ABSTRACT

Inverse Limits of Set-Valued Functions

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Much is known about inverse limits of compact spaces with continuous bonding maps. When the requirement that the bonding maps be continuous functions is relaxed, to allow for upper semi-continuous set-valued functions, many of the standard tools used to study inverse limit spaces are no longer available. Also, the structure of the inverse limit becomes more complicated.

In this dissertation, the structure of compact subsets of the inverse limit is studied. Necessary and sufficient conditions are given for a compact subset of an inverse limit to be the inverse limit of its projections. Weak crossovers are introduced and used to describe the relationship between a set and the inverse limit of its projections. In the process of examining weak crossovers, tools with potentially broader applications are developed.

When set-valued bonding maps are continuous and can be written as a union of continuous functions, sufficient conditions to make the inverse limit a Hausdorff continuum are given. It is also shown that, with another assumption, the inverse limit is in fact decomposable.

When the bonding map can be written as a union of set-valued functions between compact subsets of the factor space, a description of the inverse limit is given. This uses a generalization of the itinerary of a point.

Inverse Limits of Set-Valued Functions

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In Loving Memory of

Marie Cornelius

1951-2008

CHAPTER ONE

Introduction and Background

The road to inverse limits with set-valued bonding maps starts in the study of continuum theory. In Section 1.1, some basic definitions and examples from continuum theory are given. The examples illustrate the diversity of spaces which occur in continuum theory.

Basic definitions and results for inverse limits with continuous bonding functions are given in Section 1.2. Important results from inverse limits and relationships between inverse limits and both continuum theory and dynamical systems are also presented. Finally, the examples from Section 1.1 are discussed in the context of inverse limits.

A new development in the study of inverse limit spaces is to use bonding maps that are set-valued relations instead of functions. In this setting, many of the fundamental results from classical inverse limit theory fail to hold, but some extend with little or no modification. In Section 1.3 we review the existing literature on this new subject.

In Section 1.4, we compare and contrast inverse limits with set-valued bonding maps to inverse limits with continuous bonding functions, which motivates the research presented in this dissertation.

1.1 Continuum Theory

We start with some basic definitions and examples from continuum theory. These definitions and descriptions are standard. A good general reference is *Continuum Theory* by Nadler [30] (see also [19], [28], and [32]).

Definition 1.1. Two sets are *mutually separated* if and only if they are mutually exclusive and neither contains a limit point of the other. A topological space is *connected* if and only if it is not the union of two mutually separated subsets.

An *open cover* of a set X is a collection of open sets whose union is X. A *subcover*, S, of an open cover, C, is a subcollection of C such that S is a cover. A topological space is *compact* if and only if every open cover has a finite subcover.

A Hausdorff (metric) space is a *Hausdorff (metric) continuum* if and only if it is both compact and connected. In this dissertation, the word *continuum* refers to a Hausdorff continuum. A *(proper) subcontinuum* of a continuum X is a (proper) subset of X that is also a continuum.

Definition 1.2. [30] A continuum X is *decomposable* if and only if there are two proper subcontinua, H and K, of X such that $H \cup K = X$. A continuum X is *indecomposable* if and only if it is not decomposable. A continuum is *hereditarily indecomposable* if and only if each of its subcontinua is indecomposable.

Definition 1.3. [32] Let X be a continuum and $x \in X$. The *composant of x,* denoted by $Comp(x)$, is the union of all proper subcontinua of X which contain x.

A classical example of an indecomposable continuum is the Brouwer-Janiszewski-Knaster (B-J-K) continuum, also known as the buckethandle continuum. The B-J-K continuum is the closure of the ray in Figure 1.1.

The B-J-K continuum may be defined as follows. Let C be the Cantor Ternary set. Let C_0 be the union of all semicirlces in the upper half-plane with endpoints on C, which are symmetric about the line $x = 1/2$. For each $i \in \mathbb{N}$, let C_i be the union of all semicircles in the lower half-plane with endpoints on C, which are symmetric about the line $x = 5/(3^i * 2)$. Then the B-J-K continuum is $\cup_0^\infty C_i$; see [23] or [8]. A metric continuum is indecomposable if and only if it has uncountably many disjoint composants; see [22] and [28]. The ray in Figure 1.1 is sometimes called the visible composant of the B-J-K continuum.

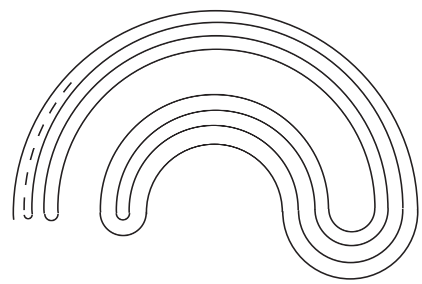

Figure 1.1. The dense ray of the B-J-K continuum

The Pseudo-Arc is an example of a hereditarily indecomposable continuum. To describe the Pseudo-Arc, we need several more definitions. Theorem 1.1 lists characterizations of the Pseudo-Arc that are less complicated than the following construction.

Definition 1.4. A *chain* is a finite collection $\mathcal{C} = \{C_1, ..., C_n\}$ of open sets such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The members of $\mathcal{C}$ are called *links.*

The *mesh* of a chain is $\max\{\text{diam}(C)|C$ is a link of $\mathcal{C}\}$. An ϵ-*chain* is a chain whose mesh is not greater then ϵ.

A continuum X is *chainable* if and only if it can be covered by an ϵ-chain for each $\epsilon > 0$.

Chainable continua are an important and widely studied class of continua, about which more will be said later. Chainability can be characterized with inverse limits, which will be seen in Section 1.2.

Definition 1.5. The *subchain* of a chain $\mathcal{C} = \{C_i, C_{i+1}, ..., C_{j-1}, C_j\}$ of $\mathcal{C}$ is denoted by $\mathcal{C}(i, j)$. In particular, $\mathcal{C}(i)$ denotes the link C_i.

A chain $\mathcal{D}$ *refines (strongly refines)* a chain $\mathcal{C}$ if and only if every link (the closure of every link) of $\mathcal{D}$ is a subset of some link of $\mathcal{C}$.

Two links C_i and C_j are *adjacent* if and only if $|i-j| = 1$. A chain $\mathcal{D} = \{D_1, ..., D_n\}$ is *crooked* in a chain $\mathcal{C} = \{C_1, ..., C_m\}$ if and only if it refines $\mathcal{C}$ and for each subchain $\mathcal{D}(i, j)$ such that D_i and D_j are contained in C_h and C_k respectively and $|h-k| \geq 2$, then there are links D_r and D_s where $i < r < s < j$ that are contained in links of $\mathcal{C}(h, k)$ adjacent to C_k and C_h respectively.

Figure 1.2 shows a chain $\mathcal{C} = \{C_m, C_{m+1}, ..., C_n\}$ with a crooked refinement $\mathcal{D} = \{D_{i-1}, ..., D_{j+1}\}$.

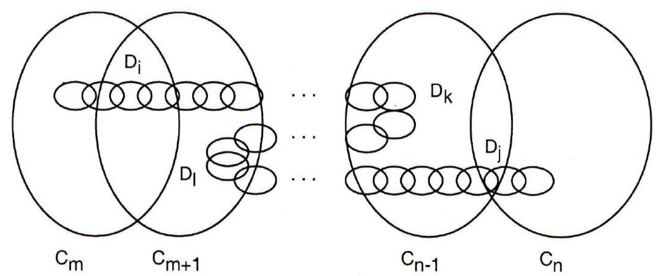

Figure 1.2. A crooked refinement $\mathcal{D}$ of a chain $\mathcal{C}$ ([24, Figure 2])

Consider the following description of the Pseudo-Arc. Let x and y be two distinct points of a compact metric space, and let $\mathcal{C}_1, \mathcal{C}_2, ...$ denote a sequence of chains from x to y that satisfies each of the following for each $i \in \mathbb{N}$.

(1) $\mathcal{C}_{i+1}$ strongly refines $\mathcal{C}_i$,

(2) $\mathcal{C}_{i+1}$ is crooked in $\mathcal{C}_i$,

(3) $\text{mesh}[\mathcal{C}_i] \to 0$.

Then the common part of $\bigcup \mathcal{C}_1, \bigcup \mathcal{C}_2, \bigcup \mathcal{C}_3, ...$ is a Pseudo-Arc. All Pseudo-Arcs are unique up to homeomorphism; see [29] or [6]. Theorem 1.1 characterizes the Pseudo-Arc. First, we need to define homogeneity.

Definition 1.6. A continuum X is *homogeneous* if and only if, for each pair x and y of points in X, there is a homeomorphism of X onto itself that takes x to y.

For example, the unit circle is homogeneous, whereas the unit interval is not.

Theorem 1.1 ([9]; [7], Example 1 and Theorem 1). *Suppose M is a nondegenerate metric continuum. The following are equivalent.*

(1) M is homeomorphic to the Pseudo-Arc.

(2) M is homogeneous and chainable.

(3) M is hereditarily indecomposable and chainable.

Definition 1.7. A continuum is an *arc* if and only if it is homeomorphic to $[0, 1]$.

A continuum is a *simple triod* if and only if it contains a point O and is the union of three arcs each having O as an end point and such that the common part of each two of them is O.

A continuum, X, is a *triod* if and only if it has a subcontinuum, H, such that $X - H$ has at least three components. A continuum is *atriodic* if and only if it fails to contain a triod.

A continuum, X, is an *n-od*, where $n \in \mathbb{N}$, if and only if it has a subcontinuum, H, such that $X - H$ has at least n components. A continuum, X, is an ∞-*od* if and only if there is a subcontinuum, H, such that $X - H$ has infinitely many components.

The collection of triods and n-ods is a subclass of the class decomposable continua. A simple triod is homeomorphic to the letter "T." The continuum $[0, 1] \times [0, 1]$ is an example of a triod that fails to be a simple triod. It is also an ∞-od. Chainable continua are atriodic; see [8].

The definition of a continuum requires only that a space be Hausdorff, compact, and connected. It is not surprising that the spectrum of continua is rich. Continua vary widely from the simplicity of the arc to the complexity of the Pseudo-Arc. Inverse limits give an efficient way to study continua using simpler descriptions and spaces as we shall see in the next section.

1.2 Inverse Limits

In this section, the definition of an inverse limit and some of the basic tools used in the study of inverse limits are given. We will discuss the examples from Section 1.1 in the context of inverse limits and consider the role of inverse limits in both continuum theory and dynamical systems. A good reference on inverse limits is *Inverse Limits* by W. T. Ingram [19] (see also [30] or [32]); a good reference for one-dimensional topological dynamics is *Topics from One-Dimensional Dynamics* [11] (see also [14].) Some fundamental results in inverse limits involve inverse limits with surjective bonding maps, subsets and inverse limits of their projections, connectedness of inverse limits, and the types of spaces that can arise in inverse limits. In this section, some results on these topics are given.

Definition 1.8. A *relation* on a set A is a subset of $A \times A$ such that each member of A is a first term of some pair in the relation. If $\preceq$ is a relation on a set A and (x, y) is in $\preceq$ then we write $x \preceq y$.

A *directed set* is a pair $(A, \preceq)$ where $\preceq$ is a relation on A such that

(1) if $a \in A$ then $a \preceq a$,

(2) if a, b, and c are in A and $a \preceq b$ and $b \preceq c$, then $a \preceq c$, and

(3) if a and b are in A then there is a member c in A such that $a \preceq c$ and $b \preceq c$.

Let A be a directed set. If $B \subset A$, then B *is cofinal in* A if and only if for each $\alpha \in A$, there is a $\beta \in B$ such that $\alpha \preceq \beta$.

Definition 1.9. Let D be a directed set, and for each $\alpha \in D$, let X_α be a topological space. Suppose that for each $\alpha, \beta \in D$, with $\alpha \preceq \beta$, $f_\alpha^\beta : X_\beta \to X_\alpha$ is a continuous function such that f_α^α is the identity on X_α, and if $\alpha \preceq \beta \preceq \gamma$, then $f_\alpha^\gamma = f_\alpha^\beta \circ f_\beta^\gamma$. Then $\{X_\alpha, f_\alpha^\beta, D\}$ is an *inverse system.* The spaces X_α are called *factor spaces,* and the mappings f_α^β are called *bonding maps.*

When the directed set is the natural numbers, $\mathbb{N}$, the inverse system is denoted $\{X_n, f_n\}$, instead of $\{X_n, f_n^m, \mathbb{N}\}$, and is called an *inverse sequence.* When each factor space, X_n, is some fixed space, X, and, each map f_n is some fixed mapping, f, the inverse sequence is written $\{X, f\}$.

Usually the directed set for inverse systems is the natural numbers. All new results in this dissertation are formulated in the context in which the directed set is the natural numbers.

Definition 1.10. Let $\{X_\alpha, f_\alpha^\beta, D\}$ be an inverse system. The *inverse limit* of the inverse system, denoted by $\varprojlim\{X_\alpha, f_\alpha^\beta, D\}$, is the subset of the product space, $\prod_{\alpha\in D} X_\alpha$, to which x belongs if and only if $f_\alpha^\beta(x_\beta) = x_\alpha$.

Theorem 1.2. *If $\{X_\alpha, f_\alpha^\beta, D\}$ is an inverse system and, for each $\alpha \in D$, X_α is a Hausdorff continuum, then $\varprojlim\{X_\alpha, f_\alpha^\beta, D\}$ is connected.*

As a consequence of Theorem 1.2, inverse limits are often used in the study of continuum theory.

Theorem 1.3. *Suppose $M = \varprojlim\{X_i, f_i\}$ and H and K are closed subsets of M with a common point. Set $H_i = \pi_i[H]$ and $K_i = \pi_i[K]$. Then $H\cap K = \varprojlim\{H_i\cap K_i, f_i|_{H_i\cap K_i}\}$. In particular, $H = \varprojlim\{H_i, f_i|_{H_i}\}$.*

This theorem is of great importance to the study of inverse limits as it allows one to regard closed subsets of an inverse limit space as inverse limits in their own right which retain the structure provided by the original inverse limit. Since many of the definitions in continuum theory are defined in terms of subcontinua, this theorem is crucial for studying continua in the setting of inverse limits.

Theorem 1.4. *Suppose D is a directed set, E is a cofinal subset of D, and $\{X_\alpha, f_\alpha^\beta, D\}$ is an inverse system where each X_α is a compact Hausdorff space. Then the inverse limit of $\{X_\alpha, f_\alpha^\beta, E\}$ is homeomorphic to the inverse limit of $\{X_\alpha, f_\alpha^\beta, D\}$.*

When the directed set is the natural numbers, this theorem is known as the Subsequence Theorem. However, in this general case, it should probably be known as the Subnet Theorem.

Theorem 1.5. *Suppose* $\{X, f\}$ *is an inverse sequence and* $M = \varprojlim\{X, f\}$*. Then,* $h : M \to M$*, given by* $h(x) = (x_2, x_3, x_4, ...)$*, where* $x = (x_1, x_2, ...)$*, is a homeomorphism.*

Definition 1.11. Suppose $\{X, f\}$ is an inverse sequence and $M = \varprojlim\{X, f\}$. The *shift map*, $h : M \to M$, is given by $h(x) = (x_2, x_3, x_4, ...)$, for each $x = (x_1, x_2, ...) \in M$.

The shift homeomorphism is an important tool in dynamics. It is valuable for modeling certain dynamical processes. For example, Theorem 1.6 relates the shift homeomorphism for an inverse limit with a single interval map to certain global attractors in the plane [5]; similar results exist in more general settings [33].

Theorem 1.6 (Barge and Martin, [5]). *Suppose* f *is a map of* $[-1, 1]$ *onto itself, and suppose* M *is the inverse limit of* $\{[-1, 1], f\}$*. Then* M *can be realized topologically as a global attractor for the plane with respect to a homeomorphism whose restriction to* M *is conjugate to the shift homeomorphism* $\hat{f}$ *on* M*.*

Another tool for the study of inverse limits is Morton Brown's Approximation Theorem; see [10]. For this theorem we need the following definition.

Definition 1.12. If f and g are continuous functions from a space X into a compact metric space Y, then $d(f, g) = sup\{d(f(x), g(x))|x \in X\}$.

Theorem 1.7 (Morton Brown, [10]). *Let* $M = \varprojlim\{X_i, f_i\}$ *where, for each* $i \in \mathbb{N}$*,* X_i *is a compact metric space. For each* $i \in \mathbb{N}$*, let* K_i *be a non-empty collection of maps of* X_{i+1} *into* X_i*. Suppose, for each positive integer* i *and each* $\epsilon > 0$*, there is a member* g *of* K_i *such that* $d(f_i, g) < \epsilon$*. Then, there is a sequence* $g_1, g_2, g_3, ...$ *such that* $g_i \in K_i$ *and* M *is homeomorphic to* $\varprojlim\{X_i, g_i\}$*.*

Definition 1.13. Let $f : [0,1] \to [0,1]$. The map is *unimodal* if

(1) $f(0) = f(1) = 0$.

(2) f has a unique critical point c with $0 < c < 1$.

(3) $f|_{[0,c]}$ is increasing.

(4) $f|_{[c,1]}$ is decreasing.

The point c is also called the *turning point.*

Since the range of a unimodal map, f, is the interval $[0, f(c)]$, it follows that if x is an element of the inverse limit $\varprojlim\{[0,1], f\}$, $x_n \in [0, f(c)]$ for each $n \in \mathbb{N}$. It follows that $\varprojlim\{[0,1], f\}$ can be rewritten as an inverse limit with surjective bonding maps, namely $\varprojlim\{[0, f(c)], f|_{[0,f(c)]}\}$. For any inverse limit with a continuous bonding map, the Surjective Map Theorem gives a subset of the domain that is f-invariant and the map restricted to this subset gives the inverse limit.

Theorem 1.8. *Suppose α is an interval and $f : \alpha \to \alpha$. If $\beta = \bigcap_{n\in\mathbb{N}} f^n[\alpha]$, then $\varprojlim\{\alpha, f\} = \varprojlim\{\beta, f|_\beta\}$.*

In dynamics, the itinerary of a point, x, of an inverse limit gives an easy, and effective, way to encode information about the orbit of x. Sometimes this encoding completely determines the composants of an indecomposable continuum [12].

Definition 1.14. Let $f : [0,1] \to [0,1]$ be unimodal with turning point c. For each $x \in \varprojlim\{[0,1], f\}$, the *itinerary of* x is given by $I(x) = (I_1(x), I_2(x), I_3(x), ...)$ where

$$I_j(x) = \begin{cases} 0 & \text{if } x_j < c, \\ * & \text{if } x_j = c, \\ 1 & \text{if } x_j > c. \end{cases}$$

The standard definition of itinerary is not that which is given in Definition 1.14. However, the definition given is often called the backwards itinerary of x.

Related to both itineraries and indecomposability is the "two-pass" condition.

Theorem 1.9. *If* $f : [a,b] \to [a,b]$ *is surjective and has the property that there exist two subintervals, whose intersection is at most a point,* α *and* β*, of* $[a,b]$ *such that* $f[\alpha] = f[\beta] = [a,b]$ *then* $\varprojlim\{[a,b], f\}$ *is an indecomposable continuum.*

The hypotheses of Theorem 1.9 are satisfied when $[a,b] = [0,1]$ and f is the full tent-map. Figure 1.3 is an illustration of the tent map. The inverse limit, $\varprojlim\{[0,1], f\}$, is homeomorphic to the B-J-K continuum.

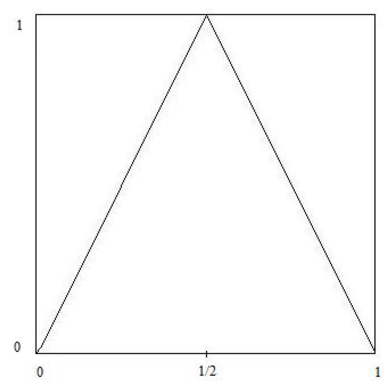

Figure 1.3. The Full Tent Map

The Pseudo-Arc is homeomorphic to the inverse limit, $\varprojlim\{[0,1], g\}$, where g is the Henderson Map in Figure 1.4; see [15] and [27]. The construction of the Henderson map is nontrivial, but "its construction may be described roughly as starting with $g(x) = x^2$ and notching its graph with an infinite set of non-intersecting v's which accumulate at (1,1)," [15].

As seen in the previous two examples, it is possible to construct complicated continua using inverse limits with relatively simple factor spaces. Other examples demonstrating this can be found in [1] and [17].

Theorem 1.10 (J. R. Isbell, [21]). *A metric continuum is chainable if and only if it is homeomorphic to an inverse limit on* $[0,1]$.

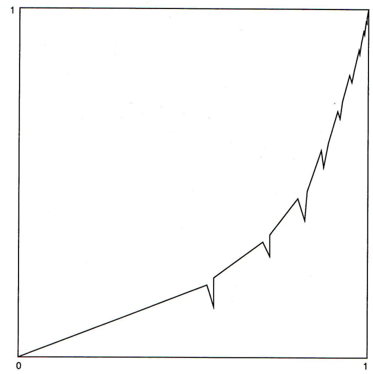

Figure 1.4. The Henderson Map ([24, Figure 5])

Since chainable continua are atriodic, it is not possible to obtain an n-od, or in particular a simple triod, as an inverse limit on $[0, 1]$.

