beta_4\right]f'\right.\right.\\
&\left.\left.\left.-30r^4\left[\beta_2+\beta_4\right]f''\right)(g')^2-20r^4\left[\beta_2+\beta_4\right]f'g'g''\right)+\frac{15r^4}{f^2}\left(\beta_2+\beta_4\right)(f')^2(g')^2\right\}\\
&+\frac{30r^3}{g^6}\left\{-\beta_6+\left[\beta_2+\beta_4\right]\left(z+r\frac{f'}{f}\right)\right\}(g')^3
\end{aligned}
\tag{3.50}
$$

$$
0 = -r^z\zeta^2 f\gamma_0+\frac{r^z}{g^2}f\left(2\left[z+\frac{f'}{f}\right]\gamma_1+\left[z^2+2zr\frac{f'}{f}+r^2\frac{(f')^2}{f^2}\right]\beta\right)
$$

$$
\begin{aligned}
&+\frac{r^z}{\zeta^2 g^4}\Big\{ \big(4(1-2z)\gamma_2 - 2z(z^2-1)\beta_6 + 2z^2(z-2)\gamma_4 + 3\beta_1 z^4 \\
&+z^2(2z-1)\beta_2 - z^3(z-2)\beta_3 + 3\beta_4 z^2\big)\, f \\
&+2r\left(\left[-4\gamma_2 + 3z(z+1)\beta_6 + z(3z-2)\gamma_4 + 6\beta_1 z^3\right.\right. \\
&\left.-4\beta_2 z^2 - z^2(2z-3)\beta_3 - z(z-1)\beta_4\right] f' \\
&+r\left[+3(z+1)\beta_6 + 2\gamma_4 z - z(z+3)\beta_2 + z(z+4)\beta_4\right] f'' + r^2\left[\beta_6 - z(\beta_2+\beta_4)\right] f^{(3)}\Big) \\
&+\frac{r^2}{f}\Big(2\left[\gamma_4 z + 9\beta_1 z^2 + z(z-2)\beta_2 - 3z(z-1)\beta_3 + z(z+2)\beta_4\right](f')^2 \\
&+2r^3\left(\left[2\gamma_4 + (z-2)\beta_2 + (z-3)\beta_4\right] f'' - r\left[\beta_2+\beta_4\right] f^{(3)}\right) f' + r^2\left[\beta_2+\beta_4\right](f'')^2\Big) \\
&+\frac{2r^3}{f^2}\left(\left[-\gamma_4 + 6\beta_1 z - (2z-1)\beta_3 + 2\beta_4\right] f' + r\left[\beta_2+\beta_4\right] f''\right)(f')^2 \\
&+\frac{r^4}{f^3}\left[3\beta_1 - \beta_2 - \beta_3 - \beta_4\right](f')^4\Big\} \\
&+\frac{r^z}{\zeta^2 g^5}\Big\{2r\left((z+2)\left[4\gamma_2 - 2z\beta_6 + z^2(\beta_2+\beta_4)\right] g'\right. \\
&\left.+r\left[4\gamma_2 - 2\beta_6 z + z^2(\beta_2+\beta_4)\right] g''\right) f \\
&+2r^2\left(\left[4\gamma_2 - 2(3+2z)\beta_6 + 3z(z+2)(\beta_2+\beta_4)\right] f'\right. \\
&\left.+2r\left[-\beta_6 + z(\beta_2+\beta_4)\right] f''\right) g' + 4r^3\left[-\beta_6 + z(\beta_2+\beta_4)\right] g'' f' \\
&+\frac{2r^3}{f}\Big(\left((z+4)\left[\beta_2+\beta_4\right] f' + 2r\left[\beta_2+\beta_4\right] f''\right) f' g' + r\left[\beta_2+\beta_4\right](f')^2 g''\Big) \\
&-\frac{2r^4}{f^2}\left[\beta_2+\beta_4\right](f')^3 g'\Big\} \\
&+\frac{5r^2 r^z}{\zeta^2 g^6}\Big\{\left[-4\gamma_2 + 2\beta_6 z - z^2(\beta_2+\beta_4)\right] - 2r\left[\beta_6 - z(\beta_2+\beta_4)\right] f' \\
&-\frac{r^2}{f}\left[\beta_2+\beta_4\right](f')^2\Big\}(g')^2
\end{aligned}
\tag{3.51}
$$

The Lifshitz spacetime corresponds to

$$
f = f_0, \quad g = g_0 \tag{3.52}
$$

where f_0 and g_0 are two constant. Then, the corresponding metric can be cast in the form,

$$
ds^2 = L^2\left\{-\left(\frac{r}{\ell}\right)^{2z} dt^2 + \left(\frac{\ell}{r}\right)^2 dr^2 + \left(\frac{r}{\ell}\right)^2 dx^2\right\} \tag{3.53}
$$

where $L \equiv (f_0 g_0^z)^{1/(z+1)}$, $\ell \equiv (g_0/f_0)^{1/(1+z)}$. Inserting Eq.(3.52) into above expanded equations of motion, we obtain

