ABSTRACT

Inhomogeneous Diophantine Approximation

Brian King

Director: Daniel Herden, Ph.D.

Diophantine approximation is a topic of number theory concerned with rational approximations of real, typically irrational, numbers. In other words, we seek $q \in \mathbb{Z}$ for a given $\alpha \in \mathbb{R}$ such that $||q\alpha||$ (the distance from $q\alpha$ to the nearest integer) is small. This thesis serves as a primer on many of the famous results in this field. First, the fundamental result of Diophantine approximation, Dirichlet's theorem, is presented along with several methods of proof. Other theorems involving limits of accuracy are given before moving into inhomogeneous approximations (of the form $||q\alpha - \beta||$ for some $\beta \in \mathbb{R}$). Finally, recent results in the inhomogeneous case are used to prove a theorem on irrational circle rotations with connections to ergodic theory.

APPROVED BY DIRECTOR OF HONORS THESIS:

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Dr. Daniel Herden, Department of Mathematics

APPROVED BY THE HONORS PROGRAM:

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Dr. Elizabeth Corey, Director

DATE: ______________________

INHOMOGENEOUS DIOPHANTINE APPROXIMATION

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Brian King

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Contents

CHAPTER ONE

Dirichlet's Theorem

A major area in the field of Diophantine approximation involves how well we can approximate a real number by rational numbers. A classic first result in this subject is Dirichlet's Approximation Theorem.

Theorem (Dirichlet)**.** *For each $\alpha \in \mathbb{R}$ and $Q \in \mathbb{N} \setminus \{1\}$, there exists $p, q \in \mathbb{Z}$ with $1 \leq q < Q$ such that*

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{qQ}\ . \tag{1}$$

Furthermore, with α irrational, there exist infinitely many solutions $\frac{p}{q}$ to the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}\ . \tag{2}$$

Pigeonhole Principle Proof

We multiply both sides of (1) by q to arrive at the inequality $|q\alpha - p| \leq \frac{1}{Q}$. Now consider numbers of the form $s\alpha - \lfloor s\alpha \rfloor$. Letting s range from 1 to $Q-1$, we get a list of $Q-1$ numbers, and then add to our list 0 and 1. All of these can be written in the form $m\alpha - n$ for some $m, n \in \mathbb{Z}$ with $m = s, n = \lfloor s\alpha \rfloor$ for $1 \leq s \leq Q-1$ and $0 = 0\alpha - 0$ and $1 = 0\alpha - (-1)$. Furthermore, all fall in the interval $[0, 1]$. Divide this interval into Q segments each of length $\frac{1}{Q}$. By the Pigeonhole Principle, at least two of the $Q+1$ numbers created above must fall in one of the intervals. Call m_1, n_1, m_2, n_2 the corresponding integers. Note that $m_1 \neq m_2$ by our choice of the

m, n. Let $m_1 > m_2$ without loss of generality. Then we have

$$|(m_1\alpha - n_1) - (m_2\alpha - n_2)| = |(m_1 - m_2)\alpha - (n_1 - n_2)| \leq \frac{1}{Q}\,.$$

Taking $p = m_1 - m_2$ and $q = n_1 - n_2$, we have the first part of the theorem.

Now take α irrational. Let's say we have some number of solutions to (2): $\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}$. Then take Q big enough so that $\frac{1}{Q} < \left|\alpha - \frac{p_i}{q_i}\right|$ for all $i \in \{1, \ldots, n\}$. Note that if we have no solutions yet we can take Q to be anything. By the first part of the theorem, there exists a solution $\frac{p}{q}$ (with $1 \leq q < Q$) such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{qQ} < \frac{1}{q^2}\,.$$

Note that this solution is distinct from all of the previous ones (since $\frac{1}{qQ} \leq \frac{1}{Q}$). Using this method, we can obtain infinitely many solutions. ■

Farey Fractions

A useful tool in the study of Diophantine approximation is that of Farey fractions, named after the British geologist John Farey, Sr. The *Farey fractions* F_n of order n consist of all fractions $\frac{p}{q} \in [0, 1]$ with denominator less than or equal to n, reduced to lowest terms. For example:

$$F_1 = \{\frac{0}{1}, \frac{1}{1}\}, F_2 = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}, F_3 = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}, F_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}\,.$$

When put in increasing order as above, these fractions are also referred to as *Farey sequences*. These easily defined sequences turn out to have useful properties which will allow for a second proof of the second part of Dirichlet's theorem as well as a proof of Hurwitz's Theorem (cf. Chapter 2). This latter result deals with tightening the bound given by Dirichlet and shows that there is a limit to how far we can restrict our bound before we lose the guarantee of infinitely many integers with the desired property.

