ABSTRACT

Forcing $\aleph_1$-Free Groups to Be Free

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$\aleph_1$-free groups, abelian groups whose countable subgroups are free, are objects of both algebraic and set-theoretic interest. Illustrating this, we note that $\aleph_1$-free groups, and in particular the question of when $\aleph_1$-free groups are free, were central to the resolution of the Whitehead problem as undecidable. In elucidating the relationship between $\aleph_1$-freeness and freeness, we prove the following result: an abelian group G is $\aleph_1$-free in a countable transitive model of ZFC (and thus by absoluteness, in every transitive model of ZFC) if and only if it is free in some generic model extension. We would like to answer the more specific question of when an $\aleph_1$-free group can be forced to be free while preserving the cardinality of the group. For groups of size $\aleph_1$, we establish a necessary and sufficient condition for when such forcings are possible. We also identify both existing and novel forcings which force such $\aleph_1$-free groups of size $\aleph_1$ to become free with cardinal preservation. These forcings lay the groundwork for a larger project which uses forcing to explore various algebraic properties of $\aleph_1$-free groups and develops new set-theoretical tools for working with them.

Forcing $\aleph_1$-Free Groups to Be Free

by

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To Selina Montague Foster

CHAPTER ONE

Introduction

$\aleph_1$-free groups, abelian groups whose countable subgroups are free, are objects of both algebraic and set-theoretic interest. They played a critical role in Shelah's celebrated resolution of the Whitehead problem, the question of whether there exist any non-free abelian groups A with $\mathrm{Ext}(A, \mathbb{Z}) = 0$ [11]. Such groups are known as Whitehead groups, and the Whitehead problem asks whether every Whitehead group is free. It can be shown algebraically that every Whitehead group is $\aleph_1$-free. Shelah proved that the Whitehead problem was undecidable within ZFC by demonstrating that it is undecidable whether every $\aleph_1$-free Whitehead group of size $\aleph_1$ is free.

This remarkable result represented the first time a seemingly purely algebraic problem was proved to be undecidable, and came as a great surprise to many in the mathematical community. $\aleph_1$-free groups and their properties, particularly those pertaining to the relationship between $\aleph_1$-freeness and freeness, seem to lie at the heart of this undecidability. It is from this observation that we launch the investigations that comprise this work.

The core question that we investigate in this work is when an $\aleph_1$-free group can be forced to be free. Forcing is a set-theoretical technique developed by Paul Cohen in 1963 to prove the independence of the Continuum Hypothesis [5], and which is the standard method of generating consistency and independence results. The general method of forcing involves constructing, from some suitable ground model $\mathbf{M}$ of ZFC, a model extension $\mathbf{M}[G]$ having specific prescribed properties, or in which a desired proposition holds. The central result of this dissertation provides a necessary and sufficient condition under which $\aleph_1$-free groups of size $\aleph_1$ can be forced to be free while preserving the cardinality of the group. The question of cardinal preservation is an important one, as cardinality can sometimes vary between the ground model and the extension depending upon the forcing. In the case of forcing an $\aleph_1$-free group to be free, this can be accomplished rather trivially by forcing the cardinality of the group to become countable. However, we seek a forcing which adds a basis to the $\aleph_1$-free group while making minimal changes to the ground model. We explore both established and novel forcings which make such $\aleph_1$-free groups free with cardinal preservation, thus providing a number of different ways in which to generate various model extensions which add a basis to an $\aleph_1$-free group.

The results contained here constitute the groundwork of a larger proposed program investigating various forcings related to $\aleph_1$-free groups. Due to the unique positioning of $\aleph_1$-free groups in the intersection between algebra and set theory, such an investigation could point to the development of new set-theoretical tools for constructing algebraic objects with various prescribed properties.

1.1 Two Perspectives: $\aleph_1$-Free Groups in Algebra and Set Theory

From an algebraic point of view, the class of $\aleph_1$-free groups displays a significant degree of complexity, as demonstrated by the test of ring realization. However, from a set-theoretical perspective, $\aleph_1$-free groups are rather simple objects, as evidenced by the absoluteness of $\aleph_1$-freeness, which will be proved in this work. We propose forcing as a tool through which to bridge the gap between the algebraic and set-theoretical perspectives. In particular, in identifying and developing a number of different forcings related to $\aleph_1$-free groups, we can explore the algebraic diversity of $\aleph_1$-free groups within the set-theoretical framework provided by forcing. For example, the characterization that we give of "turbid groups" provides an algebraic description of a class of $\aleph_1$-free groups which can be thought of as "universally non-free" (subject to the requirement of cardinal preservation) in the sense that they cannot be made free in any model extension without collapsing the size of the group to countable. This dissertation in particular focuses on forcings which make certain $\aleph_1$-free groups free, but future work would broaden this investigation to other properties of $\aleph_1$-free groups.

The class of $\aleph_1$-free groups exhibits a high degree of diversity. One standardized test of the algebraic complexity of a class is ring realization: Dugas and Göbel [7] and Corner and Göbel [6] show that any ring with free additive structure can be realized as the endomorphism ring of some $\aleph_1$-free group. This can be interpreted as a strong statement of algebraic complexity: almost any property that does not contradict $\aleph_1$-freeness outright will be realized by some $\aleph_1$-free group. In particular, we can construct arbitrarily large $\aleph_1$-free groups A such that $\mathrm{End(A)} \cong \mathbb{Z}$, indicating that the class of $\aleph_1$-free groups is significantly different from the class of free groups, for which every endomorphism is uniquely determined by the images of its basis elements (and thus the endomorphism ring of a rank κ free group will have size 2^κ).

Yet set-theoretically, $\aleph_1$-free groups can be thought of as somewhat simple objects. One important set-theoretic property of $\aleph_1$-freeness which demonstrates this is its absoluteness, which is proved in Chapter 2 of this work. The absoluteness of $\aleph_1$-freeness states that if an abelian group H is $\aleph_1$-free in some transitive model of ZFC, then H is $\aleph_1$-free in any transitive model of ZFC containing H. It is properties such as this absoluteness that make $\aleph_1$-free groups particularly susceptible to set-theoretic techniques.

In the wake of Shelah's result that the question of whether there exist any non-free Whitehead groups is undecidable from ZFC, set-theoretic prediction principles and axioms have become widely accepted tools for $\aleph_1$-free constructions. We propose that a thorough investigation into the family of forcings relating to $\aleph_1$-free groups could yield novel set-theoretic principles which can be used for constructing $\aleph_1$-free groups with prescribed properties.

1.2 The Relationship Between $\aleph_1$-Freeness and Freeness

Clearly, any free group is also $\aleph_1$-free, as subgroups of free groups are free. However, the converse is not true, as the direct product of countably many infinite cyclic groups, known as the Baer-Specker group $\mathbb{Z}^\omega$, provides an example of a group which is $\aleph_1$-free but not free, as proved by Baer and Specker [2, 12].

However, an $\aleph_1$-free group can be made free in a suitable model extension. The absoluteness of $\aleph_1$-freeness offers this novel, set-theoretic characterization of $\aleph_1$-freeness: that G is $\aleph_1$-free in a countable transitive model $\mathbf{M}$ of ZFC if and only if G is free in some transitive model extension of $\mathbf{M}$. This result is the main result established in Chapter 2.

We conclude Chapter 2 by presenting other applications and consequences of the absoluteness of $\aleph_1$-freeness and of the central characterization presented in that chapter. In particular, we observe that this absoluteness result and characterization offers a new method of proof, one which can be used to prove novel results, or to offer novel proofs of established theorems. To illustrate this potential, we present simple new proofs of two established results relating to particular algebraic properties of $\aleph_1$-free groups, using the absoluteness of $\aleph_1$-freeness.

1.3 Forcing an $\aleph_1$-Free Group to be Free

The novel characterization of $\aleph_1$-freeness given in Chapter 2 provides a jumping-off point from which we begin our deeper investigation into model extensions with the property that any particular $\aleph_1$-free group becomes free. Indeed, the chapters that follow concern the construction of such model extensions using various established and novel forcing techniques. As previously mentioned, such a forcing can be achieved by collapsing the cardinality of the group to be countable. However, much of the structure of the group and of the ground model is lost in such a cardinal collapse. We would like to know under what conditions such a forcing is possible without collapsing the cardinality of the group. In Chapter 3, we explore two natural candidates for such forcings defined using filters over posets of partial bases, $\mathcal{P}_1$ and $\mathcal{P}_2$, and some of their various properties. Specifically, we focus on properties related to cardinal preservation such as chain conditions and closure conditions.

1.4 Forcing $\aleph_1$-Free Groups to Be Free with Cardinal Preservation

In Chapter 4 of this work, we establish a necessary and sufficient condition, for groups of size $\aleph_1$, for when an $\aleph_1$-free group can be forced to be free. In particular, for an $\aleph_1$-free group H of size $\aleph_1$, we show that such a forcing is possible if and only if $\Gamma(H) \neq [\aleph_1]$. The Γ-invariant, denoted $\Gamma(H)$, was introduced by Eklof and Mekler in [8]. Given an $\aleph_1$-filtration of H, $\Gamma(H) := [\{\alpha < \aleph_1 : H/H_\alpha \text{ is not } \aleph_1\text{-free}\}]$, where $[\cdot]$ designates the equivalence class defined by intersection with closed unbounded subsets of $\aleph_1$. Under this definition, H is free if and only if $\Gamma(H) = [\varnothing]$.

We call such groups for which $\Gamma(H) = [\aleph_1]$ "turbid groups." In particular, $\aleph_1$-preserving forcings which make an $\aleph_1$-free group free can be achieved for non-turbid groups by forcing a club into a stationary subset of $\aleph_1$ which is defined by the Γ-invariant of the group. We show that a forcing presented by Baumgartner, Harrington, and Kleinberg [4] achieves this while preserving the cardinality of the group, while a forcing given by Abraham and Shelah [1] achieves this while preserving cardinality generally. Finally, we demonstrate that the novel poset $\mathcal{P}_2$ presented in chapter three forces non-turbid $\aleph_1$-free groups to be free while preserving the cardinality of the group. This forcing, unlike those in [4] and [1] which merely demonstrate that a basis exists in the extension, provides a clear picture of the basis added to the generic extension in terms of the partial bases in the ground model.

1.5 Other Forcings and Further Work

As pertains to the application of the forcings described here which force non-turbid $\aleph_1$-free groups to become free, the natural question arises of how to characterize a turbid group, or perhaps more pertinently, how to characterize a non-turbid group. A simple characterization or test of when a group is turbid would allow for easier application of these forcings in an algebraic context.

More broadly, future investigations into $\aleph_1$-free forcings could branch-off in many different directions. One quite natural off-shoot of the work in this dissertation would be a generalization of the results found here to the setting of iterated forcing, in which two (or more) $\aleph_1$-free groups could be simultaneously forced to be free. Note that it would then be trivial to force an isomorphism between the groups by subsequently forcing the cardinality of the two groups to be the same. (Such isomorphisms could potentially also be forced without the groups becoming free using a partial isomorphism approach similar to the partial basis approach we use in the forcing $\mathcal{P}_2$, and this method might be favored as a more minimalist approach which does not require such drastic changes to the ground model.) Such iterated forcings could also provide a setting in which results from homological algebra related to free groups could be applied to $\aleph_1$-free groups.

Another class of $\aleph_1$-free forcings which could prove fruitful to investigate is forcings which force an $\aleph_1$-free group to have an endomorphism ring with prescribed properties. Given the ring realization property described above (recall that any ring with free additive structure can be realized as the endomorphism ring of some $\aleph_1$-free group), this could provide a powerful tool for algebraists to generate a wide variety of examples and counterexamples.

As an eventual goal, $\aleph_1$-free forcings could provide the foundation for more readily producing consistency and independence results in algebra, as well as guiding the development of predictive principles or axioms which can be used in the construction of algebraic objects with various prescribed properties.

CHAPTER TWO

Absoluteness and $\aleph_1$-Free Groups

2.1 Introduction to Absoluteness

We begin in this section by introducing and collecting some general definitions and results on absoluteness. We refer the reader to [10, Chapter IV.2–5] for further proofs and details.

We first define relativization, which allows us to explore the notion of truth in a given model $\mathbf{M}$.

2.1.1 Definition. *Let* $\mathbf{M}$ *be any class. Then for any formula* ϕ*, we define* $\phi^{\mathbf{M}}$*, the* relativization *of* ϕ *to* $\mathbf{M}$*, inductively as follows:*

1. $(x = y)^{\mathbf{M}}$ *is* $x = y$.

2. $(x \in y)^{\mathbf{M}}$ *is* $x \in y$.

3. $(\phi \wedge \psi)^{\mathbf{M}}$ *is* $\phi^{\mathbf{M}} \wedge \psi^{\mathbf{M}}$.

4. $(\neg\phi)^{\mathbf{M}}$ *is* $\neg(\phi^{\mathbf{M}})$.

5. $(\exists x\ \phi)^{\mathbf{M}}$ *is* $\exists x\ (x \in \mathbf{M} \wedge \phi^{\mathbf{M}})$.

2.1.2 Definition. *Let* $\mathbf{M}$ *be any class. For a sentence* ϕ*, "*ϕ is true in $\mathbf{M}$*" means that* $\phi^{\mathbf{M}}$ *is true. For a set of sentences* S*, "*S is true in $\mathbf{M}$*" means that each sentence in* S *is true in* $\mathbf{M}$.

We are now ready to give a definition for absoluteness.

2.1.3 Definition. *Let ϕ be a formula with free variables $x_1, \ldots, x_n$. If $\mathbf{M} \subseteq \mathbf{N}$, ϕ is* absolute for $\mathbf{M}, \mathbf{N}$ *if and only if*

$$\forall x_1, \ldots, x_n \in \mathbf{M}\, (\phi^{\mathbf{M}}(x_1, \ldots, x_n) \longleftrightarrow \phi^{\mathbf{N}}(x_1, \ldots, x_n)).$$

We say that ϕ is absolute for $\mathbf{M}$ *if and only if ϕ is absolute for $\mathbf{M}, \mathbf{V}$. That is,*

$$\forall x_1, \ldots, x_n \in \mathbf{M}\, (\phi^{\mathbf{M}}(x_1, \ldots, x_n) \longleftrightarrow \phi(x_1, \ldots, x_n)).$$

Intuitively, the absoluteness of ϕ for $\mathbf{M}, \mathbf{N}$ means that $\phi(x_1, \ldots, x_n)$ is true in $\mathbf{M}$ if and only if it is true in $\mathbf{N}$. Note that if ϕ is absolute for both $\mathbf{M}$ and $\mathbf{N}$, and $\mathbf{M} \subseteq \mathbf{N}$, then ϕ is absolute for $\mathbf{M}, \mathbf{N}$. The following lemma states that absoluteness is preserved under logical equivalence.

2.1.4 Lemma. *Suppose $\mathbf{M} \subseteq \mathbf{N}$, and both $\mathbf{M}$ and $\mathbf{N}$ are models for a set of sentences S such that*

$$S \vdash \forall x_1, \ldots, x_n\, (\phi(x_1, \ldots, x_n) \longleftrightarrow \psi(x_1, \ldots, x_n)).$$

Then ϕ is absolute for $\mathbf{M}, \mathbf{N}$ if and only if ψ is absolute for $\mathbf{M}, \mathbf{N}$.

The following definition introduces a family of formulas, the Δ_0 formulas, which is foundational to our results on absoluteness.

2.1.5 Definition. *A formula is Δ_0 if it is built inductively according to the following:*

1. *$x \in y$ and $x = y$ are Δ_0.*

2. *If ϕ, ψ are Δ_0, then $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$ and $\phi \leftrightarrow \psi$ are Δ_0.*

3. *If ϕ is Δ_0, then $\exists x\, (x \in y \wedge \phi)$ is Δ_0.*

We use $\exists x \in y\ \phi$ as abbreviation for $\exists x\,(x \in y \wedge \phi)$ and $\forall x \in y\ \phi$ as abbreviation for $\forall x\,((x \in y) \rightarrow \phi)$, and we call $\exists x \in y$ and $\forall x \in y$ *bounded quantifiers.* According to Definition 2.1.5, a formula in which all quantifiers are bounded of type $\exists x \in y$ is automatically Δ_0. The next lemma connects Δ_0 formulas to absoluteness.

2.1.6 Lemma. *If* $\mathbf{M}$ *is transitive and* ϕ *is* Δ_0*, then* ϕ *is absolute for* $\mathbf{M}$*.*

Note that $(\forall x \in y\ \phi) \longleftrightarrow \neg(\exists x \in y\ (\neg\phi))$. Thus, the above lemmas tell us that formulas in which all quantifiers are bounded are logically equivalent to Δ_0 formulas and hence absolute.

In addition to the previous exposition, we need also to account for the absoluteness of defined notions which take the form of functions. This gives rise to the following definition.

2.1.7 Definition. *If* $\mathbf{M} \subseteq \mathbf{N}$*, and* $F(x_1, \ldots, x_n)$ *is a well-defined function both for* $\mathbf{M}$ *and* $\mathbf{N}$*, we say* F *is absolute for* $\mathbf{M}, \mathbf{N}$ *if the formula* $F(x_1, \ldots, x_n) = y$ *is absolute for* $\mathbf{M}, \mathbf{N}$*.*

More formally, suppose that $F(x_1, \ldots, x_n)$ *was defined as the unique* y *such that* $\phi(x_1, \ldots, x_n, y)$*. Then* $F(x_1, \ldots, x_n)$ *is a well-defined function for* $\mathbf{M}, \mathbf{N}$ *only if*

$$\forall x_1, \ldots, x_n\, \exists! y\ \phi(x_1, \ldots, x_n, y)$$

is true in both $\mathbf{M}$ *and* $\mathbf{N}$*. Assuming this,* F *is absolute for* $\mathbf{M}, \mathbf{N}$ *if and only if* ϕ *is absolute.*

This definition allows us to make full sense of the following lemma, which states that absolute notions are closed under composition.

