ABSTRACT

Diagrams and Reduced Decompositions
for Cominuscule Flag Varieties and Affine Grassmannians

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We develop a system of canonical reduced decompositions of minimal coset representatives of quotients corresponding to cominuscule flag varieties and affine Grassmannians. This canonical decomposition allows, in the first case, an abbreviated computation of relative R-polynomials. From this, we show that these polynomials can be obtained from unlabelled intervals, and more generally, that Kazhdan-Lusztig polynomials associated to cominuscule flag varieties are combinatorially invariant. In the second case, we are able to provide a list of the rationally smooth Schubert varieties in simply laced affine Grassmannians corresponding to types A, D, and E. The results in this case were obtained independently by Billey and Mitchell in 2008.

Diagrams and Reduced Decompositions
for Cominuscule Flag Varieties and Affine Grassmannians

by

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A Dissertation

Approved by the Department of Mathematics

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Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

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Accepted by the Graduate School
May 2010

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Page bearing signatures is kept on file in the Graduate School.

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ACKNOWLEDGMENTS

I begin by thanking my wife Taylor: without your patience, love, and support, this work would never have been possible. I thank Emmy and Patrick for keeping me sane though this process.

I am grateful to my advisor, Markus Hunziker, who started me on a path of most interesting questions, and who always had a listening ear for my ideas.

I am thankful for Brian Williams, Casey Sherman, Britney Hopkins, and most of all Leslie Jones, for their support and friendship. This would not have been nearly so fun without you all by my side.

I thank my comittee for their time and suggestions.

Finally, I am thankful for my parents, who laid the groundwork for my scientific curiosity. Without a good foundation, none of this could have happened.

CHAPTER ONE

Introduction

1.1 Overview

In this thesis, we will present a canonical combinatorial presentation of reduced decompositions of minimal coset representatives associated with cominuscule flag varieties and affine Grassmannians. These have been described previously as normal form decompositions, but our presentation allows one to read critical information pertaining to geometry and representation theory directly from reduced decompositions. We will use this technique to solve two problems in disparate areas of representation theory. First, for cominuscule Schubert varieties, we will construct an algorithm that produces relative R-polynomials uniformly, that is, without regard to the type of the variety. Second, we will categorize for the sets of rationally smooth Schubert varieties of affine Grassmannians.

As an application of the first, we are able to prove that the relative R polynomials, and thus the relative Kazhdan-Lusztig polynomials, are combinatorially invariant. That is to say, if order intervals from distinct cominuscule varieties are isomorphic, then they share a relative R-polynomial. This confirms a conjecture of Kazhdan, Lusztig, and Dyer in the special case of generalized flag manifolds of cominuscule type.

As an application of the second, we categorize those elements of affine Grassmannians corresponding to rationally smooth Schubert varieties. Our method is to determine which intervals $[e, w]$ are palindromic, and then apply the Carrell-Peterson criterion. The second problem was solved simultaneously with Billey and Mitchell [2], using very similar methods.

1.2 *History of the Problems*

1.2.1 *Kazhdan-Lusztig R-polynomials and Combinatorial Invariance*

In 1979, David Kazhdan and George Lusztig proved the existence of a set of polynomials, henceforth called the Kazhdan-Lusztig polynomials, indexed by $W \times W$ for a Coxeter group W. These polynomials indicated a change of basis from the natural basis in the Hecke algebra over W to a basis fixed by the bar involution $^{-}$ (see Section 2.4.2). These polynomials are used in the construction of irreducible representations of Hecke algebras, and they stand at the intersection of the core concepts of Bruhat order and the algebraic geometry of Schubert varieties. Due to their importance, they have been extensively studied; see [10] for an overview.

An auxiliary set of polynomials is used to define the Kazhdan-Lusztig polynomials; they are referred to as R-polynomials. By construction, knowledge of R-polynomials is equivalent to knowledge about Kazhdan-Lusztig polynomials, but they are more amenable to combinatorial exploration. In particular, they can be computed from a simple recursion based on Bruhat order [31, Equations 2.0 a,b,c]. Furthermore, a variable transform turns R-polynomials into Brenti's $\widetilde{R}$ polynomials, which have numerous combinatorial interpretations, e.g. [7], [8], [9], [29].

One of the more important results obtained by Kazhdan and Lusztig was that, in the flag varieties G/B for G an algebraic group and B a Borel subgroup, the intersection cohomology of Schubert varieties is given by the coefficients of the Kazhdan-Lusztig polynomials. Deodhar [18] considered the generalized flag varieties G/P and replicated Kazhdan and Lusztig's results there. The result was a set of relative Kazhdan-Lusztig and R-polynomials indexed by W/W_J, where W_J is the Weyl group of the parabolic subgroup P. In the case that the parabolic subgroup W_J is the trivial group, then Deodhar's relative R-polynomials become the standard R-polynomials.

One of the most tantalizing conjectures about Kazhdan-Lusztig polynomials is that they are combinatorially invariant, that is, that $P_{u,v}(q)$ depends only on the abstract poset $[u, v]$, and not on the group $[u, v]$ lies within. This was conjectured independently by Lusztig [32] and Dyer [19]. The calculation of the Kazhdan-Lusztig polynomial $P_{u,v}(q)$ for $u \leq v$ in some Coxeter group W is known to be dependent on the edge-labelled Hasse diagram for the interval $[u, v]$ in the poset $(W, \leq_{\text{Bruhat}})$. Combinatorial invariance would mean that the edge labeling could be dropped, thus rendering the Kazhdan-Lusztig polynomials dependent solely on the poset structure. The same conjecture is an open question for Deodhar's relative polynomials.

Much of the difficulty in pursuing a proof of combinatorial invariance is that no general combinatorial interpretation is known for the Kazhdan-Lusztig polynomials. In specific subcases, notably the relative Kazhdan-Lusztig polynomials associated to cominuscule generalized flag varieties, combinatorial bijections have been found in [33] and [5]. The methods used in the various groups, though similar in spirit, were disparate and thus of no use to proof of the conjecture in this case.

A few cases of combinatorial invariance have been solved. Brenti verified the conjecture within Type A Coxeter groups in [11]. In Kazhdan and Lusztig's original paper [31] it was shown that if $\ell(u, v) \leq 4$, then the R-polynomials do not depend on edge labellings. Furthermore, if $[u, v]$ is a lattice, then $R_{u,v}(q)$ depends only on the covering relations as was noted in [6]. The largest class of R-polynomials known to be combinatorial invariants are those of the form $R_{e,v}(q)$, where e is the identity element of W. This was shown by Delanoy in [15] using the technique of special matchings. In this paper, we gain a handhold on the problem by using a bijection between minimal coset representatives of W/W_J in the case that G/P is cominuscule with certain classes of partitions. This bijection in addition to a marking schema yield an algorithm to determine relative R-polynomials explicitly, and this technique gives us the main theorems from the first half of this work:

Theorem 1.1. *If $[u, v]$ is a Bruhat interval in W/W_J, and is poset isomorphic to $[u', v']$ in $W'/W_{J'}$, with W/W_J and $W'/W'_{J'}$ associated to cominuscule flag varieties, then $R^J_{u,v}(q) = R^{J'}_{u',v'}(q)$.*

As the relative Kazhdan-Lusztig polynomials are derived directly from the relative R-polynomials, this theorem has a clear corollary.

Corollary 1.1. *If $[u, v]$ is poset isomorphic to $[u', v']$, and both posets are subposets of Bruhat intervals associated with cominuscule flag varieties, then $P^J_{u,v}(q) = P^J_{u',v'}(q)$.*

Thus we will resolve the combinatorial invariance conjecture for the case in which G/P is a cominuscule flag variety.

1.2.2 Rationally Smooth Schubert Varieties

We give here an overview of rational smoothness of Schubert varieties. The underlying idea is that one can approximate smoothness using purely cohomological criteria. What we mean by this is that there are cohomological characteristics of smooth points, and any point for which these characteristics hold is called rationally smooth. Rational smoothness is weaker than smoothness; indeed in $\widetilde{A}_n/A_n$, we will construct an infinite class of Schubert varieties that are rationally smooth, but are not smooth.

We call a point x in an irreducible variety X of pure dimension d rationally smooth provided there is some open set $U \ni x$ in the analytic topology such that for all $y \in U$, the singular cohomology

$$H^j(X, X\backslash\{y\}, \mathbb{Q}) = \begin{cases} 0 & j \neq 2d \\ \mathbb{Q} & j = 2d. \end{cases}$$

If each point in X is rationally smooth, then X is said to be rationally smooth. Note that smooth points are rationally smooth, and the set of rationally smooth points is open in the Zariski topology.

If X is a complex projective variety, as Schubert varieties are, then it is due to McCrory [34] that X is rationally smooth if and only if the ordinary cohomology $H^*(X)$ over $\mathbb{C}$ admits Poincaré duality. Equivalently, the intersection cohomology and ordinary cohomology groups of X over $\mathbb{C}$ coincide, which implies that the Poincaré polynomial of X must be symmetric.

The basic test of rational smoothness is due to Kazhdan and Lusztig. Begin by noting the coefficients of the Kazhdan-Lusztig polynomial $P_{u,w}(q)$ are the dimensions of the intersection cohomology of the Schubert varieties X_w taken at the point e_u [32]. Combining this with the definition of rational smoothness and the properties of irreducible complex projective varieties given above, yields a succinct test for rational smoothness: $v \in X_w$ is rationally smooth provided $P_{v,w}(q) = 1$. Because all affine Weyl groups are Weyl groups of Kac-Moody groups, these results apply to our situation.

The difficulty with this particular test for rational smoothness is that the Kazhdan-Lusztig polynomials are difficult to compute. Indeed, no combinatorial characterization is known; even the nonnegativity of their coefficients cannot currently be proven for general Coxeter groups. Carrell and Peterson [13] obtained a criterion (see Theorem 2.8) that is simpler from the computational perspective. Namely, the Schubert variety X_w is rationally smooth provided its Poincar'é polynomial is palindromic. Recall that the Poincaré polynomial of a Schubert variety X_w can be computed as the length generating function for the interval $[e, w]$ in the Bruhat order, and that such a polynomial $p_w(t)$ is palindromic provided $p_w(t) = t^{\ell(w)} p_w(t^{-1})$.

Thus the converse of McCrory's observation is in fact true, assuming the nonnegativity conjecture.

We will use the equivalence of the rational smoothness of X_w with the palindromicity of $p_w(q)$ to establish the results in Chapters 6-8. In particular, we obtain the following results:

Theorem 1.2. *In types $\widetilde{D}_n/D_n$ and $\widetilde{E}_n/E_n$ for $n \in \{6, 7, 8\}$ small partitions with one outside corner parametrize exactly the rationally smooth subvarieties of the affine Grassmanian. In $\widetilde{A}_n/A_n$, the small partitions with one outside corner completely parametrize a finite set of rationally smooth points, and large rectangles of shape* (1^{nk}) *or* (n^k), $k \in \{1, 2, \ldots\}$ *parametrize all other rationally smooth subvarieties.*

It should be noted that this criterion was used by Billey and Mitchell in [2] to simultaneously establish all of the results that appear in Chapters 6 through 8.

Our methods differ significantly in an important regard: how we detected elements of “small” length. Billey and Mitchell used a concept called allowable pairs, while we provide an explicit model of the intervals $[e, w]$ for w small, a concept that will be defined for Type $\widetilde{A}_n$ in Chapter 6, $\widetilde{D}_n$ in Chapter 7, and $\widetilde{E}_n$ in Chapter 8. All palindromic polynomials are found in the small element setting, except in $\widetilde{A}_n$, so this is the case that matters most. In addition, Billey and Mitchell’s work encompasses all affine Weyl group types.

CHAPTER TWO

Background

2.1 Coxeter Groups

2.1.1 Groups Generated by Reflections

If V is a finite dimensional Euclidean vector space with inner product $(*, *)$, we can define a reflection with respect to some nonzero vector α to be the map $s_\alpha \in V^*$ that operates by

$$s_\alpha(\beta) = \beta - 2 \cdot \frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

for any β a vector in V. Note that s_α is an involution of V. Also note that s_α is an orthogonal map, i.e. $(\lambda, \nu) = (s_\alpha(\lambda), s_\alpha(\nu))$. Thus, given a collection of vectors $\{\alpha_1, \ldots, \alpha_n\}$ in V, we can discuss the subgroup of $O(V)$ generated by $\{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$. Examples include the dihedral groups, the symmetric groups, and the symmetry group of the icosohedron, to name a few. These groups were classified by H.S.M. Coxeter in his seminal paper [14], and so are also called Coxeter groups.

2.1.2 Root Systems

A root system of a real vector space V is a set of vectors Φ in V such that

(1) Φ is finite and spans V,

(2) If $\alpha \in \Phi$, then the only multiples of α in Φ are $\pm\alpha$,

(3) If $\alpha \in \Phi$, then the reflection s_α sending α to $-\alpha$, and that fixes the hyperplane through the origin perpendicular to α, stabilizes the set Φ,

(4) If $\alpha, \beta \in \Phi$, then $s_\alpha(\beta) - \beta$ is an integer multiple of α.

Because Φ spans V, we have a well defined dimension of V, and so we call $\dim(V)$ the rank of Φ.

2.1.3 Positive and Simple Systems

The proofs in this section are basic, but technical. We direct the interested reader to [27] for full details.

Let Φ be a root system with associated vector space V, and let W denote the finite reflection group generated by all s_α with $\alpha \in \Phi$. We can define a total order on V in the following way: every vector in V can be written as a linear combination of elements in Φ. Fix a basis $\lambda_1, \ldots, \lambda_r$ of V, and use the lexicographic order, $\sum_{k=1}^{r} a_k\lambda_k < \sum_{k=1}^{r} b_k\lambda_k$ provided $a_i < b_i$ for the first $1 \leq i \leq r$ such that $a_i \neq b_i$. If we call a nonzero vector v positive if all of its coefficients in the basis $\{\lambda_1, \ldots, \lambda_r\}$ are nonnegative, we see that all of the basis elements are positive. We say that a subset Π of Φ is a positive system if all of its elements are positive, similarly for negative system. By the well-ordering principle, given a positive system, there is a minimal set $S = \{\lambda_{i_1}, \ldots, \lambda_{i_m}\}$ such that every element of Π is a nonnegative linear combination formed from Σ. We call Σ a simple system.

Theorem 2.1 ([27], page 8). *If Σ is a simple system in Φ, there is a unique positive system containing Σ. Conversely, every positive system contains a unique simple system.*

The reason we defined simple root systems was to find a generating set for our Coxeter group W.

Theorem 2.2 ([27], page 11). *For a fixed simple system Σ for root system Φ associated to Coxeter group W, the set $S := \{\alpha : s_\alpha \in \Sigma\}$ is a generating set for W.*

The effect of these results is that Coxeter groups are intimately related to geometric objects, a fact that will be further detailed in Section 2.1.9, below. We finish this section with a technical lemma showing that all simple systems of a given root system Φ are conjugate through the action of W on Φ.

Theorem 2.3. *For Σ a simple system in positive system Π, and $\alpha \in \Sigma$, $s_\alpha(\Pi\backslash\{\alpha\}) = \Pi\backslash\{\alpha\}$. If $\Sigma \subset \Pi$, $\Sigma' \subset \Pi'$, there is a $w \in W$ such that $w\Pi = \Pi'$.*

Proof. Begin by noting that if $\beta \in \Pi$, $\beta \neq \alpha$, then $\beta = \sum_{\gamma \in \Sigma} c_\gamma \gamma$ with $c_\gamma \geq 0$ for each γ. But $\gamma \in \Phi$ implies the only multiples of γ in Φ are $\pm\gamma$. Applying s_α to both sides of the sum above, we see $s_\alpha \beta = \beta - c_\alpha \alpha$. The root system is preserved under multiplication by s_α, so $s_\alpha \beta \in \Phi$. Also, as $\beta \neq \alpha$, there is some $c_\gamma > 0$ for some $\gamma \neq \alpha$, implying that $s_\alpha \beta \in \Pi$. Thus the coefficient of α in $s_\alpha \beta$ must be nonnegative, and $s_\alpha \beta \in \Pi \backslash \{\alpha\}$.

The second statement is proved by induction on $m = |\{\Pi \cap -\Pi'\}|$; if $m = 0$, $\Pi = \Pi'$. Suppose $m > 0$; then $\Sigma \not\subset \Pi'$, so there is some $\alpha \in \Sigma$ such that $\alpha \in -\Pi'$. As $s_\alpha(\Pi \backslash \{\alpha\}) = \Pi \backslash \{\alpha\}$, and $s_\alpha(\alpha) = -\alpha \in \Pi'$, then $|\{s_\alpha \Pi \cap -\Pi'\}| = m - 1$ and the proof is done. □

2.1.4 Reduced Expressions and the Length Function

Given a Coxeter group W with generating set S, any $w \in W$ can be expressed as a product of elements of S. Among all such expressions, any expression of w using a minimal number of generators counting repetitions is called a reduced decomposition of w. By the well-ordering principle, reduced expressions exist. We can define a function $\ell_\Sigma : W \to \mathbb{N} \cup \{0\}$ by letting $\ell_\Sigma(w)$ be the number of elements from S associated to Σ used to write w as a reduced expression.

Theorem 2.4 ([27], pages 10-14). *Define a function $n_\Sigma : W \to \mathbb{N} \cup \{0\}$ as follows: let $n_\Sigma(w)$ be the number of positive roots in the system Π associated to Σ that are made negate by r_w, the reflection associated to w. The*

(1) $n_\Sigma(ws_\alpha) = n_\Sigma(w) \pm 1$ for any $\alpha \in \Sigma$,

(2) $n_\Sigma(s_\alpha w) = n_\Sigma(w) \pm 1$ for any $\alpha \in \Sigma$,

(3) $\ell_\Sigma(w) = n_\Sigma(w)$.

In this paper, we work with Coxeter pairs (W, S), that is, we have a fixed simple set S chosen for W, corresponding to a fixed simple system $\Sigma \subset \Pi$, and thus we will refrain from showing the dependence of ℓ_Σ on the simple set chosen, writing ℓ instead.

2.1.5 Reflections, Length, and the Exchange Property

We now examine the relationship between reflections and the length function. We begin by stating one of the fundamental properties of Coxeter groups, the Exchange condition.

Theorem 2.5 ([27], page 14). *Suppose* $w = s_1 \dots s_k$ *for* $s_i \in S$*, and* t *is a reflection, i.e. an element of the form* $(s'_1 \cdots s'_r)s'_k(s'_1 \cdots s'_r)^{-1}$*. If* $\ell(tw) < \ell(w)$*, then* $tw = s_1 \cdots \hat{s}_i \cdots s_k$*, where the hat denotes deletion.*

A fact that will be assumed throughout this work is the following:

Theorem 2.6. *The number of reflections that decrease the length of* w *is equal to* $\ell(w)$*.*

Proof. If $w = s_1 \cdots s_k$ is reduced, so $\ell(w) = k$, then the set of reflections that decrease the length of w is easily seen to be

$$\{t_i = s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1 : 1 \leq i \leq k\}.$$

If $t_i = t_j$ for some $i \neq j$, then

$$w = t_i t_j w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$$

contradicting the minimality of w. ☐

Definition 2.1. *Let* $\mathcal{R}$ *denote the set of reflections (involutions) in* W*. Define*

$$T_L(w) := \{r \in \mathcal{R} \text{ such that } rw < w\},$$

and similarly define $T_R(w)$*. Then define* $D_R(w) := T_R(w) \cap S$ *and* $D_L(w) := T_L(w) \cap S$*.*

Corollary 2.1. *If* $s \in D_L(w)$*, then there is some reduced decomposition for* w *such that* $w = ss_1 \cdots s_k$*. A similar statement holds for* $s \in D_R(w)$*.*

2.1.6 Bruhat Order

Definition 2.2. *If* $u, v \in W$ *with* $\ell(u) < \ell(v)$*, and* t *is a reflection in* w *such that* $ut = v$*, then we write* $u \xrightarrow{t} v$*. Define a relation* $\mathcal{R}$ *on* $W \times W$ *by* $u \sim v$ *if there is a reflection* t *such that* $u \xrightarrow{t} v$*. The transitive closure of* $\mathcal{R}$ *determines a partial order on* W *called the Bruhat order.*

This definition is derived from the inclusion order of Bruhat cells, discussed in Section 2.3.2. This form of the Bruhat order is not amenable to combinatorial calculation. Therefore we use an equivalent formulation:

Theorem 2.7 (Subword Criterion). *Let* $w = s_1 \cdots s_r$ *be a reduced expression. Then* $u \leq w$ *if and only if there is a subset* $\{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$, $i_1 < \cdots < i_k$ *such that* $u = s_{i_1} s_{i_2} \cdots s_{i_k}$.