1.3 Inverse Limits and Set-Valued Functions

In 2004, Mahavier pioneered the study of inverse limits with set-valued bonding maps in "Inverse Limits with Subsets of $[0, 1] \times [0, 1]$," [26]. He introduces inverse limits on [0,1], denoted by I, where the bonding maps need not be functions but are relations whose graphs are closed subsets of $I \times I$. He defines the inverse limit in the following way.

Definition 1.15. [26] Let $M = (M_1, M_2, M_3, ...)$ be a sequence of closed subsets of I^2. Then the *inverse limit*, $\varprojlim M$, is the subspace of the Hilbert cube such that $x \in \varprojlim M$ if and only if $(x_{i+1}, x_i) \in M_i$ for each $i \in \mathbb{N}$.

Definition 1.16. Let X and Y be compact Hausdorff spaces. Denote the set of compact subsets of Y by 2^Y. A function $F : X \to 2^Y$ is *upper semi-continuous* at $x \in X$ if and only if for each open set V in Y containing $F(x)$, there is an open set U in X containing x such that if $u \in U$, then $F(u) \subset V$. Then F is *upper semi-continuous* if and only if F is upper semi-continuous for each $x \in X$.

In Theorem 1 of [26], Mahavier shows that if M is a closed subset of I^2, such that $M_x = I$, then M has a related upper semi-continuous set-valued function, F, such that $G(F) = M$, where $G(F)$ is the graph of F. The next theorem is Mahavier's sufficient condition for connectedness.

Definition 1.17. Let X and Y be compact Hasudorff spaces and $F : X \to 2^Y$ be upper semi-continuous. A *vertical section of* F is a set of the form $F(x)$ for some $x \in X$.

Theorem 1.11. *[26] Let* $M = (M_1, M_2, M_3, ...)$ *be a sequence of closed subsets of* I^2. *Then* $\varprojlim M$ *is a continuum if each vertical section of* M *is connected.*

He also conjectures that $\varprojlim M$ either contains a Hilbert cube or is one dimensional.

In 2006, in "Inverse Limits of Upper Semi-continuous Set-Valued Functions," [20], W. T. Ingram and Mahavier generalized Mahavier's paper, [26]. They expand the theory to include inverse limits of compact Hausdorff spaces with upper semi-continuous set-valued bonding maps.

Definition 1.18. For each $n \in \mathbb{N}$, let X_n be a topological space and $F_n : X_{n+1} \to 2^{X_n}$ be an upper semi-continuous set-valued function. Then $\{X_n, F_n\}$ is an *inverse system*, and the *inverse limit*, $\varprojlim\{X_n, F_n\}$, is the subset of $\prod_{n \in \mathbb{N}} X_n$ such that $x \in \varprojlim\{X_n, F_n\}$ if and only if $x_i \in F_i(x_{i+1})$ for each $i \in \mathbb{N}$.

The following theorem is Theorem 1 in [20]. It is a useful theorem for studying inverse limits with set-valued functions since it states that a set-valued function is upper semi-continuous if and only if its graph is closed.

Theorem 1.12 (Ingram and Mahavier, [20]). *Suppose each of* X *and* Y *is a compact Hausdorff space and* M *is a subset of* $X \times Y$ *such that if* $x \in X$ *then there is a point* $y \in Y$ *such that* $(x, y) \in M$. *Then* M *is closed if and only if there is an upper semi-continuous set-valued function* $F : X \to 2^Y$ *such that* $M = G(F)$.

Ingram and Mahavier then give a sequence of theorems which yield their sufficient condition for connectedness.

Theorem 1.13 (Ingram and Mahavier, [20]). *Suppose that for each $n \in \mathbb{N}$, X_n is a Hausdorff continuum, $F_n : X_{n+1} \to 2^{X_i}$ is an upper semi-continuous set-valued function, and for each $x \in X_{n+1}$, $F_n(x)$ is connected. Then $\varprojlim\{X_n, F_n\}$ is a Hausdorff continuum.*

Theorem 1.14 (Ingram and Mahavier, [20]). *Suppose $X_1, X_2, X_3, ...$ and $Y_1, Y_2, Y_3, ...$ are sequences of compact Hausdorff spaces and, for each positive integer i, $f_i : X_{i+1} \to 2^{X_i}$ and $g_i : Y_{i+1} \to 2^{Y_i}$ are upper semi-continuous functions. Let $\phi_i : X_i \to Y_i$ be a continuous function for each $i \in \mathbb{N}$. Suppose further that $\phi_i \circ f_i = g_i \circ \phi_{i+1}$. The function $\phi : \varprojlim\{X_n, f_n\} \to \varprojlim\{Y_n, g_n\}$ given by $\phi(x) = (\phi_1(x_1), \phi_2(x_2), ...)$ is continuous. Furthermore, ϕ is injective (surjective) if each ϕ_i is injective (surjective).*

Ingram and Mahavier conclude the paper by giving many examples where $X_i = [0,1]$ for each $i \in \mathbb{N}$. One example, [20, Example 5], answers Mahavier's dimension conjecture that $\varprojlim M$ either contains a Hilbert cube or is one dimensional in the negative. Moreover, it implies that for each $n \in \mathbb{N}$, there is a closed subset of I^2 such that the inverse limit is n-diminsional.

In 2006, Iztok Banič began publishing a series of papers [2], [3], and [4], which further develop the theory of inverse limits with set-valued bonding maps. The main focus of Banič's work, motivated by Ingram and Mahavier's work, is on the dimension of the inverse limits [2].

In 2008 in "Inverse Limits with Set-Valued Functions," [31], Van Nall shows that it is not possible to obtain a disk as an inverse limit on $[0,1]$. In doing so, he introduces a new concept, the crossover between two points.

Definition 1.19. [31] For each $n \in \mathbb{N}$, let X_n be a topological space. Let $x, y \in \prod_{n\in\mathbb{N}} X_n$ such that $x_i = y_i$. The i^{th}*-crossover between* x *and* y, denoted by $Cr_i(x, y)$, is given by $Cr_i(x, y) = (x_1, ..., x_{i-1}, x_i, y_{i+1}, y_{i+2}, ...)$.

Let $A \subset \prod_{n\in\mathbb{N}} X_n$. Then $Cr(A) = \{z \in \prod_{n\in\mathbb{N}} X_n |$ there are $j \in \mathbb{N}$ and $x, y \in A$ such that $z = Cr_j(x, y)\}$. The set $Cr(A)$ is called the *set of all crossovers of* A.

Nall gives only one main result, [31, Theorem 3.2], using crossovers. Even though it is the only theorem dealing with crossovers appearing in his paper, it is the starting point for much of the study of inverse limits presented in this dissertation.

Theorem 1.15. *[31] Suppose each* X_i *is a compact space, and* M *is a compact subset of* $\prod_{i\in\mathbb{N}} X_i$*, and* $M_i = \pi_i[M]$ *for each* $i \in \mathbb{N}$*. Then the following are equivalent:*

(1) There exist set-valued functions $g_i : M_{i+1} \to 2^{M_i}$ *such that* $M = \varprojlim\{M_i, g_i\}$.

(2) There exist upper semi-continuous set-valued functions $f_i : M_{i+1} \to 2^{M_i}$ *such that* $M = \varprojlim\{M_i, f_i\}$*. (In fact,* $f_i = \pi_i \circ \pi_{i+1}^{-1}$*.)*

(3) $Cr(M) = M$*, in other words,* M *contains all crossovers.*

Nall then continues work on the dimension of inverse limits focusing on inverse limits on compact separable metric spaces with a single bonding map.

The following example illustrates that an inverse limit with simple factor spaces and simple bonding maps can be complicated in this setting.

Example 1.1. Let $X = \{0, 1\}$ and denote $\prod_{\mathbb{N}} X$ by $\{0, 1\}^\omega$. Let $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. To see that the inverse limit of X with F is the Cantor set of points $\{0, 1\}^\omega$, let $x \in \{0, 1\}^\omega$. Since $F(z) = X$ for each $z \in X$, $x_i \in F(x_{i+1})$ for each $i \in \mathbb{N}$. So $\{0, 1\}^\omega \subset \varprojlim\{X, F\}$. Now, let $x \in \varprojlim\{X, F\}$. Then $x_i \in X$ for each $i \in \mathbb{N}$, and so $x \in \{0, 1\}^\omega$. Hence $\varprojlim\{X, F\} \subset \{0, 1\}^\omega$. Thus $\varprojlim\{X, F\} = \{0, 1\}^\omega$.

1.4 Motivations

Theorem 1.2 states that the inverse limit of Hausdorff continua is connected. In fact, an inverse limit of compact Hausdorff spaces with continuous bonding functions is connected if and only if the factor spaces are connected. Necessary and sufficient conditions for an inverse limit with set-valued bonding maps to be connected remain elusive. In [20, Theorem 7], Ingram and Mahavier show that an inverse limit with set-valued bonding maps is connected when each vertical section of each bonding map is connected.

Theorem 1.3 states that a closed subset of an inverse limit with continuous bonding functions is the inverse limit of its projections. It is not true that a closed subset of an inverse limit with upper semi-continuous set-valued bonding maps is the inverse limit of its projections.

Theorem 1.4 states that the inverse limit of an inverse system along a directed set D and the inverse limit along a cofinal subset of D are homeomorphic. This fails to extend to the setting in which the bonding maps are set-valued, even for the special case when the directed set is the natural numbers; see Example 2.4 and [20, Example 3]. However, in Theorem 2.4, sufficient conditions are given to guarantee the inverse limits along the directed set and cofinal subset are homeomorphic.

Theorem 1.5 states that the shift map is a homeomorphism for inverse limits with continuous bonding functions. It is not a homeomorphism for inverse limits with set-valued bonding maps. It is still a continuous and surjective function; see [20, Theorem 13].

As noted after Theorem 1.10, it is not possible to obtain a simple triod, or any n-od, as the inverse limit along I with continuous bonding functions. It is not known if it is possible to obtain a simple triod as an inverse limit with a single set-valued bonding map. It is possible to obtain an n-od, and indeed an ∞-od, from an inverse limit with a single bonding map on I. For example Mahavier constructs a cone over a Cantor set in [26, Example 4].

The following table contrasts some properties of inverse limits with bonding maps that are functions to those with bonding maps that are set-valued.

Table 1.1. Properties of Inverse Limits: Functions vs. Set-Valued Relations

Properties of Inverse Limits	Function Bonding Maps	Set-Valued Bonding Maps
Inverse limits of Hausdorff spaces are Hausdorff	Yes	Yes
Inverse limits of compact Hausdorff spaces are compact	Yes	Yes
Inverse limits of continua are necessarily connected	Yes	No
Subsequence Theorem holds	Yes	No
Closed sets are always inverse limits of their projections	Yes	No
Shift map is necessarily 1-1	Yes	No
Simple triod from a single bonding map on I	No	?
Morton Brown's Approximation Theorem holds	Yes	?

The following questions are motivated by these differences and are addressed in this dissertation.

Question. Is a subcompactum of an inverse limit with upper semi-continous set-valued bonding maps an inverse limit of its projections? If not, what is the relationship between the set and the inverse limit of its projections?

These are important questions because many concepts in continuum theory are defined in terms of subcompacta. In general, it is not true that a closed subset of an inverse limit space is the inverse limit of its projections when the bonding maps are set-valued. Theorem 2.9 provides necessary and sufficient conditions for a closed set to be the inverse limit of its projections. Theorem 4.4 describes, for the general case, the relationship between a set and the inverse limit of its projections.

Question. What spaces arise as an inverse limit on I with only one set-valued bonding map? What is possible with a sequence of set-valued bonding maps?

When the bonding maps of an inverse sequence can be written as a union of continuous functions Corollary 5.1 states sufficient conditions for an inverse limit to be an n-od. In this setting Theorem 5.3 also provides sufficient conditions for inverse limits to be decomposable. Theorem 6.1 provides sufficient conditions for an inverse limit with a single bonding to be an ∞-od and have a structure similar to a fan.

Question. Is an inverse limit, using set-valued bonding maps, of a continuum a continuum?

In Theorem 5.2, sufficient conditions for inverse limits to be connected are given, however, these conditions restrict the family of set-valued bonding maps.

In Chapters 2 through 4, necessary and sufficient conditions are given for which a set is the inverse limit of its projections using restrictions of the original bonding maps. When a set fails to satisfy these properties, a description of the inverse limit of its projections is given. Central to the relationship between a subcompactum of an inverse limit and the inverse limit of its projections is the notion of weak crossovers. This notion, motivated by Theorem 1.15, is defined and many properties of it are explored.

In Chapter 2, weak crossovers are defined and it is shown that a compact set is the inverse limit of its projections, with natural restrictions on the bonding maps, if and only if the set contains all weak crossovers. This motivates the study, in Chapter 3, of the set $Cw(A)$, which is the union of A with all weak crossovers, and the operator Cw. In Chapter 4, properties of multiple weak crossovers are given. This allows one to give an explicit description of the inverse limit of the projections of a compact set.

In Chapter 5 we consider bonding maps that can be written as unions of continuous functions. This restriction on the bonding maps is useful for addressing connectedness, decomposability, and the presence of n-ods.

In Chapter 6 inverse limits with a single factor space and set-valued bonding map are considered. When the factor space and the bonding map have a particular structure, the inverse limit in shown to be an ∞-od and to have a structure similar to a cone over a Cantor set.

CHAPTER TWO

Inverse Limits of Projections

In Section 1, preliminary concepts are developed. Theorem 2.2 states that an inverse limit with an upper semi-continuous set-valued bonding map, F, is the inverse limit of F restricted to its core. Theorem 2.3 shows that, unlike in conventional inverse limits, a closed subset of the inverse limit space is not necessarily the inverse limit of its projections. This leads to the investigation of this structure in Sections 2 and 3.

In Section 2, Property P and crossovers are defined. Theorem 2.6 states that a closed subset of the inverse limit is the inverse limit of its projections when it has Property P and contains all crossovers. Lemma 2.6 gives a method for constructing a sequence of points that converge to a given point of the inverse limit space, which turns out to be a useful lemma in several results.

In Section 3, the weak crossover between two points is defined. It follows that a set contains all weak crossovers if and only if it has Property P and contains all crossovers (Theorem 2.8). It is shown that a compact set is the inverse limit of its projections with the defined restrictions if and only if it contains all weak crossovers (Theorem 2.9). Weak crossovers are a useful tool as they allow a description of subsets of the inverse limit; see Chapter 4.

2.1 Preliminary Definitions

Notation. Let X be a compact Hausdorff space. Then 2^X is the set of compact subsets of X. Let $F : X \to 2^X$ be a set-valued function. Then, for each $n \in \mathbb{N}$, $F^n(x) = F[F^{n-1}(x)]$, where $F^0(x) = X$.

Definition 2.1. [20] Let X and Y be compact Hausdorff spaces. A function $F : X \to 2^Y$ is *upper semi-continuous* at $x \in X$ if and only if for each open set V in Y

containing $F(x)$, there is an open set U in X containing x such that if $u \in U$, then $F(u) \subset V$. Then F is *upper semi-continuous* if and only if F is upper semi-continuous for each $x \in X$.

Definition 2.2. For each $n \in \mathbb{N}$, let X_n be a topological space and $F_n : X_{n+1} \to 2^{X_n}$ be an upper semi-continuous set-valued function. Then $\{X_n, F_n\}$ is an *inverse system*, and the *inverse limit*, $\varprojlim\{X_n, F_n\}$, is the subset of $\prod_{n\in\mathbb{N}} X_n$ such that $x \in \varprojlim\{X_n, F_n\}$ if and only if $x_i \in F_i(x_{i+1})$ for each $i \in \mathbb{N}$.

Definition 2.3. An upper semi-continuous function $F : X \to 2^Y$ is *surjective* provided for each $y \in Y$, there is a $x \in X$ such that $\{y\} \subset F(x)$.

Lemma 2.1. *Let $\{X_n, F_n\}$ be an inverse sequence where for each $n \in \mathbb{N}$, X_n is a compact Hausdorff space and $F_n : X_{n+1} \to 2^{X_n}$ is upper semi-continuous. Let $M = \varprojlim\{X_n, F_n\}$ and set $M_n = \pi_n[M]$. Then F_n is surjective for each $n \in \mathbb{N}$ if and only if $M_n = X_n$ for each $n \in \mathbb{N}$.*

Proof. Suppose $M_n = X_n$ for each $n \in \mathbb{N}$. Let $x_N \in X_N$. Then $x_N \in M_N$ and so there is an $x \in M$ such that $\pi_N(x) = x_N$. Since $x \in M$, $\pi_{N+1}(x) = x_{N+1} \in M_{N+1}$. So $x_{N+1} \in X_{N+1}$ and $x_N \in F_N(x_{N+1})$. Thus F_N is surjective.

Let $N \in \mathbb{N}$, and let $x_N \in X_N$. Since F_n is surjective for each $n \in \mathbb{N}$, for each $n \geq N$, there is an $x_n \in X_n$ such that $x_{n-1} \in F_n(x_n)$. Since $F_n : X_{n+1} \to 2^{X_n}$ for each $0 < n \leq N$, there is an $x_n \in F_n(x_{n+1})$ for each $0 < n \leq N$ with $x_{N-1} \in F_N(x_N)$. Let $x = (x_1, x_2, ..., x_{N-1}, x_N, ...)$. Then $x \in M$ and $\pi_N(x) = x_N \in M_N$. Thus $M_N = X_N$. Therefore $M_n = X_n$ for each $n \in \mathbb{N}$. □

Definition 2.4. Let X be a compact Hausdorff space and $F : X \to 2^X$ be a set-valued function. The *core of F* is given by $C_F := \bigcap_{\mathbb{N}_0} F^n(X)$.

Lemma 2.2. *Let X be a compact Hausdorff space and $F : X \to 2^X$ be surjective. Then $C_F = X$.*

Proof. Since F is surjective, $F(X) = X$. Then $F^2(X) = F(F(X)) = F(X) = X$. Proceeding inductively, we see that $F^n(X) = X$ for each $n \in \mathbb{N}$. Hence, $\bigcap_{\mathbb{N}_0} F^n(X) = X = C_F$. □

Lemma 2.3. *Let X be a compact Hausdorff space and $F : X \to 2^X$ be upper semi-continuous. Set $M = \varprojlim\{X, F\}$. Then $M_i = C_F$ for each $i \in \mathbb{N}$.*

Proof. To see that $M_i \subset C_F$ for each $i \in \mathbb{N}$, fix $j \in \mathbb{N}$ and let $z \in M_j$. Then there is a $\hat{z} \in M$ such that $\hat{z}_j = z$. Then $\hat{z}_n \in F(\hat{z}_{n+1})$ for each $n \in \mathbb{N}$. Then $\hat{z}_j \in F(\hat{z}_{j+1})$, $\hat{z}_j \in F^2(\hat{z}_{j+2})$, ..., and in general, $\hat{z}_j \in F^n(\hat{z}_{j+n})$ for each $n \in \mathbb{N}$. Hence $\hat{z}_j = z \in C_F$. So $M_j \subset C_F$. Hence $M_i \subset C_F$ for each $i \in \mathbb{N}$.

To see that $C_F \subset M_i$ for each $i \in \mathbb{N}$, fix $j \in \mathbb{N}$. For each $n \in \mathbb{N}$, set $A_n = \{z \in \prod_{\mathbb{N}} X | z_k \in F(z_{k+1}) \text{for } k < n\}$. Then $A_n = \bigcap_{k=1}^{n-1}\{z \in \prod_{\mathbb{N}} X | z_k \in F(z_{k+1})\}$. Note that A_n is closed for each $n \in \mathbb{N}$ and that $M = \bigcap_{\mathbb{N}} A_n$. Let $x \in C_F$. Then, for each $n \geq j$, there is an $\hat{x}^n \in A_n$ such that $\hat{x}^n_j = x$. Note that $\hat{x}^n \in A_k$ for each $k \leq n$. Then $\hat{x}^n \to \hat{x}$ and $\hat{x} \in M$. By the construction of $\hat{x}$, $\hat{x}_j = x$, and so $x \in M_j$. Hence $C_F \subset M_j$. Thus $C_F \subset M_i$ for each $i \in \mathbb{N}$. □

Theorem 2.1. *Let X be a compact Hausdorff space and $F : X \to 2^X$ be upper semi-continuous. Then $F(C_F) = C_F$.*

Proof. To see that $F(C_F) \subset C_F$, let $z \in C_F$. Then $z \in F^n(X)$ for each $n \in \mathbb{N}_0$. So $F(z) \subset F^{n+1}(X)$ for each $n \in \mathbb{N}_0$. Hence $F(z) \subset C_F$.

Now, to see that $C_F \subset F(C_F)$, let $z \in C_F$. By Lemma 2.3, $C_F = M_i$ for each $i \in \mathbb{N}$. So there is a $\hat{z} \in M$ such that $\hat{z}_1 = z$. Since $\hat{z} \in M$, $\hat{z}_1 \in F(\hat{z}_2)$, and so $z \in F(\hat{z}_2)$. By Lemma 2.3, $\hat{z}_2 \in C_F$. Thus $z \in F(C_F)$.

Therefore, $F(C_F) = C_F$. □

Definition 2.5. For each $i \in \mathbb{N}$, let X_i be a topological space and let $F_i : X_{i+1} \to 2^{X_i}$ be a set-valued function. Set $M = \varprojlim\{X_i, F_i\}$, and let $K \subset M$. For each $i \in \mathbb{N}$, let

$K_i = \pi_i[K]$. The *restriction of F_i to the projections of K*, denoted by $F_i|_K$, is given by $F_i|_K(x) = K_i \cap F_i(x)$ for each $x \in K_{i+1}$. Note that $F_i|_K : K_{i+1} \to 2^{K_i}$. Also note that $K \subset \varprojlim\{K_i, F_i|_K\}$.