2.1.8 Lemma. *Let* $\mathbf{M} \subseteq \mathbf{N}$*, and suppose that formula* $\phi(x_1, \ldots, x_n)$ *and functions* $F(x_1, \ldots, x_n)$*,* $G_i(y_1, \ldots, y_m)$ $(i = 1, \ldots, n)$ *are absolute for* $\mathbf{M}, \mathbf{N}$*. Then so are the formula*

$$\phi(G_1(y_1, \ldots, y_m), \ldots, G_n(y_1, \ldots, y_m))$$

and the function

$$F(G_1(y_1, \ldots, y_m), \ldots, G_n(y_1, \ldots, y_m)).$$

Using the definitions and results above, we can establish the absoluteness of a number of defined notions and formulas from set theory which produce useful results on the absoluteness of properties of finite sets.

2.1.9 Lemma. *The following are absolute for any transitive model* $\mathbf{M}$ *of* ZFC*:*

1. *ordered pairs* (x, y)*,*
2. *set union* $\bigcup x$*,*
3. *set inclusion* $x \subseteq y$*,*
4. x *is an ordinal,*
5. $\alpha + \beta$*,* $\alpha \cdot \beta$ *for ordinals* α, β*,*
6. ω *and* $(\mathbb{Z}, +, \cdot)$*.*

Proof. Note that $\mathbb{Z}$ is formally defined as a set of ordered pairs from $\omega \times \omega$.

We need to check the absoluteness of some formal definition of $\mathbb{Z}$. For definiteness let us take

$$\mathbb{Z} = ((\omega \times \{0\}) \cup (\omega \times \{1\})) \backslash \langle 0, 0 \rangle,$$

where $\langle n, 1 \rangle$ represents the integer n and $\langle n, 0 \rangle$ represents $-n$. The operations $+$ and $\cdot$ on $\mathbb{Z}$ are defined appropriately and are primarily determined by the ordinal arithmetic of ω. $\square$

The absoluteness of finite sets will be of particular importance.

2.1.10 Lemma. *If* $\mathbf{M}$ *is a transitive model of* ZFC*, then every finite subset of* $\mathbf{M}$ *is in* $\mathbf{M}$*, and "x is finite" is absolute for* $\mathbf{M}$*.*

Our last lemma in this section addresses the absoluteness of formulas which are built recursively over ω from absolute formulas. While the result can be generalized to include transfinite recursion over well-founded and set-like relations on an arbitrary class $\mathbf{A}$, cf. [10, Chapter IV, Theorem 5.6], for simplicity we present only the version involving standard induction over ω. We will use this to establish the absoluteness results of the next section.

2.1.11 Lemma. *Suppose* $F : \mathbf{V} \to \mathbf{V}$*, and let* $G : \omega \to \mathbf{V}$ *be defined so that*

$$\forall n \in \omega\, [G(n) = F(G \upharpoonright n)],$$

where $G \upharpoonright n$ *denotes the restriction of* G *to the domain* $n = \{0, 1, \ldots, n-1\}$*.*

Let $\mathbf{M}$ *be a transitive model of* ZFC *and assume that* F *is absolute for* $\mathbf{M}$*. Then also* G *is absolute for* $\mathbf{M}$*.*

2.2 The Absoluteness of $\aleph_1$-Freeness

We will now apply the absoluteness results in Section 2.1 to some algebraic notions. After relating in Theorem 2.2.10 the $\aleph_1$-freeness of a group to the freeness of the pure subgroups generated by its finite subsets, we will be ready to establish the main result of this section, Theorem 2.2.11, namely the absoluteness of $\aleph_1$-freeness.

2.2.1 Basic Absoluteness Results for Abelian Groups

In this section, we collect some first basic absoluteness results on abelian groups. We will follow the algebraic convention of using G as a shorthand to denote the abelian group $(G, +)$.

2.2.1 Lemma. *Suppose* $\mathbf{M}$ *is a transitive model of* ZFC*. Then "G is an abelian group" is absolute for* $\mathbf{M}$*.*

Proof. Suppose $(G, +) \in \mathbf{M}$. Note the logical equivalence

$$\text{“}G \text{ is an abelian group”} \leftrightarrow \phi_1 \wedge \phi_2 \wedge \phi_3,$$

where ϕ_1, ϕ_2 and ϕ_3 denote the sentences

1. $\forall x \in G \; \forall y \in G \; \forall z \in G \; (x + (y + z) = (x + y) + z)$,

2. $\exists u \in G \; ((\forall x \in G \; (x + u = x)) \wedge (\forall x \in G \; \exists y \in G \; (x + y = u)))$,

3. $\forall x \in G \; \forall y \in G \; (x + y = y + x)$.

Note that each of ϕ_1, ϕ_2 and ϕ_3 involves only bounded quantifiers and logical symbols, so each property is equivalent to a Δ_0 statement, and thus is absolute for $\mathbf{M}$. So as each of ϕ_1, ϕ_2 and ϕ_3 is absolute, "G is an abelian group" is absolute for $\mathbf{M}$. □

2.2.2 Lemma. *Suppose* $\mathbf{M}$ *is a transitive model of* ZFC *and* $(G, +) \in \mathbf{M}$ *is abelian. The defined notions "0_G" and "nx"* $(n \in \mathbb{Z}, x \in G)$ *are absolute for* $\mathbf{M}$*.*

Proof. To see that "0_G" is an absolute defined notion, note that 0_G is uniquely defined by

$$z = 0_G \iff (z \in G \wedge \forall x \in G \; (x + z = x)),$$

where the right-hand side of the above equivalence is a Δ_0 formula. As the right-hand side of the above equivalence is easily seen to be equivalent to a Δ_0 statement, "0_G" is an absolute defined notion.

For the absoluteness of "nx", note that nx for $n \geqslant 0$ is formally defined recursively on $n \in \omega$ by $0x = 0_G$ and $nx = (n-1)x + x$ for all $n > 0$ (cf. Lemma 2.1.10). Absoluteness easily extends to $n < 0$ as nx is the additive inverse of $(-n)x$. $\square$

2.2.3 Lemma. *Suppose* $\mathbf{M}$ *is a transitive model of* ZFC *and* $(G, +) \in \mathbf{M}$ *is abelian. Then "G is torsion-free" is absolute for* $\mathbf{M}$.

Proof. Note that

$$\text{"}G \text{ is torsion-free"} \leftrightarrow \forall x \in G\ \forall n \in \omega\ (nx = 0_G \rightarrow (x = 0_G \vee n = 0)),$$

where the right-hand side above is obtained by substituting absolute notions "0_G" and "nx" into a Δ_0 sentence. $\square$

2.2.2 Establishing the Absoluteness of $\aleph_1$-Freeness

We are nearly ready to establish our main result concerning the absoluteness of $\aleph_1$-freeness. We will review finite rank pure subgroups and Pontryagin's Criterion, the main ingredients of the proof of Theorem 2.2.10. The interested reader may refer to [9] for more detail.

2.2.4 Definition. *A subgroup H of an abelian group G is said to be a* pure *subgroup if for any $x \in H$, $0 \neq n \in \mathbb{Z}$, $n \mid x$ in G implies $n \mid x$ in H. In particular, a subgroup H of a torsion-free group G is pure if and only if $x = ny$ implies $y \in H$ for all $x \in H, y \in G$ and $0 \neq n \in \mathbb{Z}$.*

The intersection of pure subgroups of a torsion-free group is again pure. Therefore if G is a torsion-free abelian group, and S is a subset of G, the intersection of all pure subgroups containing S is the minimal pure subgroup containing S, *which we denote by $\langle S\rangle_*$.*

Explicitly, we may write

$$\langle S\rangle_* = \{y \in G \mid \exists n, n_1, \ldots, n_m \in \mathbb{Z}, n \neq 0 \, \exists s_1, \ldots, s_m \in S : ny = n_1 s_1 + \ldots + n_m s_m\}.$$

2.2.5 Lemma. *Suppose $\mathbf{M}$ is a transitive model of* ZFC. *Suppose $(G,+) \in \mathbf{M}$ is abelian and S is a finite subset of G. Then "$\langle S\rangle_*$" is an absolute notion for $\mathbf{M}$.*

Proof. Let $S = \{s_1, \ldots, s_m\}$. Recall that

$$\langle S\rangle_* = \{y \in G \mid \exists n, n_1, \ldots, n_m \in \mathbb{Z}, n \neq 0 : \; ny = n_1 s_1 + \ldots + n_m s_m\}.$$

Then we have the logical equivalence

$$z = \langle S\rangle_* \iff$$

$$\Big[\Big[\forall y \in z \, \big[y \in G \wedge \exists n, n_1, \ldots, n_m \in \mathbb{Z} \, (\neg(n=0) \wedge ny = n_1 s_1 + \ldots + n_m s_m)\big]\Big]$$

$$\wedge \Big[\forall y \in G \, \big[(\exists n, n_1, \ldots, n_m \in \mathbb{Z} \, (\neg(n=0) \wedge ny = n_1 s_1 + \ldots + n_m s_m)) \rightarrow y \in z\big]\Big]\Big],$$

where the statement on the right-hand side above involves only bounded quantifiers, logical symbols, "$\mathbb{Z}$", and various multiples of elements. So it can be seen that the right-hand side is obtained by substituting the absolute notions "$\mathbb{Z}$", "ny", "$n_1 s_1$", $\ldots$, "$n_m s_m$" into a sentence which is logically equivalent to a Δ_0 statement. Thus by the closure under composition of absolute notions, "$\langle S\rangle_*$" is absolute. □

2.2.6 Remark. *The proof of Lemma 2.2.5 can easily be modified to drop the finiteness condition. In particular, "$\langle S\rangle_*$" is an absolute notion for any subset S of G in $\mathbf{M}$. Note also that for any finite set $S \in \mathbf{V}$ with $S \subseteq G$, we have $\langle S\rangle_* \in \mathbf{M}$.*

In order to establish our result on the absoluteness of $\aleph_1$-freeness, we need a simple estimate for the torsion-free rank of $\langle S \rangle_*$. Recall that the *torsion-free rank* $\mathrm{rk}_0(G)$ of a torsion-free abelian group G is defined as the size of a maximal linearly independent subset $S \subseteq G$. We must first show that if S is a finite subset of an abelian group G, then $\mathrm{rk}_0(\langle S \rangle_*) \leqslant |S|$.

To this end, we prove the following lemma.

2.2.7 Lemma. *If S is a finite subset of a torsion-free abelian group, then*

$$\mathrm{rk}_0(\langle S \rangle_*) \leqslant |S|.$$

If S' is a maximal linearly independent subset of a finite subset S of a torsion-free abelian group, then

$$\langle S' \rangle_* = \langle S \rangle_*$$

Proof. Let $S = \{s_1, \ldots, s_m\}$ and let S' be a maximal linearly independent subset of the finite set S.

Suppose $S = \{s_1, \ldots, s_m\}$ and $S' = \{s_1, \ldots, s_k\}$ with $k \leqslant m$. Clearly, $\langle S' \rangle_* \subseteq \langle S \rangle_*$. Let $t \in \langle S \rangle_*$. Then there exist $N, n_1, \ldots, n_m \in \mathbb{Z}$, $N \neq 0$ such that $Nt = \sum_{i=1}^{m} n_i s_i$. So $Nt \in \langle S \rangle$.

For any $i > k$, $S' \cup \{s_i\}$ is linearly dependent, so there exists $0 \neq N_i \in \mathbb{Z}$ such that $N_i s_i \in \langle S' \rangle$.

Note $NN_{k+1}N_{k+2} \ldots N_m t \in \langle (N_{k+1} \ldots N_m s_1), \ldots, (N_{k+1} \ldots N_m s_m) \rangle$. We claim that

$$\langle (N_{k+1} \ldots N_m s_1), \ldots, (N_{k+1} \ldots N_m s_m) \rangle \subseteq \langle s_1, \ldots, s_k \rangle = \langle S' \rangle,$$

and thus, that $t \in \langle S' \rangle_*$. To see this, note that

$$N_{k+1}N_{k+2}\dots N_m s_{k+1} = N_{k+2}\dots N_m(N_{k+1}s_{k+1}) \in \langle S' \rangle.$$

Similarly, for any $k < i \leqslant m$,

$$N_{k+1}N_{k+2}\dots N_m s_i = N_{k+1}\dots N_{i-1}N_{i+1}\dots N_m(N_i s_i) \in \langle S' \rangle.$$

Finally, for any $i \leqslant k$, it is clear that $N_{k+1}\dots N_m s_i \in \langle S' \rangle$. This establishes the claim, and completes the proof.

Choose S' to be a maximal linearly independent subset of S. We wish to show $\mathrm{rk}_0(\langle S \rangle_*) = |S'|$. By way of contradiction, suppose there exists some $t \in \langle S \rangle_*$ such that $S' \cup \{t\}$ is linearly independent. Then $t \in \langle S \rangle_* = \langle S' \rangle_*$. So there exists some $0 \neq n \in \mathbb{Z}$ such that $nt \in \langle S' \rangle$, and t is linearly dependent on S', contradicting our assumption. Thus, S' is a maximal linearly independent subset of $\langle S \rangle_*$, and so $\mathrm{rk}_0(\langle S \rangle_*) = |S'| \leqslant |S|$. $\square$

2.2.8 Remark. *An alternative proof of Lemma 2.2.7 uses the divisible hull* $\mathbb{Q} \otimes \langle S \rangle$ *and* $\mathrm{rk}_0(\langle S \rangle_*) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes \langle S \rangle) = \mathrm{rk}_0(\langle S \rangle)$. *We have chosen a more elementary approach to emphasize the underlying aspects of set theory.*

More generally, using the axiom of choice (AC), $\mathrm{rk}_0(\langle S \rangle_*) \leqslant |S|$ *holds for any subset of a torsion-free group.*

We recall Pontryagin's Criterion next. For a proof, see [9, Chapter 3, Theorem 7.1].

2.2.9 Theorem (Pontryagin's Criterion). *A countable torsion-free abelian group is free if and only if each of its finite rank subgroups is free.*

The theorem below gives a number of equivalent characterizations of $\aleph_1$-freeness. We will use the last of these alternative characterizations to prove the absoluteness of $\aleph_1$-freeness. Note that $\mathrm{rk}(G)$ denotes the rank of G, and $\mathrm{rk}_0(G)$ denotes its torsion-free rank.

2.2.10 Theorem. *Let* $\mathbf{M}$ *be a transitive model of* ZFC *and* G *be an abelian group in* $\mathbf{M}$. *The following statements are equivalent:*

(*i*) *G is $\aleph_1$-free, that is, for all subgroups H of G, if $|H| \leqslant \aleph_0$, then H is free.*

(*ii*) *For all subgroups H of G, if* $\mathrm{rk}(H)$ *is finite, then H is free.*

(*iii*) *G is torsion-free and for all pure subgroups H of G, if* $\mathrm{rk}(H)$ *is finite, then H is free.*

(*iv*) *G is torsion-free and for all finite subsets S of G, $\langle S\rangle_*$ is free.*

Proof. For $(i) \to (ii)$, let H be a subgroup of G with $\mathrm{rk}(H)$ finite. By (i), all cyclic subgroups of G are free. Hence G is torsion-free, and $H \subseteq G$ is torsion-free, too. This implies $\mathrm{rk}(H) = \mathrm{rk}_0(H)$ and $|H| \leqslant \aleph_0 \cdot \mathrm{rk}(H)$. Hence $|H| \leqslant \aleph_0$, and H is free.

The direction $(ii) \to (iii) \to (iv)$ is easy. For $(ii) \to (iii)$, note that with (ii) every cyclic subgroup of G is free, hence G is torsion-free. For $(iii) \to (iv)$, note that if S is finite, then $\langle S\rangle_*$ is of finite rank by Lemma 2.2.7.

To see that $(iv) \to (i)$, let H be a subgroup of G with $|H| \leqslant \aleph_0$. If $H = 0$, H is free, so suppose that H is non-trivial. Then as G is torsion-free, $|H| = \aleph_0$. We wish to use Pontryagin's Criterion to show that H is free, so let K be a finite rank subgroup of H. Choose S to be a maximal linearly independent subset of K. Then as S is finite, $\langle S \rangle_*$ is free, and $K \subseteq \langle S \rangle_*$ is free, too. So by Pontryagin's Criterion, H is free. $\square$

Finally, we are ready to establish the absoluteness of $\aleph_1$-freeness below.

2.2.11 Theorem. *Suppose* $\mathbf{M}$ *is a transitive model of* ZFC*, and* G *is an abelian group in* $\mathbf{M}$*. Then "*G *is* $\aleph_1$*-free" is absolute.*

Proof. Let ϕ denote the statement

$$G \text{ is torsion-free } \wedge \ \forall S((S \subseteq G \wedge S \text{ is finite}) \rightarrow \langle S \rangle_* \text{ is free}).$$

By Theorem 2.2.10, "G is $\aleph_1$-free" is equivalent to ϕ. To establish the absoluteness of $\aleph_1$-freeness, we will show that $\forall G \in \mathbf{M}\ (\phi^{\mathbf{M}} \longleftrightarrow \phi)$.