An immediate corollary of the subword criterion is that Coxeter groups are chained, that is to say, if $u < w$ with $\ell(w) - \ell(u) = r+1$, there is a sequence $v_1, \ldots, v_r$ and reflections $t_1, \ldots t_r$ such that $\ell(v_i) + 1 = \ell(v_{i+1})$ for $1 \leq i \leq r-1$, and

$$u \xrightarrow{t_0} v_1 \xrightarrow{t_1} \cdots \xrightarrow{t_r} w.$$

In addition, a weaker ordering, called the weak Bruhat ordering, can be constructed by restricting the usual Bruhat ordering. We say that u is left weakly covered by v if there is a simple reflection s such that $su = v$; this forces $\ell(v) - \ell(u) = 1$. An analogous definition can be made for right weakly covering. We can construct the left weak order on W by taking the transitive closure of all right weak covering relations.

2.1.7 Quotients by Parabolic Subgroups

Given a Coxeter group and generating set (W, S), suppose that $J \subsetneq S$, and let W_J denote the subgroup of W formed by all generators in J, with relations induced from W. Clearly W_J is a subgroup of W, and by considering Coxeter groups of Type A_n for $n \geq 5$, it is clear that W_J is typically not normal. We call W_J the parabolic subgroup associated to J.

Let W^J denote the set of cosets W/W_J, and similarly ${}^J W$ the set of cosets $W_J \backslash W$. We identify for this paper W^J and ${}^J W$ with their set of minimal coset representatives with respect to the Bruhat order. From these definitions, it is easy to see that W^J consist of those elements of W for which $D_R(w) \cap J = \emptyset$, and similarly for ${}^J W$. If J is S with one simple reflection deleted, we say that W_J is maximal parabolic. Both the Bruhat and weak Bruhat orders descend to W^J and ${}^JW$, where J is a parabolic subgroup of W.

2.1.8 Dynkin Diagrams

We can graphically depict a Coxeter pair (W, S) in the form of Dynkin diagrams. A Dynkin diagram is graph $\Gamma(W, S)$ with a node for each generator in S, with an edge between the nodes associated to s_i and s_j provided the order of the product s_is_j is greater than or equal to 3. If the order of the product exceeds three, the edge is marked with the order. If no edge is written, s_is_j is of order two, which means $s_is_j = s_js_i$. If all edges in $\Gamma(W, S)$ are unmarked, then we say that (W, S) is a simply laced Coxeter group.

2.1.9 Tits' Representation

Recall that a linear representation of a group W is a map $\phi : W \to GL(V)$ for some vector space V, and $GL(V)$ denotes the group of invertible linear transformations of V. We always take $V = \mathbb{R}^{|S|}$, where $W := (W, S)$ is a Coxeter group. Let $M^{(W,S)}$ denote the Coxeter matrix of (W, S). If the group is clear, we write M for $M^{(W,S)}$.

For each pair $(s_i, s_j) \in S \times S$ such that $M_{i,j} \geq 3$, choose an integer $k_{i,j}$ such that

(1) $k_{i,j} > 0$,

(2) $k_{i,j}k_{j,i} = 4\cos^2\left(\frac{\pi}{M_{i,j}}\right)$ if $M_{i,j} \neq \infty$,

(3) $k_{i,j}k_{j,i} \geq 4$ if $M_{i,j} = \infty$.

Let $\{\alpha^*_s\}_{s\in S}$ denote the canonical basis vectors of $\mathbb{R}^{|S|}$, i.e. $\alpha^*_s(s') = \delta_{s,s'}$, and for each $s \in S$, define $\sigma^*_s : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$ by

$$\sigma^*_s(p) = p + p_s \sum_{s'\in S} k_{s,s'}\alpha^*_{s'},$$

letting $p = \sum_{s'\in S} p_{s'}\alpha^*_{s'} \in \mathbb{R}^{|S|}$. In this usage, we write $V^* := \mathbb{R}^{|S|}$.

Proposition 2.1. *For all $s_i, s_j \in S$, we have*

(1) $(\sigma^*_{s_i})^2 = Id$, and

(2) the order of $\sigma^*_{s_i}\sigma^*_{s_j}$ is $M_{i,j}$.

A proof may be found in [4, Section 4.1].

Using the fact that S generates W, this representation has a unique extension to W.

Now let V be the real vector space spanneded by $\{\alpha_s\}_{s\in S}$, the canonical dual to $\{\alpha^*_s\}_{s\in S}$ defined above, i.e. $\langle \alpha^*_s, \alpha^*_{s'}\rangle = \delta_{s,s'}$. Note that, as vector spaces, $V \simeq V^* \simeq \mathbb{R}^{|S|}$, and that we can use the representation $W \to GL(V^*)$ to induce a representation $W \to GL(V)$. Explicitly, using the choices of $k_{s,s'}$ from before, we define a bilinear form on V by

$$(\alpha_s, \alpha_{s'}) = -\frac{k_{s,s'}}{2}, \tag{2.1}$$

which allows us to define

$$\sigma_s(\beta) = \beta - 2(\alpha_s, \beta)\alpha_s. \tag{2.2}$$

As $(\alpha_s, \alpha_s) = 1$, we have that $\sigma_s(\alpha_s) = -\alpha_s$ implying $\sigma_s^2 = \mathrm{Id}$, and that the order of $\sigma_{s_i}\sigma_{s_j} = M_{i,j}$. Thus this definition extends to a representation $W \to GL(V)$ that we call the geometric representation.

By abuse of notation, in both cases, we write the action of W on a vector as $w.p$.

We have the following consequences of these definitions. Call a vector $v \in \mathbb{R}^{|S|}$ positive if all of its components are positive, and negative if all of its components are negative.

Proposition 2.2. *For all $w \in W$ and $s \in S$,*

(1) $\ell(ws) > \ell(w)$ implies $w.\alpha_s$ is positive,

(2) $\ell(ws) < \ell(w)$ implies $w.\alpha_s$ is negative.

Proof. Suppose $ws > w$, let $s' \in D_R(w)$, $J = s, s'$, and for $w \in W$, use the unique decomposition $w = w^J w_J$. Then $ws > w$ implies $w_J s > w_J$, so $w_J = (\ldots ss'ss')$ (m elements), with m smaller than the order of ss'. Then

$$w(\alpha_s) = w^J(\ldots ss')(\alpha_s) = w^J(as + bs')$$

(as the group generated by J is dihedral). By definition, $w^J s > w^J$, and $w^J s' > w^J$, so by induction, $w^J.\alpha_s$ is positive, as is $w^J.\alpha_{s'}$. Thus $w^J(a\alpha_s + b\alpha_{s'})$ is positive. The other statement is similar. □

Corollary 2.2. *If p is a positive vector in V^* and $w \in W$, then*

$$D_L(u) = \{s \in S : \langle w.p, \alpha\rangle \textit{ is negative}\}$$

Proof. All coefficients of p are positive, so the previous result implies

$$\langle w.p, \alpha_s\rangle = \langle p, w^{-1}\alpha_s\rangle$$

and the right side is negative if and only if $w^{-1}s < w^{-1}$. □

2.2 Permutation Representations

2.2.1 Definition

Definition 2.3. *Fix a set R. A bijection $R \to R$ is called a permutation of R. Such bijections form a group under composition, and the group is denoted $S(R)$. The subgroups of $S(R)$ are called permutation groups.*

Definition 2.4. *A permutation representation of a group W is a map $W \to S(R)$ for some R.*

All permutations in this paper will have R an m-tuple of integers taken from some set B. Hence a bijection can be defined by specifying what happens to each entry in the tuple. Thus we can write a permutation in complete notation, that is to say, $w \in W$ corresponds to $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(m)]$, where $\sigma(i)$ describes what σ does to the i^{th} entry in the tuple.

The permutation groups associated to the Weyl groups are well known. We present the details of the affine groups salient to this paper. Eriksson and Eriksson provide in [23] a class of representation of the affine Weyl groups as affine transformations of $\mathbb{Z}$, which can then be used to give permutation representations of the affine Weyl groups of type ABCD. We do this only for the simply laced affine Weyl groups. In addition, we give Henrik Eriksson's permutation representations of the groups of type $\widetilde{E}_n$, $n \in \{6, 7, 8\}$ from [22].

2.2.2 Type A_n

The Weyl group of type A_n has a permutation representation as S_{n+1}, realized by mapping the nodes s_i to the adjacent transpositions switching i and $i + 1$.

2.2.3 Type B_n

The Weyl group of type B_n has a permutation representation as a rearrangement of the $2n$-tuple $(-n, \ldots, -1, 1, \ldots, n)$ such that $\sigma(-i) = \sigma(i)$. This condition means that a permutation is uniquely identified by noting how the second half of a tuple is rearranged. We refer to the positions in this fundamental domain as positions 1 through n, and the positions in the "first half" as $-n$ through -1. The generating set is the set of adjacent transpositions s_i interchanging positions i and $i + 1$, and thus positions $-i$ and $-i - 1$ as well, and the generator s_0 which sends the entry in the first position of the tuple to its negative. The Weyl group of type C_n has the same permutation representation.

2.2.4 Type D_n

The Weyl group of type D_n has as permutation representation the rearrangements of $(-n, \ldots, -1, 1, 2, \ldots, n)$ such that $\sigma(-i) = \sigma(i)$, and there are an even number of negative entries included in the last n entries of σ. As above, the fundamental domain is positions $1, 2, \ldots, n$. The generators are adjacent transpositions, and the generator s_0 which acts on a tuple $(r_1, \ldots, r_n)$ to yield $(-r_2, -r_1, r_3, \ldots, r_n)$.

2.2.5 Type E_6

A permutation representation of the group E_6 may be obtained in the following way. Take $R := [-8, -7, \ldots, 8]$, and let $e = [3, 4, 5, 6, 7, 8]$. Have s_1 act on the right by interchanging the first two coordinates, and s_m act on the right by interchanging the $m-1$ and m coordinates, for $m \geq 3$. For $w \in E_6$, define

$$\gamma(w) := \left(\frac{1}{3} \sum_{i=1}^{6} w(i) \right) - w(1) - w(2) - w(3),$$

and let s_2 act on $w = [w(1), w(2), w(3), w(4), w(5), w(6)]$ by

$$w.s_2 = [w(1) + \gamma(w), w(2) + \gamma(w), w(3) + \gamma(w), w(4), w(5), w(6)].$$

then one can easily check that the Coxeter relations of E_6 are satisfied by the generators $s_1, \ldots, s_6$, and that they are involutions. Thus this is indeed a representation of E_6 modeled by permutations.

2.2.6 Type E_7

For E_7, take $e = [9, 10, 11, \ldots, 15]$, and consider the orbit of e under all combinations of reflections $s_1, \ldots, s_7$, in which s_1 acts by interchanging the first two coordinates, s_i for $3 \leq 7$ act by interchanging coordinates $i-1$ and i, and s_0 acts by adding

$$\gamma(w) = \frac{1}{3} \left(\sum x_i \right) - w(1) - w(2) - w(3)$$

to the first three coordinates of w.

2.2.7 Type $\widetilde{A}_n$

Ordinarily, we can define $\widetilde{A}_n$ to be the group of affine transformations of $\mathbb{Z}$ such that if $w \in \widetilde{A}_n$, and $w(i)$ denotes the action of w on the integer i, w possesses the following properties:

(a) $w(k + n + 1) = w(k)$ for all $k \in \mathbb{Z}$,

(b) the sum $w(1) + \cdots + w(n+1) = \binom{n+2}{2}$, and

(c) the set of residues modulo $n+1$ of $w(1), w(2), \ldots, w(n+1)$ exhaust the set $\{0, 1, \ldots, n\}$.

Then given a collection of $n+1$ integers satisfying (b) and (c), demanding that (a) also be satisfied we can realize an affine transformation of $\mathbb{Z}$ from this $n+1$-tuple. Here R is the set of $n+1$ tuples of integers subject to the constraints that each class in $\mathbb{Z}$ modulo $n+1$ appears exactly once in each tuple, and the sum of all entries in a tuple is $\binom{n+2}{2}$. Such a tuple extends to an affine $\mathbb{Z}$-transformation by using (a) to extend the action to all of $\mathbb{Z}$. The window $[w(1), \ldots, w(n+1)]$ is the *complete notation* of the permutation in question. By taking the permutations of the $n+1$-tuple $(1, \ldots, n+1)$ and extending them to permutations of $\mathbb{Z}$ using (a), we see that the Weyl group A_n is embedded as a subgroup of this representation. We refer to the extentions of the adjacent transpositions of A_n as adjacent class transpositions.

To realize this permutation representation as a Coxeter system, for $1 \leq i \leq n$, denote by s_i the adjacent class transposition associated with the transposition $(i \quad i+1)$. Each s_i is clearly an involution, and $\{s_1, \ldots, s_n\}$ generate $A_n < \widetilde{A}_n$. Denote by s_0 the permutation (written in complete notation) $[0, 2, \ldots, n-1, n+1]$, and let $S = \{s_0, s_1, \ldots, s_n\}$. Let W be the group generated by the s_i. It is proved in [23], that (W, S) is a Coxeter system of type $\widetilde{A}_n$, and we identify the group $\widetilde{A}_n$ with the representation defined above.

The length function $\ell : \widetilde{A}_n \to \mathbb{N}$ is a rank function on the poset $(\widetilde{A}_n, \leq_{\text{Bruhat}})$. We will typically drop the subscript on the order. For $1 \leq i < j \leq n+1$, define functions $\eta_i^j : \widetilde{A}_n \to \{0, 1\}$ by $\eta_i^j(v) = \begin{cases} 1 & v(i) > v(j) \\ 0 & v(j) > v(i) \end{cases}$. Then a simple formula for length in $\widetilde{A}_n$, derived in [4], is

$$\ell(w) = \sum_{1 \leq i < j \leq n+1} \left\lfloor \frac{|w(j) - w(i)|}{n+1} \right\rfloor + \sum_{1 \leq i < j \leq n+1} \eta_i^j(w). \tag{2.3}$$

2.2.8 Type $\widetilde{D}_n$

A permutation representation of $\widetilde{D}_n$ is constructed in the following way: an n-tuple

$$w = [w(1), w(2), \ldots, w(n)]$$

is an element of $\widetilde{D}_n$ provided

(1) $w(k) = -w(k)$,

(2) $w(k + 2n + 1) = w(k) + 2n + 1$, and

(3) w is locally even at 0 and at n, in the terminology of [23].

Given an n-tuple, we can extend to a $2n$-tuple, and if this satisfies (a) and (c), we can use (b) to induce a $\mathbb{Z}$-permutation as in Type A. We refer to the n-tuple $[w(1), \ldots, w(n)]$ as the complete notation for the permutation, and alternately, refer to it as the fundamental window for the permutation. By taking the signed permutations on $(-n, \ldots, -1, 1, \ldots, n)$ and extending via (2), we see D_n is embedded in a natural way as a subgroup.

For $1 \leq i \leq n-1$, let s_i denote the class transposition $(n-i \;\; n-i+1)$, let s_n denote the permutation $(1 \quad -2)(-1 \;\; 2)$, and s_0 the affine permutation $(n-1 \;\; n+1)(n \;\; n+2)$ extended via Property (b), above, to a $\mathbb{Z}$-permutation. Then $\{s_1, \ldots, s_n)$ is a generating set of type D, and $S = \{s_0, s_1, \ldots, s_n\}$ generates $\widetilde{D}_n$ [4].

Define $\eta_i^j : \widetilde{D}_n \to \{0, 1\}$ to count class inversions as in 2.2.7. It was shown in [4] that the length function on $\widetilde{D}_n$ is given by

$$\ell(w) = \sum_{1 \leq i < j \leq n} \eta_i^j(w) + \mathrm{nsp}(w) + \sum_{1 \leq i < n} \sum_{i < j \leq n} \left\lfloor \frac{|w(j) - w(i)|}{2n+1} \right\rfloor + \sum_{1 \leq i < n} \sum_{i < j \leq n} \left\lfloor \frac{|w(j) + w(i)|}{2n+1} \right\rfloor$$

where $\mathrm{nsp}(w)$ is the total number of pairs (i, j) with $q \leq i < j \leq n$ such that $w(i) + w(j) < 0$.

Remark 2.1. *In all cases that follow, $\eta_i^j(w)$ will be the characteristic function for the property "$i < j$ and $w(i) > w(j)$" on the permutation representatives of the group being considered.*

2.2.9 Type $\widetilde{E}_7$

The group $\widetilde{E}_7$ has a Coxeter presentation of type $A_7 = \langle s_0, s_1, \ldots, s_6 \rangle$ augmented with an involution s_7 commuting with s_i for $i \neq 3$ such that $(s_3 s_7)^3 = (s_7 s_3)^3 = e$. Letting s_i^A denote the usual generating set for type A_7, then we see the the permutation realization of $\widetilde{E}_7$ is obtained by taking the right action of $s_0, \ldots, s_6$ to be $s_i = s_{i+1}^A$ on the "identity 8-tuple" $(1, 2, 3, \ldots, 8)$ as in A_7. The action of s_7 on an 8-tuple w is to add

$$\gamma(w) = \frac{1}{2}\sum_{i=1}^{8} w_i - 9 - \sum_{i=1}^{4} w_i = \frac{1}{2}\sum_{i=5}^{8} w_i - 9 - \frac{1}{2}\sum_{i=1}^{4} w_i$$

to the first four entries in w, and to leave the rest alone. The fact that this is indeed a faithful representation is proved in [22].

Set $g(w) = \frac{1}{2}\sum_{i=1}^{8} w_i$, and define $N_4(w)$ to be the number of four element subsets $\{A_{\hat{j}}\}_{\hat{j}=\{i_1,i_2,i_3,i_4\}}$ of the entries of w such that $\sum_{k=1}^{4} w_{i_k} - g(w) + 9$ is negative. Then the length function for this representation of $\widetilde{E}_7$ is

$$\begin{aligned} \ell(w) \quad = \quad & \sum_{1 \le i < j \le 8} \eta_i^j(w) + \sum_{1 \le i < j \le 8} \left\lfloor \frac{|w_i - w_j|}{18} \right\rfloor \\ & + N_4(w) + \sum_{1 \le i < j < l < k \le 8} \left\lfloor \frac{|g(w) - 9 - (w_i + w_j + w_l + w_k)|}{18} . \right\rfloor \end{aligned}$$

The length function in $\widetilde{E}_n / E_n$, $n \in \{6, 7, 8\}$, follows from an analysis of hyperplane arrangements conducted in [22] and is a corollary to Theorem 2.4.