Remark 2.1. For each $i \in \mathbb{N}$, let X_i be a compact Hausdorff space. Let $A \subset \prod_{i\in\mathbb{N}} X_i$. In [31], $f_i : A_{i+1} \to 2^{A_i}$ is defined to be $f_i = p_i \circ p_{i+1}^{-1}$, where $p_i = \pi_i|_A$, where $\pi_i|_A$ is the projection map restricted to A, for each $i \in \mathbb{N}$.

Notation. Let X be a topological space and $x, y \in X$. Then x^j is x repeated j-many times, $x^j = (x, x, ..., x)$, and $x^\infty = (x, x, ...)$. Then $x^j y^i$ is x^j concatenated with y^i, $x^j y^i = (x, x, ..., x, y, y, ..., y)$.

Example 2.1. Let $X = \{0, 1, 2\}$ and denote $\prod_{\mathbb{N}} X$ by $\{0, 1, 2\}^\omega$. Let $F : X \to 2^X$ be defined by $F(0) = F(1) = F(2) = X$. To see that the inverse limit of X with F is the Cantor set of points $\{0, 1, 2\}^\omega$, let $x \in \{0, 1, 2\}^\omega$. Since $F(z) = X$ for each $z \in X$, $x_i \in F(x_{i+1})$ for each $i \in \mathbb{N}$. So $\{0, 1, 2\}^\omega \subset \varprojlim\{X, F\}$. Now, let $x \in \varprojlim\{X, F\}$. Then $x_i \in X$ for each $i \in \mathbb{N}$, and so $x \in \{0, 1, 2\}^\omega$. Hence $\varprojlim\{X, F\} \subset \{0, 1, 2\}^\omega$. Thus $\varprojlim\{X, F\} = \{0, 1, 2\}^\omega$.

Let $A = \{0^\infty\} \cup \{1^\infty\}$. Then A is compact.

We now consider the maps in the preceding definition and remark in the context of this example. For each $i \in \mathbb{N}$, $F_i|_A(0) = A_i \cap F_i(0) = \{0, 1\} \cap X = \{0, 1\}$ and $F_i|_A(1) = A_i \cap F_i(1) = \{0, 1\} \cap X = \{0, 1\}$. So, $\varprojlim\{A_i, F_i|_A\} = \{0, 1\}^\omega$. Evidently A is not the inverse limit of its projections with the restricted bonding map.

On the other hand, for each $i \in \mathbb{N}$, $f_i(0) = p_i(0^\infty) = 0$ and $f_i(1) = p_i(1^\infty) = 1$. So in this example, for each $i \in \mathbb{N}$, f_i is the identity map on A. Hence $\varprojlim\{A_i, f_i\} = \{0^\infty, 1^\infty\}$. So A is the inverse limit of its projections with these particular bonding maps. Indeed, in [31], Nall shows that, for any compact set A, A is the inverse limit of its projections with the bonding maps f_i if and only if $Cr(A) = A$.

However, there are simple examples of compact sets which are the inverse limit of neither of the maps $F_i|_A$ nor f_i, as seen in the following example.

Example 2.2. Let $K \subset M$ such that $K = \{0^\infty, 1^\infty, (01)^\infty\}$. For each $i \in \mathbb{N}$, let $F'_i : K_{i+1} \to 2^{K_i}$ be upper semi-continuous such that $G(F'_i) \subset G(F_i)$. Since $0^\infty, 1^\infty \in K$, $F'_i(0) \ni 0$ and $F'_i(1) \ni 1$ for each $i \in \mathbb{N}$. Since $(01)^\infty \in K$, there is an $N \in \mathbb{N}$ such that $F'_N(1) \ni 0$. Then the point $0^{N-1}1^\infty \in \varprojlim\{K_i, F'_i\}$. Thus $K \subsetneq \varprojlim\{K_i, F'_i\}$.

From these examples it is apparent that the structure of compact subsets in inverse limit spaces with upper semi-continuous bonding maps is not easily ascertained. In general, $F_i|_A$ is more readily obtained than f_i since the latter involves a deeper understanding of the complexities of A and the former requires only knowledge of the projections of A. In contrast, the general structure of the inverse limit with $F_i|_A$ preserves more of the complexity of the ambient inverse limit space than f_i.

Lemma 2.4. *Let $A \subset M$. Then $G(f_i) \subset G(F_i|_A) \subset G(F_i)$.*

Proof. Let $(x, y) \in G(f_i)$. Then $x \in A_{i+1}, y \in A_i$ such that $y \in f_i(x)$. Since $f_i = p_i \circ p_{i+1}^{-1}$, and since $y \in f_i(x)$, there is an $\hat{x} \in p_{i+1}^{-1}(x) \subset A$ such that $p_i(\hat{x}) = y$. Since $\hat{x} \in A$, $\hat{x}_i \in A_i$, $\hat{x}_{i+1} \in A_{i+1}$, and $\hat{x}_i \in F_i|_A(\hat{x}_{i+1})$. So $(\hat{x}_{i+1}, \hat{x}_i) \in G(F_i|_A)$. Hence $(x, y) \in G(F_i|_A)$. So $G(f_i) \subset G(F_i|_A)$.

Let $(x, y) \in G(F_i|_A)$. Then $x \in A_{i+1}$, $y \in A_i$, and $y \in F_i|_A(x)$. So $y \in F_i(x)$. Thus $(x, y) \in G(F_i)$. So $G(F_i|_A) \subset G(F_i)$. □

Theorem 2.2. *Let X be a compact Hausdorff space and $F : X \to 2^X$ be upper semi-continuous. Then $M = \varprojlim\{X, F\} = \varprojlim\{C_F, F|_{C_F}\}$.*

Proof. Clearly $\varprojlim\{C_F, F|_{C_F}\} \subset M$. Let $z \in M$. By Lemma 2.3, $M_i = C_F$ for each $i \in \mathbb{N}$. So $z_i \in C_F$ for each $i \in \mathbb{N}$. By Theorem 2.1, $F(C_F) \subset C_F$, so for each $x \in C_F$, $F|_{C_F}(x) = F(x)$. Hence $z_i \in F|_{C_F}(z_{i+1})$ for each $i \in \mathbb{N}$. Thus $z \in \varprojlim\{C_F, F|_{C_F}\}$. □

Theorem 2.3. *Suppose $\{X_n, F_n\}$ is an inverse sequence where, for each $n \in \mathbb{N}$, X_n is a compact Hausdorff space and F_n is an upper semi-continuous function from X_{n+1} into 2^{X_n}. Suppose H and K are closed subsets of $M = \varprojlim\{X_n, F_n\}$. For each $n \in \mathbb{N}$, define G_n to be the upper semi-continuous function from $H_{n+1} \cap K_{n+1}$ into $2^{H_n \cap K_n}$ such that $y \in G_n(x)$ if and only if $x \in H_{n+1} \cap K_{n+1}$, $y \in H_n \cap K_n$, and $y \in F_n(x)$. Then $H \cap K \subset \varprojlim\{H_n \cap K_n, G_n\}$.*

Proof. Let $x \in H \cap K$. Then for each $n \in \mathbb{N}$, $x_n \in H_n$ and $x \in K_n$. Hence $x \in H_n \cap K_n$ for each $n \in \mathbb{N}$. Since $x \in M$, $x_n \in F_n(x_{n+1})$ for each $n \in \mathbb{N}$. So, for each $n \in \mathbb{N}$, $x_n \in G_n(x_{n+1})$, and so $x \in \varprojlim\{H_n \cap K_n, G_n\}$. Hence $H \cap K \subset \varprojlim\{H_n \cap K_n, G_n\}$. $\square$

Remark 2.2. Theorem 2.3 gives a subset relationship between a compact set and the inverse limit of its projections when $H = \varprojlim\{X_n, F_n\}$. In general, this is the best one can hope for as the next example shows. What structure, if any, allows a compact set to be the inverse limit of its projections? The rest of Chapter 2 answers this question.

Example 2.3. Let $X = \{0, 1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Notice that $\varprojlim\{X, F\}$ is the set of all sequences of $0's$ and $1's$. Let $K = \{0^\infty, 1^\infty\}$. Then $K_i = X$ for each $i \in \mathbb{N}$ and so the inverse limit of the projections of K is just $\varprojlim\{X, F\}$.

Remark 2.3. The Subsequence Theorem from inverse limits does not hold when the bonding maps are set-valued. The following lemma generalizes the Subsequence Theorem to an extent.

Theorem 2.4. *Let $\{X_n, f_n\}$ be an inverse system, where, for each $n \in \mathbb{N}$, X_n is a compact Hausdorff space. Let f_n be an upper semi-continuous set-valued function for each $n \in \mathbb{N}$. Let $\{n_i\}$ be an increasing sequence of positive integers. Suppose f_n is a continuous function if and only if $n \notin \{n_i\}_{\mathbb{N}}$. Then $\varprojlim\{X_{n_i}, f_{n_i}^{n_{i+1}}\}$ is homeomorphic to $\varprojlim\{X_n, f_n\}$.*

Proof. Let $M = \varprojlim\{X_n, f_n\}$ and $M' = \varprojlim\{X_{n_i}, f_{n_i}^{n_{i+1}}\}$. Define $h : M' \to M$ by

$$h(x) = (f_1^{n_1}(x_{n_1}), ..., f_{n_1-1}^{n_1}(x_{n_1}), x_{n_1}, f_{n_1+1}^{n_2}(x_{n_2}), ...,$$

$$f_{n_2-1}^{n_2}(x_{n_2}), x_{n_2}, f_{n_2+1}^{n_3}(x_{n_3}), ...,$$

$$x_{n_i}, f_{n_i+1}^{n_{i+1}}(x_{n_{i+1}}), ..., f_{n_{i+1}-1}^{n_{i+1}}(x_{n_{i+1}}), x_{n_{i+1}}, ...).$$

Then for $i \in \mathbb{N}$, h_{n_i} is the identity. For each $j \notin \{n_i\}_{\mathbb{N}}$, f_j is a continuous function. So $f_{n_i+j}^{n_{i+1}}$ is a continuous function for each $i \in \mathbb{N}_0$, where $n_0 = 0$, and where $n_i < j < n_{i+1}$. Hence h is a well-defined function and is also continuous.

To see that h is surjective, let $z \in M$. Let $x = (z_{n_1}, z_{n_2}, z_{n_3}, ...)$.

Then

$$h(x) = (f_1^{n_1}(z_{n_1}), ..., f_{n_1-1}^{n_1}(z_{n_1}), z_{n_1}, f_{n_1+1}^{n_2}(z_{n_2}), ...,$$

$$f_{n_2-1}^{n_2}(z_{n_2}), z_{n_2}, f_{n_2+1}^{n_3}(z_{n_3}), ...,$$

$$z_{n_i}, f_{n_i+1}^{n_{i+1}}(z_{n_{i+1}}), ..., f_{n_{i+1}-1}^{n_{i+1}}(z_{n_{i+1}}), z_{n_{i+1}}, ...).$$

By construction of h, $h(x) = (z_1, z_2, ..., z_{n_1}, ...) = z$. Hence h is surjective.

To see that h is injective, let $x, y \in M'$ where $x \neq y$. Then there is an $m \in \mathbb{N}$ such that $x_{n_m} \neq y_{n_m}$. Set $\hat{x} = h(x)$ and $\hat{y} = h(y)$. Then $\hat{x}_{n_m} \neq \hat{y}_{n_m}$. Hence h is injective.

Since M' is compact and M is Hausdorff, h is a homeomorphism. □

Remark 2.4. To see that the Subsequence Theorem does not generalize completely, consider the following example.

Example 2.4. Let $X = \{0, 1\}$, $F : X \to 2^X$ be given by $F(0) = F(1) = X$. Let $f : X \to X$ be given by $f(X) = \{0\}$. If $n \in \mathbb{N}$ is odd, let $G_n = f$ and if $n \in \mathbb{N}$ is even, let $G_n = F$. Then $F \circ f = F$ and $f \circ F = f$. Theorem 2.4 gives that the inverse limit, $\varprojlim\{X, F \circ f\}$, is homeomorphic to the inverse limit $\varprojlim\{X, G_n\}$, which is a Cantor set. The inverse limit $\varprojlim\{X, f \circ F\}$ is the single point 0^∞. So the inverse limit along different subsequences may yield quite different structures.

2.2 Crossovers and Property P

Definition 2.6. A set $K \subset M = \varprojlim\{X_i, F_i\}$ is said to *have Property P with respect to* $\{F_i\}$ provided for each $i \in \mathbb{N}$, if $x \in K_{i+1}$ and $y \in K_i$ such that $y \in F_i(x)$, then there is a $z \in K$ such that $z_i = y$ and $z_{i+1} = x$. K will be said to have *Property P* when it is clear to which collection of set-valued functions the property refers.

Remark 2.5. Let $A \subset M$. Then A has Property P with respect to $\{F_i\}$ if and only if A has Property P with respect to $\{F_i|_A\}$.

Lemma 2.5. *Let $A \subset M$. Then A has Property P with respect to $\{f_i\}$.*

Proof. Let $x \in A_i$, $y \in A_{i+1}$, such that $x \in f_i(y)$. Since $x \in f_i(y)$, there is a $z \in A$ such that $z_i = x$ and $z \in p_{i+1}^{-1}(y)$. Hence there is a $z \in A$ such that $z_{i+1} = y$ and $z_i = x$. Thus A has Property P with respect to $\{f_i\}$. □

Theorem 2.5. *Let $A \subset M$. Then A has Property P with respect to $\{F_i\}$ if and only if $G(F_i|_A) = G(f_i)$.*

Proof. Suppose A has Property P with respect to $\{F_i\}$. By Lemma 2.4, $G(f_i) \subset G(F_i|_A)$. Let $(x, y) \in G(F_i|_A)$. Then $x \in A_{i+1}$, $y \in A_i$, such that $y \in F_i|_A(x)$. Since A has Property P, there is a $z \in A$ such that $z_i = y$ and $z_{i+1} = x$. So $z \in p_{i+1}^{-1}(x)$ and $p_i(z) = y$. Hence $y \in f_i(x)$. Thus $(x, y) \in G(f_i)$.

Now suppose $G(F_i|_A) = G(f_i)$. Since the graph of $F_i|_A$ is the same as the graph of f_i, by [20], $F_i|_A = f_i$. By Lemma 2.5, A has Property P with respect to $\{f_i\}$. Thus A has Property P with respect to $\{F_i|_A\}$. □

Remark 2.6. If K has Property P with respect to $\{F_i\}$, then K does not necessarily contain all crossovers; see Definition 1.19. Consider the following example.

Example 2.5. Let $X = \{0, 1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. For each $i \in \mathbb{N}_0$, let $\hat{x}^i = 0^i 1^\infty$ and $\hat{y}^i = 1^i 0^\infty$. Set $K = \{\hat{x}^i | i \in \mathbb{N}_0\} \cup \{\hat{y}^i | i \in \mathbb{N}_0\}$. Let $n \in \mathbb{N}$. Then $\hat{x}^n_{n+1} = 1 = \hat{y}^{n+1}_{n+1}$. Let $z = Cr_{n+1}(\hat{x}^n, \hat{y}^{n+1})$. By construction of $\hat{x}^n$ and

$\hat{y}^{n+1}$, $\hat{x}^n_i = 0$ for each $i < n$ and $\hat{y}^{n+1}_i = 0$ for each $i > n$. So $z = 0^n10^\infty$, and hence $z \in Cr(K) - K$.

To see that K has Property P, let $i \in \mathbb{N}$ and let $x, y \in X$. Then $y \in F(x)$. If $x = y = 0$ or $x = y = 1$, then $\hat{x}^{i+1}$ or $\hat{y}^{i+1}$ satisfies, respectively, the necessary conditions. If $x = 0$ and $y = 1$, then $\hat{x}^i$ satisfies the necessary conditions. If $x = 1$ and $y = 0$, then $\hat{y}^i$ satisfies the necessary conditions. So K has Property P, but fails to contain all crossovers.

Remark 2.7. A set can contain all crossovers, but fail to have Property P.

Example 2.6. Let $X = \{0, 1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Let $K = \{0^\infty, 1^\infty\}$. Since $(0^\infty)_i \neq (1^\infty)_i$ for each $i \in \mathbb{N}$, $Cr_i(x, y)$ is defined if and only if $x = y$. Hence $Cr(K) = K$.

Notice that for $i = 1$, both $0, 1 \in X$, and $0 \in F(1)$. If z is a point of the inverse limit where $z_1 = 0$ and $z_2 = 1$, then $z \notin K$. So K fails to have Property P.

Lemma 2.6. *Let $A \subset M$ such that A has Property P with respect to $\{F_i\}$ and that $Cr(A) = A$. Let $z \in M$ such that $z_i \in A_i$ for each $i \in \mathbb{N}$. Then there is $\{z^i\} \subset A$ such that $z^i \to z$. Moreover, $z^i_r = z_r$ for each $r \leq i + 1$.*

Proof. Since A has Property P, for each $i \in \mathbb{N}$ there is a $\hat{z}^i \in A$ such that $\hat{z}^i_i = z_i$ and $\hat{z}^i_{i+1} = z_{i+1}$. Let $z^1 = \hat{z}^1$. Define $z^2 = Cr_2(z^1, \hat{z}^2)$. Since $z^1, \hat{z}^2 \in A$, and since $Cr(A) = A$, $z^2 \in A$. Note that $z^2_i = z_i$ for each $i \leq 3$. Proceeding in this manner, for each $i > 1$, define $z^i = Cr_i(z^{i-1}, \hat{z}^i)$. Since $Cr(A) = A$, $z^i \in A$ for each $i \in \mathbb{N}$. Since $z^i_r = z_r$ for each $r \leq i + 1$, $z^i \to z$. □

Theorem 2.6. *Let $K \subset M = \varprojlim\{X_i, F_i\}$ be closed, where X_i is compact Hausdorff and F_i is upper semi-continuous for each $i \in \mathbb{N}$. Then $K = \varprojlim\{K_i, F_i|_K\}$ if and only if K has Property P and K contains all crossovers.*

Proof. Suppose K has Property P with respect to $\{F_i\}$ and that $Cr(K) = K$. It suffices to show that $\varprojlim\{K_i, F_i|_K\} \subset K$. Let $z \in \varprojlim\{K_i, F_i|_K\}$. Then, for each

$i \in \mathbb{N}$, $z_i \in K_i$ and $z_i \in K_i \cap F_i|_K(z_{i+1})$. Then, by Lemma 2.6, there is $\{z^i\} \subset K$ such that $z^i \to z$. Since K is closed, $z \in K$. So $K = \varprojlim\{K_i, F_i|_K\}$.

Now, suppose $K = \varprojlim\{K_i, F_i|_K\}$. Then K is compact and $Cr(K) = K$. Let $i \in \mathbb{N}$, $x \in K_{i+1}$, and $y \in K_i$ such that $y \in F_i(x)$. Then there is a $z, w \in K$ such that $z_{i+1} = x$ and $w_i = y$. Set $p = (w_1, w_2, ..., w_{i-1}, y, x, z_{i+2}, z_{i+3}, ...)$.

Since $w \in K$, $w_j \in K_j$ and $w_j \in F_j|_{K_{j+1}}(w_{j+1})$ for each $1 \leq j < i$. Since $z \in K$, $z_j \in K_j$, and $z_j \in F_j|_{K_{j+1}}(z_{j+1})$ for each $j > i$. By hypothesis, $y \in F_j|_{K_{j+1}}(x)$. So $p_j \in K_j$ and $p_j \in F_j|_{K_{j+1}}(p_{j+1})$ for each $j \in \mathbb{N}$. Hence $p \in \varprojlim\{K_i, F_i|_K\} = K$. Thus K has Property P with respect to $\{F_i|_K\}$, and hence with respect to $\{F_i\}$. □

Remark 2.8. By [31, Theorem 3.2], if a compact subset, H, of a product space of compact sets, X_i, can be written as the inverse limit of its projections, H_i with some upper semi-continuous set-valued bonding maps, $\{F_i\}$, then H contains all crossovers. This also implies that H has Property P with respect to $\{F_i\}$.

Corollary 2.1. *For each $i \in \mathbb{N}$, let X_i be compact. If $H \subset \prod_{i\in\mathbb{N}} X_i$ is compact, and there is a sequence of upper semi-continuous functions, $F_i : H_{i+1} \to 2^{H_i}$ such that $H = \varprojlim\{H_i, F_i\}$, then H has Property P with respect to $\{F_i\}$.*

Lemma 2.7. *Let $X = \varprojlim\{X_i, F_i\}$ and $Y = \varprojlim\{Y_i, G_i\}$. Suppose there is a bijection, h, of X onto Y such that $h(x) = (h_1(x), h_2(x), ...)$ where, for each $i \in \mathbb{N}$, $h_i : X_i \twoheadrightarrow Y_i$ is a bijection. Then $h_i \circ F_i = G_i \circ h_{i+1}$ for each $i \in \mathbb{N}$.*

Proof. Let $x \in X$ and set $y = h(x)$. Let $i \in \mathbb{N}$. Then $h_{i+1}(x_{i+1}) = y_{i+1}$ and $h_i(x_i) = y_i$. Let $z \in F_i(x_{i+1})$. Then there is a $p \in X$ such that $p_i = z$ and $p_{i+1} = x_{i+1}$. Let $q = h(p)$. Then $q_i = h_i(p_i) = h_i(z)$ and $q_{i+1} = h_{i+1}(x_{i+1}) = y_{i+1}$. Since $q \in Y$, $h_i(z) \in G_i(y_{i+1})$. So $h_i(z) \in G_i \circ h_{i+1}(x_{i+1})$. Hence $h_i \circ F_i(x_{i+1}) \subset G_i \circ h_{i+1}(x_{i+1})$.