$$\begin{aligned}&2\zeta^2\Lambda g_0^4 - \zeta^2 g_0^2\left[z(2+z)\beta + 2\gamma_1\right] - z^3(4+3z)\beta_1 + 4\gamma_2\\&+z\Big[z(3+2z)\beta_2 + z\left(z^2-2\right)\beta_3 - (2+z)\left(\beta_4 - 2\beta_5 + 2\beta_6\right)\Big] = 0\end{aligned} \quad (3.54)$$

$$\begin{aligned}&2\zeta^2\Lambda g_0^4 - z\zeta^2 g_0^2\left(z\beta + 2\gamma_1\right) - 4\gamma_2 + 2z\left(4\gamma_2 + \beta_6\right)\\&-z^2\left\{\beta_2 + 3\beta_4 - 4\beta_5 + 4\beta_6 + z\Big[3z\beta_1 - 2\beta_2 - (z-2)\beta_3 + 2\beta_5\Big]\right\} = 0\end{aligned} \quad (3.55)$$

In the IR limit, all the fourth-order terms become negligible, and the above equations reduce to

$$2\Lambda g_0^2 - \left[z(2+z)\beta + 2\gamma_1\right] = 0 \quad (3.56)$$

$$2\Lambda g_0^2 - z\left(z\beta + 2\gamma_1\right) = 0 \quad (3.57)$$

which have the solutions,

$$z = \frac{\gamma_1}{\gamma_1 - \beta}, \quad \Lambda = \frac{\gamma_1^2(2\gamma_1 - \beta)}{2g_0^2(\gamma_1 - \beta)^2} \quad (3.58)$$

These are exactly what were obtained in [45]. When the higher-order operators are not negligible, the sum of Eqs.(3.54) and (3.55) yields,

$$\begin{aligned}\Lambda \;=\; &\frac{\zeta^2\left[z\beta + (1-z)\gamma_1\right]}{\Delta}\Big\{z^4\left[z\beta - (1+3z)\gamma_1\right]\beta_1 + z^2\left[z\beta + \left(2z^2+z+1\right)\gamma_1\right]\beta_2\\&+z^4\left[\beta + (z-1)\gamma_1\right]\beta_3 + z^2\left[z(z+2)\beta + (1-z)\gamma_1\right]\beta_4\\&+z^3\left[(z+2)(z-1)\beta + 4\gamma_1\right]\beta_5 + z\left[z(z+2)(z+1)\beta - 2\gamma_1(z^2+1)\right]\beta_6\\&-4\left[z\left(z^2+z-1\right)\beta + (z-1)\gamma_1\right]\gamma_2\Big\}\end{aligned} \quad (3.59)$$

where we denote,

$$\begin{aligned}\Delta \;=\; &2\Big\{2z^3\beta_1 - 2z^2\beta_2 - z(z-3)\beta_6 + (1-z)\left[z^2\beta_3 + z\beta_4 - 4\gamma_2\right]\\&-z\left[2 + z(z-1)\right]\beta_5\Big\}^2\end{aligned} \quad (3.60)$$

The difference of Eqs.(3.54) and (3.55), on the other hand, yields,

$$az^3 + bz^2 + cz + d = 0 \tag{3.61}$$

where

$$\begin{aligned}
a &= -2\beta_1 + \beta_3 + \beta_5, \\
b &= 2\beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6, \\
c &= -\alpha^2(\beta - \gamma_1) - 4\gamma_2 - \beta_4 + 2\beta_5 - 3\beta_6, \\
d &= 4\gamma_2 - \alpha^2\gamma_1, \quad \alpha \equiv \zeta g_0
\end{aligned} \tag{3.62}$$

which can be used to determine the dynamical exponent z in terms of the coupling constants. In general, it has three different solutions for any given set of the coupling constants. On the other hand, Eq.(3.61) can be also used to determine the integration constant g_0 for any given z and a set of the coupling constants. In this case, we have

$$g_0^2 = \frac{az^3 + bz^2 + \hat{c}z + 4\gamma_2}{\zeta^2[\gamma_1 - (\gamma_1 - \beta)z]} \tag{3.63}$$

where $\hat{c} \equiv -4\gamma_2 - \beta_4 + 2\beta_5 - 3\beta_6$. Clearly, for the metric to have a proper signature, z has to be chosen so that $g_0^2 > 0$ for any given set of the coupling constants (β_i, γ_j).

When the fourth-order corrections are small, we can expand z near its IR fixed point, z_0, given by Eq.(3.58). Writing the fourth-order coupling constants in the form $s = s_0 + \epsilon\hat{s}$ $\epsilon \ll 1$, we find that

$$\begin{aligned}
z &= z_0 + \epsilon\delta z \\
a &= \epsilon(-2\hat{\beta}_1 + \hat{\beta}_3 + \hat{\beta}_5) \\
b &= \epsilon(2\hat{\beta}_2 - \hat{\beta}_3 + \hat{\beta}_4 - \hat{\beta}_5 + \hat{\beta}_6) \\
c &= c_0 + \epsilon(-4\hat{\gamma}_2 - \hat{\beta}_4 + 2\hat{\beta}_5 - 3\hat{\beta}_6) \\
d &= d_0 + 4\epsilon\hat{\gamma}_2
\end{aligned} \tag{3.64}$$

where

$$z_0 = \frac{\gamma_1}{\gamma_1 - \beta}, \quad c_0 = -\alpha^2(\beta - \gamma_1), \quad d_0 = -\alpha^2\gamma_1.$$