2.2.10 Type $\widetilde{E}_8$

The group $\widetilde{E}_8$ has a Coxeter presentation of type $A_8 = \langle s_0, s_1, \ldots, s_7 \rangle$ augmented with an involution s_8 commuting with s_i for $i \neq 3$ such that $(s_3 s_8)^3 = (s_8 s_3)^3 = e$. Letting s_i^A denote the usual generating set for type A_8, then we see the the permutation realization of $\widetilde{E}_8$ is obtained by taking the right action of $s_0, \ldots, s_7$ to be $s_i = s_{i+1}^A$ on the "identity 9-tuple" $(1, 2, 3, \ldots, 9)$ as in A_8. The action of s_8 on a 9-tuple w is to add

$$\gamma(w) = \frac{1}{2}\sum_{i=1}^{9} w_i - 10 - \sum_{i=1}^{3} w_i = \frac{1}{2}\sum_{i=4}^{9} w_i - 10 - \frac{1}{2}\sum_{i=1}^{3} w_i$$

to the first four entries in w, and to leave the rest alone. The fact that this is indeed a faithful representation is proved in [22].

Define $g(w) = \frac{1}{3}\sum_{i=1}^{9} w(i)$. Then the length function is given by

$$\begin{aligned} \ell(w) &= \sum_{1\le i<j\le 9} \eta_i^j(w) + \sum_{1\le i<j\le 9} \left\lfloor \frac{|w_i - w_j|}{30} \right\rfloor \\ &\quad + N_3(w) + \sum_{1\le i<j<k\le 9} \left\lfloor \frac{|g(w) - 10 - (w_i + w_j + w_k)|}{30} \right\rfloor \end{aligned}$$

where $N_3(w)$ is the number of 3 element subsets $\{w(i_1), w(i_2), w(i_3)\}$ such that

$$\sum_{j=1}^{3} w(i_j) < g(w).$$

2.2.11 Type $\widetilde{E}_6$

The permutation representation $\widetilde{E}_6$ is again realized by extending a symmetric group representation with a pair of new involutions. We take as generators for $\widetilde{E}_6$ elements $s_1, \ldots, s_5$ acting as in A_5 on 6-tuples, and define the action of s_6 on w to be " add $\gamma(w) = \frac{1}{3}\sum_{1\le i\le 6} w_i - \sum_{1\le i\le 3} w_i$ to w_1, w_2 and w_3". Similarly, multiplying by s_7 adds $\mu = -12 + \frac{1}{3}\sum_{1\le i\le 6} w_i$. For the purposes of this representation, the identity element is $[3, 4, 5, 6, 7, 8]$, and $\widetilde{E}_6$ can be realized as the forward orbit of the identity element under all words in the alphabet $\{s_2, \ldots, s_7\}$.

The length function is

$$\begin{aligned} \ell(w) &= \sum_{1\le i<j\le 6} \eta_i^j(w) + \sum_{1\le i<j\le 6} \left\lfloor \frac{|w_i - w_j|}{12} \right\rfloor + \left\lfloor \frac{\sigma(w)}{12} \right\rfloor + \frac{1 + \operatorname{sgn}(\sigma(w))}{2} \\ &\quad + N_3(w) + \sum \left\lfloor \frac{|\sigma(w) - x_{i_1} - x_{i_2} - x_{i_3}|}{12} \right\rfloor \end{aligned}$$

where $\sigma(w) = \frac{1}{3}\sum_{1\le i\le 6} w_i$ and $N_3(w)$ is the number of 3 element subsets $\{w(i_1), w(i_2), w(i_3)\}$ of entries in w such that

$$\sum_{j=1}^{s} w(i_j) < \frac{1}{3}\sum_{j=1}^{6} w(j).$$

2.3 Algebraic Geometry of Simple Lie Groups

2.3.1 Flag Varieties

Fix a simply connected simple Lie group G with Borel subgroup B containing maximal torus T. Then G/B is a homogeneous space, and can be associated in a canonical way to a flag variety. Explicitly, let $V = \mathbb{C}^N$, where N is determined by the rank and type of G, and define a full flag in V to be a sequence of subspaces

$$0 = V_0 \subset V_1 \subset \cdots V_n = V$$

with $\dim(V_i) = i$. If V is spanned by orthonormal vectors $\{e_1, \dots, e_N\}$, call

$$F := 0 \subset \mathrm{Span}(e_1) \subset \mathrm{Span}(e_1, e_2) \cdots \subset \mathrm{Span}(e_1, \dots, e_n) = V$$

the standard flag of V. Let $\mathcal{F}(V)$ denote the set of all full flags of V. Then G acts transitively on $\mathcal{F}(V)$ by matrix multiplication, and the isotropy group of the standard flag is B. Hence we can identify G/B with $\mathcal{F}(V)$, and $\mathcal{F}(V)$ is a projective variety.

2.3.2 Schubert Varieties

Let G be as in Section 2.3.1, and let $W = N(T)/T$, where $N(T)$ is the normalizer of T. Then we can decompose G as a disjoint union $G = \bigcup_{w \in W} BwB$, the Bruhat decomposition of G. We can instead consider the flag manifold G/B, which decomposes as a disjoint union

$$G/B = \bigcup_{w \in W} Bw.B, \tag{2.4}$$

where we use the convention $Bw.B$ to distinguish from the double cosets in the Bruhat decomposition of G. We call the set $Bw.B$ the Schubert cell of wB in G/B. It is well known that $Bw.B$ is isomorphic to an affine space Ω^n of dimension $\ell(w)$, the Bruhat length of w in G. The Zariski closure $X(w) := \overline{Bw.B}$ is a projective variety, called the Schubert variety of w, and it is well known that $X(w)$ is the disjoint union of Schubert cells. We can use this to induce a partial order on W by $w \leq y$ if and only if $Bw.B \subset X(y)$, called the Bruhat order on W.

We consider in this paper a generalization of Schubert varieties. Let $P \supset B$ be a parabolic subgroup of G with respect to B, and consider the Grassmanian corresponding to G/P. We are justified in using the terminology of Grassmanian for this case as it generalizes the usual identification of the Grassmanian $Gr(k,n)$ with $GL_n(\mathbb{C})/P$, where P is a parabolic subgroup of $GL_n(\mathbb{C})$. P determines a subset $J \subset S$, for (W,S) the Weyl group of G, and the following decomposition generalizes the Bruhat decomposition:

$$P_J = \bigcup_{w \in W^J} BwB,$$

writing W^J for a set of coset representatives of W/W_J. Then we have the decomposition

$$G/P = \bigcup_{w \in W^J} Bw.P$$

where the union is disjoint, generalizing 2.4.

Here, let $X_w := \bigcup_{u \leq w} Bu.P$. Then X_w can be given the structure of projective variety in the following way: let U be the enveloping algebra of the Lie algebra associated to G, and let U_J denote the subalgebra generated by the positive roots in the span of J. Then let $L(\Lambda)$ denote an irreducible highest weight U_J module with highest weight Λ dominant integral, so that for any root α, $(\alpha, \Lambda) = 0$ if and only if $\alpha \in J$. Then there is a finite dimensional subspace $L' \subset L$ [17] such that if X_w is a subset of $\mathbb{P}(L')$, then X_w is closed in $\mathbb{P}(L')$, so it is a projective variety. By abuse of notation, we call X_w the Schubert variety of w.

2.3.3 The Carrell-Peterson Criterion

Because of its importance in later chapters, we wish to include a proof of the Carrell-Peterson criterion for rational smoothness in Schubert varieties. We begin with an overview of the ideas behind this test.

Let α be a positive root, and let Z_α denote the $SL_2(\mathbb{C})$ copy in G corresponding to α. If T is the maximal torus in G determined by B, which is itself determined by a choice of simple roots, then Z_α is the subgroup generated by the T-stable oneparameter subgroups U_α and $U_{-\alpha}$. For $x \in W$, exactly one of U_α and $U_{-\alpha}$ fixes e_x, the coset representative of x in G/B. It can be shown that $Z_\alpha \cdot e_x$ is a T-stable curve in G/B, and is isomorphic to $\mathbb{P}^1$, and all T-stable curves are of this form [13]. As $s_\alpha \in Z_\alpha$, $e_{s_\alpha \cdot x} \in Z_\alpha \cdot e_x$, showing that the number of T-stable curves in G/B corresponding to each α is $\frac{1}{2}|W|$, and so the total number of T-stable curves is $\frac{1}{2}|W| \cdot N$, where N is the number of positive roots.

A T-stable curve $Z_\alpha \cdot e_x$ is contained in a Schubert variety X_w if and only if e_x and $e_{s_\alpha x}$ are both in X_w. If $y \le w$, define

$$r(y, w) = |\{r \text{ a reflection such that } ry \le w\}.$$

Thus there are $r(y, w)$ T-stable curves in X_w through the point e_y. But the length of a Coxeter group element w is the number of reflections that shorten it in the Bruhat order, and Deodhar's inequality, proved by Dyer [20], states that $r(y, w) \ge \ell(w)$. Hence there are at least $\ell(w)$ T-stable curves in X_w passing though a T-fixed point e_y.

In its full strength, Deodhar's inequality is that

$$|\{r \text{ a reflection} : x \le ry \le w\}| \ge \ell(w) - \ell(x),$$

a fact that will be necessary in the proof of the rational smoothness criterion below.

Before we state the theorem that gives us our test, recall that the Bruhat graph of W is the graph with vertex set $w \in W$, with an edge $[v, w]$ provided there is some reflection r such that $vr = w$. For $w \in W$, the Bruhat graph Γ_w is the Bruhat graph of the interval $[e, w]$.

Theorem 2.8 (Carrell, Peterson). *Assume that the coefficients of all Kazhdan-Lusztig polynomials are nonnegative (this is known for the affine Weyl groups). For a Schubert variety X_w, the following statements are equivalent:*

(1) X_w is rationally smooth.

(2) The Poincaré polynomial of X_w is palindromic.

(3) The Bruhat graph Γ_w is regular: i.e. the degree of each vertex is $\ell(w)$.

Proof. . The proof we give follows that of [3]. The Poincaré polynomial of X_w is $p_w(q) = \sum_{y \leq w} q^{\ell(y)}$. The Kazhdan-Lusztig polynomial for the intersection cohomology is $\sum_{v \leq w} P_{v,w}(q) q^{\ell(v)}$ by a basic result of Kazhdan and Lusztig's original paper. Intersection cohomology has Poincaré duality, so this polynomial is symmetric. Thus if $P_{v,w}(q) = 1$ for each $v \leq w$, the polynomials agree, so $p_w(q)$ is symmetric. Thus (1) implies (2).

Now assume condition (2). If $y \in W$, $\ell(y)$ is the number of reflections that shorten y. Let R denote the full set of reflection in W. Each edge of the Bruhat graph meets two vertices y and ry, implying

$$\sum_{y \leq w} \ell(y) = \sum_{y \leq w} |\{r \in R : ry < y \leq w\}| = \sum_{y \leq w} |\{r \in R : y < ry \leq w\}|$$

(by reindexing). As the Poincaré polynomial is symmetric by assumption,

$$\sum_{y \leq w} \ell(e) - \ell(y) = \sum_{y \leq w} \ell(y) = \sum_{y \leq w} |\{r \in R : y < ry \leq w\}|.$$

Applying Deodhar's inequality with $x = y$ shows, for each $y \leq w$, that

$$|\{r \in R : y < ry \leq w\}| \geq \ell(w) - \ell(y).$$

But this shows that $\ell(w) - \ell(y) = r(y, w)$, implying the Bruhat graph is regular.

Now suppose condition (3). Induct on $\ell(w) - \ell(y)$: If the difference is zero, $w = y$, and $P_{w,w}(q) = 1$, implying the theorem. If $y < w$ and $P_{x,w}(q) = 1$ for all x such that $\ell(w) - \ell(x) < \ell(w) - \ell(y)$. If we select a function

$$f(q) = q^{\ell(w) - \ell(y)} P_{y,w}(q^{-2}) - 1$$

and note that the degree of $P_{y,w}(q) \leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$ and $P_{y,w}(0) = 1$ for all $y \leq w$, we see that $f(q)$ is a polynomial with no constant term. But

$$\frac{d}{dq}\left(q^{\ell(w)-\ell(y)} P_{y,w}(q^{-2})\right)|_{q=1} = \sum_{r \in R, y < ry \leq w} P_{ry,w}(1)$$

due to Deodhar in [16]. Thus

$$f'(1) = \sum_{r \in R, y < ry \leq w} P_{ry,w}(q) - \ell(w) + \ell(y).$$

By the induction hypothesis, $P_{ry,w}(1) = 1$ for $y < ry$, and by assumption $\ell(w) - \ell(y) = |\{r \in R : y < ry \leq w\}|$, so $f'(1) = 0$. If the coefficients of $P_{u,v}(q)$ are nonnegative for all pairs $u \leq v$, then $f'(1) = 0$ implies that $f(q)$ is a constant, so $P_{y,w}(q) = 1$, and X_w is rationally smooth. □

2.4 Hecke Algebras and R-polynomials

2.4.1 The Hecke Algebra

Let $\mathcal{A}$ be a commutative ring, (W, S) a Coxeter group, and define for each $s \in S$ a pair of parameters a_s, b_s. Let $\mathcal{E}$ be a free $\mathcal{A}$-module over W with basis elements $T_w, w \in W$.

Theorem 2.9. *Given parameters a_s, b_s as above, there is a unique associative $\mathcal{A}$-algebra structure on $\mathcal{E}$, with T_e acting as identity, such that*

$$T_s T_w = T_{sw} \text{ if } \ell(sw) < \ell(w) \tag{2.5}$$

$$T_s T_w = a_s T_w + b_s T_{sw} \text{ if } \ell(sw) > \ell(w) \tag{2.6}$$

for all $s \in S, w \in W$.

The proof of the above theorem involves realizing the structure satisfying equations 2.5 and 2.6 in End $\mathcal{E}$, the endomorphism algebra over $\mathcal{E}$, then proving the operators of left and right multiplication commute with one another, allowing us to "export" the structure in End $\mathcal{E}$ back to $\mathcal{E}$ itself.

We call the above the generic algebra $\mathcal{E}_A(a_s, b_s)$.

Example 2.1. *Take $b_s = 1, a_s = 0$ for all $s \in S$ to obtain the group algebra.*

A right handed form of Equations 2.5 and 2.6 exists, and the theorem is true from that point of view as well. Additionally, we can realize these equations in an alternate format that will be necessary in the sequel. Namely,

$$T_sT_w = T_{sw} \text{ if } \ell(sw) < \ell(w) \tag{2.5$'$}$$

$$T_s^2 = (q-1)T_s + qT_e \tag{2.6$'$}$$

Define the Hecke algebra $\mathcal{H}(W)$ to be the generic algebra with $\mathcal{A} := \mathbb{Z}[q, q^{-1}]$, and $a_s := q-1$, $b_s = q$ for all $s \in S$.

2.4.2 R-polynomials

Note that Equation 2.5$'$ implies the existence of inverses in $\mathcal{H}(W)$:

$$T_s^{-1} = q^{-1}\left(T_s - (q-1)T_e\right).$$

Use of Equation 2.5 allows us to extend inverses to T_w, $\ell(w) > 1$. The problem is that it becomes an increasingly difficult problem to write out an explicit inverse in each case, due to Equation 2.6. Thus we have the following Theorem-Definition. As R-polynomials are the substance of this paper, we provide a proof of this result, although the result is well-known in the field. The proof follows that of [31], with details supplied by [27].

Theorem 2.10. *For all $w \in W$,*

$$T_{w^{-1}}^{-1} = (-1)^{\ell(w)} q^{-\ell(w)} \sum_{x \le w} (-1)^{\ell(x)} R_{x,w}(q) T_x$$

where $R_{x,w}(q)$ is a polynomial of degree $\ell(w) - \ell(x)$ in q such that $R_{w,w}(q) = 1$ and $R_{x,w}(q) = 0$ if $x \not\le w$ (in the Bruhat order).

Proof. The proof will be inductive on the length of w. The result is clear if $\ell(w) = 0$, and by Equation 2.6$'$, it is clear that $R_{e,s}(q) = q-1$ if $s \in S$.

If we suppose $\ell(w) > 0$ and $w = sv$ for some $v \in W$ with $\ell(v) < \ell(w)$, then we have that $(-1)^{\ell(v)} = -(-1)^{\ell(w)}$ and $qq^{-\ell(w)} = q^{-\ell(v)}$. Then we can use the Hecke algebra relations above to directly calculate $T_{w^{-1}}^{-1}$. Explicitly, we see that

$$\begin{aligned}
T_{w^{-1}}^{-1} &= T_{v^{-1}s}^{-1} \\
&= T_s^{-1} T_{v^{-1}}^{-1} \\
&= q^{-1}\left(T_s - (q-1)T_e\right)(-1)^{\ell(v)} q^{-\ell(v)} \sum_{x \leq v} (-1)^{\ell(x)} R_{x,v}(q) T_x \\
&= (-1)^{\ell(w)} q^{-\ell(w)} \left((q-1) \sum_{x \leq v} (-1)^{\ell(x)} R_{x,v}(q) T_x - \sum_{x \leq v} (-1)^{\ell(x)} R_{x,v}(q) T_s T_x \right)
\end{aligned}$$

Consider the final summation: if $sx > x$, the term inside is $(-1)^{\ell(x)} R_{x,v}(q) T_{sx}$, but if $sx < x$, the term is $(q-1)(-1)^{\ell(x)} R_{x,v}(q) + q(-1)^{\ell(x)} R_{x,v}(q) T_{sx}$. Note that the first of this pair cancels a term from the first type of summation. Thus we can write the final pair of sums as sums over three types of terms, parametrized by pairs $(u, v) \in W \times W$ such that

(1) $u \leq v$, $u \leq sv$, yielding $(q-1)(-1)^{\ell(u)} R_{u,v} T_u$,

(2) $u \leq v$, yielding $-(-1)^{\ell(u)} R_{u,v} T_{su}$, or

(3) $u \leq v$, $u > su$, yielding $-1(-1)^{\ell(u)} T_{su}$

In each of the above situations, $x < w$, and $sx \leq w$. Furthermore, each and every terms fits in exactly one of the above cases. Thus we check coefficients to prove the result.

If $s \leq w$, $x \geq sx$, then T_x occurs in Case (2) (with $x = su$) and the resulting coefficient is

$$-(-1)^{\ell(u)} R_{u,v} = (-1)^{\ell(x)} R_{sx,v} = (-1)^{\ell(x)} R_{sx,sw}$$

with degree $\ell(sw) - \ell(sx) = \ell(w) - \ell(x)$. If $x = w$, then $u = v$, and $R_{y,v}(q) = 1$. Thus $R_{x,w}(q) = R_{sx,sw}(q)$ satisfies the requirements.

If $x < w$, and $x < sx$, we have two cases:

(a) If $sx < v$, then T_x occurs in Case (1) above, with $x = u \leq v$ and in Case (3) above, with $x = su$, $u = sx \leq v$. Then the coefficient of this pair of occurrences is

$$(q-1)(-1)^{\ell(x)}R_{x,v}(q) - q(-1)^{\ell(sx)}R_{sx,v}(q).$$

the degree of the second term is $\ell(v)-\ell(sx)+1 = (\ell(w)-1)-(\ell(x)+1)+1 = \ell(w)-\ell(x)-1$, but the degree of the first term is $\ell(v)-\ell(x)+1 = \ell(w)-1+\ell(x)+1 = \ell(w)-\ell(x)$. Thus the combined term has the proper degree, and we see

$$R_{x,w}(q) = (q-1)R_{x,sw}(q) + qR_{sx,sw}(q).$$

(b) If $sx \not\leq v$, T_x occurs only in Case (1) with coefficient $(q-1)(-1)^{\ell(x)}R_{x,v}(q)$. As $R_{sx,v}(q) = 0$ if $sx \neq\leq v$, we can define $R_{x,w}(q)$ as in Case (a).

This completes the induction. □

Define an involution $^{-} : H_q(W) \to H_q(W)$ by $\bar{T_w} = T_{w^{-1}}^{-1}$.

