The GF(2) General Linear Group for Dimensions 2, 3, 4, and 5

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1 Introduction

This report contains some data about the General Linear Groups of GF(2) for dimensions 2, 3, 4, and 5. These groups are groups of $n \times n$ matrices over GF(2), the integers modulo 2. The General Linear Group of order n is the set of all non-singular $n \times n$ matrices over some field, in this case, GF(2). There is also the Special Linear Group, which is the set of all $n \times n$ matrices of determinant 1, but since the only possible determinants for GF(2) matrices are 0, and 1, and since all non-singular matrices must have a non-zero determinant, the General Linear Groups and the Special Linear Groups are the same.

The observations we make here will be supported using short programs that use the GF2Matrices package, which is available as a Baylor Computer Science Technical report. (See http://beardocs.baylor.edu).

2 Dimension 2

The two-dimensional general linear group is the simplest. There are only 16 2×2 matrices over GF(2), six of which are non-singular. These are as follows.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The General Linear group is isomorphic to S_3, the symmetric group of order 3. The three matrices following the identity matrix are of order two, and the last two are of order 3. The three subgroups of order two are conjugate to one another.

If we make the following identification:

1—(0,1)
2—(1,0)
3—(1,1)

Then the matrices given above correspond to the following permutations given in cycle notation.

I,(1,2),(1,3),(2,3),(1,3,2),(1,2,3)

This group is irreducible (it has no conjugates other than itself) and is one of the two irreducible representations of S_3 and S_4 over GF(2). (The other is (1), the trivial representation.)

The construction of this group is somewhat interesting and we will return to it in a later section.

Note that the standard representation of S_2 is reducible, because

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Is conjugate to:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Which is reducible to (1).

3 Dimension 3

3.1 The Matrices

The three-dimensional general linear group is significantly more complex than the two dimensional group. It is of size 168, and has the following spectrum. We use the term spectrum to denote the list of matrix orders from 1 to $2^n - 1$ along with the number of matrices of each order.

1 -- 1
2 -- 21
3 -- 56
4 -- 42
5 -- 0
6 -- 0
7 -- 48

The following GF2Matrices program was used to generate this list.

```
#include <iostream>
#include "GF2Matrices.h"

using namespace std;

int main()
{
	int Spectrum[8] = {0,0,0,0, 0,0,0,0 };
	GenerateGLG3 G;
	for (int i=G.First() ; i ; i=G.Next())
	{
		Spectrum[G.Order()]++;
	}
	for (int i=1 ; i<8 ; i++)
	{
		cout<<i<<" -- "<<Spectrum[i]<<endl;
	}
}
```

All 21 order 2 matrices are conjugate to one another. Each of them represents a permutation over the set {1,2,3,4,5,6,7} of the form (1,2) (5,6). Each has the eigenpolynomial (characteristic polynomial) of $x^3+x^2+x+1=(x+1)^3$

By the same token, all 56 order-3 matrices are conjugate to one another. They each represent a permutation of the form (1,2,4) (3,6,5), and have an eigenpolynomial of $x^3+1=(x+1)(x^2+x+1)$. Note that (x^2+x+1) is irreducible.

The 42 order four matrices are also all conjugate to one another. They represent permutations of the form (1,2,4,7) (3,6), and all have the eigenpolynomial $x^3+x^2+x+1=(x+1)^3$.

The 48 order 7 matrices form two conjugacy classes with 24 matrices each. All of these matrices represent 7-cycles, but the two different classes of matrices have different eigenpolynomials. These are x^3+x^2+1 and x^3+x+1, both of which are irreducible.

The following GF2Matrices program can be used to determine the number of conjugate classes of a particular order of matrices, and show the permutation structure and the eigenpolynomial of each class.

```
#include <iostream>
#include "GF2Matrices.h"

using namespace std;

int main()
{
	GenerateGLG3 M;
	Group3 Bucket, Gems;
	for (int i=M.First(7) ; i ; i=M.Next(7))
	{
		if (!Bucket.Contains(M))
		{
			Gems.Add(M);
			GenerateGLG3 W;
			for (int j=W.First() ; j ; j=W.Next())
			{
				Matrix3 Z;
```

```
                        Z = W.Invert();
                        Matrix3 K = W * M * Z;
                        if (!Bucket.Contains(K))
                        {
                              Bucket.Add(K);
                        }
                  }
            }
      }
      for (Matrix3 * Temp = Gems.GetFirstMatrix() ; Temp ;
                              Temp=Gems.GetNextMatrix())
      {
            cout<<Temp->GetPermutation()<<endl;
            cout<<Temp->GetEigenPoly()<<endl;
            cout<<Temp->SerializeNL()<<endl;
      }
}
```

3.2 The Groups

There are 21 S_2 isomorphs, one for each of the order-2 matrices. This class is reducible to the trivial representation (1). There are 28 S_3 isomorphs, all of which are conjugate to the standard representation of S_3. This class is reducible to the 2×2 general linear group.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

In addition to these obvious groups, there are two conjugacy classes of S_4 isomorphs. These are constructed by adding rows and columns of 1's to the standard representation.

The standard representation is constructed by starting with the following three rows, and constructing every possible matrix from them: 001, 010, and 100. We could do the same with three different columns containing a single 1. There are obviously 3! such matrices.

If we add a fourth row, 111 (or a column of all 1's) we get a representation of S_4. There are 4 ways to select 3 distinct rows from 001, 010, 100, and 111, but order is important, so from each selection of 3 rows we can create 3! matrices for a total of 4! matrices. Each of the resultant matrices is a permutation of the vector set {(0,0,1), (0,1,0), (1,0,0), (1,1,1) }. The following is the set of matrices created in this fashion.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&1&1\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&1&1\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&1&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\1&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\1&1&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&1&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&1&1\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\1&1&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\1&1&1\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&1&1\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\1&1&1\end{pmatrix}$$

The following is the group created by adding an extra column of 1's

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\1&0&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}1&1&0\\1&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&1\\1&0&0\\1&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0\\0&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}0&1&1\\0&1&0\\1&1&0\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&1&1\\1&1&0\end{pmatrix}\begin{pmatrix}0&1&1\\1&1&0\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}0&1&1\\1&0&1\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}0&1&1\\0&0&1\\1&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&1\\0&1&1\end{pmatrix}$$

It is not necessarily obvious that these sets of matrices are closed, but we will prove that this in a later section, and we will also prove the permutation properties of these matrices.

The two groups given above are not conjugate to one another, but they are transposes of one another. Each has seven conjugates, including itself. Both of these groups are reducible to the 2×2 general linear group. (That is, they have a reducible conjugate.)

The GF2Matrices program given below can be used to find all conjugate classes for a particular set of matrices. This particular program searches for all conjugacy classes of matrices isomorphic to S_3. Because S_3 is generated by the two permutations (1,2,3) and (1,2), any group isomorphic to S_3 must be generated by an order 2 and an order 3 matrix. The program enumerates all such pairs, and computes the smallest group containing the pair of matrices. If this group is of order 6, then all of its conjugates are computed and saved for future reference. Any time a new order 6 group is found, it is compared against all groups and conjugates found so far to determine if this is a member of a new class. If so, the generators of the group are saved, and the group and all of its conjugates are saved for future reference.

This procedure will generate all groups of order 6, because every group of order 6 must have a subgroup of order 3 and a subgroup of order 2. These groups must be cyclic, and must consist of order 2 and order 3 elements plus the identity. Unfortunately, the generated group may not be isomorphic to S_3. It may be the cyclic group of order 6. This is not a possibility for 3x3 matrices because there are no 3×3 matrices of order 6, however larger matrices require more care.