Thus, to the first-order of ϵ Eq.(3.61) yields,

$$\begin{aligned}(-2\hat{\beta}_1 + \hat{\beta}_3 + \hat{\beta}_5)z_0^3 &+ (2\hat{\beta}_2 - \hat{\beta}_3 + \hat{\beta}_4 - \hat{\beta}_5 + \hat{\beta}_6)z_0^2 \\ &+ c_0\delta z + (-4\hat{\gamma}_2 - \hat{\beta}_4 + 2\hat{\beta}_5 - 3\hat{\beta}_6)z_0 + 4\hat{\gamma}_2 = 0 \quad (3.65)\end{aligned}$$

from which we find that,

$$\begin{aligned}\delta z &= \frac{1}{\alpha^2(\beta-\gamma_1)^4}\Big\{\gamma_1[\beta^2\left(\beta_4 - 2\beta_5 + 3\beta_6\right) - \beta\gamma_1\left(-2\beta_2 + \beta_3 + \beta_4 - 3\beta_5 + 5\beta_6\right) \\ &\quad +2\gamma_1^2\left(\beta_1 - \beta_2 - \beta_5 + \beta_6\right)] + 4\beta\gamma_2\left(\beta - \gamma_1\right)^2\Big\} \quad (3.66)\end{aligned}$$

Note that in writing the above expression, without causing any confusions, we had dropped hats from all fourth-order parameters. To study the behavior of z in the UV, let us consider some particular cases.

Consider solutions with softly-breaking detailed balance condition.

When the softly-breaking detailed balance condition is imposed, we have $\gamma_2 = \beta_i = 0, \ (i \geq 2)$. Then, Eqs.(3.61) and (3.59) reduce, respectively, to

$$z^3 + \frac{\alpha^2}{2\beta_1}\left(\beta - \gamma_1\right)z + \frac{\alpha^2}{2\beta_1}\gamma_1 = 0 \quad (3.67)$$

$$\Lambda = \frac{\zeta^2}{4z^2\beta_1}\left[z\beta + (1-z)\gamma_1\right]\left[z\beta - (1+3z)\gamma_1\right] \quad (3.68)$$

Eq.(3.67) in general has three roots, and depending on the signature of $\mathcal{D}$, the nature of these roots are different, where

$$\mathcal{D} \equiv \frac{\alpha^4}{16\beta_1^2}\left[\gamma_1^2 - \frac{2\alpha^2\left(\gamma_1 - \beta\right)^3}{27\beta_1}\right] \quad (3.69)$$

Let us consider the cases $\mathcal{D} = 0, \ \mathcal{D} > 0$ and $\mathcal{D} < 0$, separately.

When $\mathcal{D} = 0$, we find that

$$\beta_1 = \frac{2\alpha^2\left(\gamma_1 - \beta\right)^3}{27\gamma_1^2} \quad (3.70)$$

and Eq.(3.67) has three real roots, two of which are equal and given by

$$z_1 = \frac{3\gamma_1}{\beta - \gamma_1}, \quad z_2 = z_3 = -\frac{3\gamma_1}{2\,(\beta - \gamma_1)} \tag{3.71}$$

Clearly, by properly choosing β and γ_1, they can take any real values, $z_i \in (-\infty, \infty)$.

In this case $\mathcal{D} > 0$, Eq.(3.67) has only one real root, which can be written as

$$z = \sqrt[3]{\mathcal{D}^{1/2} - \frac{q}{2}} - \sqrt[3]{\mathcal{D}^{1/2} + \frac{q}{2}} \tag{3.72}$$

where $q \equiv \alpha^2\gamma_1/(2\beta_1)$. In this it is clear that z can also take any real values for different choices of $(\beta, \gamma_1, \beta_1)$. In particular, it has an extreme at $\beta = \gamma_1$, given by $z_m = -q^{1/3}$.

In this case: $\mathcal{D} < 0$, Eq.(3.67) has three real and different roots, given by

$$z_n = \sqrt{\frac{2\alpha^2\,(\gamma_1 - \beta)}{3\beta_1}} \cos\left(\theta + \frac{2n\pi}{3}\right), (n = 0, 1, 2) \tag{3.73}$$

where θ is defined as

$$\theta = \frac{1}{3}\text{arcos}\left[\frac{\alpha^2\gamma_1}{4\beta_1}\left(\frac{6\beta_1}{\alpha^2(\gamma_1 - \beta)}\right)^{3/2}\right] \tag{3.74}$$

Again, similar to the last two subcases, by choosing different values of the coupling constants, we can have different values of z_n. For example, taking $\alpha^2 = 4,\ \beta = -1,\ \beta_1 = 0.00001,\ \gamma_1 = 1$, we obtain $z_1 \simeq 632.205$.