We need to determine $\phi^{\mathbf{M}}$ first. Recall that torsion-freeness, set inclusion, finiteness and "$\langle S \rangle_*$" (for finite subsets S of a torsion-free abelian group) are absolute. Unfortunately, however, freeness is not absolute. Thus $\phi^{\mathbf{M}}$ is the statement

$$G \text{ is torsion-free } \wedge \ \forall S \in \mathbf{M}\ ((S \subseteq G \wedge S \text{ is finite}) \rightarrow \langle S \rangle_* \text{ is free}^{\mathbf{M}}).$$

For $\phi^{\mathbf{M}} \rightarrow \phi$, let $S \in \mathbf{V}$ such that $S \subseteq G$ and S is finite, and assume $\phi^{\mathbf{M}}$. Note that by our previous result on the absoluteness of finite sets, as $G \in \mathbf{M}$, if S is a finite subset of G then $S \in \mathbf{M}$. Thus, $\langle S \rangle_*$ has a basis in $\mathbf{M}$, which is automatically a basis of $\langle S \rangle_*$ in $\mathbf{V}$.

For $\phi \rightarrow \phi^{\mathbf{M}}$, let $S \in \mathbf{M}$ such that $S \subseteq G$ and S is finite, and assume ϕ. Then $\langle S \rangle_*$ is free in $\mathbf{V}$. Choose a basis $B \in \mathbf{V}$ of $\langle S \rangle_*$. Then, still in $\mathbf{V}$, $\mathrm{rk}_0(\langle S \rangle_*) = |B|$ and $\mathrm{rk}_0(\langle S \rangle_*) \leqslant |S|$. So $|B| \leqslant |S|$, and thus $B \subseteq \langle S \rangle_* \subseteq G$ is a finite set. Suppose $B = \{x_1, ..., x_n\}$. As each $x_i \in B$ is in H, by the transitivity of $\mathbf{M}$, each $x_i \in B$ is in

$\mathbf{M}$. Then using union and finite recursion, we have that $B \in \mathbf{M}$, and B will witness that $\langle S \rangle_*$ is free in $\mathbf{M}$.

Thus, $\aleph_1$-freeness is absolute. $\square$

2.3 Proofs with Model Extensions

In this section, we discuss some major applications and consequences of the absoluteness of $\aleph_1$-freeness. We start with a general observation concerning the relationship between $\aleph_1$-freeness and freeness in different models of set theory. We then demonstrate how this observation can be turned into a quick and elegant routine for generating and simplifying proofs concerning $\aleph_1$-free groups.

We will repeatedly reference the use of forcing to collapse the cardinality of a given $\aleph_1$-free group G to countable. Such a forcing may be defined by the partial order consisting of all injective functions from finite subsets of ω into G which results in a model extension in which G is countable. Thus, by the absoluteness of $\aleph_1$-freeness, G is free in this model extension, as it is a countable subgroup of itself. For more detail on this type of forcing, we refer the reader to Remark 3.1.18. However, the technical details of forcing are not necessary in order to understand the applications below. Rather, we simply reference forcing as a method for producing a model extension with some required properties, namely with the property that G is free in this model extension.

The following result is another take on Theorem 2.2.11 and highlights how $\aleph_1$-freeness as an initially algebraic property can be interpreted and understood in the context of model extensions in set theory.

2.3.1 Theorem. *Let* $\mathbf{M}$ *be a transitive model of* ZFC*, and* G *an abelian group in* $\mathbf{M}$*. Then the following are equivalent:*

(i) *G is $\aleph_1$-free in* $\mathbf{M}$.

(ii) *G is $\aleph_1$-free in* $\mathbf{V}$.

(iii) *G is $\aleph_1$-free in any transitive model* $\mathbf{N}$ *with* $G \in \mathbf{N}$.

In addition, if $\mathbf{M}$ *is a countable transitive model of* ZFC*, we may add to the above list of equivalent statements:*

(iv) *G is free in some generic extension* $\mathbf{N}$ *of* $\mathbf{M}$.

Proof. The equivalence of (i), (ii) and (iii) is an immediate consequence of the absoluteness of $\aleph_1$-freeness, Theorem 2.2.11.

We have $(iv) \rightarrow (i)$ as G is free in $\mathbf{N}$ implies G is $\aleph_1$-free in $\mathbf{N}$, and thus by the absoluteness of $\aleph_1$-freeness, G is $\aleph_1$-free in $\mathbf{M}$. Finally, $(i) \rightarrow (iv)$ can be seen by letting $\mathbf{N}$ be the generic extension obtained by collapsing the cardinality of G to countable. $\square$

Theorem 2.3.1 provides a new approach to proving statements about $\aleph_1$-free groups. To illustrate the utility of such an approach, we give a remarkably simple proof of the well-known transitivity of $\aleph_1$-free groups. We will be using countable transitive models of ZFC as is common convention for forcing arguments. It should be understood that countable transitive models only exist for finite lists of axioms, and we will provide in Remark 2.3.4 an explanation of how these proofs with countable transitive models translate into a formal proof within the metatheory.

2.3.2 Theorem. *If H and G/H are $\aleph_1$-free for abelian groups $H \subseteq G$, then G is $\aleph_1$-free.*

Proof. We will assume a countable transitive model $\mathbf{M}$ with $H, G \in \mathbf{M}$. Let $\mathbf{N}$ be the generic extension of $\mathbf{M}$ produced by collapsing the cardinality of G to countable. Then G/H is countable and $\aleph_1$-free in $\mathbf{N}$, thus it is free in $\mathbf{N}$. In particular, we have $G^{\mathbf{N}} \cong H \oplus G/H$, and as H is also countable and free in $\mathbf{N}$, G is free in $\mathbf{N}$. With Theorem 2.3.1, $(iv) \rightarrow (i)$, G is $\aleph_1$-free in $\mathbf{M}$. □

2.3.3 Theorem. *Let G be $\aleph_1$-free and let $H \subseteq G$ be a finite rank pure subgroup of G. Then G/H is $\aleph_1$-free.*

Proof. Without loss of generality, assume $G \in \mathbf{M}$ for some countable transitive model $\mathbf{M}$ of ZFC. Suppose that $\mathbf{N}$ is some generic model extension of $\mathbf{M}$ in which G is countable. Then G is free in $\mathbf{N}$. So in $\mathbf{N}$, as G is free, we can choose a basis B of G.

Let H be a finite rank pure subgroup of G (recall that this is an absolute property). Then by the Pontryagin Criterion, H is free of finite rank, so we can choose $B' \subseteq B$ finite with $H \subseteq \langle B' \rangle$.

Now H is a pure subgroup of G, and thus, H is pure in $\langle B' \rangle$. So $\langle B' \rangle / H$ is torsion-free, and as it is also finitely generated, $\langle B' \rangle / H$ is free by the Fundamental Theorem of Abelian Groups.

So H is a direct summand of $\langle B' \rangle$, which is a direct summand of $\langle B \rangle = G$. Thus G/H is free in $\mathbf{N}$, and by the absoluteness of $\aleph_1$-freeness, G/H is $\aleph_1$-free in $\mathbf{M}$. □

2.3.4 Remark. *We will discuss a more formal argument for why we can restrict ourselves to countable transitive models* $\mathbf{M}$ *with* $H, G \in \mathbf{M}$ *in our proof of Theorem 2.3.2 and Theorem 2.3.3.*

Let ψ *denote the first-order logical sentence which expresses the statement of Theorem 2.3.2* (*or Theorem 2.3.3*) *and note that the proofs of Theorem 2.3.2 and Theorem 2.3.3 can be formalized using a finite list of axioms* $\varphi_1, \ldots, \varphi_n$ *of* ZFC. *If* ψ *were not provable on the basis of* ZFC, *then Gödel's Completeness Theorem implies the existence of a model for* ZFC + $\neg\psi$. *In particular, the finite list of axioms* $\varphi_1, \ldots, \varphi_n, \neg\psi$ *is consistent and a standard procedure using the Reflection Theorem, Löwenheim-Skolem Theorem, and Mostowski Collapse Lemma produces a countable transitive model* $\mathbf{M}$ *for* $\varphi_1, \ldots, \varphi_n, \neg\psi$. *In particular, in* $\mathbf{M}$ *we can find abelian groups* G, H *for which* ψ *fails and going from* $\mathbf{M}$ *to a generic model extension* $\mathbf{N}$ *where* $G^{\mathbf{N}}$ *is countable we can reproduce the proofs of Theorem 2.3.2 and Theorem 2.3.3 for a contradiction.*

CHAPTER THREE

Forcing an $\aleph_1$-Free Group to Become Free

Forcing is a general technique used to produce models of set theory satisfying a variety of different properties. It is an indispensable tool for producing relative consistency and independence results.

The key idea behind forcing is to begin with a countable transitive model $\mathbf{M}$ of ZFC called the ground model, and to construct from it another countable transitive model of ZFC called $\mathbf{M}[G]$ which extends $\mathbf{M}$. To build $\mathbf{M}[G]$, we begin by choosing a partially ordered set in $\mathbf{M}$, the properties of which will determine what propositions hold in $\mathbf{M}[G]$ beyond ZFC.

One technical issue is that there cannot be a proof from ZFC that there exists a countable transitive model of ZFC, for the existence of such a proof would be a violation of Gödel's Second Incompleteness Theorem.

We may resolve this by noting that while we may not be able to build set models of ZFC from ZFC, we can build a countable transitive model of any finite fragment of ZFC from ZFC. So we only need $\mathbf{M}$ to satisfy enough of ZFC to carry out the given argument. For further discussion of this approach, see [10, Chapter IV.1].

3.1 Forcing Basics

We will now discuss briefly how the extensions $\mathbf{M}[G]$, called *generic extensions*, are built.

3.1.1 Posets, Filters, and Generic Extensions

3.1.1 Definition. *A* partially ordered set, *or a "poset", is a pair $\langle P, \leqslant \rangle$ such that $P \neq \varnothing$ and $\leqslant$ is a relation on P such that the following properties hold:*

1. $\forall p \in P \; (p \leqslant p)$

2. $\forall p, q \in P \; ((p \leqslant q \wedge q \leqslant p) \rightarrow p = q)$

3. $\forall p, q, r \in P \; ((p \leqslant q \wedge q \leqslant r) \rightarrow p \leqslant r)$

In a slight abuse of notation, we may speak of a poset P to indicate a pair $\langle P, \leqslant \rangle$.

For $p, q \in P$, if there exists an $r \in P$ such that $r \leqslant p$ and $r \leqslant q$, we call such an r a *common extension* of p and q in P. If there does not exist a common extension of p and q in P, we say that p and q are *incompatible*, and write $p \perp q$.

For forcing purposes, we will restrict our attention to partial orders with a *maximal element*, that is, an element 1 such that $\forall p \in P \; (p \leqslant 1)$. When we refer to a partial order in this work, we will be referring specifically to one with a maximal element, and refer to this element as "1".

3.1.2 Definition. *Let $\langle P, \leqslant \rangle$ be a partial order. $G \subseteq P$ is a* filter *in P if and only if the following conditions hold:*

1. $\forall p, q \in G \; \exists r \in G \; (r \leqslant p \wedge r \leqslant q)$

2. $\forall p \in G \; \forall q \in P \; (p \leqslant q \rightarrow q \in G)$

3.1.3 Definition. *Let $\langle P, \leqslant \rangle$ be a partial order. $D \subseteq P$ is* dense *in P if and only if $\forall p \in P \; \exists q \leqslant p \; (q \in D)$.*

3.1.4 Definition. *Let* $\langle P, \leqslant \rangle$ *be a partial order.* $G \in \mathbf{V}$ *is* P-generic *over* $\mathbf{M}$ *if and only if* G *is a filter in* P *and for all dense* $D \subseteq P$ *with* $D \in \mathbf{M}$, $G \cap D \neq \varnothing$.

The following lemmas give properties of P-generic filters which will help us construct our desired model extensions and establish the relevant forcing notions.

3.1.5 Lemma. *If* $\mathbf{M}$ *is countable and* $p \in P$*, there is a* P*-generic* $G \in \mathbf{V}$ *over* $\mathbf{M}$ *with* $p \in G$.

3.1.6 Lemma. *If* $\mathbf{M}$ *is a transitive model of* $ZF - P$*, and* $\langle P, \leqslant, 1 \rangle \in \mathbf{M}$ *is such that*

$$\forall p \in P \; \exists q, r \in P \; (q \leqslant p \wedge r \leqslant p \wedge q \perp r)$$

and $G \in \mathbf{V}$ *is* P*-generic over* $\mathbf{M}$*, then* $G \notin \mathbf{M}$.

We are now ready to introduce generic model extensions. We will set aside many details here, but the interested reader is referred to [10, Chapter IV.2] for further details.

Beginning with a countable transitive model $\mathbf{M}$ of ZFC, a partial order $P \in \mathbf{M}$, and a P-generic filter $G \in \mathbf{V}$, we define the *generic extension* $\mathbf{M}[G]$. $\mathbf{M}[G]$ is the smallest model of ZFC such that $\mathbf{M} \subseteq \mathbf{M}[G]$ and $G \in \mathbf{M}[G]$, and can be thought of as the set of all sets which can be built from G using processes definable in $\mathbf{M}$.

Formally, this is done by defining, through transfinite recursion, names for every element of $\mathbf{M}[G]$ which describe how the element is constructed. These are called *P-names*, and they do not make any explicit reference to a particular G. Thus the P-names can be understood from the perspective of $\mathbf{M}$. However, the objects in $\mathbf{M}[G]$ to which the P-names refer cannot in general be identified from the perspective of

$\mathbf{M}$, as G does not exist in $\mathbf{M}$. If τ is a P-name, we will use τ_G to refer to the object in $\mathbf{M}[G]$ which is named by τ.

3.1.7 Definition. *τ is a P-name if and only if τ is a relation and*

$$\forall\langle\sigma, p\rangle \in \tau \; (\sigma \textit{ is a } P\textit{-name} \wedge p \in P).$$

This is a definition by transfinite recursion on the (set-theoretic) rank of τ. In particular, $\tau = \varnothing$ is trivially a P-name.

3.1.8 Definition. *$\mathbf{V}^P$ denotes the class of P-names in $\mathbf{V}$. If $\mathbf{M}$ is a transitive model of* ZFC *with $P \in \mathbf{M}$, then $\mathbf{M}^P = \mathbf{V}^P \cap \mathbf{M}$ is the class of P-names in $\mathbf{M}$.*

3.1.9 Definition. *For τ a P-name and $G \in \mathbf{V}$ with $G \subseteq P$, let*

$$\tau_G = \{\sigma_G : \exists p \in G \, (\langle\sigma, p\rangle \in \tau)\}$$

denote the valuation *of τ with respect to G. This is again a definition by transfinite recursion on the rank of τ.*

3.1.10 Definition. *If $\mathbf{M}$ is a transitive model of* ZFC*, $P \in \mathbf{M}$, and $G \in \mathbf{V}$ with $G \subseteq P$, then $\mathbf{M}[G] = \{\tau_G : \tau \in \mathbf{M}^P\}$.*

Using the definitions above, it is possible to represent any element $x \in \mathbf{M}$ in a canonical way by a P-name called $\check{x}$.

3.1.11 Definition. *Define the P-name $\check{x}$ recursively by $\check{x} = \{\langle\check{y}, 1\rangle : y \in x\}$.*

3.1.12 Lemma. *If $\mathbf{M}$ is a transitive model of* ZFC*, P a poset in $\mathbf{M}$, and G a non-empty filter on P, then $\forall x \in \mathbf{M} \, (\check{x} \in \mathbf{M}^P \wedge \check{x}_G = x)$.*

We can then construct a forcing language which uses the P-names to make statements about $\mathbf{M}[G]$ which are understandable from the perspective of $\mathbf{M}$. Generally, we will not know from the perspective of $\mathbf{M}$ whether a given proposition is true in $\mathbf{M}[G]$. However, we will still be able to give some surprisingly explicit conditions on G under which such a proposition holds.

3.1.13 Definition. *Let $\phi(x)$ be a formula, $\mathbf{M}$ a countable transitive model of* ZFC*, $\langle P, \leqslant \rangle \in \mathbf{M}$, τ a P-name, and $p \in P$. We say that p* forces *$\phi(\tau)$, written $p \Vdash \phi(\tau)$ if and only if for all $G \in \mathbf{V}$ such that G is P-generic over $\mathbf{M}$ and $p \in G$, $\phi(\tau_G)$ holds in $\mathbf{M}[G]$.*

From the definition above, it is clear that if $G \in \mathbf{V}$ is a P-generic filter containing some p which forces ϕ, then ϕ holds in $\mathbf{M}[G]$. Amazingly, the converse also holds. That is, if G is P-generic over $\mathbf{M}$ and ϕ holds in $\mathbf{M}[G]$, then there exists some $p \in G$ such that $p \Vdash \phi$. This is known as the Fundamental Theorem of Forcing.

Importantly, and perhaps surprisingly, it can be decided from $\mathbf{M}$ whether $p \Vdash \phi$, although $\mathbf{M}$ will in general not know of any P-generic filters.

In order to decide such statements from within $\mathbf{M}$, we must define a new notion, $\Vdash^*$, such that for all ϕ, $(p \Vdash \phi) \leftrightarrow (p \Vdash^* \phi)^{\mathbf{M}}$. There are many equivalent definitions of $\Vdash^*$, and the details of one construction may be found in [10, Chapter IV]. Summarizing, we have the following critical result about forcing.