Now, let $z \in G_i \circ h_{i+1}(x_{i+1})$. Then there is a $p \in Y$ such that $p_i = z$ and $p_{i+1} = y_{i+1}$. Let $q = h^{-1}(p)$.

Then $q_i = h_i^{-1}(z)$ and $q_{i+1} = h_{i+1}^{-1}(p_{i+1}) = h_{i+1}^{-1}(y_{i+1}) = x_{i+1}$. Since $q \in X$, $q_i \in F_i(q_{i+1})$, so $h_i^{-1}(z) \in F_i(x_{i+1})$. Hence $z \in h_i \circ F_i(x_{i+1})$. So $G_i \circ h_{i+1}(x_{i+1}) \subset h_i \circ F_i(x_{i+1})$.

Thus $h_i \circ F_i = G_i \circ h_{i+1}$ for each $i \in \mathbb{N}$. □

Theorem 2.7. *Let* $X = \varprojlim\{X_i, F_i\}$ *and* $Y = \varprojlim\{Y_i, G_i\}$. *Let* $K \subset X$. *Suppose there is a bijection,* h, *of* X *onto* Y *such that* $h(x) = (h_1(x), h_2(x), ...)$ *where, for each* $i \in \mathbb{N}$, $h_i : X_i \to Y_i$ *is a bijection. Set* $H = h[K]$. *Then* $Cr(K) = K$ *if and only if* $Cr(H) = H$, *and* K *has Property* P *with respect to* $\{F_i\}$ *if and only if* H *has Property* P *with respect to* $\{G_i\}$.

Proof. Suppose $Cr(K) = K$. Let $w \in Cr(H)$. Set $z = h^{-1}(w)$. Since $w \in Cr(H)$, there is $i \in \mathbb{N}$, $p, q \in H$ such that $w = Cr_i(p, q)$ where $p_i = q_i$. Set $x = h^{-1}(p)$ and $y = h^{-1}(q)$. Then $x, y \in K$ and $x_i = h_i^{-1}(p_i) = h_i^{-1}(q_i) = y_i$. Then $z = Cr_i(x, y)$ and so $z \in K$. Hence $w \in H$. Thus $Cr(H) = H$.

Now suppose K has Property P with respect to $\{F_i\}$. Let $z \in H_{i+1}$ and $w \in H_i$ such that $w \in G_i(z)$. Let $x = h_{i+1}^{-1}(z)$ and $y = h_i^{-1}(w)$. Then $x \in K_{i+1}$ and $y \in K_i$. Since $w \in G_i(z)$, $h_i(y) \in G_i(h_{i+1}(x))$. So $h_i(y) \in h_i \circ F_i(x)$ by Lemma 2.7. Hence $y \in F_i(x)$. Since K has Property P with respect to $\{F_i\}$, there is an $\alpha \in K$ such that $\alpha_i = y$ and $\alpha_{i+1} = x$. Let $\beta = h(\alpha)$. Then $\beta \in H$, $\beta_i = h_i(\alpha_i) = w$, and $\beta_{i+1} = h_{i+1}(\alpha_{i+1}) = z$. Hence H has Property P with respect to $\{G_i\}$. □

In this section, Property P is defined. This property is compared and contrasted with the crossovers of a set. It is shown that these properties are distinct from each other. Theorem 2.6 states that a closed set is the inverse limit of its projections if and only if it has both properties. Theorem 2.7 states that if there are bijections between the factor spaces of two inverse systems then Property P and crossovers are preserved by the induced bijection.

2.3 Weak Crossovers

Theorem 2.6 states that for a set K, $K = \varprojlim\{K_i, F_i|_K\}$ if and only if K has Property P and K contains all crossovers. The following definition is motivated by this relationship between Property P and crossovers.

Definition 2.7. Let $M = \varprojlim\{X_i, F_i\}$. Let $i \in \mathbb{N}$, $x, y \in M$, where $x_i \in F_i(y_{i+1})$. The i^{th} *weak crossover between* x *and* y, is given by $Cw_i(x, y) = (x_1, ..., x_i, y_{i+1}, y_{i+2}, ...)$.

For $K \subset M$, the *set of all weak crossovers of* K, denoted by $Cw(K)$, is given by $C_w(K) = \{z \in M|$ there is an $i \in \mathbb{N}, x, y \in K$ such that $z = Cw_i(x, y)\}$. If $Cw(K) = K$, then K is said to *contain all weak crossovers.*

Remark 2.9. For a subset K of $M = \varprojlim\{X_i, F_i\}$, $K \subset Cw(K)$ since for each $z \in K$, $Cw_1(z, z) = z$. Thus $Cw(K) = K$ if and only if $Cw(K) \subset K$.

Remark 2.10. Let A be a subset of an inverse limit space. Then $Cr(A) \subset Cw(A)$ since, if $z \in Cr(A)$, then $z = Cr_i(x, y)$, where $x, y \in A$, and so $z = Cw_i(x, y)$.

Theorem 2.8. *Let* $M = \varprojlim\{X_i, F_i\}$*, and let* $K \subset M$*. Then* $Cw(K) = K$ *if and only if* $Cr(K) = K$ *and* K *has Property P.*

Proof. First suppose $Cw(K) = K$ and let $z \in Cr(K)$. Then there is an $i \in \mathbb{N}, x, y \in K$ such that $z = Cr_i(x, y) = (x_1, ..., x_i, y_{i+1}, ...)$ where $x_i = y_i$. Since $x_i = y_i$, $x_i \in F_i(y_{i+1})$, and so $z = Cw_i(x, y) \in K$. Hence $Cr(K) = K$.

To see that K has Property P, let $i \in \mathbb{N}, x' \in K_{i+1}$, and $y' \in K_i$ such that $y' \in F_i(x')$. Since $y' \in K_i$ and $x' \in K_{i+1}$, there are $x, y \in K$ such that $y_i = y'$ and $x_{i+1} = x'$. Let $z = Cw_i(y, x)$. Then $z = (y_1, ..., y_i, x_{i+1}, x_{i+2}, ...) \in K$, $z_i = y_i = y'$, and $z_{i+1} = x_{i+1} = x'$. Hence K has Property P.

Now suppose $Cr(K) = K$ and K has Property P. Let $z \in Cw(K)$. Then there is an $i \in \mathbb{N}$, and $p, q \in K$ such that $z = Cw_i(p, q) = (p_1, ..., p_i, q_{i+1}, ...)$ where $p_i \in F_i(q_{i+1})$. Since K has Property P, there is an $x \in K$ such that $x_i = p_i$

and $x_{i+1} = q_{i+1}$. Let $\hat{z} = Cr_{i+1}(x, q) = (x_1, ..., x_i, q_{i+1}, q_{i+2}, ...)$. Then $Cr_i(p, \hat{z}) = (p_1, ..., p_i, \hat{z}_{i+1}, ...) = (p_1, ..., p_i, q_{i+1}, ...) = z$. Since $Cr(K) = K$, $\hat{z} \in K$, and so $z \in K$. □

Remark 2.11. In Example 2.5, K has Property P but $Cr(K) \neq K$. So $K \subsetneq Cw(K)$. In Example 2.6, $Cr(K) = K$ but K fails to have Property P and so $K \subsetneq Cw(K)$.

Theorem 2.9. *Let $K \subset M = \varprojlim\{X_i, F_i\}$ be closed. Then $K = \varprojlim\{K_i, F_i|_K\}$ if and only if $Cw(K) = K$.*

Proof. This follows from Theorem 2.6 and Theorem 2.8. □

Remark 2.12. In an inverse limit space, $M = \varprojlim\{X_i, F_i\}$, the structure of a compact subset, K, can be nice, in the sense that $Cw(K) = K$, so that K is the inverse limit of its projections with the restricted bonding maps. However, if K is a compact subset of M such that $K \subsetneq Cw(K)$, there does not necessarily exist a collection of upper semi-continuous functions, $\{F_i'\}$, where $G(F_i') \subset G(F_i)$ for each i, such that $K = \varprojlim\{K_i, F_i'\}$. Example 2.2 demonstrates this. By [31, Theorem 3.2], in the special case when K is compact and $Cr(K) = K$, $K = \varprojlim\{K_i, f_i\}$ where $f_i = p_i \circ p_{i+1}^{-1}$, as seen in Example 2.1.

Remark 2.13. In [31] Theorem 3.2, if a compact subset, H, of a product space of compact sets, X_i, can be written as the inverse limit of its projections, H_i, with some upper semi-continuous set-valued bonding maps, $\{F_i\}$, then H contains all crossovers. In light of Theorem 2.8, this also implies that H contains all weak crossovers.

Corollary 2.2. *For each $i \in \mathbb{N}$, let X_i be compact. If $H \subset \prod_{i \in \mathbb{N}} X_i$ is compact, and there is a sequence of upper semi-continuous functions, $F_i : H_{i+1} \to 2^{H_i}$, such that $H = \varprojlim\{H_i, F_i\}$, then $Cw(H) = H$.*

Corollary 2.3. *Let* $K \subset M = \varprojlim\{X_i, F_i\}$ *be closed. Then* $K = M$ *if and only if* $K_i = X_i$ *for each* $i \in \mathbb{N}$ *and* $Cw(K) = K$.

Theorem 2.10. *Let* $X = \varprojlim\{X_i, F_i\}$ *and* $Y = \varprojlim\{Y_i, G_i\}$. *Let* $K \subset X$. *Suppose there is a bijection* h *of* X *onto* Y *such that* $h(x) = (h_1(x), h_2(x), ...)$ *where, for each* $i \in \mathbb{N}$, $h_i : X_i \twoheadrightarrow Y_i$ *is a bijection. Set* $H = h[K]$. *Then* $Cw(K) = K$ *if and only if* $Cw(H) = H$.

Proof. Suppose $Cw(K) = K$. By Theorem 2.8, $Cr(K) = K$ and K has Property P with respect to $\{F_i\}$. By Theorem 2.7, $Cr(H) = H$ and H has Property P with respect to $\{G_i\}$. By Theorem 2.8 $Cw(H) = H$.

Since h is a homeomorphism, $Cw(H) = H$ implying $Cw(K) = K$ follows similarly. □

CHAPTER THREE

Properties of Weak Crossovers

In the previous chapter, it was shown that a compact set, K, is the inverse limit of its projections if and only if $Cw(K) = K$. That result motivates the study, in this chapter, of the Cw operator and of sets which contain all weak crossovers. Theorem 3.1 gives properties of sets that are preserved by Cw. Theorem 3.2 gives properties of sets which contain all weak crossovers. These results play a pivotal role in Chapter 4, where the relationship between a set and the inverse limit of its projections is explicitly described.

Lemma 3.1. *If $\{x_\alpha\}_A$ is a net and B is cofinal in A, then $\{x_\beta\}_B$ is a subnet of $\{x_\alpha\}_A$.*

Proof. Let $i : B \to A$ be the inclusion map. Then i is a cofinal function since B is cofinal in A. Note that i is increasing. □

Lemma 3.2. *If A is a directed set and is the union of finitely many sets, A_n, then, for at least one n, A_n is cofinal in A.*

Proof. Let $A = \bigcup_{n=1}^{N} A_n$ and suppose that for each $1 \leq n \leq N$, A_n is not cofinal in A. Then for each $1 \leq n \leq N$, there is an $\lambda_n \in A$ such that $\alpha \leq \lambda_n$ for each $\alpha \in A_n$. Then there is a $\lambda \in A$ such that $\lambda \geq \lambda_n$ for each $1 \leq n \leq N$. There is an $1 \leq M \leq N$ such that $\lambda \in A_M$. Then for each $\alpha \in A$, $\alpha \leq \lambda_n \leq \lambda$ for some $1 \leq n \leq N$. So A_M is cofinal in A. □

Theorem 3.1. *Let M be an inverse limit space and let $A \subset M$ be non-empty. Then*

(1) $A_i = \pi_i[Cw(A)]$ for each $i \in \mathbb{N}$.

(2) $Cw(A)$ has Property P.

(3) If M is Hausdorff, and if A is compact, then $Cw(A)$ is compact.

(4) If A is a sub-basic open set, then $Cw(A) = A$.

(5) If A is a basic open set, then $Cw(A) = A$.

Proof. (1) Let $i \in \mathbb{N}$. Since $A \subset Cw(A)$, $\pi_i(A) \subset \pi_i[Cw(A)]$. Let $x \in \pi_i[Cw(A)]$. Then there is a $z \in Cw(A)$ such that $z_i = x$. Then there are $j \in \mathbb{N}, p, q \in A$ such that $z = Cw_j(p,q) = (p_1, ..., p_j, q_{j+1}, ...)$. So $z_r \in \pi_r(A)$ for each $r \in \mathbb{N}$. In particular $x = z_i \in \pi_i(A)$. Thus $\pi_i(A) = \pi_i[Cw(A)]$.

(2) Let $x \in \pi_{i+1}[Cw(A)], y \in \pi_i[Cw(A)]$ such that $y \in F_i(x)$. Then $x \in A_{i+1}$ and $y \in A_i$. Then there is an $\hat{x} \in A$ and $\hat{y} \in A$ such that $\hat{x}_{i+1} = x$ and $\hat{y}_i = y$. Then $z = Cw_i(\hat{y}, \hat{x})$ exists and $z \in Cw(A)$. Thus $Cw(A)$ has Property P.

(3) Suppose A is compact. Let z be a limit point of $Cw(A)$. Then there is a net $\{z^\alpha\}_{\mathcal{A}} \subset Cw(A)$ such that $z^\alpha \to z$. Then for each $\alpha \in \mathcal{A}$, there is $i_\alpha \in \mathbb{N}$, $p^\alpha, q^\alpha \in A$, such that $z^\alpha = Cw_{i_\alpha}(p^\alpha, q^\alpha)$. Consider the net $\{i_\alpha\}_{\mathcal{A}}$.

Case 1: There is a subnet, $\{i_\mu\}_{\mathcal{A}'}$, of $\{i_\alpha\}_{\mathcal{A}}$ such that $i_\mu \to \infty$.

Then $\{z^\mu\}_{\mathcal{A}'}$ is a subnet of $\{z^\alpha\}_{\mathcal{A}}$ and so $z^\mu \to z$. Note that $z^\mu = Cw_{i_\mu}(p^\mu, q^\mu)$; so $z^\mu_r = p^\mu_r$ for each $r \leq i_\mu$. Since $\{p^\mu\}_{\mathcal{A}'} \subset A$ is a net and since A is compact, there is a subnet $\{p^\gamma\}_{\mathcal{A}''}$ of $\{p^\mu\}_{\mathcal{A}'}$ and a $p \in A$ such that $p^\gamma \to p$. Since $i_\mu \to \infty$, $i_\gamma \to \infty$ and since $z^\gamma_r = p^\gamma_r$ for each $r \leq i_\gamma$, $z^\gamma \to p$. Since M is Hausdorff, and since $\{z^\gamma\}_{\mathcal{A}''}$ is a net, $p = z$. Hence $z \in A$ and so $z \in Cw(A)$. Thus $Cw(A)$ is compact.

Case 2: If $\{i_\mu\}_{\mathcal{A}'}$ is a subnet of $\{i_\alpha\}_{\mathcal{A}}$, then there is an $N \in \mathbb{N}$ such that $i_\mu \leq N$ for each $\mu \in \mathcal{A}'$.

Let $\{i_\mu\}_{\mathcal{A}'}$ be a subnet of $\{i_\alpha\}_{\mathcal{A}}$. Let $n \in \mathbb{N}$ and let $U_n = \{\mu \in \mathcal{A}' | i_\mu = n\}$. Then $\bigcup_{n=1}^{N} U_n = \mathcal{A}'$. By Lemma 3.2, there is an $M \leq N$ such that U_M is cofinal in $\mathcal{A}'$. Then, by Lemma 3.1, $\{i_\gamma\}_{U_M}$ is a subnet of $\{i_\alpha\}_{\mathcal{A}}$ and $i_\gamma = M$ for each $\gamma \in U_M$. Also, $\{z^\gamma\}_{U_M}$ is a subnet of $\{z^\alpha\}_{\mathcal{A}}$, and so $z^\gamma \to z$. Since $\{p^\gamma\}_{U_M} \subset A$ is a net, and since A is compact, there is a subnet $\{p^\delta\}_{U'_M}$ and $p \in A$ such that $p^\delta \to p$. Then $\{q^\delta\}_{U'_M} \subset A$ is a net. So there is a subnet $\{q^\epsilon\}_{U''_M}$ and $q \in A$ such that $q^\epsilon \to q$.

Then $\{p^\epsilon\}_{U''_M}$ is a subnet of $\{p^\delta\}_{U'_M}$ and $\{z^\epsilon\}_{U''_M}$ is a subnet of $\{z^\delta\}_{U'_M}$, and so $z^\epsilon \to z$. Note that $z^\epsilon = Cw_M(p^\epsilon, q^\epsilon)$. So, for each $\epsilon \in U''_M$, $z^\epsilon_r = p^\epsilon_r$ for each $r \leq M$ and $z^\epsilon_r = q^\epsilon_r$ for each $r > M$. Since $p^\epsilon \to p$, $p_r = z_r$ for each $r \leq M$. Similarly, $q_r = z_r$ for each $r > M$. So $Cw_M(p, q) = z$. Hence $Cw(A)$ is compact.

(4) Since S is a sub-basic open set, there is an $n \in \mathbb{N}$ and set $O \subset X_n$ such that $S = \pi_n^{-1}[O]$. Let $z \in Cw(S)$. Then there is an $i \in \mathbb{N}$, and $x, y \in S$ such that $z = Cw_i(x, y)$. If $i < n$, then $z_n = y_n \in O$, so $z \in S$. If $i \geq n$, then $z_n = x_n \in O$, so $z \in S$. Thus $Cw(S) = S$.

(5) Suppose A is a basic open set. Then $A = \bigcap \mathcal{S}$ where $\mathcal{S}$ is a finite collection of sub-basic open sets. Let $z \in Cw(A)$. Then there is an $i \in \mathbb{N}$ nd $x, y \in \bigcap \mathcal{S}$ such that $z = Cw_i(x, y)$. Then $x, y \in S$ for each $S \in \mathcal{S}$ and so $z \in Cw(S)$ for each $S \in \mathcal{S}$. By (4), $Cw(S) = S$ for each $S \in \mathcal{S}$, so $z \in S$ for each $S \in \mathcal{S}$. Thus $z \in \bigcap \mathcal{S} = A$. Hence $Cw(A) = A$. $\square$

Example 3.1. It is possible for a set K to be such that $Cw(K) = K$, $Cr(K) = K$, and K has Property P, but K fails to be compact.

Let $X = \{0, 1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Set $K = M - \{z \in M | z = A0^\infty \text{ where } |A| < \infty\}$. Then K is not compact.

Let $z \in Cw(K)$. Then there is $x, y \in K$, $i \in \mathbb{N}$ such that $z = Cw_i(x, y)$. Since $y \in K$, for each $j \in \mathbb{N}$, there is an $r > j$ such that $y_r = 1$. So for each $j > i \in \mathbb{N}$, there is an $l > j$ such that $z_l = 1$. So $z \in K$. Hence $Cw(K) = K$.

Remark 3.1. Let $K \subset M$ such that $Cw(K)$ has Property P. Then it need not be the case that K has Property P. A set K need not be compact even if $Cw(K)$ is.

Example 3.2. Let $X = \{0, 1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Let $K = \{0^\infty, 1^\infty\}$. By Example 2.6, K fails to have Property P. Hence K is a set such fails to have Property P and $Cw(K)$ has Property P.

Example 3.3. To see the converse of (3) fails, let $X = \{0,1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Let $M = \varprojlim\{X, F\}$. Set $K = M - \{0^\infty\}$. Then K is not compact and $Cw(K) = M$ is compact.

Remark 3.2. If $K \subset M$, then $Cw(K)$ has Property P. However, $Cw(K)$ need not contain all crossovers, even if K does contain all crossovers. So, in general, $Cw(K)$ does not contain all weak crossovers.

Example 3.4. Let $X = \{0,1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Let $M = \varprojlim\{X, F\}$. Set $K = \{0^\infty, 1^\infty\}$. Then $Cr(K) = K$, see Example 2.6, and $Cw(K) = \{0^i1^\infty, 1^j0^\infty | i, j \in \mathbb{N}_0\}$, which fails to contain all crossovers; see Example 2.5.

Remark 3.3. In (4) and (5) it is shown that basic and sub-basic open sets contain all weak crossovers. However, open sets do not necessarily contain all weak crossovers.