The following program can be modified slightly to look for conjugacy classes of other sizes.

```
#include <iostream>
#include "GF2Matrices.h"

using namespace std;

int main()
{
	Group3 O2,O3;
	GroupList3 ClassGenerators,Bucket;

	O2.GenerateOrder(2);
	O3.GenerateOrder(3);
	for (Matrix3 * M2 = O2.GetFirstMatrix() ; M2 ; M2 = O2.GetNextMatrix())
	{
		for (Matrix3 * M3 = O3.GetFirstMatrix() ; M3 ;
								M3 = O3.GetNextMatrix())
		{
			Group3 G;
			G.Clear();
			G.Add(*M2);
			G.Add(*M3);
			if (G.Close(6))
			{
				G.Sort();
				if (!Bucket.Contains(G))
				{
					Group3 H;
					H.Add(*M2);
					H.Add(*M3);
					ClassGenerators.Add(H);
					GenerateGLG3 T;
					for (int i = T.First() ; i ; i=T.Next())
					{
						Group3 K = G.Conjugate(T);
						K.Sort();
						if (!Bucket.Contains(K))
						{
							Bucket.Add(K);
						}
					}
				}
			}
```

```
            }
        }
        for (Group3 *TG=ClassGenerators.GetFirstGroup() ; TG ;
                                TG=ClassGenerators.GetNextGroup())
        {
            char *x = TG->SerializeNL();
            cout<<x<<endl;
            delete [] x;
        }
}
```

The following GF2Matrices program can be used to determine the reducibilities of a particular group.

```
#include <iostream>
#include "GF2Matrices.h"

using namespace std;

int main()
{
        GroupList3 GL;

        GL.GenerateS4HConj();
        for (Group3 * TG=GL.GetFirstGroup() ; TG ; TG=GL.GetNextGroup())
        {
            if (TG->Reducible1())
            {
                char * x;
                cout<<"Reducible 1\n";
                x = TG->SerializeNL();
                cout<<x;
                delete [] x;
                Group2 G;
                G = TG->GetLeftComponent();
                x = G.SerializeNL();
                cout<<x;
                delete [] x;
            }
            else if (TG->Reducible2())
            {
                char * x;
                cout<<"Reducible 2\n";
                x = TG->SerializeNL();
                cout<<x;
                delete [] x;
                Group2 G;
                G = TG->GetLeftComponent();
                x = G.SerializeNL();
                cout<<x;
                delete [] x;
            }
        }
}
```

The orbits of the groups in each conjugacy class will have the same structure. That is to say that the number and size of the vector sets will be the same. In the following, we use the term "orbits" to denote the orbits created by pre-multiplying a matrix with a row-vector and the term "v-orbits" to denote the orbits created by post-multiplying a matrix with a column vector. In both cases, the orbits are displayed as row-vectors.

To illustrate we list the orbits and v-orbits for the groups generated by the GenerateS2, GenerateS3, GenerateS4H, and GenerateS4V functions of the GroupList3 class from the GF2Matrices package.

GenerateS2

```
000
001
010 100                       VOrbits are the same
011 101
110
111
```

GenerateS3

```
000
001 010 100                   VOrbits are the same
011 110 101
111
```

GenerateS4H **(v-orbits)**

```
000                           000
001 010 100 111               001 010 100 110 101 011
011 110 101                   111
```

GenerateS4V **(v-orbits)**

```
000                           000
001 010 100 110 101 011       001 010 100 111
111                           011 110 101
```

4 Dimension 4

4.1 The Matrices

The 4×4 general linear group has the following spectrum.

1 -- 1
2 -- 315
3 -- 1232
4 -- 3780
5 -- 1344
6 -- 5040
7 -- 5760
8 -- 0
9 -- 0
10 -- 0

11 -- 0
12 -- 0
13 -- 0
14 -- 0
15 -- 2688

There are two classes of order 2 matrices. The following is an instance of each, along with the permutation form of the matrix, its eigenpolynomial and the size of the conjugacy class.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} (1,8)\ (2,4)\ (3,12)\ (5,10)\ (7,14)\ (11,13) \qquad x^4+1=(x+1)^4 \qquad \text{Size}=210$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} (1,8)\ (3,10)\ (5,12)\ (7,14) \qquad x^4+1=(x+1)^4 \qquad \text{Size}=105$$

The following GF2Matrices program can be used to obtain the size of the conjugacy classes.

```
#include <iostream>
#include "GF2Matrices.h"

using namespace std;

int main()
{
	Matrix4 M(0,0,0,1, 0,1,0,0, 0,0,1,0, 1,0,0,0);
	Group4 Cj;
	GenerateGLG4 G;
	for (int i=G.First() ; i ; i=G.Next())
	{
		Matrix4 Gi;
		Gi = G.Invert();
		Matrix4 K = G * M * Gi;
		if (!Cj.Contains(K))
		{
			Cj.Add(K);
		}
	}
	cout<<Cj.Order()<<endl;
}
```

There are two conjugacy classes of order-3 matrices. The following is an instance of each, along with the permutation form, eigenpolynomial and conjugacy class size.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} (1,9,8)\ (2,6,4)\ (3,15,12)\ (5,11,14)\ (7,13,10) \qquad x^4 + x^2 + 1 \qquad \text{Size=112}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} (1,14,8)\ (2,11,4)\ (3,5,12)\ (6,9,15) \qquad x^4 + x^3 + x + 1 \qquad \text{Size=1120}$$

There are two conjugacy classes of order-4 matrices. The following is an instance of each, along with the permutation form, eigenpolynomial and conjugacy class size.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} (1,10,5,8)\ (2,4)\ (3,14,7,12)\ (9,11,15,13) \quad x^4 + 1 = (x+1)^4 \qquad \text{Size=2520}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} (1,14,7,8)\ (2,4)\ (3,10,5,12)\ (9,15)\ x^4 + 1 = (x+1)^4 \qquad \text{Size=1260}$$

There is one conjugacy classe of order-5 matrices. The following is an instance of it, along with the permutation form, eigenpolynomial and conjugacy class size.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} (1,11,15,13,8)\ (2,5,9,10,4)\ (3,14,6,7,12) \quad x^4 + x^3 + x^2 + x + 1 \qquad \text{Size=1344}$$

There are two conjugacy classes of order-6 matrices. The following is an instance of each, along with the permutation form, eigenpolynomial and conjugacy class size.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} (1,9,8)\ (2,4)\ (3,13,10,5,11,12)\ (7,15,14) \quad x^4 + x^3 + x + 1 \ \text{Size=3360}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} (1,10,4,2,5,8)\ (3,15,12)\ (6,7,13,9,11,14) \quad x^4+x^2+1 \quad \text{Size}=1680$$

There are two conjugacy classes of order-7 matrices. The following is an instance of each, along with the permutation form, eigenpolynomial and conjugacy class size.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} (1,14,6,7,9,15,8)\ (2,5,12,3,11,10,4) \quad x^4+x^2+x+1 \quad \text{Size}=2880$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} (1,15,9,14,6,7,8)\ (2,5,13,12,3,10,4) \quad x^4+x^3+x^2+1 \quad \text{Size}=2880$$

There are two conjugacy classes of order-15 matrices. The following is an instance of each, along with the permutation form, eigenpolynomial and conjugacy class size.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} (1,12,3,9,13,15,10,4,2,5,14,6,7,11,8) \quad x^4+x+1 \quad \text{Size}=1344$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} (1,13,14,6,7,10,4,2,5,15,11,9,12,3,8) \quad x^4+x^3+1 \quad \text{Size}=1344$$

4.2 The Groups

The 4×4 groups exhibit a surprisingly rich structure. There are two classes of S_2 isomorphs, three classes of S_3 isomorphs, nine classes of S_4 isomorphs, and two classes of S_5 isomorphs.

4.2.1 The S2 classes

The two S_2 classes are generated by the two classes of 2×2 matrices. The class created by embedding the standard representation in the upper left corner has the permutation structure {I, (4,8) (5,9) (6,10) (7,11)}, while the other class has the permutation structure {I, (1,8) (2,4) (3,12) (5,10) (7,14) (11,13)}.

The first class has the following orbit structure. The v-orbits are the same.

0000
0001
0010
0011
0100 1000
0101 1001
0110 1010
0111 1011
1100
1101
1110
1111

The second class has the following orbit structure. Again, the v-orbits are the same.

0000
0001 1000
0010 0100
0011 1100
0101 1010
0110
0111 1110
1001
1011 1101
1111

Both classes of groups are reducible to the trivial representation (1). The first class is directly reducible to each of the 3×3 order-2 groups and to the 3×3 identity matrix. The second class is directly reducible to each of the 3×3 order-2 groups but not to the 3×3 identity matrix. The first class has 105 members, the second has 210.

4.2.2 The S3 Classes

The S_3 classes start with the standard 3×3 representation in the upper left, and continue with the other 2. Representatives of the three groups are given below.

4.2.3 Standard S3 Embedding

The following is a representative of the class, along with the cycle structure for the group, and its orbits. The v-orbits are the same.

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$$

I
(4,8) (5,9) (6,10) (7,11)
(2,8) (3,9) (6,12) (7,13)
(2,4) (3,5) (10,12) (11,13)
(2,8,4) (3,9,5) (6,10,12) (7,11,13)
(2,4,8) (3,5,9) (6,12,10) (7,13,11)

0000
0001
0010 1000 0100
0011 1001 0101
0110 1010 1100
0111 1011 1101
1110
1111

This class has 560 members. It is directly reducible to each of the 28 3×3 S_3 representations, which are in turn reducible to the 2×2 general linear group. It is also directly reducible to a direct product of the 2×2 general linear group with the 2×2 identity matrix.