3.1.14 Theorem. *Let $\mathbf{M}$ be a countable transitive model for* ZFC*, $\langle P, \leqslant \rangle \in \mathbf{M}$, τ a P-name, and $G \in \mathbf{V}$ P-generic over $\mathbf{M}$. Then*

$$\left[\exists p \in G\ (p \Vdash^* \phi(\tau))^{\mathbf{M}}\right] \leftrightarrow \left[\exists p \in G\ (p \Vdash \phi(\tau))\right] \leftrightarrow (\phi(\tau_G))^{\mathbf{M}[G]}.$$

In other words, a proposition holds in $\mathbf{M}[G]$ if and only if some $p \in G$ forces it. This establishes the relationship between forcing and truth in the generic extension.

In the following example, we will show how to use forcing to construct a function from ω to 2 which is not in the ground model. In essence, we can think of this as building a generic extension in which there exist more real numbers than in the ground model.

3.1.15 Example. *Let* $\mathbf{M}$ *be a countable transitive model of* ZFC. *Let* $\langle P, \leqslant \rangle$ *be the poset with*

$$P = \{p \mid p \textit{ is a function } \wedge |p| < \omega \wedge \operatorname{dom}(p) \subseteq \omega \wedge \operatorname{range}(p) \subseteq 2\},$$

where $p \leqslant q$ *means* $q \subseteq p$, *i.e., the function* p *is an extension of the function* q. *Finally, let* $G \in \mathbf{V}$ *be a* P*-generic filter over* $\mathbf{M}$.

Then $\bigcup G$ *is a function from a subset of* ω *to 2.* $\bigcup G$ *is indeed a function, as the filter properties of* G *guarantee that any two elements of* G *will agree in value where their domains overlap due to their having a common extension in* G.

Furthermore, the domain of the function $\bigcup G$ *is* ω *by the* P*-genericity of* G. *To see this, let* $D_n = \{p \in P \mid n \in \operatorname{dom}(p)\}$. *Then for all* $n < \omega$, D_n *is dense, as any* $p \in P$ *which is not in* D_n *can be extended to one which is by extending its domain to include* n. *Thus* $G \cap D_n \neq \varnothing$ *for all* n, *and so* $\operatorname{dom}(\bigcup G) = \omega$.

To see that this new function $\bigcup G$ *does not exist in the ground model* $\mathbf{M}$, *suppose by way of contradiction that* $\bigcup G \in \mathbf{M}$ *and let* $D = \{p \in P \mid p \nsubseteq \bigcup G\}$. D *is dense, as given any* $p \in P$ *we can extend* p *to a function* q *such that* $q(n) \neq (\bigcup G)(n)$ *for some* n. *However,* $G \cap D = \varnothing$. *Note that by absoluteness, if* $\bigcup G \in \mathbf{M}$, *then* $D \in \mathbf{M}$, *but then* $G \cap D = \varnothing$ *contradicts the* P*-genericity of* G. *Thus* $\bigcup G \notin \mathbf{M}$. *However, note that* $\bigcup G \in \mathbf{M}[G]$ *as* $G \in \mathbf{M}[G]$.

This is a special case of a more general class of forcing posets, given below, of partial functions.

3.1.16 Definition. *For any infinite cardinal* λ, *let*

$$\mathrm{Fn}(I, J, \lambda) = \{p \mid |p| < \lambda \wedge p \textit{ is a function} \wedge \mathrm{dom}(p) \subseteq I \wedge \mathrm{range}(p) \subseteq J\}.$$

Order $\mathrm{Fn}(I, J, \lambda)$ *by* $p \leqslant q \leftrightarrow q \subseteq p$.

Note that when $\lambda > \omega$, $\mathrm{Fn}(I, J, \lambda)$ is not absolute for $\mathbf{M}$.

3.1.17 Lemma. *If* $I, J, \lambda \in \mathbf{M}$, *(*$\lambda$ *is an infinite cardinal)*$^{\mathbf{M}}$, $J \neq \varnothing$, $(|I| \geqslant \lambda)^{\mathbf{M}}$, *and* $G \in \mathbf{V}$ *is* $\mathrm{Fn}(I, J, \lambda)^{\mathbf{M}}$*-generic over* $\mathbf{M}$, *then* $\bigcup G$ *is a function from* I *onto* J.

3.1.18 Remark. *In creating such new functions in our generic extensions, we introduce the possibility of changing the cardinals in the model extension. In fact, the forcing described above can be used to collapse the cardinality* κ *of a group to be countable as discussed in the first chapter, by taking* $I = \omega$, $J = \kappa$, *and* $\lambda = \omega$.

3.1.2 Cardinal Preservation

Recall the following properties and definitions concerning cardinals.

3.1.19 Definition. *The* cardinality $|A|$ *of a set* A *is the least ordinal* α *such that there exists a bijection between* A *and* α. *We say that* α *is a* cardinal *if and only if* α *is an ordinal with* $|\alpha| = \alpha$.

The cofinality *of* β, $\mathrm{cf}(\beta)$, *is the least ordinal* α *such that there is a map from* α *into* β *whose range is unbounded in* β. *We say* β *is* regular *if and only if* β *is a limit ordinal and* $\mathrm{cf}(\beta) = \beta$.

We now define what it means for a forcing poset P to preserve cardinals.

3.1.20 Definition. *If* $\langle P, \leqslant \rangle \in \mathbf{M}$, P preserves cardinals *if whenever* $G \in \mathbf{V}$ *is* P-*generic over* $\mathbf{M}$, *then*

$$\forall \beta \in \mathbf{M}\ [(\beta \text{ is a cardinal})^{\mathbf{M}} \leftrightarrow (\beta \text{ is a cardinal})^{\mathbf{M}[G]}].$$

If a cardinal κ is not preserved by a poset P, we say that P *collapses* κ, that is, forcing with P introduces to the generic extension $\mathbf{M}[G]$ bijections between κ and some sets in $\mathbf{M}$ of smaller size.

We will now give two conditions under which a poset P preserves certain cardinals. These two conditions taken together will describe an interval of cardinals in $\mathbf{M}$ which may possibly be collapsed by P. Cardinal preservation is guaranteed outside this interval. For proofs and further information, we refer the reader to [10, Chapter IV.5-6].

3.1.21 Definition. *An* antichain *in* $\langle P, \leqslant \rangle$ *is a subset* $A \subseteq P$ *such that*

$$\forall p, q \in A\ (p \neq q \rightarrow p \perp q).$$

3.1.22 Definition. *A partial order* $\langle P, \leqslant \rangle$ *has the* θ-chain condition *if and only if every antichain in* P *has cardinality* $< \theta$.

3.1.23 Theorem. *Assume* $\langle P, \leqslant \rangle \in \mathbf{M}$, *and that in* $\mathbf{M}$, θ *is a cardinal,* P *has the* θ*-chain condition, and* θ *is regular. Then* P *preserves cardinals* $\geqslant \theta$.

While the θ-chain condition provides a sufficient condition under which P preserves cardinals $\geqslant \theta$, it is not a necessary condition. Pikry forcing provides one example which demonstrates this fact.

Note that $\mathrm{Fn}(I, J, \lambda)$ has the $(|J|^{<\lambda})^{+}$-chain condition. Thus we have the following result.

3.1.24 Lemma. *Assume* $I, J \in \mathbf{M}$, *and that in* $\mathbf{M}$, λ *is regular,* $|J| \leqslant 2^{<\lambda}$, *and* $\theta = (2^{<\lambda})^{+}$. *Then* $\mathrm{Fn}(I, J, \lambda)^{\mathbf{M}}$ *preserves cardinals* $\geqslant \theta$.

3.1.25 Definition. *A partial order* $\langle P, \leqslant, 1 \rangle$ *is* λ-closed *if and only if whenever* $\alpha < \lambda$ *and* $\{p_\beta : \beta < \alpha\}$ *is a decreasing sequence of elements of* P, *then*

$$\exists q \in P \ \forall \beta < \alpha \ (q \leqslant p_\beta).$$

3.1.26 Theorem. *Assume* $P \in \mathbf{M}$, *and that in* $\mathbf{M}$, λ *is a cardinal, and* P *is* λ*-closed. Then* P *preserves cardinals* $\leqslant \lambda$.

If λ is regular, then $\mathrm{Fn}(I, J, \lambda)^{\mathbf{M}}$ is λ-closed. Thus we have the following result.

3.1.27 Lemma. *Assume* $I, J \in \mathbf{M}$, *and that in* $\mathbf{M}$, λ *is regular,* $2^{<\lambda} = \lambda$, *and* $|J| \leqslant \lambda$. *Then* $\mathrm{Fn}(I, J, \lambda)^{\mathbf{M}}$ *preserves all cardinals.*

The θ-chain condition provides an upper bound on the set of cardinals which could potentially fail to be preserved by a given poset P, while λ-closure provides a lower bound on the set of cardinals which may fail to be preserved by P.

Note that the θ-chain condition relates to the size of the largest antichain in P, called the *width of* P, while λ-closure relates to the smallest size of a maximal chain in P.

Combining the two conditions, we have the following:

3.1.28 Theorem. *Assume $P \in \mathbf{M}$, and that in $\mathbf{M}$, λ and θ are cardinals, θ is regular, and P is λ-closed and has the θ-chain condition. If $\lambda < \theta$ in $\mathbf{M}$, then P preserves all cardinals κ such that $\kappa \notin (\lambda, \theta)$.*

If $\lambda^+ \geqslant \theta$ in $\mathbf{M}$, then P preserves all cardinals.

3.2 Adding a Basis to an $\aleph_1$-Free Group

In this section, we begin by presenting the simple example of adding a basis to a vector space using forcing with partial bases, and then explore the analogous case of adding a basis to an $\aleph_1$-free group using partial basis forcing.

3.2.1 A Simple Forcing Example: Adding a Basis to a Vector Space

We will now give an example of a forcing notion which provides a new basis for a vector space in the generic extension.

In the next sections, we will generalize this forcing to free and $\aleph_1$-free groups.

Note that in this example, we refer to fields F and F-vector spaces V in the ground model $\mathbf{M}$. It is easily seen from our discussion in Chapter 2 that being a field or an F-vector space is an absolute notion for $\mathbf{M}$ provided $F, V \in \mathbf{M}$, so we need not specify in which model these sets are fields and vector spaces.

3.2.1 Example. *Let $\mathbf{M}$ be a countable transitive model of* ZFC*. Let $F \in \mathbf{M}$ be a field and $V \in \mathbf{M}$ be an F-vector space with $(\dim V = \lambda \geqslant \aleph_0)^{\mathbf{M}}$.*

Define the poset $\mathcal{P} = \{S \subset V : S \text{ linearly independent}, |S| < \lambda\}^{\mathbf{M}}$, ordered by $S \subseteq S' \leftrightarrow S' \leqslant S$.

Let $G \in \mathbf{V}$ be a $\mathcal{P}$-generic filter over $\mathbf{M}$, and define $B = \bigcup G$.

Clearly B is linearly independent, for if B was linearly dependent, there would be some dependence relation

$$n_1 x_1 + \ldots + n_m x_m = 0 \text{ with distinct } x_1, \ldots, x_m \in \bigcup G.$$

Then there exist $S_i \in G$ with $x_i \in S_i$ for $i \leqslant m$. Let $S \in G$ be a common extension of all of the S_i. Then $x_1, \ldots, x_m \in S$. But then $x_1, \ldots, x_m$ must be linearly independent.

Furthermore, B spans V. To see this, define for each $x \in V$,

$$D_x = \{S \in \mathcal{P} : x \in \langle S \rangle\}.$$

To see that D_x is dense in $\mathcal{P}$, let $S \in \mathcal{P}$ and suppose $S \notin D_x$. Then S is linearly independent, but $x \notin \langle S \rangle$, so x is not linearly dependent on S. Thus $S \cup \{x\} \in D_x$, and $S \cup \{x\} \leqslant S$. So because G is $\mathcal{P}$-generic, $G \cap D_x \neq \varnothing$ for each $x \in V$, and thus for each $x \in V$, $x \in \langle \bigcup G \rangle = \langle B \rangle$.

Thus B is a basis for V. It remains to see that $B \notin \mathbf{M}$, that is, that B is in fact a new basis not in the ground model.

To see this, assume by way of contradiction that $B \in \mathbf{M}$, and let $G' = \{S \subset B : |S| < \lambda\}^{\mathbf{M}}$. We can see that $G \subseteq G'$, for if $S \in G$, then $(|S| < \lambda)^{\mathbf{M}}$ and $S \subseteq \bigcup G = B$, and therefore $S \in G'$.

Note that $G' \in \mathbf{M}$ and thus, $\mathcal{P} - G' \in \mathbf{M}$.

We now wish to show that $\mathcal{P} - G'$ is dense. Let $S \in \mathcal{P}$, and assume $S \in G'$. Then $|S| < \lambda$, so S is a proper subset of B. Thus we can choose some $b \in B - S$. There are now two cases, depending on the characteristic of the field F. If F does not have characteristic 2, then we can take $S \cup \{-b\}$ to be an extension of S which is in $\mathcal{P} - G'$. If F does have characteristic 2, then we can choose some $b_1, b_2 \in B - S$, and note that $S \cup \{b_1, b_1 + b_2\} \in \mathcal{P} - G'$ is an extension of S. Thus in either case, $\mathcal{P} - G'$ is dense.

Finally, note that $(\mathcal{P} - G') \cap G \subseteq (\mathcal{P} - G) \cap G = \varnothing$. So $(\mathcal{P} - G') \cap G = \varnothing$, which contradicts the genericity of G.

We have shown in the above example that we may use forcing to add a basis to a vector space in the generic extension. We may wish to know whether cardinals are preserved in such a forcing extension. As we establish in the following lemma, the λ-closure of the poset in the above example ensures that such a forcing preserves cardinals less than or equal to the dimension λ of the vector space V.

3.2.2 Lemma. *Let $\mathbf{M}$ be a countable transitive model of* ZFC. *Let $F \in \mathbf{M}$ be a field and $V \in \mathbf{M}$ be an F-vector space with $(\dim V = \lambda \geqslant \aleph_0)^{\mathbf{M}}$ where $(\lambda \text{ is regular})^{\mathbf{M}}$.*

Define the poset $\mathcal{P} = \{S \subset V : S \text{ linearly independent}, |S| < \lambda\}^{\mathbf{M}}$, ordered by $S \subseteq S' \leftrightarrow S' \leqslant S$.

Then $\mathcal{P}$ is λ-closed, and thus preserves cardinals $\leqslant \lambda$.

Proof. Let $\gamma < \lambda$, and suppose we have a decreasing sequence $\{S_\alpha : \alpha < \gamma\} \subseteq \mathcal{P}$. Thus, for all $\alpha \leqslant \beta < \gamma$, $S_\alpha \subseteq S_\beta$. Let $S = \bigcup_{\alpha<\gamma} S_\alpha$. We wish to show that $S \in \mathcal{P}$, for this would demonstrate that $\mathcal{P}$ is λ-closed.

To see that $S \in \mathcal{P}$, note that for all α, as $S_\alpha \in \mathcal{P}$, $|S_\alpha| < \lambda$. Thus, as $|\gamma| < \lambda$ and λ is regular in $\mathbf{M}$, we must have $|S| < \lambda$. Furthermore, S must be linearly independent, as if it were not, there would be some dependence relation

$$n_1x_1 + \ldots + n_mx_m = 0 \text{ with distinct } x_1, \ldots, x_m \in S.$$

Then there would exist $\alpha_i < \gamma$ with $x_i \in S_{\alpha_i}$ for $i \leqslant m$. For $\alpha' = \max\{\alpha_i : i \leqslant m\} < \gamma$, $S_{\alpha'}$ is a common extension of all of the S_{α_i}. Then $x_1, \ldots, x_m \in S_{\alpha'}$. But then $x_1, \ldots, x_m$ must be linearly independent.

Thus $S \in \mathcal{P}$, and therefore $\mathcal{P}$ is λ-closed. □

Expanding on the previous example in which we use forcing to add a new basis to a vector space, it is natural to ask whether we can use a similar forcing notion to add a basis to a free group.

3.2.2 Finding a Suitable Poset

Let $\mathbf{M}$ be a countable transitive model of ZFC, and let $H \in \mathbf{M}$ be a free group of rank $\lambda \geqslant \aleph_0$.

Recall that the poset which we used to force a new basis for a vector space V was the collection of all linearly independent subsets of V of size less than the dimension of V. Without adjustments, this poset clearly will not work for free groups.

As an example, take the free group to be $H = \bigoplus_{i\in\omega} \mathbb{Z}e_i$ with $\lambda = \aleph_0$, and note that $\{2e_0\} \subset H$ is a linearly independent set of size less than λ. However, we cannot extend this subset to a linearly independent set B with $e_0 \in \langle B \rangle$. Thus, the union of any generic filter G over the poset of finite, linearly independent subsets of H with $2e_0 \in G$ will not, as it did with vector spaces, produce a basis, as it fails to generate H.

In order to ensure that such a situation does not arise, we must insist that the subsets of H which we include in the poset generate pure subgroups of H.

3.2.3 Definition. *Let*

$$\mathcal{P}_1 = \{S \subset H : S \textit{ is linearly independent } \wedge |S| < \lambda \wedge \langle S \rangle \subseteq_* H\}^{\mathbf{M}}$$

ordered by $S' \leqslant S \leftrightarrow S \subseteq S'$.