Consider solutions with $\mathcal{L}_V$ containing only the pure R contributions: an interesting case is the $\mathcal{F}(R)$ models [52], for which we have

$$\mathcal{L}_V = \mathcal{F}(R) \tag{3.75}$$

where $\mathcal{F}(R)$ can be any function of R (possibly subjected to some stability and ghost-free conditions). In particular, one can take the form,

$$\mathcal{F}(R) = 2\Lambda + \gamma_1 R + \beta\mathcal{A}^2 + \frac{\gamma_2}{\zeta^2}R^2 \tag{3.76}$$

which corresponds to the potential given by Eq.(3.12) with $\beta_i = 0$, $(i = 1, ..., 6)$, where $\mathcal{A}^2 \equiv a_i a^i$. Note that in writing the above expression, we had kept the $a_i a^i$ term, in order to have a healthy IR limit for any given coupling constant λ [45, 46].

In this case, Eqs.(3.54) and (3.55) have the solutions,

$$
\begin{aligned}
z &= 1 - \frac{\alpha^2 \beta}{4\gamma_2 - \alpha^2(\gamma_1 - \beta)} \\
\Lambda &= \frac{\zeta^2}{2\alpha^4} \left\{ \alpha^2 \left[z(2+z)\beta + 2\gamma_1 \right] - 4\gamma_2 \right\}
\end{aligned}
\tag{3.77}
$$

Consider solutions of $\mathcal{L}_V$ purely with A contributions: similar to the last case, a function $\mathcal{G}(\mathcal{A})$ can take any form in terms of $\mathcal{A}$. A particular case is the potential given by Eq.(3.12) with $\gamma_1 = \gamma_2 = \beta_5 = \beta_6 = 0$, for which we have

$$
\mathcal{G}(A) = 2\Lambda + \beta a_i a^i + \frac{1}{\zeta^2} \left[\beta_1 \left(a_i a^i \right)^2 + \beta_2 \left(a^i{}_i \right)^2 + \beta_3 a_i a^i a^j{}_j + \beta_4 a^{ij} a_{ij} \right] \tag{3.78}
$$

In this case, Eq.(3.61) reduces to

$$
a z^2 + b z + c = 0 \tag{3.79}
$$

but now with

$$
\begin{aligned}
a &= -2\beta_1 + \beta_3 \\
b &= 2\beta_2 - \beta_3 + \beta_4 \\
c &= -\alpha^2 \beta - \beta_4
\end{aligned}
\tag{3.80}
$$

Thus, in general there are two solutions,

$$
z_{\pm} = \frac{1}{2(2\beta_1 - \beta_3)} \left[(2\beta_2 - \beta_3 + \beta_4) \pm \sqrt{D} \right] \tag{3.81}
$$

where $D \equiv (2\beta_2 - \beta_3 + \beta_4)^2 + 4(\alpha^2\beta + \beta_4)(\beta_3 - 2\beta_1)$. Clearly, for $z_{\pm}$ to be real, we must assume that $D \geq 0$.

3.4 Scalar Field in The Lifshitz Spacetime

The action of a scalar field in the HL theory takes the form [53],

$$S_M = \int dtd^2xN\sqrt{g}\Big\{\frac{1}{2N^2}[\dot{\varphi} - N^i\nabla_i\varphi]^2 - V(\varphi) -\frac{1}{2}\left[1+2V_1(\varphi)\right](\nabla\varphi)^2 - V_2(\varphi)\left(\nabla^2\varphi\right)^2 - V_4(\varphi)\nabla^4\varphi\Big\} \quad (3.82)$$

Do the variation of the action with respect to φ,

$$S_M = \int dtd^2xN\sqrt{g}\Big[\frac{1}{2N}(\dot{\phi} - N^i\nabla_i\phi)^2 - V(\phi) - \left(\frac{1}{2}+V_1(\phi)\right)^2 -V_2(\phi)(\nabla^2\phi)^2 - V_4(\phi)(\nabla^4\phi)^2\Big] \quad (3.83)$$

$$\begin{aligned}\delta\int dtd^2x\frac{N\sqrt{g}}{2N^2}(\dot{\phi}-N^i\nabla_i\phi)^2 &= \int dtd^2x\frac{\sqrt{g}}{N}(\dot{\phi}-N^i\nabla_i\phi)(\delta\dot{\phi}-N^k\nabla_k\delta\phi)\\ &= -\int dtd^2x\partial t[\frac{\sqrt{g}}{N}(\dot{\phi}-N^i\nabla_i\phi)]\delta\phi\\ &\quad +\int dtd^2x\sqrt{g}\nabla_k[\frac{N^k}{N}(p\dot{h}i - N^i\nabla_i\phi)]\delta\phi\end{aligned}$$

$$-\delta\int dtd^2xN\sqrt{g}V(\phi) = -\delta\int dtd^2xN\sqrt{g}V'\delta\phi = -2\int dtd^2xN\sqrt{g}m^2\phi\delta\phi$$

$$\begin{aligned}-\delta\int dtd^2xN\sqrt{g}(\frac{1}{2}+V_1)(\nabla\phi)^2 &= -\delta\int dtd^2xN\sqrt{g}V'_1(\nabla\phi)^2\delta\phi\\ &\quad -\int dtd^2xN\sqrt{g}(1+2V_1)(\nabla_i\phi)(\nabla^i\phi)\end{aligned}$$

