ABSTRACT

The Left-Definite Spectral Analysis of the Legendre Type Differential Equation

Davut Tuncer, Ph.D.

Advisor: Lance L. Littlejohn, Ph.D.

Littlejohn and Wellman developed a general abstract left-definite theory for a self-adjoint operator A that is bounded below in a Hilbert space $(H, (\cdot,\cdot))$. More specifically, they construct a continuum of Hilbert spaces $\{(H_r, (\cdot,\cdot)_r)\}_{r>0}$ and, for each $r > 0$, a self-adjoint restriction A_r of A in H_r. The Hilbert space H_r is called the r^{th} left-definite Hilbert space associated with the pair (H, A) and the operator A_r is called the r^{th} left-definite operator associated with (H, A). We apply this left-definite theory to the self-adjoint Legendre type differential operator generated by the fourth-order formally symmetric Legendre type differential expression

$$\ell[y](x) := ((1-x^2)^2 y''(x))'' - ((8 + 4A(1-x^2))y'(x))' + \lambda y(x),$$

where the numbers A and λ are, respectively, fixed positive and non-negative parameters and where $x \in (-1, 1)$.

The Left-Definite Spectral Analysis of the Legendre Type Differential Equation

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Davut Tuncer, B.S., M.S.

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Approved by the Dissertation Committee

Lance L. Littlejohn, Ph.D., Chairperson

John M. Davis, Ph.D.

Johnny Henderson, Ph.D.

Qin Sheng, Ph.D.

Dean M. Young, Ph.D.

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J. Larry Lyon, Ph.D., Dean

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DEDICATION

To my beloved family:

my mother Ayşe,

my late blessed father Hikmet,

my sister Emine,

and

my brother Turan

CHAPTER ONE

Bochner-Krall Orthogonal Polynomials

1.1 Introduction

This chapter provides background material on some aspects of orthogonal polynomials in general, and Bochner-Krall orthogonal polynomials in particular. In 1929, Bochner [7] classified all orthogonal polynomial solutions (see Section 1.4) to a second-order differential equation of the form

$$a_2(x)\,y''(x) + a_1(x)\,y'(x) + a_0(x)\,y(x) = \lambda y(x)\,, \tag{1.1.1}$$

where $a_2(x)$, $a_1(x)$, and $a_0(x)$ are polynomials and λ is a real parameter independent of x. Up to a complex linear change of variable, we have only the classical polynomials of Jacobi, Laguerre, Hermite, Bessel as well as the monomials $\{x^n\}_{n=0}^{\infty}$ (which cannot form an orthogonal sequence with respect to a positive measure).

Bochner's result naturally leads to a question of classifying all orthogonal polynomial solutions to higher-order differential equations of the form

$$L_N[y](x) = \sum_{i=0}^{N} a_i(x)\,y^{(i)}(x) = \lambda_n y(x)\,,\ N \in \mathbb{N}\ \text{, and } n \in \mathbb{N}_0, \tag{1.1.2}$$

where (necessarily) $a_i(x) = \sum\limits_{j=1}^{i} a_{ij}x^j$ for some real constants a_{ij} and

$$\lambda_n = a_{11}n + a_{22}(n-1) + ... + a_{NN}n(n-1)...(n-N+1).$$

In Section 1.2, we will state the important classification theorem given by H. L. Krall in 1938. In Section 1.3, we will review some fundamental properties of orthogonal polynomials. Sections 1.4, 1.5, and 1.6 are devoted to the study of the known *Bochner-Krall orthogonal polynomial sequences,* that is, sequences $\{p_n(x)\}_{n=0}^{\infty}$, of orthogonal polynomials which satisfy a differential equation of the form (1.1.2). The name *"Bochner-Krall"* for such polynomials was introduced by A. M. Krall and L.

L. Littlejohn [41] in honor of the many contributions of S. Bochner and H. L. Krall to the theory of orthogonal polynomials. In Section 1.7, we will focus on five particular examples of Bochner-Krall polynomials which we will call the *Legendre*$^{(r)}$ polynomials for the reasons explained in the section. In the final section, we will briefly discuss the Koornwinder polynomials since they are generalizations of all known Bochner-Krall polynomials.

1.2 Krall's Classification Theorem

In 1938, H. L. Krall proved his important classification theorem:

Theorem 1.2.1. **(a)** Suppose, for each $n \in \mathbb{N}_0$, $y = p_n(x)$ is a polynomial solution of degree n of the equation $L_N[y](x) = \lambda_n y(x)$. In addition, suppose $\{\mu_n\}_{n=0}^{\infty}$ is a sequence of real numbers satisfying the conditions

$$\text{(i) } \Delta_n := \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \neq 0 \text{ for each } n \in \mathbb{N}_0, \ \ (\Delta_0 := \mu_0)\,; \text{ and}$$

$$\text{(ii) } S_k(m) := \sum_{i=2k+1}^{r} \sum_{u=0}^{i} \binom{i-k-1}{k} P(m-2k-1, i-2k-1)\, a_{i,i-u}\mu_{m-u} = 0 \tag{1.2.1}$$

for all integers $m \geq 2k+1$ and all integers k satisfying $2k+1 \leq r$, and where $P(m,i) = m(m-1)\cdots(m-i+1)$. Then, $\{p_n\}_{n=0}^{\infty}$ is an orthogonal polynomial sequence with respect to some measure μ whose moments are given by $\{\mu_n\}_{n=0}^{\infty}$; that is,

$$\int_{\mathbb{R}} x^n d\mu = \mu_n \ \ (n \in \mathbb{N}_0)\,.$$

Furthermore, N is necessarily even.

(b) Conversely, suppose $\{p_n\}_{n=0}^{\infty}$ is an orthogonal polynomial sequence with respect to some measure μ having moment sequence $\{\mu_n\}_{n=0}^{\infty}$ (so $\Delta_n \neq 0$ for all $n \in \mathbb{N}_0$). In addition, suppose there exist a positive integer N and $\frac{(N+1)(N+2)}{2}$ real constants $a_{r,j}$ $(r = 0, 1, \ldots, N,\ j = 0, 1, \ldots, r)$ such that these numbers are solutions of the above equations $S_k(m) = 0$ for all integers $m \geq 2k + 1$ and all integers k satisfying $2k + 1 \leq r$. Then, for each $n \in \mathbb{N}_0$, $y = p_n(x)$ is a solution of the differential equation $L_N[y](x) = \lambda_n y(x)$. Furthermore, N is necessarily even.

Using his 1938 classification theorem, H. L. Krall characterized all fourth-order differential equations having orthogonal polynomial solutions. In this case, there are seven such equations (four of which are the iterations of the classical second-order equations); the nonclassical equations have orthogonal polynomial solutions which have been named, by A. M. Krall, as the Jacobi type, Laguerre type, and Legendre type orthogonal polynomials. A. M. Krall [36] studied these three sets of orthogonal polynomials in detail in 1981.

1.3 Definition and Properties of Orthogonal Polynomials

In this section, we discuss some properties of orthogonal polynomials. The interested reader should consult [8] and [67] for an in-depth study of this subject.

Let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. A complex-valued linear functional $\mathcal{L}$ defined on the vector space of all polynomials with complex coefficients by $\mathcal{L}[x^n] = \mu_n$, $n = 0, 1, 2, \ldots,$ is called *the moment functional* determined by the moment sequence $\{\mu_n\}_{n=0}^{\infty}$. The number μ_n is called the *moment of order n.* A sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials is called *an orthogonal polynomial sequence* with respect to $\mathcal{L}$ if for all non-negative integers m and n,

(i) $p_n(x)$ is a polynomial of degree n,

(ii) $\mathcal{L}[p_n(x)\, p_m(x)] = 0$ when $m \neq n$, and

(iii) $\mathcal{L}[p_n^2(x)] \neq 0$.

If, in addition, we also have $\mathcal{L}[p_n^2(x)] = 1$, $n \geq 0$, then $\{p_n(x)\}_{n=0}^{\infty}$ will be called *an orthonormal polynomial sequence.* Of course, not every sequence of complex numbers determines a moment functional having an orthogonal polynomial sequence. Let Δ_n be as in (1.2.1). Then, a moment functional $\mathcal{L}$ is called *quasi-definite* if $\Delta_n \neq 0$, for $n \geq 0$. We have the following existence theorem:

Theorem 1.3.1. Let $\mathcal{L}$ be a moment functional with moment sequence $\{\mu_n\}_{n=0}^{\infty}$. A necessary and sufficient condition for the existence of an orthogonal polynomial sequence for $\mathcal{L}$ is $\Delta_n \neq 0$, $n = 0, 1, 2, \ldots$.

A moment functional $\mathcal{L}$ is called *positive-definite* if $\mathcal{L}[p(x)] > 0$ for every polynomial $p(x)$ that is not identically zero and is non-negative for all real x. When $\mathcal{L}$ is positive-definite, we have the following important characterization of positive-definite moment functionals:

Theorem 1.3.2. $\mathcal{L}$ is positive-definite if and only if all of its moments are real and $\Delta_n > 0$, $(n \geq 0)$.

One of the most important characteristics of orthogonal polynomials is the fact that any three consecutive polynomials are connected by a very simple relation that is called a *three-term recurrence formula*. More specifically, we have the following theorem:

Theorem 1.3.3. Let $\mathcal{L}$ be a quasi-definite moment functional with monic orthogonal polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$. Then, there exist constants c_n and $\lambda_n \neq 0$ such that $p_n(x) = (x - c_n)\, p_{n-1}(x) - \lambda_n p_{n-2}(x)$, $(n \geq 0)$, where we define $p_{-1}(x) = 0$.

We next take up the very important converse to Theorem 1.3.3.. This was first announced by J. Favard in 1935. It was apparently discovered at about the same time independently by J. Shohat and I. Natanson. However, the result is actually contained implicitly in earlier known results from the theory of continued fractions, and a form of it goes back to Stieltjes. We will, however, refer to it as Favard's theorem.

Four years later in 1939, Boas [6] proved the following, rather surprising, representation theorem for moment functionals:

Theorem 1.3.4. Let $\{\mu_n\}_{n=0}^{\infty}$ be an arbitrary sequence of real numbers. Then, there is a function φ of bounded variation on $(-\infty, \infty)$ such that for $n = 0, 1, 2, \ldots$

$$\int_{-\infty}^{\infty} x^n d\varphi(x) = \mu_n.$$

It should be noted that the function φ in Theorem 1.3.4. is not unique since we can always add a function of bounded variation to φ with the property that all of its moments are zero. For example, define

$$g(x) := \begin{cases} \exp\left(-x^{1/4}\right) \sin x^{1/4} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases};$$

then,

$$\int_R x^n g(x)\, dx = 0, \ \ n = 0, 1, 2, \ldots.$$

Although Boas' theorem is an important theoretical result, its proof is not constructive. In practice, it is usually a difficult matter to find a weight function for a given moment sequence.

We have the following theorem ([67], Section 3.1):

Theorem 1.3.5. Let $\alpha(x)$ be a nondecreasing function which is not constant on the compact interval $[a, b]$. Assume $\{p_n(x)\}_{n=0}^{\infty}$ is an orthogonal polynomial sequence with respect to the distribution $d\alpha(x)$ on $[a, b]$. Then, $\{p_n(x)\}_{n=0}^{\infty}$ is a complete orthogonal polynomial sequence in $L_\alpha^2[a, b]$ where

$$L_\alpha^2[a,b] = \left\{ f : [a,b] \to \mathbb{C} \mid f \text{ is } \alpha - \text{measurable and } \int_a^b |f(x)|^2 d\alpha(x) < \infty \right\}.$$

1.4 Bochner-Krall Orthogonal Polynomials of Order 2

In 1929, Bochner [7] classified all orthogonal polynomial solutions to the second-order equation of the form (1.1.1). He observed that if (1.1.1) has a polynomial solution of degree n, $n = 0, 1, 2$, then a_2, a_1, and a_0 are necessarily of degrees at most 2, 1, and 0, respectively.

By considering the possible locations of the roots of $a_2(x)$, Bochner concluded that the only polynomial solutions (up to a complex linear change of variable) are:

(i) $\left\{P_n^{(\alpha,\beta)}(x)\right\}$, the Jacobi polynomials,

(ii) $\left\{L_n^{(\alpha)}(x)\right\}$, the Laguerre polynomials,

(iii) $\{H_n(x)\}$, the Hermite polynomials,

(iv) $\left\{Y_n^{(\alpha)}(x)\right\}$, the Bessel polynomials, and

(v) $\{x^n\}$.

Although Bochner knew the existence of the Bessel orthogonal polynomial sequence, Bessel polynomials were not officially discovered until 1949; and the polynomials in (v) cannot form an orthogonal polynomial sequence with respect to any moment functional $\mathcal{L}$ since then $0 \neq \mathcal{L}[x^2x^2] = \mathcal{L}[xx^3] = 0$. Thus, the only orthogonal polynomials that are solutions to a second-order differential equation of the form (1.1.1) are the classical orthogonal polynomials of Jacobi, Laguerre, and Hermite together with the Bessel polynomials. We call these four sequences of polynomials the *Bochner-Krall orthogonal polynomials of order 2.*

Another important classification theorem was given by Hahn [30] in 1935. He showed that if $\{p_n(x)\}_{n=0}^{\infty}$ and $\{q_n(x)\}_{n=0}^{\infty}$ are orthogonal polynomial sequences with respect to positive-definite moment functionals, then $\{p_n(x)\}_{n=0}^{\infty}$ is (up to a complex linear change of variable) one of the three classical orthogonal polynomials. It was later observed by H. L. Krall [44] and Beale [4] that the only orthogonal polynomial sequences whose derivatives form an orthogonal polynomial sequence with respect to a quasi-definite moment functional are the classical orthogonal polynomials and the Bessel polynomials.

A third characterization of these polynomials was suggested by Tricomi [68], and a complete proof was given by Ebert [14] and Cryer [12]. They showed that the only polynomial sequences that have Rodrigues formulas are the Jacobi, the Laguerre, the Hermite, and the Bessel polynomials. By a Rodrigues formula, we mean a formula of the form

$$p_n(x) = \frac{1}{K_n w(x)} \cdot \frac{d^n}{dx^n} \left(\rho^n(x) w(x)\right), \ n = 0, 1, 2, \ldots, \text{ where}$$

(i) K_n is independent of x;

(ii) $\rho(x)$ is a polynomial independent of n;

(iii) $w(x)$ is positive and integrable over some interval (a, b).

Several orthogonal polynomial sequences can be found through *generating functions. A generating function* for $\{p_n(x)\}_{n=0}^{\infty}$ is a function F of two variables such that

$$F(x, t) = \sum_{n=0}^{\infty} c_n p_n(x) t^n,$$

where convergence is in some region of the plane $\mathbb{R}^2$ and $\{c_n\}_{n=0}^{\infty}$ is a known sequence of constants.

We conclude this section by listing formulas for and properties of the Bochner-Krall orthogonal polynomials of order 2. The reader is referred to [2] and [62] for further properties of these polynomials.

The Jacobi Polynomials

Notation:

$\left\{P_n^{(\alpha,\beta)}(x)\right\}$, where $\alpha > -1$ and $\beta > -1$.

Explicit Formula:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k}.$$

Differential Equation:

$$(1-x^2) y''(x) + (\beta - \alpha - (\alpha+\beta+2) x) y'(x) + n(n+\alpha+\beta+1) y(x) = 0.$$

Orthogonality:

The Jacobi polynomials are orthogonal on $[-1,1]$ with respect to the weight function $w(x)=(1-x)^{\alpha}(1+x)^{\beta}$ and

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(x)\, w(x)\, dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\, n!}\delta_{mn},$$

where Γ is the Gamma function.

Rodrigues Formula:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{(-2)^n n!\,(1-x)^{\alpha}(1+x)^{\beta}} \cdot \frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right).$$

Generating Function:

$$\frac{2^{\alpha+\beta}}{R(1-t+R)^{\alpha}(1+t+R)^{\beta}} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n,$$

where $R=(1-2xt+t^2)^{\frac{1}{2}}$.

Recurrence Relation:

$P_{-1}^{(\alpha,\beta)}(x)=0,\ P_0^{(\alpha,\beta)}(x)=1$, and for $n\geq 1$

$$2n(n+\alpha+\beta)(2n+\alpha+\beta-2)\, P_n^{(\alpha,\beta)}(x)$$
$$=(2n+\alpha+\beta-1)\left((2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2\right)P_{n-1}^{(\alpha,\beta)}(x)$$
$$-2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)\, P_{n-2}^{(\alpha,\beta)}(x).$$

The Laguerre Polynomials

Notation:

$\{L_n^{\alpha}(x)\}$, where $\alpha>-1$.

Explicit Formula:

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}.$$

Differential Equation:

$$xy''(x)+(\alpha+1-x)\, y'(x)+ny(x)=0.$$

Orthogonality:

The Laguerre polynomials are orthogonal on $[0,\infty)$ with respect to the weight function $w(x) = x^{\alpha}e^{-x}$ and

$$\int_0^{\infty} L_m^{\alpha}(x)\, L_n^{\alpha}(x)\, w(x)\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}.$$

Rodrigues Formula:

$$L_n^{\alpha}(x) = \frac{e^x}{n!x^{\alpha}} \cdot \frac{d^n}{dx^n}\left(x^{n+\alpha}e^{-x}\right).$$

Generating Function:

$$\frac{\exp\left(\frac{-xt}{1-t}\right)}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} L_n^{\alpha}(x)\, t^n.$$

Recurrence Relation:

$$L_{-1}^{\alpha}(x) = 0,\ \ L_0^{\alpha}(x) = 1,$$

and

$$nL_n^{\alpha}(x) = (2n+\alpha-1-x)\, L_{n-1}^{\alpha}(x) - (n+\alpha-1)\, L_{n-2}^{\alpha}(x)\,,\ \ n \geq 1.$$

The Hermite Polynomials

Notation:

$\{H_n(x)\}$.

Explicit Formula:

$$H_n(x) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2x)^{n-2k}}{(n-2k)!k!},$$

where $\left[\frac{n}{2}\right]$ denotes the greatest integer less than or equal to $\frac{n}{2}$.

Differential Equation:

$$xy''(x) - 2xy'(x) + 2ny(x) = 0.$$

Orthogonality:

The Hermite polynomials are orthogonal on $(-\infty, \infty)$ with respect to the weight function $w(x) = \exp(-x^2)$ and

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) w(x) \, dx = 2^n n! \pi^{1/2} \delta_{mn}.$$

Rodrigues Formula:

$$H_n(x) = (-1)^n \exp\left(x^2\right) \frac{d^n}{dx^n} \exp\left(-x^2\right).$$

Generating Function:

$$\exp\left(2xt - t^2\right) = \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!}.$$

Recurrence Relation:

$$H_{-1}(x) = 0, \; H_0(x) = 1, \text{ and}$$

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1) H_{n-2}(x), \; n \geq 1.$$

The Bessel Polynomials

Notation:

$\left\{Y_n^{(\alpha)}(x)\right\}$, where $\alpha \neq -2, -3, -4, \ldots$.

Explicit Formula:

$$Y_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} (n+\alpha+1)_k \left(\frac{x}{2}\right)^k.$$

Differential Equation:

$$x^2 y''(x) + ((\alpha+2)x + 2) y'(x) - n(n+\alpha+1) y(x) = 0.$$

Orthogonality:

H. L. Krall and Frink [47] give the orthogonality relation

$$\frac{1}{2\pi i} \int_C Y_m^{(\alpha)}(z) Y_n^{(\alpha)}(z) \rho^{(\alpha)}(z) \, dz = \frac{2(-1)^{n+1} n!}{(2n+\alpha+1)(\alpha+1)_n} \delta_{mn},$$

where

$$\rho^{(\alpha)}(z) = \sum_{k=0}^{\infty} \frac{1}{(1+\alpha)_k} \left(-\frac{2}{z}\right)^k,$$

and integration is around the unit circle C. Morton and A. M. Krall [43] found that the distribution

$$w(x) = \sum_{n=0}^{\infty} \frac{2^{n+1}\delta^{(n)}(x)}{n!\,(n+1)!}$$

formally makes the Bessel polynomials orthogonal on $(-\infty, \infty)$. We refer the reader to [29] for a detailed study of these enigmatic polynomials.

Rodrigues Formula:

$$Y_n^{(\alpha)}(x) = \frac{e^{2/x}}{2^n x^{\alpha}} \cdot \frac{d^n}{dx^n}\left(x^{2n+\alpha} e^{-2/x}\right).$$

Generating Function:

$$\frac{1}{\sqrt{1-2xt}}\left(\frac{2}{1+\sqrt{1-2xt}}\right)^{\alpha} \exp\left(\frac{2t}{1+\sqrt{1-2xt}}\right) = \sum_{n=0}^{\infty} Y_n^{(\alpha)}(x)\,\frac{t^n}{n!}.$$

Recurrence Relation:

$$Y_0^{(\alpha)}(x) = 1,\ Y_1^{(\alpha)}(x) = \left(\frac{\alpha+2}{2}\right)x + 1,\ \text{and for } n \geq 1$$

$$2n(2n+\alpha+2)\,Y_{n-1}^{(\alpha)}(x) + (2n+\alpha+1)\left((2n+\alpha)(2n+\alpha+2)x + 2\alpha\right)Y_n^{(\alpha)}(x)$$
$$= 2(n+\alpha+1)(2n+\alpha)\,Y_{n+1}^{(\alpha)}(x).$$

We list some of the properties of the Legendre polynomials which are the special case of the Jacobi polynomials determined by letting the parameters $\alpha = \beta = 0$.

The Legendre Polynomials

Notation:

$\{P_n(x)\} = \{P_{1,n}(x)\}$; an explanation of the notation $\{P_{1,n}(x)\}$ is given in the final section.

Explicit Formula:

$$P_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2n-2k)!\,x^{n-2k}}{2^n k!\,(n-k)!\,(n-2k)!},$$

where $\left[\frac{n}{2}\right]$ denotes the greatest integer less than or equal to $\frac{n}{2}$.

Differential Equation:

$$\left(1-x^2\right) y''(x) - 2xy'(x) + n(n+1)\, y(x) = 0.$$

Orthogonality:

The Legendre polynomials are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = 1$ and

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, w(x)\, dx = \frac{2}{(2n+1)} \delta_{mn}.$$

Rodrigues Formula:

$$P_n(x) = \frac{(-1)^n}{2^n n!} \cdot \frac{d^n}{dx^n} \left(1-x^2\right)^n.$$

Generating Function:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n.$$

Recurrence Relation:

$$P_{-1}(x) = 0,\ P_0(x) = 1, \text{ and}$$

$$nP_n(x) = (2n-1)\, xP_{n-1}(x) - (n-1)\, P_{n-2}(x),\ \ n \geq 1.$$

1.5 Bochner-Krall Orthogonal Polynomials of Order 4

Krall's classification theorem gives necessary and sufficient conditions for when an orthogonal polynomial sequence satisfies a differential equation of the form (1.1.2).

H. L. Krall, in his 1938 paper [45], included the first example of an orthogonal polynomial sequence (which we will denote by $\{P_{n,A}(x)\}_{n=0}^{\infty}$) satisfying a fourth-order differential equation; he also found an orthogonalizing weight function for these polynomials. Because of the relationship

$$\lim_{A\to\infty} \frac{P_{n,A}(x)}{A} = P_n(x),$$

where $P_n(x)$ is the n^{th} Legendre polynomial, A. M. Krall [36] named these polynomials the Legendre type polynomials and studied them in 1981. In 1940, H. L. Krall completed the classification of all fourth-order differential equations having orthogonal polynomial solutions [46]; in all, he found three new fourth-order differential equations that have orthogonal polynomial solutions. Besides the Legendre type, there are also the Laguerre type and the Jacobi type orthogonal polynomials. A. M. Krall [36] studied these three sets of orthogonal polynomials in detail in 1981; we will list their properties at the end of this section.

In view of the contributions made by Bochner and H. L. Krall to the theory of orthogonal polynomials and differential equations, we call a sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ a Bochner-Krall orthogonal polynomial sequence of order $2n$ if $p_n(x)$, $n = 0, 1, \ldots,$ satisfies a differential equation of the form

$$L_N[y](x) = \sum_{i=0}^{N} a_i(x)\, y^{(i)}(x) = \lambda_n y(x), \; N = 2n, \tag{1.5.1}$$

where (necessarily)

$$a_i(x) = \sum_{j=1}^{i} a_{ij} x^j \text{ with } a_{ij} \text{ real constants and}$$

$\lambda_n = a_{11}n + a_{22}(n-1) + ... + a_{NN}n(n-1)...(n-N+1)$ for some $n \geq 1$ [41]. In particular, the Jacobi type, the Laguerre type, and the Legendre type polynomials are the Bochner-Krall orthogonal polynomials of order 4.

The Bochner-Krall polynomials of order 4 were studied in detail by A. M. Krall [36]. We will list some of their properties below.

The Legendre Type Polynomials

Notation:

$\{P_{n,A}(x)\}$, $n = 0, 1, \ldots,$ and $A > 0$.

Explicit Formula:

$$P_{n,A}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2n-2k)! \left(A + \frac{1}{2}n(n-1) + 2k\right) x^{n-2k}}{A 2^n k! (n-k)! (n-2k)!}.$$

Differential Equation:

$$\left(x^2-1\right)^2 y^{(4)}(x) + 8x\left(x^2-1\right) y^{(3)}(x) + (4A+12)\left(x^2-1\right) y''(x)$$
$$+8Axy'(x) - n(n+1)\left(n^2+n+4A-2\right) y(x) = 0.$$

Orthogonality:

The Legendre type polynomials are orthogonal on $[-1,1]$ with respect to the weight function

$$w(x) = \frac{1}{A}\delta(x-1) + \frac{1}{A}\delta(x+1) + 1,$$

where δ is Dirac's $\delta-$ function and

$$\int_{-1}^{1} P_{n,A}(x) P_{m,A}(x) w(x)\,dx = \frac{\left(A+\frac{1}{2}n(n-1)\right)\left(A+\frac{1}{2}(n+1)(n+2)\right)}{A(2n+1)}\delta_{mn}. \tag{1.5.2}$$

Rodrigues Type Formula:

$$P_{n,A}(x) = \frac{1}{2^n A n!}\left(A - x\frac{d}{dx} + \frac{1}{2}n(n+1)\right)\frac{d^n}{dx^n}\left(x^2-1\right)^n.$$

Generating Function:

$$\frac{1}{A}\left(A - x\frac{\partial}{\partial x} + \frac{1}{2}t\frac{\partial^2}{\partial t^2}t\right)\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_{n,A}(x)\,t^n.$$

Recurrence Relation:

$$P_{0,A}(x) = 1,\ P_{1,A}(x) = x,\ \text{and for } n \geq 1,$$

$$P_{n+1,A}(x) = \frac{(2n+1)\left(A+\frac{1}{2}n(n+1)\right)}{(n+1)\left(A+\frac{1}{2}n(n-1)\right)} x P_{n,A}(x)$$

$$-\frac{n\left(A+\frac{1}{2}(n+1)(n+2)\right)}{(n+1)\left(A+\frac{1}{2}n(n-1)\right)} P_{n-1,A}(x).$$

The Laguerre Type Polynomials

Notation:

$\{R_{n,r}(x)\}$, $n = 0, 1, \ldots,$ and $r > 0.$

Explicit Formula:

$$R_{n,r}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!} \binom{n}{k} (k(r+n+1)+r) x^k.$$

Differential Equation:

$$x^2 y^{(4)}(x) - (2x^2 - 4x) y^{(3)}(x) + (x^2 - (2r+6)x) y''(x)$$
$$+ ((2r+2)x - 2r) y'(x) - ((2r+2)n + n(n-1)) y(x) = 0.$$

Orthogonality:

The Laguerre type polynomials are orthogonal on $[0, \infty)$ with respect to the weight

$$w(x) = \frac{1}{r}\delta(x) + e^{-x}, \text{ and}$$

$$\int_0^\infty R_{n,r}(x) R_{m,r}(x) w(x) dx = (r+n+1)(r+n)\delta_{mn}.$$

Rodrigues Type Formula:

$$\exp\left(\frac{-xt}{1-t}\right) \cdot \frac{\left(r(1-t)^2 - xt\right)}{(1-t)^3} = \sum_{n=0}^{\infty} R_{n,r}(x) t^n.$$

Recurrence Relation:

$$R_{0,r}(x) = r, \ R_{1,r}(x) = r - (r+1)x, \text{ and for } n \geq 1,$$

$$R_{n,r}(x) = \frac{(2n-1)r^2 + n(n-1)r + (2n-1)n(n-1)}{n(r+n-1)^2} R_{n-1,r}(x)$$

$$- \frac{r+n}{n(r+n-1)} x R_{n-1,r}(x) - \frac{(n-1)(r+n)^2}{n(r+n-1)^2} R_{n-2,r}(x).$$

The Jacobi Type Polynomials

Notation:

$\{S_n(A,\alpha;x)\} = \{S_n(x)\}$, where $\alpha > -1$ and $A > 0$.

Explicit Formula:

$$S_n(x) = \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} (1+\alpha)_{n+k} (k(n+\alpha)(n+1) + (k+1)A) x^k}{(k+1)!\,(1+\alpha)_n}.$$

Differential Equation:

$$\begin{aligned}
&(1-x^2)^2 y^{(4)}(x) - 2(1-x^2)((\alpha+4)x+\alpha)\,y^{(3)}(x) \\
&+(1+x)((\alpha^2-3\alpha-10-4A2^{\alpha}) + (\alpha^2+9\alpha+14+4A2^{\alpha})x)\,y''(x) \\
&+((2^{\alpha+2}\alpha A + 2^{\alpha+3}A + 2\alpha^2+6\alpha+4)x + 2^{\alpha+2}\alpha A + 2\alpha^2 + 6\alpha + 4)\,y'(x) \\
&-((\alpha+2)(2\alpha+2+2A)n + (\alpha^2+9\alpha+14+2A)n(n-1) \\
&+2(\alpha+4)n(n-1)(n-2) + n(n-1)(n-2)(n-3))y(x) = 0.
\end{aligned}$$

Orthogonality:

The Jacobi type polynomials are orthogonal on $[0,1]$ with respect to the weight

$$w(x) = \frac{1}{A}\delta(x) + (1-x)^{\alpha}.$$

Rodrigues Type Formula:

$$S_n(x) = \left((1-x)\frac{d}{dx} + \left(n^2 + (\alpha+1)n + A\right)\right)(-1)^n (1-x)^{-\alpha} \frac{d^n}{dx^n}\left((1-x)^{n+\alpha} x^n\right).$$

Generating Function:

$$F(G(x,t)) = \sum_{n=0}^{\infty} S_n(x)\,t^n, \text{ where } F \text{ is the operator}$$

$$F = \left((1-x)\frac{\partial}{\partial x} + t^2\frac{\partial^2}{\partial t^2} + (\alpha+2)\,t\frac{\partial}{\partial t} + A\right),$$

$$G(x,t) = \rho^{-1}\left(\frac{2}{1-t+\rho}\right)^{\alpha}, \text{ and}$$

$$\rho(x,t) = \left(1 - 2(2x-1)t + t^2\right)^{\frac{1}{2}}.$$

Recurrence Relation:

$S_n(x) = (A_{n-1}x + B_{n-1})S_{n-1}(x) - C_{n-1}S_{n-2}(x)$, where

$$A_{n-1} = \frac{(2n+\alpha)(2n+\alpha-1)(n^2+\alpha n+A)}{(n+\alpha)n(2n+\alpha-2)\left((n-1)^2+\alpha(n-1)+A\right)^2},$$

$$B_{n-1} = \frac{(2n+\alpha-1)P(n)}{(n+\alpha)n(2n+\alpha-2)\left((n-1)^2+\alpha(n-1)+A\right)^2}, \text{ where}$$

$$\begin{aligned}P(n) &= -2n^6 - 6(\alpha+1)n^5 \\ &\quad + \left(-6\alpha^2+15\alpha-8-4A\right)n^4 + \left(-2\alpha^3+12\alpha^2-16\alpha-8\alpha A+8A+6\right)n^3 \\ &\quad + (3\alpha^3-9\alpha^2+9\alpha-4\alpha^2A+12\alpha A-4A-2A^2-2)n^2 \\ &\quad + \left(-\alpha^3+3\alpha^2-2\alpha+4\alpha^2A-4\alpha A-2\alpha A^2+2A^2\right)n + \alpha A^2, \text{ and}\end{aligned}$$

$$C_{n-1} = \frac{(n+\alpha-1)(n-1)(2n+\alpha)(n^2+\alpha n+A)^2}{(n+\alpha)n(2n+\alpha-2)\left((n-1)^2+\alpha(n-1)+A\right)^2}.$$

The left and right Jacobi type polynomials are special cases ($\alpha = 0$) of the Jacobi polynomials. Below, we list some of the properties of these polynomials. Although A. M. Krall's Jacobi type polynomials are defined on $[0,1]$, we have redefined the left and right Jacobi type polynomials so that they are orthogonal on $[-1,1]$ as well.

The Left Jacobi Type Polynomials

Notation:

$\{S_n(A,0;x)\} = \{SL_n(x)\}$, where $A > 0$.

Explicit Formula:

$$SL_n(x) = \sum_{k=0}^{n} \frac{(-1)^k (n+k)!\,(n^2+n+2A-k)}{(k!)^2 (n-k)!\,(n^2+n+2A)} \left(\frac{1-x}{2}\right)^k.$$

Differential Equation:

$$\begin{aligned}&(1-x^2)^2 y^{(4)}(x) - 8x(1-x^2)y^{(3)}(x) \\ &+ (4(3x^2-1) - 2(1-x)((2A+1)x+2A+3))y''(x) \\ &+ (2((2A+1)x+2A+3) - 2(1-x)(2A+1))y'(x) \\ &- (n^4+2n^3+(4A+1)n^2+4An)y(x) = 0.\end{aligned}$$

Orthogonality:

The left Jacobi type polynomials are orthogonal on $[-1,1]$ with respect to the weight $w(x) = \frac{1}{A}\delta(x+1) + 1$, and

$$\int_{-1}^{1} SL_n(x)\, SL_m(x)\, w(x)\, dx = \frac{2\left(n^2+2A\right)\left(n^2+2n+2A+1\right)}{(2n+1)\left(n^2+n+2A\right)^2}\delta_{mn}.$$

Rodrigues Type Formula:

$$SL_n(x) = \frac{(n+1)\left(n^2+2A\right)(-1)^n D^n\left((1-x)^n (1+x)^{n+1}\right)}{(2n+1)\left(n^2+n+2A\right)2^n n!\,(1+x)}$$

$$+\frac{n\left(n^2+2n+2A+1\right)(-1)^{n-1} D^{n-1}\left((1-x)^{n-1}(1+x)^n\right)}{(2n+1)\left(n^2+n+2A\right)2^{n-1}(n-1)!\,(1+x)}.$$

Recurrence Relation:

$$SL_0(x) = 1,\ SL_1(x) = \frac{(2A+1)(x+1)}{(2A+2)},\ \text{and for } n \geq 2.$$

$$SL_n(x) = \left(\frac{(2n-1)\left(n^2+2A\right)\left(n^2-n+2A\right)}{n\left(n^2+n+2A\right)\left(n^2-2n+2A+1\right)}x\right.$$

$$\left.-\frac{(2n-1)\left(n^2-n-2A\right)\left(n^2-n+2A\right)}{n\left(n^2+n+2A\right)\left(n^2-2n+2A+1\right)^2}\right) SL_{n-1}(x)$$

$$-\frac{(n-1)\left(n^2-3n+2A+2\right)\left(n^2+2A\right)^2}{n\left(n^2+n+2A\right)\left(n^2-2n+2A+1\right)^2} SL_{n-2}(x).$$

The Right Jacobi Type Polynomials

Notation:

$\{S_n(B,0;x)\} = \{SR_n(x)\}$, where $B > 0$.

Explicit Formula:

$$SR_n(x) = \sum_{k=0}^{n} \frac{(-1)^k (n+k)!\left(n^2+n+2B-k\right)}{(k!)^2 (n-k)!\left(n^2+n+2B\right)}\left(\frac{1+x}{2}\right)^k.$$

Differential Equation:

$$
\begin{aligned}
&\left(1-x^2\right)^2 y^{(4)}(x) + 8x\left(1-x^2\right) y^{(3)}(x) \\
&+ \left(4\left(3x^2-1\right) - 2(1+x)\left((-2B-1)x + 2B+3\right)\right) y''(x) \\
&- \left(2\left((-2B-1)x + 2B+3\right) - 2(1+x)(2B+1)\right) y'(x) \\
&- \left(n^4 + 2n^3 + (4B+1)n^2 + 4Bn\right) y(x) = 0.
\end{aligned}
$$

Orthogonality:

The right Jacobi type polynomials are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = \frac{1}{B}\delta(1-x) + 1$, and

$$
\int_{-1}^{1} SR_n(x)\, SR_m(x)\, w(x)\, dx = \frac{2\left(n^2+2B\right)\left(n^2+2n+2B+1\right)}{(2n+1)\left(n^2+n+2B\right)^2}\delta_{mn}.
$$

Rodrigues Type Formula:

$$
\begin{aligned}
SR_n(x) = &\frac{(n+1)\left(n^2+2B\right)(-1)^n D^n\left((1+x)^n (1-x)^{n+1}\right)}{(2n+1)\left(n^2+n+2B\right) 2^n n!\,(1-x)} \\
&+\frac{n\left(n^2+2n+2B+1\right)(-1)^{n-1} D^{n-1}\left((1+x)^{n-1}(1-x)^n\right)}{(2n+1)\left(n^2+n+2B\right) 2^{n-1}(n-1)!\,(1-x)}.
\end{aligned}
$$

Recurrence Relation:

$$
SR_0(x) = 1,\ SR_1(x) = \frac{(2B+1)(1-x)}{(2B+2)},\ \text{and for } n \geq 2
$$

$$
\begin{aligned}
SR_n(x) = &\left(-\frac{(2n-1)\left(n^2+2B\right)\left(n^2-n+2B\right)}{n\left(n^2+n+2B\right)\left(n^2-2n+2B+1\right)}x\right. \\
&\left.-\frac{(2n-1)\left(n^2-n-2B\right)\left(n^2-n+2B\right)}{n\left(n^2+n+2B\right)\left(n^2-2n+2B+1\right)^2}\right) SR_{n-1}(x) \\
&-\frac{(n-1)\left(n^2-3n+2B+2\right)\left(n^2+2B\right)^2}{n\left(n^2+n+2B\right)\left(n^2-2n+2B+1\right)^2} SR_{n-2}(x).
\end{aligned}
$$

Notice that the weight for the Legendre type orthogonal polynomial sequence has equal jumps at ± 1. It is natural to ask whether an orthogonal polynomial sequence exists when unequal jumps at ± 1 are considered. In 1981, using techniques due to Shore [66] and H. L. Krall [46], Littlejohn found that such an orthogonal polynomial sequence exists and called them the Krall polynomials. These polynomials satisfy a sixth-order differential equation. The properties of the Krall polynomials are given in [49].

Littlejohn [50] proved the following theorem. Below, the notation $\langle \Lambda, \varphi \rangle$ denotes the action of a distribution on a test function φ.

Theorem 1.6.1. Let $\{p_n(x)\}_{n=0}^{\infty}$ be an orthogonal polynomial sequence which satisfies an equation of the form (1.1.2). Suppose that Λ is a distribution that satisfies the system

$$\left\langle \sum_{i=2k+1}^{2n} (-1)^i \binom{i-k-1}{k} (a_i \Lambda)^{(i-2k-1)}, \varphi \right\rangle = 0, \; k = 0, 1, \ldots, n-1 \qquad (1.6.1)$$

for all polynomials φ, where $a_1(x) = \sum_{j=0}^{i} l_{ij} x^j$. Suppose further that $\Lambda(x) \to 0$ as $|x| \to \infty$. Then Λ is an orthogonalizing weight distribution of $\{p_n(x)\}_{n=0}^{\infty}$.

The distributional differential equations (1.6.1) are called *the weight equations* for $\{p_n(x)\}_{n=0}^{\infty}$. Since $\langle L(\Lambda), \varphi \rangle = \langle \Lambda, L^+(\varphi) \rangle$, where L is any linear differential expression and L^+ denotes the formal Lagrange adjoint of L, we see that the system (1.6.1) is equivalent to the system

$$\left\langle \Lambda, \sum_{i=2k+1}^{2n} \binom{i-k-1}{k} a_i \varphi^{(i-2k-1)} \right\rangle = 0, \; k = 0, 1, \ldots, n-1. \qquad (1.6.2)$$

If the coefficients of a differential equation having an orthogonal sequence of polynomials as eigenfunctions are known, then Theorem 1.6.1. can be used to find an orthogonalizing weight function. Conversely, it is possible to start with a weight function and then use the weight equations along with symmetry properties to construct a differential equation having orthogonal polynomial solutions. Littlejohn [51] used this latter approach to construct the second example of a sixth-order differential equation having a sequence of orthogonal polynomials as eigenfunctions. The orthogonal polynomials Littlejohn found by this method are the generalized Laguerre type polynomials for $\alpha = 1$. Now, we list some properties of the two Bochner-Krall orthogonal polynomials of order 6.

Krall Polynomials

Notation:

$\{P_{3,n}(A,B,C;x)\} = \{P_{3,n}(x)\}$, where $A,\ B,\ C > 0$.

Explicit formula:

$$\{P_{3,n}(x)\} = \sum_{j=0}^{n} \frac{(-1)^{[j/2]}(2n-j)!Q(n,j)\,x^{n-j}}{2^{n+1}(n-[(j+1)/2])!\,[j/2]!\,(n-j)!\,(n^2+n+AC+BC)},$$

where

$$Q(n,j) = \frac{1+(-1)^j}{2}(n^4 + (2AC+2BC-1)\,n^2 + 4ABC^2)$$

$$+\frac{1+(-1)^j}{2}2j\left(n^2+n+AC+BC\right) + \frac{1-(-1)^j}{2}(4BC-4AC)$$

and $[x]$ denotes the greatest integer function.

Differential Equation:

$$\begin{aligned}
&(x^2-1)^3\,y^{(6)}(x) + 18x\,(x^2-1)^2\,y^{(5)}(x)\\
&+((3AC+3BC+96)\,x^4-(6AC+6BC+132)\,x^2+(3AC+3BC+36))\,y^{(4)}(x)\\
&+((24AC+24BC+168)\,x^3-(24AC+24BC+168)\,x)\,y^{(3)}(x)\\
&+(\,(12ABC^2+42AC+42BC+72)\,x^2+(12BC-12AC)\,x\\
&-(12ABC^2+30AC+30BC+72)\,)y''(x)\\
&+((24ABC^2+12AC+12BC)\,x+(12BC-12AC))\,y'(x)\\
&=\lambda_n y(x),
\end{aligned}$$

where

$$\lambda_n = \left(24ABC^2 + 12AC + 12BC\right) n + \{\left(12ABC^2 + 42AC + 42BC + 72\right)$$
$$+ \left(24AC + 24BC + 168\right)(n-2) + \left(3AC + 3BC + 96\right)\left(n^2 - 5n + 6\right)$$
$$+18(n-2)(n-3)(n-4) + (n-2)(n-3)(n-4)(n-5)\}\left(n^2 - n\right).$$

Orthogonality:

The Krall polynomials are orthogonal on $[-1, 1]$ with respect to the weight function

$$w(x) = \frac{1}{A}\delta(x+1) + \frac{1}{B}\delta(x-1) + C, \text{ and}$$

$$\int_{-1}^{1} P_{3,m}(x)\, P_{3,n}(x)\, w(x)\, dx = \delta_{mn} \cdot M,$$

where

$$M = \frac{C\left(n^4 + (2AC + 2BC - 1)\, n^2 + 4ABC^2\right)(2AC + 2BC - 1)(n+1)^2}{2(2n+1)\left(n^2 + n + AC + BC\right)^2}$$

$$+ \frac{C\left(n^4 + (2AC + 2BC - 1)\, n^2 + 4ABC^2\right)\left((n+1)^4 + 4ABC^2\right)}{2(2n+1)\left(n^2 + n + AC + BC\right)^2}$$

Rodrigues Type Formula:

$$P_{3,n}(x) = \left(\frac{AC + BC}{2} + \frac{n(n+1)}{2}\right) \frac{(-1)^n D^n\left(\left(1 - x^2\right)^n\right)}{2^n n!}$$

$$- \left(xD + \frac{2(AC - BC)^2}{n^2 + n + AC + BC}\right) \frac{(-1)^n D^n\left(\left(1 - x^2\right)^n\right)}{2^n n!}$$

$$+ \frac{(-1)^{n-1}(n+1)(BC - AC)\, D^{n-1}\left(\left(1 - x^2\right)^n\right)}{2^n\left(n^2 + n + AC + BC\right)(n-1)!\left(1 - x^2\right)},$$

where $D = \frac{d}{dx}$.

Generating Function:

$$\sum_{n=0}^{\infty}\left(n^2+n+AC+BC\right)P_{3,n}\left(x\right)t^n$$

$$=\frac{\left(BC-AC\right)t}{\left(1-2xt+t^2\right)^{3/2}}-\frac{\left(AC-BC\right)^2}{\left(1-2xt+t^2\right)^{1/2}}\quad+\frac{M_1\circ M_2}{\left(1-2xt+t^2\right)^{1/2}},$$

where

$$M_1=\left(\left(AC+BC\right)+t\frac{\partial^2}{\partial t^2}\right),\ \text{and } M_2=\left(\frac{AC-BC}{2}-x\frac{\partial}{\partial x}+\frac{1}{2}t\frac{\partial^2}{\partial t^2}\right).$$

Recurrence Relation:

$$P_{3,-1}\left(x\right)=0,\ P_{3,0}\left(x\right)=\frac{ABC^2}{AC+BC},\ \text{and for } n\geq 1,$$

$$P_{3,n}\left(x\right)=\frac{\left(2n-1\right)A\left(n\right)B\left(n-1\right)x}{nA\left(n-1\right)B\left(n\right)}P_{3,n-1}\left(x\right)$$

$$+\frac{\left(2n-1\right)\left(2BC-2AC\right)C\left(n\right)B\left(n-1\right)}{n\left(A\left(n-1\right)\right)^2B\left(n\right)}P_{3,n-1}\left(x\right)$$

$$-\frac{\left(n-1\right)\left(A\left(n\right)\right)^2B\left(n-2\right)}{n\left(A\left(n-1\right)\right)^2B\left(n\right)}P_{3,n-2}\left(x\right),\ \text{where}$$

$A\left(n\right)=n^4+\left(2AC+2BC-1\right)n^2+4ABC^2,$

$B\left(n\right)=n^2+n+AC+BC,$ and

$C\left(n\right)=-3n^4+6n^3+\left(-2AC-2BC-3\right)n^2+\left(2AC+2BC\right)n+4ABC^2.$

Generalized Laguerre Polynomials $(\alpha=1)$

Notation:

$\left\{Q_n\left(x;R\right)\right\}=\left\{Q_n\left(x\right)\right\}$, where $R>0$.