Theorem 2.11 (Kazhdan,Lusztig). *For each $w \in W$, there is a unique $C_w \in H_q(W)$ such that*

(1) $\bar{C}_w = C_w$

(2) $C_w = (q^{-\ell(w)/2}\sum_{v\leq w} P_{v,w}(q)T_v,$

where $P_{w,w}(q) = 1$, $P_{v,w}(q) \in \mathbb{Z}[q]$ has degree lass than or equal to $\frac{1}{2}(\ell(w)-\ell(v)-1)$ provided $v < w$, and $P_{v,w}(q) = 0$ if $v \not\leq w$.

The polynomials $P_{v,w}(q)$ are the (standard) Kazhdan-Lusztig polynomials discussed in [31].

2.4.3 Deodhar's Results

The parabolic Bruhat decomposition described in Section 2.3.2 was first considered by Deodhar in [17]. His goal was to show that Kazhdan and Lusztig's [32] geometric observation that the intersection cohomology of Schubert varieties is given by the Kazhdan-Lusztig polynomials remained true in the parabolic Bruhat decomposition. The first step in doing this is to realize Hecke algebras over quotients of Weyl groups. Then if W is the Weyl group of a semisimple or affine Kac-Moody group, one can appeal to canonically chosen parameters and reinterpret Kazhdan-Lusztig theory with regards to these choices.

Deodhar's technique was to create an operator acting like left multiplication on M^J, a free $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ module with basis $\{m_w, w \in W^J\}$. A kink in this development is that there are two parallel sets of polynomials that satisfy the necessary recursions. So let u be a solution of

$$\tau^2 = q + (q-1)\tau, \tag{2.7}$$

and for $s \in S$, let $L(s)$ denote the element of $\hom_{\mathcal{A}}(M^J)$ that acts as

$$L(s)(m_w) = \begin{cases} qm_{ws} + (q-1)m_w & \ell(ws) < \ell(w) \\ qm_w & \ell(ws) > \ell(w), ws \in W^J \\ \tau m_w & \ell(ws) > \ell(w), ws \notin W^J \end{cases}.$$

Then $L(s)$ is seen to create an action of W on M^J by extending linearly. We see that it is possible for a left ascent of $w \in W^J$ to move w out of the quotient W^J. This will give rise to two (related) structures, one for each solution of 2.7.

Lemma 2.1. *(1) $L(s)^2 = qL(e) + (q-1)L(s)$, and*
$(L(s)L(s'))^n L(s) = (L(s')L(s))^n L(s')$ for all $s, s' \in S$, with $m(s,s') = n$,

(2) M^J is a left $\mathcal{H}$-module under $T_w m_u = (L(s_1) \circ L(s_2) \circ \cdots \circ L(s_k))(m_u)$, if $w = s_1 \cdots s_k$ is a reduced decomposition, and $\mathcal{H}$ is the Hecke algebra of the pair (W,S).

Proof. $\mathcal{H}$ is an $\mathcal{A}$ algebra by the set $\{T_s, s \in S\}$ with relations $T_s^2 = (q-1)T_s + qT_e$ and $(T_sT_{s'})^nT_s = (T_{s'}T_s)^nT_{s'}$. If $\phi^J : \mathcal{H} \to M^J$ by $\phi^J(T_w) = u^{\ell(w_J)}m_w$, where $w = w^Jw_J$ is the decomposition into parabolic/quotient parts, then ϕ_J intertwines right multiplication in $\mathcal{H}$ by T_s with $L(s)$, and is surjective onto M^J. The result follows immediately. $\square$

If we define an involution on M^J as in the Hecke algebra, by $\bar{}: M^J \to M^J$ as $\bar{m}_w = \bar{T}_w m_e$, we see that as $\phi^J(T_w) = T_w m_e$, we have $\phi^J(\bar{T}_w) = \bar{T}_w m_e = \bar{m}_w$ for $w \in W^J$, which allows us to see immediately that

$$\overline{T_s m_w} = \overline{T_s T_w m_e} = \overline{T_s T_w} m_e = \bar{T}_s \bar{T}_w m_e = \bar{T}_s \bar{m}_w$$

and that

$$\overline{\overline{m_w}} = \overline{\overline{T_w m_e T_w}} \bar{m}_e = \overline{\overline{T_w}} m_e = T_w m_e = m_w$$

so $\bar{}$ is indeed an involutive homomorphism. This leads to the definition of the relative R-polynomials for W^J: we write

$$\bar{m}_w = \sum_{u \in W^J} (-1)^{\ell(w)+\ell(u)} q^{-\ell(w)} R^J_{u,w} m_u.$$

Theorem 2.12 (Deodhar). *The relative R-polynomials $R^J_{u,w}(q)$ satisfy the following recursions: for $s \in D_R(w)$,*

$$R^J_{u,w}(q) = \begin{cases} R^J_{su,sw}(q) & \ell(us) < \ell(u) \\ (q-1)R^J_{u,ws}(q) + qR^J_{us,ws}(q) & \ell(us) > \ell(u), us \in W^J \\ (q-1-\tau)R^J_{u,ws}(q) & \ell(us) > \ell(u), us \notin W^J. \end{cases} \tag{2.8}$$

Furthermore, $R^J_{u,w}(q) = 0$ if and only if $u \not\leq w$, $R^J_{u,w}(q) \in \mathbb{Z}[q]$, and the degree of $R^J_{u,w}(q)$ is $\ell(w) - \ell(u)$ in the case $\tau = -1$, and is smaller than or equal to $\ell(w) - \ell(u)$ in the case $\tau = q$. Finally, the relative R polynomials satisfy an orthogonality relation:

$$\sum_{u \leq v \leq w} q^{-\ell(v)} R^J_{u,v}(q) R^J_{v,w}\left(\frac{1}{q}\right) = \delta_{u,w} \cdot q^{-\ell(w)}.$$

Proof. The proof descends from the theorem-definition of the standard R-polynomials, and from the fact that $\bar{m}_w$ corresponds to $T^{-1}_{w^{-1}}$ in $\mathcal{H}$. $\square$

Furthermore, we can relate ordinary R polynomials to their parabolic counterparts as follows:

Theorem 2.13 (Deodhar). *Let (W, S) be a Coxeter system, and $J \subseteq S$, and $u, w \in W^J$. Then for any $\tau \in \{-1, q\}$,*

$$R^{J,\tau}_{u,w}(q) = \sum_{v \in W_J} (-\tau)^{\ell(w)} R_{vu,v}(q).$$

Remark 2.2. *In this paper, we will be working only with the polynomials for which $\tau = -1$.*

2.5 Combinatorics

2.5.1 Young's Lattice and its Generalizations

Recall that a partition λ is a weakly decreasing sequence of positive numbers. We can form a poset $\mathcal{Y}$ on the set of all partitions using containment as the order. This is called Young's lattice, and is well known.

An offset partition is a sequence $o = (o_2, \dots, o_m, \dots)$, together with a list of tuples of the form $(\lambda_1, \dots, \lambda_k)$ with

$$\lambda_1 \geq \lambda_2 + o_2 \geq \cdots \geq \lambda_k + o_k \geq 0.$$

The lattice of o-offset partitions $\mathcal{Y}_o$ is the set of all o-offset partitions using containment order as in Young's lattice. Hence we write $(\lambda_1, \dots, \lambda_j) \leq (\mu_1, \dots, \mu_k)$ if $k \geq j$ and $\lambda_i \leq \mu_i$ for $1 \leq i \leq j$.

2.5.2 The Mozes Game

The following realization of the contragredient representation $W \to GL(V^*)$ first appeared in [35], and in the form here in [24]. It will appear in various contexts through this paper, and is especially useful in studying groups of type E and $\widetilde{E}$

We set the integers $k_{s,s'}$ used earlier equal to $2\cos(\pi/M_{s,s'})$, where we use $M_{s,s'}$ to denote the order of ss'. Note that $k_{s,s'} = k_{s',s}$. Any choice that meets the criterion in the previous section will yield a usable game, but this choice keeps all entries integral.

Let Γ_S denote the Coxeter graph of (W, S), and let $p := (p_{s_1}, p_{s_2}, \dots, p_{s_n})$ denote a distribution of integer values to the nodes of Γ_S.

Definition 2.5. *To fire node s, we change p in the following way:*

(1) p_s turns into $-p_s$.

(2) If node s' is adjacent to node s, we add $k_{s,s'}p_s$ to p'_s.

(3) If node s' is not adjacent to node s, we leave p_s alone.

Note that firing node s corresponds to adding p_s times row s of $M^{(W,S)}$ to Γ_S.

A play sequence on Γ_S with initial distribution p is a sequence of integers $i_1, i_2, \dots i_m$, corresponding to a sequence of states on Γ_S, with the initial state being p, the next state being p after node s_{i_1} is fired, and so on. A positive play sequence is one in which, if node s is fired on the state p, then p_s is positive. The terminal position for a play sequence is the state of Γ_S after all nodes in the play sequence have been fired.

The usefulness of this game is seen in the following theorem.

Theorem 2.14. *Let p be a fixed initial position for the numbers game, and assume that all components of p are positive.*

(1) Two play sequence $s_{i_1} \dots s_{i_x}$ and $s_{j_1} \dots s_{j_y}$ have the same terminal position if and only if $s_{i_1} \dots s_{i_x} = s_{j_1} \dots s_{j_y}$ as elements of W. Thus for $w \in W$, we can write $w.p$ for the terminal state of any play sequence that forms a decomposition of w.

(2) $D_R(w)$ is the set of all $s \in S$ such that, if $(w.p)_s$ is negative.

(3) $s_{i_1} \dots s_{i_x}$ is a reduced decomposition for w if and only if $s_{i_1} \dots s_{i_x}$ describes a positive play sequence.

If we take the integers $k_{s,s'}$ as above, the image of $\{\alpha_s\}_{s \in S}$ under the action of W in $GL(V)$ forms a root system. To see this, note that (1) follows from the definition of the geometric representations, (2) follows from the first part of Theorem 2.14, and (3) and (4) from Equation 2.2. We will use this identification without further mention.

2.5.3 *Descent Lemma*

Let (W, S) be a Coxeter system, with $J = S\backslash\{s\}$. We will have need of the following well-known theorem to determine when the product of a simple element t with an element in W^J also lies in W^J.

Lemma 2.2. *Let (W, S) and J be as above. Let w be an element of the quotient W^J. If $t \in S$ is a left descent of w, then $tw \in W^J$ as well, and if t is a left ascent, then either tw is in the quotient, or $tw = ws'$ for some $s' \neq s$.*

Proof. If t is a left descent of w, and $s' \neq t$, then $\ell(tw) = \ell(w)-1$ and $\ell(ws') = \ell(w)+1$, and $\ell(tws') = \ell(w) > \ell(tw)$, and this is true for each $s' \neq t$, implying that s is the lone right descent of tw, implying that tw is an element of the quotient. The second statement follows from the unique decomposition of any element in W into $w^J w_J$, with w^J in the quotient, and w_J in the parabolic subgroup. Then $w^J < w$ by the subword property, and this is true for any w in the W. But then $tw < w$ is a covering relation, so either w is in the quotient, or $(tw)s'$ for $s' \neq s$ is the unique decomposition described above. □

2.5.4 *Braid Lemma*

We have need of a tool to determine when the product of two reduced words is again reduced.

Lemma 2.3. *Let W be a simply laced reflection group, and Q a (right) quotient of W by a maximal parabolic subgroup. Let $w \in Q$, with $w = uv$ with $v \in Q$, and assume that both u and v are reduced. Define $R_u = \{s_{r_2}s_{r_1} | u = (\text{prefix})s_{r_2}s_{r_1}\}$, and $L_v = \{s_{l_1}s_{l_2} | v = s_{l_1}s_{l_2}(\text{suffix})\}$. If there is no pair $s_{r_2}s_{r_1} \in R_u$ and $s_{l_1}s_{l_2} \in L_v$ such that*

$$r_1 = l_1, \quad r_2 = l_1, \quad r_1 = l_2, \quad s_{r_1}s_{l_1}s_{l_2} = s_{l_1}s_{l_2}s_{r_1}, \text{ or } \quad s_{r_2}s_{r_1}s_{l_1} = s_{l_1}s_{r_2}s_{r_1},$$

then w is reduced.

Proof. By induction on $\ell(u) + \ell(v)$. If the combined length is 4, then the statement is clear. So write $w = s_{i_1} \cdots s_{i_k} s_{j_1} \cdots s_{j_m}$, and assume that the theorem is true for all combined lengths less than $k + m$. If w is not reduced, then there exist indices i' and j' such that

$$w = s_{i_1} \cdots \hat{s}_{i'} \cdots s_{i_k} s_{j_1} \cdots \hat{s}_{j'} \cdots s_{j_m},$$

as we have assumed that both u and v are reduced. For $a = 1, 2, \ldots, m$, define $u_a = us_{j_1} \cdots s_{j_a}$. As $u_m = w$ is not reduced, $\ell(w) < k + m$, so there must be a first index a in the given decomposition for w for which $\ell(u_a) > \ell(u_{a+1})$. Define $v_a = s_{j_1} \cdots s_{j_a}$, we have $a \leq m$. In fact, we have $a + 1 < m$, as v has a unique right descent, and we are assuming that both v and w are elements of the quotient. By the inductive hypothesis, we can write uv_{a+1} in such a way that one of the five conditions is broken, and appending $v'_a = s_{j_{a+2}} \cdots s_{j_m}$ to this reduced decomposition does not alter the fact that a commutation relation or braid relation exists. Thus by contrapositive, if u and v are as above, then their product is reduced. □

As stated, this theorem holds for any simply laced group, but a simple modification allows it to be used on any reflection group.

CHAPTER THREE

Diagrams of Hermitian Type

3.1 Essential Definitions

3.1.1 Cominuscule Flag Varieties

Definition 3.1. *Let G be a complex connected simple algebraic group with parabolic subgroup P, with $\mathfrak{g}$ its simple complex Lie algebra and maximal parabolic subalgebra $\mathfrak{p}$ with Levi decomposition $\mathfrak{p} = \mathfrak{k} + \mathfrak{u}$ such that one of the following equivalent conditions is satisfied:*

(1) $(\mathfrak{g}, \mathfrak{k})$ *is a (complexified) Hermitian symmetric pair;*

(2) $\mathfrak{u}$ *is abelian;*

(3) the coefficient of α in the highest root of $\mathfrak{g}$ is 1.

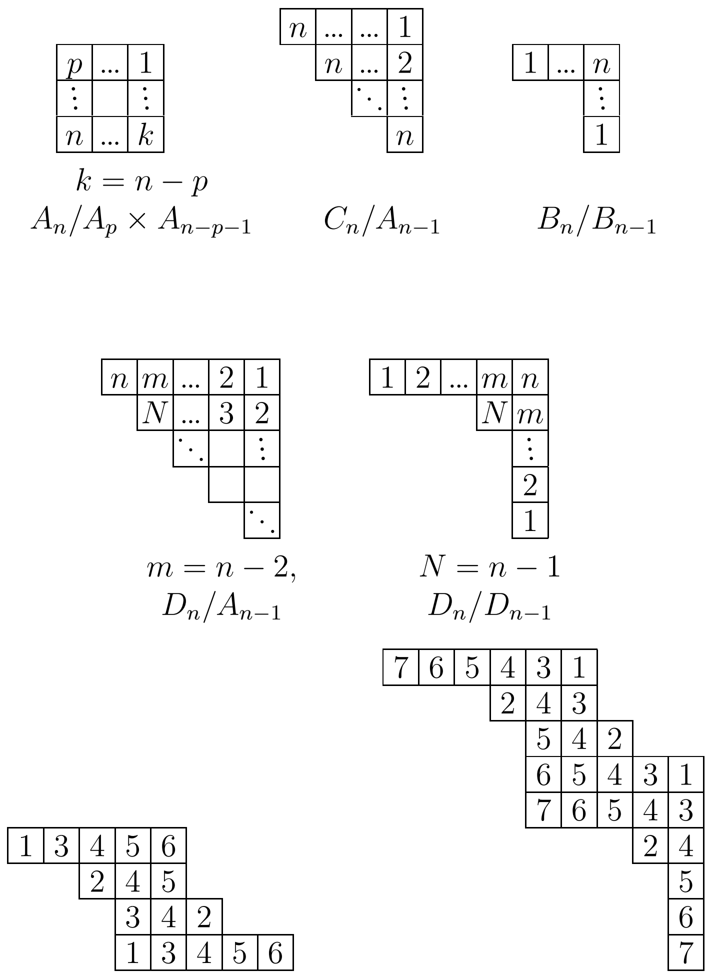
The generalized flag variety G/P is called cominuscule provided it satisfies one of the above conditions.

By abuse of notation, we refer to the Weyl group quotient W/W_J, with W the Weyl group of G, and W_J the Weyl group of P, as cominuscule, or Hermitian symmetric. It is well known that the Bruhat and weak orders coincide for W/W_J cominuscule. This fact can be deduced from Eriksson's game as well.

3.1.2 Diagrams of Hermitian Type

Definition 3.2. *We refer to the diagrams in Figure 3.1 as the diagrams of Hermitian type:*

Turning any diagram in Figure 3.1 counterclockwise by 135° yields a lattice that is the Hasse diagram of the Bruhat order of the specified quotient. This fact was shown by Proctor, and the following is essentially a restatement of [36], Proposition 3.2.



$k = n - p$

$A_n/A_p \times A_{n-p-1}$ $\quad C_n/A_{n-1}$ $\quad B_n/B_{n-1}$

$m = n - 2,$ $\quad N = n - 1$

D_n/A_{n-1} $\quad D_n/D_{n-1}$

E_6/D_5 $\quad E_7/E_6$

Figure 3.1. The diagrams of Hermitian type.

Proposition 3.1 (Proctor). *The subdiagrams of W/W_J correspond bijectively to the elements of W/W_J, for each cominuscule W/W_J listed above.*

Proctor provided a different proof of this result in [37, Theorem A] using a modification of a numbers game of Mozes. Alternatively, we could show the result using a sorting game that is essentially dual to Proctor's method. We give the details in $A_n/A_{p-1} \times A_{n-p}$ in the following section.

These Hasse diagrams appeared almost simultaneously in [30, Section 4] as the noncompact root lattices associated with unitarizable highest weight modules.

3.1.3 Diagrams and Partitions

The following definition is fundamental to this paper.

Definition 3.3. *A subdiagram of one of the above Hermitian type diagrams is a subset of the given diagram such that, if a box b is in the subdiagram, every box located above and left of b is also in the subdiagram.*

Importantly, diagrams retain the labels, which we think of as corresponding to simple reflections in the standard (Bourbaki) ordering of the generators S of the Weyl group W, imagining $J \subset W$ as lying in S in the canonical way.

Definition 3.4. *A partition is a diagram with its labels removed. We refer to the partition λ underlying a diagram Λ as the shape of Λ. We refer to the size of a partition as the number of boxes that appear in it.*

Then the subdiagrams of the above can be interpreted as offset partitions as in Table 3.1. In particular, $w \in W^J$ corresponds to a partition of type W^J and of size $\ell(w)$. Furthermore, because we require subdiagrams to be upper left justified, knowing a partition and its type is enough to construct the diagram for that partition.