4.2.4 S3 Representation 2

The following is a representative of the class, along with the cycle structure of the group and its orbits. The v-orbits are the same.

$$\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

I
(1,8) (2,4) (3,12) (5,10) (7,14) (11,13)
(1,9) (2,6) (3,15) (5,13) (7,11) (10,14)
(4,6) (5,7) (8,9) (10,11) (12,15) (13,14)
(1,8,9) (2,4,6) (3,12,15) (5,14,11) (7,10,13)
(1,9,8) (2,6,4) (3,15,12) (5,11,14) (7,13,10)

0000
0001 1000 1001
0010 0100 0110

0011 1100 1111
0101 1010 1011 1101 0111 1110

This class has 560 members. It is directly reducible to a direct product of two copies of the 2×2 general linear group.

4.2.5 S3 Representation 3

The following is a representative of the class, along with the cycle structure of the group and its orbits. The v-orbits have the same structure.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

I
(1,8) (2,12) (3,4) (5,11) (6,15) (10,13)
(1,14) (2,5) (3,11) (4,12) (6,9) (10,13)
(2,3) (4,5) (8,14) (9,15) (10,13) (11,12)
(1,14,8) (2,11,4) (3,5,12) (6,9,15)
(1,8,14) (2,4,11) (3,12,5) (6,15,9)

0000
0001 1110 1000
0010 1011 1100 0101 0011 0100
0110 1001 1111
0111
1010 1101

This class has 1680 members. It is directly reducible to each of the 28 3×3 S_3 representations, which are in turn reducible to the 2×2 general linear group. It is also directly reducible to a direct product of the 2×2 general linear group with the 2×2 standard representation of S_2.

4.2.6 The S4 representations

The S_4 representations start with the standard representation. When using the above algorithm to find all S_4 isomorph conjugacy classes, ten classes of order-24 groups will be found. Nine of these contain S_4 isomorphs while the tenth contains order-6 elements. To keep things short we will not exhibit an entire group of each class. Instead we will list only a set of generators for the group, along with the cycle structure, orbits and v-orbits.

Each of these groups can be reduced in three different ways depending on the shape of the zero matrix in the upper left. This matrix can be a 1×3, a 2×2, or a 3×1. Reducibilities are given in the following form. S1 is the 2×2 identity matrix, S2 is the standard representation of S_2, along with its conjugates, and S3 is the 2×2 general linear group. The set A designates the

3×3 representation of S_4 with a horizontal row of 1's added to the standard representation of S_3, while B designates the S_4 representation with a vertical row of 1's. Q designates the standard representation of S_3 and it's conjugates.

The fifth and eighth classes are decomposable, the others are not.

4.2.7 The standard S4 representation

The following are the generators, cycle-structure, and orbits for this class. The orbits and v-orbits are the same. The size of the class is 420.

$$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{pmatrix}$$

I
(1,2) (5,6) (9,10) (13,14)
(2,4) (3,5) (10,12) (11,13)
(1,4) (3,6) (9,12) (11,14)
(4,8) (5,9) (6,10) (7,11)
(2,8) (3,9) (6,12) (7,13)
(1,8) (3,10) (5,12) (7,14)
(1,2) (4,8) (5,10) (6,9) (7,11) (13,14)
(1,8) (2,4) (3,12) (5,10) (7,14) (11,13)
(1,4) (2,8) (3,12) (6,9) (7,13) (11,14)
(1,4,2) (3,5,6) (9,12,10) (11,13,14)
(1,2,4) (3,6,5) (9,10,12) (11,14,13)
(2,8,4) (3,9,5) (6,10,12) (7,11,13)
(1,8,4) (3,10,6) (5,9,12) (7,11,14)
(2,4,8) (3,5,9) (6,12,10) (7,13,11)
(1,8,2) (3,9,10) (5,12,6) (7,13,14)
(1,4,8) (3,6,10) (5,12,9) (7,14,11)
(1,2,8) (3,10,9) (5,6,12) (7,14,13)
(1,8,4,2) (3,9,12,6) (5,10) (7,11,13,14)
(1,2,8,4) (3,10,12,5) (6,9) (7,11,14,13)
(1,4,8,2) (3,5,12,10) (6,9) (7,13,14,11)
(1,8,2,4) (3,12) (5,9,10,6) (7,13,11,14)
(1,2,4,8) (3,6,12,9) (5,10) (7,14,13,11)
(1,4,2,8) (3,12) (5,6,10,9) (7,14,11,13)

0000
0001 0010 0100 1000
0011 0101 0110 1001 1010 1100
0111 1011 1101 1110
1111

Reducibilities		
1×3	2×2	3×1
A	None	B

Not 2x,420, R1 Any A, R2 Any B

4.2.8 The second S4 representation

The following are the generators, cycle-structure, orbits and v-orbits for this class. This class is of size 840.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

I
(1,5) (3,7) (8,10) (9,15) (11,13) (12,14)
(2,4) (3,5) (8,15) (9,14) (10,11) (12,13)
(1,12) (2,4) (3,8) (5,14) (7,10) (9,15)
(1,9) (2,4) (3,13) (5,11) (7,15) (10,12)
(1,3) (5,7) (8,14) (9,13) (10,12) (11,15)
(1,13) (2,4) (3,9) (5,15) (7,11) (8,14)
(1,14) (2,4) (3,10) (5,12) (7,8) (11,13)
(1,7) (2,4) (8,13) (9,10) (11,14) (12,15)
(1,7) (3,5) (8,12) (9,11) (10,14) (13,15)
(1,12,13) (3,14,15) (5,8,9) (7,10,11)
(1,9,14) (3,11,12) (5,13,10) (7,15,8)
(1,11,8) (3,9,10) (5,15,12) (7,13,14)
(1,15,10) (3,13,8) (5,11,14) (7,9,12)
(1,8,11) (3,10,9) (5,12,15) (7,14,13)
(1,13,12) (3,15,14) (5,9,8) (7,11,10)
(1,14,9) (3,12,11) (5,10,13) (7,8,15)
(1,10,15) (3,8,13) (5,14,11) (7,12,9)
(1,10,5,8) (2,4) (3,14,7,12) (9,11,15,13)
(1,3,7,5) (2,4) (8,11,12,9) (10,15,14,13)
(1,8,5,10) (2,4) (3,12,7,14) (9,13,15,11)
(1,15,3,11) (2,4) (5,13,7,9) (8,10,14,12)
(1,11,3,15) (2,4) (5,9,7,13) (8,12,14,10)
(1,5,7,3) (2,4) (8,9,12,11) (10,13,14,15)

0000
0001 1010 1011 0101 0011 1000 1111 1101 1100 1001 1110 0111
0010 0100
0110

0000
0001 1000 1001
0010 0101 0011 1010 1101 0100 1011 1100
0110 0111 1111 1110

Reducibilities		
1×3	2×2	3×1
B	S2xS3	None

4.2.9 The third S4 representation

The following are the generators, cycle-structure, orbits and v-orbits for this class. This class is of size 420.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

I
(1,5) (3,7) (8,10) (9,15) (11,13) (12,14)
(1,8) (2,13) (3,5) (4,11) (7,14) (10,12)
(1,3) (5,7) (8,14) (9,13) (10,12) (11,15)
(1,12) (2,4) (3,8) (5,14) (7,10) (9,15)
(1,10) (2,15) (3,5) (4,9) (7,12) (8,14)
(1,7) (2,9) (3,14) (4,15) (5,8) (10,12)
(1,7) (3,5) (8,12) (9,11) (10,14) (13,15)
(1,7) (2,11) (3,12) (4,13) (5,10) (8,14)
(1,14) (2,4) (3,10) (5,12) (7,8) (11,13)
(2,11,9) (3,10,8) (4,13,15) (5,12,14)
(1,12,5) (2,11,15) (3,7,10) (4,13,9)
(2,9,11) (3,8,10) (4,15,13) (5,14,12)
(1,5,14) (2,13,9) (3,8,7) (4,11,15)
(1,12,14) (2,15,13) (4,9,11) (7,10,8)
(1,5,12) (2,15,11) (3,10,7) (4,9,13)
(1,14,5) (2,9,13) (3,7,8) (4,15,11)
(1,14,12) (2,13,15) (4,11,9) (7,8,10)
(1,10,5,8) (2,4) (3,14,7,12) (9,11,15,13)
(1,8,5,10) (2,4) (3,12,7,14) (9,13,15,11)
(1,3,14,10) (2,13,4,11) (5,8,12,7) (9,15)
(1,8,12,3) (2,9,4,15) (5,7,14,10) (11,13)
(1,10,14,3) (2,11,4,13) (5,7,12,8) (9,15)
(1,3,12,8) (2,15,4,9) (5,10,14,7) (11,13)