3.2.4 Remark. *Note that for torsion-free groups H the statement of purity $\langle S \rangle \subseteq_* H$ is equivalent to $H/\langle S \rangle$ being torsion-free.*

To see this, note that in general, A/G torsion-free implies G is pure in A, cf. [9, Section 5.1]. If A is torsion-free with G pure in A, then assuming $n(a + G) = G$ implies $na \in G$ and, by purity of G, $a + G = G$.

For λ a regular cardinal, $\mathcal{P}_1$ is λ-closed, which means that forcing with $\mathcal{P}_1$ preserves cardinals $\leqslant \lambda$.

Naively, one might hope that adding the purity condition will rule out the situation we had above in which the poset "dead ends" with certain elements which cannot be extended to a full basis. However, the condition turns out not to be strong enough. As we will see, any subset S of H for which $H/\langle S \rangle$ is not $\aleph_1$-free will also fail to extend to a full basis of H in $\mathbf{V}$. To make matters worse, the set of all such S is dense in $\mathcal{P}_1$, which means that for any $\mathcal{P}_1$-generic filter G, $\bigcup G$ will fail to be a basis of H.

In fact, the condition that $H/\langle S\rangle$ be $\aleph_1$-free is, in conjunction with the condition that S be linearly independent, both necessary and sufficient to guarantee that S can be extended to a full basis of H in $\mathbf{V}$. Thus we may define a second poset $\mathcal{P}_2$ by further restricting the conditions on $\mathcal{P}_1$ as follows:

3.2.5 Definition. *Let*

$$\mathcal{P}_2 = \{S \subset H : S \textit{ is linearly independent } \wedge |S| < \lambda \wedge H/\langle S\rangle \textit{ is } \aleph_1\textit{-free}\}^{\mathbf{M}}$$

ordered by $S' \leqslant S \leftrightarrow S \subseteq S'$.

If G is $\mathcal{P}_2$-generic, then $\bigcup G$ is a basis for H.

Note that $\mathcal{P}_2$ does not satisfy the λ-chain condition. To see this, take as an example the free group $H = \bigoplus_{\alpha\in\lambda} \mathbb{Z}e_\alpha$, and consider the set $\{\{e_\alpha, e_\alpha + e_0\} : 0 < \alpha < \lambda\}$. Then this set forms an antichain of size λ in $\mathcal{P}_2$. Note that this antichain also demonstrates that $\mathcal{P}_1$ does not have the λ-chain condition. So we cannot use chain conditions to demonstrate cardinal preservation (in particular, preservation of the cardinality λ of the group H).

It is also the case that $\mathcal{P}_2$ is not λ-closed, as we will prove at the end of this chapter, which means that we cannot assess cardinal preservation using this closure condition either. In particular, we are concerned with the preservation of cardinals less than or equal to λ. Recall that as $\mathbf{M}$ is a countable model of ZFC, H is countable in $\mathbf{V}$. So trivially, we could add a basis to H by collapsing the size of H to be countable in a forcing extension. Then since H is countable and $\aleph_1$-free in the forcing extension (by the absoluteness of $\aleph_1$-freeness), H itself must be free in the forcing extension. Ideally, we would like to avoid such trivial cases by ensuring that $|H| > \aleph_0$ in $\mathbf{M}[G]$.

Note that G being $\aleph_1$-free implies G is torsion-free, for the countable subgroup $\langle g \rangle$ will be free and thus torsion-free for every $g \in G$. Thus we have that $H/\langle S \rangle$ $\aleph_1$-free implies $H/\langle S \rangle$ torsion-free, which in turn implies $\langle S \rangle$ is pure in H (cf. Remark 3.2.4). Thus $\mathcal{P}_2 \subseteq \mathcal{P}_1$. For further results concerning the nature of the relationship between $\mathcal{P}_1$ and $\mathcal{P}_2$, the interested reader is referred to the appendix.

3.2.3 Forcing with $\mathcal{P}_1$

Let $\mathbf{M}$ be a countable transitive model of ZFC, and let $H \in \mathbf{M}$ be an $\aleph_1$-free group of rank $\lambda > \aleph_0$, and let $\mathcal{P}_1$ be defined as in Definition 3.2.3.

As in our Example 3.2.1 with vector spaces, if we let $G \in \mathbf{V}$ be a $\mathcal{P}_1$-generic filter, and define $B = \bigcup G$, then B is clearly linearly independent. Furthermore, if B is a basis for H in $\mathbf{M}[G]$, then it must not be in $\mathbf{M}$. This is clear to see if H is not free in $\mathbf{M}$. However, to see this in the case where H is free in $\mathbf{M}$, assume by way of contradiction that $B \in \mathbf{M}$, and define in $\mathbf{M}$ the set

$$D = \{S : |S| < \lambda \wedge S \text{ is linearly independent} \wedge S \subseteq \text{some basis of } H \wedge S \nsubseteq B\}.$$

We claim that D is dense in $\mathcal{P}_1$. To see this, assume $T \subseteq B$ with $|T| < \lambda$, and let $b_1, b_2 \in B - T$. Then $(B \cup \{b_1 + b_2\}) - \{b_2\}$ is a basis of H, so $T \cup \{b_1 + b_2\}$ is an element of D extending T. So then D is dense in $\mathcal{P}_1$, and $G \cap D \neq \varnothing$ by the genericity of G, but this contradicts that $B = \bigcup G$. Furthermore, for regular cardinals λ, $\mathcal{P}_1$ also preserves cardinals $\leqslant \lambda$, as the union of a chain of pure subgroups is pure [9, Section 5.1], and thus $\mathcal{P}_1$ is λ-closed. However, even with these adjustments to the poset, B will still fall short of providing us with a new basis of H. In particular, B fails to generate the group H. This is demonstrated using the lemmas below.

3.2.6 Lemma. *Let $S \in \mathcal{P}_1$. Then S can be extended to a basis of H in $\mathbf{V}$ if and only if $H/\langle S\rangle$ is $\aleph_1$-free.*

Proof. First, suppose that there is some basis B of H in $\mathbf{V}$ with $S \subseteq B$. Then $H/\langle S\rangle = \langle B\rangle/\langle S\rangle \cong \langle B - S\rangle$ is free in $\mathbf{V}$, and thus $H/\langle S\rangle$ is $\aleph_1$-free in $\mathbf{V}$. So by absoluteness of $\aleph_1$-freeness, $H/\langle S\rangle$ is $\aleph_1$-free in $\mathbf{M}$.

In the other direction, suppose $H/\langle S\rangle$ is $\aleph_1$-free. Recall that H is countable in $\mathbf{V}$. Then if $H/\langle S\rangle$ is $\aleph_1$-free, it is free in $\mathbf{V}$ as a countable subgroup of itself (again using the absoluteness of $\aleph_1$-freeness). So $H \cong \langle S\rangle \oplus H/\langle S\rangle$, cf. [9, Chapter 3, Theorem 1.5], and we can extend S to a full basis of H in $\mathbf{V}$. $\square$

In spite of the previous result, there is at least one such $S \in \mathcal{P}_1$ which does not extend to a full basis in $\mathbf{V}$.

3.2.7 Example. *Let H be an $\aleph_1$-free group of cardinality $\lambda > \aleph_0$. Fix a well-ordering of H in $\mathbf{M}$. We will construct a set of elements $e_i \in H$, $i \in \omega$ with the property that for each $i \in \omega$, e_i is minimal with respect to this well-ordering such that $\{e_j : j \leqslant i\}$ is linearly independent, and $H/\langle e_j : j \leqslant i\rangle$ is $\aleph_1$-free. This construction is done by induction as follows.*

Assume that we have already constructed $e_0, e_1, \ldots, e_{i-1}$ satisfying the properties above. As $H/\langle e_j : j \leqslant i-1\rangle$ is $\aleph_1$-free in $\mathbf{M}$, it is free in $\mathbf{V}$, so $\langle e_j : j \leqslant i-1\rangle$ is a direct summand of H in $\mathbf{V}$. And as $\{e_j : j \leqslant i-1\}$ is linearly independent and H is free in $\mathbf{V}$, we can extend $\{e_j : j \leqslant i-1\}$ to a basis B of H in $\mathbf{V}$, as in the previous lemma.

Now choose any $e_i \in B - \{e_j : j \leqslant i-1\}$. Then $\{e_j : j \leqslant i\}$ is linearly independent. Moreover, as $\{e_j : j \leqslant i\}$ extends to a basis B of H in $\mathbf{V}$, by the previous lemma, $H/\langle e_j : j \leqslant i\rangle$ is $\aleph_1$-free. This demonstrates that a pick for $e_i \in H$ is possible such that $e_0, e_1, \ldots, e_i$ satisfy the above properties. We will proceed to choose e_i minimal with respect to our fixed well-ordering, possibly discarding our initial pick for e_i.

If $H/\langle e_i : i < \omega\rangle$ is not $\aleph_1$-free, then we are done, as $\{e_i : i < \omega\}$ must not extend to a basis. So assume $H/\langle e_i : i < \omega\rangle$ is $\aleph_1$-free. Let $H' = \langle e_i : i < \omega\rangle = \bigoplus_{i\in\omega} \mathbb{Z}e_i$, and define the surjective group homomorphism $\phi : H' \to \mathbb{Q}$ by $\phi : e_i \mapsto \frac{1}{i+1}$ and extend by linearity. Let $K = \ker\phi$. Then as H' is free and K is a subgroup of H', K is free. And as $K \subseteq H'$, $|K| \leqslant \aleph_0$. Let S be a basis of K.

By the first isomorphism theorem, we have $\mathbb{Q} = \operatorname{Im}\phi \cong H'/\ker\phi = H'/\langle S\rangle$. So we have H/H' and $H'/\langle S\rangle$ both torsion-free, and therefore by Remark 3.2.4, $\langle S\rangle$ is pure in H' and H' is pure in H. By the transitivity of purity [9, Chapter 5, Theorem 1.3(ii)], $\langle S\rangle$ is pure in H. To see that $H/\langle S\rangle$ is torsion-free, note that if $nh + \langle S\rangle = 0$, then $h \in \langle S\rangle$, and so by the purity of $\langle S\rangle$, $h \in \langle S\rangle$ and thus $h + \langle S\rangle = 0$. So $S \in \mathcal{P}_1$.

However, $H/\langle S\rangle$ cannot be $\aleph_1$-free, for it contains a copy of $\mathbb{Q}$ as a subgroup $H'/\langle S\rangle \subseteq H/\langle S\rangle$, which is countable but not free. Thus S does not extend to a basis of H in $\mathbf{V}$. In particular, if $G \in \mathbf{V}$ is $\mathcal{P}_1$-generic with $S \in G$, then $\bigcup G$ is not a basis for H.

Note that the contradiction above was generated by showing that $H'/\langle S\rangle \cong \mathbb{Q}$, which is not free. This gives us a clue into the nature of such counterexamples. More specifically, we see that $H'/\langle S\rangle$ is a countable subgroup of $H/\langle S\rangle$ which is not free, and thus $H/\langle S\rangle$ is not $\aleph_1$-free. However, $H/\langle S\rangle$ is $\aleph_1$-free if and only if S can be extended to a basis of H in $\mathbf{V}$ (cf. Lemma 3.2.6). Thus the failure of $H/\langle S\rangle$ to be $\aleph_1$-free is what lies at the heart of the issue.

Despite the result above, one might naively hope that there is at least some $\mathcal{P}_1$-generic filter G which avoids all such S which do not extend to a full basis. However, as the following lemma shows, any $\mathcal{P}_1$-generic filter must contain at least one such S.

3.2.8 Lemma. *Let D be the set of all $S \in \mathcal{P}_1$ such that $H/\langle S\rangle$ is not $\aleph_1$-free. Then D is dense in $\mathcal{P}_1$.*

Proof. Let $S \in \mathcal{P}_1$. If $S \notin \mathcal{P}_2$, then we are done. So assume $S \in \mathcal{P}_2$, that is, that $H/\langle S\rangle$ is $\aleph_1$-free.

Note that $|H/\langle S\rangle| = |H| = \lambda$. Then by taking $H/\langle S\rangle$ to be our uncountable $\aleph_1$-free group in the construction in the preceding example, we can find K' (with appropriate corresponding set K of representatives) such that $K' = \{x + \langle S\rangle : x \in K\}$ with K' countable and linearly independent, and $(H/\langle S\rangle)/\langle K'\rangle$ torsion-free but not $\aleph_1$-free.

Let $T = S \cup K$. Then as K' and S are linearly independent, $T = S \cup K$ is linearly independent. Moreover, $|T| < \lambda$. So it remains to see that $H/\langle T\rangle$ is torsion-free, but not $\aleph_1$-free, in order to establish that T is an extension of S in D.

We have

$$H/\langle T\rangle = H/\langle S \cup K\rangle \cong (H/\langle S\rangle)/(\langle S \cup K\rangle/\langle S\rangle) = (H/\langle S\rangle)/\langle K'\rangle$$

with the isomorphism above given by the third isomorphism theorem. Recall that $(H/\langle S\rangle)/\langle K'\rangle$ is torsion-free but not $\aleph_1$-free. Thus T is an extension of S in D. □

Thus as D is dense in $\mathcal{P}_1$, any $\mathcal{P}_1$-generic filter $G \in \mathbf{V}$ will contain some element which cannot be extended to a full basis of H. So $\bigcup G$ will not provide a basis for H in our generic extension $\mathbf{M}[G]$.

However, we can strengthen our conditions on $\mathcal{P}_1$ to form the poset $\mathcal{P}_2$ which, we will show, does furnish a basis for H in any generic extension.

3.2.4 Forcing with $\mathcal{P}_2$

As seen in the last section, we will need to strengthen our conditions on the poset if we wish for the union $\bigcup G$ of a generic filter G on that poset to span H. To that end, we will use $\mathcal{P}_2$.

3.2.9 Theorem. *Let $\mathbf{M}$ be a countable transitive model of* ZFC*, and let $H \in \mathbf{M}$ be an $\aleph_1$-free abelian group with $(\mathrm{rk}(H) = \lambda > \aleph_0)^{\mathbf{M}}$. Let G be $\mathcal{P}_2$-generic, and let $B = \bigcup G$. Then B is a basis of H in $\mathbf{M}[G]$, with $B \notin \mathbf{M}$.*

Proof. B is linearly independent and $B \notin \mathbf{M}$ by the same arguments given for the case involving $\mathcal{P}_1$.

To see that B is a basis for H, we must show that $H = \langle B \rangle$. To this end, define for all $x \in H$ the set $D_x = \{S \in \mathcal{P}_2 : x \in \langle S \rangle\}$. As $\mathcal{P}_2 \in \mathbf{M}$, $D_x \in \mathbf{M}$. We wish to show that D_x is dense in $\mathcal{P}_2$. Suppose $S \in \mathcal{P}_2$ with $x \notin \langle S \rangle$. To see that D_x is dense, we must find some S' with $S \subseteq S' \in D_x$. Consider the group $T = \langle S \cup \{x\} \rangle_* \subseteq H$. We wish to show that T is generated by a linearly independent set S' with $S \subseteq S'$, and that $H/T = H/\langle S' \rangle$ is $\aleph_1$-free, thus establishing the denseness of D_x.

To see that T has a basis S' which extends S, note that $\langle S \rangle_* = \langle S \rangle$. This is because $H/\langle S \rangle$ is $\aleph_1$-free, and thus every countable subgroup is free, in particular, the subgroup $\langle y + \langle S \rangle \rangle \subseteq H/\langle S \rangle$ is free and thus torsion-free for all $y \in H$. So $ny \in \langle S \rangle$ with $n \in \mathbb{Z}$, $n \neq 0$ and $y \in H$ implies $n(y + \langle S \rangle) = \langle S \rangle$, hence $y + \langle S \rangle = \langle S \rangle$, and $y \in \langle S \rangle$. Or in other words, $\langle S \rangle$ contains all elements of H of which its elements are nonzero multiples, and thus $\langle S \rangle$ is a pure subgroup of H. Therefore, $\langle S \rangle_* = \langle S \rangle$.

Clearly, $T/\langle S \rangle_* \neq 0$ as $x \in T - \langle S \rangle = T - \langle S \rangle_*$. We now show that $\mathrm{rk}(T/\langle S \rangle_*) = 1$. Let $a, b \in \langle S \cup \{x\} \rangle_* - \langle S \rangle_*$. We wish to show that $a + \langle S \rangle_*$ and $b + \langle S \rangle_*$ are linearly dependent in $T/\langle S \rangle_*$. As $a \in \langle S \cup \{x\} \rangle_*$, there exist some $n \in \mathbb{Z}$ with $n > 0$ and $na \in \langle S \cup \{x\} \rangle$. Thus we can write $na = s + mx$ for some $m \in \mathbb{Z}$ and some $s \in \langle S \rangle$. We have $m \neq 0$ as $a \notin \langle S \rangle_*$. Similarly, there exist $n', m' \in \mathbb{Z}$ with $n' > 0$, $m' \neq 0$, and $s' \in \langle S \rangle$ such that $n'b = s' + m'x$. Then $m'na - mn'b = m'(s + mx) - m(s' + m'x) = m's - ms' \in \langle S \rangle = \langle S \rangle_*$. So $m'n(a + \langle S \rangle_*) - mn'(b + \langle S \rangle_*) = \langle S \rangle_*$ in $T/\langle S \rangle_*$, thus demonstrating the linear dependence of $a + \langle S \rangle_*$ and $b + \langle S \rangle_*$. So $\mathrm{rk}(T/\langle S \rangle_*) = 1$.