Explicit Formula:

$$Q_n\left(x\right)=\sum_{k=0}^{n}\frac{\left(-1\right)^k\left(n+1\right)!\left(\left(n+1\right)\left(n+2\right)k+2R\left(k+2\right)\right)x^k}{k!\left(n-k\right)!\left(k+2\right)!}.$$

Differential Equation:

$$x^3y^{(6)}(x) + (-x^3+12x^2)\,y^{(5)}(x) + (3x^3-30x^2+30x)\,y^{(4)}(x)$$
$$+(-x^3+24x^2-60x)\,y^{(3)}(x) + (-6x^2+(36+6R)\,x)\,y''(x)$$
$$+((-6-6R)\,x+12R)\,y'(x) + (n^3+3n^2+(6R+2)\,n)\,y(x) = 0.$$

Orthogonality:

The generalized Laguerre polynomials with $\alpha = 1$ are orthogonal on $[0, \infty)$ with respect to the weight function

$$w(x) = xe^{-x} + \frac{1}{R}\delta(x)\,, \text{ and}$$

$$\int_0^\infty Q_m(x)\,Q_n(x)\,w(x)\,dx = \left(n^3+3n+2R+2\right)\left(n^2+n+2R\right)(n+1)\,\delta_{mn}.$$

Rodrigues Type Formula:

$$Q_n(x) = \frac{e^x}{n!}\cdot\frac{d^n}{dx^n}((n^2+n+2R)\,e^{-x}x^n + \left(-n^3+n^2-2Rn+2n\right)e^{-x}x^{n-1}$$

$$+\left(-2n^3+2n\right)e^{-x}x^{n-2}).$$

Generating Function:

$$\sum_{n=0}^{\infty} Q_n(x)\,t^n = \frac{\exp\left(\frac{-xt}{1-t}\right)}{(1-t)^6}(2Rt^4 + (2x-8R)\,t^3 + \left(x^2+12R\right)t^2$$

$$+(-2x-8R)\,t+2R).$$

Recurrence Relation:

$Q_{-1}(x) = 0$, $Q_0(x) = 2R$, and for $n \geq 1$,

$$Q_n(x) = \left(\frac{-(n^2+n+2R)\,x}{n\,(n^2-n+2R)} + \frac{2n^4+(8R-2)\,n^2+8R\,(R-1)}{(n^2-n+2R)^2}\right)Q_{n-1}(x)$$

$$-\left(\frac{n^2+n+2R}{n^2-n+2R}\right)^2 Q_{n-2}(x)\,.$$

The only other orthogonal polynomial sequence known to satisfy a differential equation of the form (1.5.1) is the generalized Laguerre type orthogonal polynomial sequence for $\alpha = 2$. A. M. Krall and L. L. Littlejohn showed that these polynomials satisfy an eight-order differential equation and studied their properties in [41].

1.7 Legendre$^{(r)}$ Orthogonal Polynomials

Let $\hat{\varphi}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\mu}$, and $\hat{\kappa}$ be monotic functions on $\mathbb{R}$ defined by

$$\hat{\varphi}(x) := \begin{cases} -1 & if \ -\infty < x \leq -1 \\ x & if \ -1 < x < 1 \\ 1 & if \ 1 \leq x < \infty, \end{cases}$$

$$\hat{\alpha}(x) := \begin{cases} -1 - \frac{1}{A} & if \ -\infty < x \leq -1 \\ x & if \ -1 < x < 1 \\ 1 & if \ 1 \leq x < \infty, \end{cases}$$

$$\hat{\beta}(x) := \begin{cases} -1 & if \ -\infty < x \leq -1 \\ x & if \ -1 < x < 1 \\ 1 + \frac{1}{B} & if \ 1 \leq x < \infty, \end{cases}$$

$$\hat{\mu}(x) := \begin{cases} -1 - \frac{1}{A} & if \ -\infty < x \leq -1 \\ x & if \ -1 < x < 1 \\ 1 & if \ 1 \leq x < \infty, \text{ and} \end{cases}$$

$$\hat{\kappa}(x) := \begin{cases} -1 - \frac{1}{A} & if \ -\infty < x \leq -1 \\ x & if \ -1 < x < 1 \\ 1 + \frac{1}{B} & if \ 1 \leq x < \infty, \end{cases}$$

where it is assumed that both A and B are positive parameters. Let φ, α, β, μ, and κ denote the regular Borel measures generated by $\hat{\varphi}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\mu}$, and $\hat{\kappa}$ respectively.

Notice that

(*i*) $d\varphi(x) = w(x)\,dx$ where $w(x)$ is the weight for the Legendre polynomials;

(*ii*) $d\alpha(x) = w(x)\,dx$ where $w(x)$ is the weight for the left Jacobi type polynomials on $[-1,1]$ with $\alpha = 0$;

(*iii*) $d\beta(x) = w(x)\,dx$ where $w(x)$ is the weight for the right Jacobi type on $[-1,1]$ with $\alpha = 0$;

(*iv*) $d\mu(x) = w(x)\,dx$ where $w(x)$ is the weight for the Legendre type polynomials, and

(*v*) $d\kappa(x) = w(x)\,dx$ where $w(x)$ is the weight for the Krall polynomials in the case $C = 1$.

All of these five measures have the properties that all Borel sets in the complement of $[-1,1]$ measure to zero and all Borel sets contained in the open interval $(-1,1)$ are measured to Lebesque measure. Because of the similarity of these measures, we call the five sets of Bochner-Krall polynomials which are orthogonal with respect to these measures the *Legendre*$^{(r)}$ *polynomials.*

As mentioned earlier in Section 1.3, when $\alpha = \beta = 0$, the Jacobi type polynomials, $\left\{P_n^{(\alpha,\beta)}\right\}$, are known as the classical Legendre polynomials and are commonly denoted by $\{P_n\}$. Since the Legendre polynomials satisfy a second-order differential equation, we will call them the *Legendre*$^{(1)}$ *polynomials* and use the notation $\{P_{1,n}(x)\}$ to emphasize their relationship to the other Legendre$^{(r)}$ polynomials.

The Jacobi type polynomials orthogonal with respect to the measures generated by $\alpha(x)$ and $\beta(x)$ are given by the formulas;

$$PL_{2,n}(A;x) = PL_{2,n}(x) = \frac{1}{n^2+n+2A} S_n\left(2A, 0; \frac{1+x}{2}\right)$$

and

$$PR_{2,n}(A;x) = PR_{2,n}(x) = \frac{1}{n^2+n+2B} S_n\left(2B, 0; \frac{1-x}{2}\right), \text{ respectively.}$$

We shall refer to $\{PL_{2,n}(A;x)\}$ and $\{PL_{2,n}(B;x)\}$ as, respectively, the *left* and *right Jacobi type sequences of orthogonal polynomials.* Since they are all eigenfunctions of fourth-order differential equations, the left and right Jacobi type polynomials and the Legendre type polynomials will be referred as the *Legendre*$^{(2)}$ *polynomials.*

The Krall polynomials, in the case $C = 1$, are called the *Legendre*$^{(3)}$ *polynomials* since they satisfy a sixth-order differential equation.

1.8 Koornwinder's Orthogonal Polynomials

In 1984, Koornwinder [35] defined the following generalization of the Jacobi polynomials $\left\{P_n^{(\alpha,\beta)}\right\}$:
$P_n^{(\alpha,\beta,M,N)}(x) :=$

$$\left(\frac{(\alpha+\beta+1)_n}{n!}\right)^2 \left(\frac{B_n M(1-x) - A_n N(1+x)}{\alpha+\beta+1}\frac{d}{dx} + A_n B_n\right) P_n^{(\alpha,\beta)}(x), \quad (1.8.1)$$

where

$$A_n = \frac{(\alpha+1)_n n!}{(\beta+1)_n (\alpha+\beta+1)_n} + \frac{n(n+\alpha+\beta+1)M}{(\beta+1)(\alpha+\beta+1)},$$

$$B_n = \frac{(\beta+1)_n n!}{(\alpha+1)_n (\alpha+\beta+1)_n} + \frac{n(n+\alpha+\beta+1)N}{(\alpha+1)(\alpha+\beta+1)}, \ \alpha,\ \beta > -1, \text{ and } M,\ N \geq 0.$$

We shall call these polynomials the *Koornwinder-Jacobi polynomials.* These polynomials are orthogonal on $[-1,1]$ with respect to the weight function

$$w(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} (1-x)^\alpha (1+x)^\beta + M\delta(x+1) + N\delta(x-1).$$

Each of the Legendre$^{(r)}$ orthogonal polynomials is a special case of Koornwinder-Jacobi polynomials.

In particular,

$$P_{1,n}(x) = P_n^{(0,\,0,\,0,\,0)}(x),$$

$$PL_{2,n}(A;x) = \frac{2A}{n^2+n+2A} P_n^{\left(0,\,0,\,\frac{1}{2A},\,0\right)}(x),$$

$$PR_{2,n}(B;x) = \frac{1}{n^2+n+2B} P_n^{\left(0,\,0,\,0,\,\frac{1}{2B}\right)}(x),$$

$$P_{2,n}(A;x) = \frac{A}{A+\frac{n(n+1)}{2}} P_n^{\left(0,\,0,\,\frac{1}{2A},\,\frac{1}{2A}\right)}(x), \text{ and}$$

$$P_{3,n}(A,B;x) = \frac{2A}{n^2+n+2A} P_n^{\left(0,\,0,\,\frac{1}{2A},\,\frac{1}{2B}\right)}(x).$$

In the same paper [35], Koornwinder defined a generalization of the Laguerre polynomials $\left\{L_n^{(\alpha)}(x)\right\}$:

$$L_n^{(\alpha,N)}(x) := \left\{1 + \frac{N(\alpha+1)_n}{n!}\left(\frac{d}{dx} + \frac{n}{\alpha+1}\right)\right\} L_n^{(\alpha)}(x), \tag{1.8.2}$$

where $\alpha > -1$ and $N > 0$. We shall call these polynomials the *Koornwinder-Laguerre polynomials.* They are orthogonal on $[0,\infty)$ with respect to the weight function

$$w(x) = \frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)} + N\delta(x).$$

Although Koornwinder's polynomials generalize all known Bochner-Krall polynomials, it is not known if they are always themselves Bochner-Krall polynomials. That is, it is unknown whether there are differential equations of the form (1.1.1) of fixed order having the Koornwinder-Jacobi polynomials or the Koornwinder-Laguerre polynomials as solutions. More specifically, if α and β vary in (1.8.1) or if α varies in (1.8.2), the order of the differential equation seems to vary.

CHAPTER TWO

Theory of Self-Adjoint Differential Operators

2.1 Introduction

The field of orthogonal polynomials is a field that touches many areas of mathematics, both pure and applied. However, there is another subject that has received surprisingly little attention in the circles of orthogonal polynomials: the spectral theory of differential operators. On the other hand, it is difficult to understand why this is the case. After all, the second-order differential equations associated with the classical orthogonal polynomials of Jacobi, Laguerre and Hermite have long earned their niche as important models in various areas of mathematical physics and they do serve as excellent examples to fit the theory of singular self-adjoint differential operators. Furthermore, the mere existence of these equations implicitly suggests that there may be more differential equations with orthogonal polynomial eigenfunctions. On the other hand, the spectral theory of differential operators is a very rich subject with literally thousands of contributions in mathematical literature. If one wants to learn this subject to understand the role that orthogonal polynomials will play, where should one look? To the expert, the answer is obvious: the texts of Naimark ([55], Part II, Sections 14-18), Akhieszer and Glazman ([1], Vol. 2, Chapter 7 and Appendix 2)

and to a certain extent, Dunford and Schwartz ([13], Part II, Chapters 12 and 13), Weidmann ([70], Chapters 8 and 10) and, with specific applications to the second-order Sturm-Liouville problem, the text of Hellwig ([32], Parts 4 and 5) are recommended. Of course, herein may lie the problem for the non-expert: the amount of recommended reading is rather significant.

In this chapter, we merely are seizing the opportunity to collect results from several sources. In Sections from 2.2 to 2.8, we shall discuss the necessary rudimentary theory of self-adjoint extensions of formally symmetric differential expressions. No proofs shall be given but exact references will be given.

In Section 2.9, we shall apply this theory to discuss some of the five Legendre differential expressions:

$$M_k^{(1)}[y](x) := -\left(\left(1-x^2\right)y'(x)\right)' + ky(x), \tag{2.1.1}$$

$$\begin{gathered} ML_k^{(2)}[y](x) := \\ \left(\left(1-x^2\right)^2 y''(x)\right)'' - 2\left(\left(1-x\right)\left(\left(2A+1\right)x+2A+3\right)y'(x)\right)' + ky(x), \end{gathered} \tag{2.1.2}$$

$$\begin{gathered} MR_k^{(2)}[y](x) := \\ \left(\left(1-x^2\right)^2 y''(x)\right)'' - 2\left(\left(1+x\right)\left(\left(-2B+1\right)x+2B+3\right)y'(x)\right)' + ky(x), \end{gathered} \tag{2.1.3}$$

$$M_k^{(2)}[y](x) := \left(\left(1-x^2\right)y''(x)\right)'' - \left(\left(4A\left(1-x^2\right)+8\right)y'(x)\right)' + ky(x), \tag{2.1.4}$$

$$\begin{gathered} M_k^{(3)}[y](x) := \\ -\left(\left(1-x^2\right)^3 y^{(3)}(x)\right)^{(3)} + \left(\left(1-x^2\right)\left(12+\left(3A+3B+6\right)\left(1-x^2\right)\right)y''(x)\right)'' \\ -\left(\left(\left(-6A-6B-12AB\right)x^2+12\left(A-B\right)x+12AB+18A+18B+24\right)y'(x)\right)' \\ +ky(x), \end{gathered} \tag{2.1.5}$$

where $x \in (-1,1)$; A, B are fixed positive parameters and k is a fixed non-negative constant. Of course, expression (2.1.1) is the classical Legendre expression having the Legendre polynomials as eigenfunctions but (2.1.2)-(2.1.5) are also called the Legendre expressions. Indeed, these four expressions also have orthogonal polynomial eigenfunctions and, in a certain asymptotic sense, they behave much like the classical Legendre polynomials. Furthermore, there are no other differential expressions of the above type that have "the Legendre type" orthogonal polynomial eigenfunctions. We shall restrict our study of (2.1.1)-(2.1.5) to the right-definite setting; i.e., in the spaces $L^2(-1,1)$ and $L^2(-1,1;w)$ where w is an orthogonalizing weight distribution associated with the corresponding orthogonal eigenfunctions.

For a complete study of these five differential expressions, the reader is encouraged to consult the Ph.D. thesis of S. M. Loveland [54].

2.2 Singular Differential Expressions

In this section, we shall assume that $a_k \in C^k(I,\mathbb{R})$, $k = 0,1,\ldots,n$ with $a_k(x) \neq 0$ for all $x \in I$ and n is some positive integer. Here, we assume that $I = (a,b)$ is an open interval on the real line with $-\infty \leq a < b \leq \infty$. We shall study certain linear operators in the Lebesgue space $L^2(I)$ generated from the ordinary differential expression $\ell[\cdot]$ of order $2n$ defined by:

$$\ell[y] := \sum_{j=0}^{n} (-1)^j \left(a_j(x)\, y^{(j)}(x)\right)^{(j)}, \ x \in I. \tag{2.2.1}$$

We shall consider further conditions on these coefficients below in Definition 2.2.1. Notice that expressions (2.1.1)-(2.1.5) are all of the form (2.2.1).

Two operators associated with $\ell[\cdot]$ that we define and discuss below are the maximal and the minimal operators generated by $\ell[\cdot]$. With this, we shall seek to find self-adjoint extensions (respectively, restrictions) of the minimal operator $\mathcal{L}_0$ (maximal operator $\mathcal{L}$). We shall also be interested in the spectra of these extensions. In particular, we shall study the eigenvalue problem

$$S[y] = \lambda y,$$

where S is one of these self-adjoint operators. It is precisely this problem that is interesting from the viewpoint of orthogonal polynomials.

The expression (2.2.1) is called a *formally symmetric differential expression.* At this point, we note that both Naimark [55], and Akhieszer and Glazman [1] consider differential expressions with far less smoothness conditions on the a_k's than we have assumed here. Indeed, the less restrictive hypotheses assumed by these authors leads them to the important concept of *quasi-derivative.* However, for the sake of simplicity, we shall keep with our above assumptions. Furthermore, in the case of the eigenvalue problem $\ell[y] = \lambda y$ having a sequence of orthogonal polynomial solutions $\{p_n(x)\}_{n=0}^{\infty}$, where $p_n(x)$ has degree exactly n, it is always the case that $a_j \in C^j(I, \mathbb{R})$.

At this point, we distinguish between the important concepts of *regular* and *singular* differential expressions.

Definition 2.2.1. The expression $\ell[\cdot]$, given in (2.2.1), is called a *regular differential expression* if I has finite length and the coefficients $\frac{1}{a_n}, a_{n-1}, \ldots, a_0 \in L(I)$. If $\ell[\cdot]$ is not regular, it is called a *singular differential expression.* The right endpoint b of I is called a *regular* point of $\ell[\cdot]$ if $b < +\infty$ and if there exists an $\varepsilon > 0$ sufficiently small so that $\frac{1}{a_n}, a_{n-1}, \ldots, a_0 \in L(b-\varepsilon, b)$; otherwise, the point b is a *singular* point of $\ell[\cdot]$. There is a similar definition for the left endpoint a of I.

We make the assumption, for the rest of this chapter unless otherwise stated, that $\ell[\cdot]$ is a singular differential expression. The reader can consult the texts [55] (pages 62-67 and 77-78) and [1] (pages 166-170) for the analysis of regular differential expressions and their self-adjoint extensions. Our assumption is based on the fact that equations (2.1.1)-(2.1.5) are singular differential expressions on $(-1, 1)$.

Definition 2.2.2. Let $\ell[\cdot]$ be given as in (2.2.1). The operator $\mathcal{L} : L^2(I) \to L^2(I)$ defined by:

$$\mathcal{L}[y] = \ell[y]$$

$$\mathcal{D}(\mathcal{L}) := \left\{ y : I \to \mathbb{C} \,\middle|\, y^{(k)} \in AC_{loc}(I),\ k = 0, 1, \ldots, 2n-1;\ y,\ \ell[y] \in L^2(I) \right\} \tag{2.2.2}$$

is called the *maximal operator* generated by $\ell[\cdot]$ in $L^2(I)$.

The name "maximal" is actually quite appropriate: the space $\mathcal{D}(\mathcal{L})$ is the largest subspace in which $\mathcal{L}$ can be defined as an operator from $L^2(I)$ into $L^2(I)$.

2.3 Green's Formula

For $f,\ g \in \mathcal{D}(\mathcal{L})$ and $[\alpha, \beta] \subset I$, the following formula can be easily verified using integration by parts:

$$\int_{\alpha}^{\beta} \left(\ell[f]\,\overline{g} - \ell[\overline{g}]\,f\right) dx = [f, g](x) \Big|_{\alpha}^{\beta}, \tag{2.3.1}$$

where $[f, g](\cdot)$ is sesquilinear form defined by:

$$[f, g](x) :=$$

$$\sum_{j=1}^{n} \sum_{m=1}^{j} (-1)^{m+j} \left\{ \left(a_j(x)\,\overline{g}^{(j)}(x)\right)^{(j-m)} f^{(m-1)}(x) - \left(a_j(x)\,f^{(j)}(x)\right)^{(j-m)} \overline{g}^{(m-1)}(x) \right\} \tag{2.3.2}$$

Notice that $[g, f](x) = -\overline{[f, g]}(x)$ for all $f,\ g \in \mathcal{D}(\mathcal{L})$ and $a < x < b$. Observe, by definition of $\mathcal{D}(\mathcal{L})$ and Hölder's inequality, that the limits

$$[f, g](b) := \lim_{x \to b^-} [f, g](x) \ \text{ and } \ [f, g](a) := \lim_{x \to a^+} [f, g](x)$$

both exist and are finite, for all $f,\ g \in \mathcal{D}(\mathcal{L})$.

Equation (2.3.1) is known as *Green's formula* for $\ell[\cdot]$. As we shall see, Green's formula is essential in the determination of all self-adjoint extensions in $L^2(I)$ of the minimal operator generated by $\ell[\cdot]$. We list Green's formula for expressions (2.1.1)-(2.1.5), respectively.

$$\int_{-1}^{1} \left\{ M_k^{(1)}[f](x)\,\overline{g}(x) - M_k^{(1)}[\overline{g}](x)\,f(x) \right\} dx = [f, g]_{(1)}(x) \Big|_{-1}^{1}, \text{ where}$$

$$[f, g]_{(1)}(x) = \left(1 - x^2\right)\left(f(x)\,\overline{g}'(x) - f'(x)\,\overline{g}(x)\right), \tag{2.3.3}$$

$$\int_{-1}^{1} \left\{ ML_k^{(2)}[f](x)\,\overline{g}(x) - ML_k^{(2)}[\overline{g}](x)\,f(x) \right\} dx = [f, g]_L(x) \Big|_{-1}^{1}, \text{ where}$$

$$[f,g]_L(x) :=$$

$$\left\{\left(\left(1-x^2\right)^2 f''(x)\right)' - 2(1-x)\left((2A+1)x+(2A+3)\right)f'(x)\right\}\overline{g}(x)$$

$$-\left\{\left(\left(1-x^2\right)^2 \overline{g}''(x)\right)' - 2(1-x)\left((2A+1)x+(2A+3)\right)\overline{g}'(x)\right\}f(x)$$

$$-\left(1-x^2\right)^2\left(f''(x)\,\overline{g}'(x)-\overline{g}''(x)\,f'(x)\right), \tag{2.3.4}$$

$$\int_{-1}^{1}\left\{MR_k^{(2)}[f](x)\,\overline{g}(x)-MR_k^{(2)}[\overline{g}](x)\,f(x)\right\}dx=[f,g]_R(x)\,\Big|_{-1}^{1},\ \text{where}$$

$$[f,g]_R(x) :=$$

$$\left\{\left(\left(1-x^2\right)^2 f''(x)\right)' - 2(1+x)\left((-2B-1)x+(2B+3)\right)f'(x)\right\}\overline{g}(x)$$

$$-\left\{\left(\left(1-x^2\right)^2 \overline{g}''(x)\right)' - 2(1+x)\left((-2B-1)x+(2B+3)\right)\overline{g}'(x)\right\}f(x)$$

$$-\left(1-x^2\right)^2\left(f''(x)\,\overline{g}'(x)-\overline{g}''(x)\,f'(x)\right), \tag{2.3.5}$$

$$\int_{-1}^{1}\left\{M_k^{(2)}[f](x)\,\overline{g}(x)-M_k^{(2)}[\overline{g}](x)\,f(x)\right\}dx=[f,g]_2(x)\,\Big|_{-1}^{1},\ \text{where}$$

$$[f,g]_2(x) := \left\{\left(\left(1-x^2\right)^2 f''(x)\right)' - \left(8+4A\left(1-x^2\right)\right)f'(x)\right\}\overline{g}(x)$$

$$-\left\{\left(\left(1-x^2\right)^2 \overline{g}''(x)\right)' - \left(8+4A\left(1-x^2\right)\right)\overline{g}'(x)\right\}f(x)$$

$$-\left(1-x^2\right)^2\left(f''(x)\,\overline{g}'(x)-\overline{g}''(x)\,f'(x)\right), \tag{2.3.6}$$

$$\int_{-1}^{1}\left\{M_k^{(3)}[f](x)\,\overline{g}(x)-M_k^{(3)}[\overline{g}](x)\,f(x)\right\}dx=[f,g]_3(x)\,\Big|_{-1}^{1},$$

where

$$[f,g]_3(x) :=$$

$$\left\{-\left((1-x^2)^3 f^{(3)}(x)\right)'' + ((1-x^2)(12+\alpha(1-x^2))f''(x))' - \pi(x) f'(x)\right\} \overline{g}(x)$$

$$-\left\{-\left((1-x^2)^3 \overline{g}^{(3)}(x)\right)'' + ((1-x^2)(12+\alpha(1-x^2))\overline{g}''(x))' - \pi(x) \overline{g}'(x)\right\} f(x)$$

$$-\left\{-\left((1-x^2)^3 f^{(3)}(x)\right)' + (1-x^2)(12+\alpha(1-x^2))f''(x)\right\} \overline{g}'(x)$$

$$+\left\{-\left((1-x^2)^3 \overline{g}^{(3)}(x)\right)' + (1-x^2)(12+\alpha(1-x^2))\overline{g}''(x)\right\} f'(x)$$

$$-\left\{f^{(3)}(x)\overline{g}''(x) - \overline{g}^{(3)}(x) f''(x)\right\}(1-x^2)^{(3)}, \text{ where} \tag{2.3.7}$$

$$\pi(x) = (-6A - 6B - 12AB)x^2 + (12A - 12B)x + (12AB + 18A + 18B + 24)$$

and $\alpha = 3A + 3B + 6$.

2.4 Operators in Hilbert Space

There are two basic types of linear operators in a Hilbert space H with inner product $(\cdot,\cdot)$: bounded and unbounded. A linear unbounded operator is actually discontinuous at every point. A differential operator is invariably of the unbounded type.

At this point, we recall a number of fundamental definitions and facts concerning linear operators.

Definition 2.4.1. An operator $T : H \to H$ with domain $\mathcal{D}(T)$ is said to be *densely defined* if its domain $\mathcal{D}(T)$ is dense in H.

Since a Hilbert space is isometrically isomorphic to its dual space, the concept of a *Hilbert space adjoint* plays an important role.

Definition 2.4.2. The *adjoint operator* of T, denoted by T^*, is the operator in H whose domain, $\mathcal{D}(T^*)$, consists of those $x \in H$ for which there is a *unique* element $x^* \in H$ that satisfies $(Tf, x) = (f, x^*)\ \forall\ f \in \mathcal{D}(T)$.

In this case, we define T^* by $T^*x = x^*$. T and its adjoint T^* are related by $(Tf, g) = (f, T^*g)\ \forall\ f \in \mathcal{D}(T)$ and $\forall\ g \in \mathcal{D}(T^*)$.

Proposition 2.4.1. See ([48], page 308). The adjoint T^* of a linear operator $T : H \to H$ exists and is unique if and only if T is densely defined.

Since $\mathcal{D}(\mathcal{L})$ is clearly dense in $L^2(I)$, the adjoint operator $\mathcal{L}^*$ for the maximal operator $\mathcal{L}$ exists. If T is any densely defined linear operator in $L^2(I)$ satisfying $T \subset \mathcal{L}$, then $\mathcal{L}^* \subset T^*$; consequently, it is natural to call $\mathcal{L}_0 := \mathcal{L}^*$ the *minimal operator* generated by $\ell[\cdot]$. It is possible, in fact, to give a better description of $\mathcal{D}(\mathcal{L}_0)$ (see Theorems 2.4.6. and 2.4.7. below).

Definition 2.4.3. The restriction of the maximal operator $\mathcal{L}$ to the (densely defined) subspace $\mathcal{D}_0'$ of all functions $f \in \mathcal{D}(\mathcal{L})$ which have compact support in I will be denoted by $\mathcal{L}_0'$.

Definition 2.4.4. Let H be a Hilbert space with inner product $(\cdot, \cdot)$. A linear operator $T : H \to H$ is symmetric in H if

(i) $\mathcal{D}(T)$ is dense in H (T is densely defined),

(ii) $(Tx, y) = (x, Ty)$ for all $x, y \in \mathcal{D}(T)$.

An operator T which satisfies property (ii) above, but not necessarily property (i), is called *Hermitian*. A characterization of a symmetric operator in a (complex) Hilbert space is given in:

Proposition 2.4.2. See ([48], page 534). Let T be a densely defined operator in a complex Hilbert space H. Then T is symmetric if and only if $(Tx, x) \in \mathbb{R}\ \forall\ x \in \mathcal{D}(T)$.

Proposition 2.4.3. The adjoint T^* of a linear operator $T : H \to H$ exists and is unique if and only if T is densely defined.

In particular, it follows from Proposition 2.4.3. that if T is symmetric then T^* exists. Evidently, a densely defined operator T is symmetric in H if and only if $T \subset T^*$ (see [48], page 533).

Theorem 2.4.1. The operator $\mathcal{L}_0'$ is symmetric in $L^2(I)$.

Proof. See ([55], page 61). □

Definition 2.4.5. Let H be a Hilbert space and $T : H \to H$ be a linear operator with domain $\mathcal{D}(T)$. Then T is *closed* if whenever $\{x_n\}_{n=0}^{\infty} \subset \mathcal{D}(T)$ satisfies $x_n \to x$ and $Tx_n \to y$, then $x \in \mathcal{D}(T)$ and $Tx = y$.

For example, it is easy to see that the adjoint T^* of a densely defined operator T is a closed operator. In particular, the minimal operator $\mathcal{L}_0$ is closed.

Proposition 2.4.4. See ([48], page 300). the Hilbert adjoint T^* of a linear operator $T : H \to H$ is always closed.

Questions concerning extensions of a given operator will invariably arise. We shall adhere to the following convention:

Definition 2.4.6. Given T, $S : H \to H$ the statement $S \subseteq T$ (T is an extension of S or S is a restriction of T) is taken to mean

(i) $T \subseteq \overline{T} \subseteq T^*$,

(ii) $T^* = \left(\overline{T}\right)^*$, and

(iii) $\overline{T} = (T^*)^*$.

Theorem 2.4.2. Suppose that T is a densely defined operator on a Hilbert space H.

(i) T is symmetric if and only if $T \subset T^*$.

(ii) If S is an extension of T, then

T^* is an extension of S^*, i.e., $T \subset S$ implies $S^* \subset T^*$.

(iii) If T is symmetric, then

every symmetric extension S of T satisfies $T \subset S \subset S^* \subset T^*$.

Definition 2.4.7. Let H be a Hilbert space and $T : H \to H$ be a linear operator. Then T is said to be *closable* if there exists a closed, linear extension S of T.

The smallest such closed extension is called the *closure* of T and T is said to *admit a closure*. The standard notation for the closure of an operator T is $\overline{T}$.

Theorem 2.4.3. A symmetric operator T is closable. Moreover, its closure $\overline{T}$ is unique, symmetric, and satisfies:

(i) $T \subseteq \overline{T} \subseteq T^*$,

(ii) $T^* = \left(\overline{T}\right)^*$, and

(iii) $\overline{T} = (T^*)^*$.

Proposition 2.4.5. The closure $\overline{T}$, of a symmetric operator T, is the minimal closed extension of T. That is to say, if $T \subseteq S$ and S is closed then it must be the case that $\overline{T} \subseteq S$.

The closure of an operator T is sometimes referred to as the *graph* closure of T since the process of closing a given operator T is equivalent to constructing the topological closure of the *graph* of T, $G(T)$, in the Hilbert space $H \oplus H$.

Theorem 2.4.4. Let H be a Hilbert space. A symmetric operator $T : H \to H$ admits a closure. Furthermore, this closure $\overline{T}$ is also symmetric in H.

Proof. See ([55], page 13). □

As a consequence of this theorem, $\mathcal{L}_0'$ has a symmetric closure $\overline{\mathcal{L}_0'}$. In fact,

Theorem 2.4.5. $\overline{\mathcal{L}_0'} = \mathcal{L}_0$, where $\mathcal{L}_0$ is the adjoint of the maxial operator $\mathcal{L}$ and the so-called minimal operator.

Proof. See ([55], page 68). □

It is well-known (e.g., [48], page 541) that a closed, densely defined operator A in H has the property that $A^{**} = A$. Hence, this fact, combined with Theorem 2.4.5., yields:

Theorem 2.4.6. $\mathcal{L}_0 = \overline{\mathcal{L}_0'}$ and $\mathcal{L}_0^* = \mathcal{L}$. In particular, the minimal operator $\mathcal{L}_0$ and the maximal operator $\mathcal{L}$ are closed operators, each being the adjoint of the other.

One of the most important type of linear operators in a Hilbert space is the one that is *self-adjoint*.

Definition 2.4.8. A densely defined operator $T : H \to H$ is said to be *self-adjoint* when $T = T^*$.

Property (i) of Theorem 2.4.2. shows that every self-adjoint operator is necessarily symmetric; however, not every symmetric operator is self-adjoint (in the case of a bounded operator $T : H \to H$ the concepts of symmetry and self-adjointness are identical). For example, define $T : L^2 [0, 1] \to L^2 [0, 1]$ by

$$Tf := if'$$

$$\mathcal{D}(T) := \left\{ f : [0,1] \to \mathbb{C} \,\middle|\, f \in AC\,[0,1]\,;\; f' \in L^2\,[0,1]\,;\; f(0) = f(1) = 0 \right\}. \tag{2.4.1}$$

Then for any $f,\, g \in \mathcal{D}(T)$,

$$\begin{aligned}
(Tf, g) &= \int_0^1 if'(t)\,\overline{g}(t)\,dt \\
&= if(t)\,\overline{g}(t)\Big|_0^1 - \int_0^1 i\overline{g}'(t)\,f(t)\,dt \\
&= if(1)\,\overline{g}(1) - if(0)\,\overline{g}(0) + \int_0^1 i\overline{g}'(t)\,f(t)\,dt \\
&= \int_0^1 i\overline{g}'(t)\,f(t)\,dt = (f, Tg).
\end{aligned}$$

It follows that T is symmetric because $\mathcal{D}(T)$ is dense in $L^2 [0, 1)$. However, it can be shown that

$$\mathcal{D}(T^*) := \left\{ f : [0,1) \to \mathbb{C} \,\middle|\, f \in AC\,[0,1]\,;\; f' \in L^2[0,1) \right\}$$

so that T is not self-adjoint.

From property (iii) of Theorem 2.4.2., we see that the most general symmetric extension in H (in particular the most general self-adjoint extension) of a symmetric operator T is suitably chosen restriction of the adjoint T^* of T.

For example, the operator $S : L^2[0,1) \to L^2\ [0,1)$ defined by

$$Sf := if'$$

$$\mathcal{D}(S) := \{f : [0,1) \to \mathbb{C} \mid f \in AC\,[0,1]\,;\ f' \in L^2[0,1);\ f(0) = f(1)\}$$

is a self-adjoint extension of the operator T defined in (2.4.1).

Proposition 2.4.6. See ([48], page 524). A self-adjoint operator T is both closed and symmetric.

Proposition 2.4.7. See ([48], page 535). A self-adjoint operator T is *maximally symmetric*. That is to say, if S is a symmetric operator and $T \subseteq S$, then it must be the case $T = S$.

The spectrum of a self-adjoint operator T is described by:

Proposition 2.4.8. The spectrum $\sigma(T)$ of a self-adjoint operator $T : H \to H$ consists of approximate eigenvalues; that is to say, $\forall\ \lambda \in \sigma(T)\ \exists\ \{x_n\}_{n=0}^{\infty} \subset \mathcal{D}(T)$ such that

$$\|x_n\| = 1 \text{ and } \|Tx_n - \lambda x_n\| \to 0 \text{ as } n \to \infty.$$

Morever, the spectrum of T is real: $\sigma(T) \subseteq \mathbb{R}$.

The following theorem is a very useful criterion for determining whether or not an element $f \in \mathcal{D}(\mathcal{L})$ is in the minimal domain $\mathcal{D}(\mathcal{L}_0)$. It involves the sesquilinear form defined in equation (2.3.2).

Theorem 2.4.7. The domain $\mathcal{D}(\mathcal{L}_0)$ of the minimal operator $\mathcal{L}_0$ in $L^2(I)$ consists of all $f \in \mathcal{D}(\mathcal{L})$ satisfying $[f,g](x)\Big|_a^b = 0$, for all $g \in \mathcal{D}(\mathcal{L})$.

Proof. See ([55], page 70). $\square$

We remark that if one or both of the endpoints of I are regular, then the condition stated in Theorem 2.4.7. simplifies further. For example, if the left endpoint a is regular and the right endpoint b is singular, then the condition stated in

Theorem 2.4.7. may be restated as: $f \in \mathcal{D}(\mathcal{L})$ is in the minimal domain $\mathcal{D}(\mathcal{L}_0)$ whenever

(i) $f^{(k)}(a) = 0, 1, 2, \ldots, 2n-1$, and

(ii) $[f, g](b) = 0$, for all $g \in \mathcal{D}(\mathcal{L})$.

The interested reader can find the proof of this in ([55], page 71).

Remark 1.

Observe that if A is a symmetric extension of the minimal operator $\mathcal{L}_0$ in $L^2(I)$, then $A \subset \mathcal{L}$, where $\mathcal{L}$ is the maximal operator. Indeed, this is an immediate consequence of Theorem 2.4.6. : $\mathcal{L}_0 \subset A \subset A^* \subset \mathcal{L}_0^* = \mathcal{L}$.

In particular, $A[y] = \ell[y]$ for all $y \in \mathcal{D}(A)$; i.e., A has the same form as the expression $\ell[\cdot]$ and A is the restriction of the maximal operator $\mathcal{L}$.

Remark 2.

We note that all the theory that we present in this section can be applied *mutatis mutandis* to expressions of the form

$$\eta[y](x) = \frac{1}{f(x)} \sum_{j=0}^{n} (-1)^j \left(a_j(x) y^{(j)}(x)\right)^{(j)}, \ x \in I$$

where $f(x) \in C^{2n}(I)$ and $f(x) > 0$ for all $x \in I$. Observe that $f(x)\eta[y](x)$ is formally symmetric; in this case, we call such a function $f(x)$ a *symmetry factor* for $\eta[\cdot]$ (see [52]). The appropriate Hilbert space setting for the self-adjoint extension theory would be $L^2(a, b; f)$.

We note that the maximal operator $\mathcal{L}$ in $L^2(a, b; f)$, generated by $\eta[\cdot]$, is defined to be

$$\mathcal{L}[y] = \eta[y]$$

$$\mathcal{D}(\mathcal{L}) = \left\{ y : (a, b) \to \mathbb{C} \,\middle|\, y^{(j)} \in AC_{loc}(a, b),\ j = 0, 1, \ldots, 2n-1; \right.$$
$$\left. y,\ \eta[y] \in L^2(a, b; f) \right\}.$$

1. the *Jacobi expression* is defined by

$$\tau_J[y] := -\left(1-x^2\right)y'' + (\alpha - \beta + (\alpha+\beta+2)x)\,y' + ky,\ x \in (-1,1),\ \alpha,\ \beta > -1.$$

the n^{th} Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ satisfies

$$\tau_J\left[P_n^{(\alpha,\beta)}\right] = (k + n(n+\alpha+\beta+1))\,P_n^{(\alpha,\beta)}.$$

Although this expression cannot be directly put into the form (2.2.1), multiplication of τ_J by the symmetry factor $f(x) = (1-x)^{\alpha}(1+x)^{\beta}$ yields:

$$\begin{aligned}\ell_J[y] &:= (1-x)^{\alpha}(1+x)^{\beta}\,\tau_J[y](x)\\ &= -\left((1-x)^{\alpha}(1+x)^{\beta}y'(x)\right)' + k(1-x)^{\alpha}(1+x)^{\beta}y(x).\end{aligned} \tag{2.5.1}$$

Observe that when $\alpha = \beta = 0$, then $\tau_J[\cdot] \equiv M_k^{(1)}[\cdot]$, where $M_k^{(1)}[\cdot]$ is defined in (2.1.1). For the Jacobi expression, the proper right-definite setting is the weighted Lebesgue space $L^2\left(-1,1;(1-x)^{\alpha}(1+x)^{\beta}\right)$ (not $L^2(-1,1)$ unless $\alpha=\beta=0$) and the maximal and minimal operators in this space are generated from

$$\tau_J[\cdot] = (1-x)^{-\alpha}(1+x)^{-\beta}\,\ell_J[\cdot].$$

2. the *Laguerre expression* is defined by

$$\tau_L[y] := -xy'' + (x-1-\alpha)\,y' + ky,\ x \in (0,\infty),\ \alpha > -1.$$

the n^{th} Laguerre polynomial $L_n^{(\alpha)}(x)$ satisfies $\tau_L\left[L_n^{(\alpha)}\right] = (n+k)\,L_n^{(\alpha)}$. In this case, a symmetry factor for τ_L is $f(x) = x^{\alpha}e^{-x}$. Indeed, we have

$$\begin{aligned}\ell_L[y] &:= x^{\alpha}e^{-x}\tau_L[y](x)\\ &= -\left(x^{\alpha+1}e^{-x}y'(x)\right)' + kx^{\alpha}e^{-x}y(x).\end{aligned} \tag{2.5.3}$$

For this expression, the maximal and minimal operators associated with the Laguerre expression are generated by $\tau_L[\cdot] = x^{\alpha}e^{-x}\ell_L[\cdot]$ and studied in the Lebesgue space $L^2(0,\infty;x^{\alpha}e^{-x})$.

3. the *Hermite expression* is defined by

$\tau_H[y] := -y'' + 2xy' + ky,\ x \in (-\infty, \infty)$.

the n^{th} Hermite polynomial $H_n(x)$ satisfies $\tau_H[H_n] = (k+2n)H_n$. In this case, a symmetry factor for τ_H is $f(x) = e^{-x^2}$; when multiplied by this factor, we get

$$\begin{aligned} \ell_H[y] &:= e^{-x^2}\tau_H[y](x) \\ &= -\left(e^{-x^2}y'(x)\right)' + ke^{-x^2}y(x). \end{aligned} \tag{2.5.4}$$

The proper Hilbert space setting is $L^2\left(-\infty, \infty; e^{-x^2}\right)$ and the correct expression there to study in this context is $\tau_H[\cdot] = e^{x^2}\ell_H[\cdot]$.

In 1929, von Neumann considered and solved the problem of when a symmetric operator in a Hilbert space H had self-adjoint extensions in H. The motivation for this study came from his interest in several unbounded operators that appear quite naturally in the theory of quantum mechanics. In 1939, Calkin presented his method for determining necessary and sufficient conditions when such self-adjoint extensions exist and proceeded to characterize the domains of each of these extensions interms of general "boundary conditions". A well-written account of this elegant theory can be found in [13] (see pages 1222-1239 and 1268-1274). For our study, this theory has particularly important applications to the subject of symmetric differential operators. Indeed, the Russian mathematicians M. A. Naimark and I. M. Glazman are credited for applying and refining both von Neumann's theory and Calkin's method to the minimal operator $\mathcal{L}_0$ generated by $\ell[\cdot]$. We now briefly describe von Neumann's results and follow this by the Glazman-Naimark theory of self-adjoint extensions of $\mathcal{L}_0$.

2.6 von Neumann's Formula

Definition 2.6.1. Let A be a symmetric operator in a Hilbert space H. Let

$$\mathcal{D}_+ := \{f \in \mathcal{D}(A^*) \mid A^*f = if\} \text{ and}$$
$$\mathcal{D}_- := \{f \in \mathcal{D}(A^*) \mid A^*f = -if\},$$

where $i = \sqrt{-1}$. The space $\mathcal{D}_+$ is called the *positive deficiency space* of A and $\mathcal{D}_-$ is called the *negative deficiency space* of A. The dimensions of these spaces are called, respectively, the *positive* and *negative deficiency indices* of A. We shall write $n_\pm := \dim(\mathcal{D}_\pm)$. The *deficiency index* of A in $L^2(I)$ is the ordered pair (n_+, n_-).

As shown in [13] (see page 1232), there is nothing special about using the complex number i in this definition: If $\lambda \in \mathbb{C}$ and $\operatorname{Im}(\lambda) > 0$, then it is the case that

$$\dim\{f \in \mathcal{D}(A^*) \mid A^* f = \lambda f\} = n_+.$$

A similar result holds for n_- and any $\lambda \in \mathbb{C}$ with $\operatorname{Im}(\lambda) < 0$. This result was actually proved by Weyl in 1910 (see [32] and ([72], Chapter 13)) in the case of the classical second-order Sturm-Liouville differential expression.

If A is a symmetric operator in a Hilbert space H, we define a new inner product on $\mathcal{D}(A^*)$ by $(x, y)^* := (x, y) + (A^* x, A^* y)$. It can be shown (see [13], page 1225) that $\mathcal{D}(A^*)$ is a Hilbert space when equipped with this inner product. At this point, we can now state the following important theorem. Equation (2.6.1) below is known as *von Neumann's formula.*

Theorem 2.6.1. Let A be a symmetric operator in a Hilbert space H. Then $\mathcal{D}\left(\overline{A}\right)$, $\mathcal{D}_+$, $\mathcal{D}_-$ are closed orthogonal subspaces in

$$(\mathcal{D}(A^*), (x, y)^*), \text{ and } \mathcal{D}(A^*) = \mathcal{D}\left(\overline{A}\right) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-.$$

Proof. See ([13], page 1227). □

In this case of $A = \mathcal{L}_0$, the minimal operator in $L^2(I)$ generated by $\ell[\cdot]$, von Neumann's formula becomes

$$\mathcal{D}(\mathcal{L}) = \mathcal{D}\left(\overline{\mathcal{L}_0}\right) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-. \tag{2.6.1}$$

Consequently, in view of Remark 1, it is not too surprising that the positive and negative deficiency spaces play a major role in the determination of self-adjoint extensions of $\mathcal{L}_0$ in $L^2(I)$. In fact, we state the following theorem (see [13], page 1228) to illustrate this influence.

Theorem 2.6.2. Let A be a symmetric operator in a Hilbert space H. Let $\mathcal{D}'$ be a closed subspace of $\mathcal{D}_+ \oplus \mathcal{D}_-$ and set $\mathcal{D} = \mathcal{D}\left(\overline{A}\right) \oplus \mathcal{D}'$. The restriction of A^* to $\mathcal{D}$ is self-adjoint if and only if $\mathcal{D}'$ is the graph of an isometry mapping $\mathcal{D}_+$ onto $\mathcal{D}_-$.

From Theorem 2.6.2., the key result follows:

Theorem 2.6.3. Let A be a symmetric operator in a Hilbert space H. Then A has self-adjoint extensions in H if and only if its deficiency indices n_+ and n_- are equal. Furthermore, if $n_+ = n_- = 0$, then the only self-adjoint extension of A is its closure $\overline{A} = A^*$.

Proof. See ([13], page 1230). $\square$

Although much more can be said about the characterizations of self-adjoint extensions of general symmetric operators in a Hilbert space, we instead return to our discussion of finding self-adjoint extensions of the minimal operator $\mathcal{L}_0$ in $L^2(I)$. Since for any complex number λ, the equation $\ell[y] = \lambda y$ has a basis of $2n$ solutions, the deficiency indices of $\mathcal{L}_0$ in $L^2(I)$ are both finite. In fact, these two indices are equal. Indeed, because the coefficients a_k of $\ell[\cdot]$ are real-valued, the function f is a solution of $\ell[y] = iy$ if and only if $\overline{f}$ is a solution of $\ell[y] = -\lambda y$. This same argument shows, in fact, that if $\{f_1, f_2, \ldots, f_m\}$ is a basis for the positive deficiency space $\mathcal{D}_+$, then $\left\{\overline{f_1}, \overline{f_2}, \ldots, \overline{f_m}\right\}$ is a basis for the negative deficiency space $\mathcal{D}_-$. However we note that, in general, the deficiency indices $n_\pm$ need not to be equal when the coefficients of $\ell[\cdot]$ are complex-valued. We state the following theorem:

Theorem 2.6.4. Let $\mathcal{L}_0$ denote the minimal operator in $L^2(I)$ generated by $\ell[\cdot]$, where $I = (a, b)$. Then,

(i) If both endpoints a and b are regular endpoints, then $n_\pm = 2n$.