0000
0001 1010 0101 1000 1100 0011 1110 0111
0010 0100 1011 1101 1111 1001
0110

0000
0001 1000 1111 1110 0110 0111
0010 0101 1010 0100 1011 1100 0011 1101
1001

Reducibilities		
1×3	2×2	3×1
A	None	B

4.2.10 The fourth S4 representation

The following are the generators, cycle-structure, orbits and v-orbits for this class. This class is of size 840.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

I
(1,5) (3,7) (8,10) (9,15) (11,13) (12,14)
(1,8) (3,10) (4,6) (5,14) (7,12) (13,15)
(1,12) (2,4) (3,8) (5,14) (7,10) (9,15)
(1,12) (2,6) (3,10) (5,8) (7,14) (11,15)
(1,3) (5,7) (8,14) (9,13) (10,12) (11,15)
(1,10) (2,6) (3,12) (5,14) (7,8) (9,13)
(1,14) (2,4) (3,10) (5,12) (7,8) (11,13)
(1,12) (3,14) (4,6) (5,10) (7,8) (9,11)
(1,7) (3,5) (8,12) (9,11) (10,14) (13,15)
(1,3,5) (2,6,4) (9,11,13) (10,14,12)
(2,6,4) (3,7,5) (8,14,10) (9,15,11)
(1,5,3) (2,4,6) (9,13,11) (10,12,14)
(1,7,3) (2,6,4) (8,10,12) (9,13,15)
(1,7,5) (2,4,6) (8,14,12) (11,13,15)
(2,4,6) (3,5,7) (8,10,14) (9,11,15)
(1,3,7) (2,4,6) (8,12,10) (9,15,13)
(1,5,7) (2,6,4) (8,12,14) (11,15,13)
(1,10,5,8) (2,4) (3,14,7,12) (9,11,15,13)
(1,14,7,10) (3,12,5,8) (4,6) (9,13,11,15)
(1,8,5,10) (2,4) (3,12,7,14) (9,13,15,11)
(1,14,3,8) (2,6) (5,10,7,12) (9,15,13,11)
(1,8,3,14) (2,6) (5,12,7,10) (9,11,13,15)
(1,10,7,14) (3,8,5,12) (4,6) (9,15,11,13)

0000
0001 1010 0011 0101 1110 1000 0111 1100

0010 0100 0110
1001 1011 1111 1101

0000
0001 1000
0010 0101 0111 1010 1111 0100 1110 0110 0011 1100 1101 1011
1001

Reducibilities		
1×3	2×2	3×1
None	S3xS2	A

4.2.11 The fifth S4 representation

The following are the generators, cycle-structure, orbits and v-orbits for this class. This class is of size 420 and is decomposable.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

I
(1,7) (3,5) (8,14) (10,12)
(2,3) (4,5) (8,9) (14,15)
(1,14) (3,12) (5,10) (7,8)
(1,9) (2,10) (4,12) (7,15)
(1,7) (2,4) (9,15) (10,12)
(1,15) (2,12) (4,10) (7,9)
(1,8) (3,10) (5,12) (7,14)
(2,5) (3,4) (8,15) (9,14)
(2,4) (3,5) (8,14) (9,15)
(1,15,8) (2,5,12) (3,10,4) (7,9,14)
(1,9,14) (2,5,10) (3,12,4) (7,15,8)
(1,8,15) (2,12,5) (3,4,10) (7,14,9)
(1,15,14) (2,3,12) (4,5,10) (7,9,8)
(1,8,9) (2,10,3) (4,12,5) (7,14,15)
(1,14,9) (2,10,5) (3,4,12) (7,8,15)
(1,14,15) (2,12,3) (4,10,5) (7,8,9)
(1,9,8) (2,3,10) (4,5,12) (7,15,14)
(1,14,7,8) (2,4) (3,10,5,12) (9,15)
(1,7) (2,3,4,5) (8,15,14,9) (10,12)
(1,8,7,14) (2,4) (3,12,5,10) (9,15)
(1,9,7,15) (2,12,4,10) (3,5) (8,14)
(1,15,7,9) (2,10,4,12) (3,5) (8,14)
(1,7) (2,5,4,3) (8,9,14,15) (10,12)

0000

0001 1110 1111 0111 1000 1001
0010 0100 0101 0011 1100 1010
0110
1011
1101

0000
0001 1000 1111
0010 0101 1011 1100
0011 1101 1010 0100
0110
0111 1110 1001

Reducibilities		
1×3	2×2	3×1
B	S1xS3	B,Q

4.2.12 The sixth S4 representation

The following are the generators, cycle-structure, orbits and v-orbits for this class. This class is of size 420.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

I
(1,7) (3,5) (8,14) (10,12)
(1,7) (3,5) (9,15) (11,13)
(8,14) (9,15) (10,12) (11,13)
(2,4) (3,5) (8,15) (9,14) (10,11) (12,13)
(1,14) (2,4) (3,10) (5,12) (7,8) (11,13)
(1,15) (2,4) (3,11) (5,13) (7,9) (10,12)
(1,9) (2,4) (3,13) (5,11) (7,15) (10,12)
(1,8) (2,4) (3,12) (5,10) (7,14) (11,13)
(2,4) (3,5) (8,9) (10,13) (11,12) (14,15)
(1,9,8) (3,11,10) (5,13,12) (7,15,14)
(1,15,14) (3,13,12) (5,11,10) (7,9,8)
(1,8,9) (3,10,11) (5,12,13) (7,14,15)
(1,9,14) (3,11,12) (5,13,10) (7,15,8)
(1,8,15) (3,10,13) (5,12,11) (7,14,9)
(1,14,15) (3,12,13) (5,10,11) (7,8,9)
(1,14,9) (3,12,11) (5,10,13) (7,8,15)
(1,15,8) (3,13,10) (5,11,12) (7,9,14)
(1,7) (2,4) (8,15,14,9) (10,11,12,13)
(1,7) (2,4) (8,9,14,15) (10,13,12,11)

(1,8,7,14) (2,4) (3,12,5,10) (9,15)
(1,14,7,8) (2,4) (3,10,5,12) (9,15)
(1,15,7,9) (2,4) (3,11,5,13) (8,14)
(1,9,7,15) (2,4) (3,13,5,11) (8,14)

0000
0001 1110 1001 0111 1000 1111
0010 0100
0011 1010 1011 0101 1100 1101
0110

0000
0001 1000 1001
0010 0101 1011 1100 1101 1010 0011 0100
0110
0111 1110 1111

Reducibilities		
1×3	2×2	3×1
Q	S2xS3	B

4.2.13 The seventh S4 representation

The following are the generators, cycle-structure, orbits and v-orbits for this class. This class is of size 420.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

I
(1,7) (3,5) (8,14) (10,12)
(1,5) (3,7) (8,12) (10,14)
(1,3) (5,7) (8,10) (12,14)
(1,8) (3,10) (4,6) (5,14) (7,12) (13,15)
(1,10) (2,4) (3,14) (5,8) (7,12) (9,15)
(1,10) (2,6) (3,12) (5,14) (7,8) (9,13)
(1,14) (2,6) (3,8) (5,10) (7,12) (9,13)
(1,12) (2,4) (3,8) (5,14) (7,10) (9,15)
(1,10) (3,8) (4,6) (5,12) (7,14) (13,15)
(1,5,7) (2,6,4) (9,13,15) (10,14,12)
(2,6,4) (3,7,5) (8,12,14) (9,13,15)
(1,7,5) (2,4,6) (9,15,13) (10,12,14)
(1,3,5) (2,6,4) (8,14,10) (9,13,15)
(1,3,7) (2,4,6) (8,12,10) (9,15,13)
(2,4,6) (3,5,7) (8,14,12) (9,15,13)

(1,5,3) (2,4,6) (8,10,14) (9,15,13)
(1,7,3) (2,6,4) (8,10,12) (9,13,15)
(1,14,7,8) (2,4) (3,10,5,12) (9,15)
(1,12,3,14) (4,6) (5,10,7,8) (13,15)
(1,8,7,14) (2,4) (3,12,5,10) (9,15)
(1,12,5,8) (2,6) (3,10,7,14) (9,13)
(1,8,5,12) (2,6) (3,14,7,10) (9,13)
(1,14,3,12) (4,6) (5,8,7,10) (13,15)

0000
0001 1110 0101 0111 1100 1000 0011 1010
0010 0100 0110
1001 1111 1101
1011

0000
0001 1000
0010 0101 0110 1011 1100 1111
0011 1101 0111 1010 1110 0100
1001

Reducibilities		
1×3	2×2	3×1
A	S3xS2	Q

4.2.14 The eighth S4 representation

The following are the generators, cycle-structure, orbits and v-orbits for this class. This class is of size 420 and is decomposable.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