Note that $T/\langle S \rangle_*$ is torsion-free. For if $z + \langle S \rangle_* \neq \langle S \rangle_*$ in $T/\langle S \rangle_*$, this means that $z \notin \langle S \rangle_*$, which implies that $nz \notin \langle S \rangle_*$ for all $n \in \mathbb{Z}$, $n \neq 0$ by purity. Thus $T/\langle S \rangle = T/\langle S \rangle_*$ must be countable, as it is a torsion-free, abelian group of rank 1, and thus embeds in $\mathbb{Q}$ [9, Section 3.4]. Thus, since $T/\langle S \rangle \subseteq H/\langle S \rangle$ is $\aleph_1$-free, $T/\langle S \rangle$ is a rank 1 free group. So we can write T as the direct sum of free groups, as $T \cong \langle S \rangle \oplus T/\langle S \rangle$. Therefore T is free, and we can write $T = \langle S' \rangle$, for some linearly independent S' with $S \subset S'$. Let $y \in T$ be such that $S' = S \cup \{y\}$.

In order to see that B is a basis for H, it remains now to show that $H/T = H/\langle S' \rangle$ is $\aleph_1$-free. That is, we must show that $\langle x_i + T : i \in \omega \rangle$ is a free subgroup of H/T for any countable set of elements $x_i \in H$, $i \in \omega$. To this end, let $U = \langle S \cup \{x_i, y : i \in \omega\}\rangle \subseteq H$. Then $U/\langle S\rangle = \langle x_i + \langle S\rangle, y + \langle S\rangle : i \in \omega\rangle$ is a countable subgroup of $H/\langle S\rangle$, which is $\aleph_1$-free. Thus, $U/\langle S\rangle$ is free, and we can write U as the direct sum of free groups, $U \cong \langle S\rangle \oplus U/\langle S\rangle$. Thus U is free, and we can write $y = a + b$, with $a \in \langle S\rangle$ and $b \in U/\langle S\rangle$.

We claim now that $\langle b\rangle$ is pure in $U/\langle S\rangle$. To see this, suppose $c \in U/\langle S\rangle$, and $nc = mb$ for some $n, m \in \mathbb{Z}, n > 0$, that is, $c \in \langle b\rangle_*$. We wish to show that $c \in \langle b\rangle$. Note that $nc = mb \in \langle b\rangle \subseteq \langle S \cup \{b\}\rangle = \langle S'\rangle = \langle S \cup \{x\}\rangle_*$, so nc is an element of a pure subgroup of H, namely $\langle S'\rangle$. Thus, $c \in \langle S'\rangle = \langle S \cup \{b\}\rangle = \langle S\rangle \oplus \langle b\rangle$. Thus, as $c \in U/\langle S\rangle$ and $U = \langle S\rangle \oplus U/\langle S\rangle$, it follows that $c \in \langle b\rangle$.

Now, as $U/\langle S\rangle$ is free, and $b \in U/\langle S\rangle$, b can be written as the finite linear combination of basis elements $b_1, \ldots, b_n$ with respect to some basis of $U/\langle S\rangle$. So we can write $U/\langle S\rangle = A \oplus B'$, where $B' = \bigoplus_{1 \leqslant i \leqslant n} \mathbb{Z}b_i$. So B' is finitely generated and A is free. Also, $\langle b\rangle = \langle b\rangle_*$ in B', as purity is preserved in subgroups. Now $B'/\langle b\rangle_*$ is finitely generated and by purity, torsion-free. Therefore $B'/\langle b\rangle_*$ is free. So $B' \cong \langle b\rangle \oplus B'/\langle b\rangle$.

Thus,

$$U \cong \langle S\rangle \oplus U/\langle S\rangle = \langle S\rangle \oplus A \oplus B \cong \langle S\rangle \oplus A \oplus \langle b\rangle \oplus B/\langle b\rangle$$

$$= \langle S \cup \{b\}\rangle \oplus (A \oplus B/\langle b\rangle) = \langle S \cup \{y\}\rangle \oplus (A \oplus B/\langle b\rangle)$$

$$= \langle S'\rangle \oplus (A \oplus B/\langle b\rangle) = T \oplus (A \oplus B/\langle b\rangle)$$

Therefore, $U/T \cong A \oplus B/\langle b\rangle$. And as A and $B/\langle b\rangle$ are free, U/T is free. Thus H/T is $\aleph_1$-free.

This establishes that $\bigcup G$ is a basis for H. $\square$

3.2.10 Remark. *Note that we can dramatically simplify the preceding proof using the proof technique introduced in Chapter 2 which utilizes the absoluteness of $\aleph_1$-freeness. In particular, we can prove the denseness of $D_x = \{S \in \mathcal{P}_2 : x \in \langle S \rangle\}$ as follows.*

Proof. Let $q \in \mathcal{P}_2$. Then by Lemma 3.2.6, q can be extended to a basis B in $\mathbf{V}$. So we can write $H = \langle q \rangle \oplus \langle B - q \rangle$ in $\mathbf{V}$. Then we have $x = a + b$, with $a \in \langle q \rangle$ and $b \in \langle B - q \rangle$. We can then write b in terms of $B - q$, say $b = n_1 x_1 + \ldots + n_m x_m$. So we have $H = \langle q \rangle \oplus \langle x_1, \ldots, x_m \rangle \oplus C$ in $\mathbf{V}$, with C free. Thus $H = \langle q \cup \{x_1, \ldots, x_m\} \rangle \oplus C$, and so $q \cup \{x_1, \ldots, x_m\} \in D_x$, and thus D_x is dense. □

3.2.5 Cardinal Preservation

The question remains as to whether this forcing preserves cardinals less than or equal to the size λ of the group (note that as we are concerned here with uncountable free groups, the size and rank are the same). A standard method for demonstrating such properties of cardinal preservation is the test of λ-closure, as this provides a simple condition which guarantees the preservation of cardinals $\leqslant \lambda$, cf. Theorem 3.1.26. However, as we will demonstrate in the next lemma, $\mathcal{P}_2$ even fails to be $\aleph_1$-closed, and thus if we wish to settle the question of cardinal preservation, we must do so by alternative means.

3.2.11 Lemma. *If H is an $\aleph_1$-free group of cardinality $\lambda > \aleph_0$, then $\mathcal{P}_2$ is not $\aleph_1$-closed.*

Proof. We begin by demonstrating that $\mathcal{P}_2$ is not $\aleph_1$-closed if H is free of countable rank, with $H = \bigoplus_{i \in \omega} \mathbb{Z} e_i$.

Define $s_i = e_i - (i+1)e_{i+1}$, $S_i = \{s_j : j \leqslant i\}$ for all $i \in \omega$. We wish to show that S_i is linearly independent for all i. By way of induction, suppose S_j is linearly independent for $j < i$. We have

$$S_i = S_{i-1} \cup \{e_i - (i+1)e_{i+1}\}.$$

Observing that the element e_{i+1} does not contribute to $S_{i-1} \subseteq \bigoplus_{0 \leqslant j \leqslant i} \mathbb{Z}e_j$, it is obvious that S_{i-1} and $e_i - (i+1)e_{i+1}$ are linearly independent from each other. Thus S_i is linearly independent.

Define $B_0 = \{e_0, e_1, \ldots\}$, $B_{i+1} = \{s_0, s_1, \ldots, s_i, e_{i+1}, e_{i+2}, \ldots\}$ for $i \geqslant 0$. Note that B_0 is obviously a basis of H while B_{i+1} results from B_i by replacing the basis element e_i with the element $s_i = e_i - (i+1)e_{i+1}$. Thus, an easy induction shows that each B_i is a basis for H. Then we have $H = \langle S_i \rangle \oplus \bigoplus_{j>i} \mathbb{Z}e_j$ for each $i \in \omega$. So each $\langle S_i \rangle$ is a pure subgroup of H (as direct summands are pure subgroups cf. [9, Section 5.1]), and $H/\langle S_i \rangle \cong \bigoplus_{j>i} \mathbb{Z}e_j$ is free for each i. So $S_i \in \mathcal{P}_2$ for all $i \in \omega$.

Define $S = \bigcup_{i \in \omega} S_i = \{s_i : i \in \omega\}$, and note that the S_i form a countable descending sequence in $\mathcal{P}_2$. Defining the homomorphism $\phi : H \to \mathbb{Q}$ by mapping $e_i \mapsto \frac{1}{i!}$, note that $\langle S \rangle \subseteq \ker \phi$ while $\phi(e_0) = 1$. Thus $e_0 \notin \langle S \rangle$. To show that $S \notin \mathcal{P}_2$, it suffices to show that $e_0 + \langle S \rangle$ is divisible in $H/\langle S \rangle$ and that $H/\langle S \rangle$ is torsion-free. For if $H/\langle S \rangle$ contains a torsion-free divisible subgroup, then it contains a copy of $\mathbb{Q}$ (cf. [9, Chapter 4, Theorem 3.1]), which is countable and not free. Thus $H/\langle S \rangle$ cannot be $\aleph_1$-free.

To see that $H/\langle S \rangle$ is torsion-free, suppose that $n(h + \langle S \rangle) = 0$ for some $n \in \mathbb{Z}$, $n \neq 0$. Then there exist $n_i \in \mathbb{Z}$, $0 \leqslant i \leqslant k$, such that $nh = \sum_{0 \leqslant i \leqslant k} n_i s_i \in \langle S_k \rangle$, and by the purity of $\langle S_k \rangle$, $h \in \langle S_k \rangle \subseteq \langle S \rangle$. Thus $h + \langle S \rangle = \langle S \rangle$ in $H/\langle S \rangle$. The construction of the s_i gives us that $e_0 + \langle S \rangle = i!e_i + \langle S \rangle$ for all $i \in \omega$. Thus e_0 is divisible in $H/\langle S \rangle$. So $S \notin \mathcal{P}_2$.

It remains to check that there is no $p \leqslant S$ with $p \in \mathcal{P}_2$. Suppose by way of contradiction that such a p exists. Then $H/\langle p \rangle$ is $\aleph_1$-free. As $p \in \mathcal{P}_2$, it is linearly independent, and since $S \subseteq p$, $\langle p \rangle / \langle S \rangle$ is free. By the third isomorphism theorem, we have $H/\langle p \rangle \cong (H/\langle S \rangle)/(\langle p \rangle / \langle S \rangle)$. Thus $H/\langle S \rangle$ is $\aleph_1$-free by Theorem 2.3.2. But then $S \in \mathcal{P}_2$, which is a contradiction.

Let us now consider the general case of an $\aleph_1$-free group H of size $\lambda > \aleph_0$, and pick a strictly decreasing sequence $\{p_i : i \in \omega\}$ of finite sets $p_i \in \mathcal{P}_2$. If $\bigcup_{i \in \omega} p_i \notin \mathcal{P}_2$, then by the argument in the preceding paragraph, there is no $p \leqslant \bigcup_{i \in \omega} p_i$ with $p \in \mathcal{P}_2$. So assume $\bigcup_{i \in \omega} p_i \in \mathcal{P}_2$ and let $H' = \langle \bigcup_{i \in \omega} p_i \rangle$. As $\bigcup_{i \in \omega} p_i \in \mathcal{P}_2$, H/H' is $\aleph_1$-free. Furthermore, H' is free of countable rank (with basis $\bigcup_{i \in \omega} p_i$). So we can write $H' = \bigoplus_{i \in \omega} \mathbb{Z} e_i$, and perform the construction given in the first half of this proof for countable rank free groups to construct s_i, S_i, and $S = \bigcup_{i \in \omega} S_i$ as above. Then $H'/\langle S_i \rangle$ is free and H/H' is $\aleph_1$-free, so again by the third isomorphism theorem and Theorem 2.3.2, we have that $H/\langle S_i \rangle$ is $\aleph_1$-free, so $S_i \in \mathcal{P}_2$.

Now by the divisibility argument above, $\mathbb{Q} \subseteq H'/\langle S \rangle \subseteq H/\langle S \rangle$. So $H/\langle S \rangle$ is not $\aleph_1$-free, and thus $S \notin \mathcal{P}_2$. We can complete the proof by carrying through the argument that there is no $p \leqslant S$ in $\mathcal{P}_2$. $\square$

CHAPTER FOUR

Forcing $\aleph_1$-Free Groups to Be Free with Cardinal Preservation

4.1 Preliminaries

Here, we give a formal definition of the Γ-invariant and state a theorem which characterizes the freeness of an $\aleph_1$-free group of size $\aleph_1$ using the Γ-invariant.

We also discuss two established forcings which, given a stationary subset A of $\aleph_1$, add a closed unbounded subset to A while preserving $\aleph_1$.

4.1.1 The Γ-Invariant

We now define the Γ-invariant of an $\aleph_1$-free group of cardinality $\aleph_1$, and state a theorem of Eklof and Mekler relating the freeness of the group to its Γ-invariant. For further details and proofs, see [8, Section IV.1]. The definition of the Γ-invariant requires that we first define an $\aleph_1$-filtration, and an equivalence relation on subsets of $\aleph_1$. These definitions are given below.

4.1.1 Definition. *Let H be an abelian group of cardinality $\aleph_1$. An* $\aleph_1$-filtration *of H is a sequence $\{H_\alpha : \alpha < \aleph_1\}$ of subgroups of H whose union is H and which satisfies for all $\alpha, \beta < \aleph_1$:*

1. *$|H_\alpha| \leqslant \aleph_0$;*

2. *if $\alpha \leqslant \beta$, then $H_\alpha \subseteq H_\beta$;*

3. *if α is a limit ordinal, then $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$.*

4.1.2 Definition. *If X and Y are subsets of $\aleph_1$, we can define an equivalence relation by $X \sim Y$ if and only if there exists some closed (with respect to the order topology) and unbounded set (club) $C \subseteq \aleph_1$ such that $X \cap C = Y \cap C$. Denote the equivalence class of X by $[X]$.*

We may now give the definition of the Γ-invariant of an $\aleph_1$-free group of cardinality $\aleph_1$ as follows:

4.1.3 Definition. *Let H be an $\aleph_1$-free abelian group of cardinality $\aleph_1$, and let $\{H_\alpha : \alpha < \aleph_1\}$ be an $\aleph_1$-filtration of H. Let $E = \{\alpha < \aleph_1 : H/H_\alpha$ is not $\aleph_1$-free$\}$. The Γ-invariant of H, denoted $\Gamma(H)$, is defined to be the equivalence class of E, $[E]$.*

Note that the Γ-invariant $\Gamma(H)$ does not depend on the choice of filtration.

We now state the key result of Eklof and Mekler relating the freeness of an $\aleph_1$-free group to its Γ-invariant.

4.1.4 Theorem. *If H is an $\aleph_1$-free group of size $\aleph_1$, then H is free if and only if $\Gamma(H) = [\varnothing]$.*

It is of interest to note that given any subset E of $\aleph_1$, an $\aleph_1$-free group H of cardinality $\aleph_1$ with $\Gamma(H) = [E]$ can be constructed.

We can see that H is free if and only if the representative E of the equivalence class $[E]$ defining the Γ-invariant $\Gamma(H)$ is not stationary. So if we wish to force a non-free $\aleph_1$-free group H of cardinality $\aleph_1$ to become free, we must force E to become non-stationary, i.e., we must add a club to $\aleph_1 - E$.

4.1.2 Forcing a Club into a Stationary Set

In [4], Baumgartner, Harrington, and Kleinberg describe a forcing which, for any stationary subset A of $\aleph_1$, forces a closed unbounded subset of $\aleph_1$ into A. Furthermore, this forcing preserves the cardinality of $\aleph_1$. In particular, if $\mathbf{M}$ is a countable transitive model of ZFC, and $A \subseteq \aleph_1$ is stationary, then there exists a generic extension $\mathbf{N}$ of $\mathbf{M}$ which has the same reals as $\mathbf{M}$ and in which there exists some club C with $C \subseteq A$. The poset defining this forcing consists of all closed subsets of A of successor order-type with $q \leqslant p$ if and only if p is a subset of q and $(q - p) \cap (\bigcup p) = \varnothing$.

If we wish to preserve all cardinals, we may use the forcing described by Abraham and Shelah in [1, Theorem 3]. The poset which describes this forcing adds a closed unbounded subset to a given stationary subset A of $\aleph_1$ while preserving the cardinality of $\aleph_1$, and the poset itself has size $\aleph_1$ and thus preserves all cardinals. However, it does add new reals to the base model $\mathbf{M}$.

We will show that if $\Gamma(H) \neq [\aleph_1]$, then H is free of size $\aleph_1$ in some generic extension produced by either of these forcings.

4.2 Forcing an $\aleph_1$-Free Group to Become Free with Cardinal Preservation

4.2.1 Lemma. *Let $\mathbf{M}$ be a transitive model of* ZFC *and H a (non-free) $\aleph_1$-free abelian group with cardinality $\aleph_1$ and $\Gamma(H) = [\aleph_1]$ in $\mathbf{M}$. If $\mathbf{N}$ is any transitive model of* ZFC *containing $\mathbf{M}$ with H free in $\mathbf{N}$, then ${\aleph_1}^{\mathbf{M}} \neq {\aleph_1}^{\mathbf{N}}$.*

Proof. Note first that if $\Gamma(H) = [E] = [\aleph_1]$, where E is any representative of the Γ-invariant, then there exists some club C in $\aleph_1$ with $E \cap C = \aleph_1 \cap C = C$, and thus $C \subseteq E$. So we will prove the result under this assumption by contrapositive.