$$\begin{aligned}&-\delta\int dtd^2xN\sqrt{g}V'_1(\nabla\phi)^2\delta\phi + \int dtd^2x\sqrt{g}\nabla^i[N(1+2V_1)(\nabla_i\phi)]\delta\phi\\ &= \int dtd^2xN\sqrt{g}[-V'_1(\nabla\phi)^2 + a^i(1+2V_1)(\nabla_i\phi) + 2(\nabla^1V_1)(\nabla_i\phi) + (1+2V_1)\nabla^2\phi]\delta\phi\end{aligned}$$

$$\begin{aligned}&-\quad \delta\int dtd^2xN\sqrt{g}V_2(\nabla^2\phi)^2\\ &= \quad -\delta\int dtd^2xN\sqrt{g}(\nabla^2\phi)^2V'_2\delta\phi - 2\int dtd^2xN\sqrt{g}V_2\nabla^2\phi\nabla^i\nabla_i\delta\phi\\ &= \quad -\delta\int dtd^2xN\sqrt{g}(\nabla^2\phi)^2V'_2\delta\phi + 2\int dtd^2x\sqrt{g}[NV_2\nabla^2\phi]\nabla^i\delta\phi\end{aligned}$$

$$
\begin{aligned}
&= -\int dtd^2xN\sqrt{g}(\nabla^2\phi)^2{V'}_2\delta\phi \\
&\quad -2\int dtd^2x\sqrt{g}\nabla^i[Na_iV_2(\nabla^2\phi)+N(\nabla_iV_2)(\nabla^2\phi)+NV_2\nabla_i\nabla^2\phi]
\end{aligned}
$$

$$
\begin{aligned}
-\delta\int dtd^2xN\sqrt{g}V_4\nabla^4\phi &= -\int dtd^2xN\sqrt{g}{V'}_4(\nabla^4\phi)\delta\phi-\int dtd^2xN\sqrt{g}V_4\nabla^4\delta\phi \\
&= \int dtd^2xN\sqrt{g}{V'}_4(\nabla^4\phi)\delta\phi-\int dtd^2xN\sqrt{g}\nabla^4(NV_4)\delta\phi \quad (3.84)
\end{aligned}
$$

Eventually it yields,

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\partial_t\left[\frac{\sqrt{g}}{N}(\dot{\varphi}-N^i\nabla_i\varphi)\right]=\nabla_i\left[\frac{N^i}{N}(\dot{\varphi}-N^k\nabla_k\varphi)\right] \\
&\quad+\nabla^i[N(\nabla_i\varphi)(1+2V_1)]-\nabla^2[2NV_2(\nabla^2\varphi)] \\
&\quad-\nabla^4[NV_4]-N[V'+V_1'(\nabla\varphi)^2+V_2'(\nabla^2\varphi)^2+V_4'(\nabla^4\varphi)] \qquad (3.85)
\end{aligned}
$$

To compare with the results obtained in [37], we first set $L=\ell=1,\ z=2$ and $u=1/r$. Then, the metric (3.53) becomes,

$$
ds^2=-\frac{1}{u^4}dt^2+\frac{1}{u^2}(dx^2+du^2) \qquad (3.86)
$$

In the probe limit, the backreaction of the scalar field is neglected. Hence, taking the above space-time as the background, and choosing

$$
\begin{aligned}
V &= m^2\varphi^2,\ V_1=a_1,\ V_2=\frac{\hat{a}_2}{M_*^2}\equiv a_2 \\
V_4 &= \frac{\hat{a}_4}{M_*^2}\varphi\equiv a_4\varphi \qquad (3.87)
\end{aligned}
$$

where a_n are constants, we find that Eq.(3.85) reduces to,

$$
\begin{aligned}
u^2\partial_t^2\varphi &= (1+2a_1)\left(\partial_x^2\varphi+\partial_u^2\varphi-\frac{2}{u}\partial_u\varphi\right)-\frac{2}{u^2}m^2\varphi \\
&\quad -2u^2(a_2+a_4)(\partial_x^4\varphi+2\partial_x^2\partial_u^2\varphi+\partial_u^4\varphi) \\
&\quad -a_4\left[8\partial_x^2\varphi+16\partial_u^2\varphi-\frac{32}{u}\partial_u\varphi+\frac{36\varphi}{u^2}\right] \qquad (3.88)
\end{aligned}
$$

At the boundary $u=0$, the scalar field takes the asymptotical form,

$$
\varphi\sim u^{\triangle}\varphi_1(t,x) \qquad (3.89)
$$

where $\triangle$ is one of the real roots of the equation,

$$(1+2a_1)(\triangle^2 - 3\triangle) - 2m^2 - a_4(16\triangle^2 - 48\triangle + 36)$$
$$-2(a_2+a_4)\triangle(\triangle-1)(\triangle-2)(\triangle-3) = 0 \tag{3.90}$$

From the action (3.82), integrating it by parts and discarding boundary terms, we find that it takes the form,

$$S_M = \int dt d^2x N\sqrt{g}\Bigg\{ -\frac{\varphi}{N\sqrt{g}}\partial_t(\frac{\sqrt{g}\dot{\varphi}}{2N}) - m^2\varphi^2 + \frac{(1+2a_1)\varphi}{2N}\nabla_i(N\nabla^i\varphi) - \frac{a_2\varphi}{N}\nabla^2\left(N\nabla^2\varphi\right) - a_4\varphi\nabla^4\varphi\Bigg\} \tag{3.91}$$

Both actions (3.82) and (3.89) are finite for

$$\triangle > \frac{3}{2} \tag{3.92}$$

with the asymptotic condition (3.89).