(ii) If one of the endpoints is singular, then $0 \leq n_+ = n_- \leq 2n$.

In fact, it is possible to construct $\ell[\cdot]$ so that $n_{\pm} = m$ for any integer m, $0 \leq m \leq 2n$. If exactly one of the endpoints is singular, then $n \leq n_{+} = n_{-} \leq 2n$.

Proof. For the proof of (i), (see [55], page 66). For the proof of (ii), (see [55], pages 69 and 71). Furthermore, in [28], Glazman constructs examples to show that $m = n_{\pm}$ can actually take on all possible integer values between 0 and $2n$. □

Let $c \in I$; necessarily, c is a regular point of $\ell[\cdot]$. Let $\mathcal{L}_0^-$ denote the minimal operator generated by $\ell[\cdot]$ on (a, c) and let $\mathcal{L}_0^+$ denote the minimal operator generated by $\ell[\cdot]$ on (c, b). Let (m_-, m_-) and (m_+, m_+) denote, respectively, the deficiency indices of $\mathcal{L}_0^-$ in $L^2(a, c)$ and $\mathcal{L}_0^+$ in $L^2(c, b)$. We state the following theorem:

Theorem 2.6.5. The deficiency index of the minimal operator $\mathcal{L}_0$ in $L^2(I)$ is (m, m) where:

$$m = m_+ + m_- - 2n, \tag{2.6.2}$$

and $2n$ is the order of the expression $\ell[\cdot]$. Furthermore, m is independent of the choice of $c \in I$.

Proof. See ([28], page 353). □

The importance of this theorem may need some explanation. Since the point c is a regular point, all solutions of $\ell[y] = \pm iy$ will belong to $L^2(c - \varepsilon, c]$ for all $0 < \varepsilon < c - a$. Consequently, the number m_- is precisely equal to the number of solutions of $\ell[y] = \pm iy$ that are in $L^2(a, a + \delta]$ for some sufficiently small $\delta > 0$. Similarly, the number m_+ is equal to the number of solutions of $\ell[y] = \pm iy$ that are in $L^2[b - \delta, b)$ for small enough $\delta > 0$. This motivates the following definition:

Definition 2.6.2. The differential expression $\ell[\cdot]$ is said to be in the *limit-p case at x=a* in $L^2(I)$ if there exist exactly p solutions of $\ell[y] = \pm iy$ belong to $L^2(a, a + \epsilon)$ for some sufficiently small $\epsilon > 0$. Similarly, $\ell[\cdot]$ is said to be in the *limit-q case at x=b* in $L^2(I)$ if there exist exactly q solutions of $\ell[y] = \pm iy$ belong to $L^2(b - \epsilon, b)$ for some sufficiently small $\epsilon > 0$.

Since $\mathcal{L}[\cdot]$ is of order $2n$, it is clear that $0 \leq p$ and $q \leq 2n$. If the order of $\ell[\cdot]$ is two, the limit-2 case is more commonly called the *limit-circle case* while the limit-1 case is often referred to as the *limit-point case*. This notation goes back to Weyl's seminal paper [72]. His analysis of the number of Lebesgue square-integrable solutions of the second-order Sturm-Liouville equation involved some key geometric arguments. The terms "limit-point" and "limit-circle" reflect the geometry used in his solution.

Remark 3.

In the second-order case, Weyl showed that if $\ell[y] = \lambda_0 y$ is limit-point (respectively, limit-circle) at a or b for a certain complex number λ_0, then $\ell[y] = \lambda y$ is limit-point (limit-circle) at a or b for all complex numbers $\lambda \in \mathbb{C}$.

From Definition 2.6.2. and Theorem 2.6.5., it is clear that once we can determine the limit case for each of the endpoints, then the deficiency index of the minimal operator $\mathcal{L}_0$ in $L^2(I)$ can be determined. Fortunately, there is a method available for determining the limit case of an endpoint when that endpoint is a *regular singular point* in the sense of Frobenius. Indeed, the so-called *Method of Frobenius* from ordinary differential equations (see [33], pages 396-404) can sometimes be used to determine the number of Lebesgue square-integrable solutions near this singular endpoint.

Definition 2.6.3. Consider the differential equation

$$L[y](x) = \sum_{j=0}^{n} b_j(x)\, y^{(j)}(x) = 0,\ x \in J \tag{2.6.3}$$

where $J \subset \mathbb{R}$ is some open interval, $b_j : J \to \mathbb{R}$, $j = 0, 1, \ldots, n$, $b_n(x) \neq 0$ for all $x \in J$. Suppose a, $b \in J$ with $a < b$. If $x = a > -\infty$, then $x = a$ is called a *regular singular point* of $L[\cdot]$ if

$$\frac{(x-a)^n L[y](x)}{b_n(x)} = \sum_{j=0}^{n} (x-a)^j c_j(x)\, y^{(j)}(x),$$

where $c_n(x) \equiv 1$ and where each $c_j(x)$ is analytic in some neighbourhood of $x = a$, $j = 0, 1, \ldots, n-1$. The definition of $x = b < \infty$ as a regular singular point is similar. If $a = -\infty$ or $(b = \infty)$ and $L[\cdot]$ can be put into the form

$$\sum_{j=0}^{n} t^j c_j(t)\, y^{(j)}(t),$$

under the transformation $x = \frac{1}{t}$, where again $c_n(t) \equiv 1$ and each $c_j(t)$ is analytic in some neighbourhood of $t = 0$, then we say $x = \infty$ is a regular singular point of $L[\cdot]$. If an endpoint is not a regular singular endpoint, it is called an *irregular singular point*.

Based on earlier work of Fuchs, Frobenius developed an ingenious tool for determining a basis of n solutions of the homogeneous equation (2.6.3), where each solution is expanded about a regular singular point. A key ingredient in this method is the *indicial equation* at $x = a$ associated with (2.6.3):

$$\sum_{j=0}^{n} P(r, j)\, c_j = 0, \tag{2.6.4}$$

where $c_j = c_j(a)$ and $P(r, j) = \frac{r!}{(r-j)!}$, $j = 0, 1, \ldots, n$. Evidently, this is a polynomial of degree exactly n. For lack of space, we do not describe this method; it suffices to say that each of the n roots of the indicial equation (2.6.4) determines a solution of (2.6.3), even in the case of roots having multiplicity greater than one. The examples below, we hope, will help the reader in understanding this important method.

2.7 Examples

1. Consider the Legendre differential expression $M_k^{(1)}[\cdot]$ defined in (2.1.1). The endpoints $x = \pm 1$ are both regular singular points of $M_k^{(1)}[\cdot]$. All $x \in (-1, 1)$ are regular singular points in the sense of Definition 2.2.1. The reader can check that the indicial equation at both $x = \pm 1$ is given by $r^2 = 0$. Around $x = 1$, for example, Frobenius' method yields a basis $\{y_1, y_2\}$ of solutions of $M_k^{(1)}[y] = 0$,

where y_1 and y_2 have the form:

$$y_1(x) = \sum_{j=0}^{\infty} a_j (x-1)^j, \ a_0 \neq 0 \text{ and}$$

$$y_2(x) = \ln|x-1| \sum_{j=0}^{\infty} a_j (x-1)^j + \sum_{j=0}^{\infty} b_j (x-1)^j, \ b_0 \neq 0$$

where both series converge for $|x+1| < 2$. Similarly, by replacing -1 by $+1$ in the above series, we obtain a basis of solutions about $x = -1$, valid for $|x-1| < 2$.

It is clear from these representations that all solutions of $M_k^{(1)}[y] = 0$ are Lebesgue square-integrable near the endpoints $x = \pm 1$. Hence, referring to Remark 3, we conclude that the Legendre differential expression is limit-circle at both $x = \pm 1$. Consequently, from Theorem 2.6.5., the deficiency index of the minimal operator generated by the Legendre expression $M_k^{(1)}[\cdot]$ is $(2,2)$.

2. The Krall-Legendre differential expression $M_k^{(2)}[\cdot]$ is defined by (2.1.4). We note that the n^{th} Krall-Legendre polynomial $P_{n,A}(x) = P_{2,n}(x)$ (Legendre type, $P_{2,n}(x)$ is in the original notation of A. M. Krall, see [36]) is a solution of the equation

$$M_k^{(2)}[y] = \left(n(n+1)\left(n^2+n+4A-2\right)+k\right) y, \ n = 0, 1, \ldots.$$

As with the Legendre expression, the endpoints $x = \pm 1$ are both regular singular endpoints and all $x \in (-1,1)$ are regular points of this expression. The indicial equation at both $x = \pm 1$ associated with

$$M_k^{(2)}[y] = \pm iy \tag{2.7.1}$$

is given by $r(r-1)(r-2)(r+1) = 0$. If we apply the Frobenius' method to (2.7.1) with particular attention paid to ([33], Section 16.33), we see that a basis of solutions about $x = 1$ is given by $\{y_1, y_2, y_3, y_4\}$, where

$$y_1(x) = \sum_{j=0}^{\infty} a_j (x-1)^{j+2}, \ a_0 \neq 0,$$

$$y_2(x) = \sum_{j=0}^{\infty} b_j (x-1)^{j+1}, \ b_0 \neq 0,$$

$$y_3(x) = \sum_{j=0}^{\infty} c_j (x-1)^{j}, \quad c_0 \neq 0,$$

$$y_4(x) = \sum_{j=0}^{\infty} d_j (x-1)^{j-1}, \ d_0 \neq 0;$$

with each of these series converging for $|x-1| < 2$. It is clear that y_1, y_2, y_3 are all Lebesgue square-integrable near $+1$ but that y_4 is not. Hence, $M_k^{(2)}[\cdot]$ is in the limit-3 case at $x = 1$. It can also be shown that $M_k^{(2)}[\cdot]$ is in the limit-3 case at $x = -1$. Hence, from Theorem 2.6.5., the deficiency index of the minimal operator $\mathcal{L}_0$ generated from $M_k^{(2)}[\cdot]$ in $L^2(-1,1)$ is $(2,2)$.

Remark 4.

Fortunately, the endpoints $x = \pm 1$ are regular singular endpoints of the Legendre expressions (2.1.1) and (2.1.5) so that the Frobenius analysis can be carried out. We note that if an endpoint is not a regular singular endpoint of $\ell[\cdot]$, quite a different analysis must be applied to determine the limit classification of $\ell[\cdot]$ at this point. In the second-order case, the usual procedure that is employed is to first transform the expression into its *Liouville normal form* (see [32], page 42) and then to try to apply some known limit-point criteria (*e.g.,* the Levinson criteria, see [11], pages 229-230) to the transformed equation. For example, the Laguerre expression (2.5.3) has $x = \infty$ as an irregular singular point but the Levinson condition readily shows that this expression is in the limit-point case at $x = \infty$.

We are now almost in the position to state the important Glazman-Naimark theorem. We first need the following definition:

Definition 2.7.1. Let X be a vector space and $M_1 \subset M_2$ be subspaces of X. We say that $\{x_1, x_2, \ldots, x_n\} \subset M_2$ is *linearly independent modulo* M_1 if the condition

$$\sum_{j=1}^{n} \alpha_j x_j \in M_1$$

implies $\alpha_j = 0,\ j = 1, 2, \ldots, n$. If $A \subset M_2$ is a maximal linearly independent set modulo M_1 and $\beta = card\,(A)$, we say that the dimension of M_2 is β modulo M_1.

It is not difficult to see that if $\{x_1, x_2, \ldots, x_n\} \subset M_2$ is a linearly independent set, then it is a maximal linearly independent set modulo M_1 if and only if

$$M_2 = M_1 \dotplus sp\,\{x_1, x_2, \ldots, x_n\}\,. \tag{2.7.2}$$

Of course, any set of linearly independent vectors modulo M_1 is a linearly independent set in X; but the converse of this is not necessarily true.

This concept of linear independence modulo subspace plays an important role in characterizing all self-adjoint extensions of $\mathcal{L}_0$ in $L^2\,(I)$. In view of (2.7.2) and the importance that von Neumann's formula (2.6.1) plays, this statement is not too surprising.

By suitably modifying Theorem 2.6.2. with the minimal operator $\mathcal{L}_0$, we can obtain the Glazman-Naimark theorem. The proof of this theorem can be found in ([55], pages 75-76).

2.8 The Glazman-Naimark Theorem

Theorem 2.8.1. Suppose the deficiency index of the minimal operator $\mathcal{L}_0$ in $L^2\,(a, b)$ generated by the expression $\ell\,[\cdot]$ is (m, m).

(i) Let S be a self-adjoint extension of $\mathcal{L}_0$ in $L^2\,(a, b)$. Then there exists a set

$\{w_1, w_2, \ldots, w_m\} \subset \mathcal{D}(S)$ that is linearly independent modulo $\mathcal{D}(\mathcal{L}_0)$ such that

$$S[y] = \ell[y],$$
$$\mathcal{D}(S) = \left\{ y \in \mathcal{D}(\mathcal{L}) \;\middle|\; [w_j, y] \Big|_a^b = 0,\ j = 1, 2, \ldots, m \right\}. \tag{2.8.1}$$

Here $[\cdot, \cdot]$ is the sesquilinear form defined in (2.3.2).

(ii) Suppose $\{w_1, w_2, \ldots, w_m\} \subset \mathcal{D}(\mathcal{L})$ is linearly independent modulo $\mathcal{D}(\mathcal{L}_0)$ with

$$[w_j, w_k] \Big|_a^b = 0,\ j,\ k = 1, 2, \ldots, m.$$

Define an operator S in $L^2(a, b)$ by

$$S[y] = \ell[y],$$
$$\mathcal{D}(S) := \left\{ y \in \mathcal{D}(\mathcal{L}) \mid\ [w_j,\ y] \Big|_a^b = 0,\ j = 1, 2, \ldots, m \right\}.$$

Then S is a self-adjoint extension of $\mathcal{L}_0$.

Before proceeding to the next section, we note that the conditions $[w_j, y] \Big|_a^b = 0$ given in (2.8.1) are known as the Glazman-Naimark *boundary conditions* and the functional

$[w_j, \cdot] \Big|_a^b : \mathcal{D}(\mathcal{L}) \to \mathbb{C}$ is called a *boundary value* for $\mathcal{L}_0$. If for some j, $[w_j, y] \Big|_a^b = 0$ is independent of a or b for all $y \in \mathcal{D}(S)$, it is called a *separated boundary condition*; otherwise it is a *mixed boundary condition*.

In ([13], page 1234), a boundary value for a symmetric operator A is defined to be a continuous linear functional on $(\mathcal{D}(A^*), (\cdot, \cdot)^*)$ that vanishes on $\mathcal{D}(A)$. As can be seen by Theorem 2.4.7., our notion of a boundary value is in complete agreement with that in [13].

Finally we note, for the reader's sake, that there is a generalization of Theorem 2.8.1. for arbitrary symmetric operators; such a theorem can be found in ([13], page 1239).

2.9 Applications to the Legendre Differential Expressions

In this section, among other results, we show how the Glazman-Naimark theorem may be applied directly to the Legendre expression (2.1.1). We shall produce *all* of the self-adjoint extensions in $L^2(-1,1)$ of the associated minimal operator, including that extension having the Legendre polynomials as a complete set of eigenfunctions.

The reader will notice that in all these classical second-order cases, the symmetry factor f for the expression is identical with the orthogonalizing weight function for the associated orthogonal polynomials. Consequently, in each of these examples, the Glazman-Naimark theory of self-adjoint extensions of the minimal operator $\mathcal{L}_0$ will yield, as a special case, that self-adjoint extension having the corresponding orthogonal polynomials as eigenfunctions.

However, the situation is quite different for the higher-order equations having orthogonal polynomial eigenfunctions (including (2.1.2)-(2.1.5)). In these cases, the symmetry factor $f(x)$ differs from the associated orthogonalizing weight $w(x)$, $x \in I$. As a result, while the Glazman-Naimark theory picks up all of the self-adjoint extensions of the minimal operator in $L^2(I; f)$ generated from each of these higher-order equations, none of these extensions will have the associated set of orthogonal polynomials as eigenfunctions. Indeed, because eigenfunctions of a self-adjoint operator corresponding to distinct eigenvalues are necessarily orthogonal in $L^2(I; w)$, the self-adjoint operator with the set of orthogonal polynomials as eigenfunctions must be in $L^2(I; w)$.

It may appear, then, that the Glazman-Naimark theory cannot be used to find this particular self-adjoint operator in $L^2(I; w)$. However, due to a method of W. N. Everitt [21] based on earlier work of A. M. Krall [36], the Glazman-Naimark theory is very cleverly used to find this particular operator in $L^2(I; w)$.

Details of the self-adjoint extension theory in $L^2(-1,1)$ for expressions (2.1.2)-(2.1.5) can be found in the Loveland's thesis [54]. We shall not discuss this in here; instead, we concentrate on that self-adjoint extension in $L^2(I;w)$ having the orthogonal polynomials as eigenfunctions.

Examples

1. The first example that we discuss is the expression $M_k^{(1)}[\cdot]$, defined in (2.1.1); this expression has been studied extensively by Everitt in [15]; (see also [54], Chapter 3). From Section 2.5, we found that $M_k^{(1)}[\cdot]$ is limit-circle at both singular endpoints $x=\pm 1$ and that the deficiency index of the minimal operator $\mathcal{L}_0$ in $L^2(-1,1)$ generated by $M_k^{(1)}[\cdot]$ is $(2,2)$. According to the Glazman-Naimark theorem, each self-adjoint extension S of $\mathcal{L}_0$ in $L^2(-1,1)$ has the form

$$\begin{aligned} S[y] &= M_k^{(1)}[y], \\ \mathcal{D}(S) &= \left\{ y \in \mathcal{D}(\mathcal{L}) \mid \; [w_1,y]_{(1)} \Big|_{-1}^{1} = [w_2,y]_{(1)} \Big|_{-1}^{1} = 0 \right\}, \end{aligned} \tag{2.9.1}$$

where $\{w_1,w_2\} \subset \mathcal{D}(S)$ is linearly independent modulo $\mathcal{D}(\mathcal{L}_0)$, $[\cdot,\cdot]_{(1)}$ is the sesquilinear form defined in (2.3.2), and $\mathcal{D}(\mathcal{L})$ is the domain of the maximal operator $\mathcal{L}$ in $L^2(-1,1)$ generated by $M_k^{(1)}[\cdot]$. We now describe a procedure for determining w_1 and w_2.

It is easy to see that a basis of solutions for $M_k^{(1)}[y]=ky$ is given by $\{y_1,y_2\}$ where

$$\begin{aligned} y_1(x) &= 1, \text{ and} \\ y_2(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad -1<x<1. \end{aligned}$$

These solutions may be found by directly solving this equation. Alternatively, we could have applied Frobenius' method and obtained a basis around both endpoints $x=\pm1$; asymptotically, these solutions would behave much like y_1 and y_2 above. We use these two solutions to find a basis for the quotient space

$$\mathcal{D}(\mathcal{L})/\mathcal{D}(\mathcal{L}_0)$$

(this space being isomorphic to $\mathcal{D}_+ \oplus \mathcal{D}_-$), which according to von Neumann's formula (2.6.1) is of dimension 4. More specifically, define $\varphi_i \in C^2(-1,1)$, $i = 1,2,3,4$ such that

$$\varphi_1(x) = \begin{cases} 0 & \text{near } x = -1 \\ 1 & \text{near } x = 1, \end{cases}$$

$$\varphi_2(x) = \begin{cases} 1 & \text{near } x = -1 \\ 0 & \text{near } x = 1, \end{cases}$$

$$\varphi_3(x) = \begin{cases} 0 & \text{near } x = -1 \\ \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) & \text{near } x = 1, \end{cases} \tag{2.9.2}$$

$$\varphi_4(x) = \begin{cases} \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) & \text{near } x = -1 \\ 0 & \text{near } x = 1. \end{cases}$$

By this construction, it is clear that $\varphi_i \in \mathcal{D}(\mathcal{L})$, $i = 1,2,3,4$. The reader can readily check that

$$\begin{aligned} &[\varphi_1, \varphi_2]_{(1)}(\pm 1) = [\varphi_1, \varphi_4]_{(1)}(\pm 1) = 0, \\ &[\varphi_2, \varphi_3]_{(1)}(\pm 1) = [\varphi_3, \varphi_4]_{(1)}(\pm 1) = 0, \\ &[\varphi_1, \varphi_3]_{(1)}(1) = [\varphi_2, \varphi_4]_{(1)}(-1) = 1, \\ &[\varphi_1, \varphi_3]_{(1)}(-1) = [\varphi_2, \varphi_4]_{(1)}(1) = 0. \end{aligned} \tag{2.9.3}$$

It is also easy to see that the set $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ is linearly independent modulo $\mathcal{D}(\mathcal{L}_0)$, and hence, there exist linearly independent vectors $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\{\beta_1, \beta_2, \beta_3, \beta_4\} \in \mathbb{C}^4$ such that

$$w_1 = \sum_{j=1}^{4} \alpha_j \varphi_j \text{ and} \tag{2.9.4}$$

$$w_2 = \sum_{j=1}^{4} \beta_j \varphi_j, \tag{2.9.5}$$

where w_1 and w_2 are defined in (2.9.1). Of course, the requirement that $w_1,\ w_2 \in \mathcal{D}(S)$ forces restriction on the vectors $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\{\beta_1, \beta_2, \beta_3, \beta_4\} \in \mathbb{C}^4$. That is to say, w_1 and w_2 must satisfy the Glazman-Naimark *symmetry conditions*

$$[w_1, w_1]_{(1)} \Big|_{-1}^{1} = [w_1, w_2]_{(1)} \Big|_{-1}^{1} = [w_2, w_2]_{(1)} \Big|_{-1}^{1} = 0.$$

From (2.9.3), these symmetry conditions yield the following necessary and sufficent conditions on the linearly independent vectors $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\{\beta_1, \beta_2, \beta_3, \beta_4\}$:

$$\alpha_1 \overline{\alpha_3} - \overline{\alpha}_1 \alpha_3 - \alpha_2 \overline{\alpha_4} + \overline{\alpha_2} \alpha_4 = 0,$$

$$\overline{\alpha}_1 \beta_3 - \overline{\alpha}_3 \beta_1 - \overline{\alpha}_2 \beta_4 + \overline{\alpha}_4 \beta_2 = 0, \tag{2.9.6}$$

$$\beta_1 \overline{\beta}_3 - \overline{\beta}_1 \beta_3 - \beta_2 \overline{\beta}_4 + \overline{\beta}_2 \beta_4 = 0.$$

In the case of separated boundary conditions, we can take $\beta_1 = \alpha_2 = \beta_3 = \alpha_4 = 0$. In this case, the symmetry conditions are

$$\alpha_1 \overline{\alpha_3} = \overline{\alpha}_1 \alpha_3,$$

$$\beta_2 \overline{\beta}_4 = \overline{\beta}_2 \beta_4.$$

It is easy to see, in this case, that it suffices to take (α_1, α_3), $(\beta_2, \beta_4) \in \mathbb{R}^2$ to be both nonzero. In summary, we state the following theorem:

Theorem 2.9.1. All self-adjoint extensions S in $L^2(-1,1)$ of the minimal operator $\mathcal{L}_0$ generated by the Legendre expression $M_k^{(1)}[\cdot]$ have the form

$$S[y] = M_k^{(1)}[y],$$

$$\mathcal{D}(S) = \left\{ y \in \mathcal{D}(\mathcal{L}) \mid \ [w_1, y]_{(1)} \Big|_{-1}^{1} = [w_2, y]_{(1)} \Big|_{-1}^{1} = 0 \right\}.$$

Here, w_1, w_2 are defined in (2.9.4) and (2.9.5), respectively, φ_1, φ_2, φ_3, φ_4 are defined in (2.9.2), and the linearly independent vectors $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\{\beta_1, \beta_2, \beta_3, \beta_4\} \in \mathbb{C}^4$ satisfy the symmetry conditions in (2.9.6). In the special case that S is determined by separated boundary conditions, then S has the form

$$S[y] = M_k^{(1)}[y],$$

$$\mathcal{D}(S) = \left\{ y \in \mathcal{D}(\mathcal{L}) \,\Big|\, [\hat{w}_1, y]_{(1)}(1) = [\hat{w}_2, y]_{(1)}(-1) = 0 \right\};$$

In this case, $\hat{w}_1 = a_1\psi_1 + a_2\psi_2$ and $\hat{w}_2 = b_1\phi_1 + b_2\phi_2$, where $\psi_1 := \varphi_1$, $\psi_2 := \phi_3$, $\phi_1 := \varphi_2$ and $\phi_2 := \varphi_4$, and (a_1, a_2), $(b_1, b_2) \in \mathbb{R}^2$ are both nonzero vectors.

We now focus our attention on determining the self-adjoint extension(s) S of $M_k^{(1)}[\cdot]$ in $L^2(-1,1)$ that have the Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ as eigenfunctions. To emphasize the relationship this sequence of orthogonal polynomials has with the other Legendre polynomials discussed in this chapter, we shall write $P_n(x) = P_{1,n}(x)$. Notice that if S is such an extension, then necessarily we must have

$$[w_1, P_{1,0}]_{(1)} \Big|_{-1}^{1} = 0, \ i = 1, 2,$$

where w_1, w_2 are given by (2.9.4) and (2.9.5), respectively. Since $P_{1,0}(x) = 1$, these conditions yield

$$\alpha_3 = \alpha_4 \text{ and } \beta_3 = \beta_4. \tag{2.9.7}$$

In addition, since $P_{1,1}(x) = x$, we must also have

$$[w_1, x]_{(1)} \Big|_{-1}^{1} = 0, \ i = 1, 2,$$

which yields

$$\alpha_3 = -\alpha_4 \text{ and } \beta_3 = -\beta_4. \tag{2.9.8}$$

From (2.9.7) and (2.9.8), we find that $\alpha_3 = \alpha_4 = \beta_3 = \beta_4 = 0$. Consequently, we have:

$$w_1 = \alpha_1\varphi_1 + \alpha_2\varphi_2 = \begin{cases} \alpha_2 \text{ near } x = -1 \\ \alpha_1 \text{ near } x = 1, \end{cases}$$

$$w_2 = \beta_1\varphi_1 + \beta_2\varphi_2 = \begin{cases} \beta_2 \text{ near } x = -1 \\ \beta_1 \text{ near } x = 1, \end{cases}$$

where (α_1, α_2), (β_1, β_2) are linearly independent vectors in $\mathbb{C}^2$. However, it is easy to see that

$$[w_1, y]_{(1)} \Big|_{-1}^{1} = \alpha_1 [1, y]_{(1)} (1) - \alpha_2 [1, y]_{(1)} (-1),$$

$$[w_2, y]_{(1)} \Big|_{-1}^{1} = \beta_1 [1, y]_{(1)} (1) - \beta_2 [1, y]_{(1)} (-1),$$

for all $y \in \mathcal{D}(S)$. From these conditions, it is clear that $[y, w_i]_{(1)} \Big|_{-1}^{1} = 0$, $i = 1, 2$, if and only if $[y, 1]_{(1)} (1) = [y, 1]_{(1)} (-1) = 0$. Consequently, we have the following theorem:

Theorem 2.9.2. The self-adjoint operator S in $L^2(-1, 1)$ which extends the minimal operator $\mathcal{L}_0$ generated by the Legendre differential expression $M_k^{(1)}[\cdot]$ and has the Legendre polynomials as eigenfunctions is given by

$$S[y] = M_k^{(1)}[y], \tag{2.9.9}$$

$$\mathcal{D}(S) = \left\{ y \in \mathcal{D}(\mathcal{L}) \,\middle|\, [y, 1]_{(1)} (1) = [y, 1]_{(1)} (-1) = 0 \right\}.$$

Furthermore, the spectrum of S is $\sigma(S) = \{n(n+1) + k \mid n = 0, 1, \ldots\}$.

Proof. Details about the spectrum can be found in [54]. □

2. the n^{th} Legendre$^{(2)}$-left polynomial $PL_{2,n}(x)$ (see [54]) is a solution of

$$ML_k^{(2)}[y] = \left(n^4 + 2n^3 + (4A+1)n^2 + 4An + k\right)y, \ n = 0, 1, \ldots,$$

where the expression $ML_k^{(2)}[\cdot]$ is given by (2.1.2). They form a complete orthogonal set in the Hilbert space $L_\alpha^2[-1,1)$ generated from the inner product

$$(f,g)_\alpha := \int_{[-1,1)} f(x)\overline{g}(x)\,d\alpha(x) = \frac{f(-1)\overline{g}(-1)}{A} + \int_{-1}^{1} f(x)\overline{g}(x)\,dx.$$

The endpoints $x = \pm 1$ are regular singular points of $ML_k^{(2)}[\cdot]$. At $x = 1$, $ML_k^{(2)}[\cdot]$ is limit-4 in $L^2(-1,1)$ while $ML_k^{(2)}[\cdot]$ is limit-3 in $L^2(-1,1)$ at $x = -1$; this can easily be verified using the method of Frobenius. Hence, from Theorem 2.6.5., the deficiency index of the minimal operator generated by $ML_k^{(2)}[\cdot]$ in $L^2(-1,1)$ is $(3,3)$. Consequently, each self-adjoint extension in $L^2(-1,1)$ of this minimal operator can be obtained by imposing three linearly independent boundary conditions as required by Theorem 2.8.1.. We leave it to the reader to apply this theorem to find all of these self-adjoint extensions in $L^2(-1,1)$. *We note, however, that none of these self-adjoint extensions will have the set of Legendre*$^{(2)}$*-left polynomials as eigenfunctions.* Indeed, if such a self-adjoint extension exists, the appropriate setting for this operator is $L_\alpha^2[-1,1)$ and not $L^2(-1,1)$. The proof of the following theorem can be found in ([54], Chapter 5):

Theorem 2.9.3. The self-adjoint operator S in $L_\alpha^2[-1,1)$ which has the Legendre$^{(2)}$-left polynomials as eigenfunctions is defined by

$$S[y](x) := \begin{cases} -8Ay'(-1) + ky(-1) & \text{if } x = -1 \\ ML_k^{(2)}[y](x) & \text{if } -1 < x < 1, \end{cases} \tag{2.9.10}$$

$$\mathcal{D}(S) = \{y \in \mathcal{D}(\mathcal{L}) \mid \ [y,w_1]_L(1) = [y,w_2]_L(1) = 0\}.$$

Here, $\mathcal{D}(\mathcal{L})$ refers to the domain of the maximal operator in $L^2(-1,1)$ generated by $ML_k^{(2)}[\cdot]$, $[\cdot,\cdot]_L$ is the sesquilinear form defined by (2.3.4), and $\{w_1, w_2\} \subset \mathbb{C}^4[-1,1]$

are defined by

$$w_1(x) = \begin{cases} 1 & \text{for } x \text{ near } 1 \\ 0 & \text{for } x \text{ near } -1, \end{cases}$$

$$w_2(x) = \begin{cases} 1-x & \text{for } x \text{ near } 1 \\ 0 & \text{for } x \text{ near } -1. \end{cases}$$

Furthermore, the spectrum of S is given by

$$\sigma(S) = \left\{ n^4 + 2n^3 + (4A+1)\, n^2 + 4An + k \mid n = 0, 1, 2, \ldots \right\}.$$

Remark 5.

In proving Theorem 2.9.3., we found that each $y \in \mathcal{D}(S)$ has the property that $y'' \in L^2(-1,1)$; consequently, by redefining y at $x = \pm 1$, we may assume that y, $y' \in AC[-1,1]$ for all $y \in \mathcal{D}(S)$. Moreover, it is true that each $y \in \mathcal{D}(\mathcal{L})$ has the property that $y'' \in L^2(-1,0]$; in general, however, $y'' \notin L^2(-1,1)$ for $y \in \mathcal{D}(\mathcal{L})$. The definition of the maximal domain $\mathcal{D}(\mathcal{L})$, given by (2.2.2), does not suggest that functions in the maximal domain of the expression $ML_k^{(2)}[\cdot]$ enjoy this smoothness condition at $x = -1$. We can attribute this remarkable property to the smoothness of the coefficients of $ML_k^{(2)}[\cdot]$ and to the *discontinuity* of the monotonically increasing function $\hat{\alpha}$ at $x = -1$ which generates the orthogonalizing measure α. Indeed, $\hat{\alpha}$ may be defined by:

$$\hat{\alpha}(x) = \begin{cases} -1 - \frac{1}{A} & \text{if } \; x \leq -1 \\ x & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1; \end{cases}$$

notice that $\alpha(\{-1\}) = 1/A$. We shall find that this pattern persists with the other higher-order Legendre expressions: whenever the measure of the endpoint -1 (respectively, $+1$) is nonzero, we pick up a certain degree of smoothness of functions near $x = -1$ $(+1)$ in the corresponding maximal domain in $L^2(-1,1)$ and possibly even more smoothness for functions in the domain of the appropriate self-adjoint operator having the orthogonal polynomials as eigenfunctions.

Does this same pattern hold for the classical Legendre expression? In this case, the orthogonalizing measure is Lebesgue measure restricted to the interval $[-1,1]$; *i.e.,* the measure of both $x = \pm 1$ is zero. In general, functions in the maximal domain in $L^2(-1,1)$ associated with this expressions have no special smoothness conditions. For example,

$$f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \in \mathcal{D}(\mathcal{L}) \ ; \text{ but } f'(x) = \frac{1}{1-x^2} \notin L^2(-1,1).$$

However, any function y in the domain of the self-adjoint operator S, defined by (2.9.9), has the property that $y' \in L^2(-1,1)$; this result is due to Everitt and Marić [23]. Consequently, each $y \in \mathcal{D}(S)$ is absolutely continuous on $[-1,1]$.

As we noted earlier, each self-adjoint extension of $ML_k^{(2)}[\cdot]$ in $L^2(-1,1)$ is determined by three linearly independent boundary conditions. In the special case of separated conditions, two appropriately chosen boundary conditions would have to be prescribed at $x = 1$ and one boundary condition would have to be assigned at $x = -1$; this is due to the limit cases at $x = \pm 1$. It is interesting to note that S in $L^2_\alpha[-1,1)$, given in (2.9.10), is defined using two separated boundary conditions at $x = 1$ and *no* boundary conditions at $x = -1$. This is due, in part, to the continuity of the function $\hat{\alpha}$ at $x = 1$ and the discontinuity of $\hat{\alpha}$ at $x = -1$. *As we shall see with all of the Legendre expressions, whenever an endpoint has nonzero measure (the measure being the orthogonalizing measure), we need one less boundary condition than the Glazman-Naimark theorem requires to define the self-adjoint operator having the corresponding orthogonal polynomials as eigenfunctions while the same number of boundary conditions would be needed at an endpoint which measures to zero through the orthogonalizing measure.*

3. Through the transformation $(x, A) \rightarrow (-x, B)$, the expression (2.1.2) changes to expression $MR_k^{(2)}[\cdot]$ given by (2.1.3); this later expression has the Legendre$^{(2)}$-right polynomials as eigenfunctions. Properties of these polynomials can be found in ([54], Chapter 6). They form a complete orthogonal set in $L^2_\beta(-1, 1]$, where β is the measure generated by the monotonically increasing function $\hat{\beta}$ defined by

$$\hat{\beta}(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ x & \text{if } -1 < x < 1 \\ 1 + \frac{1}{B} & \text{if } x \geq 1. \end{cases}$$

The following theorem is the analog of Theorem 2.9.3.:

Theorem 2.9.4. The self-adjoint operator S in $L^2_\beta(-1, 1]$ which has the Legendre$^{(2)}$-right polynomials as eigenfunctions is defined by

$$S[y](x) := \begin{cases} MR_k^{(2)}[y](x) & \text{if } -1 < x < 1 \\ 8By'(1) + ky(1) & \text{if } x = 1, \end{cases}$$

$$\mathcal{D}(S) = \{y \in \mathcal{D}(\mathcal{L}) \mid \ [y, w_1]_R(-1) = [y, w_2]_R(-1) = 0\}.$$

In this case, $\mathcal{D}(\mathcal{L})$ is the domain of the maximal operator in $L^2(-1, 1)$ generated by $MR_k^{(2)}[\cdot]$, $[\cdot, \cdot]_R$ is the sesquilinear form defined by (2.3.5), and $\{w_1, w_2\} \subset \mathbb{C}^4[-1, 1]$ are defined by

$$w_1(x) = \begin{cases} 0 & \text{for } x \text{ near } 1 \\ 1 & \text{for } x \text{ near } -1, \end{cases}$$

$$w_2(x) = \begin{cases} 0 & \text{for } x \text{ near } 1 \\ 1 + x & \text{for } x \text{ near } -1. \end{cases}$$

Furthermore, the spectrum of S is given by

$\sigma(S) = \{n^4 + 2n^3 + (4B + 1)n^2 + 4Bn + k \mid n = 0, 1, 2, \ldots\}$.

Proof. See ([54], Chapter 6). $\square$

4. the Legendre$^{(2)}$ polynomials, also called the Legendre type or the Krall-Legendre polynomials, have been extensively written up in [36], [21], [17], and [22]; we also refer the reader to ([54], Chapter 4) where a complete account of these polynomials and the associated self-adjoint theory is given. These polynomials are also the main subject of this thesis.

the n^{th} Legendre$^{(2)}$ polynomial $P_{2,n}(x)$ is a solution of

$$M_k^{(2)}[y] = \left(n\left(n+1\right)(n^2+n+4A-2\right)+k\right)y,\ n = 0, 1, \ldots$$

where $M_k^{(2)}[\cdot]$ is defined by (2.1.4.). They form a complete orthogonal set in $L^2_\mu[-1,1]$ where μ is the orthogonalizing weight generated from the monotonically increasing function $\hat{\mu}$ defined by

$$\hat{\mu}(x) = \begin{cases} -1 - \frac{1}{A} & \text{if } x \leq -1 \\ x & \text{if } -1 < x < 1 \\ 1 + \frac{1}{A} & \text{if } x \geq 1. \end{cases} \tag{2.9.11}$$

By the method of Frobenius, it can be shown that both endpoints $x = \pm 1$ are in the limit-3 case in $L^2(-1,1)$. Hence, by Theorem 2.6.5., the deficiency index of the minimal operator in $L^2(-1,1)$ generated by $M_k^{(2)}[\cdot]$ is $(2,2)$. Since the deficiency index is $(2,2)$, each of the self-adjoint extensions is determined by two appropriately chosen boundary conditions. In the special case of separated boundary conditions, there would be one Glazman-Naimark boundary condition needed at each endpoint $x = \pm 1$. We note, however, that none of these extensions will have the Legendre$^{(2)}$ polynomials as eigenfunctions; indeed, we must look for such a self-adjoint operator in $L^2_\mu[-1,1]$. We list the following theorem which gives the appropriate self-adjoint operator in $L^2_\mu[-1,1]$:

Theorem 2.9.5. Define the operator $S : L^2_\mu[-1,1] \to L^2_\mu[-1,1]$ by

$$S[y](x) := \begin{cases} -8Ay'(-1) + ky(-1) & \text{if } x = -1 \\ M_k^{(2)}[y](x) & \text{if } -1 < x < 1 \\ 8Ay'(1) + ky(1) & \text{if } x = 1, \end{cases} \tag{2.9.12}$$

$\mathcal{D}(S) = \mathcal{D}(\mathcal{L})$, where $\mathcal{D}(\mathcal{L})$ is the maximal domain of $M_k^{(2)}[\cdot]$ in $L^2(-1,1)$. Then S is a self-adjoint operator in $L^2_\mu[-1,1]$ having the Legendre$^{(2)}$ polynomials as eigenfunctions. The spectrum of S is given by

$\sigma(S) = \{(n(n+1)(n^2+n+4A-2)+k) \mid n = 0,1,2,\ldots\}$.

Proof. See ([21] and [17]). $\square$

Remark 6.

It is remarkable that $\mathcal{D}(\mathcal{L})$ is the domain of the self-adjoint operator S given by (2.9.12). Indeed, there is no reason to believe *apriori* that functions $f \in \mathcal{D}(\mathcal{L})$ should have the property that their derivatives at $x = \pm 1$ exist and are finite (which is required in the definition of $S[\cdot]$). Quite surprisingly, it is true that any $f \in \mathcal{D}(\mathcal{L})$ has the property that $f'' \in L^2(-1,1)$; thus, we may assume that such functions f satisfy the condition that $f,\ f' \in AC[-1,1]$. Notice that, this example fits the pattern alluded to in Remark 5: since the endpoints $x = \pm 1$ both have nonzero μ-measure, functions in the maximal domain have certain smoothness properties at $x = \pm 1$. Furthermore, while the Glazman-Naimark theory says that one separated boundary condition must be prescribed at each endpoint to obtain a self-adjoint extension in $L^2(-1,1)$, there are no boundary conditions needed in the definition of S above. Again, this fits the pattern suggested in Remark 5.

5. Our last example concerns the Legendre$^{(3)}$ polynomials $\{P_{3,n}(x)\}$ which are solutions of

$$M_k^{(3)}[y] = \lambda_n y,$$

where

$$\lambda_n = n(n+1)\left(n^4 + 2n^3 + (3A+3B-1)n^2 + (3A+3B-2)n + 12AB\right) + k,$$

and where $M_k^{(3)}[\cdot]$ is given by (2.1.5). These polynomials form a complete orthogonal set in the space $L_\kappa^2[-1,1]$
where κ is the mesaure generated from the monotonically increasing function $\hat{\kappa}$ defined by

$$\hat{\kappa}(x) = \begin{cases} -1 - \frac{1}{A} & \text{if } x \leq -1 \\ x & \text{if } -1 < x < 1 \\ 1 + \frac{1}{B} & \text{if } x \geq 1. \end{cases}$$

Properties of these polynomials can be found in [49] and [54]. The reader will notice, upon comparison of $\hat{\kappa}$ and $\hat{\mu}$ (defined in (2.9.11)), that $P_{3,n}(x)$ is a generalization of the Krall-Legendre polynomial $P_{2,n}(x)$. However, as we see below, the domain of the self-adjoint operator in $L_\kappa^2[-1,1]$ having the polynomials $\{P_{3,n}(x)\}$ as eigenfunctions is significantly different than the operator S defined in (2.9.12).

Both of the endpoints $x = \pm 1$ are regular singular endpoints of $M_k^{(3)}[\cdot]$. By applying the method of Frobenius, we find that each endpoint is in the limit-5 case in $L^2(-1,1)$. From Theorem 2.6.5., the deficiency index of the minimal operator in $L^2(-1,1)$ generated by $M_k^{(3)}[\cdot]$ is $(4,4)$. Consequently, every self-adjoint extension in $L^2(-1,1)$ of the minimal operator generated by $M_k^{(3)}[\cdot]$ is determined by four linearly independent Glazman-Naimark boundary conditions, as described by Theorem 2.8.1.. Moreover, in the case of separated boundary conditions, there would be *two* separated boundary conditions required at each endpoint $x = \pm 1$.

While no boundary conditions are required to define the self-adjoint operator S given by (2.9.12), we shall see that the domain of the appropriate self-adjoint operator in $L_\kappa^2[-1,1]$ having the set $\{P_{3,n}(x)\}$ as eigenfunctions involves *one* boundary condition at each endpoint $x = \pm 1$.

Notice that, this fits the pattern mentioned at the end of Remark 5.

Theorem 2.9.6. Define $S : L^2_\kappa[-1,1] \to L^2_\kappa[-1,1]$ by

$$S[y](x) = \begin{cases} 24Ay''(-1) + (-24AB - 24A)\,y'(-1) + ky(-1) & \text{if } x = -1 \\ M_k^{(3)}[y](x) & \text{if } -1 < x < 1 \\ 24By''(1) + (24AB + 24B)\,y'(1) + ky(1) & \text{if } x = 1, \end{cases}$$

$$\mathcal{D}(S) = \{y \in D(\mathcal{L}) \mid \ [y, w_1]_3(1) = [y, w_2]_3(-1) = 0\}.$$

In this case, $\mathcal{D}(\mathcal{L})$ is the maximal domain of $M_k^{(3)}[\cdot]$ in $L^2(-1,1)$, $[\cdot,\cdot]_{(3)}$ is the sesquilinear form defined in (2.3.7) and w_1, w_2 are defined by:

$$w_1(x) = \begin{cases} 0 & \text{for } x \text{ near } -1 \\ \frac{1}{2}(1-x^2) + \frac{1}{8}(A+2)(1-x^2)^2 & \text{for } x \text{ near } 1, \end{cases}$$

$$w_2(x) = \begin{cases} -\frac{1}{2}(1-x^2) - \frac{1}{8}(B+2)(1-x^2)^2 & \text{for } x \text{ near } -1 \\ 0 & \text{for } x \text{ near } 1. \end{cases}$$

Then S is a self-adjoint operator having the Legendre$^{(3)}$ polynomials $\{P_{3,n}(x)\}$ as eigenfunctions. Moreover, the spectrum of S is given by

$$\sigma(S) = \{n(n+1)\,n^4 + 2n^3 + (3A + 3B - 1)\,n^2 \\ + (3A + 3B - 2)\,n + 12AB + k \mid n = 0, 1, 2, \ldots\}.$$

Proof. See ([54], Chapter 7). □

CHAPTER THREE

A General Left-Definite Spectral Theory

3.1 Introduction

The history of left-definite spectral theory - as it relates to differential operators - can be traced to the work of Weyl [72] who, in his landmark analysis of second-order Sturm-Liouville differential equations, coined the term *polare-Eigenwertaufgabe* for the study of second-order equations in the left-definite setting. The terminology *left-definite* (actually, the German *Links-definit*) first appeared in the literature in 1965 in a paper by Schäfke and Schneider [65]. In his book [34], Kamke uses the term *F-definit* in his study of the differential equation $Fy = \lambda Gy$ (he also uses *G-definit* for his right-definite study of this equation). In ([56], [57], and [58]), Niessen and Schneider considered general left-definite singular systems and left-definite s-hermitian problems. Pleijel ([60] and [61]) provided one of the first concrete examples of such a left-definite setting for a self-adjoint differential operator with his analysis of the classical second-order Legendre equation. His work was followed soon after by the work of Atkinson *et al.* [3] who examined left-definite square-integrable homogeneous solutions. Later, Everitt [15] gave a complete (first) left-definite analysis of the classical Legendre equation and his student, Onyango-Otieno [59], extended these results by analyzing the appropriate right-definite and first left-definite spectral settings for the differential equations having the classical orthogonal polynomials (Jacobi, Laguerre, Hermite) as solutions. Everitt, in [16], and Bennewitz and Everitt [5] further the general theory of left-definite operators associated with second-order differential equations.