I
(1,7) (3,5) (8,14) (10,12)
(4,11) (5,10) (6,9) (7,8)
(1,7) (2,4) (9,15) (10,12)
(2,11) (3,10) (6,15) (7,14)
(1,8) (3,10) (5,12) (7,14)
(1,8) (2,11) (5,12) (6,15)
(2,4) (3,5) (8,14) (9,15)
(1,14) (3,12) (4,11) (6,9)
(1,14) (3,12) (5,10) (7,8)
(1,14,8) (2,11,4) (3,5,12) (6,9,15)
(1,7,14) (2,11,4) (3,12,10) (6,9,15)

(1,8,14) (2,4,11) (3,12,5) (6,15,9)
(1,8,7) (2,11,4) (5,10,12) (6,9,15)
(2,4,11) (3,5,10) (6,15,9) (7,14,8)
(1,14,7) (2,4,11) (3,10,12) (6,15,9)
(1,7,8) (2,4,11) (5,12,10) (6,15,9)
(2,11,4) (3,10,5) (6,9,15) (7,8,14)
(1,14,7,8) (2,4) (3,10,5,12) (9,15)
(1,8,14,7) (3,10,12,5) (4,11) (6,9)
(1,8,7,14) (2,4) (3,12,5,10) (9,15)
(1,7,8,14) (2,11) (3,12,10,5) (6,15)
(1,14,8,7) (2,11) (3,5,10,12) (6,15)
(1,7,14,8) (3,5,12,10) (4,11) (6,9)

0000
0001 1110 0111 1000
0010 0100 1011
0011 1010 0101 1100
0110 1001 1111
1101

0000
0001 1000 1010 0100 0011 1101
0010 0101 0111 1011 1110 1100
0110
1001
1111

Reducibilities		
1×3	2×2	3×1
A	S3xS1	A,Q

4.2.15 The ninth S4 representation

The following are the generators, cycle-structure, orbits and v-orbits for this class. This class is of size 420.

$$\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&1&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix}$$

I
(1,5) (3,7) (8,10) (9,15) (11,13) (12,14)
(4,6) (5,7) (8,11) (9,10) (12,13) (14,15)
(1,14) (2,4) (3,10) (5,12) (7,8) (11,13)
(1,13) (2,6) (3,11) (5,9) (7,15) (10,14)
(1,7) (3,5) (8,12) (9,11) (10,14) (13,15)

(1,15) (2,6) (3,9) (5,11) (7,13) (8,12)
(1,12) (2,4) (3,8) (5,14) (7,10) (9,15)
(1,3) (4,6) (8,15) (9,12) (10,13) (11,14)
(1,3) (5,7) (8,14) (9,13) (10,12) (11,15)
(1,9,8) (2,6,4) (3,15,12) (5,11,14) (7,13,10)
(1,11,10) (2,6,4) (3,13,14) (5,9,12) (7,15,8)
(1,8,9) (2,4,6) (3,12,15) (5,14,11) (7,10,13)
(1,15,14) (2,6,4) (3,9,10) (5,13,8) (7,11,12)
(1,12,13) (2,4,6) (3,8,11) (5,10,15) (7,14,9)
(1,10,11) (2,4,6) (3,14,13) (5,12,9) (7,8,15)
(1,14,15) (2,4,6) (3,10,9) (5,8,13) (7,12,11)
(1,13,12) (2,6,4) (3,11,8) (5,15,10) (7,9,14)
(1,10,5,8) (2,4) (3,14,7,12) (9,11,15,13)
(1,7,3,5) (4,6) (8,9,14,13) (10,11,12,15)
(1,8,5,10) (2,4) (3,12,7,14) (9,13,15,11)
(1,11,7,9) (2,6) (3,13,5,15) (8,10,12,14)
(1,9,7,11) (2,6) (3,15,5,13) (8,14,12,10)
(1,5,3,7) (4,6) (8,13,14,9) (10,15,12,11)

0000
0001 1010 1001 0101 0111 1000 1011 1111 1110 1101 1100 0011
0010 0100 0110

0000
0001 1000 1001
0010 0101 0110 1010 0111 0100 1111 1110 1011 1101 1100 0011

Reducibilities		
1×3	2×2	3×1
None	S3xS3	None

4.2.16 The first S5 representation

The first S_5 representation is created by adding either a row or a column of all 1's to the standard S_4 representation. Although two different groups can be created in this fashion, they are conjugate to one another. The first group is generated by creating every possible matrix from the five rows 0001, 0010, 0100, 1000, and 1111. The second group is generated by creating every possible matrix from the five columns. This class of groups is not reducible. It is of size 168.

0	0	0	1	1
0	0	1	0	1
0	1	0	0	1
1	0	0	0	1

I

(1,2) (5,6) (9,10) (13,14)
(2,4) (3,5) (10,12) (11,13)
(1,4) (3,6) (9,12) (11,14)
(4,8) (5,9) (6,10) (7,11)
(2,8) (3,9) (6,12) (7,13)
(1,8) (3,10) (5,12) (7,14)
(1,9) (2,10) (4,12) (7,15)
(1,5) (2,6) (8,12) (11,15)
(1,3) (4,6) (8,10) (13,15)
(2,3) (4,5) (8,9) (14,15)

(1,2) (4,8) (5,10) (6,9) (7,11) (13,14)
(1,4) (2,8) (3,12) (6,9) (7,13) (11,14)
(1,8) (2,4) (3,12) (5,10) (7,14) (11,13)
(1,10) (2,9) (4,12) (5,6) (7,15) (13,14)
(1,9) (2,12) (3,5) (4,10) (7,15) (11,13)
(1,6) (2,5) (8,12) (9,10) (11,15) (13,14)
(1,12) (2,10) (3,6) (4,9) (7,15) (11,14)
(1,5) (2,12) (3,9) (6,8) (7,13) (11,15)
(1,12) (2,6) (3,10) (5,8) (7,14) (11,15)
(1,6) (3,4) (8,10) (9,12) (11,14) (13,15)
(1,3) (4,10) (5,9) (6,8) (7,11) (13,15)
(1,10) (3,8) (4,6) (5,12) (7,14) (13,15)
(2,5) (3,4) (8,9) (10,12) (11,13) (14,15)
(2,3) (4,9) (5,8) (6,10) (7,11) (14,15)
(2,9) (3,8) (4,5) (6,12) (7,13) (14,15)

(1,4,2) (3,5,6) (9,12,10) (11,13,14)
(1,2,4) (3,6,5) (9,10,12) (11,14,13)
(2,8,4) (3,9,5) (6,10,12) (7,11,13)
(1,8,4) (3,10,6) (5,9,12) (7,11,14)
(2,4,8) (3,5,9) (6,12,10) (7,13,11)
(1,8,2) (3,9,10) (5,12,6) (7,13,14)
(1,4,8) (3,6,10) (5,12,9) (7,14,11)
(1,2,8) (3,10,9) (5,6,12) (7,14,13)
(1,9,5) (2,10,6) (4,8,12) (7,11,15)
(1,9,3) (2,8,10) (4,12,6) (7,13,15)
(1,8,9) (2,10,3) (4,12,5) (7,14,15)
(1,5,3) (2,4,6) (8,12,10) (11,13,15)
(1,4,5) (2,6,3) (8,12,9) (11,14,15)
(1,5,9) (2,6,10) (4,12,8) (7,15,11)
(1,2,3) (4,6,5) (8,10,9) (13,14,15)
(1,3,5) (2,6,4) (8,10,12) (11,15,13)
(1,3,9) (2,10,8) (4,6,12) (7,15,13)
(1,3,2) (4,5,6) (8,9,10) (13,15,14)

(1,5,4) (2,3,6) (8,9,12) (11,15,14)
(1,9,8) (2,3,10) (4,5,12) (7,15,14)