For any two fractions $\frac{p}{q}, \frac{r}{s}$ with $\frac{p}{q} < \frac{r}{s}$, it is clear that $qr - ps > 0$, i.e. $qr - ps \geq 1$. We can also write this as $\frac{r}{s} - \frac{p}{q} \geq \frac{1}{qs}$. The equality case in both expressions signifies that two fractions are *as close as possible*. As it turns out, any two consecutive Farey fractions are as close as possible, a conjecture originally stated and proved by Cauchy, the proof of which we demonstrate now. As a historical note, an immediate corollary of this theorem is that given any three consecutive Farey fractions, $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$, the middle fraction is the mediant of its neighbors (i.e. $\frac{c}{d} = \frac{a+e}{b+f}$). This was actually the original conjecture of John Farey. We take the theorem in its following form from Klazar [7].

Theorem (Cauchy-Farey). *If $\frac{p}{q}, \frac{r}{s}$ with $\frac{p}{q} < \frac{r}{s}$ are consecutive Farey numbers in F_n, then they are as close as possible and thus $qr - ps = 1$.*

Proof. We seek to show that the pair r, s is a solution to the Diophantine equation $qx - py = 1$ where $x, y \in \mathbb{Z}$ are the unknowns. The equation has a solution, since $\gcd(p, q) = 1$, and thus there are any number of solutions formed by $x - pa, y - qa$ for any $a \in \mathbb{Z}$. So there exist x_1, y_1 with the constraint that $0 \leq n - q < y_1 \leq n$, where

n is the order of our Farey sequence F_n.

We know $qx_1 - py_1 = 1$, and thus

$$\frac{x_1}{y_1} = \frac{1}{qy_1} + \frac{p}{q} .$$

Note that $\frac{x_1}{y_1} \in F_n$: $\gcd(x_1, y_1) = 1$ from the fact that x_1, y_1 solves the Diophantine equation above, $0 < y_1 \leq n$ by how we defined y_1 and finally $0 < x_1 \leq y_1$ (since $qx_1 - py_1 = 1$ and $0 < p < q$).

From above, $\frac{x_1}{y_1} > \frac{p}{q}$ and thus $\frac{x_1}{y_1} \geq \frac{r}{s}$. We show that equality must be the case, and thus prove the theorem, by showing that a strict inequality ($\frac{x_1}{y_1} > \frac{r}{s}$) leads to a contradiction.

We know, based on the ordering of the fractions, that:

$$\frac{r}{s} - \frac{p}{q} \geq \frac{1}{qs} \text{ and } \frac{x_1}{y_1} - \frac{r}{s} \geq \frac{1}{sy_1} .$$

Adding the two together, we arrive at

$$\frac{x_1}{y_1} - \frac{p}{q} \geq \frac{q + y_1}{qsy_1} .$$

Furthermore, using the equation from earlier, we know the left hand side is equal to $\frac{1}{qy_1}$. Thus we have

$$\frac{1}{qy_1} \geq \frac{q + y_1}{qsy_1} .$$

But this then implies that $s \geq q + y_1$, which from the earlier restriction on y_1, implies

$s > n$. This is a contradiction (since $\frac{r}{s} \in F_n$). Thus $\frac{x_1}{y_1} = \frac{r}{s}$ and the result follows. $\blacksquare$

Using this knowledge that consecutive Farey fractions are as close as possible, we can construct a simple proof of the second part of Dirichlet's Theorem.

Let $\alpha \in (0,1)$ be an irrational number. We find the Farey fractions closest to α: $\frac{p}{q}, \frac{r}{s} \in F_n$ for some fixed n with $\frac{p}{q} < \alpha < \frac{r}{s}$. Since the two Farey fractions are consecutive, the theorem above tells us that $\frac{r}{s} - \frac{p}{q} = \frac{1}{qs}$. We have two cases.

Case 1: $q \leq s$

$$\alpha - \frac{p}{q} < \frac{r}{s} - \frac{p}{q} = \frac{1}{qs} \leq \frac{1}{q^2}$$

Case 2: $q > s$

$$\left|\alpha - \frac{r}{s}\right| < \left|\frac{r}{s} - \frac{p}{q}\right| = \frac{1}{qs} < \frac{1}{s^2}$$

So it is clear we have at least one solution. From here we refine our bounding of α by looking at Farey fractions of a higher order ($\in F_m$). We use m large enough so that $\frac{1}{m} < \min(\alpha - \frac{p}{q}, \frac{r}{s} - \alpha)$. Note that $\frac{1}{m}$ is the maximum distance between any two consecutive fractions in F_m. From here we start the process anew, finding consecutive $\frac{a}{b}, \frac{c}{d} \in F_m$ such that $\frac{a}{b} < \alpha < \frac{c}{d}$. Our definition of m guarantees that $\frac{a}{b} \neq \frac{p}{q}$ and that $\frac{c}{d} \neq \frac{r}{s}$. Thus we can find a new approximation as we did above. Since we can always find a higher m that gives us a closer approximation, we can obtain infinitely many solutions $\frac{p}{q}$ to the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$.