Thus, let $\mathbf{N}$ be a transitive model of ZFC with $\mathbf{M} \subseteq \mathbf{N}$, and assume ${\aleph_1}^{\mathbf{M}} = {\aleph_1}^{\mathbf{N}}$. Then if $\{H_\alpha : \alpha < \aleph_1\}$ is an $\aleph_1$-filtration of H in $\mathbf{M}$, it is also an $\aleph_1$-filtration in $\mathbf{N}$, by absoluteness and because ${\aleph_1}^{\mathbf{M}} = {\aleph_1}^{\mathbf{N}}$ with $\mathbf{M} \subseteq \mathbf{N}$.

So by the absoluteness of $\aleph_1$-freeness,

$$E := \{\alpha \in \aleph_1 : H/H_\alpha \text{ is not } \aleph_1\text{-free}\}^{\mathbf{M}} = \{\alpha \in \aleph_1 : H/H_\alpha \text{ is not } \aleph_1\text{-free}\}^{\mathbf{N}},$$

and thus, $(\Gamma(H) = [E])^{\mathbf{N}}$. We note that while the definition of the set E as a representative of the Γ-invariant is absolute, the equivalence class of E is not absolute.

As $\Gamma(H) = [E] = [\aleph_1]$ in $\mathbf{M}$ there exists some club C in $\aleph_1$ in $\mathbf{M}$ with $C \subseteq E$. Note that as C is a club in $\aleph_1$ in $\mathbf{M}$, then C is also a club in $\aleph_1$ in $\mathbf{N}$. Now, let C' be any club in $\mathbf{N}$. Then as the intersection of two clubs is nonempty, $\varnothing \neq C \cap C' \subseteq E \cap C'$. So E is stationary in $\mathbf{N}$, and thus H is not free in $\mathbf{N}$.

Thus, if H is free in $\mathbf{N}$, then $\aleph_1$ is not preserved in the model extension. $\square$

4.2.2 Lemma. *Let* $\mathbf{M}$ *be a countable transitive model of* ZFC *and in* $\mathbf{M}$*, let* H *be an* $\aleph_1$*-free abelian group of cardinality* $\aleph_1$ *with* $\Gamma(H) \neq [\aleph_1]$*. Then there exists a generic extension* $\mathbf{N}$ *of* $\mathbf{M}$ *which preserves the cardinality of* H *with* H *free in* $\mathbf{N}$*.*

Proof. Let $\mathbf{M}$ be a countable transitive model of ZFC and H be an $\aleph_1$-free abelian group of cardinality $\aleph_1$ with $\Gamma(H) \neq [\aleph_1]$ in $\mathbf{M}$. Let E be a representative of the Γ-invariant of H resulting from some $\aleph_1$-filtration $\{H_\alpha : \alpha < \aleph_1\}$ of H in $\mathbf{M}$.

We argue first in $\mathbf{M}$ that $\aleph_1 - E$ is stationary. Proceeding by contrapositive, if $\aleph_1 - E$ is not stationary, then there exists some club C in $\aleph_1$ with $C \subseteq E$. Thus $E \cap C = C = \aleph_1 \cap C$, and so $[E] = [\aleph_1]$. So $\aleph_1 - E$ is stationary, and thus we can use the forcings described by [4] and [1, Theorem 3] to produce generic extensions $\mathbf{N}$ of $\mathbf{M}$ which preserve the cardinality of H in which there exists some club C in $\aleph_1$ with $C \subseteq \aleph_1 - E$.

Thus in $\mathbf{N}$, E is not stationary. And as $(\Gamma(H) = [E])^{\mathbf{N}}$, as shown in the previous proof, $(\Gamma(H) = [E] = [\varnothing])^{\mathbf{N}}$. Thus H is free in $\mathbf{N}$. □

Combining the previous two lemmas gives the following necessary and sufficient condition under which an $\aleph_1$-free group of size $\aleph_1$ can be forced to be free while preserving the cardinality of the group.

4.2.3 Theorem. *Let* $\mathbf{M}$ *be a countable transitive model of* ZFC *and* H *an* $\aleph_1$*-free abelian group of size* $\aleph_1$ *in* $\mathbf{M}$. *Then there exists some transitive model* $\mathbf{N}$ *of* ZFC *extending* $\mathbf{M}$ *in which the cardinality of* H *is preserved and* H *is free if and only if* $\Gamma(H) \neq [\aleph_1]$ *in* $\mathbf{M}$.

Note that the backwards direction of this result can in fact be strengthened, as we do not require $\mathbf{M}$ to be countable. That is, if we begin with an arbitrary transitive model $\mathbf{M}$ of ZFC containing H in which $\Gamma(H) = [\aleph_1]$, then there is no transitive extension of $\mathbf{M}$ in which $\aleph_1$ is preserved and H is free, cf. Lemma 4.2.1.

4.2.1 The Baer-Specker Group is Turbid Assuming CH

Theorem 4.2.3 motivates the following definition.

4.2.4 Definition. *We call an* $\aleph_1$*-free group* H turbid *if* $|H| = \aleph_1$ *and* $\Gamma(H) = [\aleph_1]$.

We note that the Baer-Specker group, $\mathbb{Z}^\omega = \prod_{i\in\omega} \mathbb{Z}e_i$, is $\aleph_1$-free of cardinality $2^{\aleph_0}$. If we assume the Continuum Hypothesis (CH), the Baer-Specker group has cardinality $\aleph_1$, and so it makes sense to ask whether it is a turbid group. It is, in fact, turbid, owing to the fact that it is strongly $\aleph_1$-free, which we define below. See [8, Chapter IV.0] for further details regarding strongly $\aleph_1$-free groups.

4.2.5 Definition. *We call a group* H strongly $\aleph_1$-free *if every countable subset of* H *is contained in some countable free subgroup* $K \subseteq H$ *with* H/K $\aleph_1$*-free.*

4.2.6 Remark. *Note that every strongly* $\aleph_1$*-free group is automatically* $\aleph_1$*-free. Moreover, a group* H *of size* $\aleph_1$ *is strongly* $\aleph_1$*-free if and only if it allows an* $\aleph_1$*-filtration* $\{H_\alpha : \alpha \in \aleph_1\}$ *such that* $\aleph_1 - E$ *is unbounded. Thus, if* H *is not strongly* $\aleph_1$*-free, then* $\aleph_1 - E$ *is bounded for every filtration, in which case* $\Gamma(H) = [E] = [\aleph_1]$*, i.e.,* H *is turbid. Note that the converse is not true as there exist strongly* $\aleph_1$*-free turbid groups.*

We now cite the following result concerning the uncountable product of copies of the integers [8, Theorem 2.8].

4.2.7 Theorem. *For any infinite cardinal* κ*,* $\mathbb{Z}^\kappa = \prod_{\alpha<\kappa} \mathbb{Z}e_\alpha$ *is not strongly* $\aleph_1$*-free.*

As the Baer-Specker group is not strongly $\aleph_1$-free, it is turbid (assuming the Continuum Hypothesis). Thus, if H is the Baer-Specker group in a ground model $\mathbf{M}$ for ZFC + CH, there is no $\aleph_1$-preserving transitive extension of $\mathbf{M}$ in which H is free.

4.2.2 $\mathcal{P}_2$ Revisited

In an even more algebraic fashion, we may also explicitly add a basis to our non-turbid $\aleph_1$-free group H of size $\aleph_1$. Since the final goal is to add a full basis, it is appropriate that our forcing set should be a set of 'partial bases,' i.e., sets of linearly independent elements of H.

Recall the definition of the partial order $\mathcal{P}_2$ given in Definition 3.2.5.

$$\mathcal{P}_2 = \{p \subset H : p \text{ is linearly independent}, |p| < \aleph_1, \text{ and } H/\langle p\rangle \text{ is } \aleph_1\text{-free}\}$$

There is a natural ordering on the elements of $\mathcal{P}_2$ by $p \leqslant q \Leftrightarrow q \subseteq p$. Recall also that forcing with $\mathcal{P}_2$ produces a basis for H, cf. Theorem 3.2.9. That is:

4.2.8 Theorem. *Let* $\mathbf{M}$ *be a countable transitive model of* ZFC*, and let H be an $\aleph_1$-free group of size $\aleph_1$ in* $\mathbf{M}$*. Let $G \in \mathbf{V}$ be a $\mathcal{P}_2$-generic filter and define $B = \bigcup G$. Then B is a basis of H in* $\mathbf{M}[G]$*, with $B \notin \mathbf{M}$.*

If we restrict our attention to non-turbid groups, we are able to settle the question of cardinal preservation. In particular, if H is a non-turbid $\aleph_1$-free group of size $\aleph_1$, then forcing with $\mathcal{P}_2$ preserves the cardinality of H.

4.2.9 Theorem. *Let* $\mathbf{M}$ *be a countable transitive model of* ZFC *and let H be a non-turbid $\aleph_1$-free group of size $\aleph_1$ in* $\mathbf{M}$*. Let $(\mathcal{P}, \leqslant)$ be defined to be $\mathcal{P}_2$ as above. Then forcing with $\mathcal{P}$ preserves $\aleph_1$, and H is free of size $\aleph_1$ in* $\mathbf{M}[G]$*.*

Proof. Fix some filtration $\{H_\alpha : \alpha < \aleph_1\}$ of H in $\mathbf{M}$ (without loss of generality we may choose this to be a strictly increasing filtration with $\alpha < \beta \to H_\alpha \subset H_\beta$). Let $E = \{\alpha < \aleph_1 : H/H_\alpha \text{ is not } \aleph_1\text{-free}\}$. Since H is not turbid, we have that the set

$\aleph_1 - E = \{\alpha < \aleph_1 : H/H_\alpha \text{ is } \aleph_1\text{-free}\}$ is stationary. Then define

$$\mathcal{P}' = \{p \in \mathcal{P} : \exists \alpha \in \aleph_1 - E \text{ with } p \text{ a basis for } H_\alpha\}.$$

We claim that $\mathcal{P}'$ is dense in $\mathcal{P}$.

Let $p \in \mathcal{P}$. As p is finite or countable, it is contained in some H_α. Since $\aleph_1 - E$ is unbounded, we may, without loss of generality, choose this α to be in $\aleph_1 - E$. Now, as $p \in \mathcal{P}$,$H/\langle p\rangle$ is $\aleph_1$-free. Furthermore, because $H_\alpha/\langle p\rangle$ is a countable subgroup of $H/\langle p\rangle$, $H_\alpha/\langle p\rangle$ is free. So we can extend p to some basis q of H_α. Thus we have $q \in \mathcal{P}'$ with $q \leqslant p$, so $\mathcal{P}'$ is dense in $\mathcal{P}$.

Let $p \in \mathcal{P}'$. Since the filtration is strictly increasing, we can define the "height" of p to be the unique ordinal $h(p)$ such that $\langle p\rangle = H_{h(p)}$.

We will prove that $\mathcal{P}$ preserves $\aleph_1$, that is, that there is no bijection $f : \omega \to \omega_1^{\mathbf{M}}$ in $\mathbf{M}[\mathbf{G}]$. So suppose by way of contradiction that $\mathbf{M}[G]$ contains such a function $f : \omega \to \omega_1^{\mathbf{M}}$. Then let τ be a $\mathcal{P}$-name such that $\tau_G = f$. Then by the Fundamental Theorem of Forcing, there must be some $p \in \mathcal{P}$ (hence some $p \in \mathcal{P}'$) which forces this. That is, there is some $p \in \mathcal{P}'$ such that

$$p \Vdash \text{“}\tau \text{ is a bijection from } \check{\omega} \text{ to } \widecheck{\omega_1^{\mathbf{M}}}.\text{”} \tag{4.1}$$

Within $\mathbf{M}$, we will define an ascending sequence $\{A_\alpha : \alpha < \aleph_1\}$ of subsets $A_\alpha \subseteq \mathcal{P}' \times \omega \times {}^{<\omega}\omega_1$ as follows: if $(q, n, g) \in A_\alpha$ then $\mathrm{dom}(g) = n = \{0, 1, 2, \ldots, n-1\}$ and $q \Vdash$ "τ is a bijection from $\check{\omega}$ to $\widecheck{\omega_1^{\mathbf{M}}}$ with $\tau \restriction \check{n} = \check{g}$.". In other words, this sequence consists of partial functions which approximate this bijection, and elements of $\mathcal{P}'$ which force this approximation. We will also define, for each A_α, a (countable) ordinal $h_\alpha = \sup\{h(q) : (q, n, g) \in A_\alpha\}$. We construct the A_α's inductively as follows.

First define $A_0 = \{(p, 0, \varnothing)\}$ where p is the element of $\mathcal{P}'$ given in (4.1). For the successor step, let A_α be defined with $h_\alpha < \aleph_1$, and suppose that for each $q \Vdash$ "τ is a bijection from $\check{\omega}$ to $\widetilde{\omega_1{}^{\mathbf{M}}}$ with $\tau \restriction \check{n} = \check{g}$." Now, within $\mathbf{M}[G]$, τ evaluates to such a bijection, i.e., $\tau_G : \omega \to \omega_1{}^{\mathbf{M}}$. Because τ_G is defined in $\mathbf{M}[G]$, $\tau_G(n) \in \omega_1{}^{\mathbf{M}}$. Define a function $g' \in {}^{n+1}\omega_1$ by $g' = g \cup \{\langle n, \tau_G(n)\rangle\}$. Evidently, both g' and g are finite functions such that $g \subseteq g'$. Therefore $g' \in \mathbf{M}$ and $\tau_G \restriction \widetilde{n+1} = \check{g}'$. By the Fundamental Theorem of Forcing there exists some $r \in \mathcal{P}$ such that

$$r \Vdash \text{"}\tau \text{ is a bijection from } \check{\omega} \text{ to } \widetilde{\omega_1{}^{\mathbf{M}}} \text{ with } \tau \restriction \widetilde{n+1} = \check{g}'.\text{"}$$

Because q and r are contained in the filter G, choose $s \in G$ such that $s \leqslant q, r$ (by taking a common extension of q and r if necessary), and let $q' \leqslant s$ with $q' \in \mathcal{P}'$ and $h(q') > h_\alpha$. Then define

$$A_{\alpha+1} = A_\alpha \cup \{(q', n+1, g') : (q, n, g) \in A_\alpha\}, \tag{4.2}$$

completing the successor step. Note that a fixed choice for $(q', n+1, g')$ can be obtained by choosing it to be minimal with respect to some fixed well-ordering of $\mathcal{P}' \times \omega \times {}^{<\omega}\omega_1$ in $\mathbf{M}$. Note also that this procedure is well-defined and decidable within $\mathbf{M}$, as we can replace $\Vdash$ by $\Vdash^*$.

Finally, for α a limit ordinal define $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$.

From the construction, it is clear that for $\beta < \alpha < \aleph_1$ we have that $A_\beta \subset A_\alpha$. Furthermore, for each $\alpha < \aleph_1$, A_α is countable. This can be seen by first noting that $|A_0| < \aleph_1$, and if $|A_\alpha| < \aleph_1$ then the successor step construction produces some $A_{\alpha+1}$ which is also countable. Finally, if α is a countable limit ordinal, it may be written as $\alpha = \bigcup_{i\in\omega} \beta_i$ for ordinals $\beta_i < \alpha$, hence $A_\alpha = \bigcup_{i\in\omega} A_{\beta_i}$ is a countable union of countable sets. So A_α is countable for any ordinal $\alpha < \aleph_1$.

As A_α is countable, by the regularity of $\aleph_1$, we have that $h_\alpha < \aleph_1$ is countable for each α. Note that $\beta < \alpha \to h_\beta < h_\alpha$, and for α a limit ordinal, $h_\alpha = \sup_{\beta<\alpha} h_\beta$.

Then the set $C = \{h_\alpha : \alpha < \aleph_1\}$ is a club in $\aleph_1$, as is the set $C^* = \{h_\alpha : \alpha < \aleph_1, \alpha \text{ is a limit ordinal}\}$, which is also a club in $\aleph_1$.

Since $\aleph_1 - E$ is a stationary set, it has nontrivial intersection with C^* so choose some $h_{\alpha^*} \in (\aleph_1 - E) \cap C^*$. Since α^* is a countable limit ordinal, $\mathrm{cf}(\alpha^*) = \omega$, so there is a set of strictly increasing ordinals $\{\alpha_n : n \in \omega\}$ with supremum α^*. Thus

$$h_{\alpha^*} = \sup_{\beta<\alpha^*} h_\beta = \sup_{n\in\omega} h_{\alpha_n}.$$

We will now construct within $\mathbf{M}$ a sequence $\{(q_n, n, g_n) : n \in \omega\}$ such that $(q_{n+1}, n+1, g_{n+1}) \in A_{\alpha_{n+1}} - A_{\alpha_n}$. First define $(q_0, 0, g_0) = (p, 0, \varnothing) \in A_0 \subseteq A_{\alpha_0}$. Then, assume that $(q_n, n, g_n) \in A_{\alpha_n}$ is given, and let $(q_{n+1}, n+1, g_{n+1}) = (q'_n, n+1, g'_n) \in A_{\alpha_n+1} - A_{\alpha_n} \subseteq A_{\alpha_{n+1}} - A_{\alpha_n}$ as in (4.2).