(1,8,4,2) (3,9,12,6) (5,10) (7,11,13,14)
(1,2,8,4) (3,10,12,5) (6,9) (7,11,14,13)
(1,4,8,2) (3,5,12,10) (6,9) (7,13,14,11)
(1,8,2,4) (3,12) (5,9,10,6) (7,13,11,14)
(1,2,4,8) (3,6,12,9) (5,10) (7,14,13,11)
(1,4,2,8) (3,12) (5,6,10,9) (7,14,11,13)
(1,9,5,3) (2,8,12,6) (4,10) (7,11,13,15)
(1,8,12,5) (2,10,6,3) (4,9) (7,11,14,15)
(1,9,3,5) (2,12) (4,8,10,6) (7,13,11,15)
(1,8,10,3) (2,9) (4,12,6,5) (7,13,14,15)
(1,12) (2,10,3,6) (4,8,9,5) (7,14,11,15)
(1,10) (2,8,9,3) (4,12,5,6) (7,14,13,15)
(1,4,6,3) (2,5) (8,12,10,9) (11,13,14,15)
(1,6) (2,4,5,3) (8,12,9,10) (11,14,13,15)
(1,5,3,9) (2,12) (4,6,10,8) (7,15,11,13)
(1,12) (2,6,3,10) (4,5,9,8) (7,15,11,14)
(1,5,9,3) (2,4,12,10) (6,8) (7,13,15,11)
(1,4,12,9) (2,6,10,3) (5,8) (7,14,15,11)
(1,6) (2,3,5,4) (8,10,9,12) (11,15,13,14)
(1,2,6,5) (3,4) (8,10,12,9) (11,14,15,13)
(1,3,9,5) (2,10,12,4) (6,8) (7,11,15,13)
(1,3,5,9) (2,6,12,8) (4,10) (7,15,13,11)
(1,10) (2,3,9,8) (4,6,5,12) (7,15,13,14)
(1,2,10,9) (3,8) (4,6,12,5) (7,14,15,13)
(1,5,6,2) (3,4) (8,9,12,10) (11,13,15,14)
(1,3,6,4) (2,5) (8,9,10,12) (11,15,14,13)
(1,9,12,4) (2,3,10,6) (5,8) (7,11,15,14)
(1,9,10,2) (3,8) (4,5,12,6) (7,13,15,14)
(1,5,12,8) (2,3,6,10) (4,9) (7,15,14,11)
(1,3,10,8) (2,9) (4,5,6,12) (7,15,14,13)

(1,8,12,6,3) (2,9,4,10,5) (7,11,13,14,15)
(1,10,4,9,6) (2,8,12,5,3) (7,11,14,13,15)
(1,12,2,9,6) (3,5,4,8,10) (7,13,14,11,15)
(1,8,10,6,5) (2,12,3,4,9) (7,13,11,14,15)
(1,10,5,2,12) (3,6,4,8,9) (7,14,13,11,15)
(1,12,3,4,10) (2,8,9,5,6) (7,14,11,13,15)
(1,12,2,5,10) (3,9,8,4,6) (7,15,11,13,14)
(1,6,9,2,12) (3,10,8,4,5) (7,15,11,14,13)
(1,4,12,10,3) (2,5,8,6,9) (7,13,14,15,11)
(1,12,3,8,6) (2,4,5,9,10) (7,13,15,11,14)
(1,6,8,5,10) (2,4,12,9,3) (7,14,13,15,11)
(1,4,6,10,9) (2,12,3,8,5) (7,14,15,11,13)

(1,10,5,8,6) (2,3,9,12,4) (7,11,15,13,14)
(1,2,10,12,5) (3,8,6,9,4) (7,11,14,15,13)
(1,6,9,4,10) (2,3,5,12,8) (7,15,13,14,11)
(1,10,4,3,12) (2,6,5,9,8) (7,15,13,11,14)
(1,2,6,12,9) (3,4,10,5,8) (7,14,15,13,11)
(1,6,8,3,12) (2,10,9,5,4) (7,14,11,15,13)
(1,9,12,6,2) (3,8,5,10,4) (7,11,13,15,14)
(1,3,10,12,4) (2,9,6,8,5) (7,11,15,14,13)
(1,5,12,10,2) (3,4,9,6,8) (7,13,15,14,11)
(1,9,10,6,4) (2,5,8,3,12) (7,13,11,15,14)
(1,3,6,12,8) (2,5,10,4,9) (7,15,14,13,11)
(1,5,6,10,8) (2,9,4,3,12) (7,15,14,11,13)

(1,10,5,2,9,6) (4,8,12) (7,11,15) (13,14)
(1,12,2,9,4,10) (3,5,6) (7,15) (11,13,14)
(1,10,4,9,2,12) (3,6,5) (7,15) (11,14,13)
(1,12,3,4,9,6) (2,8,10) (7,13,15) (11,14)
(1,8,9) (2,12,3,4,10,5) (7,14,15) (11,13)
(1,6,9,2,5,10) (4,12,8) (7,15,11) (13,14)
(1,12,2,5,8,6) (3,9,10) (7,13,14) (11,15)
(1,4,5) (2,12,3,8,6,9) (7,13) (11,14,15)
(1,6,8,5,2,12) (3,10,9) (7,14,13) (11,15)
(1,12,3,8,5,10) (2,4,6) (7,14) (11,13,15)
(1,2,3) (4,10,5,8,6,9) (7,11) (13,14,15)
(1,10,4,3,8,6) (5,9,12) (7,11,14) (13,15)
(1,6,9,4,3,12) (2,10,8) (7,15,13) (11,14)
(1,6,8,3,4,10) (5,12,9) (7,14,11) (13,15)
(1,10,5,8,3,12) (2,6,4) (7,14) (11,15,13)
(1,3,2) (4,9,6,8,5,10) (7,11) (13,15,14)
(2,9,4,3,8,5) (6,10,12) (7,11,13) (14,15)
(2,5,8,3,4,9) (6,12,10) (7,13,11) (14,15)
(1,5,4) (2,9,6,8,3,12) (7,13) (11,15,14)
(1,9,8) (2,5,10,4,3,12) (7,15,14) (11,13)

0000
0001 0010 0100 1000 1001 1010 1100 0101 0110 0011
0111 1011 1101 1110 1111

0000
0001 0010 0100 1000 1111
0011 0101 0110 1001 1010 1100 1110 1101 1011 0111

4.2.17 The second S5 representation

There are two non-conjugate representations of S5 in 4×4 matrices. The second representation is generated by the following pair of matrices. The cycle form of the group is given below along with its orbits and v-orbits. The class of groups is of size 168 and is not reducible.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

I

(1,8) (2,12) (3,4) (5,11) (6,15) (10,13)
(1,11) (2,14) (3,5) (4,8) (7,13) (9,15)
(4,6) (5,7) (8,9) (10,11) (12,15) (13,14)
(1,10) (2,13) (3,7) (5,14) (6,9) (8,12)
(1,9) (2,15) (3,6) (4,12) (7,10) (11,14)
(1,9) (2,6) (3,15) (5,13) (7,11) (10,14)
(1,2) (4,7) (8,13) (9,15) (10,12) (11,14)
(1,14) (2,11) (3,5) (4,13) (7,8) (10,12)
(1,3) (4,8) (5,11) (6,10) (7,9) (13,15)
(1,5) (2,14) (3,11) (6,10) (7,15) (9,13)
(1,4) (2,12) (3,8) (6,13) (7,9) (10,15)
(1,6) (2,15) (3,9) (4,14) (5,8) (11,12)
(1,3) (4,11) (5,8) (6,9) (7,10) (12,14)
(2,3) (4,15) (5,14) (6,12) (7,13) (8,9)
(4,7) (5,6) (8,10) (9,11) (12,13) (14,15)
(2,3) (4,12) (5,13) (6,15) (7,14) (10,11)
(1,7) (2,13) (3,10) (4,11) (5,12) (8,14)
(4,5) (6,7) (8,11) (9,10) (12,14) (13,15)
(1,11) (2,5) (3,14) (4,9) (6,12) (8,15)
(1,15) (2,9) (3,6) (4,10) (7,12) (8,13)
(1,13) (2,10) (3,7) (5,9) (6,14) (11,15)
(1,12) (2,8) (3,4) (5,15) (6,11) (9,14)
(1,2) (5,6) (8,12) (9,14) (10,13) (11,15)
(1,10) (2,7) (3,13) (4,15) (6,8) (9,12)
(1,8) (2,4) (3,12) (5,10) (7,14) (11,13)

(1,7,6) (2,13,15) (3,10,9) (4,12,8) (5,11,14)
(1,4,5) (2,12,14) (3,8,11) (6,9,15) (7,13,10)
(1,15,14) (2,9,11) (3,6,5) (4,12,8) (7,10,13)
(1,13,12) (2,10,8) (3,7,4) (5,14,11) (6,9,15)
(1,5,4) (2,14,12) (3,11,8) (6,15,9) (7,10,13)
(1,6,7) (2,15,13) (3,9,10) (4,8,12) (5,14,11)
(1,8,9) (2,4,6) (3,12,15) (5,14,11) (7,10,13)
(1,3,2) (4,15,11) (5,12,9) (6,14,8) (7,13,10)
(1,12,13) (2,8,10) (3,4,7) (5,11,14) (6,15,9)

(1,2,3) (4,10,14) (5,8,13) (6,9,15) (7,11,12)
(1,3,2) (4,12,8) (5,15,10) (6,13,11) (7,14,9)
(1,2,3) (4,11,15) (5,9,12) (6,8,14) (7,10,13)
(1,14,15) (2,11,9) (3,5,6) (4,8,12) (7,13,10)
(1,2,3) (4,9,13) (5,11,14) (6,10,12) (7,8,15)
(1,11,10) (2,5,7) (3,14,13) (4,12,8) (6,9,15)
(1,10,11) (2,7,5) (3,13,14) (4,8,12) (6,15,9)
(1,2,3) (4,8,12) (5,10,15) (6,11,13) (7,9,14)
(1,3,2) (4,14,10) (5,13,8) (6,15,9) (7,12,11)
(1,3,2) (4,13,9) (5,14,11) (6,12,10) (7,15,8)
(1,9,8) (2,6,4) (3,15,12) (5,11,14) (7,13,10)