CHAPTER TWO

Limits of Accuracy

In addition to providing an alternative proof of Dirichlet's Theorem, Farey fractions are key in the proof of Hurwitz's Theorem, a result which improves upon the upper bound laid out by Dirichlet.

Theorem (Hurwitz)**.** *For every irrational α, there exist infinitely many solutions $\frac{p}{q}$ to the inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}\ . \tag{1}$$

Furthermore, with $\phi = \frac{1+\sqrt{5}}{2}$ and for every constant $C > \sqrt{5}$, the inequality

$$\left|\phi - \frac{p}{q}\right| < \frac{1}{Cq^2} \tag{2}$$

has only finitely many solutions $\frac{p}{q}$.

Proof. To prove the first part of this theorem, let $\alpha \in (0,1)$ without loss of generality. Given two consecutive members $\frac{p}{q}, \frac{r}{s} \in F_n$ for some n such that $\frac{p}{q} < \alpha < \frac{r}{s}$, we wish to show that either

$$\frac{p}{q}, \frac{r}{s}, \text{ or their mediant } \frac{a}{b} = \frac{p+r}{q+s}$$

satisfies (1). If this can be shown, the theorem follows simply by employing the same refining argument used in the Farey fraction proof of Dirichlet's theorem.

For the purpose of arriving at a contradiction, we first assume that none of the fractions satisfy (1). Furthermore, we assume $\frac{a}{b} > \alpha$ (the proof given $\frac{a}{b} < \alpha$ is similar). We thus know:

$$\alpha - \frac{p}{q} \geq \frac{1}{\sqrt{5}q^2}, \quad \frac{a}{b} - \alpha \geq \frac{1}{\sqrt{5}b^2}, \text{ and } \frac{r}{s} - \alpha \geq \frac{1}{\sqrt{5}s^2} \; .$$

We then add the first and second inequalities along with the first and third inequalities, obtaining:

$$\frac{a}{b} - \frac{p}{q} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{b^2} + \frac{1}{q^2}\right) \text{ and } \frac{r}{s} - \frac{p}{q} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{s^2} + \frac{1}{q^2}\right) \; .$$

From the Cauchy-Farey theorem, we have the equalities $\frac{a}{b} - \frac{p}{q} = \frac{1}{bq}$ and $\frac{r}{s} - \frac{p}{q} = \frac{1}{sq}$. We multiply the first inequality by $\sqrt{5}b^2q^2$ and the second one by $\sqrt{5}s^2q^2$ to obtain

$$\sqrt{5}bq \geq b^2 + q^2 \text{ and } \sqrt{5}sq \geq s^2 + q^2 \; .$$

Adding them together and substituting in $b = q + s$ gives

$$\sqrt{5}q(q + 2s) \geq 2s^2 + 3q^2 + 2sq$$

or equivalently

$$(3 - \sqrt{5})q^2 + 2s^2 + (2 - 2\sqrt{5})sq \leq 0$$

which can finally be rewritten as

$$\frac{1}{2}\left((1 - \sqrt{5})q + 2s\right)^2 \leq 0 \; .$$

Since a square can't be negative, this implies $(1-\sqrt{5})q + 2s = 0$, and thus $\sqrt{5} = 1 + \frac{2s}{q} \in \mathbb{Q}$, which is clearly not true. Therefore, we have arrived at a contradiction and our assumption that none of the fractions satisfy (1) must be false. This proves the first part of the theorem.