In this light, it makes sense to make the following definition:

Definition 3.5. *The partition of a permutation w is the partition underlying the diagram associated to w through the bijection detailed in the following section.*

Table 3.1. Offset Partition Descriptions by Weyl Group Type

Quotient	Partition Description
$A_n/A_{p-1} \times A_{n-p}$	$\lambda = (\lambda_1, \ldots, \lambda_k)$, $k \leq n+1-p$, $\lambda_i \leq p$ where $S\backslash J = \{s_p\}$.
C_n/A_{n-1}	$\lambda = (\lambda_1, \ldots, \lambda_m)$, $n \geq \lambda_1 > \lambda_2 \cdots > \lambda_m > 0$.
B_n/B_{n-1}	$\lambda = (m)$, $m \leq n+1$, or $\lambda = (n+1, 1, 1, \ldots, 1)$.
D_n/A_{n-1}	$\lambda = (\lambda_1, \ldots, \lambda_m)$, $n \geq \lambda_1 > \lambda_2 \cdots > \lambda_m > 0$.
D_n/D_{n-1}	$\lambda = (m)$, $m \leq n$, or $\lambda = (n, m)$, $m \in \{1, 2\}$ or $\lambda = (n, 2, 1, 1, \ldots, 1)$.
$E_6/D_5, E_7/E_6$	irregular, best considered individually.

The next definitions will only be used in Section 4.5, but we state them now for completeness.

Definition 3.6. *We say that a partition is standard if it is a Young tableau under the usual definition (i.e. consists of a top and left justified set of rows, weakly decreasing in length). We say that a partition is standard skew if it can be written as a skew tableau of two standard tableaus. Note that all standard tableaus μ are standard skew, realized as $\mu\backslash\emptyset$.*

Definition 3.7. *We say that W^J supports a diagram of shape $\lambda\backslash\mu$ if the skew shape $\lambda\backslash\mu$ corresponds to $\Lambda\backslash M$, with Λ the diagram of v in W^J, and M the diagram of u. Given $u < v$ in W^J with diagrams M and Λ, respectively, we refer to $\Lambda\backslash M$ as a skew diagram of type W^J, or of type W, when J is clear from context. The support for the skew diagram for $v\backslash u$ where u and v are specific elements of some quotient W^J is the set of simple reflections that appear in the filling of $v\backslash u$.*

3.2 The Sorting Game

3.2.1 The Idea of the Algorithm

An alternate realization of the diagrams of Hermitian type can be had by modifying a sorting game developed by Eriksson and Eriksson [23, Section 9]. We begin by noting that each Weyl group can be associated to a permutation group: A_n with the symmetric group on $n+1$ letters, etc. We realize each of the above quotients in their permutation representation, using the permutation representation described in [22, Section 6.2] for E_6/D_5 and E_7/E_6 and the standard representations described in

[4, Chapter 8] for the others, then use a sorting argument to construct tableau. The rows in the constructed tableau correspond to rows in the above diagrams in a natural way, and because we are working in the cominuscule setting, the Bruhat and weak Bruhat orders coincide, allowing us to read the left descent sets of each element from the diagram (see Proposition 3.2). Furthermore, because the sorting game is modeling multiplication in the Weyl group, the diagrams we derive can be associated to reduced decompositions. This will be a central tool in the statement and proof of the main theorem.

3.2.2 $A_n/A_{p-1} \times A_{n-p}$

Suppose $W = A_n$ with the standard generating set S of adjacent transpositions, and $J = S\backslash\{s_p\}$. Then W^J consists of permutations increasing in positions 1 through p (called segment one), and in positions $p+1$ through $n+1$ (called segment two). Write $w(j)$ for the image of w under the action of the permutation w; if w is written in complete notation, we see this is just the j^{th} entry in w. Define a game on $w \in W^J$

in the following way: in step one, sort $w(p+1)$ into the segment $[w(1), \ldots, w(p)]$ by adjacent transpositions from the right, and record the number of transpositions used as d_1. Call the resulting permutation w_1, and sort $w_1(p+2) = w(p+2)$ into the segment $[w_1(1), \ldots, w_1(p+1)]$, again recording the number of transpositions used as d_2. As $w(p+1) < w(p+2)$ by assumption, $d_2 \leq d_1$. Continue until the permutation is completely sorted, leaving a partition $(d_1, d_2, \ldots, d_k)$. Note that $0 \leq k \leq n+1-p$, and that $k = 0$ if and only if w was sorted before any action was taken (which implies that w is the identity).

Each sort can be realized as multiplication by a specific product of simple reflections; in particular, the action of sorting position $p+k+1$ to position $p-j$ is $s_{p+k}s_{p+k-1}\cdots s_{p-j}$, corresponding to the diagram row word

$$r(p, \lambda_k) := \boxed{s_{p+k}}\boxed{\cdots}\boxed{s_{p-j+1}}\boxed{s_{p-j}}.$$

Thus we can associate to each partition an element of W^J by

$$w = r(p, \lambda_k)^{-1} r(p, \lambda_{k-1})^{-1} \cdots r(p, \lambda_1)^{-1},$$

and so we have a map

$$\phi : A_n/(A_{p-1} \times A_{n-p}) \mapsto \{\lambda = (\lambda_1, \ldots, \lambda_k), |\lambda| = \ell(w), \lambda_1 \leq p, k \leq n+1-p\}$$

where the labeling is exactly as in Section 3.2. This map is easily shown to be bijective.

Remark 3.1. *All further cases are similar, and so we give only the map*

$$\phi : W^J \mapsto \{\textit{partitions with particular conditions}\}.$$

3.2.3 C_n/A_{n-1}

Here, W can be viewed as Weyl group of type C_n, i.e. permutations of the set $\{-n, \ldots, -1, 1, \ldots, n\}$ with $w(i) = -w(-i)$, and where we take $J = A_{n-1}$. We see that

$$\begin{aligned} W^J \quad : \quad &= \{w \in C_n : \exists! 1 \leq p \leq n \text{ with } w(p) > 0, w(p+1) < 0, \\ &\text{and } w(1) < \cdots < w(p), w(p+1) < \cdots w(n) < 0\}. \end{aligned}$$

Note that $w(n) > 0$ if and only if w is the identity in this case. The algorithm is as follows: set $w'(n) = -w(n)$, and sort it into $[w(1), w(2), \ldots, w(n-1)]$, recording d_1 to be one more than the number of jumped elements, and call the resulting permutation w_1. Continue until $w_k(n) > 0$. Then $\phi(w) = (d_1, d_2, \ldots, d_k)$. As we assumed that $w(i) < w(j) < 0$, and $w(i)$ is sorted before $w(j)$, then $w(i)$ is moved more positions to the left than $w(j)$, showing that $d_j < d_i$ for $j > i$. Thus we have a map

$$\phi : C_n/A_{n-1} \mapsto \{\lambda = (\lambda_1, \ldots, \lambda_k), |\lambda| = \ell(W), \lambda_i < \lambda_j \text{ if } i < j, \lambda_1 \leq n\}$$

Using a row word argument with the labels fixed as before, this map is bijective.

3.2.4 B_n/B_{n-1}

Take $W = B_n$, and J the standard generating set of B_{n-1} chosen such that it lies within the standard generating set for B_n, with the same enumeration. Recall that as permutation groups, B_n and C_n are isomorphic, so $w \in B_n$ is a permutation of $[-n, \cdots, -1, 1, \cdots, n]$ such that $w(i) = -w(-i)$ for $1 \leq i \leq n$. Thus we can write w as $[w(1), \ldots, w(n)]$, and $W^J = \{w \in B_n : 0 < w(2) < \cdots < w(n)\}$. The only negative element, if there is one, is in position 1. The sort has two parts: first, if $w(1) > 0$, sort $w(1)$ into the segment $\{w(2), \ldots, w(n)\}$, recording the number of right jumps as d_1, and the algorithm terminates. If $w(1) < 0$, let w_1 be $(w(2), \ldots, w(n), -w(1))$, recording $d_1 = n$. Next, if necessary, sort $\{w_1(n-1), w_1(n)\}$, recording the number of left jumps of $w_1(2)$ as d_2. Continue sorting larger terminal segments in increments of one; note that as we increase the size, we can at most move an entry one position, thus $d_k \in \{0, 1\}$ for $k > 1$. The algorithm terminates when $d_k = 0$.

3.2.5 D_n/A_{n-1}

D_n is composed of permutations of $-[n] \cup [n]$ with an even number of negative entries in positions 1 through n, and with $w(i) = -w(-i)$ for $1 \leq i \leq n$. We write $w \in D_n$ as $[w(1), \ldots, w(n)]$. Then

$$\begin{aligned} W^J &:= \{w \in D_n : \exists! 1 \leq p \leq n \text{ with } w(p) < 0, w(p+1) > 0, \\ &\quad \text{and } w(1) < \cdots < w(p) < w(p+1) < \cdots w(n)\}. \end{aligned}$$

Given a permutation w , we have a two-step method of sorting: we first let $w'(n) = -w(n-1)$, and $w'(n-1) = -w(n)$. Sort $w'(n-1)$ into

$$\{w'(1), w'(2), w(3), \ldots, w(n)\},$$

noting d_1 as one more than the number of jumped elements, and denote the resulting permutation as w''. Note that $w'' \notin W^J$ as it does not have an even number of negative entries in its first n positions. Then sort $w'(n)$ into

$$\{w"(1), w''(2) \cdots, w''(n-1)\},$$

letting d_2 be the number of jumped elements. Again, by our assumptions $d_2 < d_1$ (as $w'(n) < w'(n-1)$, implying it is sorted fewer steps right).

Continue the same process with the resulting permutation. Note that d_{2k} may be zero for some k, and this may happen only if there are no remaining negative entries. The process halts when no negative entries remain at the beginning of an odd step. This gives a bijective map

$$\phi : D_n/A_{n-1} \mapsto \{\lambda = (\lambda_1, \dots, \lambda_k) : |\lambda| = \ell(w), \lambda_i < \lambda_j \text{ if } i < j, \lambda_1 \leq n-1\}.$$

3.2.6 D_n/D_{n-1}

In this case, the Bourbaki ordering of simple reflections admits no "nice" combinatorial interpretation in terms of the game we have presented so far. Instead, we present the proof using the ordering given in [4]. Explicitly, if we denote the Björner and Brenti enumeration of generators b_i, $0 \leq i \leq n-1$, and the Bourbaki ordering s_i, $1 \leq i \leq n$, we have that $b_i = s_{n-i}$ for $1 \leq i \leq n-1$, and b_0 acts as the cycles $(-1, 2)(1, -2)$, where s_n acts as the cycles $(n-1, -n)(n, -n+1)$. This is effectively a diagram bijection carrying $s_i \leftrightarrow s_{n-i}$. The difference is purely cosmetic, but it allows an easier statement of the sorting algorithm.

Take $W = D_n$, and J the generating set of D_{n-1} chosen such that it lies within the standard generating set for D_n, with the same enumeration. Then $W^J = \{w \in D_n : w(-2) < w(1) < \cdots < w(n-1)\}$. Again, the sorting process has several steps: begin by sorting $w(n)$ into $\{w(1), \dots, w(n-1)\}$. We observe that this step is trivial only in the case $w = e$, as otherwise $w(n) < w(n-1)$ as is easily verified from the definition of the quotient. Explicitly, $w(-2) < w(1) < w(2)$ implies that if any negatives appear in positions 1 through n of w , than they must be in positions 1 and n , implying that $w(n) < 0 < w(n-1)$; if no negatives appear, then $\sigma_1 = \{w(1), \dots, w(n-1)\}$ represents an ascending sequence taken from $[n]$, and it is only the trivial sequence that fails to utilize n in σ_1, showing $w(n) < n = w(n-1)$. Record the number of elements jumped by $w(n)$ as it moves left as d_1.

If there are no negative entries, then the resulting sequence is sorted, and the process terminates. Otherwise, denote the resulting permutation as w_1, and let $w'(1) = -w_1(2)$, $w'(2) = -w_1(1)$. Sort $w_1(3)$ into the segment $\{w'(1), w'(2)\}$ recording one more than the number of jumped elements as d_2 (note that it is possible that no jump occurs). We claim that $1 \leq d_2 \leq 2$: to see this, note first that $w(-2) < w(1) < w(2)$ implies $w(-2) < w(-1) = -w(1) < w(2)$, and also that, as we must assume $w(n) < 0$ to have reached this step, we know that the sets $\{w'(1), w'(2)\} = \{-w(1), -w(n)\}$. Hence sorting $w(2)$ into $\{w'(1), w'(2)\}$, we can't pass $-w(1)$, meaning we can jump at most one element.

The resulting permutation is w_2; now jump $w_2(4)$ into $\{w_2(1), w_2(2), w_2(3)\}$, recording d_3 as the number of left jumps; again, and for the same reason, $d_3 \leq 0$. Continue until no left jump is possible, at which point in time the sequence is sorted. The partition is $(d_1, d_2, \ldots, d_k)$, and labeling as in the figure realize the sorts involved, giving the bijection as before.

3.2.7 E_6/D_5 *and* E_7/E_6

In each case, since the generators J of the parabolic subgroup include a copy of A_{n-1}, we expect a necessary condition for w to be in E_n^J is that a subsequence of the n-tuple for w be increasing. Examining the definitions of the various $\widetilde{E}_n$, the specific subsequences are $w(2) < \cdots < w(n)$ for $n = 6$, and $w(1) < \cdots < w(n-1)$ for $n = 7$. In both cases, though, the s_2 generator cannot be a right descent, and thus must be positive. Analyzing the length functions for E_6 and E_7, we see that $\gamma(w) < 0$ is the extra condition.

The game is the same in both cases: Use $s_1, s_3, \ldots, s_6$ until $\gamma(w) > 0$, then use s_2. Realizing this as the Mozes game on the Dynkin diagrams, the diagrams of 3.2 appear.

Remark 3.2. *The row words that appear in each case give the labeling of the diagrams in 3.2.*

3.3 Order and Descents

3.3.1 Young's Lattice

Recall Young's lattice on partitions: two partitions $\mu = (\mu_1, \ldots, \mu_r)$ and $\lambda = (\lambda_1, \ldots, \lambda_s)$ are related by $\mu < \lambda$ provided $r \leq s$, and that for $1 \leq i \leq r$ $\mu_i \leq \lambda_i$. Visually, if the partitions are represented as Young diagrams, this just says that the diagram of μ sits inside that of λ when their upper left corners coincide.

In the current context, if M and Λ are diagrams of type W^J with shapes $\mu := (\mu_1, \ldots, \mu_r)$ and $\lambda := (\lambda_1, \ldots, \lambda_s)$ respectively, then the labels in their upper left corners again coincide, and we again say that $M \leq_{\text{diagram}} \Lambda$ provided $\mu_i \leq \lambda_i$ for each $1 \leq i \leq r \leq s$. We call this the containment order on W^J, or just the containment order when the context is clear. It is a consequence of the definition of subdiagrams that W^J is a poset with respect to this ordering.

3.3.2 Diagrams and Order

Given a diagram Λ of type W^J, we can visually distinguish the ascents and descents of $w \in W^J$ associated to Λ. Recall the definition of skew diagrams given in Definition 3.7.

Proposition 3.2. *Let $u \in W^J$ with diagram Λ, and suppose that $su \in W^J$ with diagram M. Then $s \in D_L(u)$ if and only if the skew diagram $\Lambda \backslash M$ consists of a single box, positioned in an outside corner of Λ. Similarly, if $s \notin D_L(u)$, then $M \backslash \Lambda$ is a single box that occupies an outside corner of M.*

Proof. If s is a left descent of $u \in W^J$, then as the weak Bruhat and strong Bruhat orders coincide for cominuscule flag varieties, we can write any reduced decomposition of u as asb, with $sa = as$, with s not appearing in a. Then in the lexicographically minimal reduced form, we can represent u as asb as above, and a can contain no copy of s, and $as = sa$. But then no s' labeled box appears in a, which means no s' labeled box appears in the partition for u after the final occurrence of an s-labeled box, and thus that s-labeled box is an outside corner of Λ.

Conversely, if s is the label of an outside corner of Λ, say on row j, then s commutes with all labels of boxes that appear on rows lower than j, and nothing appears right of the last s-labeled box on row j. Thus s commutes with all simple reflections that appear left of the first occurrence of s in the lexicographically minimal reduced decomposition, implying s is a left descent.

The second statement is similar. □

3.3.3 A Poset Isomorphism

We easily obtain the following from Proposition 3.2.

Corollary 3.1. *The poset $(W^J, \leq_{diagram})$ is poset isomorphic to $(W^J, \leq_{left\ weak})$. Thus $u \leq v$ in W^J if and only if the diagram of u is contained in the diagram of v.*

Proof. Covering relations in the left weak order are determined precisely by left simple descents. By the above, s is a left descent of v if and only if an s-labeled box is an outside corner in the diagram of v.

If u covers v in the left weak order, then there exists $s \in D_L(v)$ such that $su = v$. By Proposition 3.2, there is an s-labelled outside corner in the diagram Λ of v that is not contained in the diagram M of u. Then the s-labeled box can be removed from Λ, leaving a subdiagram, and considering the diagrams as products of row words forming a reduced decomposition, this corresponds to left multiplication by s. Thus the resulting diagram is M.

Similarly, if $M \subset \Lambda$, with M the diagram of u and Λ the diagram of v, then this presentation by diagrams witnesses that u is a subword of v, and so $u < v$ by the Subword Criterion (Theorem 2.7). □

CHAPTER FOUR

A Closed Form Expression for Relative R-polynomials

4.1 The Algorithm

4.1.1 Marking the Diagrams

Given $u \leq v$ in W^J, let Λ be the diagram of v, and M the diagram of u. As $u \leq v$, then $M \subset \Lambda$ by Corollary 3.1. By Proposition 3.2, $s \in D_L(u)$ if and only if an s-labeled box is an outside corner of the diagram of u, and su is a minimal coset representative covering u if and only if we can append an s-labeled box to M in such a way that the resulting diagram is a subdiagram in the sense of Definition 3.3.

The next part of the definition requires a caveat for shapes with the property that there is a diagonal not wholly contained within the shape, e.g. B_n/B_{n-1}. We imagine each diagram sitting on a grid of unlabelled, unmarked boxes. These boxes will never be marked, but they add to the length of the diagonals being considered. So for an s-labeled box b in $\Lambda \backslash M$, define the length of the diagonal containing b to be

$$\delta(b) := \#\{\text{boxes in } \Lambda \backslash M \text{ above and left of } b, \text{ including } b\}.$$

There is one exception to this rule, namely the second s_2 labelled box in E_6 and in E_7. I n particular, if the s_2 labelled box on the second row of E_6 or E_7 is an outside corner, then for b the s_2 labelled box on row 3 in E_6 or E_7, we have $\Delta(b) = 2$. Similarly, if the same s_2 labelled box is an inside corner, $\Delta(b) = 3$.

In the same way, if the s_2 box on row 3 of E_7 is an outside corner, then for b the s_2 box on row 6 we have $\Delta(b) = 2$. Again, if the same box is an inside corner, $\Delta(b) = 3$. The same relationship exists between the s_2 labelled boxes on rows three and six. Intuitively, one can think of the algorithm twisting for s_2, so s_2 sits atop the s_5 node on the sixth column, fourth row of E_7, and the s_3 in column five, row four of E_6.

We define a box b to be even provided $\delta(b)$ is even, and similarly for odd. See example 4.4 for a clear example of this phenomenon.

(1) If $s \notin D_L(u)$, $su \in W^J$, and s is a short (long) reflection, then mark every (odd) s-labeled box with a $+$.

(2) If $s \in D_L(u)$, and s is a short (long) reflection, then mark every (even) s-labeled box in $\Lambda\backslash M$ with a $-$.

For each box b in $\Lambda\backslash M$, define

$$\Delta(b) := \begin{cases} \#\{\text{boxes in } \Lambda\backslash M \text{ above and left of } b \text{ , including } b\} & \text{if } b \text{ marked} \\ 1 & \text{if otherwise} \end{cases}.$$

Suppose $u < v$, and that the diagram of v is Λ, and the diagram of u is M. Then we define $k_{u,v} := \#\{+ \text{ marked boxes in } \Lambda\backslash M\} - \#\{- \text{ marked boxes}\}$.