During the past 22 years, there have been several additional papers dealing with theory and specific examples of left-definite operators, all within the framework of differential operators. Important results related to second-order equations have been obtained by Krall ([37], [38], [39], and [40]), Krall and Littlejohn [42], Hajmirzaahmad [31]. Left-definite results for higher-order differential equations have been obtained by Loveland [54], Everitt and Littlejohn [21], Everitt *et al.* ([18], [19], [27], and [20]), Wellman [71], Vonhoff [69], Littlejohn and Wellman [53].

In this chapter, we attempt to provide a framework for a general left-definite theory of bounded below, self-adjoint operators in a Hilbert space. In Section 3.2, we define the general concept of a left-definite Hilbert space and a left-definite operator associated with a self-adjoint operator that is bounded below. In Section 3.3, we shall state the necessary theorems for the theory of left-definite space and in the final section, we recall the spectral theorem and some of its immediate consequences that we need in our work.

3.2 An Abstract Definition of A Left-Definite Space and Operator

Let V be a vector space over the complex field $\mathbb{C}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$; the resulting inner product space is denoted by $(V,(\cdot,\cdot))$. Suppose V_r (the subscripts will be made clear shortly) is a (vector) subspace (*i.e.,* a linear manifold) of V and let $(\cdot,\cdot)_r$ and $\|\cdot\|_r$ denote, respectively, an inner product (quite possibly different from $(\cdot,\cdot)$) and an associated norm on V_r.

Definition 3.2.1. Let $H=(V,(\cdot,\cdot))$ be a Hilbert space. Suppose $A:\mathcal{D}(A)\subset H\to H$ is a self-adjoint operator that is bounded below by a positive number $k>0$; *i.e.,*

$$(Ax,x)\geq k(x,x)\,,\ \ (x\in\mathcal{D}(A))\,.$$

Let $H_1=(V_1,(\cdot,\cdot)_1)$, where V_1 is a subspace of V and $(\cdot,\cdot)_1$ is an inner product on $V_1\times V_1$. Then H_1 is said to be a *left-definite (Hilbert) space* associated with the pair (H,A), if each of the following conditions holds:

(1) H_1 is a Hilbert space,

(2) $\mathcal{D}(A)$ is a subspace of V_1,

(3) $\mathcal{D}(A)$ is dense in H_1,

(4) $(x,x)_1 \geq k(x,x)$, $(x \in V_1)$, and

(5) $(x,x)_1 \geq (Ax,y)$, $(x \in \mathcal{D}(A),\ y \in V_1)$.

Given a self-adjoint operator A that is bounded below by a positive constant, it is not clear that a left-definite space H_1 exists for the pair (H,A). In fact, however, Littlejohn and Wellman proved the existence and uniqueness of this Hilbert space in [53] (see Theorem 3.1.).

If A is a self-adjoint operator in H that is bounded below by a positive number k, then, with assistance from the spectral theorem (see Section 3.4 and, in particular, Theorem 3.4.3.), we see that A^r is a self-adjoint operator bounded below by $k^r I$ for each $r > 0$. Consequently, we can extend Definition 3.2.1. to a continuum of left-definite spaces associated with (H,A).

Definition 3.2.2. Let $H = (V,(\cdot,\cdot))$ be a Hilbert space. Suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number $k > 0$. *i.e.,*

$$(Ax,x) \geq k(x,x), \quad (x \in \mathcal{D}(A)).$$

Let $r > 0$. If there exists a subspace V_r of V and an inner product $(\cdot,\cdot)_r$ on V_r such that $H_r = (V_r,(\cdot,\cdot)_r)$ is a left-definite space associated with the pair (H,A^r), we call H_r an r^{th} *left-definite space associated with the pair* (H,A). Specifically, H_r is an r^{th} left-definite space associated with the pair (H,A), if each of the following conditions holds:

(1) H_r is a Hilbert space,

(2) $\mathcal{D}(A^r)$ is a subspace of V_r,

(3) $\mathcal{D}(A^r)$ is dense in H_r,

(4) $(x,x)_r \geq k^r (x,x)$, $(x \in V_r)$, and

(5) $(x,y)_r \geq (A^r x, y)$, $(x \in \mathcal{D}(A^r),\ y \in V_r)$.

From our discussion above, we will see below in Theorem 3.3.1. that, for each $r > 0$, H_r exists and is unique. At first glance, it appears that the r^{th} left-definite space H_r depends on H, A, and the positive number k satisfying condition (4) in the above definition. In fact, however, each of the left-definite spaces H_r is independent of k.

We are now in position to define a left-definite operator associated with A.

Definition 3.2.3. Let $H = (V, (\cdot,\cdot))$ be a Hilbert space. Suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number $k > 0$. Let $r > 0$ and suppose H_r is an r^{th} left-definite space associated with the pair (H, A). If there exists a self-adjoint operator $A_r : H_r \to H_r$ that is a restriction of A; that is to say,

$$\begin{aligned} &A_r x = Ax, \\ &x \in \mathcal{D}(A^r) \subset \mathcal{D}(A), \end{aligned} \tag{3.2.1}$$

we call such an operator an r^{th} *left-definite operator associated with the pair* (H, A).

In Theorem 3.3.2. below, we see that if A is a self-adjoint operator that is, bounded below by a positive number $k > 0$, then for all $r > 0$ there exists a *unique* left-definite operator A_r in H_r associated with (H, A).

3.3 Main Theorems

There are six main theorems that we state in this section concerning left-definite Hilbert spaces and left-definite self-adjoint operators.

Theorem 3.3.1. Suppose A is a self-adjoint operator in the Hilbert space $H = (V, (\cdot, \cdot))$ that is bounded below by kI, where $k > 0$. Let $r > 0$, define $H_r = (V_r, (\cdot, \cdot)_r)$ with

$$V_r = A^{r/2} \text{ and} \tag{3.3.1}$$

$$(x, y)_r = \left(A^{r/2}x, A^{r/2}y\right), \ (x, y \in V_r). \tag{3.3.2}$$

Then, H_r is an r^{th} left-definite space associated with the pair (H, A) in the sense of Definition 3.2.2. . Moreover, suppose $H_r = (V_r, (\cdot, \cdot)_r)$ and $H_r' = \left(V_r', (\cdot, \cdot)_r'\right)$ are r^{th} left-definite spaces associated with the pair (H, A). Then, $V_r = V_r'$ and $(x, y)_r = (x, y)_r'$ for all $x, y \in V_r = V_r'$; *i.e.,* $H_r = H_r'$. Consequently, $H_r = (V_r, (\cdot, \cdot)_r)$, as defined in (3.3.1) and (3.3.2), is the unique r^{th} left-definite Hilbert space associated with the pair (H, A).

Proof. See (Section 6 in [53]). $\square$

Theorem 3.3.2. Suppose A is a self-adjoint operator in the Hilbert space $H = (V, (\cdot, \cdot))$ that is bounded below by kI, for some $k > 0$. For $r > 0$, let $H_r = (V_r, (\cdot, \cdot)_r)$ be r^{th} left-definite space associated with the pair (H, A). Then, there exists a *unique* left-definite operator A_r in H_r associated with (H, A). More specifically, if there exits a self-adjoint operator $\tilde{A}_r : H_r \to H_r$ such that $\tilde{A}_r x = Ax$ for all $x \in \mathcal{D}\left(\tilde{A}_r\right) \subset \mathcal{D}(A)$, then $A_r = \tilde{A}_r$. Furthermore,

$$\mathcal{D}(A_r) = V_{r+2}. \tag{3.3.3}$$

and A_r is bounded below by kI in $H_{r.}$

Proof. See (Section 7 in [53]). $\square$

The following corollary is an immediate consequence of Theorems 3.3.1. and 3.3.2.. It emphasizes the fact that, set-wise, the domain $\mathcal{D}(A^r)$ of the r^{th} power of A is given by V_{2r} and in particular, the domain of the positive square root of A and the domain of A. Furthermore, it describes explicitly the domain of the r^{th} left-definite operator in terms of the domain of a certain power of A. Interestingly, we note that the domains of the first and second left-definite operators, A_1 and A_2, are given by $\mathcal{D}\left(A^{3/2}\right)$ and $\mathcal{D}\left(A^2\right)$, respectively.

Corollory 3.3.1. Suppose A is a self-adjoint operator in the Hilbert space $H = (V, (\cdot, \cdot))$ that is bounded below by kI, for some $k > 0$. For each $r > 0$, let $H_r = (V_r, (\cdot, \cdot)_r)$ and A_r denote, respectively, the r^{th} left-definite space and the r^{th} left-definite operator associated with the pair (H, A). Then,

(i) $\mathcal{D}(A^r) = V_{2r}$, in particular, $\mathcal{D}\left(A^{1/2}\right) = V_1$ and $\mathcal{D}(A) = V_2$;

(ii) $\mathcal{D}(A_r) = \mathcal{D}\left(A^{(r+2)/2}\right)$, in particular, $\mathcal{D}(A_1) = \mathcal{D}\left(A^{3/2}\right)$ and $\mathcal{D}(A_2) = \mathcal{D}\left(A^2\right)$.

In the next theorem, we see that when A is a bounded, self-adjoint operator that is bounded below by a positive constant k, then the left-definite theory is trivial. However, the situation is quite different when A is unbounded.

Theorem 3.3.3. Let $H = (V, (\cdot, \cdot))$ be a Hilbert space. Suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by kI for some $k > 0$. For each $r > 0$, let $H_r = (V_r, (\cdot, \cdot)_r)$ and A_r denote the r^{th} left-definite space and the r^{th} left-definite operator, respectively, associated with the pair (H, A).

(1) Suppose A is bounded. Then, for each $r > 0$,

(i) $V = V_r$;

(ii) the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_r$ are equivalent;

(iii) $A = A_r$.

(2) Suppose A is unbounded. Then,

(i) V_r is a proper subspace of V;

(ii) V_s is a proper subspace of V_r whenever $0 < r < s$;

(iii) the inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_s$ are not equivalent for any $s > 0$;

(iv) the inner products $(\cdot, \cdot)_r$ and $(\cdot, \cdot)_s$ are not equivalent for any r, $s > 0$, $r \neq s$;

(v) $\mathcal{D}(A_r)$ is a proper subspace of $\mathcal{D}(A)$ for each $r > 0$;

(vi) $\mathcal{D}(A_s)$ is a proper subspace of $\mathcal{D}(A_r)$ whenever $0 < r < s$.

Proof. See (Section 8 in [53]). □

Since, for each $m > 0$, A^m is a self-adjoint operator that is bounded below in H by $k^m I$, we see from Theorems 3.3.1. and 3.3.2. that there are continua of left-definite spaces $\{(H^m)_r\}_{r>0}$ and left-definite operators $\{(A^m)_r\}_{r>0}$ associated with the pair (H, A^m). Furthermore, since A_m is a self-adjoint operator that is bounded below by kI in H_m, there are continua of left-definite spaces $\{(H_m)_r\}_{r>0}$ and left-definite operators $\{(A_m)_r\}_{r>0}$ associated with the pair (H, A^m). The following questions naturally arise:

(1) What is the relationship (if any) between the three continua of the left-definite spaces $\{H_r\}_{r>0}$, $\{(H^m)_r\}_{r>0}$, and $\{(H_m)_r\}_{r>0}$?

(2) Since for fixed $m > 0$, $(A_r)^m -$ the m^{th} power of the r^{th} left-definite operator A_r associated with $(H, A) -$ is a self-adjoint restriction of A^m, what is the relationship (if any) between the continuum of the left-definite operators $\{(A^m)_r\}_{r>0}$ associated with the pair (H, A^m) and the continuum of the left-definite operators $\{(A_r)^m\}_{r>0}$? In particular, is $(A_r)^m$ a left-definite operator associated with (H, A^m); that is to say, is $(A_r)^m \in \{(A_s)^m\}_{s>0}$?

(3) For fixed $m > 0$, what is the relationship (if any) between the continuum of the left-definite operators $\{(A_m)_r\}_{r>0}$ associated with the pair (H_m, A_m) and the continuum of the left-definite operators $\{A_r\}_{r>0}$ associated with (H, A)?

Each of these questions is answered in the following theorem. In essence, this theorem says that there are no left-definite spaces or left-definite operators emerging from a consideration of the above questions; that is to say, the original spaces $\{H_r\}_{r>0}$ and operators $\{A_r\}_{r>0}$ encompass all of the left-definite spaces and left-definite operators described above that are associated with the pairs (H, A^m) and (H_m, A_m).

Theorem 3.3.4. Suppose A, H, $\{H_r\}_{r>0}$ and $\{A_r\}_{r>0}$ are as in Theorem 3.3.1. and 3.3.2. above. Fix $m > 0$, and for each $r > 0$, let $(H^m)_r = ((V^m)_r, (\cdot,\cdot)_r^m)$ and $(A^m)_r$ denote, respectively, the r^{th} left-definite space and the r^{th} left-definite operator associated with the pair (H, A^m). Then,

(i) $(H^m)_r = H_{mr}$.

(ii) $(A_r)^m = (A^m)_{r/m}$ with $\mathcal{D}((A_r)^m) = V_{2m+r}$.

Equivalently, $(A^m)_r = (A_{mr})^m$ with $\mathcal{D}((A^m)_r) = V_{2m+mr}$; that is to say, the r^{th} left-definite operator associated with the pair (H, A^m) is the m^{th} power of the $(mr)^{th}$ left-definite operator associated with the pair (H, A).

Furthermore, let $(H_m)_r = \left((V_m)_r, (\cdot,\cdot)_{m,r}\right)$ and $(A_m)_r$ denote the r^{th} left-definite space and the r^{th} left-definite operator, respectively, associated with the pair (H_m, A_m). Then,

(iii) $(H_m)_r = H_{m+r}$.

(iv) $(A_m)_r = A_{m+r}$ with $\mathcal{D}((A_m)_r) = V_{m+r+2}$; in other words, the r^{th} left-definite operator associated with the pair (H_m, A_m) is the $(m+r)^{th}$ left-definite operator associated with (H, A).

Proof. See (Section 9 in [53]). □

In addition, we state the following two theorems concerning the spectra of the left-definite operators $\{A_r\}_{r>0}$.

Theorem 3.3.5. For each $r > 0$, let A_r denote the r^{th} left-definite operator associated with the self-adjoint operator A that is bounded below by kI where $k > 0$. Then,

(i) The point spectra of A and A_r coincide; *i.e.,* $\sigma_p(A_r) = \sigma_p(A)$.

(ii) The continuous spectra of A and A_r coincide; *i.e.,* $\sigma_c(A_r) = \sigma_c(A)$.

(iii) The resolvents spectra of A and A_r coincide; *i.e.,* $\sigma(A_r) = \sigma(A)$.

Proof. See (Section 10 in [53]). □

Finally, the last general result in this section is the following theorem:

Theorem 3.3.6. If $\{\varphi_n\}_{n=0}^{\infty}$ is complete orthogonal set of eigenfunctions of A in H, then for each $r > 0$, $\{\varphi_n\}_{n=0}^{\infty}$ is complete orthogonal set of eigenfunctions of the r^{th}

left-definite operator A_r in the r^{th} left-definite space H_r.

Proof. See (Section 10 in [53]). $\square$

3.4 The Spectral Theorem

If A is a self-adjoint operator in a Hilbert space H with inner product $(\cdot,\cdot)$, it is well known (see [64], Chapters 12 and 13) that there exists a unique operator-valued set function $E : \mathfrak{B} \to B(H)$, where $\mathfrak{B}$ is the σ-algebra of Borel subsets of $\mathbb{R}$ and $B(H)$ is the Banach algebra of bounded linear operators on H, called the *spectral resolution of the identity*, having the following properties:

(1) $E(\emptyset) = 0$ and $E(\mathbb{R}) = I$.

(2) $E(\Delta)$ is idempotent; that is $(E(\Delta))^2 = E(\Delta)$, for all $\Delta \in \mathfrak{B}$.

(3) $E(\Delta)$ is self-adjoint in H for all $\Delta \in \mathfrak{B}$.

(4) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2) = E(\Delta_2)E(\Delta_1)$ for all Δ_1, $\Delta_2 \in \mathfrak{B}$.

(5) $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathfrak{B}$ with $\Delta_1 \cap \Delta_2 = \emptyset$.

(6) For each x, $y \in H$, the mapping $E_{x,y} : \mathfrak{B} \to \mathbb{C}$ defined by

$$E_{x,y}(\Delta) := (E(\Delta)x, y) \tag{3.4.1}$$

is a complex, regular Borel measure. Since $E(\Delta)$ is a self-adjoint projection for each $\Delta \in \mathfrak{B}$, it follows that $\|E(\Delta)\| \leq 1$.

A spectral family (see [48] or [63]) for a self-adjoint operator A is a one-parameter family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of bounded operators in H satisfying:

(1) E_λ is self-adjoint and idempotent for each $\lambda \in \mathbb{R}$.

(2) For $\lambda < \mu$, $E_\mu - E_\lambda$ is a positive operator.

(3) $\lim_{\lambda\to\infty} E_\lambda x = x$ for each $\Delta \in H$.

(4) $\lim_{\lambda\to-\infty} E_\lambda x = 0$ for each $\Delta \in H$.

(5) $E_{\lambda+0}x := \lim_{\mu\to\lambda^+} E_\mu x = E_\lambda x$ for each $\lambda \in \mathbb{R}$ and $x \in H$.
$$\tag{3.4.2}$$

A connection between (3.4.1) and (3.4.2) lies in the following lemma; the proof is straightforward.

Lemma 3.4.1. Suppose E is a spectral resolution of the identity in the sense of (3.4.1). For $\lambda \in \mathbb{R}$, define $E_\lambda = E(-\infty, \lambda]$. Then $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is a spectral family in the sense of (3.4.2).

As mentioned earlier, the Hilbert-space spectral theorem plays a key role in proving the existence and uniqueness of the left-definite spaces $\{H_r\}_{r>0}$ and the left-definite operators $\{A_r\}_{r>0}$ associated with the pair (H, A), where A is a self-adjoint operator in H that is bounded below by kI, for some $k > 0$. In our development of these spaces and operators, we use the spectral resolution of the identity E of A rather than the one-parameter spectral family. However, properties of the spectrum $\sigma(A_r)$ and the resolvent set $\rho(A_r)$ of each left-definite operator A_r are more easily seen through the spectral family rather than the spectral resolution of the identity. Indeed, the following theorem is well known (see ([48], Section 9.11) and ([63], Section 13.2)).

Theorem 3.4.1. Suppose $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is a spectral family, satisfying the conditions of (3.4.2), of a self-adjoint operator A. For $\lambda_0 \in \mathbb{R}$, we have:

(1) $\lambda_0 \in \sigma_p(A)$ (the point spectrum) if and only if $E_{\lambda_0} \neq E_{\lambda_0 - 0}$.

(2) $\lambda_0 \in \sigma_c(A)$ (the continuous spectrum) if and only if $E_{\lambda_0} = E_{\lambda_0 - 0}$ and $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is not constant on any neighborhood of λ_0 in $\mathbb{R}$.

(3) $\lambda_0 \in \rho(A)$ (the resolution set) if and only if there exists $\varepsilon > 0$ such that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is constant on $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$.

We are now in position to state the spectral theorem in a Hilbert space (see [64], Theorems 13.24 and 13.30).

Theorem 3.4.2. (The Spectral Theorem). Let A be a self-adjoint operator (bounded or unbounded) in a Hilbert space $H = (V, (\cdot,\cdot))$. Let E be the spectral resolution of the identity associated with A. Then, for each $r > 0$, the self-adjoint operator A^r

has (densely defined) domain $\mathcal{D}(A^r)$ given by

$$\mathcal{D}(A^r) = \left\{ x \in H \;\middle|\; \int_{\mathbb{R}} \lambda^{2r} dE_{x,x} < 0 \right\}, \tag{3.4.3}$$

and is characterized by the identities

$$(A^r x, y) = \int_{\mathbb{R}} \lambda^r dE_{x,y} \quad (x \in \mathcal{D}(A^r),\ y \in H), \tag{3.4.4}$$

and

$$\|A^r x\|^2 = \int_{\mathbb{R}} \lambda^{2r} dE_{x,x} \quad (x \in \mathcal{D}(A^r)). \tag{3.4.5}$$

Conversely, suppose $F : \mathfrak{B} \to B(H)$ is a spectral resolution of the identity. Then, there exists a unique self-adjoint operator $\tilde{A}$ in H with (densely defined) domain

$$\mathcal{D}\left(\tilde{A}\right) = \left\{ x \in H \;\middle|\; \int_{\mathbb{R}} \lambda^2 dF_{x,x} < 0 \right\},$$

that is characterized by

$$\left(\tilde{A}x, y\right) = \int_{\mathbb{R}} \lambda dF_{x,y} \quad \left(x \in \mathcal{D}\left(\tilde{A}\right),\ y \in H\right),$$

and

$$\left\|\tilde{A}x\right\|^2 = \int_{\mathbb{R}} \lambda^2 dF_{x,x} \quad \left(x \in \mathcal{D}\left(\tilde{A}\right)\right).$$

Moreover, in this theorem, we can replace the interval $\mathbb{R}$ of integration in each of the above integrals with the spectrum of the self-adjoint operator. In particular, for a self-adjoint operator A that is bounded below by kI for $k > 0$, we can replace the interval of integration $\mathbb{R}$ with $[k, \infty)$ since, in this case, the spectrum $\sigma(A) \subset [k, \infty)$ (see [64], Theorem 12.32).

CHAPTER FOUR

The Legendre Type Differential Expression: Right-Definite Theory

4.1 Introduction

the Legendre type polynomials were discovered by H. L. Krall [45] in 1938 and named by A. M. Krall [36] in 1981. An explicit formula for these poynomials is

$$P_{n,A}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2n-2k)! \left(A + \frac{1}{2}n(n-1) + 2k\right) x^{n-2k}}{A 2^n k! (n-k)! (n-2k)!},$$

where $n = 0, 1, \ldots,$ $A > 0$ and $\left[\frac{n}{2}\right]$ denotes the greatest integer less than or equal to $\frac{n}{2}$. These polynomials have been normalized so that $P_{n,A}(1) = 1$ for all $n \geq 0$. Other formulas for the Legendre type polynomials can be found in Section 1.5 of this thesis. We refer the reader to [36] for further properties of these polynomials.

the Legendre type polynomials satisfy the fourth-order differential equation:

$$\ell[y](x) = \lambda_n y(x),$$

where

$$\ell[y] := \left(1-x^2\right)^2 y^{(4)} - 8x\left(1-x^2\right) y^{(3)} - (4A+12)\left(1-x^2\right) y'' + 8Axy' + ky \quad (4.1.1)$$

and $\lambda_n = n(n+1)(n^2+n+4A-2) + k$. Here, the numbers A and k are, respectively, fixed positive and nonnegative parameters. Observe that $\ell[y]$ is formally symmetric; i.e.,

$$\ell[y](x) := \left(\left(1-x^2\right)^2 y''(x)\right)'' - \left(\left(8 + 4A\left(1-x^2\right)\right) y'(x)\right)' + ky(x).$$

We remark that the Kralls studied the differential expression $\ell[y](\cdot)$ in the special case when $k = 0$. We shall study (4.1.1) for $k \geq 0$; this trivially amounts to a shift in the eigenvalue parameter k.

If we let $\hat{\mu}(x)$ denote the monotonic increasing function defined by:

$$\hat{\mu}(x) := \begin{cases} -1-\frac{1}{A} & if \quad -\infty < x \leq -1 \\ x & if \quad -1 < x < 1 \\ 1+\frac{1}{A} & if \quad 1 \leq x < \infty, \end{cases}$$

Then, $\hat{\mu}$ generates a regular positive measure μ on the Borel set of the real line (see, for example, [73], Section 11.3). the Legendre type polynomials $\{P_{n,A}\}_{n=0}^{\infty}$ are orthogonal in the Hilbert space $L^2_\mu[-1,1]$, where

$$L^2_\mu[-1,1] := \left\{ f : [-1,1] \to \mathbb{C} \,\middle|\, f \text{ is Lebesgue measurable and } \int_{-1}^{1} |f|^2 < \infty \right\} \tag{4.1.2}$$

is the Hilbert space with inner product:

$$(f,g)_\mu := \int_{-1}^{1} f(x)\,\bar{g}(x)\,d\mu(x)$$

$$= \int_{-1}^{1} f(x)\,\bar{g}(x)\,dx + \frac{f(1)\,\bar{g}(1)}{A} + \frac{f(-1)\,\bar{g}(-1)}{A}$$

and norm $\|f\|_\mu := (f,f)_\mu^{1/2}$. Specifically, the orthogonality relationship is

$$\int_{-1}^{1} P_{m,A}(x)\,P_{n,A}(x)\,d\mu(x) = \frac{\left(A+\frac{1}{2}n(n-1)\right)\left(A+\frac{1}{2}(n+1)(n+2)\right)}{A(2n+1)}\delta_{mn}. \tag{4.1.3}$$

where δ_{mn} is the Kronecker delta function.

In this chapter, we will review the study of the right-definite boundary value problem started by A. L. Krall [36] and completed by Everitt and Littlejohn in [21] and Everitt, Krall and Littlejohn in [17]. That is we will summarize the properties of the self-adjoint operator T, generated by ℓ in $L^2_\mu[-1,1]$, having the Legendre type polynomials as eigenfunctions. This is the so-called right-definite boundary value problem associated with the Legendre type polynomials. From this, we can study the left-definite boundary value problem.

In Section 4.2, we develop some essential properties of functions in the maximal domain Δ of $\ell[\cdot]$ in $L^2(-1,1)$; in this section, we also discuss *Green's formula* and *Dirichlet's formula.*

In the final section of this chapter, we define a self-adjoint operator T in $L^2_\mu[-1,1]$ having the Legendre type polynomials as eigenfunctions. Remarkably, the domain of T will be the maximal domain Δ of $\ell[\cdot]$ in $L^2(-1,1)$. This is, indeed, quite remarkable. Since the Legendre type expression $\ell[\cdot]$ is limit-3 at each endpoint $x=\pm1$ in $L^2(-1,1)$, the Glazman-Krein-Naimark (GKN) theory says that there must be a properly imposed boundary condition at each endpoint $x=\pm1$ in order to generate a self-adjoint operator in $L^2(-1,1)$. However, the setting in our case is the "jump space" $L^2_\mu[-1,1]$ and not $L^2(-1,1)$. It is the case that the discontinuity in $\hat{\mu}$ at $x=\pm1$ has the effect of eliminating a boundary condition in the domain of the self-adjoint operator T, generated by $\ell[\cdot]$, in $L^2_\mu[-1,1]$.

4.2 Properties of the Maximal Domain of ℓ

The maximal domain Δ of ℓ in $L^2(-1,1)$ is defined to be

$$\Delta := \left\{ f:(-1,1)\to\mathbb{C} \,\middle|\, f,\ f',\ f'',\ f'''\in AC_{\text{loc}}(-1,1);\ f,\ \ell[f]\in L^2(-1,1)\right\}. \tag{4.2.1}$$

Here, $AC_{\text{loc}}(-1,1)$ refers to the set of functions $f:(-1,1)\to\mathbb{C}$ that are locally absolutely continuous on $(-1,1)$, i.e., f is absolutely continuous on all compact subintervals of $(-1,1)$. Since $C_0^\infty(-1,1)$ (the space of all infinitely differentiable functions $f:(-1,1)\to\mathbb{C}$ with compact support in $(-1,1)$) is contained in Δ, and $C_0^\infty(-1,1)$ is dense in $L^2(-1,1)$ we see that Δ is dense in $L^2(-1,1)$. Now, we define, the maximal operator,

$$T_{\max}:\mathcal{D}(T_{\max})\subseteq L^2(-1,1)\to L^2(-1,1) \text{ by}$$

$$T_{\max}(f)=\ell[f],\ f\in\mathcal{D}(T_{\max}):=\Delta.$$

It follows from the classical theory (see [55], Chapter V) that $T_{\min} = T^{\star}_{\max}$ is the minimal operator with deficiency index $(3,3)$ in $L^2(-1,1)$; the GKN theory implies that there exist self-adjoint extensions of $T^{\star}_{\max}$ in $L^2(-1,1)$ but none of these can have $\{P_{n,A}\}_{n=0}^{\infty}$ as eigenfunctions since their orthogonality lives in $L^2_{\mu}[-1,1]$. So we seek a self-adjoint operator T in $L^2_{\mu}[-1,1]$ generated by $\ell[\cdot]$ that has $\{P_{n,A}\}_{n=0}^{\infty}$ as eigenfunctions. The GKN theory does not directly apply so we need to find properties of $f \in \Delta$.

For $f,\, g \in \Delta$ and $[a,b] \subset (-1,1)$, *Green's formula* is given by

$$\int_a^b \{\ell[f](x)\,\bar{g}(x) - f(x)\,\ell[\bar{g}](x)\}\,dx = [f,g]\Big|_a^b,$$

where $[f,g](\cdot)$ is the skew-symmetric sesquilinear form defined by

$$[f,g](x) := \left\{\left(\left(1-x^2\right)^2 f''(x)\right)' - \left(8+4A\left(1-x^2\right)\right) f'(x)\right\}\bar{g}(x)$$

$$-\left\{\left(\left(1-x^2\right)^2 \bar{g}''(x)\right)' - \left(8+4A\left(1-x^2\right)\right) \bar{g}'(x)\right\} f(x)$$

$$-\left(1-x^2\right)^2 f''(x)\,\bar{g}'(x) + \left(1-x^2\right)^2 \bar{g}''(x)\,f'(x),$$

where $x \in (-1,1)$, and *Dirichlet's formula,* given by

$$\int_a^b \ell[f](x)\,\bar{g}(x)\,dx$$

$$= \int_a^b \left\{\left(1-x^2\right)^2 f''(x)\,\bar{g}''(x) + \left(8+4A\left(1-x^2\right)\right) f'(x)\,\bar{g}'(x) + kf(x)\,\bar{g}(x)\right\} dx$$

$$-\left(1-x^2\right)^2 f''(x)\,\bar{g}'(x)\Big|_a^b + \left\{((1-x^2)^2 f''(x))' - \left(8+4A\left(1-x^2\right)\right) f'(x)\right\}\bar{g}(x)\Big|_a^b.$$

Of particular importance later will be *Dirichlet's formula* when $f = g$:

$$[f, f](x) := \int_a^b \ell[f](x)\,\bar{f}(x)\,dx$$

$$= \int_a^b \left\{ \left(1 - x^2\right)^2 |f''(x)|^2 + \left(8 + 4A\left(1 - x^2\right)\right) |f'(x)|^2 + k\,|f(x)|^2 \right\} dx$$

$$- \left(1 - x^2\right)^2 f''(x)\,\bar{f}'(x)\Big|_a^b + \left\{ \left(\left(1 - x^2\right)^2 f''(x)\right)' - \left(8 + 4A\left(1 - x^2\right)\right) f'(x) \right\} \bar{f}(x)\Big|_a^b.$$

From the definition of Δ, we see that the limits

$$\lim_{x \to \pm 1} [f, g](x)$$

exist and are finite, for all $f,\ g \in \Delta$. Note also that the function 1 and for all $f \in \Delta$

$$[f, 1](x) := \left(\left(1 - x^2\right)^2 f''(x)\right)' - \left(8 + 4A\left(1 - x^2\right)\right) f'(x)\,,\ x \in (-1, 1)\,. \qquad (4.2.2)$$

The main result of this section is the following theorem, proved in [21] and [17], and it contains a list of the properties of the maximal domain Δ.

Theorem 4.2.1. (Properties of $\mathbf{f} \in \Delta$**)** Let $f,\ g \in \Delta$. Then

- (i) $f',\ f'' \in L^2(-1, 1)\,;$
- (ii) $f,\ f' \in AC[-1, 1]$ in particular;

  $f(\pm 1) := \lim_{x \to \pm 1} f(x)$ and $f'(\pm 1) := \lim_{x \to \pm 1} f'(x)$ exist and are finite;
- (iii) $\lim_{x \to \pm 1} \left(1 - x^2\right)^2 f''(x)\,\bar{g}'(x) = 0$
- (iv) $\lim_{x \to \pm 1} \left(\left(1 - x^2\right)^2 f''(x)\right)' = 0$
- (v) $\lim_{x \to \pm 1} [f, g](x) = 8\left(f(\pm 1)\,\bar{g}'(\pm 1) - f'(\pm 1)\,\bar{g}(\pm 1)\right).$

In particular, we note that $\Delta \subset L^2_\mu[-1, 1]\,.$

In the next section, we will define a self-adjoint operator T in $L^2_\mu[-1, 1]$ having the Legendre type polynomials as eigenfunctions.

4.3 Self-Adjoint Legendre Type Operator in $L^2_\mu[-1,1]$

We will study the operator $T : \mathcal{D}(T) \subseteq L^2_\mu[-1,1] \to L^2_\mu[-1,1]$ given by

$$T[f](x) := \begin{cases} -8Af'(-1) + kf(-1) & if \quad x = -1 \\ \ell[f](x) & if \quad -1 < x < 1 \\ 8Af'(+1) + kf(+1) & if \quad x = 1, \end{cases}$$

$$f \in \mathcal{D}(T) := \Delta.$$

The proof of the following theorem can be found in [21].

Theorem 4.3.1. The operator T is self-adjoint in $L^2_\mu[-1,1]$.

Theorem 4.3.2. The operator T is bounded below in $L^2_\mu[-1,1]$ by kI, where I is the identity operator in $L^2_\mu[-1,1]$; i.e.,

$$(Tf,f)_\mu \geq k(f,f)_\mu \;\; \forall \; f \in \mathcal{D}(T).$$

Proof. Let f, $g \in \mathcal{D}(T)$. First, notice in light of (4.2.2), that *Green's formula* may be written as:

$$\int_{-1}^{1} \ell[f](x)\,\bar{g}(x)\,dx = \lim_{x\to 1}\left\{[f,1](x)\,\bar{g}(x) - \overline{[g,1]}(x)\,\bar{g}(x)\right\}$$

$$+ \lim_{x\to 1}\left\{-\left(1-x^2\right)^2 f''(x)\,\bar{g}'(x) + \left(1-x^2\right)^2 \bar{g}''(x)\,f'(x)\right\}$$

$$- \lim_{x\to -1}\left\{[f,1](x)\,\bar{g}(x) - \overline{[g,1]}(x)\,\bar{g}(x) - \left(1-x^2\right)^2 f''(x)\,\bar{g}'(x)\right\}$$

$$- \lim_{x\to -1}\left\{\left(1-x^2\right)^2 \bar{g}''(x)\,f'(x)\right\} + \int_{-1}^{1} \overline{\ell[g]}(x)\,f(x)\,dx,$$

where we have written

$$[\bar{g},1](x) = \overline{[g,1]}(x),$$

since the coefficient of $\ell[\cdot]$ are real-valued on $(-1,1)$. By Theorem 4.2.1., all eight terms in the above limits have individual limits; in fact, we can infer Theorem 4.2.1. that the above equation may be simplified to:

$$\int_{-1}^{1} \ell[f](x)\,\bar{g}(x)\,dx = [f,1](1)\,\bar{g}(1) - \overline{[g,1]}(1)\,f(1) - [f,1](-1)\,\bar{g}(-1)$$

$$+\overline{[g,1]}(-1)\,f(-1) + \int_{-1}^{1} \overline{\ell[g]}(x)\,f(x)\,dx.$$

Consequently,

$$(T[f],g)_{\mu} = \frac{T[f](1)\,\bar{g}(1)}{2} + \frac{A}{2}\int_{-1}^{1} \ell[f](x)\,\bar{g}(x)\,dx + \frac{T[f](-1)\,\bar{g}(-1)}{2} \tag{4.3.1}$$

$$= -\frac{A[f,1](1)\,\bar{g}(1)}{2} + \frac{kf(1)\,\bar{g}(1)}{2} + \frac{A}{2}\Big\{[f,1](1)\,\bar{g}(1) - \overline{[g,1]}(1)\,f(1)$$

$$-[f,1](-1)\,\bar{g}(-1) + \overline{[g,1]}(-1)\,f(-1) + \int_{-1}^{1} \overline{\ell[g]}(x)\,f(x)\,dx\Big\}$$

$$+\frac{A[f,1](-1)\,\bar{g}(-1)}{2} + \frac{kf(-1)\,\bar{g}(-1)}{2}.$$

Now, from *Dirichlet's formula* and Theorem 4.2.1. (iii), we see that

$$\int_{-1}^{1} \overline{\ell[g]}(x)\,f(x)\,dx = \overline{[g,1]}(1)\,f(1) - \overline{[g,1]}(-1)\,f(-1)$$

$$+\int_{-1}^{1}\Big\{\left(1-x^2\right)^2 f''(x)\,\bar{g}''(x) + \left(8+4A\left(1-x^2\right)\right) f'(x)\,\bar{g}'(x) + kf(x)\,\bar{g}(x)\Big\}\,dx$$

Combining this with equation (4.3.1) yields the identity

$$(T[f],g)_{\mu} = \frac{A}{2}\int_{-1}^{1}\Big\{\left(1-x^2\right)^2 f''(x)\,\bar{g}''(x) + \left(8+4A\left(1-x^2\right)\right) f'(x)\,\bar{g}'(x)\Big\}\,dx$$

$$+k(f,g)_{\mu},\ \text{ valid for all } f,g \in \mathcal{D}(T).$$

In particular, since

$$\left(1-x^2\right)^2\left|f''(x)\right|^2+\left(8+4A\left(1-x^2\right)\right)\left|f'(x)\right|^2\geq 0 \text{ on } (-1,1),$$

we have:

$$\begin{aligned}(T[f],f)_\mu &= \frac{A}{2}\int_{-1}^{1}\left\{\left(1-x^2\right)^2\left|f''(x)\right|^2+\left(8+4A\left(1-x^2\right)\right)\left|f'(x)\right|^2\right\}dx\\ &\quad +k(f,f)_\mu\\ &\geq k(f,f)_\mu .\end{aligned}$$

Hence, $T[\cdot]$ is bounded below by kI in $L^2_\mu[-1,1]$. This completes the proof. □

Note that, from Theorem 4.3.2., we see that the left-definite theory discussed in Chapter 3 can be applied to T. As an immediate consequence of Theorem 4.3.1. and Theorem 4.3.2., it can be shown that the spectrum of T is discrete and bounded below. In fact, the spectrum of T is known explicitly and is given in the next theorem. The proof can be found in [21].

Theorem 4.3.3.

(i) The Legendre type polynomials $\{P_{n,A}\}_{n=0}^{\infty}$ form a complete set of eigenfunctions of T in $L^2_\mu[-1,1]$.

(ii) The spectrum of T in $L^2_\mu[-1,1]$ is simple, discrete, and bounded below. In particular, $\sigma(T)=\{n(n+1)(n^2+n+4A-2)+k\,|n=0,\ 1,\ldots\}$.

CHAPTER FIVE

The Integral Power of the Legendre Type Differential Expression

5.1 Introduction

In [53], Littlejohn and Wellman developed a general abstract left-definite theory for a self-adjoint, bounded below operator A in a Hilbert space $(H, (\cdot,\cdot))$. More specifically, they construct a continuum of unique Hilbert spaces $\{(H_r, (\cdot,\cdot)_r)\}_{r>0}$ and, for each $r > 0$, a unique self-adjoint restriction A_r of A in H_r. the Hilbert space H_r is called the r^{th} *left-definite Hilbert space* associated with the pair (H, A) and the operator A_r is called the r^{th} *left-definite operator* associated with (H, A).

We apply this left-definite theory to the self-adjoint Legendre type differential operator generated by the fourth-order formally symmetric Legendre type differential expression

$$\begin{aligned}\ell[y](x) := \left(1-x^2\right)^2 y^{(4)}(x) - 8x\left(1-x^2\right)y^{(3)}(x) - (4A+12)\left(1-x^2\right)y''(x) \\ + 8Axy'(x) + ky(x).\end{aligned}$$

Where, the numbers A and k are, respectively, fixed positive and nonnegative parameters with $(x \in (-1,1))$. Since $\ell[\cdot]$ can be written as

$$\ell[y](x) := \left(\left(1-x^2\right)y''(x)\right)'' - \left(\left(8+4A\left(1-x^2\right)\right)y'(x)\right)' + ky(x). \tag{5.1.1}$$

we see that $\ell[\cdot]$ is formally symmetric. Recently in [24], Everitt, Littlejohn and Tuncer showed that if $\ell[\cdot]$ is Lagrange symmetric and has sufficiently smooth coefficients, then composite powers $\ell^j[\cdot]$ of $\ell[\cdot]$ are also Lagrange symmetric, for any $j \in \{1, 2, \ldots, m\}$.

Even though the theory obtained in [53] guarantees the existence of a continuum of left-definite spaces $\{H_r\}_{r>0}$ and left-definite operators $\{A_r\}_{r>0}$, we can only effectively determine these spaces and operators in this Legendre type situation for

$r \in \mathbb{N}$. The key to obtaining these explicit characterizations of $\{H_r\}_{r\in\mathbb{N}}$ and $\{A_r\}_{r\in\mathbb{N}}$ is in obtaining the explicit Lagrangian symmetric form for each integral power $\ell^r[\cdot]$ of the Legendre type differential expression $\ell[\cdot]$ given in (5.1.1). In turn, the key to obtaining these integral powers is a remarkable, and yet somewhat mysterious, combinatorial identity involving a function that can be viewed as a generating function for these integral powers of $\ell[\cdot]$. In our discussion of the combinatorics of these integral powers of $\ell[\cdot]$, we introduce two double sequences $\{a_j(n,k)\}$ and $\{b_j(n,k)\}$ of real numbers that we call the *Legendre type-Stirling numbers.*

In Section 5.2, we review and develop further properties of the Legendre type polynomials. In Section 5.3, we determine the Lagrangian symmetric form of each integral composite power of the fourth-order Legendre type differential expression using some new combinatorial identities. In Section 5.4, we derive the formulas for the coefficients $\{a_j(n,k)\}$ and $\{b_j(n,k)\}$. In Section 5.5, we show positivity of the coefficients $\{a_j(n,k)\}$. In Section 5.6, we remark on positivity of the coefficients $\{b_j(n,k)\}$. In the final section, we demonstrate several examples of the coefficients $\{a_j(n,k)\}$ and $\{b_j(n,k)\}$.

5.2 Further Properties of the Legendre Type Polynomials

The Legendre type polynomials $\{P_{m,A}(x)\}$, $m = 0, 1, \ldots,$ and $A > 0$, satisfy the fourth-order differential equation

$$\ell[y] = \lambda_m y, \tag{5.2.1}$$

where $\lambda_m := m(m+1)(m^2+m+4A-2)+k$, and k is a fixed, non-negative constant, and hence, they are eigenfunctions of $\ell[\cdot]$. An explicit formula for these polynomials is

$$P_{m,A}(x) = \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^k (2m-2k)! \left(A + \frac{1}{2}m(m-1) + 2k\right) x^{m-2k}}{A 2^m k! (m-k)! (m-2k)!},$$

where $\left[\frac{m}{2}\right]$ denotes the greatest integer less than or equal to $\frac{m}{2}$. Section 1.5 of this thesis contains a list of other formulas for the Legendre type polynomials. They also satisfy the recurrence relation

$$P_{m,A}(x) = \left(A + \frac{1}{2}m(m+1)\right) P_m(x) - xP_m'(x), \tag{5.2.2}$$

where $\{P_m\}_{m=0}^{\infty}$ are the classical Legendre polynomials, defined by

$$P_m(x) = \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{(-1)^k (2m-2k)! x^{m-2k}}{2^m k! (m-k)! (m-2k)!}.$$

Since

$$P_m(1) = 1, \ \ P_m(-1) = (-1)^m$$

and

$$\left(1-x^2\right) P_m''(x) - 2xP_m'(x) + m(m+1) P_m(x) = 0,$$

we see that

$$P_m'(1) = \frac{m(m+1)}{2} \tag{5.2.3}$$

and

$$P_m'(-1) = \frac{(-1)^{m+1} m(m+1)}{2}. \tag{5.2.4}$$

Now, from (5.2.2), we have

$$\begin{aligned} P_{m,A}(1) &= \left(A + \frac{1}{2}m(m+1)\right) P_m(1) - P_m'(1) \\ &= A + \frac{1}{2}m(m+1) - \frac{1}{2}m(m+1) \\ &= A. \end{aligned} \tag{5.2.5}$$

Similarly,

$$\begin{aligned} P_{m,A}(-1) &= \left(A + \frac{1}{2}m(m+1)\right) P_m(-1) - P_m'(-1) \\ &= (-1)^m \left(A + \frac{1}{2}m(m+1)\right) - \frac{(-1)^{m+1} m(m+1)}{2} \\ &= (-1)^m A. \end{aligned} \tag{5.2.6}$$

We now calculate $P_{m,A}^{(j)}(x)$. From (5.2.2), it follows that

$$P_{m,A}^{(j)}(x)=\left(A+\frac{1}{2}m(m+1)\right)P_{m}^{(j)}(x)-\left(xP_{m}'(x)\right)^{(j)},$$

Since

$$\left(xP_{m}'(x)\right)^{(j)}=xP_{m}^{(j+1)}(x)+jP_{m}^{(j)}(x),$$

we see that

$$P_{m,A}^{(j)}(x)=\left(A+\frac{1}{2}m(m+1)\right)P_{m}^{(j)}(x)-xP_{m}^{(j+1)}(x)-jP_{m}^{(j)}(x), \tag{5.2.7}$$

Moreover,

$$P_{m}^{(j)}(x)=\frac{(m+j)!}{2^{j}m!}P_{m-j}^{(j,j)}(x),$$

where $P_{m}^{(j,j)}(x)$ is the Gegenbauer polynomial of degree m, $m\in\mathbb{N}_0$, defined by

$$P_{m}^{(j,j)}(x)=\frac{k_m(j)}{2^m}\sum_{r=0}^{m}\binom{m+j}{m-r}\binom{m+j}{r}(x-1)^r(x+1)^{m-r},$$

where

$$k_m(j)=\frac{(2m+2j+1)^{1/2}\left((m+2j)!\right)^{1/2}}{2^{(2j+1)/2}(m+j)!}\quad (m,\ j\in\mathbb{N}_0);$$

see ([62], page 263, (3)). We refer the reader to [62] for various properties of the Legendre and the Gegenbauer polynomials. We write

$$P_{m}^{(0,0)}(x)=P_m(x).$$

In particular,

$$P_{m}^{(j,j)}(1)=\frac{(m+j)!}{m!j!}\text{ and }P_{m}^{(j,j)}(-1)=\frac{(-1)^m(m+j)!}{m!j!},$$

hence,

$$P_{m}^{(j)}(1)=\frac{(m+j)!}{2^{j}m!}P_{m-j}^{(j,j)}(1)=\frac{(m+j)!}{2^{j}j!(m-j)!}\text{ and}$$

$$P_{m}^{(j)}(-1)=\frac{(m+j)!}{2^{j}m!}P_{m-j}^{(j,j)}(-1)=\frac{(-1)^{m-j}(m+j)!}{2^{j}j!(m-j)!}.$$

From (5.2.7) and the above calculations,

$$P_{m,A}^{(j)}(1) = \left(A + \frac{1}{2}m(m+1)\right) P_m^{(j)}(1) - P_m^{(j+1)}(1) - jP_m^{(j)}(1) \tag{5.2.8}$$

$$= \frac{\left(A + \frac{1}{2}m(m+1)\right)(m+j)!}{2^j j!\,(m-j)!} - \frac{(m+j+1)!}{2^{j+1}(j+1)!\,(m-j-1)!} - \frac{j\,(m+j)!}{2^j j!\,(m-j)!}$$

$$= \frac{(m+j)!\left(2(j+1)\left(A + \frac{1}{2}m(m+1)\right) - (m-j)(m+j+1) - 2j(j+1)\right)}{2^{j+1}(j+1)!\,(m-j)!}$$

$$= \frac{(m+j)!\,(2Aj + m^2 j + mj + 2A - j^2 - j)}{2^{j+1}(j+1)!\,(m-j)!}.$$

Also,

$$P_{m,A}^{(j)}(-1) = \left(A + \frac{1}{2}m(m+1)\right) P_m^{(j)}(-1) + P_m^{(j+1)}(-1) - jP_m^{(j)}(-1) \tag{5.2.9}$$

$$= (-1)^{m-j}\left(\frac{(m+j)!\left(A + \frac{1}{2}m(m+1)\right) - j\,(m+j)!}{2^j j!\,(m-j)!} - \frac{(m+j+1)!}{2^{j+1}(j+1)!\,(m-j-1)!}\right)$$

$$= \frac{(-1)^{m-j}(m+j)!\,(2Aj + m^2 j + mj + 2A - j^2 - j)}{2^{j+1}(j+1)!\,(m-j)!}.$$

Moreover, we know that

$$P_{m,A}(x) = \left(A + \frac{1}{2}m(m+1)\right) P_m(x) - xP_m'(x), \text{ and} \tag{5.2.10}$$

$$xP_m'(x) = mP_m(x) + \sum_{k=0}^{\left[\frac{m-2}{2}\right]} (2m - 4k - 3)\, P_{m-2k-2}(x), \tag{5.2.11}$$

see ([2], page 156) so that

$$P_{m,A}(x) = \left(A + \frac{1}{2}m(m+1)\right)P_m(x) - mP_m(x) - \sum_{k=0}^{\left[\frac{m-2}{2}\right]} (2m-4k-3)\,P_{m-2k-2}(x),$$

equivalently,

$$P_{m,A}(x) = \left(A + \frac{1}{2}m(m-1)\right)P_m(x) - \sum_{k=0}^{\left[\frac{m-2}{2}\right]} (2m-4k-3)\,P_{m-2k-2}(x). \quad (5.2.12)$$

Hence, for $j = 1, 2, 3, \ldots$, we see that

$$P_{m,A}^{(j)}(x) = \left(A + \frac{1}{2}m(m-1)\right)P_m^{(j)}(x) - \sum_{k=0}^{\left[\frac{m-2}{2}\right]} (2m-4k-3)\,P_{m-2k-2}^{(j)}(x). \quad (5.2.13)$$

5.3 Combinatorics of the Legendre Type Differential Expression

Let $\ell[\cdot]$ be defined as in (5.1.1); Since $\ell^n[\cdot]$ is necessarily Lagrangian symmetric for any $n \in \mathbb{N}$, we know that $\ell^n[\cdot]$ has the following form.

$$\ell^n[y](x) = \sum_{j=0}^{2n} (-1)^j \left(\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) y^{(j)}(x)\right)^{(j)}. \quad (5.3.1)$$

We seek to find the coefficients $\{a_j(n,k)\}_{j=0}^{2n}$ and $\{b_j(n,k)\}_{j=0}^{2n}$ for each $n \in \mathbb{N}$. We define $b_0(n,k) = 0$ for each $n \in \mathbb{N}$. (In fact, as we see later in this section, $b_0(n,k) = 0$ for each $n \in \mathbb{N}$.)