To prove the second part of the theorem, we once again assume, for the sake of arriving at a contradiction, that there are infinitely many solutions $\frac{p}{q}$ to (2) for some fixed $C > \sqrt{5}$. If we let $\delta \in \mathbb{R}$ be such that $|\delta| < \frac{1}{C}$, then our assumption above is the same as saying there are infinitely many $\frac{p}{q}$ solving

$$\phi = \frac{p}{q} + \frac{\delta}{q^2} \ .$$

We rearrange this as $\frac{\delta}{q} = q\phi - p$. Then inserting the numerical value for ϕ and subtracting $\frac{q\sqrt{5}}{2}$, we get

$$\frac{\delta}{q} - \frac{q\sqrt{5}}{2} = \frac{q}{2} - p \ .$$

We then square both sides and subtract off $\frac{5q^2}{4}$ to get

$$\frac{\delta^2}{q^2} - \delta\sqrt{5} = -q^2 - pq + p^2 \ .$$

Since $p, q \in \mathbb{Z}$, the left-hand side of this equation must also be an integer. Furthermore, for large enough q (which we know exists given our assumption of infinite solutions),

$$\left|\frac{\delta^2}{q^2} - \delta\sqrt{5}\right| < 1 \text{ since } \lim_{q\to\infty} \frac{\delta^2}{q^2} = 0 \text{ and } \left|\delta\sqrt{5}\right| < \frac{\sqrt{5}}{C} < 1 \ .$$

Thus, $-q^2 - pq + p^2 = 0$ must have a solution and consequently we can write $q^2 - 4pq + 4p^2 = 5q^2$, i.e., $(2p-q)^2 = (q\sqrt{5})^2$. As in the proof of the first part, this leads to $\sqrt{5} = \frac{2p}{q} - 1 \in \mathbb{Q}$, which is once again a contradiction. ■

Hurwitz's Theorem provides, in some sense, a best bound on infinite rational approximations, but really only deals with a very specific type of approximation function. To make this more clear, we use the notation: $||q\alpha|| = \min_{n\in\mathbb{Z}} |q\alpha - n|$. Thus, we can say the theorems of Dirichlet and Hurwitz only consider approximations of the form $||q\alpha|| < \frac{1}{Cq}$. But what if we want to discuss other approximating sequences for the right-hand side of this inequality and the results obtained? This question is the very basis of a field called metric Diophantine approximation, of which Khinchin's (or Khintchine's) Theorem is a central result.

Theorem (Khinchin)**.** *Let (Ψ_q) be a non-increasing sequence of positive real numbers.*

(a) If $\sum_q \Psi_q$ diverges, then for almost every $\alpha \in \mathbb{R}$ there exist infinitely many integers $q > 0$ with $||q\alpha|| < \Psi_q$.

(b) If $\sum_q \Psi_q$ converges, then for almost every $\alpha \in \mathbb{R}$ there exist only finitely many integers $q > 0$ with $||q\alpha|| < \Psi_q$.

The proof of part (a) involves complex methods beyond the scope of this paper. Interested readers can consult [3, p. 324] for more details. The proof of part (b), however, uses only basic ideas from measure theory and will be provided in Chapter 3.

The earlier result of Hurwitz tells us that for $\Psi_q = \frac{1}{Cq}, C > \sqrt{5}$, the set of exceptional reals $\alpha \in \mathbb{R}$ is nonempty of measure zero. Beresnevich, Ramirez, and Velani give a full characterization of this set of exceptional reals using continued fractions [2]. Members of this set are referred to as *badly approximable numbers*. It is believed that the only badly approximable algebraic irrationals are quadratic irrationals, such as the golden ratio ϕ, but this conjecture (referred to in [2] as the Folklore Conjecture) remains unproven.

CHAPTER THREE

Inhomogeneous Approximations

Up to this point, we have been looking at bounds on $||q\alpha||$, which essentially corresponds to irrational rotation on a unit circle with start value 0 (cf. Chapter 4). A natural extension to the theory, then, might involve a shifting of that start value. In other words, for given $\beta \in \mathbb{R}$, what is the behavior of $||q\alpha - \beta||$? This type of approximation is referred to as *inhomogeneous* and has also been the subject of considerable study. As Kleinbock notes in [8], it seems natural to think that this is a simple translation of the homogeneous case, but the change in problem is actually deeper. Nevertheless, very similar results can be shown for the inhomogeneous case, which once again use a variety of methods in their proofs. The first result we demonstrate is improved from a theorem of Hua [5, p. 266], originally by Chebyshev.

Theorem (Improved Chebyshev)**.** *Let α be irrational and $\beta \in \mathbb{R}$. Then for every $\epsilon > 0$ there exist infinitely many $q \in \mathbb{Z}$ with $q > 0$ such that*

$$||q\alpha - \beta|| < \frac{3(1+\epsilon)}{2q} \ .$$

Proof. By Dirichlet's Theorem, we know there exist infinitely many $x, y \in \mathbb{Z}$ with $y > 0$ and $\gcd(x, y) = 1$ such that $|y\alpha - x| = y\left|\alpha - \frac{x}{y}\right| < \frac{1}{y}$. We can rewrite this in terms of α:

$$\alpha = \frac{x}{y} + \frac{\delta}{y^2} \text{ with } |\delta| < 1 \ .$$

For a fixed y, we now choose $t \in \mathbb{Z}$ so that $|y\beta - t| \leq \frac{1}{2}$. We then have:

$$\beta = \frac{t}{y} + \frac{\delta'}{2y} \text{ with } |\delta'| < 1 \ .$$

Next, we choose some $\epsilon' > 0$ with

$$(1+\epsilon')^2 + \frac{1+\epsilon'}{2} < \frac{3}{2}(1+\epsilon) \ .$$

Such a choice is possible as the left-hand side of this inequality tends toward $\frac{3}{2}$ for $\epsilon' \to 0$. Due to the coprimeness of x and y, we know there exist p, q such that

$$\epsilon' y \leq q \leq (\epsilon'+1)y \text{ and } qx - py = t \ .$$

With all these facts in mind, we return to our original inequality,

$$||q\alpha - \beta|| = |q\alpha - p - \beta| = \left| \frac{qx}{y} + \frac{q\delta}{y^2} - p - \frac{t}{y} - \frac{\delta'}{2y} \right| = \left| \frac{q\delta}{y^2} - \frac{\delta'}{2y} \right| < \frac{q}{y^2} + \frac{1}{2y} \ .$$

From our earlier restriction on q, we have $y \geq \frac{q}{1+\epsilon'}$ and thus $\frac{1}{y} \leq \frac{1+\epsilon'}{q}$. Therefore,

$$||q\alpha - \beta|| < \frac{q}{y^2} + \frac{1}{2y} < q\frac{(1+\epsilon')^2}{q^2} + \frac{1+\epsilon'}{2q} = \frac{1}{q}\left((1+\epsilon')^2 + \frac{1+\epsilon'}{2}\right) < \frac{3(1+\epsilon)}{2q} \ .$$

Since the start value for y can be arbitrarily large, we get the result. ■

This initial result serves as a good starting place for limits of accuracy in the inhomogeneous case. However, there exist better upper bounds.

Theorem (Minkowski)**.** *For every $\alpha, \beta \in \mathbb{R}$ with α irrational and $\beta \neq m\alpha + n$ for any $m, n \in \mathbb{Z}$, there exist infinitely many integers q with $||q\alpha - \beta|| < \frac{1}{4|q|}$.*

The original proof of this theorem, given by Minkowski in [9], is geometric in nature. An alternative proof can be seen in [4, p. 48], which relies largely on a different theorem of Minkowski on linear forms.

Note the fact that q is no longer strictly positive, but rather is allowed to range over all the integers. This differs from Hurwitz's Theorem in the homogeneous case. However, there does exist a true inhomogeneous analogue, first demonstrated by Khinchin.

Theorem (Inhomogeneous Hurwitz)**.** *For every $\alpha, \beta \in \mathbb{R}$ with α irrational and $\epsilon > 0$, there exist infinitely many integers $q > 0$ with $||q\alpha - \beta|| < \frac{1+\epsilon}{\sqrt{5}q}$.*

A proof of this theorem can be found in [5, p. 267]. There exists also an inhomogeneous version of Khinchin's Theorem.

Theorem (Inhomogeneous Khinchin)**.** *Let (Ψ_q) be a non-increasing sequence of positive real numbers and $\beta \in \mathbb{R}$.*

(*a*) *If $\sum_q \Psi_q$ diverges, then for almost every $\alpha \in \mathbb{R}$ there exist infinitely many integers $q > 0$ with $||q\alpha - \beta|| < \Psi_q$.*

(*b*) *If $\sum_q \Psi_q$ converges, then for almost every $\alpha \in \mathbb{R}$ there exist only finitely many integers $q > 0$ with $||q\alpha - \beta|| < \Psi_q$.*

Once again, the proof of (a) proves much more difficult. Consult [1, p. 67] for details. We present the proof of part (b) below. This then implies the homogeneous case presented earlier as well (simply take $\beta = 0$). Note also that the proof of part (b) does not require Ψ_q to be non-increasing.

Proof. Without loss of generality, we take $\alpha \in [0, 1)$ and $0 \leq \Psi_q \leq \frac{1}{2}$. Let $B = \{\alpha \in [0, 1) \mid ||q\alpha - \beta|| < \Psi_q \text{ for infinitely many } q\}$. We wish to show that B has measure zero.

Now fix q and look at $B_q = \{\alpha \in [0, 1) \mid ||q\alpha - \beta|| < \Psi_q\}$. Note that $||q\alpha - \beta|| < \Psi_q$ means that $|q\alpha - \beta - p| < \Psi_q$ for some $p \in \mathbb{Z}$. So we have

$$\frac{\beta + p - \Psi_q}{q} < \alpha < \frac{\beta + p + \Psi_q}{q} .$$

This creates intervals of width $\frac{2\Psi_q}{q}$ with centers $1/q$ apart from each other. Thus we know that B_q has measure $2\Psi_q$.