4.1.2 The Closed Formula

Now we can state the main result of this paper.

Theorem 4.1 (Main Theorem). *Given $u \le v$ in W^J of rank n, let Λ be the diagram of v, and M the diagram for u, and mark the skew diagram $\Lambda\backslash M$ as in Section 4.1. Define*

$$\widetilde{R}^J_{u,v}(q) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \not\le v \\ q^{\eta_{u,v}}(q-1)^{k_{u,v}} \prod_{b \in \Lambda\backslash M} [\boldsymbol{\Delta(b)}]^a & \text{if } u < v, \end{cases} \tag{4.1}$$

where the second product should be interpreted as a product over all boxes in the marked diagram Λ/M, taking a as $+1$ for $+$ marked boxes, -1 for $-$ marked boxes, and $\eta(u,v)$ to be the unique natural number so that the degree of the right hand side is $\ell(v) - \ell(u)$. Then $R^J_{u,v}(q) = \widetilde{R}^J_{u,v}(q)$.

For ease of discussion, let $\widetilde{r}^J_{u,v}(q)$ be the function defined as $R^J_{u,v}(q)$, but with $\eta_{u,v}(s) = 0$ for all s. We provide several examples to aid in the discussion:

4.1.3 Examples

Example 4.1. *Consider* $u = ((543)(65))^{-1}$, $v = ((54321)(6543)(7654)(8765)(9))^{-1}$ *in* $A_{10}/A_4 \times A_5$. *Again, we use the convention of using* i *for* s_i. *Then*

$$u = \begin{array}{ccc} 5 & 4 & 3 \\ 6 & 5 & \end{array} \qquad v = \begin{array}{ccccc} 5 & 4 & 3 & 2 & 1 \\ 6 & 5 & 4 & 3 & \\ 7 & 6 & 5 & 4 & \\ 8 & 7 & 6 & 5 & \\ 9 & & & & \end{array}$$

$$\Lambda \backslash M = \begin{array}{ccccc} X & X & X & + & \square \\ X & X & + & - & \\ + & \square & - & + & \\ \square & + & \square & - & \\ \square & & & & \end{array}$$

which yields

$$\begin{aligned} \widetilde{R}^J_{u,v}(q) &= (q-1)^{5-3} \frac{\mathbf{[2!][2!][1!]}}{\mathbf{[2!][1!]}} q^{10} \\ &= (q-1)^2 \frac{(1)(1+q)(1)(1+q)(1)}{(1)(1+q)(1)} q^{10} \\ &= (q-1)^2 q^{10}(1+q). \end{aligned}$$

Example 4.2. $u = ((86543)(7))^{-1}$, $v = ((8654321)(765432)(86543)(7654)(865)(7))^{-1}$ *in* D_8/A_7. *Then we have*

$$u = \begin{array}{ccccc} 8 & 6 & 5 & 4 & 3 \\ & 7 & & & \end{array} \qquad v = \begin{array}{ccccccc} 8 & 6 & 5 & 4 & 3 & 2 & 1 \\ & 7 & 6 & 5 & 4 & 3 & 2 \\ & & 8 & 6 & 5 & 4 & 3 \\ & & & 7 & 6 & 5 & 4 \\ & & & & 8 & 6 & 5 \\ & & & & & 7 & \end{array}$$

$$\Lambda \backslash M = \begin{array}{ccccccc} X & X & X & X & X & + & \square \\ & X & + & \square & \square & - & + \\ & & \square & + & \square & \square & - \\ & & & - & + & \square & \square \\ & & & & \square & + & \square \\ & & & & & - & \end{array}$$

so $\widetilde{R}^J_{u,v}(q) = (q-1)^{6-4} \dfrac{\mathbf{[4!][2!]}}{\mathbf{[2][4][2!]}} q^{16} = (q-1)^2(1+q+q^2)q^{16}$.

Example 4.3. *Take* $u = ((13456)(245))^{-1}$ *and* $v = ((13456)(245)(342)(1345))^{-1}$ *in* E_6/D_5. *Then*

$$u = \begin{array}{ccccc} 1&3&4&5&6 \\ &&2&4&5 \end{array} \qquad v = \begin{array}{ccccccc} 1&3&4&5&6&& \\ &&2&4&5&& \\ &&&3&4&2& \\ &&&1&3&4&5 \end{array}$$

$$\Lambda \backslash M = \begin{array}{ccccccc} X&X&X&X&X&& \\ &&X&X&X&& \\ &&&+&\square&\square& \\ &&&\square&+&\square&- \end{array}$$

and $\widetilde{R}^J_{u,v}(q) = (q-1)^{2-1}\dfrac{\mathbf{[1][2]}}{\mathbf{[2]}}q^6 = (q-1)q^6.$

Example 4.4. *Let* $u = ((13456)(2))^{-1}$ *and* $v = ((13456)(245)(342)(13456))^{-1}$ *in* E_6/D_5. *Then*

$$u = \begin{array}{ccccc} 1&3&4&5&6 \\ &&2&& \end{array} \qquad v = \begin{array}{cccccccc} 1&3&4&5&6&&& \\ &&2&4&5&&& \\ &&&3&4&2&& \\ &&&1&3&4&5&6 \end{array}$$

$$\Lambda \backslash M = \begin{array}{cccccccc} X&X&X&X&X&&& \\ &&X&+&\square&&& \\ &&&\square&+&-&& \\ &&&\square&\square&+&\square&- \end{array}$$

with $\widetilde{R}^J_{u,v}(q) = (q-1)^{3-2}\dfrac{\mathbf{[3!]}}{\mathbf{[2][3]}}q^9 = (q-1)q^9.$

4.2 Strategy of the Proof, and a Trivial Case

4.2.1 Strategy

We will show that $\widetilde{R}^J_{u,v}(q)$ obeys the recursions required by the relative R-polynomials. One case of the recursion is trivial, and will be treated case independently in Proposition 4.1. The strategy for the proof of the other recursive rules is as follows: we identify the possible diagrammatic presentations of each rule, and reduce to a trivial identity about quantum integers. The discussion in Section 4.3.1 will detail how we can restrict the skew diagram associated to the pair $u < v$ to a smaller diagram and prove the recursion there.

4.2.2 A Simple Case of the Recurrence

We can show that the $\widetilde{R}$-polynomials satisfy the second recursive relation in Deodhar's definition of relative R-polynomials without appealing to any particular cominuscule flag variety.

Proposition 4.1. *Let $u, v \in W^J$ with $u < v$, and suppose that $s \in D_L(v) \backslash D_L(u)$, and that $su \notin W^J$. Then $\widetilde{R}^J_{u,v}(q) = q\widetilde{R}^J_{u,sv}(q)$.*

Proof. If Λ and M are the diagrams for v and u respectively, the situation in the proposition is that there is an s-labeled outside corner of v that is neither an inside or outside corner of u. Thus the s-labeled boxes in $\Lambda \backslash M$ are not marked with $+$ or $-$. Similarly, if Λ' is the diagram for sv, the only difference between Λ and Λ' is a single unmarked box, implying that $\widetilde{r}^J_{u,v}(q) = \widetilde{r}^J_{u,sv}(q)$. Then the only difference between $\widetilde{R}^J_{u,v}(q)$ and $q\widetilde{R}^J_{u,sv}(q)$ is a single factor of q, yielding the result. □

4.3 The Case $s \in D_L(u) \cap D_L(v)$

4.3.1 A Remark on Markings

Suppose that $s \in D_L(v)$, and that $u < v$. Let Λ be the diagram of v, with Λ' the diagram for sv, and M the diagram for u, with M' the diagram for su if $su \in W^J$. Suppose furthermore that W^J is not E_6/D_5 or E_6.

If $s \in D_L(u)$, then there is an s-labeled outer corner in M. Then the skew diagram $\Lambda \backslash M$ differs from $\Lambda' \backslash M'$ in the diagonal containing that outside corner, and possibly in the diagonals immediately above and below this diagonal. The upper diagonal differs if $s'u > u$ and $s'u \in W^J$, where s' is the label of the box immediately right of the s-labeled outside corner in the diagram for u, and a similar condition holds for the lower diagonal.

In almost the same way, if $s \in D_L(v) \backslash D_L(u)$, and $su \in W^J$, the diagrams $\Lambda \backslash M$, $\Lambda' \backslash M$, and $\Lambda' \backslash M'$ differ at most on the diagonal of $\Lambda \backslash M$ containing the s-labeled outside corner, and on the diagonals immediately above and below it.

Thus the difference in the polynomials $\widetilde{R}^J_{u,v}(q)$, $\widetilde{R}^J_{u,sv}(q)$, and $\widetilde{R}^J_{su,sv}(q)$ is determined by markings that differ on at most three diagonals. We refer to this set of diagonals as I, and define $\widetilde{R}^J_{u,v}(q)|_I$ to be the polynomial determined by the markings on the diagonals in I.

In the cases E_6/D_5 and E_7/E_6, the above remarks are almost complete, except that the root s_2 appears on two diagonals, and so slightly complicates the above statements, but with only cosmetic effects. Here, the theorem is proved by checking finitely many cases by hand.

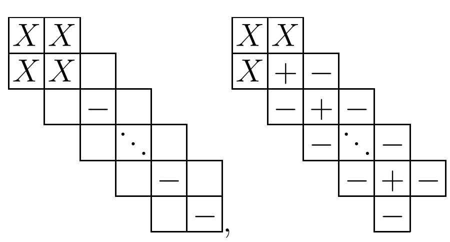
4.3.2 Proof of the Recurrence

With this remark in hand, we can take on the next case of the recursion.

Proposition 4.2. *Suppose that* $s \in D_L(u) \cap D_L(v)$. *Then* $\widetilde{R}^J_{u,v}(q) = \widetilde{R}^J_{su,sv}(q)$.

Proof. In this case, $su \leq u$, implying $su \in W^J$, so in particular the diagram for su is contained within the diagram of u, and is in fact obtained by deleting the s-labeled outside corner. Let Λ_v, M_u, and M_{su} are the diagrams associated to v, u, and su, respectively. Analyzing the diagrams of Section 3.2, we see the following marked diagrams. Note that we draw only the three relevant diagonals, and do not cover the cases E_6/D_5 or E_7/E_6, or include diagrams symmetric to given diagrams. In each pair of diagrams, the left one is taken from $\Lambda \backslash M$, and the right one is from $\Lambda' \backslash M'$, and it is a matter of definitions to see that $\widetilde{r}^J_{u,v}(q)|_I = \widetilde{r}^J_{su,sv}(q)|_I$.

To be explicit, consider the restricted diagram marked as below:

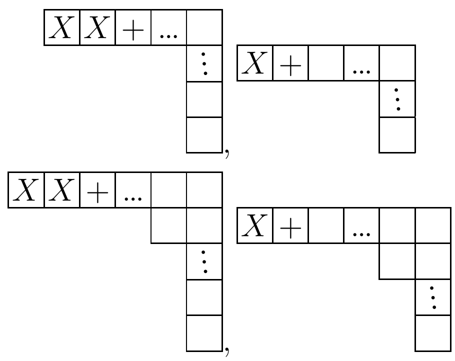


Assume that the length of the $-$ marked diagonal in the first diagram is k; then $\widetilde{r}^J_{u,v}(q) = \dfrac{1}{(q-1)^k[\boldsymbol{k!}]}$, while $\widetilde{r}^J_{su,sv}(q) = \dfrac{(q-1)^k[\boldsymbol{k!}]}{(q-1)^{2k}[\boldsymbol{k!}][\boldsymbol{k!}]}$, showing the two are equal.

As the skew diagram for the pair $su < sv$ contains exactly as many boxes as the diagram for $u < v$, $q^{\eta(u,v)} = q^{\eta(su,sv)}$, showing $\widetilde{R}^J_{u,v}(q) = \widetilde{R}^J_{su,sv}(q)$ in this case.

All other cases are similar; the relevant diagrams and their restricted polynomials appear below in Figure 4.1. In each case, the left diagram is for $u < v$, and the right diagram is for $su < sv$; we write k for the length of the $-$ marked diagonal in the first diagram. When no reduction is necessary to see that the polynomials are the same, only a single polynomial is expressed.

In the following, suppose that s_k is the ascent:



$[\mathbf{1}](q-1)q^{2(n-k)-1}$

$(q-1)q^{2(n-k)}$

By the discussion above, this shows that if $u < v$ in W^J, and $s \in D_L(v) \cap D_L(u)$, then $\widetilde{R}^J_{u,v}(q) = \widetilde{R}^J_{su,sv}(q)$ for $W^J \notin \{E_6/D_5, E_7/E_6\}$. The cases E_6/D_5 and E_7/E_6 can be verified using the computer. $\square$

4.4 The Case $s \in D_L(v) \backslash D_L(u)$, $su \in W^J$

4.4.1 Strategy, and a Reduction

As noted above, this proof will proceed by noting the possible restricted diagrams that may appear with an s-labeled outside corner of v, and an s labeled inside corner of u. In each case, we provide the diagrams associated to $u < v$, $u < sv$, and $su < sv$, respectively, and reference the part of Lemma 4.1 (below) that is used in showing that $\tilde{R}^J_{u,v}(q)|_I = (q-1)\tilde{R}^J_{u,sv}(q)|_I + q\tilde{R}^J_{su,sv}(q)|_I$.

Let Λ_x denote the diagram of x in W^J, and $\partial_{u,v}(I)$ the number of boxes in the restriction to I of $\Lambda_v \backslash \Lambda_u$. Then

$$\partial_{u,v}(I) = \partial_{u,sv}(I) + 1 = \partial_{su,sv}(I) + 2.$$

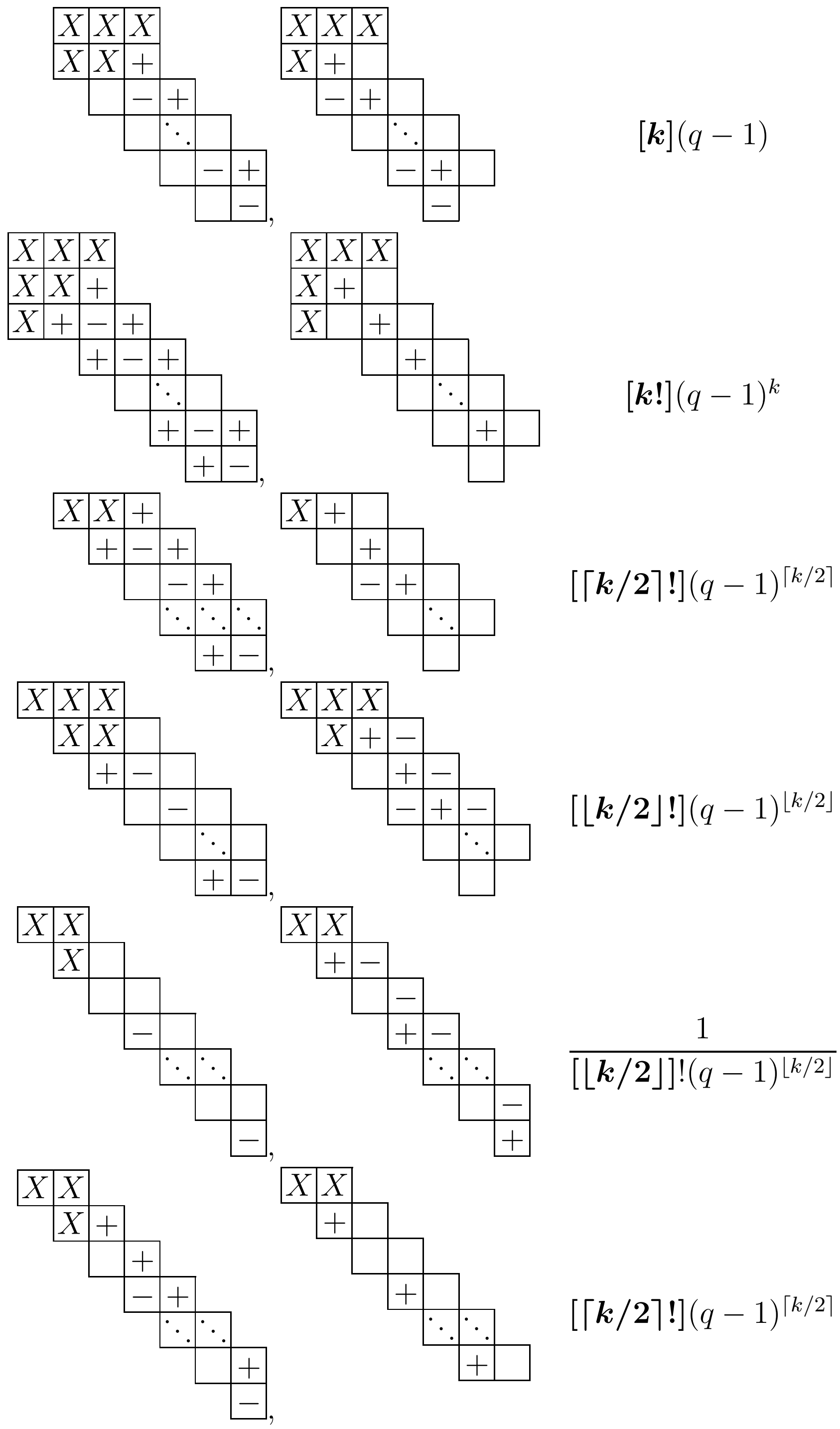


Figure 4.1. Restricted Diagrams for the case $s \in D_L(u) \cap D_L(v)$

Note that

$$\tilde{R}^J_{u,v}(q)|_I = \tilde{r}^J_{u,v}(q)q^g,$$

where g is

$$\partial_{u,v}(I) - \deg(\tilde{r}^J_{u,v}(q)|_I).$$

In particular, if we wish to show that

$$\tilde{R}^J_{u,v}(q)|_I = (q-1)\tilde{R}^J_{u,sv}(q) + q\tilde{R}^J_{su,sv}(q),$$

we can instead show

$$\tilde{r}^J_{u,v}(q)|_I = q^a(q-1)\tilde{r}^J_{u,sv}(q)|_I + q^b \cdot q\tilde{r}^J_{su,sv}(q)|_I,$$

where

$$a + \deg(\tilde{r}^J_{u,sv}(q)|_I) + 1 = \deg(\tilde{r}^J_{u,v}(q)|_I),$$

and similarly for b. Then multiplication by q^g, where g is as above, generates the desired identity. Calculating g introduces extra notation, and can be avoided in the proof that follows. Instead, we will multiply the terms coming from the diagrams for $u < sv$ and $su < sv$ by the power of q to bring those terms to degree $\partial_{u,v}(I) - 1$, resp. $\partial_{u,v}(I) - 2$.

4.4.2 A Simple Lemma

We require a lemma to make the proof.