Example 3.5. Let $X = \{0,1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Let $B_0 = \pi_1^{-1}(0) \cap \pi_2^{-1}(0) \cap \pi_3^{-1}(0)$ and $B_1 = \pi_1^{-1}(1) \cap \pi_2^{-1}(1) \cap \pi_3^{-1}(1)$. Set $O = B_0 \cup B_1$. Since $0 \in F(1)$, for each $x \in B_0$ and each $y \in B_1$, $Cw_2(x, y)$ is defined. Let $x \in B_0$ and $y \in B_1$ be such that $x = 000A$ and $y = 111B$ where $A, B \in \{0,1\}^\omega$. Then $001B = Cw_2(x, y)$ is not in O. Thus $O \subsetneq Cw(O)$. So $Cw(O) \neq O$.

Remark 3.4. Since weak crossovers preserve compactness of sets, a natural question to ask is, do weak crossovers preserve connectedness of sets? As seen in the next example, this is not the case.

Example 3.6. Let $I = [0,1]$, and let $F : I \to 2^I$ be defined as follows. For $x \in (0,1)$, $F(x) = x$. For $x \in \{0,1\}$, $F(x) = \{0,1\}$. Then $M := \varprojlim\{I, F\}$ is the union of an arc and a Cantor set. Let $A = \{z \in M | z = (x, x, x, ...) \text{ for } x \in I\}$. Then $A_i = I$ for each $i \in \mathbb{N}$, and A is an arc and hence connected. Since $0^\infty, 1^\infty \in A$, and since $0 \in F(1)$ and $1 \in F(0)$, any point of the form 0^j1^∞ or 1^j0^∞ is in $Cw(A)$ for each

$i \in \mathbb{N}$. So $Cw(A)$ is an arc, A, together with countably many points not in A. Hence $Cw(A)$ is not connected.

Theorem 3.2. *Let $\mathcal{A}$ be a non-empty collection of sets such that $Cw(A) = A$ for each $A \in \mathcal{A}$. Denote $\pi_i[\overline{A}]$ by $\overline{A}_i$. Then*

(1) $Cw(\bigcap \mathcal{A}) = \bigcap \mathcal{A}$.

(2) If for each $i \in \mathbb{N}$, and for each $A, B \in \mathcal{A}$, $A_i = B_i$ then $Cw(\bigcup \mathcal{A}) = \bigcup \mathcal{A}$.

(3) For each $i \in \mathbb{N}$, if $A, B \in \mathcal{A}$ such that $A_i = B_i$, then $Cw(B - A) = B - A$.

(4) If $A \in \mathcal{A}$ such that $A_i = \overline{A}_i$ for each $i \in \mathbb{N}$, then $Cw(\overline{A}) = \overline{A}$.

(5) If $A \in \mathcal{A}$ such that $A_i = \overline{A}_i$ for each $i \in \mathbb{N}$, then $Cw(\overline{A} - A) = \overline{A} - A$.

Proof. (1) If $\bigcap \mathcal{A} = \emptyset$, then the result is trivially true. Let $z \in Cw(\bigcap \mathcal{A})$. Then there is an $i \in \mathbb{N}$, and $x, y \in \bigcap \mathcal{A}$ such that $z = Cw_i(x, y)$. Then $x, y \in A$ for each $A \in \mathcal{A}$, so $z \in Cw(A)$ for each $A \in \mathcal{A}$. Since $Cw(A) = A$ for each $A \in \mathcal{A}$, $z \in \bigcap \mathcal{A}$. Thus $Cw(\bigcap \mathcal{A}) = \bigcap \mathcal{A}$.

(2) Let $z \in Cw(\bigcup \mathcal{A})$. Then there is an $n \in \mathbb{N}$, and $x, y \in \bigcup \mathcal{A}$ such that $z = Cw_n(x, y)$. Then there are $A, B \in \mathcal{A}$ such that $x \in A$ and $y \in B$. Then $z \in Cw(A \cup B)$. To see that $Cw(A \cup B) = A \cup B$, consider the following.

If $x, y \in A$ or $x, y \in B$ then $z \in A \cup B$, since $A, B \in \mathcal{A}$. Suppose $x \in A$ and $y \in B$. Since $A_i = B_i$ for each $i \in \mathbb{N}$, $x_i \in B_i$ for each $i \in \mathbb{N}$. By Lemma 3.2, since $Cw(B) = B$, B has Property P and contains all crossovers. Then, by Lemma 2.6, $\{x^i\} \subset B$ such that $x^i_r = z_r$ for each $r \leq i + 1$. Then $z = Cr_n(x^n, y) \in B$. So $z \in A \cup B$.

If $x \in B$ and $y \in A$, a similar argument gives $z \in A \cup B$. Hence $z \in \bigcup \mathcal{A}$, and so $Cw(\bigcup \mathcal{A}) = \bigcup \mathcal{A}$.

(3) Let $A, B \in \mathcal{A}$. If $B \cap A = \emptyset$, $Cw(B - A) = Cw(B) = B$. Suppose $B \cap A \neq \emptyset$. Let $z \in Cw(B - A)$. Then there is an $N \in \mathbb{N}$, $x, y \in B - A$ such that $z = Cw_N(x, y)$.

Then $z \in Cw(B) = B$. By (1), $Cw(A \cap B) = A \cap B$. So $A \cap B$ has Property P and contains all Crossovers, by Lemma 3.2. For the purpose of contradiction, suppose there is a $p \in Cw(A \cap B)$ and $j \in \mathbb{N}$ such that $p_i = y_i$ for all $i \geq j$. Since $y \in B$, and since $A_i = B_i$ for each $i \in \mathbb{N}$, $y_i \in A_i \cap B_i$ for each $i \in \mathbb{N}$. By Lemma 2.6, there is $\{y^i\} \subset A \cap B$ such that $y^i_r = y_r$ for each $r \leq i+1$. In particular $y^j \in A \cap B$. Then $y = Cr_j(y^j, p)$. Since $y^j, p \in A \cap B$, $y \in A \cap B$, which is a contradiction to $y \in B - A$. Thus, for each $p \in Cw(A \cap B)$ and for each $j \in \mathbb{N}$, there is an $i > j$ such that $p_i \neq y_i$. Since $z_i = y_i$ for each $i > N$, $z \notin Cw(A \cap B)$. So $Cw(B - A) \subset Cw(B) - Cw(A \cap B) = B - (A \cap B) = B - A$. Thus $Cw(B - A) = B - A$.

(4) Let $A \in \mathcal{A}$. Let $z \in Cw(\overline{A})$. By Theorem 3.1 (1), and by hypothesis, $\pi_i[Cw(\overline{A})] = \overline{A}_i = A_i$ for each $i \in \mathbb{N}$. Since $Cw(A) = A$, A has Property P and contains all Crossovers, by Lemma 3.2. Then by Lemma 2.6, there is $\{z^i\} \subset A$ such that $z^i \to z$. Hence $z \in \overline{A}$. Thus $Cw(\overline{A}) = \overline{A}$.

(5) This follows from (3) and (4). □

Remark 3.5. Theorem 3.1 parts (4) and (5) are further generalized in Corollary 3.1.

Corollary 3.1. *Let $\mathcal{S}$ be a non-empty collection of sub-basic open sets and $\mathcal{B}$ be a non-empty collection of basic open sets. Then $Cw(\bigcap \mathcal{S}) = \bigcap \mathcal{S}$ and $Cw(\bigcap \mathcal{B}) = \bigcap \mathcal{B}$.*

Proof. Let $\mathcal{S}$ be a non-empty collection of sub-basic open sets. By Theorem 3.1 (4), $Cw(S) = S$ for each $S \in \mathcal{S}$. By Theorem 3.2 (1), $Cw(\bigcap \mathcal{S}) = \bigcap \mathcal{S}$.

The proof is similar for the collection of basic open sets, $\mathcal{B}$. □

Remark 3.6. The converses to parts (1), (2), (3), and (4) of Theorem 3.2 need not be true, as seen, respectively, in the following examples.

Example 3.7. Let $X = \{0, 1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. To see that $Cw(A \cap B) = A \cap B$ does not imply $Cw(A) = A$ and $Cw(B) = B$, set $K = M - \{z \in M | z = A0^\infty \text{ where } |A| < \infty\}$. Set $H = K \cup \{0^\infty\}$.

Then $Cw(H \cap K) = H \cap K = K$. Also, $Cw(K) = K$, see Example 3.1. However $Cw(H) = M$. To see this, let $z \in M$. Suppose $z \notin H$. Then $z = A0^\infty$, where $1 \leq |A| < \infty$. Let $w = A1^\infty$, and so $w \in H$. Then $Cw_{|A|}(w, 0^\infty) = z$. Hence $Cw(H) = M$.

Example 3.8. Let $X = \{0, 1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Let $M = \varprojlim\{X, F\}$. Let $A = \{0^\infty\} \cup \{1^\infty\}$, $B = M - A$. Then $A_i = B_i$ for each $i \in \mathbb{N}$, and $Cw(A \cup B) = Cw(M) = M = A \cup B$. However $Cw(B) = M$, so $Cw(B) \neq B$, and $A \subsetneq Cw(A)$ by Example 3.4.

If $Cw(A) = A$ and $Cw(B) = B$, it need not be the case that $Cw(A \cup B) = A \cup B$. Set $K = M - \{z \in M | z = A0^\infty \text{ where } |A| < \infty\}$. Let $H = \{0^\infty\}$. Then $Cw(K) = K$ and $Cw(H) = H$, but $Cw(H \cup K) = M$.

If $Cw(A \cup B) = A \cup B$, then it need not be the case that $Cw(A) = A$ and $Cw(B) = B$. Let $A = \{0^\infty\}$ and $B = M - A$. Note that $A_i \neq B_i$. Then $Cw(A \cup B) = Cw(M) = M$ and $Cw(A) = A$. Since $01^\infty, 10^\infty \in B$, $0^\infty \in Cw(B)$. So $B \subsetneq Cw(B) = M$.

Example 3.9. Let $X = \{0, 1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Let $M = \varprojlim\{X, F\}$. Let $A \subset B \subset M$. Then $Cw(B) - Cw(A) \subset Cw(B - A)$ is not necessarily true. Let $B = \{0^\infty, 1^\infty\}$ and $A = \{0^\infty\}$. Then $Cw(B - A) = \{1^\infty\}$ and $Cw(B) - Cw(A) = \{0^i1^\infty, 1^j0^\infty\} - \{0^\infty\}$. So $Cw(B) - Cw(A) \nsubseteq Cw(B - A)$.

Also, $Cw(B - A) \subset Cw(B) - Cw(A)$ is not necessarily true. Let $B = M$ and $A = \{0^\infty, 1^\infty\}$. Then $= 01^\infty \in Cw(A)$, $01^\infty \in Cw(B)$ so $01^\infty \notin Cw(B) - Cw(A)$. $01^\infty \in B - A$, so $01^\infty \in Cw(B - A)$. So $Cw(B - A) \nsubseteq Cw(B) - Cw(A)$.

If $Cw(A) = A$ and $Cw(B) = B$, then $Cw(B) - Cw(A) \subset Cw(B - A)$, but $Cw(B - A) \subset Cw(B) - Cw(A)$ is still not necessarily true. Let $B = M$ and $A = \{0^\infty\}$. Then $Cw(B) = B$ and $Cw(A) = A$. Then $Cw(B - A) = Cw(M - \{0^\infty\}) = M$ and $Cw(B) - Cw(A) = M - \{0^\infty\}$.

Example 3.10. Let $A = M - \{0^\infty\}$. Then $A_i = M_i$ for each $i \in \mathbb{N}$ and $\overline{A} = M$ and so $Cw(\overline{A}) = \overline{A}$. However $Cw(A) = M$ since $01^\infty, 10^\infty \in A$.

CHAPTER FOUR

Inverse Limits and Higher Order Weak Crossovers

For any set, A, $Cw(A)$ is the set of all weak crossovers of A. However, $Cw(A)$ does not, in general, contain all weak crossovers of the set $Cw(A)$, as noted in the previous chapter. So $Cw(A)$ is not necessarily the inverse limit of its projections. Since $Cw(A)$ and A have the same projections, $Cw(A)$ need not be the inverse limit of the projections of A either. This leads to the investigation of what the inverse limit of the projections of such a set is by defining higher order weak crossovers.

Section 1 defines and develops a few properties of these higher order weak crossovers, as seen in Theorem 4.2. Section 2 relates this notion to inverse limits (Theorem 4.3). Theorem 4.4 describes the set $\varprojlim\{A_i, F_i|_A\}$ in terms of the set A. Theorem 4.5 successfully describes the structure of an inverse limit of projections that Theorem 2.3 in Chapter 2 could not.

4.1 Properties of Higher Order Weak Crossovers

Definition 2.7 defines the set of weak crossovers, $Cw(A)$, of a set A. Definition 4.1, the higher-order weak crossovers of a set $Cw^i(A)$ are defined. In effect, the higher-order weak crossovers can be viewed as the weak crossovers being applied multiple times to the same set.

Definition 4.1. Let $M = \varprojlim\{X_i, F_i\}$ and let $K \subset M$. The set of *i^{th}-order weak crossovers of* K is given by $Cw^i(K) = \{z \in M | \text{there is } j \in \mathbb{N}, x, y \in Cw^{i-1}(K) \text{ such that } z = Cw_j(x, y)\}$, where $Cw^1(K) = Cw(K)$. Denote $K = Cw^0(K)$. Note that $Cw^i(K) \subset Cw^j(K)$ for each $i \leq j$. Let $Cw^\infty(K) = \bigcup\{Cw^i(K) | i \in \mathbb{N}_0\}$.

Example 4.1. Let $X = \{0, 1\}$ and let $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Let $M = \varprojlim\{X, F\}$ and let $A = \{0^\infty, 1^\infty\}$. Then $Cw^2(A) = \{0^i1^j0^\infty, 1^m0^n1^\infty | i, j \in \mathbb{N}_0\} = Cw[Cw(A)]$. $Cw^\infty(A) = M - \{x \in M | x = A(01)^\infty$ or $x = A(10)^\infty$ where $A \subset \{0, 1\}^n$ for some $n \in \mathbb{N}\}$.

Remark 4.1. The following theorem, while not used in proving the main results, is useful for understanding the structure of these higher order weak crossovers.

Theorem 4.1. *Let $K \subset M$ and $z \in K$. Let $N \in \mathbb{N}$ and $x \in M$ such that $x_i = z_i$ for each $i > N$. If $x_i \in K_i$ for each $i \in \mathbb{N}$, then $x \in Cw^N(K)$.*

Proof. Suppose $x \in M$ satisfies the hypothesis. Then there is an $\hat{x}^N \in K$ such that $\hat{x}^N_N = x_N$. Set $x^1 = Cw_N(\hat{x}^N, z)$. Then $x^1 = (\hat{x}^N_1, ..., \hat{x}^N_{N-1}, x_N, z_{N+1}, ...)$ and $x^1 \in Cw(K)$. For each $i < N$, there is an $\hat{x}^i \in K$ such that $\hat{x}^i_i = x_i$. For each $i \leq N$, define $x^i = Cw_{(N+1)-i}(\hat{x}^{(N+1)-i}, x^{i-1})$. Then

$$x^i = (\hat{x}^{(N+1)-i}_1, ..., \hat{x}^{(N+1)-i}_{(N+1)-i}, x^{i-1}_{(N+2)-i}, ..., x^{i-1}_N, z_{N+1}, ...)$$

Since $x^i_j = x_j$ for all $j \geq (N + 1) - i$, we have

$$x^i = (\hat{x}^{(N+1)-i}_1, ..., \hat{x}^{(N+1)-i}_{N-i}, x_{(N+1)-i}, ..., x_N, z_{N+1}, ...)$$

and $x^i \in Cw^i(K)$. So $x^N = Cw_1(\hat{x}^1, x^{N-1}) = (\hat{x}^1_1, x_2, ..., x_N, z_{N+1}, ...)$. By construction, $\hat{x}^1_1 = x_1$, so $x^N = x$. Since $x^N \in Cw^N(K)$, $x \in Cw^N(K)$. □

Remark 4.2. It is possible for a set $K \subset M$ to be such that $K \subsetneq Cw(K)$, but $Cw^N(K) = Cw^{N+1}(K)$ for some $N \geq 1$.

Example 4.2. Let $X = \{0, 1\}$ and $F : X \to 2^X$ be defined by $F(0) = F(1) = X$. Let $M = \varprojlim\{X, F\}$. Set $H = M - \{z \in M | z = A0^\infty$ where $|A| < \infty\}$. Let $K = H - \{1^\infty\}$. Then, by Example 3.1, $Cw(H) = H$. Since $101^\infty, 01^\infty \in K$, $1^\infty = Cw_1(101^\infty, 01^\infty) \in Cw(K)$. So $K \subsetneq Cw(K)$. Since $Cw(K) = Cw(H)$, $Cw(K) = Cw^2(K)$.

Theorem 4.2. *Let M be an inverse limit space and let $A \subset M$ be non-empty. Then*

(1) If $Cw(A) = A$, then $Cw^i(A) = Cw^j(A)$ for each $i, j \in \mathbb{N}$.

(2) If $Cw^i(A) = Cw^j(A)$ for any $j > i$, then $Cw^n(A) = Cw^i(A)$ for each $n \geq i$.

(3) If $Cw(A) = A$, then $Cw(\overline{A}) = \overline{A}$ if and only if $Cw(\overline{A}) = \overline{Cw(A)}$.

(4) $Cw[Cw^\infty(A)] = Cw^\infty(A)$.

(5) $Cw(A) = A$ if and only if $Cw^\infty(A) = A$.

(6) If $Cw(A) = A$, then $Cw^\infty(\overline{A}) = \overline{A}$ if and only if $Cw(\overline{A}) = \overline{Cw(A)}$.

Proof. (1) Since $A = Cw(A)$, $Cw(A) = Cw[Cw(A)] = Cw^2(A)$. So $Cw^i(A) = Cw^{i+1}(A)$ for each $i \in \mathbb{N}$. Hence $Cw^i(A) = A$ for each $i \in \mathbb{N}$.

(2) Let $z \in Cw^{i+1}(A)$. Since $Cw^{i+1}(A) \subset Cw^j(A)$, $z \in Cw^j(A)$ and so $z \in Cw^i(A)$. So $Cw^i(A) = Cw^{i+1}(A)$.

(3) Suppose $Cw(\overline{A}) = \overline{A}$. Then $Cw(\overline{A}) = \overline{A} = \overline{Cw(A)}$ since $Cw(A) = A$.

Now suppose $Cw(\overline{A}) = \overline{Cw(A)}$. Then $Cw(\overline{A}) = \overline{A}$ since $Cw(A) = A$.

(4) Let $z \in Cw[Cw^\infty(A)]$. Then there are $x, y \in Cw^\infty(A)$, and $j \in \mathbb{N}$, such that $Cw_j(x, y) = z$. Since $x, y \in Cw^\infty(A)$, there are $r, s \in \mathbb{N}$ such that $x \in Cw^r(A)$ and $y \in Cw^s(A)$. Let $p = max\{s, r\}$. Then $x, y \in Cw^p(A)$, and so $z = Cw_j(x, y) \in Cw^{p+1}(A) \subset Cw^\infty(A)$. Thus $Cw[Cw^\infty(A)] = Cw^\infty(A)$.

(5) Suppose $Cw(A) = A$. Then $Cw^\infty(A) = A$ by definition.

Now suppose $Cw^\infty(A) = A$. Then $Cw(A) = Cw[Cw^\infty(A)] = Cw^\infty(A) = A$.

(6) Suppose $Cw^\infty(\overline{A}) = \overline{A}$. Then $Cw(\overline{A}) = Cw[Cw^\infty(\overline{A})] = Cw^\infty(\overline{A}) = \overline{A} = \overline{Cw(A)}$.

Now suppose $Cw(\overline{A}) = \overline{Cw(A)}$. Since $Cw(A) = A$, $Cw(\overline{A}) = \overline{A}$. Then $Cw^2(\overline{A}) = Cw[Cw(\overline{A})] = Cw(\overline{A}) = \overline{A}$. Then by (4), $Cw^\infty(\overline{A}) = \overline{A}$. □

Question. Let $A \subset M$ such that both A and $Cw^\infty(A)$ are compact. Does there exist an $N \in \mathbb{N}$ such that $Cw[Cw^N(A)] = Cw^N(A)$? This question remains open.

4.2 Inverse Limits

Theorem 4.3. *Let M be an inverse limit space and let $A \subset M$ be non-empty. Then*

(1) $\pi_i(A) = \pi_i[Cw^j(A)] = \pi_i[Cw^\infty(A)]$ *for each* $i, j \in \mathbb{N}$.

(2) $Cw^\infty(A) \subset \varprojlim\{A_i, F_i|_A\} \subset \overline{Cw^\infty(A)}$.

(3) If M is Hausdorff, and if $Cw^i(A)$ is compact for any $i \in \mathbb{N}_0$, then $Cw^j(A)$ is compact for all $j \geq i$.

Proof. (1) By Theorem 3.1, $\pi_i(A) = \pi_i[Cw(A)]$ for each $i \in \mathbb{N}$. So $\pi_i[Cw^2(A)] = \pi_i[Cw(A)]$ for each $i \in \mathbb{N}$. Proceeding inductively gives $\pi_i[Cw^j(A)] = \pi_i[A]$ for each $i, j \in \mathbb{N}$.

It follows that $\pi_i[Cw^\infty(A)] = \pi_i[\bigcup_{i\in\mathbb{N}} Cw^j(A)] = \bigcup_{j\in\mathbb{N}} \pi_i[Cw^j(A)] = \pi_i[A]$. So $\pi_i(A) = \pi_i[Cw^\infty(A)]$ for each $i \in \mathbb{N}$.