(2,3) (4,14,6,13) (5,15,7,12) (8,11,9,10)
(1,5,9,13) (2,11,6,7) (3,14,15,10) (4,12)
(1,4,10,15) (2,8,7,6) (3,12,13,9) (5,14)
(2,3) (4,13,6,14) (5,12,7,15) (8,10,9,11)
(1,13,9,5) (2,7,6,11) (3,10,15,14) (4,12)
(1,6,11,12) (2,9,5,4) (3,15,14,8) (7,13)
(1,7,8,14) (2,10,4,5) (3,13,12,11) (6,15)
(1,15,10,4) (2,6,7,8) (3,9,13,12) (5,14)
(1,4,6,14) (2,8,15,5) (3,12,9,11) (7,10)
(1,12,7,5) (2,4,13,11) (3,8,10,14) (6,9)
(1,14,6,4) (2,5,15,8) (3,11,9,12) (7,10)
(1,2) (4,5,7,6) (8,14,10,15) (9,12,11,13)
(1,14,13,6) (2,5,10,9) (3,11,7,15) (8,12)
(1,15,12,5) (2,6,8,11) (3,9,4,14) (10,13)
(1,13,4,6) (2,7,12,9) (3,10,8,15) (5,11)
(1,12,15,7) (2,4,9,10) (3,8,6,13) (11,14)
(1,3) (4,10,5,9) (6,8,7,11) (12,13,14,15)
(1,15,5,7) (2,6,14,10) (3,9,11,13) (4,8)
(1,5,7,12) (2,11,13,4) (3,14,10,8) (6,9)
(1,6,13,14) (2,9,10,5) (3,15,7,11) (8,12)
(1,7,15,12) (2,10,9,4) (3,13,6,8) (11,14)
(1,13,14,4) (2,7,11,8) (3,10,5,12) (9,15)
(1,6,4,13) (2,9,12,7) (3,15,8,10) (5,11)
(1,7,5,15) (2,10,14,6) (3,13,11,9) (4,8)
(1,12,11,6) (2,4,5,9) (3,8,14,15) (7,13)
(1,4,14,13) (2,8,11,7) (3,12,5,10) (9,15)
(1,2) (4,6,7,5) (8,15,10,14) (9,13,11,12)
(1,3) (4,9,5,10) (6,11,7,8) (12,15,14,13)
(1,5,12,15) (2,11,8,6) (3,14,4,9) (10,13)
(1,14,8,7) (2,5,4,10) (3,11,12,13) (6,15)

(1,11,15,13,8) (2,5,9,10,4) (3,14,6,7,12)
(1,15,8,11,13) (2,9,4,5,10) (3,6,12,14,7)
(1,13,11,8,15) (2,10,5,4,9) (3,7,14,12,6)

(1,8,13,15,11) (2,4,10,9,5) (3,12,7,6,14)
(1,8,5,6,10) (2,4,14,15,7) (3,12,11,9,13)
(1,5,10,8,6) (2,14,7,4,15) (3,11,13,12,9)
(1,9,14,12,10) (2,6,11,8,7) (3,15,5,4,13)
(1,6,8,10,5) (2,15,4,7,14) (3,9,12,13,11)
(1,10,6,5,8) (2,7,15,14,4) (3,13,9,11,12)
(1,11,4,7,9) (2,5,12,13,6) (3,14,8,10,15)
(1,9,7,4,11) (2,6,13,12,5) (3,15,10,8,14)
(1,7,11,9,4) (2,13,5,6,12) (3,10,14,15,8)
(1,10,12,14,9) (2,7,8,11,6) (3,13,4,5,15)
(1,4,9,11,7) (2,12,6,5,13) (3,8,15,14,10)
(1,13,6,4,14) (2,10,15,12,11) (3,7,9,8,5)
(1,7,12,15,5) (2,13,8,9,14) (3,10,4,6,11)
(1,6,14,13,4) (2,15,11,10,12) (3,9,5,7,8)
(1,14,10,9,12) (2,11,7,6,8) (3,5,13,15,4)
(1,5,15,12,7) (2,14,9,8,13) (3,11,6,4,10)
(1,15,7,5,12) (2,9,13,14,8) (3,6,10,11,4)
(1,14,4,6,13) (2,11,12,15,10) (3,5,8,9,7)
(1,12,5,7,15) (2,8,14,13,9) (3,4,11,10,6)
(1,4,13,14,6) (2,12,10,11,15) (3,8,7,5,9)
(1,12,9,10,14) (2,8,6,7,11) (3,4,15,13,5)

(1,9,8) (2,15,4,3,6,12) (5,10,14,13,11,7)
(1,6,2,9,3,15) (4,11,10,12,14,7) (5,13,8)
(1,7,2,10,3,13) (4,15,11) (5,8,9,14,12,6)
(1,10,11) (2,13,5,3,7,14) (4,9,12,15,8,6)
(1,11,10) (2,14,7,3,5,13) (4,6,8,15,12,9)
(1,15,3,9,2,6) (4,7,14,12,10,11) (5,8,13)
(1,8,9) (2,12,6,3,4,15) (5,7,11,13,14,10)
(1,5,2,11,3,14) (4,7,9,8,13,15) (6,12,10)
(1,4,2,8,3,12) (5,6,10,11,15,13) (7,14,9)
(1,13,3,10,2,7) (4,11,15) (5,6,12,14,9,8)
(1,10,6,3,7,9) (2,13,15) (4,14,8,11,12,5)
(1,14,3,11,2,5) (4,15,13,8,9,7) (6,10,12)
(1,11,15,2,14,9) (3,5,6) (4,13,12,7,8,10)
(1,8,13,2,12,10) (3,4,7) (5,15,14,6,11,9)
(1,9,7,3,6,10) (2,15,13) (4,5,12,11,8,14)
(1,8,5,3,4,11) (2,12,14) (6,7,15,10,9,13)
(1,12,3,8,2,4) (5,13,15,11,10,6) (7,9,14)
(1,10,12,2,13,8) (3,7,4) (5,9,11,6,14,15)
(1,9,14,2,15,11) (3,6,5) (4,10,8,7,12,13)
(1,11,4,3,5,8) (2,14,12) (6,13,9,10,15,7)

0000
0001 1000 1011 0101 1111 1001 0110 1101 0111 1010 0100 0011 0010 1110 1100

0000
0001 1100 1011 0011 1111 1110 1101 0110 1000 0111 1001 0101 1010 0010 0100

5 Dimension 5

Virtually everything about dimension 5 is extremely time consuming and difficult to verify. Thus the information given here should be considered tentative, unless noted otherwise.

5.1 The Matrices

Here is the spectrum of the general linear group. (This is confirmed.)

1 -- 1
2 -- 6975
3 -- 75392
4 -- 416640
5 -- 666624
6 -- 1249920
7 -- 476160
8 -- 624960
9 -- 0
10 -- 0
11 -- 0
12 -- 833280
13 -- 0
14 -- 1428480
15 -- 1333248
16 -- 0
17 -- 0
18 -- 0
19 -- 0
20 -- 0
21 -- 952320
22 -- 0
23 -- 0
24 -- 0
25 -- 0
26 -- 0
27 -- 0
28 -- 0
29 -- 0
30 -- 0
31 -- 1935360

5.1.1 Order 2

There are two classes (confirmed) with the following cycle-structures and eigenpolynomials.

$x^5 + x^4 + x + 1$
(1,16) (2,8) (3,24) (5,20) (6,12) (7,28) (9,18) (11,26) (13,22) (15,30) (19,25) (23,29)

$x^5 + x^4 + x + 1$
(1,16) (3,18) (5,20) (7,22) (9,24) (11,26) (13,28) (15,30)

5.1.2 Order 3

There are two classes (confirmed) with the following cycle structures and eigenpolynomials.

$x^5 + x^4 + x^3 + x^2 + x + 1$
(1,17,16) (2,10,8) (3,27,24) (5,21,20) (6,14,12) (7,31,28) (9,19,26) (11,25,18) (13,23,30) (15,29,22)

$x^5 + x^3 + x^2 + 1$
(1,26,16) (2,19,8) (3,9,24) (5,30,20) (6,23,12) (7,13,28) (10,17,27) (14,21,31)

5.1.3 Order 4

There are (apparently) three classes with the following cycle structures and eigenpolynomials.