Lemma 4.1. *For and $k \geq 1$,*

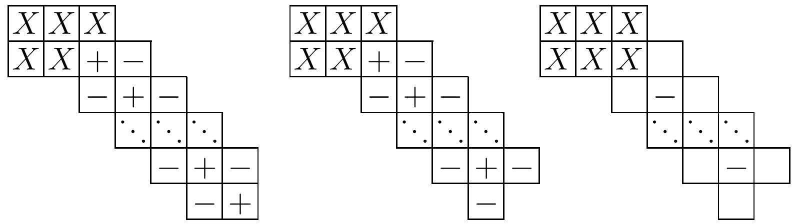
(1) $[\boldsymbol{k}] = [\boldsymbol{k-1}] + q^{k-1}$

(2) $[\boldsymbol{k}] = 1 + q[\boldsymbol{k-1}]$

Proof. The result follows from the definition of q-analogs of integers. □

4.4.3 The Final Case of the Recurrence

To finish the proof of the main theorem, consider first the restricted diagram



with outside corners north of and west of the inside s-labeled corner, occurs in the Type A_n quotients, as well as $C_n/A_{n-1}, D_n/A_{n-1}$, E_6/D_5, and E_7/E_6, with k + marked boxes. This diagram yields

$$\tilde{r}^J_{u,v}(q)|_I = (q-1)^{2-k}\frac{[\boldsymbol{k}]}{[(\boldsymbol{k-1})!]}$$

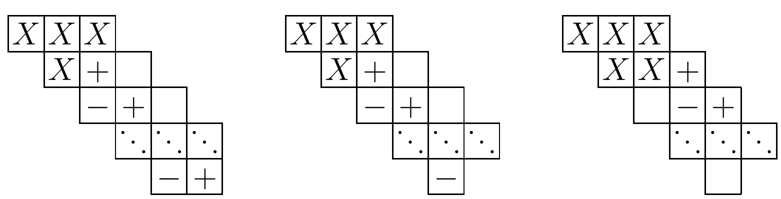
which is of degree $1-\frac{(k-2)(k-1)}{2}$. Then $\tilde{r}^J_{u,sv}(q)|_I = (q-1)^{1-k}\frac{1}{[(\boldsymbol{k-1})!]}$, which has degree $1-k-\frac{(k-2)(k-1)}{2}$, differs from expectations by $1-k$, and $\tilde{r}^J_{su,sv}(q) = (q-1)^{2-k}\frac{1}{(k-2)!}$ with degree $2-k-\frac{(k-3)(k-2)}{2}$, differs from the expected degree by 1. Thus the relevant identity to show is

$$\begin{aligned}(q-1)^{2-k}\frac{[\boldsymbol{k}]}{[(\boldsymbol{k-1})!]} &= (q-1)q^{k-1}\cdot\frac{(q-1)^{1-k}}{[(\boldsymbol{k-1})!]} + qq^{-1}\cdot\frac{(q-1)^{2-k}}{(k-2)!}\\ &= \frac{(q-1)^{2-k}}{[(\boldsymbol{k-1})!]}\left(q^{k-1}+[\boldsymbol{k-1}]\right)\end{aligned}$$

which is Lemma 4.1 (1).

From here, we describe on, we cease to explicitly calculate the expected degrees. In all cases, k is the number of + signs in the first diagram of a triple.

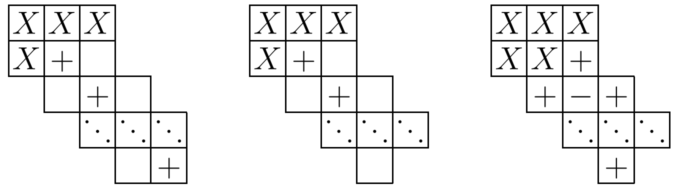
For our next diagram, take



which corresponds to the identity in Lemma 4.1, (1).

This shape appears in the Type A_n quotients, as well as $C_n/A_{n-1}, D_n/A_{n-1}$, E_6/D_5, and E_7/E_6.

Next consider



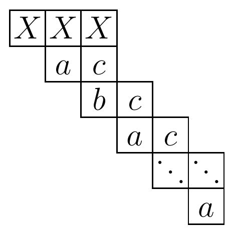
corresponds to the identity

$$(q-1)^k[\boldsymbol{k!}] = (q-1)[\boldsymbol{(k-1)!}]q^{k-1} + q(q-1)^k[\boldsymbol{k-1}][\boldsymbol{(k-1)!}]q^{-1}$$

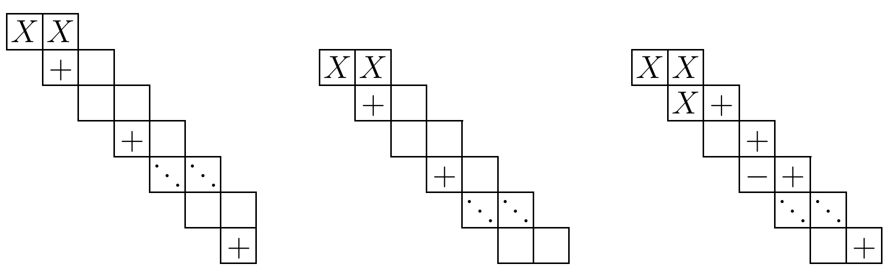
which reduces to Lemma 4.1, (1).

This appears in the Type A_n quotients, as well as $C_n/A_{n-1}, D_n/A_{n-1}$, E_6/D_5, and E_7/E_6.

We now move to restricted diagrams that appear in D_n/A_{n-1} and C_n/A_{n-1}. Consider the restricted diagram of $\Lambda\backslash M$



where a is s_n, and b is s_{n-1}, or vice versa. Then we obtain the following triple of marked diagrams



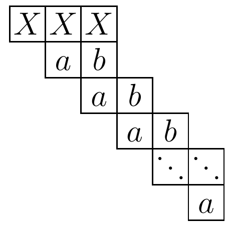
(note the first box in the bottom row of the last diagram cannot be marked, by reason of parity).

Suppose there are k + marked boxes in the first diagram. Then this restricted diagram corresponds to the identity

$$\begin{aligned}(q-1)^k[\boldsymbol{1}][\boldsymbol{3}]\cdots[\boldsymbol{2k-1}] &= (q-1)^k q^{2k-2}[\boldsymbol{1}][\boldsymbol{3}]\cdots[\boldsymbol{2k-3}] \\ &\quad +(q-1)^k[\boldsymbol{1}][\boldsymbol{3}]\cdots[\boldsymbol{2k-3}][\boldsymbol{2k-2}]\end{aligned}$$

which reduces by basic algebra to Lemma 4.1, (1). Thus the result holds in the case of D_n/A_{n-1}.

The restricted diagram of $\Lambda\backslash M$



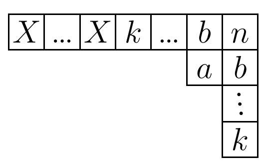
with $a = s_n$, $b = s_{n-1}$ with an odd number of copies of s_n appears as a restricted interval in C_n/C_{n-1}, and by our labeling algorithm, bears the same marks as the above case, and so satisfies the same identity.

The same restricted diagram, but with an even number of s_n labeled boxes, appears in C_n/C_{n-1} but not D_n/D_{n-1} (as we assume in that case the label in the lowest box b is a descent of v, which in D_n requires b to be an odd box). Here the identity is

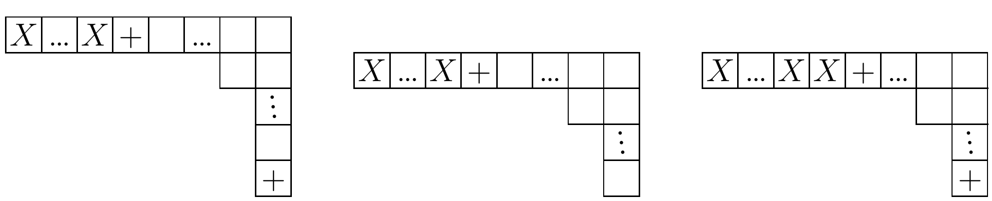
$$(q-1)^k[\mathbf{1}][\mathbf{3}]\cdots[\mathbf{2k-1}]$$
$$= q^{-1}\left((q-1)^{k+1}[\mathbf{1}][\mathbf{3}]\cdots[\mathbf{2k-1}]\right) + q^{-2}\left(q(q-1)^k[\mathbf{1}][\mathbf{3}]\cdots[\mathbf{2k-1}]\right)$$

which is true by inspection.

Consider the diagram $\Lambda\backslash M$ corresponding to $u < v$; write a for $n-1$ and b for $n-2$. Then



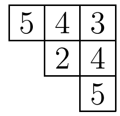
with marked triple



and note if two boxes are + marked, their weights are 1 and k, where the + marked box is labeled s_k. From this, we can read $\tilde{R}^J_{u,v}(q) = (q-1)^2[\boldsymbol{k}]q^{k-1}$, while $\tilde{R}^J_{u,v}(q)$

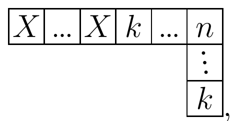
for the second diagram is $\tilde{R}^J_{u,sv}(q) = (q-1)q^{2k-1}$, and the third is $\tilde{R}^J_{su,sv}(q) = (q-1)^2[\boldsymbol{k-1}]q^{k-2}$. Substituting into the recursion, we can reduce to the identity in Lemma 4.1, (2).

This diagram appears in D_n/D_{n-1}, and can be renumbered

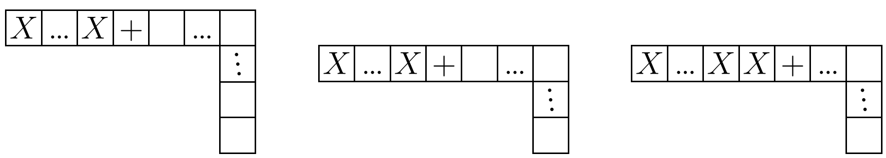


to be found in E_6/D_5 and E_7/E_6.

In B_n/B_{n-1}, the only relevant diagrams have the following shape



with marked triple



We show the full diagram, but this is only to aid comprehension. The usual three diagonals are all that is important to the calculation.

The relevant identity is

$$(q-1)q^k = (q-1)\cdot(q-1)q^{k-1} + q\cdot(q-1)q^{k-2}$$

which is trivially verified.

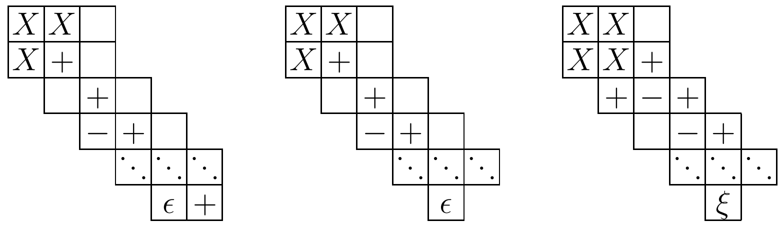
As an aside, it is straightforward to use the above identity, and induction, to show the following.

Proposition 4.3. *If $[u,v]$ is an interval in B_n/B_{n-1}, then $R^J_{u,v}(q) = (q-1)q^{\ell(v)-\ell(u)-1}$.*

This type of interval appears in B_n/B_{n-1}, C_n/C_{n-1}, E_6/D_5, and E_7/E_6.

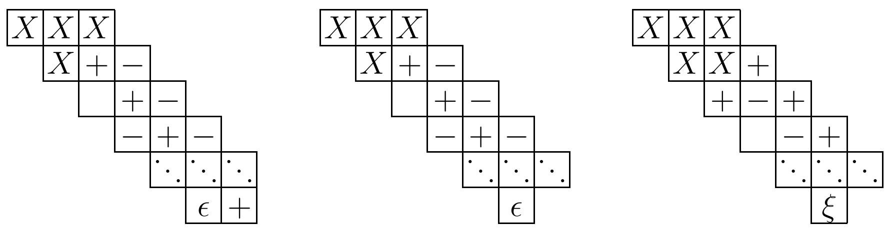
The remaining cases are similar, and we give only the diagrams which have yet to appear:

In C_n/C_{n-1} and D_n/D_{n-1} we have



where ϵ is $-$ if there are an even number of $+$ signs in the first diagram, and is unmarked otherwise, and ξ is $+$ if there are an odd number of $+$ signs in the first diagram, and unmarked otherwise.

Similarly, in C_n/C_{n-1} and D_n/D_{n-1}, we have the restricted diagrams



with ϵ and ξ as above.

Finally, there are finitely many cases arising in E_6/D_5 and E_7/E_6 that may be obtained by hand or computer program.

4.5 Combinatorial Invariance of Kazhdan-Lusztig Polynomials

4.5.1 Overview

As an application of the above algorithm, we show that the intersection cohomology of the cominuscule flag varieties is a combinatorial invariant, i.e. the cohomology is a function only of the poset. Our method is to prove that if two Bruhat intervals contained in quotients associated to cominuscule varieties are isomorphic, then they have the same relative R-polynomials.

This then proves Theorem 1.1, and because of the formula

$$q^{\ell(w)-\ell(u)}P^J_{u,w}(q^{-1}) - P_{u,w}(q) = \sum_{u<v\leq w} R^J_{u,v}(q)P^J_{v,w}(q)$$

derived as Proposition 3.1 in [18], we obtain its successor Corollary 1.1.

We proceed in three steps. First we show that two intervals with the same underlying diagram have the same relative R-polynomial. Next we show that posets are poset isomorphic if and only if their skew shapes are shape isomorphic. Finally we show that if two shapes are shape-isomorphic, they yield the same relative R-polynomial. Thus isomorphic posets yield the same relative R-polynomial

4.5.2 Shapes That Appear in Type A

We wish to show that if two diagrams have the same shape, they have the same relative R-polynomial.

Proposition 4.4. *If a skew diagram $v\backslash u$ consists of a single row of length k, it can be identified with the polynomial $(q-1)q^{k-1}$. The same statement is true for columns.*

Proof. The leftmost box in $v\backslash u$ is easily observed to be an ascent of u within v for any diagram of Hermitian type. No other box is an ascent, or descent, of u, thus the polynomial is as stated. The proof for columns is almost the same. □

Next, we note that if two restricted diagrams from Type A have the same markings, then they have the same R-polynomials.

Proposition 4.5. *If $u < v$ in Type A, with shapes λ and μ for v and u respectively, then $R^J_{u,v}(q)$ depends only on the skew shape $\lambda\backslash\mu$.*

Proof. In the proof of the main theorem for Type A above, only the length of certain diagonals was used to construct $R^J_{u,v}(q)$, and the diagonals are determined uniquely by the skew shape $v\backslash u$. □

4.5.3 Relative R-polynomials Depend Only on Shape

Recall that we say that a diagram without its labels is a partition. In particular, note that a diagram Λ is associated with a class of partitions, notably all such partitions that can contain that shape with that filling. Meanwhile the partition λ associated Λ is associated with that class, and any other diagram M that, when stripped of its filling, has shape λ. We will use upper case Greek letters for diagrams, and lower case Greek letters for partitions in this section. Recall in particular the definitions of Section 3.2.

Proposition 4.6. *If $\Lambda\backslash M$ is a standard skew diagram of type W^J, it can be identified with a skew diagram in type A, and the R-polynomial associated to $\Lambda\backslash M$ in W^J is the same as the R-polynomial of the same shape in type A.*

Proof. It is easy to see in the diagrams of Section 3.2 that a standard tableau of any type other than C_n/A_{n-1} is supported on a set of simple roots whose Coxeter graph is an induced subgraph of a Coxeter graph of type A.

For the C_n/A_{n-1} case, the standard skew diagram for $v\backslash u$ can include a single box labeled s_n in the bottom left corner (see figure below). If this corner is not an inside corner, then the box North and West of it (in the diagram for u) is an outside corner, and by the rules established for C_n/A_{n-1}, the box associated to s_n is unmarked, as it would be in type A. If it is an inside corner, it is marked with a plus, exactly as it would be in Type A. All other boxes in the diagram are marked according to type A rules, and thus the polynomial associated to $\Lambda\backslash M$ in the context of the quotient C_n/A_{n-1} is the same as the polynomial associated with the same diagram in type A. ☐

$$\begin{array}{|c|c|c|c|}\hline X & 3 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 4 & 3 & 2 \\ \cline{2-4}\end{array} = \begin{array}{|c|c|c|c|}\hline X & + & \; & \; \\ \hline \multicolumn{1}{c|}{} & \; & + & \; \\ \cline{2-4}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline X & X & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 4 & 3 & 2 \\ \cline{2-4}\end{array} = \begin{array}{|c|c|c|c|}\hline X & X & + & \; \\ \hline \multicolumn{1}{c|}{} & + & - & + \\ \cline{2-4}\end{array}$$

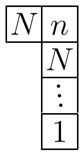
C_n/A_{n-1} standard, s_n not inside $\qquad$ C_n/A_{n-1} standard, s_n inside

We call a partition a *staircase* if it can be identified with the partition underlying a Hermitian diagram of type D_n/A_{n-1}. Note that the full list of maximal staircase partitions not explicitly belonging to D_n/A_{n-1} or C_n/A_{n-1} are

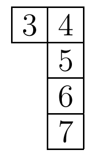
$$\begin{array}{|c|c|}\hline N & n \\ \hline \multicolumn{1}{c|}{} & N \\ \cline{2-2}\end{array} \in B_n/B_{n-1}, N := n-1 \qquad \begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & 4 \\ \cline{2-3} \multicolumn{2}{c|}{} & 5 \\ \cline{3-3}\end{array} \in E_6/D_5 \text{ or } E_7/E_6 \qquad \begin{array}{|c|c|c|}\hline 5 & 4 & 2 \\ \hline \multicolumn{1}{c|}{} & 3 & 4 \\ \cline{2-3} \multicolumn{2}{c|}{} & 5 \\ \cline{3-3}\end{array} \in E_7/E_6 \; .$$

But in all cases, the rules are those of D_n/A_{n-1}, and so we can identify polynomials again as before.

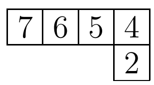
The final general form of skew partition is an L shape, of which there are two exceptional maximal examples, and one infinite class:



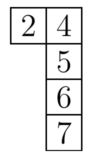
$\in B_n/B_{n-1}, N := n-1$,



$\in E_7/E_6$



$\in E_7/E_6$



$\in E_7/E_6$ and restricted in E_6/D_5 .

In type B_n/B_{n-1}, the have $R^J_{u,v}(q) = (q-1)q^{\ell(v)-\ell(u)-1}$ via Proposition 4.3, while the rule in type E is clear: one plus in the upper left box, and no other signs. Thus the relative R-polynomials are the same.

The final cases to compare are intervals contained solely in E_6/D_5 and E_7/E_6, which requires checking finitely many cases. Thus we have:

Theorem 4.2. *If $\lambda\backslash\mu$ is a skew partition, and $Lambda\backslash M$ and $\Lambda'\backslash M'$ are skew diagrams with $\lambda/\backslash\mu$ as the underlying partition, then if $[u,v]$ an interval in W^J and $[u',v']$ an interval in $W'^{J'}$ are associated with $\Lambda\backslash M$ and $\Lambda'\backslash M'$, respectively, hen $R^J_{u,v}(q) = R^J_{u',v'}(q)$.*

Hence we see that the relative R-polynomial for the interval $[u,v]$ depends on the skew shape associated to that interval, and not on the diagram for that interval.

4.5.4 Shape Isomorphism

We need a notion of isomorphism for skew diagrams. We define it in the obvious way: we let ϕ map boxes in $\Lambda\backslash M$ to boxes in $\Lambda'\backslash M'$, requiring that the map is

(1) bijective on the sets of boxes,

(2) outside corners of M and Λ are mapped to outside corners in M' and Λ', and similarly for inside corners,

(3) if two boxes are adjacent in $\Lambda\backslash M$, then their images are adjacent in $\Lambda'\backslash M'$,

(4) if two boxes in $\Lambda\backslash M$ have the same label, their images in $\Lambda'\backslash M'$ have the same label.

We refer to a map with these three properties as a diagram isomorphism.

Theorem 4.3. *Let $u < v$ in W^J with associated diagrams $M \subset \Lambda$, and $u' < v'$ in $(W')^{J'}$ with associated diagrams $M' \subset \Lambda'$. If $[u, v]$ and $[u', v']$ are isomorphic as posets then $\Lambda\backslash M$ is isomorphic to $\Lambda'\backslash M'$.*

Proof. We have already shown (Corollary 3.1) that Bruhat and partition orders are the same. Each element of the poset $[u, v]$ can be realized as a diagram $K\backslash M$. Then given a poset isomorphism $\phi : [u, v] \to [u', v']$, we can induce a diagram isomorphism Φ in the following way:

If x covers u and is an element of $[u, v]$, then x corresponds to a diagram $K_x\backslash M$ that consists of a single box b from $\Lambda\backslash M$. This box is, by necessity, an inside corner of the skew diagram. Similarly, $\phi(x) := x'$ covers u', and so corresponds to a skew diagram $K'_{x'}\backslash M'$ in $\Lambda'\backslash M'$, again consisting of a single box b' occupying an inside corner of the skew diagram. Then define $\Phi(b) = b'$; because of the identification of the set of subdiagrams of $\Lambda\backslash M$ with the poset $[u, v]$, this process is independent of the order in which boxes are chosen, as long as each step ends with a legal subdiagram.