Recall that the Legendre type polynomials $\{P_{m,A}\}_{m=0}^{\infty}$ are orthogonal on $[-1,1]$ with respect to

$$w(x) = \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1) + \frac{A}{2},$$

where δ is Dirac's $\delta-$function and $A > 0$ is the parameter in the differential equation $\ell[\cdot]$. In fact, for $m, r \in \mathbb{N}_0$,

$$\int_{[-1,1]} P_{m,A}(x)\,P_{r,A}(x)\,w(x)\,dx = \frac{A\left(A + \frac{1}{2}m(m-1)\right)\left(A + \frac{1}{2}(m+1)(m+2)\right)}{(2m+1)}\delta_{mr},$$

Now, on the one hand,

$$\int\limits_{[-1,1]} \ell^n \left[P_{m,A}(x)\right] P_{r,A}(x)\, w(x)\, dx = \lambda_m^n \int\limits_{[-1,1]} P_{m,A}(x)\, P_{r,A}(x)\, w(x)\, dx.$$

But by the previous relation, this becomes

$$\frac{\lambda_m^n A \left(A + \frac{1}{2} m (m-1)\right) \left(A + \frac{1}{2}(m+1)(m+2)\right)}{(2m+1)} \delta_{mr}\ . \tag{5.3.2}$$

However, on the other hand, by definition of $w(x)$, we also have that

$$\int\limits_{[-1,1]} \ell^n \left[P_{m,A}(x)\right] P_{r,A}(x)\, w(x)\, dx$$

$$= \frac{1}{2}\ell^n \left[P_{m,A}\right](1)\, P_{r,A}(1) + \frac{1}{2}\ell^n \left[P_{m,A}\right](-1)\, P_{r,A}(-1) + \frac{A}{2}\int\limits_{-1}^{1} \ell^n \left[P_{m,A}(x)\right] P_{r,A}(x)\, dx \tag{5.3.3}$$

$$= \frac{1}{2}\lambda_m^n \ell^n \left[P_{m,A}\right](1)\, P_{r,A}(1) + \frac{1}{2}\lambda_m^n \ell^n \left[P_{m,A}\right](-1)\, P_{r,A}(-1)$$

$$+ \frac{A}{2}\int\limits_{-1}^{1} \ell^n \left[P_{m,A}(x)\right] P_{r,A}(x)\, dx.$$

Now, from (5.2.5) and (5.2.6), (5.3.3) becomes

$$\int\limits_{[-1,1]} \ell^n \left[P_{m,A}(x)\right] P_{r,A}(x)\, w(x)\, dx$$

$$= \frac{1}{2}\lambda_m^n A^2 + \frac{1}{2}\lambda_m^n (-1)^{m+r} A^2 + \frac{A}{2}\int\limits_{-1}^{1} \ell^n \left[P_{m,A}(x)\right] P_{r,A}(x)\, dx$$

$$= \frac{1}{2}\lambda_m^n A^2 \left((-1)^{m+r} + 1\right) + \frac{A}{2}\int\limits_{-1}^{1} \ell^n \left[P_{m,A}(x)\right] P_{r,A}(x)\, dx.$$

We now calculate

$$I := \int_{-1}^{1} \ell^n \left[P_{m,A}(x)\right] P_{r,A}(x)\, dx;$$

by integration by parts and we may well assume without loss of generality that $r \leq m$. First, from (5.3.1), we see that I is the following sum:

$$\sum_{j=0}^{2n} (-1)^j \int_{-1}^{1} \left(\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(j)}(x)\right)^{(j)} P_{r,A}(x)\, dx,$$

Let

$$u = P_{r,A}(x), \;\; du = P'_{r,A}(x)\, dx \text{ and}$$

$$dv = \left(\left(a_j(n,k)(1-x^2)^j + b_j(n,k)(1-x^2)^{j-1}\right) P_{m,A}^{(j)}(x)\right)^{(j)} dx,$$

$$v = \left(\left(a_j(n,k)(1-x^2)^j + b_j(n,k)(1-x^2)^{j-1}\right) P_{m,A}^{(j)}(x)\right)^{(j-1)}.$$

We also need to compute $v \Big|_{x=\pm 1}$ in order to complete this integration by parts. Recall that

$$D^n\left(1-x^2\right)^n = (-1)^n 2^n n! P_n(x) \tag{5.3.4}$$

where $P_n(x)$ is the n^{th} degree Legendre polynomial. Expanding, we see that

$$v = \left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right)^{(j-1)} P_{m,A}^{(j)}(x)$$

$$+ \binom{j-1}{1}\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right)^{(j-2)} P_{m,A}^{(j+1)}(x)$$

$$+ \binom{j-1}{2}\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right)^{(j-3)} P_{m,A}^{(j+2)}(x)$$

$$+ \cdots + \left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(2j-1)}(x).$$

Equivalently,

$$v = \left(a_j(n,k)\, D^{j-1}\left(1-x^2\right)^j + b_j(n,k)\, D^{j-1}\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(j)}(x)$$

$$+ \binom{j-1}{1}\left(a_j(n,k)\, D^{j-2}\left(1-x^2\right)^j + b_j(n,k)\, D^{j-2}\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(j+1)}(x)$$

$$+ \binom{j-1}{2}\left(a_j(n,k)\, D^{j-3}\left(1-x^2\right)^j + b_j(n,k)\, D^{j-3}\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(j+2)}(x)$$

$$+ \cdots + \left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(2j-1)}(x).$$

Since $D^k\left(1-x^2\right)^j \Big|_{x=\pm 1} = 0$ if $k < j$, we see that from (5.3.4),

$$v = b_j(n,k)\,(-1)^{j-1}\, 2^{j-1}\,(j-1)! P_{j-1}(\pm 1)\, P_{m,A}^{(j)}(\pm 1) \qquad (j \geq 1)$$

$$= \begin{cases} b_j(n,k)\,(-1)^{j-1}\, 2^{j-1}\,(j-1)! P_{j-1}(+1)\, P_{m,A}^{(j)}(+1) & x = 1 \\ b_j(n,k)\,(-1)^{j-1}\, 2^{j-1}\,(j-1)! P_{j-1}(-1)\, P_{m,A}^{(j)}(-1) & x = -1. \end{cases}$$

Hence, from (5.2.8) and (5.2.9),

$$v(1) = b_j(n,k)\,(-1)^{j-1}\, 2^{j-1}\,(j-1)! P_{m,A}^{(j)}(+1)$$

$$= \frac{(-1)^{j-1}\, 2^{j-1}\,(j-1)!\,(m+j)!\,\left(2Aj + m^2 j + mj + 2A - j^2 - j\right)}{2^{j+1}(j+1)!\,(m-j)!} b_j(n,k)$$

$$= \frac{(-1)^{j-1}\,(m+j)!\,\left(2Aj + m^2 j + mj + 2A - j^2 - j\right)}{4j(j+1)\,(m-j)!} b_j(n,k),$$

while

$$v(-1) = b_j(n,k)\, 2^{j-1}(j-1)! P_{m,A}^{(j)}(-1)$$

$$= \frac{2^{j-1}(j-1)!\,(-1)^{m-j}(m+j)!\,(2Aj+m^2j+mj+2A-j^2-j)}{2^{j+1}(j+1)!\,(m-j)!} b_j(n,k)$$

$$= \frac{(-1)^{m-j}(m+j)!\,(2Aj+m^2j+mj+2A-j^2-j)}{4j(j+1)\,(m-j)!} b_j(n,k).$$

Hence,

$$I = \int_{-1}^{1} \ell^n [P_{m,A}(x)]\, P_{r,A}(x)\, dx$$

$$= \sum_{j=0}^{2n} (-1)^j \int_{-1}^{1} \left(\left(a_j(n,k)(1-x^2)^j + b_j(n,k)(1-x^2)^{j-1} \right) P_{m,A}^{(j)}(x) \right)^{(j)} P_{r,A}(x)\, dx$$

$$= \sum_{j=0}^{2n} (-1)^j \Bigg\{ P_{r,A}(x)\, v_j(x) \Big|_{-1}^{1}$$

$$- \int_{-1}^{1} \left(\left(a_j(n,k)(1-x^2)^j + b_j(n,k)(1-x^2)^{j-1} \right) P_{m,A}^{(j)}(x) \right)^{(j-1)} P'_{r,A}(x)\, dx \Bigg\}.$$

We now simplify

$$\int_{-1}^{1} \left(\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1} \right) P_{m,A}^{(j)}(x) \right)^{(j-1)} P'_{r,A}(x)\, dx$$

with

$u = P'_{r,A}(x)$, $du = P''_{r,A}(x)\, dx$ and

$$dv = \left(\left(a_j(n,k)(1-x^2)^j + b_j(n,k)(1-x^2)^{j-1} \right) P_{m,A}^{(j)}(x) \right)^{(j-1)} dx,$$

$$v = \left(\left(a_j(n,k)(1-x^2)^j + b_j(n,k)(1-x^2)^{j-1} \right) P_{m,A}^{(j)}(x) \right)^{(j-2)}.$$

We note that $v(\pm 1) = 0$ and so

$$\int_{-1}^{1} \left(\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(j)}(x)\right)^{(j-1)} P'_{r,A}(x)\,dx$$

$$= -\int_{-1}^{1} \left(\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(j)}(x)\right)^{(j-2)} P''_{r,A}(x)\,dx.$$

Continuing, we see that

$$I = \int_{-1}^{1} \left(\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(j)}(x)\right)^{(j-3)} P'''_{r,A}(x)\,dx$$

$$= \cdots =$$

$$(-1)^{k+1} \int_{-1}^{1} \left(\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(j)}(x)\right)^{(j-k)} \overset{(k)}{P}_{r,A}(x)\,dx.$$

In particular, setting $k = j$, we see that

$$\int_{-1}^{1} \left(\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(j)}(x)\right)^{(j-1)} P'_{r,A}(x)\,dx$$

$$= (-1)^{j+1} \int_{-1}^{1} \left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) P_{m,A}^{(j)}(x)\, \overset{(j)}{P}_{r,A}(x)\,dx.$$

Substituting this into I gives us

$$\int_{-1}^{1} \ell^{n}\left[P_{m,A}(x)\right] P_{r,A}(x)\, dx$$

$$= \sum_{j=0}^{2n} (-1)^{j} P_{r,A}(x)\, v(x) \Big|_{-1}^{1}$$

$$+ \sum_{j=0}^{2n} \int_{-1}^{1} \left(a_{j}(n,k)\left(1-x^{2}\right)^{j} + b_{j}(n,k)\left(1-x^{2}\right)^{j-1}\right) P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x)\, dx,$$

and so, we obtain that

$$\int_{-1}^{1} \ell^{n}\left[P_{m,A}(x)\right] P_{r,A}(x)\, dx$$

$$= \sum_{j=0}^{2n} \frac{-A\,(m+j)!}{4j(j+1)\,(m-j)!} \left(2Aj + m^{2}j + mj + 2A - j^{2} - j\right) b_{j}(n,k)$$

$$+ \sum_{j=0}^{2n} \frac{(-1)^{m+r+j} A\,(m+j)!}{4j(j+1)\,(m-j)!} \left(2Aj + m^{2}j + mj + 2A - j^{2} - j\right) b_{j}(n,k)$$

$$+ \sum_{j=0}^{2n} \int_{-1}^{1} \left(a_{j}(n,k)\left(1-x^{2}\right)^{j} + b_{j}(n,k)\left(1-x^{2}\right)^{j-1}\right) P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x)\, dx.$$

Hence,

$$\frac{A}{2}\int_{-1}^{1}\ell^{n}\left[P_{m,A}(x)\right]P_{r,A}(x)\,dx$$

$$=\sum_{j=0}^{2n}\frac{\left((-1)^{m+r+1}-1\right)A^{2}(m+j)!\,(2Aj+m^{2}j+mj+2A-j^{2}-j)}{8j(j+1)(m-j)!}b_{j}(n,k)$$

$$+\sum_{j=0}^{2n}\int_{-1}^{1}\left(a_{j}(n,k)\left(1-x^{2}\right)^{j}+b_{j}(n,k)\left(1-x^{2}\right)^{j-1}\right)P_{m,A}^{(j)}(x)\,P_{r,A}^{(j)}(x)\,dx.$$

We now calculate

$$(i)\qquad\int_{-1}^{1}P_{m,A}^{(j)}(x)\,P_{r,A}^{(j)}(x)\left(1-x^{2}\right)^{j}dx$$

and

$$(ii)\qquad\int_{-1}^{1}P_{m,A}^{(j)}(x)\,P_{r,A}^{(j)}(x)\left(1-x^{2}\right)^{j-1}dx.$$

Regarding (i), we note that

$$P_{m}^{(j)}(x)=\frac{(m+j)!}{2^{j}m!}P_{m-j}^{(j,j)}(x)$$

(see [62], page 263).

Furthermore, (see [62], page 260),

$$\int_{-1}^{1}P_{k}^{(j,j)}(x)\,P_{m}^{(j,j)}(x)\left(1-x^{2}\right)^{j}dx=\frac{2^{2j+1}(j+k)!\,(j+k)!}{k!\,(2k+2j+1)!\,(k+2j)!}\;\delta_{km}.$$

Hence,

$$\int_{-1}^{1} P_k^{(j)}(x)\, P_m^{(j)}(x) \left(1-x^2\right)^j dx = \frac{2\,(m+j)!}{(m-j)!\,(2m+1)}\ \delta_{km}.$$

We now see that for $0 \leq r \leq m$, from (5.2.13),

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^j dx \tag{5.3.5}$$

$$= \int_{-1}^{1} \left\{ \left(\left(A + \frac{1}{2} m\,(m-1) \right) P_m^{(j)}(x) - \sum_{k=0}^{\left[\frac{m-2}{2}\right]} (2m-4k-3)\, P_{m-2k-2}^{(j)}(x) \right) \cdot \right.$$

$$\left. \left(\left(A + \frac{1}{2} r\,(r-1) \right) P_r^{(j)}(x) - \sum_{k=0}^{\left[\frac{r-2}{2}\right]} (2r-4k-3)\, P_{r-2k-2}^{(j)}(x) \right) \left(1-x^2\right)^j \right\} dx.$$

We now consider the following two cases:

Case 1:

when $r < m$,

(i) If m is even and r is odd, then

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^j dx = 0.$$

(ii) If m and r are both even, then

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^j dx$$

$$= -(2r+1)\left(A+\frac{1}{2}r(r-1)\right)\int_{-1}^{1}\left(P_r^{(j)}(x)\right)^2\left(1-x^2\right)^j dx$$

$$+\sum_{k=0}^{\left[\frac{r-2}{2}\right]}(2r-4k-3)^2\int_{-1}^{1}\left(P_{r-2k-2}^{(j)}(x)\right)^2\left(1-x^2\right)^j dx$$

$$= -\frac{2(2r+1)\left(A+\frac{1}{2}r(r-1)\right)(r+j)!}{(r-j)!\,(2r+1)}+\sum_{k=0}^{\left[\frac{r-2}{2}\right]}\frac{2(2r-4k-3)^2(r-2k-2+j)!}{(r-2k-2-j)!\,(2r-4k-3)}$$

$$= -\frac{2\left(A+\frac{1}{2}r(r-1)\right)(r+j)!}{(r-j)!}+\sum_{k=0}^{\left[\frac{r-2}{2}\right]}\frac{2(2r-4k-3)(r-2k-2+j)!}{(r-2k-2-j)!}.$$

(iii) If m is odd and r is even, then

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x)\left(1-x^2\right)^j dx = 0.$$

(iv) If m and r are both odd, then

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^j dx$$

$$= -(2r+1)\left(A+\frac{1}{2}r(r-1)\right)\int_{-1}^{1}\left(P_r^{(j)}(x)\right)^2\left(1-x^2\right)^j dx$$

$$+\sum_{k=0}^{\left[\frac{r-2}{2}\right]}(2r-4k-3)^2\int_{-1}^{1}\left(P_{r-2k-2}^{(j)}(x)\right)^2\left(1-x^2\right)^j dx$$

$$= -\frac{2(2r+1)\left(A+\frac{1}{2}r(r-1)\right)(r+j)!}{(r-j)!\,(2r+1)}$$

$$+\sum_{k=0}^{\left[\frac{r-2}{2}\right]}\frac{2(2r-4k-3)^2(r-2k-2+j)!}{(r-2k-2-j)!\,(2r-4k-3)}$$

$$= -\frac{2\left(A+\frac{1}{2}r(r-1)\right)(r+j)!}{(r-j)!}$$

$$+\sum_{k=0}^{\left[\frac{r-2}{2}\right]}\frac{2(2r-4k-3)(r-2k-2+j)!}{(r-2k-2-j)!}.$$

Case 2:

when $m = r$, we have already calculated this; i.e.,

$$\int_{-1}^{1} \left(P_{m,A}^{(j)}(x) \right)^2 \left(1 - x^2\right)^j dx$$

$$= \frac{2\left(A + \frac{1}{2} r (r-1)\right)^2 (m+j)!}{(m-j)!\,(2m+1)}$$

$$+ \sum_{k=0}^{\left[\frac{m-2}{2}\right]} \frac{2\,(2m-4k-3)\,(m-2k-2+j)!}{(m-2k-2-j)!}.$$

So to summarize, for $0 \le r \le m$, we see that

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1 - x^2\right)^j dx$$

$$= \begin{cases} \frac{2\left(A+\frac{1}{2}r(r-1)\right)^2 (m+j)!}{(m-j)!(2m+1)} + \sum_{k=0}^{\left[\frac{m-2}{2}\right]} \frac{2(2m-4k-3)(m-2k-2+j)!}{(m-2k-2-j)!} & m = r, \\ -\frac{2\left(A+\frac{1}{2}r(r-1)\right)(r+j)!}{(r-j)!} + \sum_{k=0}^{\left[\frac{r-2}{2}\right]} \frac{2(2r-4k-3)(r-2k-2+j)!}{(r-2k-2-j)!} & \text{if } r < m \text{ and } m,\ r \text{ either both even or both odd,} \\ 0 & \text{if } r < m \text{ and one of } m \text{ and } r \text{ is even and the other is odd .} \end{cases}$$

Regarding (ii), recall that

$$P_{m,A}^{(j)}(x) = \left(A + \frac{1}{2}m(m-1)\right) \sum_{k=0}^{\left[\frac{m-1}{2}\right]} (2m-4k-1)\, P_{m-2k-1}^{(j-1)}(x)$$

$$- \sum_{k=0}^{\left[\frac{m-2}{2}\right]} \sum_{s=0}^{\left[\frac{m-2k-3}{2}\right]} (2m-4k-3)\ (2m-4k-4s-5)\, P_{m-2k-2s-3}^{(j-1)}(x).$$

Hence, for $0 \leq r \leq m$, we obtain

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^{j-1} dx \tag{5.3.6}$$

$$= \int_{-1}^{1} \left\{ \left(\left(A + \frac{1}{2}m(m-1)\right) \sum_{k=0}^{\left[\frac{m-1}{2}\right]} (2m-4k-1)\, P_{m-2k-1}^{(j-1)}(x) \right.\right.$$

$$\left. - \sum_{k=0}^{\left[\frac{m-2}{2}\right]} \sum_{s=0}^{\left[\frac{m-2k-3}{2}\right]} (2m-4k-3)\ (2m-4k-4s-5)\, P_{m-2k-2s-3}^{(j-1)}(x) \right) \cdot$$

$$\left(- \sum_{k=0}^{\left[\frac{r-2}{2}\right]} \sum_{s=0}^{\left[\frac{r-2k-3}{2}\right]} (2r-4k-3)\ (2r-4k-4s-5)\, P_{r-2k-2s-3}^{(j-1)}(x) \right.$$

$$\left.\left. + \left(A + \frac{1}{2}r(r-1)\right) \sum_{k=0}^{\left[\frac{r-1}{2}\right]} (2r-4k-1)\, P_{r-2k-1}^{(j-1)}(x) \right) \left(1-x^2\right)^{j-1} \right\} dx.$$

We now consider the following four cases:

Case 1: If m is even and r is odd, then

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^{j-1} dx = 0.$$

Case 2: If m is odd and r is even, then

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^{j-1} dx = 0.$$

Case 3: If m and r are both even, then, from (5.3.6),

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^{j-1} dx$$

$$= \left(\left(A + \tfrac{1}{2} m\,(m-1)\right) \left(A + \tfrac{1}{2} r\,(r-1)\right) \sum_{k=0}^{\left[\frac{r-1}{2}\right]} (2r-4k-1)^2 \right) P$$

$$- \left(\left(A + \tfrac{1}{2} m\,(m-1)\right) \sum_{k=0}^{\left[\frac{r-4}{2}\right]} (2r-4k-5)^2 \left(2\,(k+1)\,r - (k+1)\,(2k+3)\right) \right) Q$$

$$- \left(\left(A + \tfrac{1}{2} r\,(r-1)\right) \sum_{k=0}^{\left[\frac{r-2}{2}\right]} \tfrac{1}{2}\,(2r-4k-1)^2\,(m-r+2k)\,(m+r-2k-1) \right) P$$

$$+ \left(\sum_{k=1}^{\left[\frac{r-1}{2}\right]} k\,(k+1)\,(2r-4k-1)^2\,(2r-2k-1)\,(2r-2k+1) \right) P,$$

where

$$P = \int_{-1}^{1} \left(P_{r-2k-1}^{(j-1)}(x) \right)^2 \left(1 - x^2\right)^{j-1} dx$$

and

$$Q = \int_{-1}^{1} \left(P_{r-2k-3}^{(j-1)}(x) \right)^2 \left(1 - x^2\right)^{j-1} dx.$$

Since

$$\int_{-1}^{1} P_k^{(j)}(x) \, P_m^{(j)}(x) \left(1 - x^2\right)^j dx = \frac{2\,(m+j)!}{(m-j)!\,(2m+1)} \delta_{km}$$

we find that

$$P = \int_{-1}^{1} \left(P_{r-2k-1}^{(j-1)}(x) \right)^2 \left(1 - x^2\right)^{j-1} dx = \frac{2\,(r+j-2k-2)!}{(r-j-2k)!\,(2r-4k-1)}$$

and

$$Q = \int_{-1}^{1} \left(P_{r-2k-3}^{(j-1)}(x) \right)^2 \left(1 - x^2\right)^{j-1} dx = \frac{2\,(r+j-2k-2)!}{(r-j-2k-2)!\,(2r-4k-5)}.$$

We now use the values of P and Q to simplify

$$\int_{-1}^{1} P_{m,A}^{(j)}(x) \, P_{r,A}^{(j)}(x) \left(1 - x^2\right)^{j-1} dx.$$

After substituting P and Q in the previous integral, we see that

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^{j-1} dx$$

$$= \sum_{k=0}^{\left[\frac{r-1}{2}\right]} \frac{2\left(A+\frac{1}{2}m(m-1)\right)\left(A+\frac{1}{2}r(r-1)\right)(2r-4k-1)^2 (r+j-2k-2)!}{(r-j-2k)!\,(2r-4k-1)}$$

$$- \sum_{k=0}^{\left[\frac{r-4}{2}\right]} \frac{2\left(A+\frac{1}{2}m(m-1)\right)(2r-4k-5)^2 (k+1)(2r-2k-3)(r+j-2k-2)!}{(r-j-2k)!\,(2r-4k-5)}$$

$$- \sum_{k=0}^{\left[\frac{r-2}{2}\right]} \frac{\left(A+\frac{1}{2}r(r-1)\right)(2r-4k-1)^2 (m-r+2k)(m+r-2k-1)(r+j-2k-2)!}{(r-j-2k)!\,(2r-4k-1)}$$

$$+ \sum_{k=1}^{\left[\frac{r-1}{2}\right]} \frac{2k(k+1)(2r-4k-1)^2 (2r-2k-1)(2r-2k+1)(r+j-2k-2)!}{(r-j-2k)!\,(2r-4k-1)}.$$

After some cancellation, the integral

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^{j-1} dx$$

simplifies further.

Eventually, we get

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^{j-1} dx$$

$$= \sum_{k=0}^{\left[\frac{r-1}{2}\right]} \frac{2\left(A+\frac{1}{2}m(m-1)\right)\left(A+\frac{1}{2}r(r-1)\right)(2r-4k-1)(r+j-2k-2)!}{(r-j-2k)!}$$

$$- \sum_{k=0}^{\left[\frac{r-4}{2}\right]} \frac{2\left(A+\frac{1}{2}m(m-1)\right)(k+1)(2r-2k-3)(2r-4k-3)(r+j-2k-4)!}{(r-j-2k)!}$$

$$- \sum_{k=0}^{\left[\frac{r-2}{2}\right]} \frac{\left(A+\frac{1}{2}r(r-1)\right)(2r-4k-1)(m-r+2k)(m+r-2k-1)(r+j-2k-2)!}{(r-j-2k)!}$$

$$+ \sum_{k=1}^{\left[\frac{r-1}{2}\right]} \frac{2k(k+1)(2r-4k-1)(2r-2k-1)(2r-2k+1)(r+j-2k-2)!}{(r-j-2k)!}.$$

Case 4 : If m and r are both odd and $0 \le r \le m$, then

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^{j-1} dx = 0 \text{ for } j > r.$$

Now, from (5.3.6), we have

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x)\left(1-x^2\right)^{j-1} dx$$

$$= \int_{-1}^{1} \left\{ \left(\left(A + \frac{1}{2}m(m-1) \right) \sum_{k=0}^{\left[\frac{m-1}{2}\right]} (2m-4k-1)\, P_{m-2k-1}^{(j-1)}(x) \right.\right.$$

$$\left. - \sum_{k=0}^{\left[\frac{m-2}{2}\right]} \sum_{s=0}^{\left[\frac{m-2k-3}{2}\right]} (2m-4k-3)\ \ (2m-4k-4s-5)\, P_{m-2k-2s-3}^{(j-1)}(x) \right) \cdot$$

$$\left(- \sum_{k=0}^{\left[\frac{r-2}{2}\right]} \sum_{s=0}^{\left[\frac{r-2k-3}{2}\right]} (2r-4k-3)\ \ (2r-4k-4s-5)\, P_{r-2k-2s-3}^{(j-1)}(x) \right.$$

$$\left.\left. + \left(A + \frac{1}{2}r(r-1) \right) \sum_{k=0}^{\left[\frac{r-1}{2}\right]} (2r-4k-1)\, P_{r-2k-1}^{(j-1)}(x) \right) \left(1-x^2\right)^{j-1} \right\} dx.$$

Equivalently,

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x)\, (1-x^2)^{j-1}\, dx$$

$$= \left(A + \tfrac{1}{2} m(m-1)\right) \left(A + \tfrac{1}{2} r(r-1)\right) \sum_{k=0}^{\left[\frac{r-1}{2}\right]} (2r-4k-1)^2\, M$$

$$- \left(A + \tfrac{1}{2} m(m-1)\right) \sum_{k=1}^{\left[\frac{r-1}{2}\right]} (2r-4k-1)^2 (2kr-2k^2-k)\, M$$

$$- \left(A + \tfrac{1}{2} r(r-1)\right) \sum_{k=1}^{\left[\frac{r+1}{2}\right]} \tfrac{1}{2} (2r-4k+3)^2 (m-r+2k-2)(m+r-2k-1)\, N$$

$$+ \sum_{k=1}^{\left[\frac{r-1}{2}\right]} \tfrac{1}{2} (2r-4k-1)^2 (2kr-2k^2-k)(m-r+2k)(m+r-2k-1)\, M.$$

Where

$$M = \int_{-1}^{1} \left(P_{r-2k-1}^{(j-1)}(x)\right)^2 \left(1-x^2\right)^{j-1} dx$$

and

$$N = \int_{-1}^{1} \left(P_{r-2k+1}^{(j-1)}(x)\right)^2 \left(1-x^2\right)^{j-1} dx.$$

In the third sum,

$$\text{let } u = k-1 \text{ so } k = u+1 \text{ and when } k=1,\ u=0$$

and

$$\text{if } k = \left[\frac{r+1}{2}\right],\ \text{then } u = \left[\frac{r+1}{2}\right] - 1 = \left[\frac{r-1}{2}\right].$$

Hence,

$$-\left(A+\frac{1}{2}r\left(r-1\right)\right)\sum_{k=1}^{\left[\frac{r+1}{2}\right]}\frac{1}{2}\left(2r-4k+3\right)^{2}\left(m-r+2k-2\right)\left(m+r-2k-1\right)N$$

$$=-\left(\sum_{k=1}^{\left[\frac{r+1}{2}\right]}\frac{1}{2}\left(2r-4k+3\right)^{2}\left(m-r+2k-2\right)\left(m+r-2k-1\right)\cdot\right.$$

$$\left.\left(A+\frac{1}{2}r\left(r-1\right)\right)\int_{-1}^{1}\left(P_{r-2k+1}^{(j-1)}\left(x\right)\right)^{2}\left(1-x^{2}\right)^{j-1}dx\right)$$

$$=-\left(\sum_{u=0}^{\left[\frac{r-1}{2}\right]}\frac{1}{2}\left(2r-4u-1\right)^{2}\left(m-r+2u\right)\left(m+r-2u-1\right)\cdot\right.$$

$$\left.\left(A+\frac{1}{2}r\left(r-1\right)\right)\int_{-1}^{1}\left(P_{r-2u-1}^{(j-1)}\left(x\right)\right)^{2}\left(1-x^{2}\right)^{j-1}dx\right)$$

$$=-\left(\sum_{k=0}^{\left[\frac{r-1}{2}\right]}\frac{1}{2}\left(2r-4k-1\right)^{2}\left(m-r+2k\right)\left(m+r-2k-1\right)\cdot\right.$$

$$\left.\left(A+\frac{1}{2}r\left(r-1\right)\right)\int_{-1}^{1}\left(P_{r-2k-1}^{(j-1)}\left(x\right)\right)^{2}\left(1-x^{2}\right)^{j-1}dx\right).$$

Hence, when m and r are both odd and $0 \leq r \leq m$,

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x)\, (1-x^2)^{j-1}\, dx$$

$$= \left(A + \tfrac{1}{2}m(m-1)\right)\left(A + \tfrac{1}{2}r(r-1)\right) \sum_{k=0}^{\left[\frac{r-1}{2}\right]} (2r-4k-1)^2 M$$

$$- \left(A + \tfrac{1}{2}m(m-1)\right) \sum_{k=1}^{\left[\frac{r-1}{2}\right]} (2r-4k-1)^2 (2kr-2k^2-k) M$$

$$- \left(A + \tfrac{1}{2}r(r-1)\right) \sum_{k=0}^{\left[\frac{r-1}{2}\right]} \tfrac{1}{2}(2r-4k-1)^2 (m-r+2k)(m+r-2k-1) M$$

$$+ \sum_{k=1}^{\left[\frac{r-1}{2}\right]} \tfrac{1}{2}(2r-4k-1)^2 (2kr-2k^2-k)(m-r+2k)(m+r-2k-1) M.$$

Now suppose that $m = r$; then the third term in the last integral

$$-\left(A + \frac{1}{2}r(r-1)\right) \sum_{k=0}^{\left[\frac{r-1}{2}\right]} \frac{1}{2}(2r-4k-1)^2 (m-r+2k)(m+r-2k-1) M$$

becomes

$$-\left(A + \tfrac{1}{2}m(m-1)\right) \sum_{k=1}^{\left[\frac{r-1}{2}\right]} (2r-4k-1)^2 (2kr-2k^2-k) M.$$

Which is the second term in the last integral and the fourth term in the last integral

$$\sum_{k=1}^{\left[\frac{r-1}{2}\right]} \frac{1}{2}(2r-4k-1)^2 \left(2kr-2k^2-k\right)(m-r+2k)(m+r-2k-1) M$$

simplifies

$$\sum_{k=1}^{\left[\frac{m-1}{2}\right]} (2m-4k-1)^2 \left(2km-2k^2-k\right)\left(2km-2k^2-k\right) M.$$

Furthermore, if m is even i.e., $m = 2p$, then

$$\left[\frac{m-1}{2}\right] = \left[\frac{2p-1}{2}\right] = p - 1 = \frac{m}{2} - 1 = \frac{m-2}{2}.$$

Since

$$\int_{-1}^{1} \left(P_{r-2k-1}^{(j-1)}(x)\right)^2 \left(1-x^2\right)^{j-1} dx = \frac{2\,(r+j-2k-2)!}{(r-j-2k)!\,(2r-4k-1)},$$

we see that when both m and r are odd and $0 \le r \le m$:

$$\int_{-1}^{1} P_{m,A}^{(j)}(x)\, P_{r,A}^{(j)}(x) \left(1-x^2\right)^{j-1} dx$$

$$= \sum_{k=0}^{\left[\frac{r-1}{2}\right]} \frac{2\left(A + \frac{1}{2}m\,(m-1)\right)\left(A + \frac{1}{2}r\,(r-1)\right)(2r-4k-1)\,(r+j-2k-2)!}{(r-j-2k)!}$$

$$- \sum_{k=1}^{\left[\frac{r-1}{2}\right]} \frac{2\left(A + \frac{1}{2}m\,(m-1)\right)(2r-4k-1)\,(2kr-2k^2-k)\,(r+j-2k-2)!}{(r-j-2k)!}$$

$$- \sum_{k=0}^{\left[\frac{r-1}{2}\right]} \frac{\left(A + \frac{1}{2}r\,(r-1)\right)(2r-4k-1)\,(m-r+2k)\,(m+r-2k-1)\,(r+j-2k-2)!}{(r-j-2k)!}$$

$$+ \sum_{k=1}^{\left[\frac{r-1}{2}\right]} \frac{(2r-4k-1)\,(2kr-2k^2-k)\,(m-r+2k)\,(m+r-2k-1)\,(r+j-2k-2)!}{(r-j-2k)!}.$$

Summarizing, we see that for $0 \leq r \leq m$,

$$\frac{\lambda_m^n A\left(A+\frac{1}{2}m\left(m-1\right)\right)\left(A+\frac{1}{2}\left(m+1\right)\left(m+2\right)\right)}{\left(2m+1\right)}\delta_{mr} \tag{$\star$}$$

becomes

$$\frac{1}{2}\left(m\left(m+1\right)\left(m^2+m+4A-2\right)+k\right)^n A^2\left(\left(-1\right)^{m+r}+1\right)$$

$$+\sum_{j=1}^{2n}\frac{\left(\left(-1\right)^{m+r+1}-1\right)A^2\left(m+j\right)!\left(2Aj+m^2j+mj+2A-j^2-j\right)}{8j(j+1)\left(m-j\right)!}b$$

$$+\frac{A}{2}\sum_{j=1}^{2n}a\begin{cases}0 & \text{if } m+r \\ & \text{is odd,} \\ \frac{2\left(A+\frac{1}{2}m(m-1)\right)^2(m+j)!}{(2m+1)(m-j)!}+\sum_{k=0}^{\left[\frac{m-2}{2}\right]}\frac{2(2m-4k-3)(m-2k+j-2)!}{(m-2k-j-2)!} & \text{if } m=r, \\ \frac{-2\left(A+\frac{1}{2}r(r-1)\right)(r+j)!}{(r-j)!}+\sum_{k=0}^{\left[\frac{r-2}{2}\right]}\frac{2(2r-4k-3)(r-2k+j-2)!}{(r-2k-j-2)!} & \text{if } 0\leq r<m, \\ & m+r \text{ is even}\end{cases}$$

$$+\frac{A}{2}\sum_{j=1}^{2n}b\begin{cases}0 & \text{if } m+r \\ & \text{is odd,} \\ & \text{if} \\ \sum_{k=0}^{\left[\frac{r-1}{2}\right]}\frac{2\left(A+\frac{1}{2}m(m-1)\right)\left(A+\frac{1}{2}r(r-1)\right)(2r-4k-1)(r+j-2k-2)!}{(r-j-2k)!} & 0\leq r<m \\ -\sum_{k=0}^{\left[\frac{r-1}{2}\right]}\frac{2\left(A+\frac{1}{2}m(m-1)\right)(2r-4k-1)\left(2kr-2k^2-k\right)(r+j-2k-2)!}{(r-j-2k)!} & \text{and} \\ -\sum_{k=0}^{\left[\frac{r-1}{2}\right]}\frac{\left(A+\frac{1}{2}r(r-1)\right)(2r-4k-1)(m-r+2k)(m+r-2k-1)(r+j-2k-2)!}{(r-j-2k)!} & m+r \\ +\sum_{k=1}^{\left[\frac{r-1}{2}\right]}\frac{(2r-4k-1)\left(2kr-2k^2-k\right)(m-r+2k)(m+r-2k-1)(r+j-2k-2)!}{(r-j-2k)!} & \text{is even,}\end{cases}$$

where $a = a_j(n,k)$ and $b = b_j(n,k)$.

We now simplify the last term in the last sum:

$$\frac{(2r-4k-1)(r+j-2k-2)!}{(r-j-2k)!}\left\{2\left(A+\frac{1}{2}m(m-1)\right)\left(A+\frac{1}{2}r(r-1)\right)\right.$$

$$-2\left(A+\frac{1}{2}m(m-1)\right)\left(2kr-2k^2-k\right)$$

$$-\left(A+\frac{1}{2}r(r-1)\right)(m-r+2k)(m+r-2k-1)$$

$$\left.+\left(2kr-2k^2-k\right)(m-r+2k)(m+r-2k-1)\right\}$$

$$=\frac{(2r-4k-1)(r+j-2k-2)!\left(A+\frac{1}{2}r(r-1)-2kr+2k^2+k\right)}{(r-j-2k)!}C,$$

where

$$C=\left\{\left(A+\frac{1}{2}m(m-1)\right)-(m-r+2k)(m+r-2k-1)\right\}.$$

In particular, when $m=r$, the above term is:

$$\frac{2(2m-4k-1)(m+j-2k-2)!\left(A+\frac{1}{2}m(m-1)-2km+2k^2+k\right)^2}{(m-j-2k)!}.$$

Also,

$$\frac{\lambda_m^n\left(A+\frac{1}{2}m(m-1)\right)\left(A+\frac{1}{2}(m+1)(m+2)\right)}{(2m+1)}-\lambda_m^n A$$

$$=\frac{\lambda_m^n\left(\left(A+\frac{1}{2}m(m-1)\right)\left(A+\frac{1}{2}(m+1)(m+2)\right)-(2m+1)A\right)}{(2m+1)}.$$

Now notice that if $0\le r\le m$ and $m+r$ is odd (so $m\neq r$), the identity $(\star)$ on page 114 yields only $0=0$.

Consequently, we consider $0 \leq r \leq m$ and $m+r$ is even. We now look at the following two cases:

.Case 1 : When $m=r$, the identity $(\star)$

$$\frac{\lambda_m^n A\left(A+\frac{1}{2}m(m-1)\right)\left(A+\frac{1}{2}(m+1)(m+2)\right)}{(2m+1)}\delta_{mr}$$

on page 114 yields

$$\frac{\lambda_m^n\left(A+\frac{1}{2}m(m-1)\right)\left(A+\frac{1}{2}(m+1)(m+2)\right)}{(2m+1)}$$

$$=\lambda_m^n A+\sum_{j=0}^{2n}\frac{\left(A+\frac{1}{2}m(m-1)\right)^2(m+j)!}{(m-j)!(2m+1)}a$$

$$-\sum_{j=1}^{2n}\frac{A(m+j)!(2Aj+m^2j+mj+2A-j^2-j)}{4j(j+1)(m-j)!}b$$

$$+\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{m-2}{2}\right]}\frac{(2m-4k-3)(m-2k+j-2)!}{(m-2k-j-2)!}\ a$$

$$+\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{m-1}{2}\right]}\frac{(2m-4k-1)(m+j-2k-2)!\left(A+\frac{1}{2}m(m-1)-2km+2k^2+k\right)}{(m-j-2k)!}\ b.$$

Equivalently,

$$\frac{\left(m\left(m+1\right)\left(m^{2}+m+4A-2\right)+k\right)^{n}\left(A^{2}+m\left(m-1\right)A\right)}{\left(2m+1\right)}$$

$$+\frac{\left(m\left(m+1\right)\left(m^{2}+m+4A-2\right)+k\right)^{n}m\left(m-1\right)\left(m+1\right)\left(m+2\right)}{4\left(2m+1\right)}$$

$$=-\sum_{j=1}^{2n}\frac{A\left(m+j\right)!\left(2Aj+m^{2}j+mj+2A-j^{2}-j\right)}{4j(j+1)\left(m-j\right)!}b$$

$$+\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{m-1}{2}\right]}\frac{\left(2m-4k-1\right)\left(m+j-2k-2\right)!\left(A+\frac{1}{2}m\left(m-1\right)-2km+2k^{2}+k\right)^{2}}{\left(m-j-2k\right)!}\ b$$

$$+\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{m-2}{2}\right]}\frac{\left(2m-4k-3\right)\left(m-2k+j-2\right)!}{\left(m-2k-j-2\right)!}\ a$$

$$+\sum_{j=0}^{2n}\frac{\left(A+\frac{1}{2}m\left(m-1\right)\right)^{2}\left(m+j\right)!}{\left(m-j\right)!\left(2m+1\right)}a,$$

where $a=a_{j}\left(n,k\right)$ and $b=b_{j}\left(n,k\right).$

Case 2 : $0 \le m < r$, the identity $(\star)$ on page 114 yields

$$0 =$$

$$\left(m\left(m+1\right)\left(m^2+m+4A-2\right)+k\right)^n A^2$$

$$-\sum_{j=1}^{2n} \frac{A^2\left(m+j\right)!\left(2Aj+m^2j+mj+2A-j^2-j\right)}{4j(j+1)\left(m-j\right)!} b$$

$$-A\sum_{j=0}^{2n} \frac{\left(A+\frac{1}{2}r\left(r-1\right)\right)\left(r+j\right)!}{\left(r-j\right)!} a$$

$$+A\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{m-2}{2}\right]} \frac{\left(2r-4k-3\right)\left(r-2k+j-2\right)!}{\left(r-2k-j-2\right)!}\ a$$

$$+\frac{A}{2}\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{r-1}{2}\right]} \left\{ \frac{\left(2r-4k-1\right)\left(r+j-2k-2\right)!\left(A+\frac{1}{2}r\left(r-1\right)-2kr+2k^2+k\right)}{\left(r-j-2k\right)!} \cdot \right.$$

$$\left. \left(2A+m\left(m-1\right)-\left(m-r+2k\right)\left(m+r-2k-1\right)\right)\right\} b$$

$$=0,$$

Equivalently,

$$\lambda_m^n A = \sum_{j=1}^{2n} \frac{A^2\,(m+j)!\,(2Aj+m^2j+mj+2A-j^2-j)}{4j(j+1)\,(m-j)!}\,b$$

$$+\sum_{j=0}^{2n} \frac{\left(A+\frac{1}{2}r\,(r-1)\right)(r+j)!}{(r-j)!}\,a - \sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{r-2}{2}\right]} \frac{(2r-4k-3)\,(r-2k+j-2)!}{(r-2k-j-2)!}\,a$$

$$-\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{r-1}{2}\right]} \left\{ \frac{(2r-4k-1)\,(r+j-2k-2)!\left(A+\frac{1}{2}r\,(r-1)-2kr+2k^2+k\right)}{2\,(r-j-2k)!}\,\cdot\right.$$

$$\left.(2A+m\,(m-1)-(m-r+2k)\,(m+r-2k-1))\right\}b.$$

Summarizing, we see that we get two sets of equations if $0 \leq r \leq m$ and $m+r$ is even.