(2) Let $z \in Cw^\infty(A)$. Then by (1), $z_i \in A_i$ for each $i \in \mathbb{N}$. Since $z \in M$, $z_i \in F_i(z_{i+1})$ for each $i \in \mathbb{N}$. So $z_i \in F_i|_A(z_{i+1})$ for each $i \in \mathbb{N}$. Hence $z \in \varprojlim\{A_i, F_i|_A\}$.

Let $z \in A' := \varprojlim\{A_i, F_i|_A\}$. Since $A_i = \pi_i[Cw^\infty(A)]$ for each $i \in \mathbb{N}$, by (1), $z_i \in A_i = \pi_i[Cw^\infty(A)]$ for each $i \in \mathbb{N}$. By 4.2 (4), $Cw[Cw^\infty(A)] = Cw^\infty(A)$, and by Theorem 2.8, $Cw^\infty(A)$ has Property P and contains all crossovers. By Lemma 2.6, there is $\{z^i\} \subset Cw^\infty(A)$ such that $z^i \to z$. So $z \in \overline{Cw^\infty(A)}$. Thus $\varprojlim\{A_i, F_i|_A\} \subset \overline{Cw^\infty(A)}$

(3) This follows from Theorem 3.1 since $Cw^{i+1}(A) = Cw[Cw^i(A)]$ for each $i \in \mathbb{N}$. □

Remark 4.3. Consider (2) in the preceding theorem. It is possible for a set, A, to be such that $Cw^\infty(A) \subsetneq \varprojlim\{A_i, F_i|_A\} \subsetneq \overline{Cw^\infty(A)}$.

Example 4.3. Let $X = [0,1] \cup \{a\} \cup \{b\}$ and let $F : X \to 2^X$ be defined as follows. F restricted to [0,1] is the identity map, and $F(a) = F(b) = \{a\} \cup \{b\}$. Let $A = \{(x, x, x, ...)| x \in (0,1) \cup \{a, b\}\}$.

Then points in $Cw^\infty(A)$ have form $z = (x, x, x, ...)$ for $x \in (0, 1)$ or $z \in \{a, b\}^\omega$ where $z = Aa^\infty$ or $z = Ab^\infty$ with $|A| < \infty$. Since $(ab)^\infty \in \varprojlim\{A_i, F_i|_A\}$, as seen in Example 3.9, $Cw^\infty(A) \subsetneq \varprojlim\{A_i, F_i|_A\}$.

Since $F|_{[0,1]}$ is the identity map, the point 0^∞ is not an element of $\varprojlim\{A_i, F_i|_A\}$. Since 0^∞ is a limit point of $Cw^\infty(A)$, $0^\infty \in \overline{Cw^\infty(A)}$. So $\varprojlim\{A_i, F_i|_A\} \subsetneq \overline{Cw^\infty(A)}$.

Theorem 4.4. *Let M be Hausdorff and $A \subset M$ be such that A_i is compact for each $i \in \mathbb{N}$. Then*

(1) $\overline{Cw^\infty(A)} = \varprojlim\{A_i, F_i|_A\}$.

(2) $Cw[\overline{Cw^\infty(A)}] = \overline{Cw^\infty(A)}$ and $Cw^\infty(A)$ is dense in $\varprojlim\{A_i, F_i|_A\}$.

(3) $Cw[\overline{Cw^\infty(A)} - Cw^\infty(A)] = \overline{Cw^\infty(A)} - Cw^\infty(A)$.

(4) If K is a compactum such that $A \subset K$ and $K = \varprojlim\{K_i, F_i|_K\}$, then $\overline{Cw^\infty(A)} \subset K$. In other words, $\overline{Cw^\infty(A)}$ is the smallest compactum, K, containing A such that $K = \varprojlim\{K_i, F_i|_K\}$.

In particular, if A is compact, then $\overline{Cw^\infty(A)} = \varprojlim\{A_i, F_i|_A\}$.

Proof. (1) Let $z \in \overline{Cw^\infty(A)}$. Since $\pi_i[Cw^\infty(A)] = A_i$ for each $i \in \mathbb{N}$, $\pi_i[\overline{Cw^\infty(A)}] = A_i$ for each $i \in \mathbb{N}$. So $z_i \in A_i$ for each $i \in \mathbb{N}$ and since $z \in M$, $z_i \in F_i(z_{i+1})$ for each $i \in \mathbb{N}$. So $z \in \varprojlim\{A_i, F_i|_A\}$. Hence $\overline{Cw^\infty(A)} \subset \varprojlim\{A_i, F_i|_A\}$. By Theorem 4.3 (3), $\varprojlim\{A_i, F_i|_A\} \subset \overline{Cw^\infty(A)}$. Thus $\overline{Cw^\infty(A)} = \varprojlim\{A_i, F_i|_A\}$.

(2) Since $\overline{Cw^\infty(A)} = \varprojlim\{A_i, F_i|_A\}$, $Cw[\overline{Cw^\infty(A)}] = \overline{Cw^\infty(A)}$. Clearly $Cw^\infty(A)$ is dense in $\varprojlim\{A_i, F_i|_A\}$.

(3) Since $\pi_i[\overline{Cw^\infty(A)}] = A_i = \pi_i[Cw^\infty(A)]$ for each $i \in \mathbb{N}$, the hypothesis for Theorem 3.2 (5) is satisfied. So $Cw[\overline{Cw^\infty(A)} - Cw^\infty(A)] = \overline{Cw^\infty(A)} - Cw^\infty(A)$.

(4) Let K be a compactum such that $A \subset K$ and $K = \varprojlim\{K_i, F_i|_K\}$. Since $A \subset K$, $Cw^i(A) \subset Cw^i(K)$ for each $i \in \mathbb{N}$. Since $K = \varprojlim\{K_i, F_i|_K\}$, $Cw(K) = K$

by (1). So $Cw^i(A) \subset K$ for each $i \in \mathbb{N}$. Hence $Cw^\infty(A) \subset K$. So $\overline{Cw^\infty(A)} \subset \overline{K} = K$. $\square$

Remark 4.4. In Example 3.1, K is not compact and $Cw(K) = K$. So $Cw^\infty(K)$ is not compact.

Theorem 4.5. *Let M be a Hausdorff space and let $H, K \subset M$ be compact. Then $\overline{Cw^\infty(H \cap K)} = \overline{Cw^\infty(H)} \cap \overline{Cw^\infty(K)}$. In other words, $\varprojlim\{H_i \cap K_i, F_i|_{H\cap K}\} = \varprojlim\{H_i, F_i|_H\} \cap \varprojlim\{K_i, F_i|_K\}$*

Proof. By Theorem 4.2 (4) and Theorem 3.2 (1), $Cw^\infty(H \cap K) = Cw^\infty(H) \cap Cw^\infty(K)$. By properties of closures, $\overline{Cw^\infty(H \cap K)} \subset \overline{Cw^\infty(H)} \cap \overline{Cw^\infty(K)}$.

Let $z \in \overline{Cw^\infty(H)} \cap \overline{Cw^\infty(K)}$. Then $z_i \in H_i \cap K_i$ for each $i \in \mathbb{N}$. Then, by Theorem 4.4 (1), $z_i \in F_i|_{H\cap K}(z_{i+1})$ for each $i \in \mathbb{N}$. So $z \in \varprojlim\{H_i \cap K_i, F_i|_{H\cap K}\} = \overline{Cw^\infty(H \cap K)}$. $\square$

CHAPTER FIVE

Continuous Set-Valued Functions, Connectedness, and Decomposability

In this chapter, the structure of inverse limits with continuous set-valued functions is considered and the definition of a continuous set-valued function is given.

As seen in Example 3.6, weak crossovers do not preserve connectedness. Necessary and sufficient conditions for which an inverse limit with set-valued bonding maps is connected is still an open problem. Finding these conditions appears to be very tricky. Section 1, Conditions 5.1, gives sufficient conditions for which an inverse limit is a Hausdorff continuum (Theorem 5.2).

Another question of interest is when an inverse limit with set-valued bonding maps is decomposable. In the Section 2, Theorem 5.3 shows that Conditions 5.1 give a decomposable inverse limit space when the first factor space is a linearly ordered space. Corollary 5.1 gives progress towards answering if such an inverse limit can yield a simple triod.

5.1 Continuous Set-Valued Functions

In this section, we study, the class of continuous set-valued functions where members can be written as a union of continuous functions.

Definition 5.1. A set-valued function $F : X \to 2^Y$ is *lower semi-continuous* provided for each open subset $V \subset Y$ the set $F^{-1}(V) := \{x \in X | F(x) \cap V \neq \emptyset\} = L_V$ is open.

Remark 5.1. It can be shown that a set-valued function is upper semi-continuous if and only if the set $\{x \in X | F(x) \subset V\} = U_V$ is open for each open $V \subset Y$, [20]. This gives an interesting contrast to the definition of lower semi-continuous.

Definition 5.2. A set-valued function is *continuous* provided it is both upper and lower semi-continuous.

Theorem 5.1. *Let $F : X \to 2^Y$ be given by $G(F) = \bigcup_{\mathcal{A}} G(f_\alpha)$, where $f_\alpha : X \to Y$ is continuous for each $\alpha \in \mathcal{A}$. Then F is lower semi-continuous.*

Proof. Let $V \subset Y$ be open. Set $L_V = F^{-1}(V) = \{x \in X | F(x) \cap V \neq \emptyset\}$. To show that F is lower semi-continuous, it suffices to show that L_V is open. Set $V_\alpha = V \cap f_\alpha(X)$ for each $\alpha \in \mathcal{A}$. Then V_α is open relative to $f_\alpha(X)$ for each $\alpha \in \mathcal{A}$. Since f_α is continuous for each $\alpha \in \mathcal{A}$, $U_\alpha = f_\alpha^{-1}(V_\alpha)$ is open for each $\alpha \in \mathcal{A}$.

To see that $L_V = \bigcup_{\mathcal{A}} U_\alpha$, first let $\alpha \in \mathcal{A}$ and $x \in U_\alpha$. Then $f_\alpha(x) \in V$. So $F(x) \cap V \neq \emptyset$. So $x \in L_V$. Now, let $x \in L_V$. Then $F(x) \cap V \neq \emptyset$. So there is an $\alpha \in \mathcal{A}$ such that $f_\alpha(x) \in V$. Hence $x \in U_\alpha$. Thus $L_V = \bigcup_{\mathcal{A}} U_\alpha$. Since U_α is open for each $\alpha \in \mathcal{A}$, L_V is open. Therefore F is lower semi-continuous. $\square$

Remark 5.2. A set-valued function that satisfies Theorem 5.1 is not necessarily upper semi-continuous. Consider the following example.

Example 5.1. For each $n \in \mathbb{N}$, let $f_n(x) = (1 - \frac{1}{n})x$, set $F = \bigcup_{\mathbb{N}} f_n$, and let $z_n = (1, f_n(1))$. Then $\{z_n\} \subset G(F)$. Since $f_n(1) \to 1$ as $n \to \infty$, $z_n \to (1,1)$. Since $(1,1) \notin G(f)$, the graph of F is not closed, and hence F is not upper semi-continuous; see [20].

Remark 5.3. It is possible for a continuous surjective set-valued function, F, to be the union of continuous functions where none of the functions are surjective. It is also possible that F^n can be written as a union of continuous functions, none of which are surjective, for each $n \in \mathbb{N}$. Consider the following example.

Example 5.2. The set-valued function mentioned in Remark 5.3 can be constructed using only two continuous functions:

$$\text{Let } f_1(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1/2, \\ 1/2 & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Let $f_2(x) = \begin{cases} 1-x & \text{if } 0 \leq x \leq 1/2, \\ 1/2 & \text{if } 1/2 \leq x \leq 1. \end{cases}$

Let $F : I \to 2^I$ be given by $G(F) = G(f_1) \cup G(f_2)$. Then F is surjective and continuous. Moreover, F cannot be written as the union of continuous functions where one of the functions is surjective.

Let $f_3(x) = 1/2$ for $0 \leq x \leq 1$. Then, for each $n \in \mathbb{N}$, $F^n : I \to 2^I$ is given by $G(F^n) = G(f_1) \cup G(f_2) \cup G(f_3)$. So, for each $n \in \mathbb{N}$, F^n is surjective and continuous. Moreover, for each $n \in \mathbb{N}$, F^n cannot be written as the union of continuous functions where one of the functions is surjective.

Remark 5.4. The inverse limit, $\varprojlim\{I, F\}$, where F is given in Example 5.2, is a continuum. Moreover, it is homeomorphic to the the cone over the set $\{e_i | e_0 = 0$ and $e_i = 1/i$ for $i \in \mathbb{N}\}$ which is known as the Harmonic Fan; see [16].

Conditions 5.1. Consider the following conditions for each $n \in \mathbb{N}$:

(1) X_n is a Hausdorff continuum,

(2) there is a collection, $\{f_\alpha\}_{\mathcal{A}_n}$, such that $f_\alpha : X_{n+1} \to X_n$ is a continuous function for each $\alpha \in \mathcal{A}_n$, with $|\mathcal{A}_n| \geq 2$.

(3) there is an $a_n \in \mathcal{A}_n$ such that f_{a_n} is surjective,

(4) $F_n : X_{n+1} \to 2^{X_n}$, given by $G(F_n) = \bigcup_{\mathcal{A}_n} G(f_\alpha)$, is continuous,

(5) $G(f_{a_n}) \cap G(f_\alpha) \neq \emptyset$ for each $\alpha \in \mathcal{A}_n$.

Remark 5.5. In Conditions 5.1 (4), it is assumed that F_n is continuous for each $n \in \mathbb{N}$. By Theorem 5.1, F_n is lower semi-continuous for each $n \in \mathbb{N}$. So condition (4) only assumes that F_n is upper semi-continuous. By [20], this is equivalent to $G(F_n)$ being closed for each $n \in \mathbb{N}$. Condition (5) is always satisfied with spaces that have surjective span zero.

Definition 5.3. Suppose Conditions 5.1 (1)-(5) hold. Let $M = \varprojlim\{X_n, F_n\}$ and $S = \varprojlim\{X_n, f_{a_n}\}$. Let $x \in M$ and for each $n \in \mathbb{N}$, define $R_n(x) = \{\alpha \in \mathcal{A}_n | x_n \in f_\alpha(x_{n+1})\}$. Then $R(x) := \prod_{n\in\mathbb{N}} R_n(x)$ is the *Route of* x. Set $a = (a_1, a_2, a_3, ...)$ where a_n satisfies condition (3).

Remark 5.6. Note that $a \in R(x)$ if and only if $x \in S$.

Notation. Let $x \in M$ and $z \in R(x)$. For each $n \in \mathbb{N}$, define $F_x^{z_n} : X_{n+1} \to 2^{X_n}$ by $G(f_{a_n}) \cup G(f_{z_n})$.

Let $M_x^z = \varprojlim\{X_n, F_x^{z_n}\}$ and set $M_x = \bigcup_{R(x)} M_x^z$. Then $M_x^z \subset M_x \subset M$ for each $x \in M$ and $z \in R(x)$.

Lemma 5.1. *Let $x, y \in M$. Then $z \in R(x) \cap R(y)$ if and only if $M_x^z = M_y^z$.*

Proof. Let $z \in R(x) \cap R(y)$. Then, for each $n \in \mathbb{N}$, $F_x^{z_n} = F_y^{z_n}$. So $M_x^z = M_y^z$.

Now suppose $M_x^z = M_y^z$. Then $z \in R(x) \cap R(y)$. □

Remark 5.7. Let $p \in M_x^z$. Then it need not be the case that $z \in R(p)$. However, there is a $w \in R(p)$ such that $w_n \in \{z_n, a_n\}$ for each $n \in \mathbb{N}$.

Lemma 5.2. *Let $x, y \in M$. If $R(x) = R(y)$, then $M_x = M_y$.*

Proof. Since $R(x) \subset R(y)$, for each $z \in R(x)$, $z \in R(y)$. By Lemma 5.1, $M_x^z \subset M_y^z$. Since $M_x = \bigcup_{R(x)} M_x^z$, $M_x \subset M_y$. Similarly, $M_y \subset M_x$. □

5.2 Connected Inverse Limits

Lemma 5.3. *If Conditions 5.1 (1)-(4) hold, with $M = \varprojlim\{X_n, F_n\}$ and $S = \varprojlim\{X_n, f_{a_n}\}$, then*

(1) $\pi_n[S] = X_n$ for each $n \in \mathbb{N}$, and

(2) $\overline{Cw^\infty(S)} = M$.

Proof. (1) Since $S = \varprojlim\{X_n, f_{a_n}\}$, where f_{a_n} is surjective for each $n \in \mathbb{N}$, $S_n = X_n$.

(2) By (1), $S_n = X_n$ for each $n \in \mathbb{N}$. So $F_n|_S = F_n$ for each $n \in \mathbb{N}$. By Theorem 4.4, $\overline{Cw^\infty(S)} = \varprojlim\{S_n, F_n|_S\} = \varprojlim\{X_n, F_n\} = M$. □

Remark 5.8. In light of Lemma 5.3, the set S is called the *Spine of* M.

Lemma 5.4. *If Conditions 5.1 (1)-(5) hold, where* $\mathcal{A}_n = \{a_n, b_n\}$ *for each* $n \in \mathbb{N}$, *then* $M = \varprojlim\{X_n, F_n\}$ *is a Hausdorff continuum.*

Proof. Since X_n is a Hausdorff continuum and f_{a_n} is a continuous function for each $n \in \mathbb{N}$, $S = \varprojlim\{X_n, f_{a_n}\}$ is a non-empty Hausdorff continuum. For each $x \in M$, denote by $\hat{x}$ the member of $R(x)$ such that

$$\hat{x}_n = \begin{cases} b_n & \text{if } b_n \in R_n(x), \\ a_n & \text{if } b_n \notin R_n(x). \end{cases}$$

For each $x \in M$, set $M_{\hat{x}} = \varprojlim\{X_n, f_{\hat{x}_n}\}$. Since $M_{\hat{x}}$ is the inverse limit of continuous functions on Hausdorff continua, $M_{\hat{x}}$ is connected for each $x \in M$.

Claim: If $x \in M$ such that $\hat{x}_n = b_n$ for finitely many $n \in \mathbb{N}$, then $M_{\hat{x}}$ is a subset of the component containing S, $Comp[S]$.

To prove the claim we proceed by induction. Let $x \in M$ such that n_1 is the only natural number such that $\hat{x}_{n_1}(x) = b_n$. By Condition (5), $G(f_{a_{n_1}}) \cap G(f_{b_{n_1}}) \neq \emptyset$, so there is a $z \in X_{n_1+1}$ such that $f_{a_{n_1}}(z) = f_{b_{n_1}}(z)$. Since f_{a_n} is surjective for each $n \in \mathbb{N}$, there is a $z_{n+1} \in X_{n+1}$ such that $z_n = f_{a_n}(z_{n+1})$ for each $n > n_1$, with $z_{n_1} = f_{a_{n_1}}(z_{n_1+1})$. Then there is a $\tilde{z} = (f_{a_1}^{a_{n_1}}(z), ..., f_{a_{n_1}}(z), z, z_{n_1+1}, ...) \in M$. Note that $a \in R(\tilde{z})$ and $a' = (a_1, ..., a_{n_1-1}, b_{n_1}, a_{n_1+1}, ...) \in R(\tilde{z}) \cap R(x)$. By Lemma 5.1, $M_{\tilde{z}}^{a'} = M_{\hat{x}}$, and so $\tilde{z} \in M_{\hat{x}}$. Since $a \in R(\tilde{z})$, and by Lemma 5.1, $M_{\tilde{z}}^{a} = S$. Hence $\tilde{z} \in S$. Thus, $\tilde{z} \in S \cap M_{\hat{x}}$. Hence $S \cup M_{\hat{x}}$ is connected, and so $M_{\hat{x}} \subset Comp[S]$.

Now suppose that if $x \in M$ such that $\hat{x}_n = b_n$ for $n_1 < n_2 < ... < n_{N-1}$, then $M_{\hat{x}}$ is a subset of the component containing S, $Comp[S]$. Let $x \in M$ such that $R_{n_j}(x) = b_n$ for $j \in \{1, ..., N\}$.

By Condition (5), $G(f_{a_{n_N}}) \cap G(f_{b_{n_N}}) \neq \emptyset$, so there is a $z \in X_{n_N+1}$ such that $f_{a_{n_N}}(z) = f_{b_{n_N}}(z)$. Since f_{a_n} is surjective for each $n \in \mathbb{N}$, there is a $z_{n+1} \in X_{n+1}$ such that $z_n = f_{a_n}(z_{n+1})$ for each $n > n_N$, with $z_{n_N} = f_{a_{n_1}}(z_{n_N})$. Then there is a $\tilde{z} = (z_1, ..., z_{n_N-1}, z, z_{n_N+1}, ...) \in M$ where the bonding map between z_i and z_{i+1} is the same as that of x_i and x_{i+1} for each $i < n_N$. Then by the construction of $\tilde{z}$ there is a $\hat{\tilde{z}} \in R(\tilde{z})$ defined by

$$\hat{\tilde{z}}_n = \begin{cases} b_n & \text{if } b_n \in R_n(\tilde{z}), \\ a_n & \text{if } b_n \notin R_n(\tilde{z}). \end{cases}$$

So $\tilde{z}$ satisfies the induction hypothesis, and hence $M_{\hat{\tilde{z}}}$ is connected and $M_{\hat{\tilde{z}}} \subset Comp[S]$. Also, by the construction of $\tilde{z}$, $R(\tilde{z}) \cap R(x) \neq \emptyset$. So $M_{\hat{x}} \cap M_{\hat{\tilde{z}}} \neq \emptyset$. Thus $M_{\hat{x}} \subset Comp[S]$.