In the IR, the V_2 and V_4 terms are very small, and can be set to zero safely. In addition, in this limit the scalar field should be relativistic, so $V_1 = 0$. Hence, the above equation reduces to

$$\triangle^2 - 3\triangle - 2m^2 = 0 \tag{3.93}$$

which has the solutions,

$$\triangle_{\pm} = \frac{1}{2}\left(3 \pm \sqrt{9+8m^2}\right) \tag{3.94}$$

For

$$m^2 > -\frac{9}{8} \tag{3.95}$$

in contrast to the case considered in [37], now only the solution with $\Delta = \Delta_+$,

$$\varphi(u,t,x) \to u^{\triangle_+}(\varphi(t,x) + O(u^2)) \tag{3.96}$$

leads to a finite action either in the form of Eq.(3.82) or in the one of Eq.(3.91).

In the UV, on the other hand, the V_2 and V_4 terms dominate, and Eq.(3.90) becomes,

$$\begin{aligned}&(a_2+a_4)\bigtriangleup^4 - 6(a_2+a_4)\bigtriangleup^3 + (11a_2+27a_4)\bigtriangleup^2 \\ &-(6a_2+54a_4)\bigtriangleup + 36a_4 = 0\end{aligned} \tag{3.97}$$

In the case $a_4 = 0$, the above equation reduces to

$$\bigtriangleup^3 - 6\bigtriangleup^2 + 11\bigtriangleup - 6 = 0, \quad (a_4 = 0) \tag{3.98}$$

which has solutions

$$\bigtriangleup_1 = 1, \qquad \bigtriangleup_2 = 2, \qquad \bigtriangleup_3 = 3, \ (a_4 = 0) \tag{3.99}$$

If we choose $a_2 = -a_4$, Eq.(4.14) has the double root

$$\bigtriangleup = 6, \ (a_2 = -a_4) \tag{3.100}$$

From the above analysis, one can see that the scalar field has quite different behaviors at the boundary $u = 0$ in the two limits, IR and UV.

3.5 Two-Point Correlation Functions

The bulk field $\varphi(u, x)$ can be written in the form

$$\varphi(u,t,x) = \int d^3x'\varphi(0,t',x')G(u,t,x;0,t',x') \tag{3.101}$$

where $\varphi(0, t, x)$ is the scalar field on the boundary and $G(u, t, x; 0, t', x')$ the boundary to bulk propagator. It is easy to work in the Fourier space due to the translational invariance in t and x. In the Fourier space, we have

$$\tilde{\varphi}(u,\omega,k) = \tilde{G}(u,\omega,k)\tilde{\varphi}(0,\omega,k) \tag{3.102}$$

Consider the IR limit:

Setting $a_1 = a_2 = a_4 = 0$, Eq.(3.88) reduces to

$$-u^2\partial_\tau^2\varphi = \partial_x^2\varphi + \partial_u^2\varphi - \frac{2}{u}\partial_u\varphi - \frac{2}{u^2}m^2\varphi \tag{3.103}$$

and $\tilde{G}(u,\omega,k)$ in Fourier space satisfies the equation,

$$\partial_u^2\tilde{G} - \frac{2}{u}\partial_u\tilde{G} - (\omega^2u^2 + |k|^2)\tilde{G} = 0 \tag{3.104}$$

with the boundary conditions,

$$\begin{aligned} &(i)\ \ \tilde{G}(0,\omega,k) = 1, \\ &(ii)\ \ \tilde{G}(\infty,\omega,k)\ \text{ is finite.} \end{aligned} \tag{3.105}$$

The above conditions uniquely determine the propagator $\tilde{G}(u,\omega,k)$,

$$\tilde{G}(u,\omega,k) = \frac{2}{\sqrt{\pi}}e^{-|\omega|u^2/2}\Gamma\left(\frac{k^2}{4|\omega|} + \frac{5}{4}\right)U\left(\frac{k^2}{4|\omega|} - \frac{1}{4}, -\frac{1}{2}, |\omega|u^2\right) \tag{3.106}$$

where $U(a,b,u)$ is the confluent hypergeometric function of the second kind. Near $u=0$, $\tilde{G}$ is given by

$$\tilde{G} = 1 - \frac{k^2}{2}u^2 + \frac{8\Gamma\left(\frac{k^2}{4|\omega|} + \frac{5}{4}\right)|\omega|^{3/2}}{3\Gamma\left(\frac{k^2}{4|\omega|} - \frac{1}{4}\right)}u^3 + O\left(u^4\right) \tag{3.107}$$