The first set of equation: when $m=r$,

$$\frac{\lambda_m^n\left(A^2+m\,(m-1)\,A+\frac{1}{4}m\,(m-1)\,(m+1)\,(m+2)\right)}{(2m+1)}$$

$$+\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{m-1}{2}\right]} \frac{(2m-4k-1)\,(m+j-2k-2)!\left(A+\frac{1}{2}m\,(m-1)-2km+2k^2+k\right)^2}{(m-j-2k)!}\;b$$

$$= -\sum_{j=1}^{2n} \frac{A\,(m+j)!\,(2Aj+m^2j+mj+2A-j^2-j)}{4j(j+1)\,(m-j)!}\,b \tag{5.3.7}$$

$$+\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{m-2}{2}\right]} \frac{(2m-4k-3)\,(m-2k+j-2)!}{(m-2k-j-2)!}\;a + \sum_{j=0}^{2n} \frac{\left(A+\frac{1}{2}m\,(m-1)\right)^2(m+j)!}{(m-j)!\,(2m+1)}\,a.$$

The second set of equation: when $0 \leq r \leq m$,

$$\left(m\left(m+1\right)\left(m^2+m+4A-2\right)+k\right)^n A$$

$$=\sum_{j=1}^{2n} \frac{A^2\left(m+j\right)!\left(2Aj+m^2j+mj+2A-j^2-j\right)}{4j(j+1)\left(m-j\right)!} b$$

$$+\sum_{j=0}^{2n} \frac{\left(A+\frac{1}{2}r\left(r-1\right)\right)\left(r+j\right)!}{\left(r-j\right)!} a$$

$$-\sum_{j=1}^{2n} \sum_{k=0}^{\left[\frac{r-2}{2}\right]} \frac{\left(2r-4k-3\right)\left(r-2k+j-2\right)!}{\left(r-2k-j-2\right)!} a \tag{5.3.8}$$

$$-\sum_{j=1}^{2n} \sum_{k=0}^{\left[\frac{r-1}{2}\right]} \left\{ \frac{\left(2r-4k-1\right)\left(r+j-2k-2\right)!\left(A+\frac{1}{2}r\left(r-1\right)-2kr+2k^2+k\right)}{2\left(r-j-2k\right)!} \cdot \right.$$

$$\left. \left(2A+m\left(m-1\right)-\left(m-r+2k\right)\left(m+r-2k-1\right)\right)\right\} b.$$

We now consider the following two cases:

Case 1 : When m is even and $r=0$, (5.3.8) becomes

$$\left(m\left(m+1\right)\left(m^2+m+4A-2\right)+k\right)^n$$

$$=\sum_{j=1}^{2n} \frac{\left(m+j\right)!\left(2Aj+m^2j+mj+2A-j^2-j\right)}{4j(j+1)\left(m-j\right)!} b_j\left(n,k\right)+a_0\left(n,k\right)$$

but $a_0(n,k) = k^n$. Hence, we obtain the following identity:

$$\left(m(m+1)\left(m^2+m+4A-2\right)+k\right)^n - k^n$$

$$= \sum_{j=1}^{2n} \frac{(m+j)!\,(2Aj+m^2j+mj+2A-j^2-j)}{4j(j+1)\,(m-j)!} b_j(n,k).$$

Case 2 : When m is odd and $r=1$, (5.3.8) becomes

$$\left(m(m+1)\left(m^2+m+4A-2\right)+k\right)^n$$

$$= \sum_{j=1}^{2n} \frac{(m+j)!\,(2Aj+m^2j+mj+2A-j^2-j)}{4j(j+1)\,(m-j)!} b_j(n,k)$$

$$+ a_0(n,k) + 2a_1(n,k) - Ab_1(n,k).$$

But

$$a_0(n,k) = k^n,\ \ a_1(n,k) = 2^{3n-1}A^n \text{ and } b_1(n,k) = 2^{3n}A^{n-1}$$

so that $2a_1(n,k) - Ab_1(n,k) = 0.$

Consequently, for all $m \in \mathbb{N}_0$,

$$\sum_{j=1}^{2n} \frac{(m+j)!\,(2Aj+m^2j+mj+2A-j^2-j)}{4j(j+1)\,(m-j)!} b_j(n,k)$$

$$= \left(m(m+1)\left(m^2+m+4A-2\right)+k\right)^n - k^n. \tag{5.3.9}$$

Now, let $r=m$ in (5.3.8). Since

$$2A + m(m-1) - (m-r+2k)(m+r-2k-1)\ \Big|_{r=m}$$

$$= 2A + m(m-1) - 2k(2m-2k-1)$$

$$= 2A + m(m-1) - 4km + 4k^2 + 2k$$

$$= 2\left(A + \frac{m(m-1)}{2} - 2km + 2k^2 + k\right),$$

we see that

$$-\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{r-1}{2}\right]}\left(\frac{(2r-4k-1)(r+j-2k-2)!\left(A+\frac{1}{2}r(r-1)-2kr+2k^2+k\right)}{2(r-j-2k)!}\cdot\right.$$

$$(2A+m(m-1)-(m-r+2k)(m+r-2k-1)))\,b$$

$$=-\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{m-1}{2}\right]}\frac{(2m-4k-1)(m+j-2k-2)!\left(A+\frac{1}{2}m(m-1)-2km+2k^2+k\right)^2 b}{2(r-j-2k)!}.$$

Furthermore, from (5.3.9), we also have

$$(\lambda_m^n-k^n)\,A=\sum_{j=1}^{2n}\frac{A\,(m+j)!\,(2Aj+m^2j+mj+2A-j^2-j)}{4j(j+1)\,(m-j)!}b.$$

Hence, when $r=m$, (5.3.8) yields the following identity

$$\left(A+\frac{1}{2}m(m-1)\right)\sum_{j=1}^{2n}\frac{(m+j)}{(m-j)!}\;a$$

$$=\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{m-2}{2}\right]}\frac{(2m-4k-3)(m-2k+j-2)!}{(m-2k-j-2)!}\;a$$

$$+\sum_{j=1}^{2n}\sum_{k=0}^{\left[\frac{m-1}{2}\right]}\frac{(2m-4k-1)(m+j-2k-2)!\left(A+\frac{1}{2}m(m-1)-2km+2k^2+k\right)^2 b}{(m-j-2k)!}$$

$$=\lambda_m^n A+\lambda_m^n\frac{\left(A^2+(m^2-m)A+\frac{1}{4}m(m+2)(m^2-1)\right)}{(2m+1)}$$

$$-\sum_{j=1}^{2n}\frac{\left(A+\frac{1}{2}m(m-1)\right)^2(m+j)!}{(m-j)!\,(2m+1)}\;a.$$

Therefore, simplifying the above identity using (5.3.7), we obtain

$$\left(\left(A+\frac{1}{2}m\left(m-1\right)\right)+\frac{\left(A+\frac{1}{2}m\left(m-1\right)\right)^{2}}{\left(2m+1\right)}\right)\sum_{j=1}^{2n}\frac{\left(m+j\right)}{\left(m-j\right)!}a \tag{5.3.10}$$

$$=\frac{\lambda_{m}^{n}\left(A^{2}+m\left(m-1\right)A+\frac{1}{4}m\left(m-1\right)\left(m+1\right)\left(m+2\right)\right)}{\left(2m+1\right)}.$$

On the other hand,

$$\left(A+\frac{1}{2}m\left(m-1\right)\right)+\frac{\left(A+\frac{1}{2}m\left(m-1\right)\right)^{2}}{\left(2m+1\right)}$$

$$=\frac{\left(2m+1\right)\left(A+\frac{1}{2}m\left(m-1\right)\right)+\left(A+\frac{1}{2}m\left(m-1\right)\right)^{2}}{\left(2m+1\right)}$$

$$=\frac{A\left(2m+1\right)+\frac{1}{2}m\left(m-1\right)\left(2m+1\right)+A^{2}+m\left(m-1\right)A+\frac{1}{4}m^{2}\left(m-1\right)^{2}}{\left(2m+1\right)}$$

$$=\frac{A^{2}+m\left(m-1\right)A+\frac{1}{4}m\left(m-1\right)\left[m\left(m-1\right)+2\left(2m+1\right)\right]+A\left(2m+1\right)}{\left(2m+1\right)}$$

$$=\frac{A^{2}+m\left(m-1\right)A+\frac{1}{4}\left(m-1\right)m\left(m+1\right)\left(m+2\right)+A\left(2m+1\right)}{\left(2m+1\right)}.$$

Hence, (5.3.10) becomes

$$\left(\frac{A^{2}+m\left(m-1\right)A+\frac{1}{4}\left(m-1\right)m\left(m+1\right)\left(m+2\right)+A\left(2m+1\right)}{\left(2m+1\right)}\right)\sum_{j=1}^{2n}\frac{\left(m+j\right)}{\left(m-j\right)!}a$$

$$=\left(\frac{A^{2}+m\left(m-1\right)A+\frac{1}{4}\left(m-1\right)m\left(m+1\right)\left(m+2\right)+A\left(2m+1\right)}{\left(2m+1\right)}\right)\lambda_{m}^{n}$$

so that

$$\sum_{j=1}^{2n}\frac{\left(m+j\right)}{\left(m-j\right)!}a_{j}\left(n,k\right)=\left(m\left(m+1\right)\left(m^{2}+m+4A-2\right)+k\right)^{n}. \tag{5.3.11}$$

We will see in the following section that the identities in (5.3.9) and (5.3.11) will play key role finding the coefficients $a_{j}\left(n,k\right)$ and $b_{j}\left(n,k\right)$.

Theorem 5.4.1. Suppose $A > 0$, $k \geq 0$ and $n \in \mathbb{N}$. For each $m \in \mathbb{N}_0$, the recurrence relations

$$\sum_{j=1}^{2n} \frac{(m+j)}{(m-j)!} a_j(n,k) = \left(m(m+1)\left(m^2+m+4A-2\right)+k\right)^n \tag{5.4.1}$$

have unique, non-negative solutions $a_j(n,k)$ ($j = 0, 1, \dots, 2n$), independent of m, given explicitly by

$$a_0(n,k) = \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0, \end{cases} \tag{5.4.2}$$

and

$$a_j(n,k) := \begin{cases} a_{n,j} & \text{if } k = 0 \\ \sum_{r=0}^{n-1} \binom{n}{r} a_{n-r,j} k^r & \text{if } k > 0 \end{cases} \qquad (j \in \{1, \dots, 2n\}), \tag{5.4.3}$$

where each $a_{n,j}$ is positive and given by

$$a_{n,j} = \sum_{k=1}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^n(k^2+k+4A-2)^n}{(j+k+1)!(j-k)!}. \tag{5.4.4}$$

For the positivity of $\{a_{n,j}\}$, see the next section.

Proof . From the definition of $a_j(n,k)$ in (5.4.1), we see that

$$a_0(n,k) = k^n,$$

$$a_1(n,k) = \frac{(k+8A)^n - k^n}{2!},$$

$$a_2(n,k) = \frac{(k+24A+24)^n - 3(k+8A)^n + 2k^n}{4!}, \text{ etc;}$$

in general, it is not difficult to see that $a_j(n,k)$ is unique and given by

$$\sum_{l=0}^{j} \frac{(-1)^l}{(2j)!} \left(\binom{2j}{l} - \binom{2j}{l-1} \right) (k + (j-l)(j-l+1)\left((j-l)(j-l+1)+4A-2\right))^n$$

$$= \sum_{l=0}^{j} \frac{(-1)^{l+j}}{(2j)!} \left(\binom{2j}{j-l} - \binom{2j}{j-l-1} \right) (k + l(l+1)\left(l(l+1)+4A-2\right))^n$$

$$= \sum_{l=0}^{j} \sum_{r=0}^{n} \frac{(-1)^{l+j}}{(2j)!} \left(\binom{2j}{j-l} - \binom{2j}{j-l-1} \right) \binom{n}{r} \left((l^2+l)(l^2+l+4A-2)\right)^{n-r} k^r$$

This proves (5.4.2), (5.4.3), and (5.4.4). $\square$

Theorem 5.4.2. Suppose $A > 0$, $k \geq 0$ and $n \in \mathbb{N}$. For each $m \in \mathbb{N}_0$, the recurrence relations

$$\sum_{j=1}^{2n} \frac{(m+j)!\,(2Aj + m^2 j + mj + 2A - j^2 - j)\, b_j\,(n,k)}{4j(j+1)\,(m-j)!} \tag{5.4.5}$$

$$= \left(\left(m^2+m\right)\left(m^2+m+4A-2\right)+k\right)^n - k^n.$$

have unique, non-negative solutions $b_j(n,k)$ ($j = 0, 1, \ldots, 2n$), independent of m, given explicitly by

$$b_0(n,k) = \begin{cases} 0 & \text{if } k = 0 \\ 0 & \text{if } k > 0, \end{cases} \tag{5.4.6}$$

and

$$b_j(n,k) := \begin{cases} b_{n,j} & \text{if } k = 0 \\ \sum_{r=0}^{n-1} \binom{n}{r} b_{n-r,j} k^r & \text{if } k > 0 \end{cases} \qquad (j \in \{1, \ldots, 2n\}), \tag{5.4.7}$$

where each $b_{n,j}$ is positive and given by

$$b_{n,j} = \sum_{k=1}^{j} \frac{(-1)^{k+j}(2k+1)(k^2+k)^n(k^2+k-2+4A)^n\,(2Aj + (j+1)\,(j+k^2+k))}{(j+k+1)!(j-k)!\,(2A+(k-1)\,k)\,(2A+(k+1)\,(k+2))}. \tag{5.4.8}$$

Proof . From the definition of $b_j(n,k)$ in (5.4.5), we see that

$$b_0(n,k) = 0,$$

$$b_1(n,k) = \frac{(8A+k)^n - k^n}{A},$$

$$b_2(n,k) = \frac{(24A+24+k)^n - k^n}{6\,(A+1)} - \frac{(8A+k)^n - k^n}{2A},$$

$$b_3(n,k) = \frac{(48A+120+k)^n}{120\,(A+3)} + \frac{((8A+k)^n - k^n)\,(3A+10)}{40A\,(A+3)} - \frac{(24A+24+k)^n - k^n}{24\,(A+1)},$$

etc; in general, it is not difficult to see that $b_j(n,k)$ is unique and given by

$$\sum_{l=0}^{j}(-1)^l\left(\binom{2j}{l}-\binom{2j}{l-1}\right)\frac{4(k+(j-l)(j-l+1)((j-l)(j-l+1)-2+4A))^n(2Aj+(j+1)(j+(j-l)(j-l+1)))}{(2j)!(2A+(j-l-1)(j-l))(2A+(j-l+1)(j-l+2))}$$

$$= \sum_{l=0}^{j}(-1)^{j+l}\left(\binom{2j}{j-l}-\binom{2j}{j-l-1}\right)\frac{4(k+l(l+1)(l(l+1)-2+4A))^n(2Aj+(j+1)(j+l(l+1)))}{(2j)!(2A+(l-1)l)(2A+(l+1)(l+2))}$$

$$= \sum_{l=0}^{j}\sum_{r=0}^{n}(-1)^{l+j}\left(\binom{2j}{j-l}-\binom{2j}{j-l-1}\right)\binom{n}{r}\frac{4(2Aj+(j+1)(j+l(l+1)))\left((l^2+l)(l^2+l-2+4A)\right)^{n-r}}{(2j)!(2A+(l-1)l)(2A+(l+1)(l+2))}k^r$$

$$= \sum_{r=0}^{n}\sum_{l=0}^{j}(-1)^{l+j}\left(\binom{2j}{j-l}-\binom{2j}{j-l-1}\right)\frac{4\left((l^2+l)(l^2+l-2+4A)\right)^{n-r}(2Aj+(j+1)(j+l(l+1)))}{(2j)!(2A+(l-1)l)(2A+(l+1)(l+2))}\binom{n}{r}k^r$$

$$= \sum_{r=0}^{n}\sum_{l=0}^{j}\frac{(-1)^{l+j}}{(2j)!}\left(\left(\frac{(2j)!(2l+1)}{(j-l)!(j+l+1)!}\right)\right)\frac{4\left((l^2+l)(l^2+l-2+4A)\right)^{n-r}(2Aj+(j+1)(j+l(l+1)))}{(2j)!(2A+(l-1)l)(2A+(l+1)(l+2))}\binom{n}{r}k^r$$

$$= \sum_{r=0}^{n}\binom{n}{r}\sum_{l=0}^{j}\frac{(-1)^{l+j}(2l+1)\left((l^2+l)(l^2+l-2+4A)\right)^{n-r}(2Aj+(j+1)(j+l(l+1)))}{(j-l)!(j+l+1)!(2A+(l-1)l)(2A+(l+1)(l+2))}k^r.$$

This proves (5.4.6), (5.4.7), and (5.4.8). $\square$

We cannot, at this time, prove that $b_j(n,k) > 0$ for $j \in \mathbb{N}$; see section 5.7 below for further information. The evidence is very strong indeed that each $b_j(n,k) > 0$ for $j \in \mathbb{N}$.

5.5 *Positivity of the Coefficients* $a_j(n,k)$

Recall that the coefficients $\{a_j(n,k)\}$ are defined by

$$a_j(n,k) := \begin{cases} a_{n,j} & \text{if } k = 0 \\ \sum_{r=0}^{n-1} \binom{n}{r} a_{n-r,j} k^r & \text{if } k > 0 \end{cases} \qquad (j \in \{1, \dots, 2n\}),$$

where

$$a_{n,j} := \sum_{k=0}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^n (k^2+k+4A-2)^n}{(j-k)!(j+k+1)!}$$

$$= \sum_{k=0}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^n}{(j-k)!(j+k+1)!} \sum_{r=0}^{n} \binom{n}{r} (4A)^{n-r} (k^2+k-2)^r$$

$$= \sum_{r=0}^{n} \binom{n}{r} (4A)^{n-r} \sum_{k=0}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^n (k^2+k-2)^r}{(j-k)!(j+k+1)!}.$$

From this, it is easy to see that each $a_j(n,k) > 0$ if $a_{n,j} > 0$ and this happens if

$$\widetilde{a}_{n,j,r} := \sum_{k=0}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^n (k^2+k-2)^r}{(j-k)!(j+k+1)!} > 0 \quad (r = 0, 1, \dots, n).$$

By expanding $(k^2+k-2)^r$, we see that

$$\widetilde{a}_{n,j,r} = \sum_{m=0}^{r} \binom{r}{m} (-2)^m \left(\sum_{k=0}^{j} (-1)^{k+j} \frac{(2k+1)(k^2+k)^{n+r-m}}{(j-k)!(j+k+1)!} \right) \tag{5.5.1}$$

$$= \sum_{m=0}^{r} \binom{r}{m} (-2)^m PS^{(j)}_{n+r-m}.$$

Recall that the forward difference of a sequence of numbers $\{x_n\}_{n=0}^{\infty}$ is the sequence $\{\Delta x_n\}_{n=0}^{\infty}$ given by

$$\Delta x_n = x_{n+1} - x_n \quad (n \in \mathbb{N}_0).$$

Higher order forward differences are defined recursively by

$$\Delta^r x_n = \Delta(\Delta^{r-1} x_n) = \sum_{m=0}^{r} \binom{r}{m} (-1)^m x_{n+r-m}.$$

With this notation, we see that

$$2^{n+r}\Delta^r\left(\frac{PS_n^{(j)}}{2^n}\right) = \sum_{m=0}^{r}\binom{r}{m}(-1)^m 2^{n+r}\frac{PS_{n+r-m}^{(j)}}{2^{n+r-m}}$$

$$= \sum_{m=0}^{r}\binom{r}{m}(-1)^m \frac{PS_{n+r-m}^{(j)}}{2^{-m}}$$

$$= \sum_{m=0}^{r}\binom{r}{m}(-2)^m PS_{n+r-m}^{(j)}$$

$$= \widetilde{a}_{n,j,r} \text{ by (5.5.1);}$$

consequently, in order to show $a_{n,j} > 0$, it suffices to prove the following result.

Theorem 5.5.1. For $r \in \mathbb{N}_0$, we have

$$\Delta^r\left(\frac{PS_n^{(j)}}{2^n}\right) \geq 0 \quad (n \geq j).$$

In particular, we see that for $n \geq j$,

$$PS_{n+1}^{(j)} - 2PS_n^{(j)} \geq 0$$

$$PS_{n+2}^{(j)} - 4PS_{n+1}^{(j)} + 4PS_n^{(j)} \geq 0$$

$$PS_{n+3}^{(j)} - 6PS_{n+2}^{(j)} + 12PS_{n+1}^{(j)} - 8PS_n^{(j)} \geq 0, \text{ etc.}$$

Proof . Fix $j \in \mathbb{N}$; from the rational generating function

$$\prod_{r=1}^{j} \frac{1}{1-r(r+1)t} = \sum_{n=0}^{\infty} PS_n^{(j)} t^{n-j} \quad \left(|t| < \frac{1}{j(j+1)} \right),$$

for the Legendre-Stirling numbers, replace t by $t/2$ to obtain

$$\sum_{n=j}^{\infty} \frac{PS_n^{(j)}}{2^n} t^n = \frac{t^j}{2^j} \psi_j(t) \quad \left(|t| < \frac{2}{j(j+1)} \right), \tag{5.5.2}$$

where

$$\psi_j(t) := \frac{1}{(1-t)(1-3t)\cdots(1-\frac{j(j+1)}{2}t)}.$$

Now

$$\sum_{n=j}^{\infty} \frac{PS_n^{(j)}}{2^n} t^n = \sum_{n=j-1}^{\infty} \frac{PS_{n+1}^{(j)}}{2^{n+1}} t^{n+1} = \frac{t^j}{2^j} + \sum_{n=j}^{\infty} \frac{PS_{n+1}^{(j)}}{2^{n+1}} t^{n+1}$$

since $PS_j^{(j)} = 1$. Hence, from (5.5.2), we see that

$$\sum_{n=j}^{\infty} \frac{PS_{n+1}^{(j)}}{2^{n+1}} t^n = \frac{t^{j-1}}{2^j} \psi_j(t) - \frac{t^{j-1}}{2^j}, \tag{5.5.3}$$

and consequently

$$\begin{aligned} \sum_{n=j}^{\infty} \Delta \left(\frac{PS_n^{(j)}}{2^n} \right) t^n &= \frac{t^{j-1}}{2^j} \psi_j(t) - \frac{t^{j-1}}{2^j} - \frac{t^j}{2^j} \psi_j(t) \qquad (5.5.4) \\ &= \frac{t^{j-1}}{2^j} [\psi_j(t) - t\psi_j(t) - 1] \\ &= \frac{t^{j-1}}{2^j} [(1-t)\psi_j(t) - 1]. \end{aligned}$$

By comparing powers of t on both sides of (5.5.4), we see that

$$(1-t)\psi_j(t) - 1 = \sum_{n=1}^{\infty} a_n(1) t^n,$$

where each of the coefficients $a_n(1)$ are non-negative as can easily be seen from the Taylor expansion of

$$(1-t)\psi_j(t) - 1.$$

Indeed, the coefficients in this Taylor series are the Cauchy product of the coefficients obtained from the products of the geometric series for

$$\frac{1}{1-t}, \ \frac{1}{1-3t}, \dots, \ \frac{1}{1-\frac{j(j+1)}{2}t},$$

each of which have positive coefficients. It follows, then, from comparing coefficients on both sides of (5.5.4) that

$$\Delta\left(\frac{PS_n^{(j)}}{2^n}\right) \geq 0 \quad (n \geq j).$$

To see that

$$\Delta^2\left(\frac{PS_n^{(j)}}{2^n}\right) \geq 0 \quad (n \geq j), \tag{5.5.5}$$

we first notice that

$$\frac{t^j \psi_j(t)}{2^j} = \sum_{n=j}^{\infty} \frac{PS_n^{(j)}}{2^n} t^n = \sum_{n=j-2}^{\infty} \frac{PS_{n+2}^{(j)}}{2^{n+2}} t^{n+2}$$

$$= \frac{t^j}{2^j} + \frac{PS_{j+1}^{(j)}}{2^{j+1}} t^{j+1} + \sum_{n=j}^{\infty} \frac{PS_{n+2}^{(j)}}{2^{n+2}} t^{n+2}.$$

Thus

$$\sum_{n=j}^{\infty} \frac{PS_{n+2}^{(j)}}{2^{n+2}} t^n = \frac{t^{j-2}}{2^j} \psi_j(t) - \frac{t^{j-2}}{2^j} - \frac{PS_{j+1}^{(j)}}{2^{j+1}} t^{j-1}. \tag{5.5.6}$$

Consequently, from (5.5.2), (5.5.3), and (5.5.6), we see that

$$\sum_{n=j}^{\infty} \Delta^2 \left(\frac{PS_n^{(j)}}{2^n} \right) t^n \tag{5.5.7}$$

$$= \sum_{n=j}^{\infty} \left(\frac{PS_{n+2}^{(j)}}{2^{n+2}} - 2\frac{PS_{n+1}^{(j)}}{2^{n+1}} + \frac{PS_n^{(j)}}{2^n} \right) t^n$$

$$= \frac{t^{j-2}}{2^j}\psi_j(t) - \frac{t^{j-2}}{2^j} - \frac{PS_{j+1}^{(j)}}{2^{j+1}} t^{j-1} - 2\frac{t^{j-1}}{2^j}\psi_j(t) + \frac{2t^{j-1}}{2^j} + \frac{t^j}{2^j}\psi_j(t)$$

$$= \frac{t^{j-2}}{2^j}\left[(1-t)^2\psi_j(t) + \left(2 - \frac{PS_{j+1}^{(j)}}{2} \right) t - 1 \right].$$

Again, by comparing both sides of this identity, we see that

$$(1-t)^2\psi_j(t) + \left(2 - \frac{PS_{j+1}^{(j)}}{2} \right) t - 1 = \sum_{n=2}^{\infty} a_n(2) t^n,$$

where each $a_n(2)$ is non-negative, as can easily be seen from the Taylor series expansion of $(1-t)^2\psi_j(t)$. The inequality in (5.5.5) now follows. From (5.5.4) and (5.5.7), we can generalize to see that, for each $r \in \mathbb{N}$,

$$\sum_{n=j}^{\infty} \Delta^r \left(\frac{PS_n^{(j)}}{2^n} \right) t^n = \frac{t^{j-r}}{2^j}\left[(1-t) r \psi_j(t) + p_{r-1}(t) \right], \tag{5.5.8}$$

where $p_{r-1}(t)$ is a polynomial of degree $\leq r-1$. Moreover, by comparing both sides of (5.5.8), we see that

$$(1-t)^r\psi_j(t) + p_{r-1}(t) = \sum_{n=r}^{\infty} a_n(r) t^n$$

where each $a_n(r) \geq 0$ since the coefficients in the Taylor expansion of $(1-t)^r\psi_j(t)$ are all non-negative. Consequently, it follows that

$$\Delta^r \left(\frac{PS_n^{(j)}}{2^n} \right) \geq 0 \quad (n \geq j).$$

This completes the proof of the theorem. □

5.6 *Positivity of the Coefficients* $b_j(n,k)$

We will see from several examples of $b_j(n,k)$ below that $b_j(n,k)$ is non-negative for each $j \in \mathbb{N}$. For its proof, work still goes on.

5.7 *Examples of the coefficients* $a_j(n,k)$ *and* $b_j(n,k)$

A list of the coefficients $a_j(n,k)$

$a_0(1,k) = k$

$a_1(1,k) = 4A$

$a_2(1,k) = 1$

$a_0(2,k) = k^2$

$a_1(2,k) = 32A^2 + 8Ak$

$a_2(2,k) = 16A^2 + 48A + 2k + 24$

$a_3(2,k) = 8A + 16$

$a_4(2,k) = 1$

$a_0(3,k) = k^3$

$a_1(3,k) = 256A^3 + 96A^2k + 12Ak^2$

$a_2(3,k) = 512A^3 + 1728A^2 + 48A^2k + 1728A + 144Ak + 3k^2 + 576$

$a_3(3,k) = 64A^3 + 864A^2 + 2592A + 24Ak + 48k + 2304$

$a_4(3,k) = 48A^2 + 432A + 3k + 864$

$a_5(3,k) = 12A + 64$

$a_6(3,k) = 1$

A List of the coefficients $b_j(n,k)$

$b_0(1,k) = 0$

$b_1(1,k) = 8$

$b_2(1,k) = 0$

$b_0(2,k) = 0$

$b_1(2,k) = 64A + 16k$

$b_2(2,k) = 64A + 96$

$b_3(2,k) = 16$

$b_4(2,k) = 0$

$b_0(3,k) = 0$

$b_1(3,k) = 512A^2 + 192Ak + 24k^2$

$b_2(3,k) = 2048A^2 + 4608A + 192Ak + 288k + 2304$

$b_3(3,k) = 384A^2 + 3008A + 48k + 4224$

$b_4(3,k) = 192A + 800$

$b_5(3,k) = 24$

$b_6(3,k) = 0$

CHAPTER SIX

The Legendre Type Left-Definite Theory

6.1 Introduction

In this chapter, we will study the Legendre type left-definite theory. In Section 6.2, we define, for each $n \in \mathbb{N}$, a vector space of functions V_n and the inner product $(\cdot,\cdot)_n$ on $V_n \times V_n$. This inner product is called the n^{th} left-definite inner product. We denote the resulting inner product space $H_n = (V_n, (\cdot,\cdot)_n)$ and list several examples of this space.

In Section 6.3, we make use of the CHEL (Chisholm, Everitt, and Littlejohn) inequality to simplify the vector space V_n.

In Section 6.4, we will show that $H_n = (V_n, (\cdot,\cdot)_n)$ is a Hilbert space in the inner product $(\cdot,\cdot)_n$.

In Section 6.5., we prove that the set of complex-valued polynomials $\mathcal{P}$ in the real variable x is dense in each H_n. Of course, this will immediately imply that the Legendre type polynomials $\{P_{m,A}\}$ form a complete orthogonal set in H_n. These facts will prove useful later in this chapter when we show that H_n is, in fact, the n^{th} left-definite space associated with the pair $(T, L^2_\mu[-1,1])$.

In the final section of this chapter, we establish the left-definite theory associated with the pair $(T, L^2_\mu[-1,1])$. Specifically, we determine explicitly

(a) the sequence $\{H_n\}_{n=1}^{\infty}$ of left-definite spaces associated with the pair $(T, L^2_\mu[-1,1])$,

(b) the sequence of left-definite self-adjoint operators $\{T_n\}_{n=1}^{\infty}$, and their specific domains $\{\mathcal{D}(T_n)\}_{n=1}^{\infty}$ associated with the pair $(T, L^2_\mu[-1,1])$, and

(c) the domains $\mathcal{D}(T^n)$ of each integral power T^n of T.

These results culminate in Theorem 6.6.1.

6.2 *Definition of* V_n, $(\cdot,\cdot)_n$, *and* $H_n = (V_n, (\cdot,\cdot)_n)$

Definition 6.2.1. For each $n \in \mathbb{N}$, define

$$V_n := \{f : [-1,1] \to \mathbf{C} \,\big|\, f \in AC[-1,1]; f', f'', \ldots, f^{(2n-1)} \in AC_{\text{loc}}(-1,1) \;;$$

$$(1-x^2)^{(j-1)/2} f^{(j)} \in L^2(-1,1) \; (j = 1, 2, \ldots, 2n-1); \; (1-x^2)^n f^{(2n)} \in L^2(-1,1)\}. \tag{6.2.1}$$

We will see in Section 6.3 that the space V_n simplifies into

$$V_n := \{f : [-1,1] \to \mathbf{C} \,\big|\, f \in AC[-1,1]; f', f'', \ldots, f^{(2n-1)} \in AC_{\text{loc}}(-1,1) \;;$$

$$(1-x^2)^n f^{(2n)} \in L^2(-1,1)\}. \tag{6.2.2}$$

Let $(\cdot,\cdot)_n$ and $\|\cdot\|_n$ denote, respectively, the inner product

$$(f,g)_n := \frac{A}{2} \sum_{j=1}^{2n} \int_{-1}^{1} \left(a_j\,(n,k)\,(1-x^2)^j + b_j\,(n,k)\,(1-x^2)^{j-1}\right) f^{(j)}\,(x)\,\bar{g}^{(j)}\,(x)\,dx \tag{6.2.3}$$

$$+ \, k^n \,(f,g)_\mu \,,$$

for $f, g \in V_n$ and

$$(f,g)_\mu := \frac{A}{2} \int_{-1}^{1} f\,(x)\,\bar{g}\,(x)\,dx + \frac{1}{2} f\,(1)\,\bar{g}\,(1) + \frac{1}{2} f\,(-1)\,\bar{g}\,(-1)\,, \tag{6.2.4}$$

and the norm

$$\|f\|_n = (f,f)_n^{1/2}\,;$$

here the numbers $a_j\,(n,k)$ and $b_j\,(n,k)$ are defined in (5.4.3) and (5.4.7) respectively. Finally, let $H_n = (V_n, (\cdot,\cdot)_n)\,.$

The inner product $(\cdot,\cdot)_n$, defined in $(6.2.3)$, is a Sobolev inner product and is more commonly called the Dirichlet inner product associated with the symmetric differential expression $\ell^n\,[\cdot]$ given in $(5.3.1)$.

We now list some examples of H_n.

1. $H_1 = (V_1, (\cdot,\cdot)_1)$, where

$$V_1 = \left\{ f : [-1,1] \to \mathbf{C} \,\middle|\, f \in AC[-1,1]; f' \in AC_{\text{loc}}(-1,1)\ ; (1-x^2)f'' \in L^2(-1,1) \right\},$$

and

$$(f,g)_1 = \frac{A}{2}\int_{-1}^{1} \left\{ \left(1-x^2\right)^2 f''(x)\,\bar{g}''(x) + \left(8 + 4A\left(1-x^2\right)\right) f'(x)\,\bar{g}'(x) \right\} dx$$
$$+ k\,(f,g)_\mu .$$

2. $H_2 = (V_2, (\cdot,\cdot)_2)$, where

$$V_2 := \{ f : [-1,1] \to \mathbf{C} \,|\, f \in AC[-1,1]; f', f'', f''' \in AC_{\text{loc}}(-1,1)\ ;$$

$$(1-x^2)^2 f^{(4)} \in L^2(-1,1)\},$$

and

$$\begin{aligned}(f,g)_2 = \frac{A}{2}\int_{-1}^{1} \{ &\left(1-x^2\right)^4 f^{(4)}(x)\,\bar{g}^{(4)}(x) \\ &+ \left((8A+16)\left(1-x^2\right)^3 + 16\left(1-x^2\right)^2\right) f'''(x)\,\bar{g}'''(x) \\ &+ \left(\left(16A^2+48A+24+2k\right)\left(1-x^2\right)^2 + (64A+96)\left(1-x^2\right)\right) f''(x)\,\bar{g}''(x) \\ &+ \left(\left(32A^2+8Ak\right)\left(1-x^2\right) + 64A + 16k\right) f'(x)\,\bar{g}'(x)\} dx + k^2\,(f,g)_\mu .\end{aligned}$$

3. $H_3 = (V_3, (\cdot,\cdot)_3)$, where

$$V_3 := \{ f : [-1,1] \to \mathbf{C} \,\big|\, f \in AC[-1,1]; f', f'', \ldots, f^{(5)} \in AC_{\text{loc}}(-1,1)\ ;$$

$$(1-x^2)^3 f^{(6)} \in L^2(-1,1)\},$$

and

$$(f,g)_3 = \frac{A}{2}\int_{-1}^{1}\{\left(1-x^2\right)^6 f^{(6)}(x)\,\bar{g}^{(6)}(x)$$

$$+\left((12A+64)\left(1-x^2\right)^5+24\left(1-x^2\right)^4\right)f^{(5)}(x)\,\bar{g}^{(5)}(x)$$

$$+\left(\left(48A^2+432A+3k+864\right)\left(1-x^2\right)^4+(192A+800)\left(1-x^2\right)^3\right)f^{(4)}(x)\,\bar{g}^{(4)}(x)$$

$$+(\left(64A^3+864A^2+2592A+24Ak+48k+2304\right)\left(1-x^2\right)^3$$

$$+(384A^2+3008A+48k+4224)\left(1-x^2\right)^2)f^{(3)}(x)\,\bar{g}^{(3)}(x)$$

$$+(\left(512A^3+1728A^2+48A^2k+1728A+144Ak+3k^2+576\right)\left(1-x^2\right)^2$$

$$+\left(2048A^2+4608A+192Ak+288k+2304\right)\left(1-x^2\right))f''(x)\,\bar{g}''(x)$$

$$+(\left(256A^3+96A^2k+12Ak^2\right)\left(1-x^2\right)+\left(512A^2+192Ak+24k^2\right))f'(x)\,\bar{g}'(x)\}dx$$

$$+k^3(f,g)_\mu.$$

We now arrive at one of the main results of this section.

Theorem 6.2.1.

$$\left(\ell^n[f],g\right)_\mu=(f,g)_n \tag{6.2.5}$$

for all $f,g\in C^{2n}[-1,1]$ of all $2n$-times continuously differentiable complex-valued functions on $[-1,1]$.

Proof. Let $\ell^n[\cdot]$, $(\cdot,\cdot)_n$, and $(\cdot,\cdot)_\mu$ be defined as in (5.3.1), (6.2.3), and (6.2.4) respectively. Then, we want to compute the left-hand side of (6.2.5). From (5.3.1) and (6.2.4), we have

$$(\ell^n[f](x), g(x))_\mu$$

$$= \left(\sum_{j=0}^{2n} (-1)^j \left(\left(a_j(n,k) \left(1-x^2\right)^j + b_j(n,k) \left(1-x^2\right)^{j-1} \right) f^{(j)}(x) \right)^{(j)}, g(x) \right)_\mu$$

$$= \left(\sum_{j=1}^{2n} (-1)^j \left(\left(a_j(n,k) \left(1-x^2\right)^j + b_j(n,k) \left(1-x^2\right)^{j-1} \right) f^{(j)}(x) \right)^{(j)}, g(x) \right)_\mu$$

$$+ \left(a_0(n,k) f, g(x)\right)_\mu$$

$$= \left(\sum_{j=1}^{2n} (-1)^j \left(\left(a_j(n,k) \left(1-x^2\right)^j + b_j(n,k) \left(1-x^2\right)^{j-1} \right) f^{(j)}(x) \right)^{(j)}, g(x) \right)_\mu$$

$$+ \left(k^n f(x), g(x)\right)_\mu$$

$$= \left(\sum_{j=1}^{2n} (-1)^j \left(\left(a_j(n,k) \left(1-x^2\right)^j + b_j(n,k) \left(1-x^2\right)^{j-1} \right) f^{(j)}(x) \right)^{(j)}, g(x) \right)_\mu \tag{6.2.6}$$

$$+ k^n \left(f(x), g(x)\right)_\mu .$$

We now calculate (6.2.6): From the definition of $(\cdot,\cdot)_{\mu}$, (6.2.6) becomes

$$\frac{A}{2}\int_{-1}^{1}\sum_{j=1}^{2n}(-1)^{j}\left(\left(a\left(1-x^{2}\right)^{j}+b\left(1-x^{2}\right)^{j-1}\right)f^{(j)}\left(x\right)\right)^{(j)}\bar{g}\left(x\right)dx \qquad (\alpha)$$

$$+\int_{-1}^{1}\sum_{j=1}^{2n}\frac{(-1)^{j}}{2}\left(\left(a\left(1-x^{2}\right)^{j}+b\left(1-x^{2}\right)^{j-1}\right)f^{(j)}\left(x\right)\right)^{(j)}\underset{x=1}{|}\bar{g}\left(1\right) \qquad (\beta)$$

$$+\int_{-1}^{1}\sum_{j=1}^{2n}\frac{(-1)^{j}}{2}\left(\left(a\left(1-x^{2}\right)^{j}+b\left(1-x^{2}\right)^{j-1}\right)f^{(j)}\left(x\right)\right)^{(j)}\left(x\right)\underset{x=-1}{|}\bar{g}\left(-1\right). \qquad (\gamma)$$

Where $a=a_{j}\left(n,k\right)$ and $b=b_{j}\left(n,k\right)$.

So, to compute (6.2.6), we need to compute α, β, and γ respectively. Regarding α, we do so through integration by parts, Let

$$u=\bar{g}\left(x\right),\ dv=\left(\left(a\left(1-x^{2}\right)^{j}+b\left(1-x^{2}\right)^{j-1}\right)f^{(j)}\left(x\right)\right)^{(j)}dx$$

and

$$du=\bar{g}'\left(x\right)dx,\ v=\left(\left(a\left(1-x^{2}\right)^{j}+b\left(1-x^{2}\right)^{j-1}\right)f^{(j)}\left(x\right)\right)^{(j-1)}.$$

We also need to compute $v\underset{x=\pm 1}{|}$ in order to complete integration by parts. Recall that

$$D^{n}\left(1-x^{2}\right)^{n}=(-1)^{n}\,2^{n}n!P_{n}\left(x\right), \qquad (6.2.7)$$

where $P_{n}\left(x\right)$ is the n^{th} degree Legendre polynomial.