Thus, if $x \in M$ such that $\hat{x}_n = b_n$ for finitely many $n \in \mathbb{N}$, $M_{\hat{x}} \subset Comp[S]$.

Now, to see that M is connected, let $x \in M$ such that $\hat{x} = b_n$ for infinitely many $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $x^n \in M$ such that $x_i^n = x_i$ for $i \leq n$ and $x_i^n \in f_{a_i}(x_{i+1}^n)$ for each $i > n$. Then, by the claim, $x^n \in Comp[S]$ for each $n \in \mathbb{N}$. Since $x^n \to x$ as $n \to \infty$, $x \in \overline{Comp[S]}$. Thus M is connected, and so M is a Hausdorff continuum. □

Theorem 5.2. *If Conditions 5.1 (1)-(5) hold, then* $M = \varprojlim\{X_n, F_n\}$ *is a Hausdorff continuum.*

Proof. Let $x \in M$ and $z \in R(x)$. By Lemma 5.4, M_x^z is a Hausdorff continuum. Note that $S \subset M_x^z$ for each $x \in M$ and each $z \in R(x)$. Then M_x is connected. Since $\bigcup_{x \in M} M_x = M$, M is connected. Since F_n is upper semi-continuous for each $n \in \mathbb{N}$, M is closed. Thus M is a Hausdorff continuum. □

Remark 5.9. The proof of Theorem 5.2 shows that M is connected by building M from the spine. This construction suggests that M may be decomposable.

5.3 Decomposable Inverse Limits

Definition 5.4. A continuum M is decomposable if and only if there are two proper subcontinua, H and K, such that $H \cup K = M$.

Remark 5.10. When Conditions 5.1 (1)-(5) are satisfied, Theorem 5.2 gives that M is a continuum. To see that M is indeed decomposable, proper sub-continua H and K need to be constructed to satisfy the definition. An additional assumption that X_1 is a linearly ordered space will be used in this pursuit.

Lemma 5.5. *Let X be a Hausdorff space, and Y be linearly ordered with the Order Topology. Let f and g be continuous functions from X into Y. Then the functions $max\{f(x), g(x)\}$ and $min\{f(x), g(x)\}$ are continuous.*

Proof. Let $G = \{x \in X | g(x) \geq f(x)\}$ and $F = \{x \in X | f(x) \geq g(x)\}$. To see that $G \cup F = X$, let $x \in X$. Then $g(x), f(x) \in Y$, and so either $g(x) \geq f(x)$ or $f(x) \geq g(x)$, since Y is linearly ordered. So $x \in G$ or $x \in F$. To see that G and F are closed let $\{x_\alpha\}_\mathcal{A}$ be a convergent net in G. Let $\{x_\alpha\}_\mathcal{A}$ converge to $x \in X$. Since $x_\alpha \in G$ for each $\alpha \in \mathcal{A}$, $g(x_\alpha) \geq f(x_\alpha)$ for each $\alpha \in \mathcal{A}$. Since f and g are continuous, $g(x) \geq f(x)$. Hence G is closed. Similarly, F is closed.

Define $h(x) : X \to Y$ by

$$h(x) = \begin{cases} g(x) & \text{if } x \in G, \\ f(x) & \text{if } x \in F. \end{cases}$$

Then $h(x) = \max\{f(x), g(x)\}$. Since, for each $x \in G \cap F$, $g(x) = f(x)$, $h(x)$ is a well defined function. Then by [32, Theorem 7.6], $h(x)$ is a continuous function.

Similarly, $\min\{f(x), g(x)\}$ is continuous. □

Lemma 5.6. *Let X be a Hausdorff space and Y be linearly ordered with the order topology. Let $\mathcal{A}$ be nonempty such that $|\mathcal{A}| \geq 3$. Let $\mathcal{F} = \{f_\alpha\}_\mathcal{A}$ be such that $f_\alpha : X \to Y$ is continuous for each $\alpha \in \mathcal{A}$ and such that $F = \bigcup_\mathcal{A} f_\alpha$ is upper* *semi-continuous. If there is an $x \in X$ such that $|F(x)| \geq 3$, then there are upper semi-continuous set-valued functions F_1 and F_2 such that $G(F_1) \cup G(F_2) = G(F)$ and such that neither $G(F_1)$ nor $G(F_2)$ is a subset of the other.*

Proof. Let $x \in X$ such that $|F(x)| \geq 3$. Since Y is linearly ordered, there are β_1, β, and β_2 in $\mathcal{A}$ such that $f_{\beta_1}(x) < f_\beta(x) < f_{\beta_2}(x)$. Then for each $\alpha \in \mathcal{A}$ and each $z \in X$ define $f_{\alpha_1}(z) = \min\{f_\alpha(z), f_\beta(z)\}$ and $f_{\alpha_2}(z) = \max\{f_\alpha(z), f_\beta(z)\}$. By Lemma 5.5, f_{α_i} is continuous for each $\alpha \in \mathcal{A}$ and each $i \in \{1, 2\}$. For $i \in \{1, 2\}$, define $F_i : X \to 2^Y$ where $G(F_i) = \bigcup_{\mathcal{A}} G(f_{\alpha_i})$. Note that $G(F) = G(F_1) \cup G(F_2)$ and that $G(F_i) \neq \emptyset$ for $i \in \{1, 2\}$.

To see that F_i is upper semi-continuous for $i \in \{1, 2\}$, note that $G(F_1) = G(F) \cap \{z \in X \times Y | z_2 \leq f_\beta(x)\}$ and $G(F_2) = G(F) \cap \{z \in X \times Y | z_2 \geq f_\beta(x)\}$. Hence $G(F_i)$ is closed for $i \in \{1, 2\}$. Then by [20], $G(F_i)$ is upper semi-continuous for $i \in \{1, 2\}$.

To see that neither $G(F_1)$ nor $G(F_2)$ is a subset of the other, note that $(x, f_{\beta_1}(x)) \in G(F_1) - G(F_2)$, so $G(F_1)$ is not a subset of $G(F_2)$. Similarly, $(x, f_{\beta_2}(x)) \in G(F_2) - G(F_1)$, so $G(F_2)$ is not a subset of $G(F_1)$. □

Remark 5.11. If $Y = I$ then the previous lemma holds.

Theorem 5.3. *If Conditions 5.1 (1)-(5) hold and X_1 is linearly ordered with the Order Topology, then $M = \varprojlim\{X_n, F_n\}$ is a decomposable Hausdorff continuum.*

Proof. By Theorem 5.2, M is a Hausdorff continuum. In the proof of this theorem, we have three cases.

Case 1: $|A_1| \geq 4$.

By Lemma 5.6, there are upper semi-continuous set-valued functions, generated by $\mathcal{A}_1 - \{a_1\}$, $F_1^1 : X_2 \to 2^{X_1}$ and $F_1^2 : X_2 \to 2^{X_1}$ such that $G(F_1) = G(F_1^1) \cup G(F_1^2)$. Let F_1^a be given by $G(F_1^a) = G(F_1^1) \cup G(f_{a_1})$ and F_1^b be given by

$G(F_1^b) = G(F_1^2) \cup G(f_{a_1})$. Let $H_n = F_n = K_n$ for $n > 1$ and $H_1 = F_1^a$ and $K_1 = F_1^b$. Set $H = \varprojlim\{X_n, H_n\}$ and $K = \varprojlim\{X_n, K_n\}$.

Claim: H and K decompose M.

By Theorem 5.2, both H and K are Hausdorff continua. By the construction of H_n and K_n, $S \subset H \cap K$, and hence $H \cap K \neq \emptyset$. To see that $H \cup K = M$, let $x \in M$. Then $x_1 \in F_1(x_2)$, so $x_1 \in H_1(x_2)$ or $x_1 \in K_1(x_2)$. So $x \in H$ or $x \in K$, and hence $M \subset H \cup K$. Since $H \cup K \subset M$, we have that $H \cup K = M$. For H and K to decompose M, we need only show that H and K are distinct. By Lemma 5.6, there exist $x^1 = (x, f_{\beta_1}(x)) \in G(F_1^1)$ and $x^2 = (x, f_{\beta_2}(x)) \in G(F_1^2)$ such that $x^1 \notin G(F_1^2)$ and $x^2 \notin G(F_1^1)$. Since F_1^1 and F_1^2 were generated by using $A_1 - \{a_1\}$, $x^1 \in G(F_1^a) - G(F_1^b)$ and $x^2 \in G(F_1^2) - G(F_1^1)$. So there is a $z^1 \in H - K$ and $z^2 \in K - H$. Hence H and K are distinct. Therefore H and K decompose M and the claim is proved.

Case 2: $\mathcal{A}_1 = \{a_1, b_1, b_2\}$.

The proof of this case follows similarly to that of Case 1 with the exceptions of H_1 and K_1 are given by $G(H_1) = G(f_{b_1}) \cup G(f_{a_1})$ and $G(K_1) = G(f_{b_2}) \cup G(f_{a_1})$. Hence M is decomposable.

Case 3: $\mathcal{A}_1 = \{a_1, b_1\}$.

For $n > 1$, set $H_n = F_n = K_n$ and set $H_1 = f_{a_1}$ and $K_1 = f_{b_1}$. Set $H = \varprojlim\{X_n, H_n\}$ and $K = \varprojlim\{X_n, K_n\}$. By Lemma 5.4, H and K are Hausdorff continua. Since $G(f_{a_1}) \cap G(f_{b_1}) \neq \emptyset$, $H \cap K \neq \emptyset$. As seen in Case 1, $H \cup K = M$. Finally H and K are distinct since $f_{a_1} \neq f_{b_1}$. Therefore M is decomposable. □

Corollary 5.1. *Suppose Conditions 5.1 (1)-(5) hold where X_1 is linearly ordered with the Order Topology. Let $n \in \mathbb{N}_0$. If there is an $x \in X_2$ such that $|F_1(x)| \geq 10 + 4n$, then $M = \varprojlim\{X_n, F_n\}$ is a $(3 + n) - od$.*

Proof. Suppose $|F_1(x)| = m + 1 \geq 10 + 4n$ for some $n \in \mathbb{N}_0$. Choose $\beta_1, ..., \beta_m \in \mathcal{A}_1 - \{a_1\}$ such that $f_{\beta_i}(x) < f_{\beta_j}(x)$ if and only if $i < j$.

If m is odd, define, for each $\alpha \in \mathcal{A}_1 - \{a_1\}$, the following functions

$$f_{\alpha_1}(x) = min\{f_\alpha(x), f_{\beta_2}(x)\},$$

$$f'_{\alpha_i} = max\{f_\alpha(x), f_{\beta_{2(i-1)}}(x)\},$$

$$f_{\alpha_i} = min\{f'_{\alpha_i}(x), f_{\beta_{2i}}(x)\},$$

$$f_{\alpha_{\frac{m+1}{2}}} = max\{f_\alpha(x), f_{\beta_{m-1}}(x)\}.$$

If m is even define, for each $\alpha \in \mathcal{A}_1 - \{a_1\}$, the following functions

$$f_{\alpha_1}(x) = min\{f_\alpha(x), f_{\beta_3}(x)\},$$

$$f'_{\alpha_i} = max\{f_\alpha(x), f_{\beta_{2i-1}}(x)\},$$

$$f_{\alpha_i} = min\{f'_{\alpha_i}(x), f_{\beta_{2i+1}}(x)\},$$

$$f_{\alpha_{\frac{m}{2}}} = max\{f_\alpha(x), f_{\beta_{m-1}}(x)\}.$$

By Lemma 5.6, there are upper semi-continuous functions, $F_1^i : X_2 \to 2^{X_1}$ such that $G(F_1^i) = \bigcup_{i=1}^{\frac{m}{2}} G(f_{\alpha_i})$ if m is odd and $G(F_1^i) = \bigcup_{i=1}^{\frac{m+1}{2}} G(f_{\alpha_i})$ if m is even. Note that $F_1^i \cap F_1^j = \emptyset$ when i and j are both even or both odd. Define upper semi-continuous functions H_n^i and K_n^i similarly to H_n and K_n in the proof of Theorem 5.3, with the exception of, H_1 and K_1. Define H_1^i by $G(H_1^i) = G(F_1^i) \cup G(f_{a_1})$ for i odd, and K_1^i by $G(K_1^i) = G(F_1^i) \cup G(f_{a_1})$ for i even. Define $H_i = \varprojlim\{X_n, H_n^i\}$ and $K_i = \varprojlim\{X_n, K_n^i\}$. Note that H_i and K_i are continua. Moreover, $M = \bigcup(H_i \cup K_i)$. Set $T = \bigcup(K_i)$. Since $S \subset K_i$, T is a continuum. Then $M - T$ has $3+n$ components, and so M is a $(3+n)$-od. □

CHAPTER SIX

Inverse Limits of Decompositions

In this chapter, the notion of the itinerary of a point from dynamical systems is generalized to admit set-valued functions. Theorem 6.1 gives the structure of an inverse limit when the factor space can be decomposed as a union of compact Hausdorff spaces and the bonding map can be written as a union of surjective upper semi-continuous set-valued functions. This theorem uses the generalization of the itinerary.

Remark 6.1. In dynamical systems with interval maps, the itinerary of a point, x, in the inverse limit space, is a sequence of 0's and 1's which denote on which side of the critical point, c, of a unimodal map, $f : [0,1] \to [0,1]$, the projections of x lie. Itineraries are a very useful tool for analyzing inverse limits. Note that the unit interval can be decomposed as $[0,1] = [0,c] \cup [c,1]$. Viewing the itinerary of a point in terms of a decomposition of the factor spaces allows us to generalize the itinerary in the following way.

Definition 6.1. Let $\mathcal{A}$ be a nonempty set. Let $X = \bigcup_{\alpha \in \mathcal{A}} (X_\alpha \cup X_c)$, where X, X_c and for each $\alpha \in \mathcal{A}$, X_α are compact Hausdorff spaces such that $X_\alpha \cap X_\beta \subset X_c$ if and only if $\alpha \neq \beta$. Let $F : X \to 2^X$ be an upper semi-continuous function. Let $x \in \varprojlim\{X, F\}$ and $x_n = \pi_n(x)$. Define the *Itinerary of* x, $I(x)$, by $I(x) = (I_1(x), I_2(x), ...)$ where, for each $n \in \mathbb{N}$,

$$I_n(x) = \begin{cases} \alpha & \text{if } x_n \in X_\alpha - (X_\alpha \cap X_c), \\ c & \text{if } x_n \in X_c. \end{cases}$$

Example 6.1. The following are specific examples of the Itinerary of x.

Let $\mathcal{A} = \{0, 1\}$ and let $X = X_0 \cup X_c \cup X_1$ be a compact Hausdorff space such that $X_0 - (X_0 \cap X_c) \neq \emptyset$, $X_1 - (X_1 \cap X_c) \neq \emptyset$, and $X_0 \cap X_1 \subset X_c$. Then the itinerary of x is $I(x) = (I_1(x), I_2(x), ...)$ where, for each $n \in \mathbb{N}$,

$$I_n(x) = \begin{cases} 0 & \text{if } x_n \in X_0 - (X_0 \cap X_c), \\ c & \text{if } x_n \in X_c, \\ 1 & \text{if } x_n \in X_1 - (X_1 \cap X_c). \end{cases}$$

Let $\mathcal{A} = \{0, 1\}$ and let $X = [0, 1]$ where $X_c = \{1/2\}$. Then the itinerary of x is $I(x) = I_1(x)I_2(x)....$ where, for each $n \in \mathbb{N}$,

$$I_n(x) = \begin{cases} 0 & \text{if } x_n < 1/2, \\ c & \text{if } x_n = 1/2, \\ 1 & \text{if } x_n > 1/2. \end{cases}$$

Example 6.2. Let $X = [0, 1]$ and $F : X \to 2^X$ be the union of $f(x) = x$ with $g(x) = 1 - x$. Then $\varprojlim\{X, F\}$ is a cone over a Cantor set. Let $z \in \varprojlim\{X_i, F_i\}$ such that $z_i = g(z_{i+1})$ and $z_i \neq 1/2$ for each $i \in \mathbb{N}$. Then the possible itineraries of z are $I(z) = (01)^\infty$ or $(10)^\infty$.

Lemma 6.1. *Let $\mathcal{A}$ be a nonempty set. Let $X = \bigcup_{\alpha \in \mathcal{A}} (X_\alpha \cup X_c)$ where X, X_c, and for each $\alpha \in \mathcal{A}$, X_α are compact Hausdorff spaces such that $X_\alpha \cap X_\beta \subset X_c$ if and only if $\alpha \neq \beta$. Let $F_{\alpha\beta} : X_\beta \to 2^{X_\alpha}$ be a surjective upper semi-continuous function for each $(\alpha, \beta) \in \mathcal{A}^2$, and let $F_{cc} : X_c \to 2^{X_c}$ be a surjective upper semi-continuous function. Let $F : X \to 2^X$ be the surjective upper semi-continuous function whose graph is $G(F) = \bigcup \{G(F_{\alpha\beta}) \cup G(F_{cc}) | (\alpha, \beta) \in \mathcal{A}^2\}$. Then, for each $z \in \mathcal{A}^\omega$ there is an $x \in \varprojlim\{X, F\}$ such that $I(x) = z$.*

Proof. Let $z \in \mathcal{A}^\omega$ such that $z = (z_1, z_2, ..., z_{n-1}, z_n, ...)$. Then $F_{z_n z_{n+1}} : X_{z_{n+1}} \to 2^{X_{z_n}}$ is surjective for each $n \in \mathbb{N}$. Let $x_1 \in X_{z_1} - X_c$. Since $F_{z_n z_{n+1}}$ is surjective for each

$n \in \mathbb{N}$, there is an $x_n \in X_{z_n}$ such that $x_{n-1} \in F_{z_{n-1}z_n}(x_n)$ for each $n > 1$. Set $x = (x_1, x_2, ..., x_{n-1}, x_n, ...)$. Then $x \in \varprojlim\{X_{z_n}, F_{z_n z_{n+1}}\} \subset \varprojlim\{X, F\}$ and $I(x) = z$. □

Lemma 6.2. *Let $\mathcal{A}$ be a nonempty set. Let $X = \bigcup_{\alpha\in\mathcal{A}}(X_\alpha \cup X_c)$ where X, X_c, and for each $\alpha \in \mathcal{A}$, X_α are compact Hausdorff spaces such that $X_\alpha \cap X_\beta \subset X_c$ if and only if $\alpha \neq \beta$. Let $F_{\alpha\beta} : X_\beta \to 2^{X_\alpha}$ be a surjective upper semi-continuous function for each $(\alpha, \beta) \in \mathcal{A}^2$, and let $F_{cc} : X_c \to 2^{X_c}$ be a surjective upper semi-continuous function. Let $F : X \to 2^X$ be the surjective upper semi-continuous function whose graph is $G(F) = \bigcup\{G(F_{\alpha\beta}) \cup G(F_{cc}) | (\alpha, \beta) \in \mathcal{A}^2\}$. Suppose that $F[X_c] = F^{-1}[X_c] = X_c$. Then*

(1) If $x \in X_\beta$, then $F_{\alpha\beta}(x) = X_\alpha \cap F(x)$, and

(2) $y \in F(x)$ where $y \in X_\alpha$ and $x \in X_\beta$ if and only if $y \in F_{\alpha\beta}(x)$.

Proof. (1) Let $x \in X_\beta$. Let $y \in F_{\alpha\beta}(x)$. Then, by the definition of $F_{\alpha\beta}$, $y \in X_\alpha$. By the construction of F, $y \in F(x)$. Hence $F_{\alpha\beta}(x) \subset X_\alpha \cap F(x)$.

Now, let $y \in X_\alpha \cap F(x)$. Since $y \in F(x)$ there is a $\gamma \in (\mathcal{A} \cup \{c\})$ such that $y \in F_{\alpha\gamma}(x)$. By the definition of $F_{\alpha\gamma}$, $x \in X_\gamma$. So $x \in X_\gamma \cap X_\beta$. If $\gamma = \beta$, then $y \in F_{\alpha\beta}(x)$ and we are done. If $\gamma \neq \beta$, then $x \in X_c$ by hypothesis. Since $F(X_c) = X_c$, $y \in X_c$. Hence $y \in F_{cc}(x)$. Since $y \in X_c \cap F(x)$, we are done.

(2) Let $y \in F(x)$ such that $y \in X_\alpha$ and $x \in X_\beta$. Then by (1), $y \in F_{\alpha\beta}(x)$.

Let $y \in F_{\alpha\beta}(x)$. Then $y \in X_\alpha$ and $x \in X_\beta$. Since $y \in F_{\alpha\beta}(x)$, $y \in F(x)$. □

Remark 6.2. Requiring the space X to be written as $X = \bigcup_{\alpha\in\mathcal{A}}(X_\alpha \cup X_c)$ with X_c being strongly F-invarient can be too restrictive, and so it is desirable to have a lemma with similar results to Lemma 6.2 that does not require these restrictions. The following lemma has this desired change.