In the IR limit and $m=0$, the action Eq.(3.82) yields

$$\begin{aligned} S_M^* &\equiv \frac{i}{2}S_M \\ &= \frac{1}{2}\int d\tau d^2x N\sqrt{g}\left\{\frac{1}{N^2}\varphi'^2 + (\nabla\varphi)^2\right\} \\ &= \frac{1}{2}\int d\tau d^2x \sqrt{{}^{(3)}g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \end{aligned} \tag{3.108}$$

where $t = i\tau$, $\varphi' = \frac{\partial\varphi}{\partial\tau}$. Integrating by parts,

$$\begin{aligned} S_M' &= \int d\tau d^2x N\sqrt{g}\Big[\frac{\dot{\phi}}{2N^2} + a_2(\nabla\phi)^2\Big] \\ &= \int d\omega dk\phi(0,k)\int_\epsilon^\infty du\phi(0,-k)\Bigg[\frac{\omega^2}{2}G(u,k)G(u,-k) \\ &\quad + a_2[k^4G(u,k)G(u,-k) - k^2G(u,k)\partial_u^2G(u,-k) + \partial_u^2G(u,k)\partial_u^2G(u,-k)]\Bigg] \\ &= \int d\omega dk\phi(0,k)F(k,w)\phi(0,-k) \end{aligned} \tag{3.109}$$

where $F(k,\omega) = \int_\epsilon^\infty [\frac{\omega^2}{2} + a_2 k^4 + a_2 k^2 (k^2 + \frac{\omega}{\sqrt{2a_2}}) + a_2 (k^2 + \frac{\omega}{\sqrt{2a_2}})^2] G^2 du$ Now cut off the space at $u = \epsilon$ to regulate the bulk action, $\epsilon \to 0$, thus the on-shell bulk action is determined by the values of the field on the boundary

$$\begin{aligned} S_M^* &= \int d\tau dx [\sqrt{^{(3)}g} g^{uu} \varphi \partial_u \varphi]_\epsilon^\infty \\ &= \int d\omega dk \tilde{\varphi}(0,k,\omega) \mathcal{F}(k,\omega) \tilde{\varphi}(0,-k,-\omega) \end{aligned} \tag{3.110}$$

and the "flux factor" $\mathcal{F}$ is defined as

$$\begin{aligned} F(k,\omega) &= [\omega^2 + a_2 k^2 (k^2 + \frac{\omega}{\sqrt{2a_2}})] \int_0^\infty G^2 du \\ &= \frac{1}{2\sqrt{\frac{k^2}{|w|^4} + \frac{1}{\sqrt{2a_2}|w|^3}}} + \frac{a_2}{2}\sqrt{k^6 + \frac{|w|k^4}{\sqrt{2a_2}}} \end{aligned} \tag{3.111}$$

$$\mathcal{F}(k,\omega) = [\tilde{G}(u,k,\omega)\sqrt{^{(3)}g} g^{uu} \partial_u \tilde{G}(u,-k,-\omega)]_\epsilon^\infty \tag{3.112}$$

Since the propagator $\tilde{G}$ vanishes at $u = \infty$, $\mathcal{F}$ only receives a contribution from the cutoff at $u = \epsilon$. The momentum space two-point function for the operator $\mathcal{O}_\varphi$ dual to φ is given by differentiating Eq.(3.110) twice with respect to $\varphi(0,k,\omega)$:

$$\langle \mathcal{O}_\varphi(k,\omega) \mathcal{O}_\varphi(-k,-\omega) \rangle = \mathcal{F}(k,\omega) \tag{3.113}$$

Plugging Eq.(3.107) into Eq.(3.112), we pick out the leading non-polynomial piece in either k or ω. This gives the correlation function, after taking the limit $\epsilon \to 0$,

$$\langle \mathcal{O}_\varphi(k,\omega) \mathcal{O}_\varphi(-k,-\omega) \rangle = -\frac{8|\omega|^{3/2}\Gamma(a + \frac{3}{2})}{\Gamma(a)} \tag{3.114}$$

where $a \equiv \frac{k^2}{4|\omega|} - \frac{1}{4}$. Since $\Gamma(a \simeq 0) \to \infty$, we find that $\langle \mathcal{O}_\varphi(k,\omega) \mathcal{O}_\varphi(-k,-\omega) \rangle \simeq 0$ as $a \to 0$. When $a \gg 1$, on the other hand, we find $\langle \mathcal{O}_\varphi(k,\omega) \mathcal{O}_\varphi(-k,-\omega) \rangle \simeq -8|\omega|^{1/2}(k^2 + |\omega|)$, which gives rise to correlations between points only with temporal separation.

In general, the divergence arising as $\epsilon \to 0$ from the term proportional to u^2 is removed via local boundary terms [37, 54], and the terms $\mathcal{O}(u^4)$ and higher vanish as the cutoff is removed why taking the limit $\epsilon \to 0$.