$x^5 + x^4 + x + 1$
(1,18,9,16) (2,8) (3,26,11,24) (5,22,13,20) (6,12) (7,30,15,28) (17,19,27,25) (21,23,31,29)

$x^5 + x^4 + x + 1$
(1,20,5,16) (2,8) (3,28,7,24) (6,12) (9,22,13,18) (11,30,15,26) (17,21) (19,29) (23,25) (27,31)

$x^5 + x^4 + x + 1$
(1,26,11,16) (2,8) (3,18,9,24) (5,30,15,20) (6,12) (7,22,13,28) (17,27) (21,31)

5.1.4 Order 5

There is (apparently) one class with the following cycle structures and eigenpolynomials.

$x^5 + 1$
(1,19,27,25,16) (2,9,17,18,8) (3,26,10,11,24) (5,23,31,29,20) (6,13,21,22,12) (7,30,14,15,28)

5.1.5 Order 6

There are, apparently, two classes with the following eigenpolynomials and cycle structures.

$x^5 + x^3 + x^2 + 1$
(1,17,16) (2,8) (3,25,18,9,19,24) (5,21,20) (6,12) (7,29,22,13,23,28) (11,27,26) (15,31,30)

$x^5 + x^4 + x^3 + x^2 + x + 1$
(1,18,8,2,9,16) (3,27,24) (5,22,12,6,13,20) (7,31,28) (10,11,25,17,19,26) (14,15,29,21,23,30)

5.1.6 Order 7

There are, apparently, two classes with the following eigenpolynomials and cycle structures.

$x^5 + x^4 + x^3 + 1$
(1,26,10,11,17,27,16) (2,9,24,3,19,18,8) (5,30,14,15,21,31,20) (6,13,28,7,23,22,12)

$x^5 + x^2 + x + 1$
(1,27,17,26,10,11,16) (2,9,25,24,3,18,8) (5,31,21,30,14,15,20) (6,13,29,28,7,22,12)

5.1.7 Order 8

There is, apparently, one class with the following cycle structure and eigenpolynomial.

$x^5 + x^4 + x + 1$
(1,20,5,16) (2,9,22,12,6,13,18,8) (3,29,19,28,7,25,23,24) (10,11,31,26) (14,15,27,30) (17,21)

5.1.8 Order 12

There is, apparently, one class with the following cycle structure and eigenpolynomial.

$x^5 + x^3 + x^2 + 1$
(1,21,16) (2,9,23,25,22,12,6,13,19,29,18,8) (3,28,7,24) (5,17,20) (10,11,30,14,15,26) (27,31)

5.1.9 Order 14

There are, apparently, two classes with the following cycle structures and eigenpolynomials.

$x^5 + x^4 + x^3 + 1$
(1,30,14,15,17,31,16) (2,9,28,7,19,22,12,6,13,24,3,23,18,8) (5,26,10,11,21,27,20) (25,29)

$x^5 + x^2 + x + 1$
(1,31,17,30,14,15,16) (2,9,29,24,3,22,12,6,13,25,28,7,18,8) (5,27,21,26,10,11,20) (19,23)

5.1.10 Order 15

There are, apparently, two classes with the following cycle structures and eigenpolynomials.

$x^5 + x^4 + x^2 + 1$
(1,24,3,17,25,27,18,8,2,9,26,10,11,19,16) (5,28,7,21,29,31,22,12,6,13,30,14,15,23,20)

$x^5 + x^3 + x + 1$
(1,25,26,10,11,18,8,2,9,27,19,17,24,3,16) (5,29,30,14,15,22,12,6,13,31,23,21,28,7,20)

5.1.11 Order 21

There are, apparently, two classes with the following cycle structures and eigenpolynomials.

$x^5 + x + 1$
(1,20,4,5,17,21,16) (2,10,8) (3,30,12,7,27,29,18,11,28,6,15,25,23,26,9,22,14,13,19,31,24)

x^5+x^4+1
(1,21,17,20,4,5,16) (2,10,8) (3,31,25,22,14,13,18,11,29,19,30,12,7,26,9,23,27,28,6,15,24)

5.1.12 Order 31

There are, apparently, six classes with the following eigenpolynomials and cycle structures.

$x^5+x^3+x^2+x+1$
(1,22,15,26,8,2,11,31,27,30,13,17,23,25,21,18,10,9,20,4,5,19,28,6,14,12,7,24,3,29,16)

$x^5+x^4+x^3+x^2+1$
(1,23,24,3,28,6,14,12,7,25,20,4,5,18,10,9,21,19,29,17,22,15,27,31,26,8,2,11,30,13,16)

x^5+x^2+1
(1,28,6,14,12,7,18,10,9,30,13,27,20,4,5,25,31,17,29,26,8,2,11,21,24,3,23,19,22,15,16)

$x^5+x^4+x^2+x+1$
(1,29,27,21,25,30,13,26,8,2,11,20,4,5,24,3,22,15,17,28,6,14,12,7,19,23,18,10,9,31,16)

x^5+x^3+1
(1,30,13,25,29,24,3,21,26,8,2,11,23,17,31,19,20,4,5,27,22,15,18,10,9,28,6,14,12,7,16)

$x^5+x^4+x^3+x+1$
(1,31,18,10,9,29,25,28,6,14,12,7,17,30,13,24,3,20,4,5,26,8,2,11,22,15,19,21,27,23,16)

5.2 The Groups

No systematic study has been done of any of the conjugacy classes except those isomorphic to S_5. There are (almost certainly) four conjugacy classes. The following are the generators, orbits and v-orbits of these classes. Only a little is known about their reducibilities, but they are almost certainly all reducible to one of the two 4×4 isomorphs. Known reducibilities are given below.

5.2.1 Class 1

This class is reducible and decomposable to the first class of 4×4 S_5 isomorphs. This class contains the standard representation of S_5.

$$\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&1&0\end{pmatrix}\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&1\\1&0&0&1&1\end{pmatrix}$$

Orbits
00000
00011 11010 01011 01010 11000

00001 10010 10011 01001 01000 10000 10001 11011 11001 00010
00101 10110 10111 01101 01100 10100 10101 11111 11101 00110
00111 11110 01111 01110 11100
00100

VOrbits
00000
00010 01001 10010 01000 10001
00001 10000 10011 11000 00011 11001 11011 11010 01010 01011
00101 10100 10111 11100 00111 11101 11111 11110 01110 01111
00110 01101 10110 01100 10101
00100

5.2.2 Class 2

This class is reducible and decomposable to the second class of 4×4 S_5 isomorphs.

$$\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&1&0\end{pmatrix}\begin{pmatrix}0&0&0&0&1\\0&0&0&1&1\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&1&1\end{pmatrix}$$

Orbits
00000
00001 10010 01011 01001 10001 10000 01000 11000 00011 00010 11010 11011 11001 10011 01010
00101 10110 01111 01101 10101 10100 01100 11100 00111 00110 11110 11111 11101 10111 01110
00100

VOrbits
00000
00001 10000 11001 10011 11000 00011 00010 11010 10010 01001 01010 01011 01000 10001 11011
00101 10100 11101 10111 11100 00111 00110 11110 10110 01101 01110 01111 01100 10101 11111
00100

5.2.3 Class 3

Probably reducible to the second class of 4×4 S_5 isomorphs.

$$\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&1&0\end{pmatrix}\begin{pmatrix}0&0&0&0&1\\0&0&0&1&1\\0&0&1&0&0\\1&0&1&0&0\\0&1&1&1&1\end{pmatrix}$$

Orbits
00000

00001 10010 01111 01001 10101 10000 01100 11100 00111 00110 11110 11111 11101 10111 01010
00010 01000 10100 00011 00101 10110 01011 01101 10001 11000 11010 11011 11001 10011 01110
00100

VOrbits
00000
00100 00111 11101 10111 11100 10101
00101 10100 11110 01110 01111 11111 01101 10110 01100 00110
00001 10000 11001 10011 11000 00011 00010 11010 10010 01001 01010 01011 01000 10001 11011

5.2.4 Class 4

Probably reducible to the second class of 4×4 S_5 isomorphs. Class 3 and 4 are probably transposes of one another.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Orbits
00000
00100 00101 10110 01110 01101 10100
00110 01100 10101 10111 01111 11111 11101 11100 00111 11110
00001 10010 01011 01001 10001 10000 01000 11000 00011 00010 11010 11011 11001 10011 01010

VOrbits
00000
00001 10000 11101 10111 11100 00111 00010 11010 10010 01001 01010 01011 01000 10101 11011
00011 11001 10100 10011 00101 11000 11110 00110 10110 01110 01101 01111 01100 10001 11111
00100

6 Conclusion

There is a lot more work to be done. This analysis was done mostly by brute force. What we really need is a systematic method to count the various classes without resorting to brute force enumeration.