Recall that $\sum_q \Psi_q$ converges, and thus, for given $\epsilon > 0$, there exists $Q \in \mathbb{N}$ such that $\sum_{q \geq Q} \Psi_q < \frac{\epsilon}{2}$. Note also that if a number $\alpha \in B$, then $\alpha \in B_q$ for some $q \geq Q$. Therefore

$$B \subseteq \bigcup_{q \geq Q} B_q \text{ with } \left| \bigcup_{q \geq Q} B_q \right| \leq \sum_{q \geq Q} |B_q| \leq 2 \sum_{q \geq Q} \Psi_q < \epsilon .$$

This holds for every $\epsilon > 0$ and thus implies that B is a set with measure zero. ■

The theorems presented in Chapters 2 and 3 stand alone as strong results in the field of Diophantine approximation, but also prove useful in other areas of mathematics, such as ergodic theory. Indeed, the initial impetus for this thesis was an exploration into a certain type of ergodic transformation. We take up this topic in depth in the following chapter.

CHAPTER FOUR

Irrational Circle Rotations

As discussed in Chapter 3, inhomogeneous Diophantine approximations can be viewed in the paradigm of an irrational rotation on [0,1) (identified with the unit circle $\mathbb{T}$). In [6], Kim explores this idea and proves a rather stunning result on what he calls the *shrinking target property.*

Theorem (Kim)**.** *Let α be an irrational number. Then for almost every $\beta \in \mathbb{R}$,*

$$\liminf_{q\to\infty} q||q\alpha - \beta|| = 0 \ .$$

Kim's discovery shows how the result of Hurwitz presented in Chapter 2 is actually an outlier in some sense. From this new viewpoint of the limit inferior, we see that for the irrational ϕ,

$$\liminf_{q\to\infty} q||q\phi|| = \frac{1}{\sqrt{5}} \ .$$

To see this, note that the first part of Hurwitz implies that $\liminf_{q\to\infty} q||q\alpha|| \leq \frac{1}{\sqrt{5}}$ for any irrational α. The second part of Hurwitz's theorem implies that, for $\alpha = \phi$, no subsequence of $(q||q\phi||)$ exists which converges to a value less than $\frac{1}{C}$ for $C > \sqrt{5}$. More precisely,

$$\liminf_{q\to\infty} q||q\phi|| \geq \frac{1}{C} \text{ for all } C > \sqrt{5} \implies \liminf_{q\to\infty} q||q\phi|| \geq \frac{1}{\sqrt{5}} \ .$$

Thus $\beta = 0$ belongs to the set of all β for which Kim's result does not hold (a set of measure zero).

Kim's theorem can be used in the proof of a result related to ergodic theory, the field which first led us to the study of Diophantine approximation. The original issue at hand was a conjecture dealing with the maximal operator of ergodic theory, stated as follows:

Conjecture 1. *Let T be a measure preserving transformation on the nonatomic probability space (X, Σ, μ) and let f be a μ-measurable function on that space. If $T^*f(x)$ is finite μ-almost everywhere, where T^*f is defined by*

$$T^*f(x) = \sup_{N \geq 1} \frac{1}{N} \left| \sum_{j=0}^{N-1} f(T^j x) \right| ,$$

then the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x)$$

exists μ-almost everywhere.

The initial idea was that certain properties (e.g. a natural cancellation) of the ergodic transformation associated to an irrational rotation on [0,1) might make it an excellent candidate for a possible counterexample. For this reason, we sought to use Diophantine results, like the theorem of Hurwitz, to show that the conjecture was wrong. As it turns out, Kim's theorem makes it clear that the irrational rotation transformation provides no trivial counterexamples and is by no means an easy to handle ergodic transformation itself. Theorem 1 summarizes the results discovered.

Before presenting its statement and proof, we first prove several lemmas recalling pertinent results in analysis. Recall that for a sequence (a_n) of real numbers we have the definition

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \left(\sup_{n\geq N} a_n\right) .$$

Both $\limsup_{n\to\infty} a_n = \infty$ and $\limsup_{n\to\infty} a_n = -\infty$ are possible.

Lemma 1. *If (a_n) and (b_n) are real sequences, then*

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n) ,$$

provided the right side of this inequality is defined (i.e. not $\infty - \infty$ or $-\infty + \infty$).