The construction of the A_α's above assures that $q_{n+1} = q'_n \leqslant q_n$ and $g_n \subseteq g'_n = g_{n+1}$. Then $g = \bigcup_{n\in\omega} g_n$ defines a function from ω to ω_1 in $\mathbf{M}$. Finally, by construction, $h_{\alpha_n} < h(q_{n+1}) = h(q'_n) \leqslant h_{\alpha_n+1} \leqslant h_{\alpha_{n+1}}$. Thus $\sup_{n\in\omega} h(q_n) = \sup_{n\in\omega} h_{\alpha_n} = h_{\alpha^*}$.

Now note that $q_n \in \mathcal{P}'$ is a basis for $H_{h(q_n)}$. Then $q^* := \bigcup_{n\in\omega} q_n$ is a basis for $H_{\sup_{n\in\omega} h(q_n)} = H_{h_{\alpha^*}}$. Since $h_{\alpha^*} \in \aleph_1 - E$, $H/\langle q^*\rangle = H/H_{h_{\alpha^*}}$ is $\aleph_1$-free. Thus $q^* \in \mathcal{P}'$ with $h(q^*) = h_{\alpha^*}$ and $q^* \leqslant q_n$. Thus

$$q^* \Vdash \text{“}\tau \text{ is a bijection from } \check{\omega} \text{ to } \widetilde{\omega_1{}^{\mathbf{M}}} \text{ with } \tau = \check{g}.\text{”}$$

But this is a contradiction, as it implies that we have a bijection $g \in \mathbf{M}$. $\square$

4.2.10 Remark. *In addition to preserving $\aleph_1$, forcing with $\mathcal{P}$ also preserves reals. To see this, in the above proof we can replace “τ is a bijection from $\check{\omega}$ to $\widetilde{\omega_1{}^{\mathbf{M}}}$” with*

"τ is a function from $\check{\omega}$ to $\check{2}$ with $\tau \notin \widetilde{({}^{\omega}2)^{\mathbf{M}}}$," that is τ is a new real number not in $\mathbf{M}$.

CHAPTER FIVE

Further Work

Further work in this area could branch-out in several different directions. We would like to enable algebraists to easily apply the forcings described here, a goal which would be benefited by the development of an algebraic characterization of turbid (or more-pertinently, non-turbid) groups which does not require direct reference to the Γ-invariant, which can be quite difficult to work with in practice. One might also like to find the precise conditions under which these or similar forcings can be applied to $\aleph_1$-free groups of larger cardinality. We can also use these forcings to produce powerful new forcings through iterated forcing, which could be applied to the setting of homological algebra. Branching out even further, we can take inspiration from the forcing techniques developed here to explore new forcings which illuminate the notion of "almost isomorphism," or which could aid algebraists in the construction of objects with particular prescribed properties. Finally, it is our hope that the creation and analysis of different forcings related to $\aleph_1$-free groups will lead to the development of new predictive principles or axioms which can be used in algebraic constructions and in more easily generating independence and consistency results within algebra.

Beyond these proposed expansions upon the specific forcings described in this dissertation, the extension of these results via the method of iterated forcing could prove a powerful tool for applying these types of forcings to various questions in abelian group and ring theory and homological algebra. Iterated forcing is a method of applying multiple forcings simultaneously in order to generate a model extension in which all of the propositions forced by the individual forcings hold. Thus we could use iterated forcing to force multiple $\aleph_1$-free groups to be free simultaneously. It is important to note that it is not the case that any two non-turbid $\aleph_1$-free groups of cardinality $\aleph_1$ can be simultaneously forced to be free. As an example, suppose that H_1 and H_2 are non-turbid groups of cardinality $\aleph_1$ with $\Gamma(H_1) = [E_1]$, $\Gamma(H_2) = [E_2]$, and $(\aleph_1 - E_1) \cap (\aleph_1 - E_2) = \varnothing$. Then if we force a club into $\aleph_1 - E_1$ in order to make H_1 free, we cannot force a club into $\aleph_1 - E_2$ in order to make H_2 free, as the intersection of clubs cannot be empty. So the question of when precisely these iterated forcings can be achieved with cardinal preservation would again be a relevant one.

Such iterated forcings would open the door to applying these methods to homological algebra. As a simple example, suppose A, B, and C are $\aleph_1$-free groups in some countable transitive ground model $\mathbf{M}$, and suppose

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence. It is well-known that the functor $\mathrm{Hom}(-, D)$, where D is any abelian group, is exact on the category of free abelian groups, that is, it preserves short exact sequences. So using the iterated forcing technique above, we could find a transitive model extension in which A, B, and C are free. In this model extension then,

$$0 \to \mathrm{Hom}(A, D) \xrightarrow{f^*} \mathrm{Hom}(B, D) \xrightarrow{g^*} \mathrm{Hom}(C, D) \to 0$$

is short exact for any abelian group D. Indeed, with class forcing techniques, we could even perform iterated forcing on entire classes of $\aleph_1$-free groups (although for technical reasons, we cannot for instance, force the entire class of non-turbid $\aleph_1$-free groups of size $\aleph_1$ to become free without cardinal collapse). This could enable further investigations into independence and consistency results in homological algebra.

Iterated forcing could also be used to force isomorphisms between particular $\aleph_1$-free groups, by forcing both groups to become free, and then forcing their cardinalities to be equal. However, a more straightforward way to force an isomorphism between $\aleph_1$-free groups might be an approach by partial isomorphism similar to the forcings defined by partial bases described here. As with our partial basis forcing, this approach would be preferred over an approach which relies on cardinal collapse because it would constitute a more "minimally-invasive" forcing. It would also help illuminate the notion of partial isomorphism. Partial isomorphisms are a way to describe the degree of similarity between two algebraic structures, in which the notion of a global isomorphism is replaced by local isomorphisms between substructures which are compatible with each other and can be extended. This approach to describing algebraic similarity can be mirrored by the set-theoretic approach of considering "potential isomorphism," in which two algebraic objects are said to be "almost isomorphic" if they are isomorphic in some generic extension of the set-theoretic universe. These two approaches, that of partial isomorphism and potential isomorphism, are in fact equivalent, as shown by Barwise [3]. Forcing provides the ideal setting through which to bridge these two perspectives, and further investigation into forcings which make two $\aleph_1$-free groups isomorphic would help illuminate the mechanics behind this equivalence and enable algebraists to more easily explore and construct partial isomorphisms. The locally free structure of $\aleph_1$-free groups make them ideal targets for such investigations into partial isomorphisms, and indeed one might even expect some natural (although perhaps more limited) generalizations to the realm of κ-free groups and modules in the spirit of the generalizations described at the end of Chapter 4.

Another class of $\aleph_1$-free forcings that would be of interest to investigate is the class of forcings which force an $\aleph_1$-free group to have a particular endomorphism ring, or at least, an endomorphism ring with particular properties. Owing to the ring realization property of $\aleph_1$-free groups (that is, any ring with free additive structure can be realized as the endomorphism ring of some $\aleph_1$-free group), such forcings could provide a powerful tool for algebraic constructions.

Ultimately, as the family of $\aleph_1$-free forcings is further developed and taxonomized, we expect to see patterns and commonalities arising between the generic extensions which they produce. An analysis of the relations between these forcing extensions may, in the long-run, point to the development of new predictive principles or axioms which can be easily used by algebraists in constructions, or in producing new consistency and independence results in algebra.

APPENDIX

APPENDIX

Comparing $\mathcal{P}_1$ and $\mathcal{P}_2$

As we have seen above, while $\mathcal{P}_2$ is a subset quite naturally defined out of $\mathcal{P}_1$, these two posets have very distinct properties with regards to the properties exhibited by their forcing extensions. For example, the union of a $\mathcal{P}_2$-generic filter produces a basis for H, while the same is never true of a $\mathcal{P}_1$-generic filter. In addition, $\mathcal{P}_1$ preserves cardinals less than or equal to the cardinality of H, while $\mathcal{P}_2$ does not always preserve these cardinals.

Intuitively, there are branches in $\mathcal{P}_1$ which "dead-end," that is, which cannot be extended to produce a full basis of H. Recall that these "dead ends" cannot be avoided by a generic filter because they form a dense set, cf. Lemma 3.2.8.

In general, $\mathcal{P}_2$ involves a much stronger condition than $\mathcal{P}_1$. This is further illustrated by the following lemma which states that if an element in $\mathcal{P}_1$ is a "dead end," we cannot simply remove finitely many members of it in order to arrive at an element in $\mathcal{P}_2$.

A.0.1 Lemma. *Let $p \in \mathcal{P}_1$ and $x \in p$. If $p \in \mathcal{P}_1 - \mathcal{P}_2$, then $p - \{x\} \in \mathcal{P}_1 - \mathcal{P}_2$.*

Proof. We will prove the converse, which is that if $p \in \mathcal{P}_1$ and $x \notin p$ with $p \cup \{x\} \in \mathcal{P}_1$, then $p \in \mathcal{P}_2$ implies $p \cup \{x\} \in \mathcal{P}_2$.

Suppose $p \in \mathcal{P}_2$ with $x \notin p$ and $p \cup \{x\} \in \mathcal{P}_1$. For $p \in \mathcal{P}_1$, we have

$$\begin{aligned}
p \in \mathcal{P}_2 \quad &\longleftrightarrow \quad H/\langle p\rangle \text{ is } \aleph_1\text{-free in } \mathbf{M} \\
&\longleftrightarrow \quad H/\langle p\rangle \text{ is free in } \mathbf{V} \\
&\longleftrightarrow \quad \langle p\rangle \text{ is a direct summand of } H \text{ in } \mathbf{V} \\
&\longleftrightarrow \quad H \cong \langle p\rangle \oplus C \text{ for some C in } \mathbf{V} \\
&\longleftrightarrow \quad x \in H \text{ can be written in } \mathbf{V} \text{ as } x = x_1 \oplus x_2 \text{ for some } x_1 \in \langle p\rangle, x_2 \in C.
\end{aligned}$$

We claim that x_2 is pure in C, that is, that $\langle x_2\rangle$ is a pure subgroup of C. Assume by contradiction that x_2 is not pure in C. Then $x_2 = ny$ for some $n > 1, y \in C$. So $x = x_1 + ny$.

Consider $y + \langle p \cup \{x\}\rangle \in H/\langle p \cup \{x\}\rangle$. Then

$n(y + \langle p \cup \{x\}\rangle) = ny + \langle p \cup \{x\}\rangle = x - x_1 + \langle p \cup \{x\}\rangle = x + \langle p \cup \{x\}\rangle = \langle p \cup \{x\}\rangle$. However, $y + \langle p \cup \{x\}\rangle \neq \langle p \cup \{x\}\rangle$. Thus $H/\langle p \cup \{x\}\rangle$ fails to be torsion-free, so $p \cup \{x\} \notin \mathcal{P}_1$, which is a contradiction.

So x_2 is pure in C, and thus can be extended to a basis of C in $\mathbf{V}$ as C is free. To see this, recall that C is free in $\mathbf{V}$, as $C \cong H/\langle p\rangle$. So we can write $C = C_1 \oplus C_2$ with C_1 of finite rank and $x_2 \in C_1$. As x_2 is pure in C, it is pure in C_1. So $C_1/\langle x_2\rangle$ is torsion-free and finitely generated. Thus $C_1/\langle x_2\rangle$ is free, so $\langle x_2\rangle$ is a direct summand of C_1, which is a direct summand of C.

So we can write $H = \langle p\rangle \oplus C = \langle p\rangle \oplus \langle x_2\rangle \oplus C' = \langle p \cup \{x\}\rangle \oplus C'$ in $\mathbf{V}$, for some (free) subgroup C' of C.

Thus, as $H/\langle p \cup \{x\}\rangle$ is free in $\mathbf{V}$, $H/\langle p \cup \{x\}\rangle$ is $\aleph_1$-free in $\mathbf{M}$. Thus $p \cup \{x\} \in \mathcal{P}_2$. $\square$

As $\mathcal{P}_2$ is a subset of $\mathcal{P}_1$, and these posets define the model extensions in question, we might wish to know how these model extensions relate with each other. Specifically, if $\mathbf{M}_1$ is the model extension generated by $\mathcal{P}_1$, and $\mathbf{M}_2$ is the model extension generated by $\mathcal{P}_2$, is it necessarily the case that $\mathbf{M}_2 \subseteq \mathbf{M}_1$? In general, the answer is no. However, we can guarantee that $\mathbf{M}_2 \subseteq \mathbf{M}_1$ if there is an embedding of $\mathcal{P}_2$ into $\mathcal{P}_1$ which is "complete" (see definition below).

Thus, if we can find a complete embedding of $\mathcal{P}_2$ into $\mathcal{P}_1$, we can prove that $\mathcal{P}_2$ preserves cardinalities $\leqslant \lambda$. For if $\mathcal{P}_2$ embeds completely into $\mathcal{P}_1$, then $\mathbf{M}_2 \subseteq \mathbf{M}_1$. And as $\mathcal{P}_1$ is λ-closed, $\mathbf{M}_1$ contains no bijections collapsing cardinalities $\leqslant \lambda$, and thus $\mathbf{M}_2$ cannot contain any such bijections and must therefore also preserve cardinalities $\leqslant \lambda$.

Clearly, for turbid groups H, a complete embedding of $\mathcal{P}_2$ into $\mathcal{P}_1$ is impossible, as we know that no cardinal preserving forcing which makes H free exists, see Theorem 4.2.3. However, a complete embedding may still be possible for non-turbid $\aleph_1$-free groups H. We may ask the even more specific question of if the canonical embedding of $\mathcal{P}_2$ into $\mathcal{P}_1$ can be complete. As we will show below, this canonical embedding is not complete for any $\aleph_1$-free group H.

We first give the formal definition of a complete embedding. For further discussion of complete embeddings, see [10, Chapter IV.7].

A.0.2 Definition. *Let P and Q be partial orders and $i : P \to Q$ a function from P to Q. i is a* complete embedding *if and only if it satisfies the following three conditions:*

1. $\forall p, p' \in P \, (p' \leqslant p \to i(p') \leqslant i(p))$

2. $\forall p, p' \in P\,(p \perp p' \leftrightarrow i(p) \perp i(p'))$

3. $\forall q \in Q\, \exists p \in P\, \forall p' \in P\,(p' \leqslant p \rightarrow (i(p')$ *and* q *are compatible in* $Q))$

If p *is as in Condition* 3*, then we call* p *a* reduction *of* q *to* P*.*

While Condition 1 of the above definition trivially holds for the canonical embedding function $i : \mathcal{P}_2 \to \mathcal{P}_1$, we will now show that Condition 3 fails.

A.0.3 Lemma. *The canonical embedding* $\mathcal{P}_2 \subseteq \mathcal{P}_1$ *is not complete.*

Proof. Let $T \in \mathcal{P}_1 - \mathcal{P}_2$. Assume by way of contradiction that there is a reduction S of T to $\mathcal{P}_2$. As $S \in \mathcal{P}_2$, $H/\langle S\rangle$ is $\aleph_1$-free. Thus $H/\langle S\rangle$ is free in $\mathbf{V}$, and thus S extends to a basis B of H in $\mathbf{V}$.

By Condition 3, S and T are comparable in $\mathcal{P}_1$, so $S \cup T$ is contained in some $S' \in \mathcal{P}_1$, and thus $S \cup T$ is linearly independent. We claim that $T \subseteq S$. To see this, let $t \in T$. Then we can write t in terms of B, say

$$t = \sum_{b \in B} n_b b = \sum_{b \in S} n_b b + \sum_{b \in B - S} n_b b.$$

If $\sum_{b \in B - S} n_b b = 0$, then $t \in S$, as desired. So assume $\sum_{b \in B - S} n_b b \neq 0$. Then there exists some $c \in \langle B - S\rangle$ such that c is pure in H with $\sum_{b \in B - S} n_b b = nc$ (letting $n = \gcd(n_b : b \in B - S)$). We can extend $\{c\}$ to a basis of $\langle B - S\rangle$. To see this, note that we can write c in terms of some finite subset B' of $B - S$, and so $\langle c\rangle$ is a pure subgroup of the finite rank free summand $\langle B'\rangle$ of $\langle B - S\rangle$. Thus $\langle B'\rangle/\langle c\rangle$ is torsion-free by the purity of $\langle c\rangle$, and finitely generated. Therefore $\langle B'\rangle/\langle c\rangle$ is free, and $\langle c\rangle \sqsubseteq \langle B'\rangle$. Then $\{c\}$ can be extended to a basis $B'' \cup \{c\}$ of $\langle B'\rangle$, and so $B'' \cup \{c\} \cup (B - S - B')$ is a basis of $\langle B - S\rangle$. So we can extend $S \cup \{c\}$ to a basis of H in $\mathbf{V}$ and thus $S \cup \{c\} \in \mathcal{P}_2$.

Now by Condition 3, $S \cup \{c\}$ and T are compatible in $\mathcal{P}_1$, and thus $S \cup \{c\} \cup T$ is linearly independent, with $t = \sum_{b \in S} n_b b + nc$. So $t \in S \cup \{c\}$. But note that we can replicate the argument above with $-c$ in place of c, which would imply that $t \in S \cup \{-c\}$. And thus, we must have that $t \in S$.

So $T \subseteq S$. But this contradicts that $T \in \mathcal{P}_1 - \mathcal{P}_2$, that is, that T does not extend to a basis of H in $\mathbf{V}$. For as $S \in \mathcal{P}_2$, S extends to a basis of H in $\mathbf{V}$, but $T \subseteq S$. Thus the canonical embedding $\mathcal{P}_2 \subseteq \mathcal{P}_1$ is not complete. $\square$

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