Hence,

$$v = \left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right)^{(j-1)} f^{(j)}(x)$$

$$+\binom{j-1}{1}\left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right)^{(j-2)} f^{(j+1)}(x)$$

$$+\binom{j-1}{2}\left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right)^{(j-3)} f^{(j+2)}(x)$$

$$+\cdots+\left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right) f^{(2j-1)}(x);$$

Equivalently,

$$v = \left(aD^{j-1}\left(1-x^2\right)^j + bD^{j-1}\left(1-x^2\right)^{j-1}\right) f^{(j)}(x)$$

$$+\binom{j-1}{1}\left(aD^{j-2}\left(1-x^2\right)^j + bD^{j-2}\left(1-x^2\right)^{j-1}\right) f^{(j+1)}(x)$$

$$+\binom{j-1}{2}\left(aD^{j-3}\left(1-x^2\right)^j + bD^{j-3}\left(1-x^2\right)^{j-1}\right) f^{(j+2)}(x)$$

$$+\cdots+\left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right) f^{(2j-1)}(x).$$

Since $D^k\left(1-x^2\right)^j \Big|_{x=\pm 1} = 0$ if $k<j$, we see that from (6.2.7),

$$\begin{aligned} v &= b(-1)^{j-1}2^{j-1}(j-1)!P_{j-1}(\pm 1)f^{(j)}(\pm 1) \qquad (j\geq 1) \\ &= \begin{cases} b(-1)^{j-1}2^{j-1}(j-1)!P_{j-1}(+1)f^{(j)}(+1) & x=1 \\ \\ b(-1)^{j-1}2^{j-1}(j-1)!P_{j-1}(-1)f^{(j)}(-1) & x=-1. \end{cases} \end{aligned}$$

Since

$$P_n(1) = 1 \text{ and } P_n(-1) = (-1)^n,$$

we see that

$$v = \begin{cases} b\left(-1\right)^{j-1} 2^{j-1}\left(j-1\right)! f^{(j)}\left(+1\right) & x = 1 \\ \\ b2^{j-1}\left(j-1\right)! f^{(j)}\left(-1\right) & x = -1. \end{cases}$$

Hence, α becomes

$$\frac{A}{2}\sum_{j=1}^{2n}\left(-1\right)^{j}\int_{-1}^{1}\left(\left(a\left(1-x^{2}\right)^{j}+b\left(1-x^{2}\right)^{j-1}\right)f^{(j)}\left(x\right)\right)^{(j)}\bar{g}\left(x\right)dx$$

$$= \frac{A}{2}\sum_{j=1}^{2n}\left(-1\right)^{j}\left\{\bar{g}\left(x\right)v\left(x\right)\Big|_{-1}^{1}\right.$$

$$\left.-\int_{-1}^{1}\left(\left(a\left(1-x^{2}\right)^{j}+b\left(1-x^{2}\right)^{j-1}\right)f^{(j)}\left(x\right)\right)^{(j-1)}\bar{g}'\left(x\right)dx\right\}.$$

We now simplify

$$\int_{-1}^{1}\left(\left(a\left(1-x^{2}\right)^{j}+b\left(1-x^{2}\right)^{j-1}\right)f^{(j)}\left(x\right)\right)^{(j-1)}\bar{g}'\left(x\right)dx$$

with

$$u = \bar{g}'\left(x\right),\ dv = \left(\left(a\left(1-x^{2}\right)^{j}+b\left(1-x^{2}\right)^{j-1}\right)f^{(j)}\left(x\right)\right)^{(j-1)}dx$$

and

$$du = \bar{g}''\left(x\right)dx,\ v = \left(\left(a\left(1-x^{2}\right)^{j}+b\left(1-x^{2}\right)^{j-1}\right)f^{(j)}\left(x\right)\right)^{(j-2)},$$

we note that $v(\pm 1) = 0$ so that

$$\int_{-1}^{1} \left(\left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right) f^{(j)}(x)\right)^{(j-1)} \bar{g}'(x)\, dx$$

$$= -\int_{-1}^{1} \left(\left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right) f^{(j)}(x)\right)^{(j-2)} \bar{g}''(x)\, dx$$

$$= \int_{-1}^{1} \left(\left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right) f^{(j)}(x)\right)^{(j-3)} \bar{g}'''(x)\, dx$$

$$= \cdots =$$

$$(-1)^{k+1} \int_{-1}^{1} \left(\left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right) f^{(j)}(x)\right)^{(j-k)} \bar{g}^{(k)}(x)\, dx.$$

In particular, setting $k = j$, we see that

$$\int_{-1}^{1} \left(\left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right) f^{(j)}(x)\right)^{(j-1)} \bar{g}'(x)\, dx$$

$$= (-1)^{j+1} \int_{-1}^{1} \left(a\left(1-x^2\right)^j + b\left(1-x^2\right)^{j-1}\right) f^{(j)}(x)\, \bar{g}^{(j)}(x)\, dx.$$

Substituting this into α gives us

$$\alpha = \frac{A}{2}\sum_{j=1}^{2n}(-1)^j\int_{-1}^{1}\left(\left(a\left(1-x^2\right)^j+b\left(1-x^2\right)^{j-1}\right)f^{(j)}(x)\right)^{(j)}\bar{g}(x)\,dx$$

$$= \frac{A}{2}\sum_{j=1}^{2n}(-1)^j\left\{\bar{g}(x)\,v(x)\Big|_{-1}^{1}\right.$$

$$\left.-(-1)^{j+1}\int_{-1}^{1}\left(a\left(1-x^2\right)^j+b\left(1-x^2\right)^{j-1}\right)f^{(j)}(x)\,\bar{g}^{(j)}(x)\,dx\right\};$$

Equivalently,

$$\alpha = \frac{A}{2}\sum_{j=1}^{2n}\int_{-1}^{1}\left(a\left(1-x^2\right)^j+b\left(1-x^2\right)^{j-1}\right)f^{(j)}(x)\,\bar{g}^{(j)}(x)\,dx$$

$$+\frac{A}{2}\sum_{j=1}^{2n}(-1)^j\,\bar{g}(x)\,v(x)\Big|_{-1}^{1}$$

$$= \frac{A}{2}\sum_{j=1}^{2n}\int_{-1}^{1}\left(a\left(1-x^2\right)^j+b\left(1-x^2\right)^{j-1}\right)f^{(j)}(x)\,\bar{g}^{(j)}(x)\,dx$$

$$-\frac{A}{2}\sum_{j=1}^{2n}b2^{j-1}(j-1)!f^{(j)}(+1)\,\bar{g}(+1)$$

$$-\frac{A}{2}\sum_{j=1}^{2n}b(-1)^j2^{j-1}(j-1)!f^{(j)}(-1)\,\bar{g}(-1).$$

Now to calculate β, we first need to calculate

$$\left(\left(a_j(n,k)\left(1-x^2\right)^j+b_j(n,k)\left(1-x^2\right)^{j-1}\right)f^{(j)}(x)\right)^{(j)}. \qquad (\eta)$$

So,

$$\eta := \left(\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) f^{(j)}(x)\right)^{(j)}$$

$$= \left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right)^{(j)} f^{(j)}(x)$$

$$+ \binom{j}{1}\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right)^{(j-1)} f^{(j+1)}(x)$$

$$+ \binom{j}{2}\left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right)^{(j-2)} f^{(j+2)}(x)$$

$$+ \cdots + \left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) f^{(2j)}(x);$$

Equivalently,

$$\eta = \left(a_j(n,k)\, D^j\left(1-x^2\right)^j + b_j(n,k)\, D^j\left(1-x^2\right)^{j-1}\right) f^{(j)}(x)$$

$$+ \binom{j}{1}\left(a_j(n,k)\, D^{j-1}\left(1-x^2\right)^j + b_j(n,k)\, D^{j-1}\left(1-x^2\right)^{j-1}\right) f^{(j+1)}(x)$$

$$+ \binom{j}{2}\left(a_j(n,k)\, D^{j-2}\left(1-x^2\right)^j + b_j(n,k)\, D^{j-2}\left(1-x^2\right)^{j-1}\right) f^{(j+2)}(x)$$

$$+ \cdots + \left(a_j(n,k)\left(1-x^2\right)^j + b_j(n,k)\left(1-x^2\right)^{j-1}\right) f^{(2j)}(x).$$

Since $D^k\left(1-x^2\right)^j\Big|_{x=\pm1}=0$ if $k<j$, we see that from (6.2.7),

$$\eta\left(\pm1\right)=\Big(a\left(-1\right)^j 2^j j!P_j\left(\pm1\right)+b\left(-1\right)^{j-1}2^{j-1}\left(j-1\right)!P'_{j-1}\left(\pm1\right)\Big)f^{(j)}\left(\pm\right)$$

$$+jb\left(-1\right)^{j-1}2^{j-1}\left(j-1\right)!P_{j-1}\left(\pm1\right)f^{(j+1)}\left(\pm1\right) \tag{6.2.8}$$

and since

$$P_n\left(1\right)=1,\ P_n\left(-1\right)=\left(-1\right)^n,\ P'_n\left(1\right)=\frac{n\left(n+1\right)}{2},$$

and

$$P'_n\left(-1\right)=\frac{\left(-1\right)^{n+1}n\left(n+1\right)}{2},$$

so that

$$\beta=\frac{1}{2}\sum_{j=1}^{2n}\left(a_j\left(n,k\right)2^j j!-b_j\left(n,k\right)2^{j-1}\left(j-1\right)!\frac{\left(j-1\right)j}{2}\right)f^{(j)}\left(1\right)\bar{g}\left(1\right)$$

$$-\frac{1}{2}\sum_{j=1}^{2n}jb_j\left(n,k\right)2^{j-1}\left(j-1\right)!f^{(j+1)}\left(1\right)\bar{g}\left(1\right).$$

Likewise, from (6.2.8), we have

$$\gamma=\frac{1}{2}\sum_{j=1}^{2n}\left(-1\right)^j\left(a2^j j!-b2^{j-1}\left(j-1\right)!\frac{\left(j-1\right)j}{2}\right)f^{(j)}\left(-1\right)\bar{g}\left(-1\right)$$

$$+\frac{1}{2}\sum_{j=1}^{2n}jb2^{j-1}\left(j-1\right)!f^{(j+1)}\left(-1\right)\bar{g}\left(-1\right).$$

Putting α, β, and γ together, we see that (6.2.6) becomes

$$\frac{A}{2}\sum_{j=1}^{2n}\int_{-1}^{1}\left(a\left(1-x^2\right)^j+b\left(1-x^2\right)^{j-1}\right)f^{(j)}(x)\,\bar{g}^{(j)}(x)\,dx$$

$$+\frac{1}{2}\sum_{j=1}^{2n}\left\{\left(2ja-\frac{(j-1)\,j}{2}b\right)2^{j-1}(j-1)!f^{(j)}(1)\,\bar{g}(1)\right.$$

$$\left.-Ab2^{j-1}(j-1)!f^{(j)}(1)\,\bar{g}(1)-j2^{j-1}(j-1)!bf^{(j+1)}(1)\,\bar{g}(1)\right\} \qquad (\theta)$$

$$+\frac{1}{2}\sum_{j=1}^{2n}(-1)^j\left\{\left(2ja-\frac{(j-1)\,j}{2}b\right)2^{j-1}(j-1)!f^{(j)}(-1)\,\bar{g}(-1)\right.$$

$$\left.-Ab2^{j-1}(j-1)!f^{(j)}(-1)\,\bar{g}(-1)+j2^{j-1}(j-1)!bf^{(j+1)}(-1)\,\bar{g}(-1)\right\}. \qquad (\vartheta)$$

We now show that the terms in θ and ϑ are both zero. Hence, (6.2.6) simplifes to

$$\frac{A}{2}\sum_{j=1}^{2n}\int_{-1}^{1}\left(a_j(n,k)\left(1-x^2\right)^j+b_j(n,k)\left(1-x^2\right)^{j-1}\right)f^{(j)}(x)\,\bar{g}^{(j)}(x)\,dx,$$

and therefore,

$$\left(\ell^n[f](x)\,,g(x)\right)_\mu=$$

$$\frac{A}{2}\sum_{j=1}^{2n}\int_{-1}^{1}\left(a_j(n,k)\left(1-x^2\right)^j+b_j(n,k)\left(1-x^2\right)^{j-1}\right)f^{(j)}(x)\,\bar{g}^{(j)}(x)\,dx$$

$$+k^n\left(f(x)\,,g(x)\right)_\mu=\left(f(x)\,,g(x)\right)_n \text{ for all } f,g\in C^{2n}[-1,1]\,.\ \square$$

Claim: The terms in θ and ϑ are both zero.

Proof. Since $b_0(n,k) = b_{2n}(n,k) = 0$, θ and ϑ become

$$\sum_{j=1}^{2n}(4ja - (j^2 - j + 2A)b - b_{j-1}(n,k))2^{j-3}(j-1)!f^{(j)}(1)\,\bar{g}(1) \qquad (\theta)$$

and

$$\sum_{j=1}^{2n}(4ja - (j^2 - j + 2A)b - b_{j-1}(n,k))(-1)^j\,2^{j-3}(j-1)!f^{(j)}(-1)\,\bar{g}(-1)\,. \qquad (\vartheta)$$

Now, we show that the term $4ja_j(n,k) - (j^2 - j + 2A)b_j(n,k) - b_{j-1}(n,k)$ in (θ) and (ϑ) is zero. For simplicity, we let $\lambda_k^n = (k^2+k)^n(k^2+k-2+4A)^n$ and $k=0$ in both $a_j(n,k)$ and $b_j(n,k)$ so that $a_j(n,k) = a_{n,j}$, $b_j(n,k) = b_{n,j}$ and by definition of $a_{n,j}$ and $b_{n,j}$, we get

$$4ja_j(n,k) - (j^2 - j + 2A)b_j(n,k) - b_{j-1}(n,k)$$

$$= 4ja_{n,j} - (j^2 - j + 2A)b_{n,j} - b_{n,j-1}$$

$$= \sum_{k=1}^{j} \frac{(-1)^{k+j}4j(2k+1)\lambda_k^n}{(j-k)!(j+k+1)!}$$

$$+ \sum_{k=1}^{j} \frac{(-1)^{k+j+1}(j^2 - j + 2A)(2k+1)\left(2Aj + (j+1)\left(j + k^2 + k\right)\right)\lambda_k^n}{(j+k+1)!(j-k)!\left(2A + (k-1)k\right)\left(2A + (k+1)(k+2)\right)}$$

$$+ \sum_{k=1}^{j-1} \frac{(-1)^{k+j}4(2k+1)\left(2Aj + (j+1)\left(j + k^2 + k\right)\right)\lambda_k^n}{(j+k+1)!(j-k)!\left(2A + (k-1)k\right)\left(2A + (k+1)(k+2)\right)}.$$

When $k = j$, the above sum is

$$\frac{4j(2j+1)(j^2+j)^n(j^2+j-2+4A)^n}{(2j+1)!}$$

$$-\frac{(2j+1)(j^2-j+2A)(j^2+j)^n(j^2+j-2+4A)^n\left(2Aj+(j+1)\left(2j+j^2\right)\right)}{(2j+1)!\left(2A+(j-1)\,j\right)\left(2A+(j+1)\,(j+2)\right)}$$

$$=\frac{4j(2j+1)(j^2+j)^n(j^2+j-2+4A)^n}{(2j+1)!}$$

$$-\frac{4(2j+1)(j^2+j)^n(j^2+j-2+4A)^n(j^2-j+2A)\left(2Aj+(j+1)\left(2j+j^2\right)\right)}{(2j+1)!\left(2A+(j-1)\,j\right)\left(2A+(j+1)\,(j+2)\right)}$$

$$=\frac{4(2j+1)(j^2+j)^n(j^2+j-2+4A)^n}{(2j+1)!}\,(j-j)$$

$$=0.$$

Hence,

$$4ja_{n,j}-(j^2-j+2A)b_{n,j}-b_{n,j-1}$$

$$=\sum_{k=1}^{j-1}\frac{(-1)^{k+j}4j(2k+1)\lambda_k^n}{(j-k)!(j+k+1)!}$$

$$+\sum_{k=1}^{j-1}\frac{(-1)^{k+j+1}(2k+1)(j^2-j+2A)\left(2Aj+(j+1)\left(j+k^2+k\right)\right)\lambda_k^n}{(j+k+1)!(j-k)!\left(2A+(k-1)\,k\right)\left(2A+(k+1)\,(k+2)\right)}$$

$$+\sum_{k=1}^{j-1}\frac{(-1)^{k+j}4(2k+1)\left(2Aj+(j+1)\left(j+k^2+k\right)\right)\lambda_k^n}{(j+k+1)!(j-k)!\left(2A+(k-1)\,k\right)\left(2A+(k+1)\,(k+2)\right)},$$

and this is same as,

$$\sum_{k=1}^{j-1} \frac{(-1)^{k+j}4j(2k+1)\lambda_k^n}{(j-k)!(j+k+1)!} \left(\frac{j}{(j-k)\,(j+k+1)} \right.$$

$$- \frac{(j^2-j+2A)\,(2Aj+(j+1)\,(j+k^2+k))}{(j-k)(j+k+1)\,(2A+(k-1)\,k)\,(2A+(k+1)\,(k+2))}$$

$$\left. + \frac{2A(j-1)+j\,(j-1+k^2+k)}{(2A+(k-1)\,k)\,(2A+(k+1)\,(k+2))} \right).$$

But

$$\frac{j}{(j-k)\,(j+k+1)} - \frac{(j^2-j+2A)\,(2Aj+(j+1)\,(j+k^2+k))}{(j-k)(j+k+1)\,(2A+(k-1)\,k)\,(2A+(k+1)\,(k+2))}$$

$$\frac{2A(j-1)+j\,(j-1+k^2+k)}{(2A+(k-1)\,k)\,(2A+(k+1)\,(k+2))}$$

$$\equiv 0.$$

This completes the proof of claim. □

6.3 The CHEL Inequality and a Simplification of V_n

In 1970, W. N. Everitt and R. S. Chisholm [9] published a remarkable L^2- inequality that has proven very useful in obtaining smoothness properties of functions in the domains of certain self-adjoint differential operators. In 1999, these authors together with L. L. Littlejohn [10] generalized this result to the Banach spaces L^p and L^q where p and q are conjugate indices. Specifically, these authors proved the following theorem:

Theorem 6.3.1 (The CHEL inequality) Suppose (a, b) is an interval, bounded or unbounded, of the real line $\mathbb{R}$ and w is a real-valued, Lebesgue measurable function with $w(x) \geq 0$ for all $x \in (a, b)$. Suppose p and q are conjugate indices $(\frac{1}{p} + \frac{1}{q} = 1)$

satisfying $p, q \in (1, \infty)$. In addition, suppose φ and ψ are complex-valued functions defined on (a, b) that satisfy the conditions

(*i*) $\varphi \in L^p_{\text{loc}}((a,b);w)$, $\psi \in L^q_{\text{loc}}((a,b);w)$;

(*ii*) for some $c \in (a,b)$ (and hence all $c \in (a,b)$), we have $\varphi \in L^p((a,c];w)$ and $\psi \in L^q([c,b);w)$;

(*iii*) for all $[\alpha, \beta] \subset (a,b)$, $\int_a^\alpha |\varphi|^p \, w dx > 0$ and $\int_\beta^b |\psi|^q \, w dx > 0$.

Define the linear operators A and B on $L^p((a,b);w)$ and $L^q((a,b);w)$, respectively, by

$$(Ag)(x) := \varphi(x) \int_x^b g(t)\psi(t)w(t)dt \quad (x \in (a,b), \; g \in L^p((a,b);w)),$$

and

$$(Bg)(x) := \psi(x) \int_a^x g(t)\varphi(t)w(t)dt \quad (x \in (a,b), \; g \in L^p((a,b);w)),$$

so that

$$\begin{aligned} A &: L^p((a,b);w) \to L^p_{\text{loc}}((a,b);w) \text{ and} \\ B &: L^q((a,b);w) \to L^q_{\text{loc}}((a,b);w). \end{aligned}$$

Define $K : (a,b) \to (0,\infty)$ by

$$K(x) := \left(\int_a^x |\varphi(t)|^p \, w(t)dt \right)^{1/p} \cdot \left(\int_x^b |\psi(t)|^q \, w(t)dt \right)^{1/q} \quad (x \in (a,b)),$$

and the number $K \in (0, \infty]$ by

$$K := \sup\{K(x) \mid x \in (a,b)\}.$$

Then a necessary and sufficient condition that A, respectively B, is a bounded linear operator on $L^p((a,b);w)$, respectively on $L^q((a,b);w)$, into $L^p((a,b);w)$, respectively into $L^q((a,b);w)$, is that the number K is finite; i.e.,

$$K \in (0, \infty).$$

This theorem has proved to be remarkably useful in several areas of mathematics, including the spectral theory of differential operators for the past twenty years. For example, in [10], the authors give a new proof of the classical Hardy inequality

$$\int_0^{+\infty} \frac{1}{x^q} \left| \int_0^x g(t)\,dt \right|^q \leq \left(\frac{q}{q-1} \right)^q \int_0^{+\infty} |g(x)|^q \, dx \quad (g \in L^q(0,\infty))$$

using the CHEL inequality. In [23], the authors apply Theorem 6.3.1 to show that, in the case of the Legendre differential operator A in $L^2(-1,1)$, generated by the classical second-order Legendre differential expression

$$\ell[y](x) = -\left((1-x^2)y'(x)\right)' \quad (x \in (-1,1))$$

and having the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ as eigenfunctions, every function $f \in \mathcal{D}(A)$ has the property that $f' \in L^2(-1,1)$ so, in particular, $f \in AC[-1,1]$. This is a newly found property of this classical domain. A special case of this theorem is the following theorem.

Theorem 6.3.2. Let $\varphi(x)$ and $\psi(x)$ be complex-valued Lebesgue measurable functions with $\varphi \in L^2[0,1)$ and $\psi \in L^2_{\text{loc}}[0,1)$. Let the operators T, $S : L^2_{\text{loc}}[0,1) \to L^2_{\text{loc}}[0,1)$ be defined by

$$T[f](x) := \varphi(x) \int_0^x \psi(t) f(t)\,dt$$

and

$$S[f](x) := \varphi(x) \int_x^1 \varphi(t) f(t)\,dt \ ,$$

where $f \in L^2[0,1)$. A necessary and sufficient condition for both T and S to be bounded operators on $L^2[0,1)$ into $L^2[0,1)$ is that there exists a positive number K such that

$$\left(\int_0^x |\psi(t)|^2 f(t)\,dt \right) \left(\int_x^1 |\varphi(t)|^2\,dt \right) \leq K,$$

for $x \in [0,1]$.

In this section, we apply Theorem 6.3.1 to have a simpler characterization of the function space V_n. Recall that, for each $n \in \mathbb{N}$,

$$\begin{aligned} V_n = \{f : [-1,1] \to \mathbf{C} \,|f \in AC[-1,1]; f^{(j)} \in AC_{\text{loc}}(-1,1) \ (j = 1, 2, \ldots, 2n-1); \\ (1-x^2)^{(j-1)/2} f^{(j)} \in L^2(-1,1) \ (j = 1, 2, \ldots, 2n-1); \\ (1-x^2)^n f^{(2n)} \in L^2(-1,1)\}. \end{aligned}$$

We now prove

Theorem 6.3.3 For each $n \in \mathbb{N}$,

$$\begin{aligned} V_n = \{f : [-1,1] \to \mathbf{C} \mid f \in AC[-1,1]; f^{(j)} \in AC_{\text{loc}}(-1,1) \ (j = 1, 2, \ldots, 2n-1); \\ (1-x^2)^n f^{(2n)} \in L^2(-1,1)\}. \end{aligned} \tag{6.3.1}$$

Proof. To prove this theorem, it suffices to show

$$(1-x^2)^n f^{(2n)} \in L^2[0,1) \Rightarrow (1-x^2)^{(j-1)/2} f^{(j)} \in L^2[0,1) \ (j = 1, 2, \ldots, 2n-1);$$

a similar application of the CHEL inequality will establish these results for $L^2(-1,0]$.

<u>Step 1:</u> We first show that

$$(1-x^2)^n f^{(2n)} \in L^2(0,1) \Rightarrow (1-x^2)^{n-1} f^{(2n-1)} \in L^2[0,1).$$

Note that, since $f^{(2n-1)} \in AC_{\text{loc}}(-1,1)$, we have

$$\begin{aligned} f^{(2n-1)}(x) &= f^{(2n-1)}(0) + \int_0^x f^{(2n)}(t)dt \\ &= f^{(2n-1)}(0) + \int_0^x \frac{1}{(1-t^2)^n}(1-t^2)^{2n} f^{(2n)}(t)dt, \end{aligned}$$

so that

$$\begin{aligned} (1-x^2)^{n-1} f^{(2n-1)} &= f^{(2n-1)}(0)(1-x^2)^{n-1} \\ &\quad + (1-x^2)^{n-1} \int_0^x \frac{1}{(1-t^2)^n}(1-t^2)^n f^{(2n)}(t)dt. \end{aligned}$$

Since $f^{(2n-1)}(0)(1-x^2)^{n-1} \in L^2[0,1)$, it suffices to show that

$$(1-x^2)^{n-1}\int_0^x \frac{1}{(1-t^2)^n}(1-t^2)^n f^{(2n)}(t)dt \in L^2[0,1);$$

we use the CHEL inequality with

$$g(t) = (1-t^2)^n f^{(2n)}(t) \in L^2[0,1),\ \varphi(t) = \frac{1}{(1-t^2)^n},$$

and

$$\psi(t) = (1-t^2)^{n-1}.$$

Since

$$\frac{1}{(1+x)^{2n}} \leq 1 \text{ and } (1+x)^{2n-2} \leq 2^{2n-2},$$

we see that

$$\begin{aligned}
K^2(x) &= \left(\int_0^x \varphi^2(t)dt\right)\cdot\left(\int_x^1 \psi^2(t)dt\right) \\
&= \left(\int_0^x \frac{1}{(1-t^2)^{2n}}dt\right)\cdot\left(\int_x^1 (1-t^2)^{2n-2}dt\right) \\
&\leq 2^{2n-2}\left(\int_0^x \frac{1}{(1-t)^{2n}}dt\right)\cdot\left(\int_x^1 (1-t)^{2n-2}dt\right) \\
&= 2^{2n-2}\left[\frac{1}{(2n-1)(1-x)^{2n-1}} - \frac{1}{2n-1}\right]\cdot\left[\frac{(1-x)^{2n-1}}{2n-1}\right] \\
&= \frac{2^{2n-1}}{(2n-1)^2}\left[1-(1-x)^{2n-1}\right] \leq 1.
\end{aligned}$$

Consequently, from Theorem 6.3.1, we see that

$$(1-x^2)^{n-1}\int_0^x \frac{1}{(1-t^2)^n}(1-t^2)^n f^{(2n)}(t)dt \in L^2[0,1)$$

and hence that $(1-x^2)^{n-1}f^{(2n-1)} \in L^2(0,1)$ as required.

Step 2: We now assume that for $k = 2n-1, 2n-2, \ldots, j$

$$(1-x^2)^{(k-1)/2} f^{(k)} \in L^2[0,1) \Rightarrow (1-x^2)^{(k-2)/2} f^{(k-1)} \in L^2[0,1)$$

where $j \geq 2$. We now show that

$$(1-x^2)^{(j-1)/2} f^{(j)} \in L^2[0,1) \Rightarrow (1-x^2)^{(j-2)/2} f^{(j-1)} \in L^2[0,1).$$

The proof is similar, with some minor differences, to the proof given in Step 1. Since $f^{(j-1)} \in AC_{\text{loc}}(-1,1)$, we see that

$$f^{(j-1)}(x) = f^{(j-1)}(0) + \int_0^x f^{(j)}(t)dt,$$

so

$$(1-x^2)^{(j-2)/2} f^{(j-1)} = f^{(j-1)}(0)(1-x^2)^{(j-2)/2} + (1-x^2)^{(j-2)/2} \int_0^x f^{(j)}(t)dt$$

$$= f^{(j-1)}(0)(1-x^2)^{(j-2)/2}$$

$$+ (1-x^2)^{(j-2)/2} \int_0^x \frac{1}{(1-t^2)^{(j-1)/2}} (1-t^2)^{(j-1)/2} f^{(j)}(t)dt.$$

Similar to our argument in Step 1, it suffices to prove that

$$(1-x^2)^{(j-2)/2} \int_0^x \frac{1}{(1-t^2)^{(j-1)/2}} (1-t^2)^{(j-1)/2} f^{(j)}(t)dt \in L^2[0,1).$$

Again, with

$$g(t) = (1-t^2)^{(j-1)/2} f^{(j)}(t)dt \in L^2[0,1), \;\; \varphi(t) = \frac{1}{(1-t^2)^{(j-1)/2}},$$

and

$$\psi(t) = (1-t^2)^{(j-2)/2},$$

we see that

$$K^2(x) = \left(\int_0^x \varphi^2(t)dt\right) \cdot \left(\int_x^1 \psi^2(t)dt\right)$$

$$= \left(\int_0^x \frac{1}{(1-t^2)^{j-1}}dt\right) \cdot \left(\int_x^1 (1-t^2)^{j-2}dt\right)$$

$$\leq 2^{j-2}\left(\int_0^x \frac{1}{(1-t)^{j-1}}dt\right) \cdot \left(\int_x^1 (1-t)^{j-2}dt\right)$$

$$= \begin{cases} \dfrac{2^{j-2}}{(2-j)(1-j)}\left[1-x-(1-x)^{j-1}\right] & \text{if } j = 2 \\ (x-1)\ln(1-x) & \text{if } j \neq 2. \end{cases}$$

It is evident that $K^2(x)$ is bounded on $[0,1]$ and this proves the theorem.

6.4 The Completeness of $H_n = (V_n, (\cdot,\cdot)_n)$

Theorem 6.4.1. For each $n \in \mathbb{N}$, $H_n = (V_n, (\cdot,\cdot)_n)$ is a Hilbert space.

Proof. Let $n \in \mathbb{N}$, Suppose $\{f_m\}_{m=1}^{\infty}$ is Cauchy sequence in H_n. Since each of the numbers $a_j(n,k)$ and $b_j(n,k)$ is positive for each $j = 1, 2, \ldots, 2n-1$, we see that

$$a_j(n,k)\left(1-x^2\right) + b_j(n,k)$$

is bounded away from 0 on $(-1,1)$ for each $j = 1, 2, \ldots, 2n-1$. From the definition of $(\cdot,\cdot)_n$, it follows that

$$\left\{\left(1-x^2\right)^n f_m^{(2n)}\right\}_{m=1}^{\infty}$$

is Cauchy in $L^2(-1,1)$. Since $L^2(-1,1)$ is complete, it follows that

$$\left\{\left(1-x^2\right)^n f_m^{(2n)}\right\}_{m=1}^{\infty}$$

converges to a function in $L^2(-1,1)$ which can be written in the form $(1-x^2)^n f_{2n}$. In other words,

$$\lim_{m\to\infty}\int_{-1}^{1}\left(1-x^2\right)^n \left|f_m^{(2n)}(x) - f_{2n}(x)\right|^2 dx = 0 \tag{6.4.1}$$

and

$$\left(1-x^2\right)^n f_{2n} \in L^2\left(-1,1\right). \tag{6.4.2}$$

Let $[a,b]$ be an arbitrary compact subinterval of $(-1,1)$. Since

$$\left\{\left(1-x^2\right)^n f_m^{(2n)}\right\}_{m=1}^{\infty}$$

also converges to

$$\left(1-x^2\right)^n f_{2n} \text{ in } L^2\left[a,b\right]$$

and $\frac{1}{(1-x^2)^n}$ is bounded in $[a,b]$, it follows that

$$\left\{f_m^{(2n)}\right\}_{m=1}^{\infty} \text{ converges to } f_{2n} \text{ in } L^2\left[a,b\right]$$

(see [73], p.144). By Hölder's inequality,

$$\lim_{m\to\infty}\int_a^b f_m^{(2n)}\left(x\right)dx = \int_a^b f_{2n}\left(x\right)dx.$$

Returning to the definition of $(\cdot,\cdot)_n$, we see that

$$\left\{\left(1-x^2\right)^{\frac{2n-2}{2}} f_m^{(2n-1)}\right\}_{m=1}^{\infty}$$

is also Cauchy in $L^2\left(-1,1\right)$; hence, there exists a function

$$\left(1-x^2\right)^{\frac{2n-2}{2}} g_{2n-1} \in L^2\left(-1,1\right) \tag{6.4.3}$$

such that

$$\lim_{m\to\infty}\int_{-1}^{1}\left(1-x^2\right)^{2n-2}\left|f_m^{(2n-1)}\left(x\right)-g_{2n-1}\left(x\right)\right|^2 dx = 0. \tag{6.4.4}$$

Furthermore, we can find a subsequence

$$\left\{\left(1-x^2\right)^{\frac{2n-2}{2}} f_{m_k}^{(2n-1)}\right\}_{k=1}^{\infty} \text{ of } \left\{\left(1-x^2\right)^{\frac{2n-2}{2}} f_m^{(2n-1)}\right\}_{m=1}^{\infty}$$

such that for almost all $x \in (-1,1)$,

$$\lim_{k\to\infty}\int_a^b f_{m_k}^{(2n-1)}\left(x\right)dx = g_{2n-1}\left(x\right) \tag{6.4.5}$$

(see [73], p.85).

Choose $c \in (-1,1)$ so that

$$\left\{ f_{m_k}^{(2n-1)}(c) \right\}_{k=1}^{\infty}$$

converges. Define a function f_{2n-1} by

$$f_{2n-1}(c) := \lim_{k\to\infty} f_{m_k}^{(2n-1)}(c)$$

and for every $x \in (-1,1)$, $x \neq c$,

$$f_{2n-1}(x) := \int_c^x f_{2n}(t)\, dt + f_{2n-1}(c)$$

$$= \lim_{k\to\infty} \int_c^x f_{m_k}^{(2n)}(x)\, dx + f_{2n-1}(c)$$

$$= \lim_{k\to\infty} f_{m_k}^{(2n-1)}(x). \tag{6.4.6}$$

Notice, by definition, that

$$f_{2n-1} \in AC_{\text{loc}}(-1,1). \tag{6.4.7}$$

From (6.4.6), it follows that

$$f_{2n-1}'(x) = f_{2n}(x)$$

for almost all $x \in (-1,1)$. Therefore, from (6.4.1) and (6.4.2), we have

$$\left(1-x^2\right)^{2n} f_{2n-1}'(x) \in L^2(-1,1) \tag{6.4.8}$$

and

$$\lim_{m\to\infty} \int_{-1}^{1} \left(1-x^2\right)^{2n} \left| f_m^{(2n)}(x) - f_{2n-1}'(x) \right|^2 dx = 0. \tag{6.4.9}$$

Comparing (6.4.5) and (6.4.6), we see that

$$f_{2n-1}(x) = g_{2n-1}(x)$$

almost everywhere in $(-1,1)$. Hence, from (6.4.3),

$$\left(1-x^2\right)^{\frac{2n-2}{2}} f_{2n-1} \in L^2\left(-1,1\right) \tag{6.4.10}$$

and from (6.4.4) with $c_{2n-1} = a_{2n-1}\left(n,k\right)\left(1-x^2\right) + b_{2n-1}\left(n,k\right)$,

$$\lim_{m\to\infty} \int_{-1}^{1} \{\left(1-x^2\right)^{2n-2} c_{2n-1} \left|f_m^{(2n-1)}\left(x\right) - f_{2n-1}\left(x\right)\right|^2 dx = 0. \tag{6.4.11}$$

Since

$$\left\{\left(1-x^2\right)^{\frac{2n-2}{2}} f_m^{(2n-1)}\right\}_{m=1}^{\infty}$$

also converges to

$$\left(1-x^2\right)^{\frac{2n-2}{2}} f_{2n-1} \text{ in } L^2\left[a,b\right],$$

where $[a,b]$ is any compact subinterval of $(-1,1)$, and

$$\frac{1}{\left(1-x^2\right)^{\frac{2n-2}{2}}}$$

is bounded on $[a,b]$, it follows that

$$\left\{f_m^{(2n-1)}\right\}_{m=1}^{\infty} \text{ converges to } f_{2n-1} \text{ in } L^2\left[a,b\right].$$

By Hölder's inequality,

$$\lim_{m\to\infty} \int_a^b f_m^{(2n-1)}\left(x\right) dx = \int_a^b f_{2n-1}\left(x\right) dx. \tag{6.4.12}$$

Returning to the definition of $\left(\cdot,\cdot\right)_n$, we see that

$$\left\{\left(1-x^2\right)^{\frac{2n-3}{2}} f_m^{(2n-2)}\right\}_{m=1}^{\infty}$$

is also a Cauchy sequence in $L^2\left(-1,1\right)$; hence, there exists a function

$$\left(1-x^2\right)^{\frac{2n-3}{2}} g_{2n-2} \ \in L^2\left(-1,1\right) \tag{6.4.13}$$

such that

$$\lim_{m\to\infty} \int_{-1}^{1} \left(1-x^2\right)^{2n-3} \left|f_m^{(2n-2)}\left(x\right) - g_{2n-2}\left(x\right)\right|^2 dx = 0. \tag{6.4.14}$$

Furthermore, we can find a subsequence

$$\left\{\left(1-x^2\right)^{\frac{2n-3}{2}} f_{m_k}^{(2n-2)}\right\}_{k=1}^{\infty} \text{ of } \left\{\left(1-x^2\right)^{\frac{2n-3}{2}} f_{m}^{(2n-2)}\right\}_{m=1}^{\infty}$$

such that for almost all $x \in (-1,1)$,

$$\lim_{k\to\infty}\int_a^b f_{m_k}^{(2n-2)}(x)\,dx = g_{2n-2}(x) \tag{6.4.15}$$

(see [73], p.85).

Choose $c \in (-1,1)$ so that

$$\left\{f_{m_k}^{(2n-2)}(c)\right\}_{k=1}^{\infty}$$

converges. Define a function f_{2n-2} by

$$f_{2n-2}(c) := \lim_{k\to\infty} f_{m_k}^{(2n-2)}(c)$$

and for every $x \in (-1,1)$, $x \neq c$,

$$f_{2n-2}(x) := \int_c^x f_{2n-1}(t)\,dt + f_{2n-2}(c)$$

$$= \lim_{k\to\infty}\int_c^x f_{m_k}^{(2n-1)}(x)\,dx + f_{2n-2}(c)$$

$$= \lim_{k\to\infty} f_{m_k}^{(2n-2)}(x)\,. \tag{6.4.16}$$

Notice, by definition, that

$$f_{2n-2} \in AC_{\text{loc}}(-1,1)\,. \tag{6.4.17}$$

From (6.4.16), it follows that

$$f'_{2n-2}(x) = f_{2n-1}(x)$$

for almost all $x \in (-1,1)$. Therefore, from (6.4.7) and (6.4.8), we have

$$f'_{2n-2} \in AC_{\text{loc}}(-1,1)\,, \tag{6.4.18}$$

and

$$\left(1-x^2\right)^{2n} f''_{2n-2}(x) \in L^2(-1,1) \tag{6.4.19}$$

and from (6.4.9),

$$\lim_{m\to\infty} \int_{-1}^{1} \left(1-x^2\right)^{2n} \left|f_m^{(2n)}(x) - f''_{2n-2}(x)\right|^2 dx = 0. \tag{6.4.20}$$

From (6.4.10),

$$\left(1-x^2\right)^{\frac{2n-2}{2}} f'_{2n-2} \in L^2(-1,1) \tag{6.4.21}$$

and from (6.4.11),

$$\lim_{m\to\infty} \int_{-1}^{1} \left(1-x^2\right)^{2n-2} c_{2n-1} \left|f_m^{(2n-1)}(x) - f'_{2n-2}(x)\right|^2 dx = 0. \tag{6.4.22}$$

Comparing (6.4.15) and (6.4.16), we see that

$$f_{2n-2}(x) = g_{2n-2}(x)$$

almost everywhere in $(-1,1)$. Hence, from (6.4.13),

$$\left(1-x^2\right)^{\frac{2n-3}{2}} f_{2n-2} \in L^2(-1,1) \tag{6.4.23}$$

and from (6.4.14) with $c_{2n-2} = \left(a_{2n-2}(n,k)\left(1-x^2\right) + b_{2n-2}(n,k)\right)$,

$$\lim_{m\to\infty} \int_{-1}^{1} \left(1-x^2\right)^{2n-3} c_{2n-2} \left|f_m^{(2n-2)}(x) - f_{2n-2}(x)\right|^2 dx = 0. \tag{6.4.24}$$

Repeating above argument for each j, $j = 0, 1, \ldots, 2n-3$, we remark that

$$\{f_m\}_{m=1}^{\infty}$$

is a Cauchy sequence in $L^2_\mu[-1,1]$. Since $L^2_\mu[-1,1]$ is complete, there exists a function $g_0 \in L^2_\mu[-1,1]$ so that

$$\begin{gathered} \lim_{m\to\infty} f_m(1) = g_0(1) \\ \lim_{m\to\infty} f_m(-1) = g_0(-1) \end{gathered} \tag{6.4.25}$$

and

$$\lim_{m\to\infty}\int_{-1}^{1}|f_m(x)-g_0(x)|^2\,dx=0. \tag{6.4.26}$$

As before, we can find a subsequence

$$\{f_{m_k}\}_{k=1}^{\infty} \text{ of } \{f_m\}_{m=1}^{\infty}$$

such that for almost all $x\in(-1,1)$

$$\lim_{m\to\infty} f_{m_k}(x)=g_0(x). \tag{6.4.27}$$

Define a function f_0 by

$$f_0(x):=\int_{-1}^{x} f_1(t)\,dt+g_0(-1)$$

$$=\lim_{k\to\infty}\int_{-1}^{x} f_{m_k}^1(x)\,dx+g_0(-1)$$

$$=\lim_{k\to\infty} f_{m_k}(x) \tag{6.4.28}$$

for every $x\in[-1,1]$. By definition,

$$f_0\in AC[-1,1].$$

From (6.4.28),

$$f_0'(x)=f_1(x)$$

for almost all $x\in(-1,1)$.

Comparing (6.4.25) and (6.4.27), we see that

$$f_0(1)=g_0(1),$$

$$f_0(-1)=g_0(-1) \text{ and}$$

$$f_0(x)=g_0(x) \text{ for every } x\in(-1,1).$$

Consequently, we obtain two sets of functions $\{f_j\}_{j=0}^{2n}$ and $\{g_j\}_{j=0}^{2n-1}$ such that

(i)

$$f_m^{(j)} \to f_j \text{ in } L_{j-1}^2(-1,1) \quad (j = 1, \ldots, 2n-1);$$
$$f_j = f_k^{(j-k)} \quad (k = 1, 2, \ldots, j) \text{ and } (j = 0, 1, \ldots, 2n);$$
$$f_m^{(2n)} \to f_{2n} \text{ in } L_{2n}^2(-1,1), \ f_m \to g_0 \text{ in } L_\mu^2[-1,1] \text{ and } f_0 \in L^2(-1,1);$$
$$f_m^{(j)} \to g_j \text{ in } L_{j-1}^2(-1,1) \quad (j = 1, 2, \ldots, 2n-1) \text{ where}$$
$$L_{j-1}^2(-1,1) = \{f : (-1,1) \to \mathbb{C} \,|\, f \text{ is Lebesgue measurable and}$$
$$\textstyle\int_{-1}^1 (1-x^2)^{j-1} |f(x)|^2 \, dx < \infty\}.$$

(ii)

$$f_j = g_j \text{ a.e. } x \in (-1,1) \quad (j = 0, 1, \ldots, 2n-1);$$
$$f_0(1) = g_0(1) \text{ and } f_0(-1) = g_0(-1) \text{ and}$$
$$f_j(c) = \lim_{k\to\infty} f_{m_k}^{(j)}(c) \quad \text{for } c \in (-1,1).$$

(iii)

$$f_{j-1}(x) := \int_c^x f_j(t)\,dt + f_{j-1}(c) \quad x \in (-1,1), \ x \neq c \ (j = 1, 2, \ldots, 2n).$$

(iv)

$$f_j \in AC_{\mathrm{loc}}^{2n-j-1}(-1,1) \quad (j = 0, 1, \ldots, 2n-1) \text{ where}$$
$$AC_{\mathrm{loc}}^{2n-j-1}(-1,1) = \left\{ f : (-1,1) \to \mathbb{C} \,\middle|\, f, f', \ldots, f^{(2n-j-1)} \in AC_{\mathrm{loc}}(-1,1) \right\};$$
$$f_0 \in AC_{\mathrm{loc}}[-1,1].$$

(v)

$$f_j'(x) = f_{j+1}(x) \text{ a.e. } x \in (-1,1) \quad (j = 0, 1, \ldots, 2n-1).$$

(vi)

$$f_0^{(j)} = f_j \quad (j = 0, 1, \ldots, 2n).$$

In particular,

(i) $f_m^{(j)} \to f_0^{(j)}$ in $L^2_{j-1}(-1,1)$ $(j = 1, 2, \ldots, 2n-1)$;

(ii) $f_m^{(2n)} \to f_0^{(2n)}$ in $L^2_{2n}(-1,1)$;

(iii) $f_m \to f_0$ in $L^2_\mu[-1,1]$; and $f_0 \in V_{n.}$

Hence, we see that

$$\|f_m - f_0\|_n^2 \to 0 \text{ as } m \to \infty.$$

Thus, H_n is complete and, consequently, so is the proof of the theorem. □

6.5 The Density of Polynomials in H_n

In this section, we prove that the set of complex-valued polynomials $\mathcal{P}$ in the real variable x is dense in each H_n. Of course, this will immediately imply that the Legendre type polynomials $\{P_{m,A}\}$ form a complete orthogonal set in H_n. These facts will prove useful later in this chapter when we show that H_n is, in fact, the n^{th} left-definite space associated with the pair $(T, L^2_\mu[-1,1])$.

The proofs given in this section mimic the arguments given by Everitt, Littlejohn, and Williams in [26] in the case $n = 1$; indeed, in [26], the authors prove that the set $\mathcal{P}$ is dense in H_1. We begin with establishing the following basic lemma:

Lemma 6.5.1. If $f \in H_n$, there exists $g \in L^2(-1,1)$ such that

(i) $\int_{-1}^{1} g(t)dt = 0$;

(ii) $f^{(2n-1)}(x) = \dfrac{\int_{-1}^{x} g(t)dt}{(1-x^2)^n} = -\dfrac{\int_{x}^{1} g(t)dt}{(1-x^2)^n} \quad (x \in (-1,1))$;

(iii) $f^{(2n)}(x) = \dfrac{2nx}{1-x^2} f^{(2n-1)}(x) + \dfrac{g(x)}{(1-x^2)^n} \quad (\text{a.e.} x \in (-1,1))$;

(iv)
$$f(x) = \sum_{j=0}^{2n-1} \frac{f^{(j)}(x_0)(x-x_0)^j}{j!} + \int_{x_o}^{x}\int_{x_0}^{y_{2n-2}}\int_{x_0}^{y_{2n-1}} \cdots \int_{x_0}^{y_1} \frac{\int_{-1}^{t} g(u)du}{(1-t^2)^n} dt dy_1 \ldots dy_{2n-2}$$
$$= \sum_{j=0}^{2n-1} \frac{f^{(j)}(x_0)(x-x_0)^j}{j!} - \int_{x_o}^{x}\int_{x_0}^{y_{2n-2}}\int_{x_0}^{y_{2n-1}} \cdots \int_{x_0}^{y_1} \frac{\int_{t}^{1} g(u)du}{(1-t^2)^n} dt dy_1 \ldots dy_{2n-2}.$$

Proof. Since $(1-x^2)^{n-1} f^{(2n-1)}$ and $(1-x^2)^n f^{(2n)}$ both belong to $L^2(-1,1)$ and the function $x \to x$ is bounded, we see that

$$g(x) := (1-x^2)^n f^{(2n)}(x) - 2nx(1-x^2)^{n-1} f^{(2n-1)} \in L^2(-1,1).$$

Moreover, since $L^2(-1,1) \subset L^1(-1,1)$, we note that

$$\int_{-1}^{1} (1-x^2)^{n-1} f^{(2n-1)}(x)dx < \infty. \tag{6.5.1}$$

Now

$$(1-x^2)^n f^{(2n)}(x) - 2nx(1-x^2)^{n-1} f^{(2n-1)}(x) = \left((1-x^2)^n f^{(2n-1)}(x)\right)' \tag{6.5.2}$$

(a.e. $x \in (-1,1)$), so

$$g(x) = \left((1-x^2)^n f^{(2n-1)}(x)\right)' \in L^2(-1,1). \tag{6.5.3}$$

Since $f^{(2n-1)} \in AC_{\text{loc}}(-1,1)$, we see that for any $x_0 \in (-1,1)$,

$$(1-x^2)^n f^{(2n-1)}(x) = A + \int_{x_0}^{x} g(t)dt \quad (x \in (-1,1)), \tag{6.5.4}$$

for some $A = A(x_0) \in \mathbb{C}$. From (6.5.4) and the fact that $g \in L^2(-1,1)$, we see that the limits

$$\lim_{x \to \pm 1} (1-x^2)^n f^{(2n-1)}(x)$$

exist and are finite. We claim that each of these limits is zero. It suffices to show that

$$\lim_{x\to 1}(1-x^2)^n f^{(2n-1)}(x) = 0.$$

By way of contradiction, suppose without loss of generality that

$$\lim_{x\to 1}(1-x^2)^n f^{(2n-1)}(x) = c > 0,$$

where we assume that f is real-valued. Then, for x sufficiently close to 1^-, say all $x \in (x^*, 1)$, we have

$$(1-x^2)^n f^{(2n-1)}(x) \geq \frac{c}{2},$$

so that

$$(1-x^2)^{n-1} f^{(2n-1)}(x) \geq \frac{c}{2(1-x^2)} \quad (x \in (x^*, 1)).$$

However, this implies that

$$\int_{-1}^{1}(1-x^2)^{n-1} f^{(2n-1)}(x)dx \geq \int_{x^*}^{1}(1-x^2)^{n-1} f^{(2n-1)}(x)dx \geq \frac{c}{2}\int_{x^*}^{1}\frac{dx}{1-x^2} = \infty,$$

contradicting (6.5.1). Hence

$$\lim_{x\to \pm 1}(1-x^2)^n f^{(2n-1)}(x) = 0. \tag{6.5.5}$$

Returning to (6.5.4), we now see that

$$A = -\int_{x_0}^{1} g(t)dt = \int_{-1}^{x_0} g(t)dt. \tag{6.5.6}$$

Consequently,

$$0 = A - A = \int_{-1}^{x_0} g(t)dt + \int_{x_0}^{1} g(t)dt = \int_{-1}^{1} g(t)dt,$$

proving part (i) of the Lemma. Moreover, we see from (6.5.6) and (6.5.4) that

$$(1-x^2)^n f^{(2n-1)}(x) = \int_{-1}^{x_0} g(t)dt + \int_{x_0}^{x} g(t)dt = \int_{-1}^{x} g(t)dt$$

and, hence,

$$f^{(2n-1)}(x) = \frac{1}{(1-x^2)^n}\int_{-1}^{x} g(t)dt \quad (x \in (-1,1)). \tag{6.5.7}$$

Similarly, from (6.5.6), we see that

$$(1-x^2)^n f^{(2n-1)}(x) = \int_1^{x_0} g(t)dt + \int_{x_0}^{x} g(t)dt = -\int_x^1 g(t)dt$$

so that

$$f^{(2n-1)}(x) = \frac{-1}{(1-x^2)^n}\int_x^1 g(t)dt \quad (x \in (-1,1)). \tag{6.5.8}$$

The identities in (6.5.7) and (6.5.8) establish (ii) in the Lemma. Furthermore, from (6.5.2) and (6.5.3), we see that

$$f^{(2n)}(x) = \frac{2nx(1-x^2)^{n-1}f^{(2n-1)}(x) + g(x)}{(1-x^2)^n} \quad (\text{a.e. } x \in (-1,1)),$$

establishing (iii). From (ii), we see that

$$\begin{aligned} f^{(2n-2)}(x) &= f^{(2n-2)}(x_0) + \int_{x_0}^{x} \frac{\int_{-1}^{t} g(u)du}{(1-t^2)^n}dt \\ &= f^{(2n-2)}(x_0) - \int_{x_0}^{x} \frac{\int_{t}^{1} g(u)du}{(1-t^2)^n}dt. \end{aligned}$$

Repeated integration of this identity establishes (iv) and completes the proof of the Lemma 6.5.1. □

Definition 6.5.1. For an $f \in H_n$ and a $g \in L^2(-1,1)$, we write $f \sim g$ if each of the following conditions are satisfied:

(i) $\int_{-1}^{1} g(t)dt = 0$;

(ii) $f^{(2n-1)}(x) = \dfrac{\int_{-1}^{x} g(t)dt}{(1-x^2)^n} = -\dfrac{\int_{x}^{1} g(t)dt}{(1-x^2)^n} \quad (x \in (-1,1))$;

(iii) $f^{(2n)}(x) = \dfrac{2nx}{1-x^2}f^{(2n-1)}(x) + \dfrac{g(x)}{(1-x^2)^n} \quad (\text{a.e. } x \in (-1,1))$;

(iv)

$$f(x) = \sum_{j=0}^{2n-1} f^{(j)}(x_0)\frac{(x-x_0)^j}{j!} + \int_{x_o}^{x}\int_{x_0}^{y_{2n-2}}\int_{x_0}^{y_{2n-1}}\cdots\int_{x_0}^{y_1} \frac{\int_{-1}^{t} g(u)du}{(1-t^2)^n}dtdy_1\ldots dy_{2n-2}$$

$$= \sum_{j=0}^{2n-1} f^{(j)}(x_0)\frac{(x-x_0)^j}{j!} - \int_{x_o}^{x}\int_{x_0}^{y_{2n-2}}\int_{x_0}^{y_{2n-1}}\cdots\int_{x_0}^{y_1} \frac{\int_{t}^{1} g(u)du}{(1-t^2)^n}dtdy_1\ldots dy_{2n-2}.$$

By Lemma 6.5.1., for each $f \in H_n$ there exists $g \in L^2(-1,1)$ such that $f \sim g$.