Proof. For every $N \geq 1$ and $i \geq N$, we know that

$$a_i + b_i \leq \sup_{n\geq N} a_n + \sup_{n\geq N} b_n .$$

Thus $\sup_{n\geq N}(a_n + b_n) \leq \sup_{n\geq N} a_n + \sup_{n\geq N} b_n$. Taking $N \to \infty$, we get the result. ■

Lemma 2. *If (a_n) and (b_n) are real sequences and $b_n \to b \in \mathbb{R}$, then*

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty}(a_n) + b = \limsup_{n\to\infty}(a_n + b) .$$

Proof. By Lemma 1, we know $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(a_n) + b$. Now rewrite a_n as $a_n + b_n - b_n$. Applying Lemma 1 again, we see that $\limsup_{n\to\infty}(a_n + b_n - b_n) \leq \limsup_{n\to\infty}(a_n + b_n) - b$ (since $-b_n \to -b$). Thus we have the equality $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty}(a_n) + b$.

The equality $\limsup_{n\to\infty}(a_n + b) = \limsup_{n\to\infty}(a_n) + b$ follows from considering the constant sequence $b_n \to b$. ■

Recall that $\limsup_{n\to\infty} a_n$ can also be defined as the maximal limit (possibly $-\infty$ or ∞) of a subsequence (a_{n_k}) of (a_n).

Lemma 3. *If (a_n) and (b_n) are real sequences and $b_n \to b \in \mathbb{R}$ with $b > 0$, then*

$$\limsup_{n\to\infty}(a_n b_n) = b \limsup_{n\to\infty}(a_n) = \limsup_{n\to\infty}(a_n b) \ .$$

Proof. Note that convergent subsequences $(a_{n_k} b_{n_k})$ of $(a_n b_n)$ correspond to convergent subsequences (a_{n_k}) of (a_n), and that $\lim_{k\to\infty} a_{n_k} b_{n_k} = b \lim_{k\to\infty} a_{n_k}$. Furthermore, for $b > 0$, maximal limits of subsequences of $(a_n b_n)$ correspond to maximal limits of subsequences of (a_n), and $\limsup_{n\to\infty}(a_n b_n) = b \limsup_{n\to\infty}(a_n)$. For the second equality simply consider a constant sequence $b_n = b$. ■

Lemma 4. *If (a_n) is a real sequence, then*

$$\limsup_{n\to\infty} |a_n| = \max\{\limsup_{n\to\infty} a_n, \limsup_{n\to\infty}(-a_n)\} \ .$$

Proof. Let $\limsup_{n\to\infty} |a_n| = c$ and let (a_{n_k}) be a subsequence of (a_n) with $a_{n_k} \to c$. There will either exist infinitely many k with $a_{n_k} \geq 0$ or infinitely many k with $a_{n_k} \leq 0$. The first case results in a subsequence $(a_{n'_k})$ of (a_{n_k}) with $|a_{n'_k}| = a_{n'_k} \to c$, and thus $c \leq \limsup_{n\to\infty} a_n$. The latter case results in a subsequence $(a_{n'_k})$ of (a_{n_k}) with

$|a_{n'_k}| = -a_{n'_k} \to c$, and thus $c \leq \limsup_{n\to\infty}(-a_n)$. In both cases,

$$c \leq \max\{\limsup_{n\to\infty} a_n, \limsup_{n\to\infty}(-a_n)\} \ .$$

Note however that we also have $\pm a_n \leq |a_n|$ for all n, hence $\limsup_{n\to\infty} a_n, \limsup_{n\to\infty}(-a_n) \leq c$ and thus

$$\max\{\limsup_{n\to\infty} a_n, \limsup_{n\to\infty}(-a_n)\} \leq c \ . \qquad \blacksquare$$

Lemma 5. *If (a_n) is a sequence converging to a finite limit L, then*

$$\limsup_{n\to\infty} |a_{n+1} - a_n| = 0 \ .$$

Proof. As $a_n \to L$, we have $a_{n+1} \to L$ and $a_{n+1} - a_n \to L - L = 0$. Hence, $|a_{n+1} - a_n| \to 0$. In particular,

$$\limsup_{n\to\infty} |a_{n+1} - a_n| = \lim_{n\to\infty} |a_{n+1} - a_n| = 0 \ . \qquad \blacksquare$$

These lemmas will be used in the proof of the main theorem, stated below.