In the UV limit:

Setting $a_1 = a_4 = 0$, we find that Eq.(3.88) reduces to

$$\partial_\tau^2 \varphi = 2a_2(\partial_x^4 \varphi + 2\partial_x^2 \partial_u^2 \varphi + \partial_u^4 \varphi) \tag{3.115}$$

In the Fourier space, this becomes

$$\partial_u^4 \tilde{G} - 2k^2 \partial_u^2 \tilde{G} + (k^4 + \frac{\omega^2}{2a_2})\tilde{G} = 0 \tag{3.116}$$

with the same boundary condition as in Eq.(3.105). Then, we find that

$$\begin{aligned} \tilde{G} &= c_1 \exp -u\sqrt{\rho}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) \\ &\quad +(1-c_1)\exp -u\sqrt{\rho}(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}) \end{aligned} \tag{3.117}$$

where $\rho\cos\theta = k^2, \rho\sin\theta = \sqrt{\frac{w^2}{2a_2}}$.

Thus, with $m = 0$, the action (3.82) gives rise to,

$$\begin{aligned} S_M^{**} &\equiv iS_M \\ &= \int d\tau d^2x N\sqrt{g}\Big\{\frac{1}{2N^2}\varphi'^2 + a_2(\nabla^2\varphi)^2\Big\} \\ &= \int d\omega dk \tilde{\varphi}(0,k,\omega)\int_\epsilon^\infty du\Big\{\frac{\omega^2}{2}\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega) \\ &\quad +a_2[k^4\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega) - k^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega) \\ &\quad +\partial_u^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega)]\Big\}\tilde{\varphi}(0,-k,-\omega) \\ &= \int d\omega dk \tilde{\varphi}(0,k,\omega)\mathcal{F}(k,\omega)\tilde{\varphi}(0,-k,-\omega) \end{aligned} \tag{3.118}$$

where

$$\begin{aligned} \mathcal{F}(k,\omega) &= \int_\epsilon^\infty du\Big\{\frac{\omega^2}{2}\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega) \\ &\quad +a_2[k^4\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega) - k^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega) \\ &\quad +\partial_u^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega)]\Big\} \end{aligned} \tag{3.119}$$

Plugging Eq.(3.117) into Eq.(3.119), and taking the limit $\epsilon \to 0$, we find that

$$\mathcal{F}(k,\omega) = \frac{a_2}{4}\rho^{\frac{3}{2}}\sec\frac{\theta}{2}\Big\{(1+e^{-i\theta})\cos\theta$$

$$+c_1[5 - 3\cos 2\theta + 2(i\sin\theta - 1)\cos\theta$$
$$+c_1(2\cos\theta + 3\cos 2\theta - 5)]\Big\} \qquad (3.120)$$

We will draw a brief conclusion on this chapter.

The effects of high-order operators on the non-relativistic Lifshitz holography has been studied in the framework of the Hořava-Lifshitz (HL) theory of gravity [41], which contains all the required high-order spatial operators in order to be power-counting renormalizble. The unitarity of the theory is also preserved, because of the absence of the high-order time operators. In this sense, the HL gravity is an ideal place to study the effects of high-order operators on the non-relativistic gauge/gravity duality.

In particular, it was shown that the Lifshitz space-time (3.53) is not only a solution of the HL gravity in the IR, as first shown in [45] and later rederived in [46], but also a solution of the full theory. The effects of the high-order operators on the Lifshitz dynamical exponent z is simply to shift it to different values, as these high-order operators become more and more important. This is similar to the case studied in [48–54].

For a scalar field that has the same symmetry in the UV as the HL gravity, the foliation-preserving diffeomorphism, while in the IR, the asymptotic behavior of the scalar field near the boundary is similar to that given in the 4-dimensional spacetimes [37], its asymptotic behavior in the UV gets dramatically changed, so does the corresponding two-point correlation function. This is expected, because the high-order operators dominate the behavior of the scalar field in the UV. Then, according to the holographic correspondence, this in turn affects the two-point correlation functions.

CHAPTER FOUR

Summary and Outlook

The appearance of singularity in the traditional black hole solutions of the classical theory of general relativity predicts the limitation of the theory. The curvature and the mass density values approach infinity at the singularity.

Different ways have been attempted to circumvent the nonphysical problem of singularity. One way is to look for black hole solutions for the Einstein equation of gravity which have no singularity inside, called regular black holes. We have found the tunneling rate and the Hawking temperature for the fermion tunneling process near a regular black hole. It is possible that the existence of regular black hole still relies on the quantum processes happening within the black hole interior, probably near the location of the traditional black hole singularities.

The other way is to attempt for quantum gravity. Since the first version of quantized theory of general relativity has failed due to the non-renormaliability problem, various attempts have been proposed during the decades. Superstring theory and loop quantum gravity are the two major movements. Both of them have made great progress in the last thirty to forty years. But a successful quantum gravity theory is still not in reach. Hořava-Lifshitz theory of gravity is an another new attempt arising during this situation. Its classical version of theory is power counting renormaliable, but the price paid for this is the sacrifice of Lorentz invariance in the high energy and microscopic scales. Further work remains for the final quantization of Hořava-Lifshitz theory and various problems may occur in the process.

The problem of singularity and the quantization of gravity are expected to be solved in a combined battle. This is expected to be the main theme of the theoretical gravitational physics in the following decades.

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