We now prove an important partial converse of Lemma 6.5.1.:

Lemma 6.5.2. Suppose $g \in L^2(-1,1)$ is such that

(*i*) $\int_{-1}^{1} g(t)dt = 0$ and

(*ii*) supp $g = [-1+\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$. Then, for any $x_0 \in (-1,1)$, and any polynomial $p_{2n-2}(x)$ of degree $\leq 2n-2$, the following formula defines an $f \in H_n$:

$$f(x) = p_{2n-2}(x) + \int_{x_o}^{x} \int_{x_0}^{y_{2n-2}} \int_{x_0}^{y_{2n-1}} \cdots \int_{x_0}^{y_1} \frac{\int_{-1}^{t} g(u)du}{(1-t^2)^n} dt dy_1 \ldots dy_{2n-2} \tag{6.5.9}$$

$$= p_{2n-2}(x) - \int_{x_o}^{x} \int_{x_0}^{y_{2n-2}} \int_{x_0}^{y_{2n-1}} \cdots \int_{x_0}^{y_1} \frac{\int_{t}^{1} g(u)du}{(1-t^2)^n} dt dy_1 \ldots dy_{2n-2}, \tag{6.5.10}$$

with the understanding that, when $n = 1$,

$$f(x) = A + \int_{x_0}^{x} \frac{\int_{-1}^{t} g(u)du}{(1-t^2)} dt = A - \int_{x_0}^{x} \frac{\int_{t}^{1} g(u)du}{(1-t^2)} dt$$

where $A \in \mathbb{C}$ is arbitrary.

Proof. Since $\int_{-1}^{1} g(t)dt = 0$, we see that

$$\int_{-1}^{t} g(u)du = -\int_{t}^{1} g(u)du$$

for all $t \in (-1,1)$. Define

$$h(t) := \frac{\int_{-1}^{t} g(u)du}{(1-t^2)^n} = -\frac{\int_{t}^{1} g(u)du}{(1-t^2)^n} \quad (t \in (-1,1)). \tag{6.5.11}$$

Clearly h is continuous on $(-1,1)$; furthermore, since $g(s) = 0$ for $|s| \geq 1-\varepsilon$, we see that $h(s) = 0$ for $|s| \geq 1-\varepsilon$ so, in fact, $h \in C[-1,1]$ and hence

$h \in L^2(-1,1)$. Thus, the function $f(x)$, defined in (6.5.9) and (6.5.10), is well-defined; in fact

$$f \in AC[-1,1]. \tag{6.5.12}$$

Moreover, it is clear from (6.5.9) and (6.5.10) that

$$f^{(j)} \in AC[-1,1] \quad (j = 0, 1, \ldots, 2n-2); \tag{6.5.13}$$

indeed, for $j = 1, 2, \ldots, 2n-2$,

$$\begin{aligned}
&f^j(x) \\
&= p_{2n-2}^{(j)}(x) + \int_{x_0}^{x} \int_{x_0}^{y_{2n-j-2}} \int_{x_0}^{y_{2n-j-3}} \cdots \int_{x_0}^{y_1} \frac{\int_{-1}^{t} g(u)du}{(1-t^2)^n} dt dy_1 \ldots dy_{2n-j-2} \\
&= p_{2n-2}^{(j)}(x) - \int_{x_0}^{x} \int_{x_0}^{y_{2n-j-2}} \int_{x_0}^{y_{2n-j-3}} \cdots \int_{x_0}^{y_1} \frac{\int_{t}^{1} g(u)du}{(1-t^2)^n} dt dy_1 \ldots dy_{2n-j-2}.
\end{aligned}$$

Furthermore, from

$$f^{(2n-2)}(x) = p_{2n-2}^{(2n-2)}(x) + \int_{x_0}^{x} \frac{\int_{-1}^{t} g(u)du}{(1-t^2)^n} dt = p_{2n-2}^{(2n-2)}(x) - \int_{x_0}^{x} \frac{\int_{t}^{1} g(u)du}{(1-t^2)^n} dt,$$

we see that

$$f^{(2n-1)}(x) = \frac{\int_{-1}^{x} g(u)du}{(1-x^2)^n} = -\frac{\int_{x}^{1} g(u)du}{(1-x^2)^n} \in AC_{\text{loc}}(-1,1). \tag{6.5.14}$$

Differentiating (6.5.14), we see that

$$\begin{aligned}
f^{(2n)}(x) &= \frac{(1-x^2)^n g(x) + 2nx(1-x^2)^{n-1} \int_{-1}^{x} g(u)du}{(1-x^2)^{2n}} \\
&= \frac{g(x)}{(1-x^2)^n} + \frac{2nxh(x)}{1-x^2}
\end{aligned}$$

so that

$$(1-x^2)^n f^{(2n)}(x) = g(x) + 2nx(1-x^2)^{n-1} h(x) \in L^2(-1,1). \tag{6.5.15}$$

From the CHEL inequality and (6.5.15), it now follows that

$$(1-x^2)^{(j-1)/2} f^{(j)} \in L^2(-1,1) \quad (j = 1, 2, \ldots, 2n-1). \tag{6.5.16}$$

Hence, from (6.5.12) - (6.5.16), we see that $f \in H_n$ as required. This completes the proof of this Lemma. $\square$

The following result will prove useful near the end of this section.

Lemma 6.5.3. In addition to the hypotheses in Lemma 6.5.2., suppose $g \in C(-1,1)$. Then f, given in (6.5.9) or (6.5.10), belongs to $C^{2n}[-1,1]$ for any choice of polynomial p_{2n-2}.

Proof. From (6.5.15), we see that

$$f^{(2n)}(x) = \frac{g(x)}{(1-x^2)^n} + \frac{2nxh(x)}{1-x^2} \in C(-1,1).$$

However, $g(x) = h(x) = 0$ for $|x| \geq 1-\varepsilon$ so $f^{(2n)} \in C[-1,1]$. This implies that $f \in C^{2n}[-1,1]$.$\square$

Recall that the n^{th} left-definite inner product $(\cdot,\cdot)_n$ is defined by

$$(f,g)_n := \frac{A}{2}\sum_{j=1}^{2n}\int_{-1}^{1}\left(a_j\left(n,k\right)(1-x^2)^j + b_j\left(n,k\right)(1-x^2)^{j-1}\right)f^{(j)}(x)\overline{g}^{(j)}(x)dx \tag{6.5.17}$$

$$+\,k^n(f,g)_\mu.$$

Let

$$(f,g)_{n,-1} := \frac{k^n}{2}\left(f(1)\overline{g}(1) + f(-1)\overline{g}(-1)\right)$$
$$(f,g)_{n,0} := k^n\int_{-1}^{1} f(t)\overline{g}(t)dt$$
$$(f,g)_{n,j} := \frac{A}{2}\int_{-1}^{1}\left(a_j\left(n,k\right)(1-x^2)^j + b_j\left(n,k\right)(1-x^2)^{j-1}\right)f^{(j)}(x)\overline{g}^{(j)}(x)dx$$

where $j = 1, 2, \ldots, 2n$, so that

$$(f,g)_n = \sum_{j=-1}^{2n}(f,g)_{n,j}.$$

Furthermore, let

$$\|f\|_{n,j} := (f,f)_{n,j}^{1/2} \quad (j = -1, 0, 1, \ldots, 2n)$$

and let $\|f\|$ denote the usual L^2 norm of $f \in L^2(-1,1)$. In our next result, we estimate

$$\|f\|_{n,j} \quad (j = -1, 0, 1, \ldots, 2n)$$

for a certain class of functions $f \in H_n$. Specifically, we prove

Lemma 6.5.4. Suppose $f \in H_n$ and suppose that there exists $x_0 \in (-1,1)$ such that

$$f(x_0) = f'(x_0) = \ldots = f^{(2n-2)}(x_0) = 0.$$

Choose $g \in L^2(-1,1)$ such that $f \sim g$ (using Lemma 6.5.1.). Then there exists $C_n > 0$, independent of f and g, such that

$$\|f\|_n \leq C_n \max\left\{\left\|f^{(2n-1)}\right\|, \|g\|\right\}. \tag{6.5.18}$$

Moreover, if g is supported on $[-1+\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$, then there exists $C_\varepsilon > 0$ such that

$$\|f\|_n \leq C_\varepsilon \|g\|. \tag{6.5.19}$$

Proof. We begin by making a few estimates. First, since $f(x_0) = 0$ we have for any $x \in (-1,1)$

$$|f(x)|^2 = \left|\int_{x_0}^{x} f'(t)dt\right|^2 \leq \left(\int_{x_0}^{x} |f'(t)|\, dt\right)^2$$

$$\leq \left(\int_{-1}^{1} |f'(t)|\, dt\right)^2 \leq 2\left(\int_{-1}^{1} |f'(t)|^2\, dt\right) = 2\|f'\|^2, \tag{6.5.20}$$

where the last inequality follows from Hölder's inequality. In particular,

$$|f(\pm 1)|^2 \leq 2\|f'\|^2. \tag{6.5.21}$$

Integrating both sides of (6.5.20), we obtain

$$\|f\|^2 = \int_{-1}^{1} |f(x)|^2\, dx \leq 4\|f'\|^2$$

so that

$$\|f\| \leq 2\|f'\|. \tag{6.5.22}$$

Since $f'(x_0) = 0$, we see that for any $x \in (-1, 1)$ that

$$|f'(x)|^2 = \left| \int_{x_0}^{x} f''(t) dt \right|^2 \leq \left(\int_{x_0}^{x} |f''(t)| \, dt \right)^2$$

$$\leq \left(\int_{-1}^{1} |f''(t)| \, dt \right)^2 \leq 2 \left(\int_{-1}^{1} |f''(t)|^2 \, dt \right) = 2 \|f''\|^2 ,$$

from which it follows that

$$\|f'\| \leq 2 \|f''\| .$$

More generally, since $f^{(j)}(x_0) = 0$ $(j = 0, 1, \ldots, 2n-2)$, we see that

$$\left\| f^{(j)} \right\| \leq 2 \left\| f^{(j+1)} \right\| \quad (j = 0, 1, \ldots, 2n-2), \tag{6.5.23}$$

and, after iterating, we find that

$$\left\| f^{(j)} \right\| \leq 2^{2n-1-j} \left\| f^{(2n-1)} \right\| \quad (j = 0, 1, \ldots, 2n-2). \tag{6.5.24}$$

In particular, from (6.5.21), we see that

$$|f(\pm 1)|^2 \leq 2^{2n-1} \left\| f^{(2n-1)} \right\|^2$$

and hence that

$$\|f\|_{n,-1} \leq 2^{(2n-1)/2} k^{n/2} \left\| f^{(2n-1)} \right\| . \tag{6.5.25}$$

Furthermore, from (6.5.24), we see that

$$\|f\|_{n,0} = k^{n/2} \|f\| \leq 2^{2n-1} k^{n/2} \left\| f^{(2n-1)} \right\| . \tag{6.5.26}$$

For $1 \leq j \leq 2n-1$, we see from (6.5.24) that

$$\|f\|_{n,j} = \sqrt{\frac{A}{2}} \left(\int_{-1}^{1} \left[a_j(n,k) (1-x^2)^j + b_j(n,k) (1-x^2)^{j-1} \right] \left| f^{(j)}(x) \right|^2 dx \right)^{1/2} \tag{6.5.27}$$

$$\leq M_j \left\| f^{(j)} \right\| \leq 2^{2n-1-j} M_j \left\| f^{(2n-1)} \right\| ,$$

where

$$M_j := \sqrt{\frac{A}{2}} \max_{x\in[-1,1]} (a_j\,(n,k)\,(1-x^2)^j + b_j\,(n,k)\,(1-x^2)^{j-1}).$$

Moreover,

$$\begin{aligned}
\|f\|_{n,2n} &= \sqrt{\frac{A}{2}} \left(\int_{-1}^{1} [(1-x^2)^n \left| f^{(2n)}(x) \right|]^2 dx \right)^{1/2} \\
&= \sqrt{\frac{A}{2}} \left(\int_{-1}^{1} \left| 2nx(1-x^2)^{n-1} f^{(2n-1)}(x) + g(x) \right|^2 dx \right)^{1/2} \\
&= \sqrt{\frac{A}{2}} \left\| 2nx(1-x^2)^{n-1} f^{(2n-1)} + g \right\| \\
&\leq \sqrt{\frac{A}{2}} \left(\left\| 2nx(1-x^2)^{n-1} f^{(2n-1)} \right\| + \|g\| \right),
\end{aligned}$$

and hence

$$\|f\|_{n,2n} \leq M_{2n} \max\{ \left\| f^{(2n-1)} \right\|, \|g\| \}. \tag{6.5.28}$$

Combining (6.5.25) - (6.5.28), we obtain statement (6.5.18) in the Lemma. If, in addition, we have supp $g \subset [-1+\varepsilon, 1-\varepsilon]$, then by part (ii) of Lemma 6.5.1., we see that

$$\begin{aligned}
\left| f^{(2n-1)}(x) \right| &= \left| \frac{\int_{-1}^{x} g(t)dt}{(1-x^2)^n} \right| \leq M_{\varepsilon,n} \int_{-1}^{1} |g(t)|\, dt \\
&\leq M_{\varepsilon,n} \left(\int_{-1}^{1} |g(t)|^2\, dt \right)^{1/2} \left(\int_{-1}^{1} 1^2 dt \right)^{1/2} \\
&= \sqrt{2} M_{\varepsilon,n} \|g\|,
\end{aligned}$$

where

$$M_{\varepsilon,n} := \max_{x\in[-1+\varepsilon,1-\varepsilon]} \frac{1}{(1-x^2)^n};$$

consequently

$$\left\| f^{(2n-1)} \right\| \leq \sqrt{2} M_{\varepsilon,n} \|g\|. \tag{6.5.29}$$

Substituting (6.5.29) into (6.5.28), we obtain (6.5.19) and this completes the proof of the Lemma 6.5.4. □

We prove one more fundamental lemma before the main results of this section.

Lemma 6.5.5. Let $f \in H_n$. Choose $g \in L^2(-1,1)$ such that $f \sim g$. Then there exists sequences $\{f_m\} \subset H_n$, $\{g_m\} \subset L^2(-1,1)$ such that

(i) $f_m \to f$ in H_n and $g_m \to g$ in $L^2(-1,1)$;

(ii) $f_m \sim g_m$;

(iii) g_m has support in $(-1,1)$.

Proof. We first note that

$$\liminf_{x\to 1^-} \frac{\left|\int_x^1 g(t)dt\right|}{(1-x^2)^{1/2}} = 0. \tag{6.5.30}$$

For if

$$\liminf_{x\to 1^-} \frac{\left|\int_x^1 g(t)dt\right|}{(1-x^2)^{1/2}} = \gamma > 0,$$

then, by definition of $f \sim g$, it follows that

$$(1-x^2)^{2n-2}\left|f^{(2n-1)}(x)\right|^2 = \frac{\left|\int_x^1 g(t)dt\right|^2}{(1-x^2)^2} \geq \frac{\gamma^2}{4(1-x)} \qquad (x \in (x^*,1)),$$

where x^* is sufficiently close to 1. However, integrating this last inequality over $(x^*,1)$ implies that $(1-x^2)^{n-1}f^{(2n-1)} \notin L^2(-1,1)$, contradicting the fact that $f \in H_n$. Similarly, we see that

$$\liminf_{x\to -1^+} \frac{\left|\int_{-1}^x g(t)dt\right|}{(1-x^2)^{1/2}} = 0.$$

Hence, it follows that there exists infinite sequences $\{a_m\}$, $\{b_m\}$ such that

$$a_1 < a_2 < \ldots < a_m < \ldots, \qquad b_1 > b_2 > \ldots b_m > \ldots$$

with $a_m \to 1$ and $b_m \to -1$ satisfying

$$\frac{\left|\int_{-1}^{b_m} g(t)dt\right|}{(1-b_m^2)^{1/2}} \to 0 \quad (m\to\infty),$$

and

$$\frac{\left|\int_{a_m}^1 g(t)dt\right|}{(1-a_m^2)^{1/2}} \to 0 \quad (m \to \infty).$$

Define

$$\alpha_m := \left|\int_{a_m}^1 g(t)dt\right|, \qquad \xi_m := \operatorname{sgn} \int_{a_m}^1 g(t)dt$$
$$\beta_m := \left|\int_{-1}^{b_m} g(t)dt\right|, \qquad \eta_m := \operatorname{sgn} \int_{-1}^{b_m} g(t)dt$$

so that $\alpha_m \xi_m = \int_{a_m}^1 g(t)dt$ and $\beta_m \eta_m = \int_{-1}^{b_m} g(t)dt$.

Since $\int_{a_m}^1 g(t)dt \to 0$ and $\int_{-1}^{b_m} g(t)dt \to 0$ as $m \to \infty$, we may assume that

$$\alpha_m \le a_m \text{ and } \beta_m \ge b_m \text{ for all } m \in \mathbb{N}. \tag{6.5.31}$$

Define, for each $m \in \mathbb{N}$,

$$g_m := g\chi_{[b_m,a_m]} + \xi_m \chi_{[0,\alpha_m]} + \eta_m \chi_{[\beta_m,0]};$$

here χ_S denotes the characteristic function of the set S. It is clear that the support of g_m is contained in $[b_m, a_m]$. Furthermore,

$$\begin{aligned}
\int_{-1}^1 g_m(t)dt &= \int_{b_m}^{a_m} g(t)dt + \alpha_m\xi_m - \eta_m\beta_m \\
&= \int_{a_m}^1 g(t)dt + \int_{-1}^{b_m} g(t)dt + \int_{b_m}^{a_m} g(t)dt \\
&= \int_{-1}^1 g(t)dt = 0.
\end{aligned}$$

Consequently, by Lemma 6.5.2., there exists $f_m \in H_n$ such that $f_m \sim g_m$. Choose $x_0 = 0$ and choose

$$p_{2n-2}(x) = \sum_{j=0}^{2n-2} \frac{f^{(j)}(0)x^j}{j!}$$

in Lemma 6.5.2. so that

$$\begin{aligned}
f_m(x) &= \sum_{j=0}^{2n-2} \frac{f^{(j)}(0)x^j}{j!} + \int_0^x \int_0^{y_{2n-2}} \int_0^{y_{2n-3}} \cdots \int_0^{y_1} \frac{\int_{-1}^t g(u)du}{(1-t^2)^n} dt dy_1 \dots dy_{2n-2} \\
&= \sum_{j=0}^{2n-2} \frac{f^{(j)}(0)x^j}{j!} - \int_0^x \int_0^{y_{2n-2}} \int_0^{y_{2n-3}} \cdots \int_0^{y_1} \frac{\int_t^1 g(u)du}{(1-t^2)^n} dt dy_1 \dots dy_{2n-2}.
\end{aligned}$$

With this choice, we see that $f_m^{(j)}(0) = f^{(j)}(0)$ $(j = 0, 1, \ldots, 2n-2)$ so, by Lemma 6.5.4., there exists $C_n > 0$ such that

$$\|f - f_m\|_n \leq C_n \max\left\{\left\|f^{(2n-1)} - f_m^{(2n-1)}\right\|, \|g - g_m\|\right\}.$$

Now

$$\begin{aligned}
\|g - g_m\| &= \left\|g - g\chi_{[b_m,a_m]} - \xi_m\chi_{[0,\alpha_m]} - \eta_m\chi_{[\beta_m,0]}\right\| \\
&\leq \left\|g - g\chi_{[b_m,a_m]}\right\| + \left\|\xi_m\chi_{[0,\alpha_m]}\right\| + \left\|\eta_m\chi_{[\beta_m,0]}\right\| \\
&= \left\|g - g\chi_{[b_m,a_m]}\right\| + \alpha_m + \beta_m \\
&= \left\|g\chi_{[-1,b_m)} + g\chi_{(a_m,1]}\right\| + \alpha_m + \beta_m \\
&\to 0 \text{ as } m \to \infty \text{ since } b_m \to -1,\ a_m \to 1,\ \alpha_m \to 0 \text{ and } \beta_m \to 0.
\end{aligned}$$

We now show that $\left\|f^{(2n-1)} - f_m^{(2n-1)}\right\| \to 0$ as $m \to \infty$. Since

$$\begin{aligned}
&\limsup_{m\to\infty} \left\|f^{(2n-1)} - f_m^{(2n-1)}\right\|^2 \leq \\
&\limsup_{m\to\infty} \int_{-1}^{b_m} \left|f^{(2n-1)}(t) - f_m^{(2n-1)}(t)\right|^2 dt + \limsup_{m\to\infty} \int_{b_m}^{0} \left|f^{(2n-1)}(t) - f_m^{(2n-1)}(t)\right|^2 dt \\
&+ \limsup_{m\to\infty} \int_{0}^{a_m} \left|f^{(2n-1)}(t) - f_m^{(2n-1)}(t)\right|^2 dt \\
&+ \limsup_{m\to\infty} \int_{a_m}^{1} \left|f^{(2n-1)}(t) - f_m^{(2n-1)}(t)\right|^2 dt, \qquad (6.5.32)
\end{aligned}$$

it suffices to show that each of the four terms on the right-hand side of (6.5.32) tend to zero as $m \to \infty$. By Lemma 6.5.2.,

$$\begin{aligned}
&\limsup_{m\to\infty} \int_{-1}^{b_m} \left|f^{(2n-1)}(t) - f_m^{(2n-1)}(t)\right|^2 dt \\
&= \limsup_{m\to\infty} \int_{-1}^{b_m} \left|f^{(2n-1)}(t) - \frac{\int_{-1}^{t} g_m(u)du}{(1-t^2)^n}\right|^2 dt \\
&= \limsup_{m\to\infty} \int_{-1}^{b_m} \left|f^{(2n-1)}(t)\right|^2 dt \text{ since } g_m(t) = 0 \text{ for } t \in (-1, b_m) \\
&= 0 \text{ since } b_m \to -1.
\end{aligned}$$

Similarly, we see that

$$\limsup_{m\to\infty} \int_{a_m}^{1} \left| f^{(2n-1)}(t) - f_m^{(2n-1)}(t) \right|^2 dt = 0.$$

Now

$$\begin{aligned}
&\limsup_{m\to\infty} \int_{b_m}^{0} \left| f^{(2n-1)}(t) - f_m^{(2n-1)}(t) \right|^2 dt \\
&= \limsup_{m\to\infty} \int_{b_m}^{0} \left| \frac{\int_{-1}^{t} g(u)du}{(1-t^2)^n} - \frac{\int_{-1}^{t} g_m(u)du}{(1-t^2)^n} \right|^2 dt \\
&= \limsup_{m\to\infty} \int_{b_m}^{0} \left| \frac{\int_{-1}^{t} (g(u) - g_m(u))du}{(1-t^2)^n} \right|^2 dt \\
&= \limsup_{m\to\infty} \int_{b_m}^{0} \left| \frac{\int_{-1}^{t} (g(u)\chi_{[-1,b_m)} + g(u)\chi_{(a_m,1]} - \xi_m \chi_{[0,\alpha_m]} - \eta_m \chi_{[\beta_m,0]})}{(1-t^2)^n} \right|^2 dt \\
&= \limsup_{m\to\infty} \int_{b_m}^{0} \left| \frac{\int_{-1}^{t} (g(u)\chi_{[-1,b_m)} - \eta_m \chi_{[\beta_m,0]})}{(1-t^2)^n} \right|^2 dt
\end{aligned}$$

$\big($since $\chi_{(a_m,1]}(t) = \chi_{[0,\alpha_m]}(t) = 0$ for $t \in [b_m, 0]\big)$

$$\begin{aligned}
&\limsup_{m\to\infty} \int_{b_m}^{0} \left| f^{(2n-1)}(t) - f_m^{(2n-1)}(t) \right|^2 dt \\
&= \limsup_{m\to\infty} \int_{b_m}^{0} \left| \frac{\int_{-1}^{b_m} g(u)du - \eta_m \int_{-1}^{t} \chi_{[\beta_m,0]}(u)du}{(1-t^2)^n} \right|^2 dt.
\end{aligned}$$

Since $|\eta_m| \le 1$ and $\int_{-1}^{t} \chi_{[\beta_m,0]}(u)du \le -\beta_m$ for $t < 0$, we see that

$$\begin{aligned}
&\left| \int_{-1}^{b_m} g(u)du - \eta_m \int_{-1}^{t} \chi_{[\beta_m,0]}(u)du \right|^2 \\
&\le \left(\left| \int_{-1}^{b_m} g(u)du \right| + \beta_m \right)^2 \\
&= 4 \left| \int_{-1}^{b_m} g(u)du \right|^2 \text{ by definition of } \beta_m.
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\limsup_{m\to\infty} \int_{b_m}^{0} \left|f^{(2n-1)}(t) - f_m^{(2n-1)}(t)\right|^2 dt \\
&\le 4 \limsup_{m\to\infty} \left|\int_{-1}^{b_m} g(u)du\right|^2 \cdot \int_{b_m}^{0} \frac{dt}{(1-t^2)^{2n}} \\
&\le 4 \limsup_{m\to\infty} \left|\int_{-1}^{b_m} g(u)du\right|^2 \cdot \int_{b_m}^{0} \frac{dt}{(1+t)^{2n}} \\
&\le 4 \limsup_{m\to\infty} \left|\int_{-1}^{b_m} g(u)du\right|^2 \cdot \int_{b_m}^{0} \frac{dt}{(1+t)^{2}} \text{ since } (1+t)^{2n} \ge (1+t)^2 \text{ for } -1 < t \le 0 \\
&\le 4 \limsup_{m\to\infty} \left|\int_{-1}^{b_m} g(u)du\right|^2 \cdot \left[-1 + \frac{1}{1+b_m}\right] \\
&\le 4 \limsup_{m\to\infty} \frac{\left|\int_{-1}^{b_m} g(u)du\right|^2}{1+b_m} \\
&\le 8 \limsup_{m\to\infty} \frac{\left|\int_{-1}^{b_m} g(u)du\right|^2}{1-b_m^2} \text{ since } \frac{2}{1-b_m} \ge 1 \\
&= 0 \text{ by (6.5.30).}
\end{aligned}
$$

Similarly,

$$
\limsup_{m\to\infty} \int_{a_m}^{1} \left|f^{(2n-1)}(t) - f_m^{(2n-1)}(t)\right|^2 dt = 0
$$

and this completes the proof of the Lemma 6.5.5. □

We now arrive at one of the main results of this section.

Theorem 6.5.1. For each $n \in \mathbb{N}$, the space $C^{2n}[-1,1]$ of all $2n$-times continuously differentiable complex-valued functions on $[-1,1]$ is dense in H_n.

Proof. Let $f \in H_n$. Choose $g \in L^2(-1,1)$ such that $f \sim g$. By Lemma 6.5.5., there exist sequences $\{f_m\} \subset H_n$, $\{\widetilde{g}_m\} \subset L^2(-1,1)$ such that

(*i*) $f_m \to f$ in H_n and $\widetilde{g}_m \to g$ in $L^2(-1,1)$;

(*ii*) $f_m \sim \widetilde{g}_m$;

(*iii*) supp $\widetilde{g}_m \subset (-1,1)$.

Without loss of generality, we assume supp $g = [-1+\varepsilon, 1-\varepsilon]$ for some $0 < \varepsilon < 1$. Since $C[-1,1]$ is dense in $L^2(-1,1)$, there exists a sequence $\{g_m\} \subset L^2(-1,1)$ such that

$$\|g_m - g\| \to 0 \text{ as } m \to \infty.$$

Choose $\{\delta_m\}$ such that $\delta_1 > \delta_2 > \ldots > \delta_m > \ldots$, $\delta_m \to 0$, and $\varepsilon + \delta_m \leq 1$ for all $m \in \mathbb{N}$. Define

$$K_m(x) = \begin{cases} 0 & \text{if } |x| \geq 1-\varepsilon \\ 1 & \text{if } |x| \leq 1-\varepsilon-\delta_m \\ \frac{x+1-\varepsilon}{\delta_m} & \text{if } -1+\varepsilon < x \leq -1+\varepsilon+\delta_m \\ \frac{1-\varepsilon-x}{\delta_m} & \text{if } 1-\varepsilon-\delta_m \leq x \leq 1-\varepsilon. \end{cases}$$

Then,

$$\begin{aligned} &\|g - K_m g\|^2 \\ &= \int_{-1}^{-1+\varepsilon} |g(x) - K_m(x)g(x)|^2\, dx + \int_{-1+\varepsilon}^{-1+\varepsilon+\delta_m} |g(x) - K_m(x)g(x)|^2\, dx \\ &+ \int_{-1+\varepsilon+\delta_m}^{1-\varepsilon-\delta_m} |g(x) - K_m(x)g(x)|^2\, dx + \int_{1-\varepsilon-\delta_m}^{1-\varepsilon} |g(x) - K_m(x)g(x)|^2\, dx \\ &+ \int_{1-\varepsilon}^{1} |g(x) - K_m(x)g(x)|^2\, dx \\ &= \int_{-1+\varepsilon}^{-1+\varepsilon+\delta_m} |g(x) - K_m(x)g(x)|^2\, dx + \int_{1-\varepsilon-\delta_m}^{1-\varepsilon} |g(x) - K_m(x)g(x)|^2\, dx, \end{aligned}$$

where we use the facts that $g(x) = 0$ for $|x| \geq 1-\varepsilon$ and $g(x) - K_m(x)g(x) = 0$ for $|x| \leq 1-\varepsilon-\delta_m$. Moreover, since $0 \leq K_m(x) \leq 1$, we see that

$$\int_{-1+\varepsilon}^{-1+\varepsilon+\delta_m} |g(x) - K_m(x)g(x)|^2\, dx \leq \int_{-1+\varepsilon}^{-1+\varepsilon+\delta_m} |g(x)|^2\, dx \to 0$$

as $m \to \infty$ since $\delta_m \to 0$; similarly,

$$\int_{1-\varepsilon-\delta_m}^{1-\varepsilon} |g(x) - K_m(x)g(x)|^2\, dx \leq \int_{1-\varepsilon-\delta_m}^{1-\varepsilon} |g(x)|^2\, dx \to 0 \text{ as } m \to \infty.$$

Hence we see that

$$\|g - K_m g\|^2 \to 0 \text{ as } m \to \infty. \tag{6.5.33}$$

Moreover, since

$$\|K_m(g - g_m)\|^2 \leq \|g - g_m\|^2 \to 0 \text{ as } m \to \infty,$$

we see that

$$\|g - K_m g_m\| \leq \|g - K_m g\| + \|K_m g - K_m g_m\|$$
$$\to 0 \text{ as } m \to \infty.$$

Consequently, it follows that $K_m g_m \to g$ in $L^1(-1,1)$ as $m \to \infty$ and

$$\int_{-1}^{1} K_m(x) g_m(x) dx \to \int_{-1}^{1} g(x) dx = 0 \quad (m \to \infty). \tag{6.5.34}$$

Let

$$\varepsilon_m := \min \left\{ \varepsilon, \left| \int_{-1}^{1} K_m(x) g_m(x) dx \right| + \frac{1}{m} \right\};$$

clearly $\varepsilon_m \to 0$ as $m \to \infty$. For each $m \in \mathbb{N}$, define

$$\widehat{g}_m(x) = K_m(x) g_m(x) - \left(\int_{-1}^{1} K_m(x) g_m(x) dx \right) h_m(x),$$

where

$$h_m(x) = \begin{cases} 0 & \text{if } |x| \leq \varepsilon_m \\ \frac{x+\varepsilon_m}{\varepsilon_m^2} & \text{if } -\varepsilon_m \leq x \leq 0 \\ \frac{\varepsilon_m - x}{\varepsilon_m^2} & \text{if } 0 \leq x \leq \varepsilon_m. \end{cases}$$

It is straightforward to check that

$$\|h_m\| = \sqrt{\frac{2\varepsilon_m}{3}} \quad (m \in \mathbb{N}), \tag{6.5.35}$$

and since

$$\int_{-1}^{1} h_m(x) dx = 1,$$

we see that

$$\int_{-1}^{1} \widehat{g}_m(x) dx = 1 \quad (m \in \mathbb{N}).$$

Furthermore, from (6.5.33), (6.5.34), and (6.5.35), we see that

$$\begin{aligned} \|\widehat{g}_m - g\| &\leq \|\widehat{g}_m - K_m g_m\| + \|K_m g_m - g\| \\ &\leq \left| \int_{-1}^{1} K_m(x) g_m(x) dx \right| \|h_m\| + \|K_m g_m - g\| \\ &\to 0 \text{ as } m \to \infty. \end{aligned} \tag{6.5.36}$$

Since $\widehat{g}_m$ is continuous and supp $\widehat{g}_m \subset [-1+\varepsilon, 1-\varepsilon]$ for sufficiently large m, we see from Lemmas 6.5.2. and 6.5.3. that there exists $f_m \in C^{2n}[-1,1]$ such that, for sufficiently large m, we have

$$\widehat{g}_m \sim f_m.$$

In fact, from Lemmas 6.5.2. and 6.5.3., we can choose

$$\begin{aligned} &f_m(x) \\ &= \sum_{j=0}^{2n-2} \frac{f^{(j)}(0)x^j}{j!} + \int_0^x \int_0^{y_{2n-2}} \int_0^{y_{2n-3}} \cdots \int_0^{y_1} \frac{\int_{-1}^t \widehat{g}_m(u)du}{(1-t^2)^n} dt dy_1 \ldots dy_{2n-2} \\ &= \sum_{j=0}^{2n-2} \frac{f^{(j)}(0)x^j}{j!} - \int_0^x \int_0^{y_{2n-2}} \int_0^{y_{2n-3}} \cdots \int_0^{y_1} \frac{\int_t^1 \widehat{g}_m(u)du}{(1-t^2)^n} dt dy_1 \ldots dy_{2n-2}. \end{aligned}$$

Since $f_m^{(j)}(0) = f^{(j)}(0)$ for $j = 0, 1, \ldots, 2n-2$, we see from Lemma 6.5.4. that there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} \|f_m - f\|_n &\leq C_\varepsilon \|\widehat{g}_m - g\| \\ &\to 0 \text{ as } m \to \infty \text{ by (6.5.36).} \end{aligned}$$

This completes the proof of the theorem. □

In [25], Everitt, Littlejohn, and William showed that in certain Sobolev spaces, polynomials are dense. More specifically, if $[a,b]$ is a compact interval of the real line $\mathbb{R}$, suppose μ_j is a positive, finite measure on the Borel subsets of $[a,b]$ for each $j = 0, 1, \ldots, N$, where $N \in \mathbb{N}$. Define, for $f, g \in C^N[a,b]$, the inner product

$$(f,g)_N := \sum_{j=0}^{N} \int_a^b f^{(j)}(x)\overline{g}^{(j)}(x) d\mu_j. \tag{6.5.37}$$

Observe that $(\cdot,\cdot)_N$ induces the norm

$$\|f\|_N := \left(\sum_{j=0}^{N} \int_a^b \left|f^{(j)}(x)\right|^2 d\mu_j \right)^{1/2} \qquad (f \in C^N[a,b]).$$

Define the space $H_N[a,b]$ to be the completion of $C^N[a,b]$ in the topology generated by the norm $\|f\|_N$. In [25], the authors prove the following theorem:

Theorem 6.5.2. The space $\mathcal{P}$ of all complex-valued polynomials in the real variable x are dense in $H_N[a,b]$.

In our situation, the left-definite Legendre type inner product $(\cdot,\cdot)_n$ is a special case of the inner product in (6.5.37). And since Theorem 6.5.1. shows that the space H_n is the completion of $C^{2n}[a,b]$, we can apply Theorem 6.5.2. and conclude:

Theorem 6.5.3. For each $n \in \mathbb{N}$, the set of polynomials $\mathcal{P}$ forms a dense subset of H_n. Equivalently, the Legendre type polynomials $\{P_{m,A}\}_{m=0}^{\infty}$ form a complete orthogonal set in each H_n.

This theorem will be important in establishing that H_n is the n^{th} left-definite space associated with the pair $(T, L^2_\mu[-1,1])$ which we prove in the next section.

6.6 H_n is the n^{th} Left-Definite Space

Recall the self-adjoint operator T in $L^2_\mu[-1,1]$ discussed in Chapter 2 defined by

$$T[f](x) = \begin{cases} -8Af'(-1) + kf(-1) & \text{if } x = -1 \\ \ell[f](x) & \text{if } -1 < x < 1 \\ 8Af'(1) + kf(1) & \text{if } x = 1, \end{cases}$$

$$\begin{aligned} f \in \mathcal{D}(T) = \{ f : (-1,1) \to \mathbf{C} \mid f^{(j)} \in AC_{\text{loc}}(-1,1) \ (j = 0,1,,\ldots,2n-1); \\ f, \ell[f] \in L^2(-1,1)\}, \end{aligned}$$

where $\ell[\cdot]$ is the Legendre type differential expression, defined in (5.1.1).

We are now in position to prove the main result of this thesis.

Theorem 6.6.1. For each $n \in \mathbb{N}$, let V_n be given as in (6.2.1) or (6.2.2) and let $(\cdot,\cdot)_n$ denote the inner product defined in (6.2.3). Then, $H_n = (V_n, (\cdot,\cdot)_n)$ is the n^{th} left-definite space for the pair $(T, L^2_\mu[-1,1])$. Moreover, the Legendre type polynomials $\{P_{m,A}\}_{m=0}^{\infty}$ form a complete orthogonal set in H_n satisfying the orthogonality relation

(1.5.2). Furthermore, define $T_n : \mathcal{D}(T_n) \subseteq H_n \to H_n$ by

$$T_n[f] = \ell[f];$$
$$f \in \mathcal{D}(T_n) := V_{n+2,}$$

where $\ell[\cdot]$ is the Legendre type differential expression defined in (5.1.1). Then T_n is the n^{th} left-definite operator associated with the pair $(T, L_\mu^2[-1,1])$. Also, the Legendre type polynomials $\{P_{m,A}\}_{m=0}^{\infty}$ are eigenfunctions of T_n and the spectrum of T_n is given by

$$\sigma(T_n) = \left\{m(m+1)\left(m^2+m+4A-2\right)+k\,|m \in \mathbb{N}_0\right\}.$$

Proof. To show that $H_n = (V_n, (\cdot,\cdot)_n)$ is the n^{th} left-definite space associated with the pair $(T, L_\mu^2[-1,1])$, we must show that the five conditions in Definition 3.2.2 are satisfied.

(i) $H_n = (V_n, (\cdot,\cdot)_n)$ is a Hilbert space:

The proof of (i) is given in Theorems 6.3.3. and 6.4.1.

(ii) $\mathcal{D}(T^n) \subseteq V_n \subseteq L_\mu^2[-1,1]$:

Let $f \in \mathcal{D}(T^n)$. Since the Legendre type polynomials $\{P_{m,A}\}_{m=0}^{\infty}$ form a complete orthonormal set in $L_\mu^2[-1,1]$, we see that

$$p_j = \sum_{m=0}^{j} c_m(A)\,P_{m,A} \to f \text{ in } L_\mu^2[-1,1] \text{ as } (j\to\infty), \tag{6.6.1}$$

where $\{c_m(A)\}_{m=0}^{\infty}$ are the Fourier coefficients of f in $L_\mu^2[-1,1]$ defined by

$$c_m(A) = (f, P_{m,A})_\mu = \int_{-1}^{1} f P_{m,A}\,d\mu \ \ (m \in \mathbb{N}_0).$$

Since $T^n f \in L_\mu^2[-1,1]$, we see that

$$\sum_{m=0}^{j} \tilde{c}_m(A)\,P_{m,A} \to T^n f \text{ in } L_\mu^2[-1,1] \text{ as } (j\to\infty),$$

where

$$\begin{aligned}\tilde{c}_m(A) &= (T^n f\ , P_{m,A})_\mu \\ &= (f\ , T^n P_{m,A})_\mu \\ &= k_m^n (f\ , P_{m,A})_\mu \\ &= k_m^n c_m(A) \\ &= ((m^2+m)(m^2+m+4A-2)+k)^n c_m(A);\end{aligned}$$

that is to say,

$$T^n p_j = \sum_{m=0}^{\infty} k_m^n c_m(A) P_{m,A} \to T^n f \text{ in } L_\mu^2[-1,1] \text{ as } (j \to \infty).$$

Moreover, from Theorem 6.2.1., we see that

$$\begin{aligned}\|p_j - p_r\|_n^2 &= (p_j - p_r, p_j - p_r)_n \\ &= (T^n [p_j - p_r]\ , p_j - p_r)_\mu \to 0 \text{ as } j,\, r \to \infty;\end{aligned}$$

that is to say, $\{p_j\}_{m=0}^{\infty}$ is Cauchy in H_n. Since H_n is complete (see Theorem 6.4.1.), we see that there exists

$$g \in H_n \text{ such that } p_j \to g \text{ in } H_n \text{ as } (j \to \infty).$$

Furthermore, by definition of $(\cdot,\cdot)_n$ and the fact that

$$a_0(n,k) = k^n \text{ for } k > 0,$$

we see that

$$\|p_j - p_r\|_n^2 \geq k^n \|p_j - p_r\|_\mu^2;$$

hence,

$$p_j \to g \text{ in } L_\mu^2[-1,1] \text{ as } (j \to \infty). \tag{6.6.2}$$

Comparing (6.6.1) and (6.6.2), we see that $f = g \in H_n$.

That is to say, $f \in V_n$; this completes the proof of (ii).

(iii) $\mathcal{D}(T^n)$ is dense in H_n:

Since polynomials are contained in $\mathcal{D}(T^n)$ and are dense in H_n,we see that $\mathcal{D}(T^n)$ is dense in H_n. Furthermore, from Theorem 6.5.3., we see that the Legendre type polynomials $\{P_{m,A}\}_{m=0}^{\infty}$ form a complete orthogonal set in each H_n.

(iv) $(f,f)_n \geq k^n (f,f)_\mu$ for all $f \in V_n$:

This is clear from the definition of $(\cdot,\cdot)_n$, the positivity of the coefficients $a_j(n,k)$ and $b_j(n,k)$, and the fact that $a_0(n,k) = k^n$ and $b_0(n,k) = 0$.

(v) $(f,g)_n = (T^n f, g)_\mu$ for $f \in \mathcal{D}(T^n)$ and $g \in V_n$:

Observe that this identity is true for all f, $g \in \mathcal{P}$. Let $f \in \mathcal{D}(T^n)$ and $g \in V_n$; since polynomials are dense in both H_n and $L^2_\mu[-1,1]$ and convergence in H_n implies convergence in $L^2_\mu[-1,1]$, there exist sequences of polynomials $\{p_j\}_{m=0}^{\infty}$ and $\{q_j\}_{m=0}^{\infty}$ in polynomials $\mathcal{P}$ such that, as $j \to \infty$,

$$p_j \to f \text{ in } H_n,\ T^n p_j \to T^n f \text{ in } L^2_\mu[-1,1] \text{ as } (j \to \infty),$$

(see the proof of (ii) of this theorem), and

$$q_j \to g \text{ in } H_n \text{ and } L^2_\mu[-1,1].$$

Hence, from Theorem 6.2.1.,

$$\begin{aligned} (T^n f\ ,g)_\mu &= \lim_{j\to\infty} (T^n p_j\ ,q_j)_\mu \\ &= \lim_{j\to\infty} (p_j\ ,q_j)_n \\ &= (f,g)_n\,. \end{aligned}$$

This proves (v). The rest of the proof follows immediately from Theorems 3.3.1., 3.3.2., and 3.3.5. □

The next corollary follows immediately from Theorems 3.3.1., 6.3.3., and 6.6.1. Remarkably, it characterizes the domain of each of the integral powers of T.

Corollary 6.6.1. For each $n \in \mathbb{N}$, the domain $\mathcal{D}(T^n)$ of the n^{th} power T^n of the self-adjoint Legendre type operator T, defined in (6.6.2.), is given by

$$\mathcal{D}(T^n) = V_{2n} = \{f : [-1,1] \to \mathbf{C} \mid f \in AC[-1,1]; f^{(j)} \in AC_{\text{loc}}(-1,1)$$
$$(j = 1, 2, \ldots, 4n-1); (1-x^2)^{2n} f^{(4n)} \in L^2(-1,1)\}.$$

In particular,

$$\mathcal{D}(T) = V_2 = \{f : [-1,1] \to \mathbf{C} \mid f \in AC[-1,1]; f', f'', f''' \in AC_{\text{loc}}(-1,1)\ ;$$
$$(1-x^2)^2 f^{(4)} \in L^2(-1,1)\}.$$

This is a new characterization of $\mathcal{D}(T)$ and does not include any boundary conditions. From Theorems 3.3.2. and 6.6.1., it follows that the domain of the first left-definite operator T_1 is given by

$$\mathcal{D}(T_1) = V_3 = \{f : [-1,1] \to \mathbf{C} \mid f \in AC[-1,1]; f', f'', \ldots, f^{(5)} \in AC_{\text{loc}}(-1,1)\ ;$$
$$(1-x^2)^3 f^{(6)} \in L^2(-1,1)\}.$$

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