Theorem 1. *Let α be an irrational number, and define the measure preserving transformation T on $[0,1)$ by*

$$Tx = (x + \alpha) \bmod 1 \ .$$

Define the function f on $[0,1)$ by

$$f(x) = \frac{1}{x - \frac{1}{2}} \ .$$

If $x \in [0,1)$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x)$$

fails to converge to a finite number. Moreover for almost every $x \in [0,1)$ we have

$$T^* f(x) = \sup_{N\geq 1} \frac{1}{N} \left| \sum_{j=0}^{N-1} f(T^j x) \right| = \infty \ .$$

Proof. We first show that at no point $x \in [0,1)$ does

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x)$$

converge to a finite value. We proceed by contradiction. Suppose for a given $x \in [0,1)$ that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x) = L < \infty \ .$$

Since α is an irrational number, by the Inhomogeneous Hurwitz Theorem (using $\beta = \frac{1}{2} - x$ and $\epsilon = \sqrt{5} - 1$), we can write

$$\left\| q\alpha + x - \frac{1}{2} \right\| < \frac{1}{q} \ ,$$

and thus

$$|f(T^q x)| = |f((q\alpha + x) \bmod 1)| > q$$

for infinitely many positive integers q.

Keeping this fact in mind, note that

$$\limsup_{q\to\infty}\left|\frac{1}{q+1}\sum_{j=0}^{q}f(T^jx)-\frac{1}{q}\sum_{j=0}^{q-1}f(T^jx)\right|=\limsup_{q\to\infty}\left|\frac{q}{q+1}\cdot\frac{1}{q}\,f(T^qx)-\frac{1}{q+1}\cdot\frac{1}{q}\sum_{j=0}^{q-1}f(T^jx)\right|\ ,$$

which, applying Lemmas 2, 3 and 4,

$$=\limsup_{q\to\infty}\left|\frac{1}{q}\,f(T^qx)-\frac{1}{q+1}\,L\right|\ ,$$

and applying Lemmas 2 and 4 again,

$$=\limsup_{q\to\infty}\frac{1}{q}\,|f(T^qx)|>1\ .$$

The contrapositive of Lemma 5 thus implies that $\lim_{N\to\infty}\sum_{j=0}^{N-1}f(T^jx)$ does not converge to a finite limit L, which contradicts our supposition and proves the first part of the theorem.

We now show that for almost every $x\in[0,1)$, we have

$$T^*f(x)=\sup_{N\geq 1}\frac{1}{N}\left|\sum_{j=0}^{N-1}f(T^jx)\right|=\infty\ .$$

To accomplish this, we utilize Kim's result with $\beta=\frac{1}{2}-x$, i.e. that

$$\liminf_{q\to\infty}q\left\|q\alpha+x-\frac{1}{2}\right\|=0$$

for almost every $x \in [0,1)$. Therefore

$$\limsup_{q\to\infty} \frac{1}{q} |f(T^q x)| = \infty$$

for almost every $x \in [0,1)$. Now let $x \in [0,1)$ be such that the limit above is infinite. We show that $T^* f(x) = \infty$, again proceeding by contradiction. Suppose that

$$\sup_{N\geq 1} \frac{1}{N} \left| \sum_{j=0}^{N-1} f(T^j x) \right| = M < \infty \ .$$

Then the triangle inequality gives

$$\left| \frac{1}{q+1} \sum_{j=0}^{q} f(T^j x) - \frac{1}{q} \sum_{j=0}^{q-1} f(T^j x) \right| \leq \frac{1}{q+1} \left| \sum_{j=0}^{q} f(T^j x) \right| + \frac{1}{q} \left| \sum_{j=0}^{q-1} f(T^j x) \right| \leq 2M$$

for all $q \geq 0$. Thus,

$$\limsup_{q\to\infty} \left| \frac{1}{q+1} \sum_{j=0}^{q} f(T^j x) - \frac{1}{q} \sum_{j=0}^{q-1} f(T^j x) \right| \leq 2M \ .$$

But this is a contradiction, since by the earlier calculation

$$\limsup_{q\to\infty} \left| \frac{1}{q+1} \sum_{j=0}^{q} f(T^j x) - \frac{1}{q} \sum_{j=0}^{q-1} f(T^j x) \right| = \limsup_{q\to\infty} \frac{1}{q} |f(T^q x)| = \infty \ . \qquad \blacksquare$$

The result of Theorem 1 raises almost as many questions as it answers. For example, we proved that for almost every $x \in [0,1)$ the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x)$$

fails to converge to a finite number. However, this gives no information as to its manner of divergence. Our intuition tells us that the ergodic averages would not converge to either $+\infty$ or $-\infty$ (at least almost everywhere), but this has yet to be proven. Additionally, Conjecture 1 remains an open problem. Thus, there are still plenty of topics of ongoing research, and it appears that developments in inhomogeneous Diophantine approximation will be key to further progress.

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