Lemma 6.3. *Let $\mathcal{A}$ be a nonempty set. Let $X = \bigcup_{\alpha\in\mathcal{A}} X_\alpha$ be a compact Hausdorff space where, for each $\alpha \in \mathcal{A}$, X_α is a compact Hausdorff space. For each $(\alpha, \beta) \in \mathcal{A}^2$,* *let* $F_{\alpha\beta} : X_\beta \to 2^{X_\alpha}$ *be a surjective upper semi-continuous function. Suppose that if* $y \in F_{\alpha\beta}(x) \cap X_\gamma$ *then* $y \in F_{\gamma\beta}(x)$. *Let* $F : X \to 2^X$ *be the surjective upper semi-continuous function whose graph is* $G(F) = \bigcup\{G(F_{\alpha\beta})|(\alpha, \beta) \in \mathcal{A}^2\}$. *Then*

(1) If $x \in X_\beta$, *then* $F_{\alpha\beta}(x) = X_\alpha \cap F(x)$, *and*

(2) $y \in F(x)$ *where* $y \in X_\alpha$ *and* $x \in X_\beta$ *if and only if* $y \in F_{\alpha\beta}(x)$.

Proof. (1) Let $x \in X_\beta$. Then $F_{\alpha\beta}(x) \subset X_\alpha$. Since $G(F) \supset G(F_{\alpha\beta})$, for each $y \in F_{\alpha\beta}(x), y \in F(x)$. So $F_{\alpha\beta}(x) \subset X_\alpha \cap F(x)$.

Let $y \in X_\alpha \cap F(x)$. Since $y \in F(x)$ there is a $\gamma \in \mathcal{A}$ such that $y \in F_{\gamma\beta}(x)$. Since $y \in F_{\gamma\beta}(x) \cap X_\alpha, y \in F_{\alpha\beta}(x)$. Hence $X_\alpha \cap F(x) \subset F_{\alpha\beta}(x)$.

(2) Let $y \in F(x)$ such that $y \in X_\alpha$ and $x \in X_\beta$. Then by (1), $y \in F_{\alpha\beta}(x)$.

Let $y \in F_{\alpha\beta}(x)$. Then $y \in X_\alpha$ and $x \in X_\beta$. Since $y \in F_{\alpha\beta}(x)$, $y \in F(x)$. □

Theorem 6.1. *Let* $\mathcal{A}$ *be a nonempty set. Let* $X = \bigcup_{\alpha \in \mathcal{A}}(X_\alpha \cup X_c)$ *where* X, X_c, *and for each* $\alpha \in \mathcal{A}$, X_α *are compact Hausdorff spaces such that* $X_\alpha \cap X_\beta \subset X_c$ *if and only if* $\alpha \neq \beta$. *Let* $F_{\alpha\beta} : X_\beta \to 2^{X_\alpha}$ *be a surjective upper semi-continuous function for each* $(\alpha, \beta) \in \mathcal{A}^2$, *and let* $F_{cc} : X_c \to 2^{X_c}$ *be a surjective upper semi-continuous function. Let* $F : X \to 2^X$ *be the surjective upper semi-continuous function whose graph is* $G(F) = \bigcup\{G(F_{\alpha\beta}) \cup G(F_{cc})|(\alpha, \beta) \in \mathcal{A}^2\}$. *Suppose that* $F(X_c) = F^{-1}(X_c) = X_c$. *Then for each* $z \in \mathcal{A}^\omega \cup \{c\}^\omega$, $M_z = \varprojlim\{X_{z_n}, F_{z_n z_{n+1}}\}$, *is a proper subcompactum of* $\varprojlim\{X, F\}$, *and*

(1) $\bigcup\{M_z | z \in \mathcal{A}^\omega \cup \{c\}^\omega\} = \varprojlim\{X, F\}$,

(2) $M_z \cap M_{c^\omega} \neq \emptyset$ *for each* $z \in \mathcal{A}^\omega$ *if and only if* $X_\alpha \cap X_c \neq \emptyset$ *for* $\alpha \in \mathcal{A}$.

(3) For $z, w \in \mathcal{A}^\omega$, $M_z \cap M_w \subset M_{c^\omega}$ *if and only if* $z \neq w$.

Proof. Let $z \in \mathcal{A}^\omega \cup \{c\}^\omega$ and let $M_z = \{x \in \varprojlim\{X, F\} | I(x) = z\} \cup \{x \in \varprojlim\{X, F\} | x_n \in X_{z_n} \cap X_c$ for each $n \in \mathbb{N}\}$. Let $x \in M_z$. Then $x_n \in X_{z_n}$ and

$x_n \in F(x_{n+1})$ for each $n \in \mathbb{N}$. Then by Lemma 6.1, $x_n \in F_{z_n z_{n+1}}(x_{n+1})$ for each $n \in \mathbb{N}$. So $M_z \subset \varprojlim\{X_{z_n}, F_{z_n z_{n+1}}\}$.

Let $x \in \varprojlim\{X_{z_n}, F_{z_n z_{n+1}}\}$. There are two cases to consider, $x_n \notin X_c$ for each $n \in \mathbb{N}$ and $x_n \in X_c$ for some $n \in \mathbb{N}$.

Suppose $x_n \notin X_c$ for each $n \in \mathbb{N}$. Then $I_n(x) = z_n$ for each $n \in \mathbb{N}$ and so $I(x) = z$. Since $x_n \in F_{z_n z_{n+1}}(x_{n+1})$ for each $n \in \mathbb{N}$, $x \in \varprojlim\{X, F\}$ and so $x \in M_z$.

Suppose $x_n \in X_c$ for some $n \in \mathbb{N}$. Then, by construction of F, $x_n \in X_c$ for each $n \in \mathbb{N}$. So $x_n \in X_{z_n} \cap X_c$ for each $n \in \mathbb{N}$ and so $x \in M_z$.

Hence $\varprojlim\{X_{z_n}, F_{z_n z_{n+1}}\} \subset M_z$. Therefore $M_z = \varprojlim\{X_{z_n}, F_{z_n z_{n+1}}\}$.

(1) Let $x \in \varprojlim\{X, F\}$. If $I(x) \neq c^\omega$ then there is a $z \in \mathcal{A}^\omega$ such that $I(x) = z$. Then by (1), $x \in M_z$. If $I(x) = c^\omega$, then $x \in M_{c^\omega}$. Hence $\bigcup\{M_z | z \in \mathcal{A}^\omega \cup \{c\}^\omega\} = \varprojlim\{X, F\}$.

(2) Suppose that, for each $z \in \mathcal{A}^\omega$, $M_z \cap M_{c^\omega} \neq \emptyset$. Then for each $\alpha \in \mathcal{A}$, $M_{\alpha^\omega} \cap M_{c^\omega} \neq \emptyset$. Then $X_\alpha \cap X_c \neq \emptyset$ for each $\alpha \in \mathcal{A}$.

Suppose $X_\alpha \cap X_c \neq \emptyset$ for $\alpha \in \mathcal{A}$ and let $z \in \mathcal{A}^\omega$. Let $x_1 \in X_{z_1} \cap X_c$. Since $F^{-1}[X_c] = X_c$, $F^{-1}(x_1) \subset X_c$. Since $x_1 \in X_{z_1}$ and since $F_{z_n z_{n+1}}$ is surjective for each $n \in \mathbb{N}$, there is an $x_2 \in X_{z_2}$ such that $x_1 \in F_{z_1 z_2}(x_2)$. Note that $x_2 \in F^{-1}(x_1)$. Proceeding inductively, for each $n \in \mathbb{N}$, there is an $x_n \in X_{z_n} \cap X_c$ such that $x_{n-1} \in F(x_n)$. Hence $x \in M_z \cap M_{c^\omega}$.

(3) Let $z, w \in \mathcal{A}^\omega$ such that $z \neq w$.

Suppose that $(M_z \cap M_w) - M_{c^\omega} \neq \emptyset$. Let $x \in (M_z \cap M_w) - M_{c^\omega}$. Since $F_{z_n z_{n+1}}$ and $F_{w_n w_{n+1}}$ are surjective for each $n \in \mathbb{N}$, $M_{z_n} = X_{z_n}$ and $M_{w_n} = X_{w_n}$ for each $n \in \mathbb{N}$, by Lemma 1. So $x_n \in X_{z_n} \cap X_{w_n}$ for each $n \in \mathbb{N}$. Since $z \neq w$, there is an $N \in \mathbb{N}$ such that $z_N \neq w_N$. So $X_{z_N} \neq X_{w_N}$ and $x_N \in X_{z_N} \cap X_{w_N}$. Since $x \notin M_{c^\omega}$ and by construction of F, $x_N \notin X_c$. So $(X_{z_N} \cap X_{w_N}) - X_c \neq \emptyset$,which is a contradiction.

Suppose $M_z \cap M_w \subset M_{c^\omega}$. Since $X_\alpha - X_c \neq \emptyset$ for $\alpha \in \mathcal{A}$, $M_z - M_{c^\omega} \neq \emptyset$ and $M_w - M_{c^\omega} \neq \emptyset$. So there is an $x \in M_z - M_{c^\omega}$ such that $x \notin M_w$. So there is an $N \in \mathbb{N}$

such that $x_N \in M_{z_N}$ and $x_N \notin M_{w_N}$. By Lemma 2.1, $M_{z_N} = X_{z_N}$ and $X_{w_N} = M_{w_N}$. So $X_{z_N} \neq X_{w_N}$, and hence $z \neq w$. □

Remark 6.3. In effect, Theorem 6.1 gives a way to decompose an inverse limit into an infinite-od, even though the inverse limit need not be a continuum.

Corollary 6.1. *Let* $I_0 = [0, \frac{1}{2}]$ *and* $I_1 = [\frac{1}{2}, 1]$. *For each* $(i, j) \in \{0, 1\}^2$, *let* $f_{ij} : I_i \to I_j$ *be a continuous mapping. Let* $F : I \to 2^I$ *be the upper semi-continuous function whose graph,* G, *is the union of the graphs of the functions* f_{ij}. *Suppose that* $F(1/2) = \{1/2\}$ *and* $F^{-1}(\{1/2\}) = \{1/2\}$. *For each* $z \in \{0, 1\}^\omega \cup \{c\}^\infty$, *let* $M_z = \{x \in \varprojlim\{I, F\} | I(x) = z\} \cup \{\overrightarrow{\frac{1}{2}}\}$. *Then*

(1) $\bigcup\{M_z | z \in \{0, 1\}^\omega\} = \varprojlim\{I, F\}$,

(2) $M_z = \varprojlim\{I_{z_n}, f_{z_n z_{n+1}}\}$,

(3) for each $z \in \{0, 1\}^\omega$, M_z *is a proper subcontinuum of* $\varprojlim\{I, F\}$, *and*

(4) $M_z \cap M_w = \{\overrightarrow{\frac{1}{2}}\}$ *for* $z \neq w \in \{0, 1\}^\omega$.

Proof. Set $X_0 = [0, 1/2]$, $X_1 = [1/2, 1]$, $X_c = \{1/2\}$, and $X = I$. Each M_z is connected since $f_{z_n z_{n+1}}$ is continuous for each $n \in \mathbb{N}$. □

Remark 6.4. Corollary 6.1 shows how Theorem 6.1 can be used when the factor spaces are the interval. This gives another way to analyze inverse limits on I, specifically in the next example. The following example appeared in [26, Example 4]. Corollary 6.1 gives the structure of the inverse limit more succinctly than before.

Example 6.3. Let $I_0 = [0, 1/2]$, $I_1 = [1/2, 1]$, and $I_c = \{1/2\}$. Then $f_{ij} : I_i \to I_j$, for $(i, j) \in \{0, 1\}^2 \cup \{cc\}$. Let $f_{00} = f_{11} = f_{cc} = x$. Let $f_{01} = f_{10} = 1 - x$. Let $F : I \to 2^I$ be given by $G(F) = \bigcup\{G(F_{ij}) \cup G(F_{cc}) | (i, j) \in \{0, 1\}^2\}$. Then for each $z \in \{0, 1\}^\omega$, M_z is homeomorphic to an arc. Note that $M_c = \{(1/2)^\infty\}$. Then M a fan over a Cantor set.

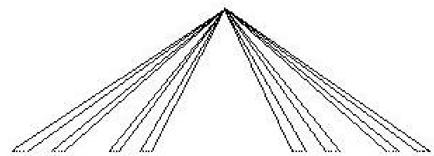

Figure 6.1. A Cone Over a Cantor Set

Remark 6.5. Lemma 6.3 extends Lemma 6.2 by allowing the space X to be written as $X = \bigcup_{\alpha\in\mathcal{A}} X_\alpha$ instead of being written as $X = \bigcup_{\alpha\in\mathcal{A}}(X_\alpha \cup X_c)$.

The definition of Itinerary can be changed to avoid defining the set X_c, but then may become not well defined. This prevents Theorem 6.1 from being completely extended in similar fashion to that of Lemma 6.3 with Lemma 6.2. However, the results of Theorem 6.1 do partially extend.

Theorem 6.2. *Let $\mathcal{A}$ be a nonempty set. Let $X = \bigcup_{\alpha\in\mathcal{A}} X_\alpha$ be a compact Hausdorff space where, for each $\alpha \in \mathcal{A}, X_\alpha$ is a compact Hausdorff space. For each $(\alpha, \beta) \in \mathcal{A}^2$, let $F_{\alpha\beta} : X_\beta \to 2^{X_\alpha}$ be a surjective upper semi-continuous function. Suppose that if $y \in F_{\alpha\beta}(x) \cap X_\gamma$ then $y \in F_{\gamma\beta}(x)$. Let $F : X \to 2^X$ be the surjective upper semi-continuous function whose graph is $G(F) = \bigcup\{G(F_{\alpha\beta})|(\alpha, \beta) \in \mathcal{A}^2\}$. Then for each $z \in \mathcal{A}^\omega \cup \{c\}^\omega$, $M_z = \varprojlim\{X_{z_n}, F_{z_nz_{n+1}}\}$ is a proper subcompactum of $\varprojlim\{X, F\}$, and*

(1) $\bigcup_{\mathcal{A}^\omega} M_z = \varprojlim\{X, F\}$,

(2) For $z, w \in \mathcal{A}^\omega$, if $M_z \cap M_w \neq \emptyset$ then $X_{z_n} \cap X_{w_n} \neq \emptyset$ for each $n \in \mathbb{N}$.

Proof. Let $z \in \mathcal{A}^\omega$ and let $M_z = \{x \in \varprojlim\{X, F\}|x_n \in X_{z_n}$ for each $n \in \mathbb{N}\}$. Let $x \in M_z$. Then $x_n \in X_{z_n}$ and $x_n \in F(x_{n+1})$ for each $n \in \mathbb{N}$. By Lemma 6.3, $x_n \in F_{z_nz_{n+1}}(x_{n+1})$ for each $n \in \mathbb{N}$. So $M_z \subset \varprojlim\{X_{z_n}, F_{z_nz_{n+1}}\}$.

Let $x \in \varprojlim\{X_{z_n}, F_{z_nz_{n+1}}\}$. Then $x_n \in X_{z_n}$ for each $n \in \mathbb{N}$. Since $x_n \in F_{z_nz_{n+1}}(x_{n+1})$ for each $n \in \mathbb{N}$, $x \in \varprojlim\{X, F\}$. So $\varprojlim\{X_{z_n}, F_{z_nz_{n+1}}\} \subset M_z$. Hence $M_z = \varprojlim\{X_{z_n}, F_{z_nz_{n+1}}\}$.

(1) Let $x \in \varprojlim\{X, F\}$. Then there is a $z \in \mathcal{A}^\omega$ such that $x_n \in X_{z_n}$ for each $n \in \mathbb{N}$ and so $x \in M_z$. Hence $\bigcup_{\mathcal{A}^\omega} M_z \supset \varprojlim\{X, F\}$. Trivially $\bigcup_{\mathcal{A}^\omega} M_z \subset \varprojlim\{X, F\}$, and so $\bigcup_{\mathcal{A}^\omega} M_z = \varprojlim\{X, F\}$.

(2) Let $z, w \in \mathcal{A}^\omega$ and suppose $x \in M_z \cap M_w$. Then, by definition of M_z and M_w, $x_n \in X_{z_n} \cap X_{w_n}$ for each $n \in \mathbb{N}$. □

In this chapter, the notion of an itinerary of a point in an inverse limit space with set-valued bonding maps is given. Theorem 6.1 uses this definition to describe the inverse limit space. Theorem 6.2 extends this by relaxing some of the restrictions imposed on the factor space.

CHAPTER SEVEN

Conclusions

7.1 Summary

Inverse limits with set-valued bonding maps have been studied only since 2004. There is much to learn about the types of spaces that can arise in this setting, and there is need for foundational results to form the basis for further study of inverse limits of set-valued bonding maps. Both of these are addressed in this dissertation.

The Subsequence Theorem does not generalize to inverse limits with set-valued bonding maps, as seen in Example 2.4. However, Theorem 2.4 states a weaker version of the theorem that holds for inverse limits with set-valued bonding maps. Since the Subsequence Theorem is useful for analyzing inverse limits with continuous bonding maps, Theorem 2.4 may be useful for studying the set-valued counterparts.

The fact that a closed set is the inverse of its projections is crucial when dealing with inverse limits in the study of continuum theory. In this dissertation, the concept of weak crossovers is introduced; in this process Property P is defined. A set contains all weak crossovers if and only if it is the inverse limit of its projections, as is stated in Theorem 2.9. Since the analogous result with continuous bonding maps, Theorem 1.3, is fundamental in the classical setting, it is reasonable to believe that this theorem will provide a similar foundation in this new setting. Weak crossovers are pivotal in this theorem, which motivates the study of weak crossovers in more depth. Theorems 3.1 and 3.2 state properties of weak crossovers.

Example 2.5 shows that not all closed sets contain all weak crossovers. This leads to the investigation, in Chapter 4, of inverse limits of the projections of sets that do not contain all weak crossovers. In this endeavor, the concept of higher-order weak crossovers is introduced. Basically, the set of higher-order weak crossovers can be thought of as the result of applying the weak crossover "operator" multiple times to a set. When this process is repeated countably many times on a set K, one obtains $Cw^\infty(K)$. Theorem 4.2 develops properties of higher-order weak crossovers. Theorem 4.3 describes the structure of the inverse limit of the projections of any subset of an inverse limit space. Of particular note is the fact that $Cw^\infty(A) \subset \varprojlim\{A_i, F_i|_A\} \subset \overline{Cw^\infty(A)}$. Theorem 4.4 shows that for a closed set, A, $\varprojlim\{A_i, F_i|_A\} = \overline{Cw^\infty(A)}$. This result is the direct extension of Theorem 1.3. Since many concepts of continuum theory are defined in terms of subcontinua, this result, like Theorem 2.9, provides a foundation on which a deeper investigation of continuum theory may proceed in the context of inverse limits with set-valued bonding maps.

Necessary and sufficient conditions for an inverse limit with set-valued bonding maps to be connected remain unknown. Chapter 5 is centered around the investigation of connectedness and decomposability by using bonding maps that are unions of functions, as prescribed by Conditions 5.1. These conditions are sufficient for the inverse limit to be connected, as is seen in Theorem 5.2. When the first factor space is linearly ordered, the inverse limit is decomposable, according to Theorem 5.3. It is not possible to obtain an n-od as an inverse limit of intervals with continuous bonding functions. However, given $n \in \mathbb{N}$, Corollary 5.1 can be used to construct an inverse limit on the interval that is an n-od. It is unknown whether an inverse limit on the interval with a single set-valued bonding map can be a simple triod.

Chapter 6 is centered around the study of inverse limits with a single factor space and single set-valued bonding map. It is assumed that the factor space is decomposed as $X = \bigcup_{\alpha \in \mathcal{A}}(X_\alpha \cup X_c)$, where $X_\alpha \cap X_\beta \subset X_c$ for $\alpha, \beta \in \mathcal{A}$. The notion of the itinerary of a point is defined in this setting and is used in Theorem 6.1 to describe the inverse limit. In this setting, the inverse limit is an ∞-od with a structure similar to a fan. Theorem 6.2 states a similar result when the structure of the factor space has the form $X = \bigcup_{\alpha \in \mathcal{A}} X_\alpha$.

7.2 Open Problems

The following are open questions and future research goals.

All results in this dissertation assume that the directed set for the inverse limit is the natural numbers.

Question. Which results extend to the setting of general directed sets?

An affirmative answer to the following question would strengthen Theorem 4.4.

Question. If both A and $Cw^\infty(A)$ are compact is there an $N \in \mathbb{N}$ such that $Cw[Cw^N(A)] = Cw^N(A)$?

Theorem 5.3 can be used to construct continua that are triods, but it does not guarantee that the triod is a simple triod.

Question. Is it possible for the inverse limit of an interval using a single bonding map to be a simple triod?

Related to this question is the following.

Question. If an inverse limit with set-valued bonding maps contains a triod, does it contain an n-od for each positive integer n?

Currently, all examples suggest that it is in fact an ∞-od.

Question. Can Theorem 5.3 be extendted to a setting in which the first factor space is not linearly ordered?

Question. What conditions are necessary and sufficient conditions for an inverse limit of continua to be connected?

Question. Can Conditions 5.1 be relaxed and still yield a connected inverse limit?

Question. Does Morton Brown's approximation theorem extend to inverse limits with set-valued bonding maps?

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