Furthermore, if b is labelled r, and Φ has already been defined on an r-labelled box b', then it must have been defined on boxes above and left of b in the diagram. This forces the label of $\Phi(b)$ to be the same as the label of $\Phi(b')$, assuming that b is supported above and left. In the case that b is unsupported on the left, e.g. as occurs in Type E_6/D_5, the adjacency of b with a box above it, as well as the possible labels for the beginning of a row in the individual cases, determines the label of $\Phi(b)$ uniquely, and guarantees that the fourth requirement is met.

Continue this process until every box in $\Lambda\backslash M$ has been sent to a unique box in $\Lambda'\backslash M'$. By finiteness, this map is bijective. Inside corners of $\Lambda\backslash M$ are mapped to inside corners of $\Lambda'\backslash M'$. To see that outside corners of Λ and Λ' correspond, note that a box b in $\Lambda\backslash M$ is an outside corner precisely when $(\Lambda - \{b\})\backslash M$ corresponds to an element y covered by v. But then the image of $(\Lambda - \{b\})\backslash M$ under Φ is a diagram corresponding to $\phi(y)$ which is covered by $\phi(v)$, implying that $(\Lambda' - \phi(b))\backslash M'$. A similar argument shows that adjacent boxes are mapped to adjacent boxes. Thus a poset isomorphism induces a diagram isomorphism.

□

4.5.5 Combinatorial Invariance

We are now to the final key lemma, which will finish the proof of combinatorial invariance.

Lemma 4.2. *If skew diagrams $\Lambda\backslash M$ and $\Lambda'\backslash M'$ are diagram isomorphic, then they yield the same relative R-polynomial.*

Proof. We proceed by induction on m, the number of boxes in $\Lambda\backslash M$. If $m = 1$, then the shape is a single box and can be identified with a two element poset $[u, v]$ with v covers u, and all such posets have the same R-polynomial: $R_{u,v}(q) = (q - 1)$.

If $m > 1$, then let $\Phi : \Lambda\backslash M \to \Lambda'\backslash M'$ be the diagram isomorphism. Thus outside and inside corners are preserved.

Let b be an inside corner of M, i.e. a box that can be appended to M, yielding a diagram M^+ contained within Λ. Then $\Phi(b)$ is an inside corner of M', and appending it to M' yields a diagram $(M')^+$ contained in Λ'. Let $\Phi' : \Lambda\backslash M^+ \to \Lambda'\backslash(M')^+$ be the restriction of ϕ. Then Φ' is a diagram isomorphism by the inductive hypothesis, and thus preserves markings .

As b and $\Phi(b)$ are outside corners in M^+ and $(M')^+$, the marked boxes in $\Lambda\backslash M^+$ and $\Lambda'\backslash(M')^+$ controlled by b and $\Phi(b)$ are $-$-marked. Examining the rules of the marking algorithm carefully, it is apparent that making b and $\Phi(b)$ inside corners (deleting them from M^+ and $(M')^+$) turns all $-$ marks controlled by b and $\phi(b)$ to $+$ marks, and marks b and $\phi(b)$ with $+$ as well. Each diagonal distance corresponding to a changed mark rises by one in both $\Lambda \backslash M$ and $\Lambda' \backslash M'$, allowing us to see that all statistics $\Delta(b)$ are preserved. Thus if $[u, v]$ is any poset with diagram $\Lambda \backslash M$ and $[u', v']$ a poset with diagram $\Lambda' \backslash M'$, then $R^J_{u,v}(q) = R^J_{u',v'}(q)$. $\square$

This lemma concludes the proof of Theorem 1.1

CHAPTER FIVE

Combinatorics of Affine Grassmannians

5.1 Combinatorial Description of the Minimal Coset Representatives

5.1.1 $\widetilde{A}_n/A_n$

Let $w \in \widetilde{A}_n$, and $s_i = (i \;\; i+1) \in S$, $1 \leq i \leq n+1$ be a simple transposition. Multiplying permutations, we see that

$$ws_i = [w(1), \ldots, w(i-1), w(i+1), w(i), \ldots, w(n+1)]$$

and that if $w(j) \equiv i \pmod{n+1}$, $w(k) \equiv i+1 \pmod{n+1}$, that

$$s_i w = [w(1), \ldots, w(j-1), w(j)+1, w(j+1), \ldots, w(k-1), w(k)-1, w(k), \ldots, w(n+1)]. \tag{5.1}$$

In a similar way, one sees that s_i acts on w from the left by adding one to the entry whose residue (modulo $n+1$) is i, and subtracting one from the entry whose reside is $i+1$.

Let J be the parabolic subgroup of $\tilde{A}_n$ generated by $\{s_1, \ldots, s_{n-1}\}$. Equation 5.1 shows that the set of minimal length coset representatives of $(\widetilde{A}_n)^J$ is exactly the set of permutations in $\widetilde{A}_n$ that are strictly increasing when written in their complete notation. Note that J generates a Coxeter group of type A_n under the relations of $\widetilde{A}_n$, and is maximal among parabolic subgroups of $\widetilde{A}_n$. In abuse of notation, we will generally refer to the set of minimal coset representatives as $\widetilde{A}_n/A_n$. Because the Coxeter diagram for $\widetilde{A}_n$ is symmetric, we can restrict our attention to the quotient $\widetilde{A}_n/A_n$; the quotient by any other maximal parabolic subgroup will have an isomorphic order poset.

5.1.2 $\widetilde{D}_n/D_n$

Consider the group $\widetilde{D}_n$; simple elements $s_0, \ldots, s_{n-1}$ act as in D_n. Thus s_1 acts on a permutation w on the left by interchanging the first and second entries in the fundamental window for w, and also interchanging the entries in positions $kn+1$ and $kn+2$, for all $k \in \mathbb{Z}$. The element s_n acts by interchanging the entries in positions $kn+(n-1)$ and $kn+n$ with those in $kn+(n+1)$ and $kn+(n+1)$, respectively, for all $k \in \mathbb{Z}$.

Take D_n to be the parabolic subgroup generated by $J = \{s_1, \ldots, s_n\}$, and form the quotient $\widetilde{D}_n^J := \widetilde{D}_n/D_n$ as in Type A (Section 5.1.1). Then the minimal length coset representatives in $\widetilde{D}_n/D_n$ are those elements w such that

$$w(-2) < w(1) < w(2) < \cdots < w(n).$$

5.1.3 $\widetilde{E}_n/E_n$

In each case, since the generators J of the parabolic subgroup include a copy of A_{n-1}, we expect a necessary condition for w to be in $\widetilde{E}_n^J$ is that a subsequence of the n-tuple for w be increasing. Examining the definitions of the various $\widetilde{E}_n$, the specific subsequences are $w(2) < \cdots < w(n)$ for $n = 7, 8$, and $w(1) < \cdots < w(n)$ for $n = 6$. An extra condition is necessary in each case, however.

For $\widetilde{E}_7/E_7$, $S \backslash J$ consists of the node s_0, so multiplication by any simple other than s_1 must correspond to an ascent. Thus s_3 through s_7 must be length increasing reflections, implying that $w(2) < \cdots < w(7)$. Examining the length function for $\widetilde{E}_7$ and the action of s_7, we see that $\gamma(w) > 0$ implies s_7 is length decreasing. Thus the elements of $\widetilde{E}7/E_7$ are those permutations in $\widetilde{E}_7$ such that $w(2) < \cdots < w(7)$, and such that $\gamma(w) < 0$.

In a similar way, we see that $w \in \widetilde{E}_8/E_8$ provided $w(2) < \cdots < w(8)$, and $\gamma(w) < 0$.

The case of $\widetilde{E}_6/E_6$ is slightly different. The node s_0 is again the sole member of $S \backslash J$, but its function in $\widetilde{E}_6$ is different than the role of s_0 in $\widetilde{E}_7$ or $\widetilde{E}_8$. Here, s_0 acts by subtracting $\mu(w)$ from all components of w, and so examining the length function on $\widetilde{E}_6$, we see that a necessary condition for w a minimal coset representative of $\widetilde{E}_6/E_6$ is that $\mu(w) > 0$, unless $w = e$. Thus the characterization of minimal coset representatives is $w(1) < \cdots < w(6)$, and $\gamma(w) < 0$. If these constraints are met, none of $s_1, s_2, \ldots, s_6$ are descents, leaving only s_0.

5.2 Erikssons' Game

We present an analog to the sorting game of Chapter 3.2 for the quotient groups associated to affine Grassmannians. For $\widetilde{A}_n/A_n$ and $\widetilde{D}_n/D_n$, the game was first put to paper in [23]. The generalization to the quotients $\widetilde{E}_m/E_m$, $6 \leq m \leq 8$, while they owe much to the permutation representations of Henrik Eriksson obtained in [22], are novel to this paper.

5.2.1 $\widetilde{A}_n/A_n$

The substance of the game is as follows: for $w \in \widetilde{A}_n/A_n$ written in complete notation

(1) Define w^+ to be the ascending sorted $n+1$-tuple $[w(1)+n+1, w(2, \ldots, w(n)]$.

(2) If $w^+(j) = w(1) + n + 1$, define $\lambda_1 = n + 1 - j$.

(3) Replace w with w^+ and repeat, terminating when $w^+ = [a, a+1, \ldots, a+n+1]$ for some integer a.

The resulting sequence $(\lambda_1, \ldots, \lambda_k)$ is a partition of $\ell(w)$ [23].

We can invert this map similarly: let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of m

(1) Let $u = [k + 1, k + 2, \ldots, k + n + 1]$.

(2) Subtract $n + 1$ from the $n + 1 - \lambda_1$ entry in u, and sort into an ascending $n + 1$-tuple, denoting the resulting tuple by u^-.

(3) Replace u with u^-, replace λ with $(\lambda_2, \ldots, \lambda_k)$, and repeat the process, terminating when $\lambda = \emptyset$.

(4) Define w to be the element of $\widetilde{A}_n/A_n$ found by extending u, using property (a) from the definition of $\widetilde{A}_n$.

Example 5.1. *Let $n = 4$, and consider the permutation $w = [-7, 2, 5, 6, 9]$. Then we have*

$$\begin{aligned} w = & \; [-7, 2, 5, 6, 9] \\ \xrightarrow{4} & \; [-2, 2, 5, 6, 9] \\ \xrightarrow{3} & \; [2, 3, 5, 6, 9] \\ \xrightarrow{1} & \; [3, 5, 6, 7, 9] \\ \xrightarrow{1} & \; [5, 6, 7, 8, 9] \end{aligned}$$

at which point the algorithm terminates. Thus the partition associated to the permutation $[-7, 2, 5, 6, 9]$ is $\lambda = (4, 3, 1, 1)$. Note that the index above the arrows gives the added entry to λ.

Conversely, given a partition $\lambda = (3, 1, 1)$, we compute

$$\begin{aligned} & \; (4, 5, 6, 7, 8) \\ \xrightarrow{1} & \; (2, 4, 5, 6, 8) \\ \xrightarrow{1} & \; (1, 2, 4, 5, 8) \\ \xrightarrow{3} & \; (-3, 1, 4, 5, 8) \end{aligned}$$

and so $[-3, 1, 4, 5, 8]$ is the associated permutation.

5.2.2 $\widetilde{D}_n/D_n$.

Eriksson and Eriksson also constructed an algorithm giving a bijection from $\widetilde{D}_n/D_n$ to a set of partitions of particular form. They assign $w \in \widetilde{D}_n/D_n$ a partition $\pi(w)$ of $\ell(w)$ such that

(a.) There is at most one part of each up to $n - 2$,

(b.) One part of $n - 1$ may be marked with a dot,

(c.) No part is larger then $2n - 2$.

I have modified the original game to correspond to Björner and Brenti's construction of $\widetilde{D}_n$ as a subgroup of A_{2n+1}, but this causes only cosmetic changes in the Erikssons' argument. In particular, the algorithm given below works on the window

$$[w(-n), \ldots, w(-1), w(1) \ldots, w(n)].$$

The algorithm proceeds in two stages. The first lasts for as long as the entries in the complete notation of w in positions $n+1$ and $3n+2$ are the smallest entries right of position n.

Phase 1: Assume $w(3n+1) < w(n+2)$.

(a) Use s_0 to interchange the entries in positions $w(n-1)$ and $w(n)$ with $w(n+1)$ and $w(n+2)$, respectively. Call the new element w'.

(b) Sort the window $[w'(-n+1), \ldots, w'(n-1)]$ using simple transpositions from $\widetilde{D}_n$, recording the number of simples required for this process in the descent sequence λ.

(c) Use s_0 to interchange $w(n-1)$ and $w(n)$ with $w(n+1)$ and $w(n+2)$, respectively. Call the new element w''.

Nb. Note $w''(n) = w(n)$.

(d) Sort $w''(n)$ into $[w''(-n+1), \ldots, w''(n-1)]$ counting the simples used and appending that number to λ.

(e) If in the second sort, $w''(n)$ is moved to position 1, mark the descent number for the sort with a dot to indicate the use of s_{n-1} as opposed to s_n.

Nb. By the assumptions on the quotient, this can only happen once.

(f) Denote the resulting element w.

Phase 2: Now $w(3n+2) > w(n+1)$.

(a) Apply s_0 to w to replace $w(n)$ with $w(n+2)$ and $w(n-1)$ with $w(n+1)$. Denote the resulting element w'.

(b) Sort $w'(n-1)$ first, recording the descent number as before.

(c) Sort $w'(n)$ similarly, recording the descent number.

Nb. The same remark as before applies with respect to dotting an element of the descent sequence.

(d) Denote the resulting element w.

The algorithm can be easily inverted:

(1) Start with the $2n$-tuple $(-n, \ldots, -1, 1, \ldots n-1, n)$.

(2) For $1 \leq i \leq k$, set $\mu_i = \lambda_i$ for $1 \leq i < n-1$ or for $i = n-1$ and λ_i marked with a dot; otherwise let $\mu_i = \lambda_i + 1$.

(3) If $\lambda_{k-1} \neq 2n-2$, proceed with the first phase of sorts:

(a) If k is odd,

(i) Shift the entry in position $\mu_k + 1$ to position $n-1$, shifting all other elements towards the center of the window to accomodate the shift, and denote by w this new element.

(ii) Interchange $w(n+1)$ with $w(n-1)$ and $w(n+2)$ with $w(n)$, and represent the new element by w.

(iii) Throw away the last entry in λ and go back to step 3.

(b) If k is even,

(i) Shift $w(n-\mu_k)$ to position $n-1$, shifting all other entries towards the center of the window to accomodate this move, and let w represent the new element.

(ii) Shift $w(\mu_{k-1}+1)$ to position n, and denote by w this new element.

(iii) Interchange $w(n-1)$ with $w(n+1)$ and $w(n)$ with $w(n+2)$ respectively, and represent the new element by w.

(iv) Throw away the last two entries in λ and restart step 3.

(4) If $\lambda_{k-1} = 2n-2$, proceed with the second phase of sorts:

(a) Move $w(n-\mu_i)$ to position n, and set the right half of the window so the element showing remains in $\widetilde{D}_n$. Denote the resulting permutation by w.

(b) Replace $w(n)$ with $2 \cdot (2n+1) - w(n)$.

(c) Repeat from Phase 2, step 1, with each remaining entry.

(d) In each step of Phase 2, if $\mu_i > n$, interchange the values of $w(-1)$ with $w(1)$ to account for the use of s_n.

Example 5.2. *Let* $n = 5$, *and consider the permutation* $w = [-2, 3, 6, 7, 21]$. *We use a vertical bar to denote the end of the fundamental window, and show "extra elements" to make clear the insertions that are occurring.*

$$\begin{aligned} w &= [-21, -7, -6, -3, 2, -2, 3, 6, 7, 21| -10, 4, \cdots, 1, \cdots] \\ &\xrightarrow{8} [-10, -4, -6, -3, -2, 2, 3, 6, 4, 10|1, 7, \cdots, 12, \cdots] \\ &\xrightarrow{4^\bullet} [-7, -6, -3, -2, -1, 1, 2, 3, 6, 7|4, 5, \cdots,] \\ &\xrightarrow{1} [-5, -4, -3, -2, -1, 1, 2, 3, 4, 5] \end{aligned}$$

Thus the partition associated with $[-2, 3, 6, 7, 21]$ *is* $(8, 4^\bullet, 1)$.

Example 5.3. *Let* $n = 5$, *and let* $\lambda = (8, 4^\bullet, 2, 1)$. *Note that steps involving an interchange from outside the permutation are marked with* $(*)$ *to make the replacement* *clear.*

$$\begin{aligned}
\sigma = \quad & [-5,-4,-3,-2,-1,1,2,3,4,5] \\
\xrightarrow{1} \quad & [-4,-5,-3,-2,-1,1,2,3,5,4] \\
\xrightarrow{2} \quad & [-4,-3,-5,-2,-1,1,2,5,3,4] \\
\xrightarrow{*} \quad & [-8,-7,-5,-2,-1,1,2,5,7,8] \\
\xrightarrow{4^\bullet} \quad & [-1,-8,-7,-5,-2,2,5,7,8,1] \\
\xrightarrow{8} \quad & [-21,-8,-7,-5,2,-2,5,7,8,21].
\end{aligned}$$

and so $[-2,5,7,8,21]$ *is the permutation of* $\widetilde{D}_5/D_5$ *associated to* $\lambda = (8,4^\bullet,2,1)$.

Example 5.4. *With* $n = 5$, *consider* $\lambda = (4,3,2,1)$. *Then*

$$\begin{aligned}
\sigma \quad = \quad & [-5,-4,-3,-2,-1,1,2,3,4,5] \\
\xrightarrow{1} \quad & [-4,-5,-3,-2,-1,1,2,3,5,4] \\
\xrightarrow{2} \quad & [-4,-3,-5,-2,-1,1,2,5,3,4] \\
\xrightarrow{(*)} \quad & [-8,-7,-5,-2,-1,1,2,5,7,8] \\
\xrightarrow{3} \quad & [-2,-8,-7,-5,-1,1,5,7,8,2] \\
\xrightarrow{4} \quad & [-2,1,-8,-7,5,-5,7,8,-1,2] \\
\xrightarrow{(*)} \quad & [-12,-9,-8,-7,5,-5,7,8,9,12]
\end{aligned}$$

so $w = [-5,7,8,9,12] = \pi^{-1}(\lambda)$.

5.2.3 $\widetilde{E}_n/E_n$

Consider first the case of $m = 7$. Our goal is to assign a partition to each permutation. As we are describing the poset $\widetilde{E}_7/E_7$, elements are 8-tuples $w = (w_1, \ldots, w_8)$ with $w_2 < w_3 < \cdots < w_8$. We play a game as follows:

(1) Sort the sequence w, recording the number of positions w_1 moves right as d_1. Call the resulting tuple w.

(2) If $w = e$, then $\lambda = (d_1)$, and the game terminates.

(3) Otherwise, add γ to the first four entries of w, call the resulting element w, and replace d_1 with d_1^+.

(4) Sort $w(5)$ left into the sequence, recording the number of positions left it moves as d_2.

(5) Continue with $w(6)$, $w(7)$, and $w(8)$, if neccessary.

(6) If w is sorted, return to step (3), apending $^+$ to the final entry in λ.

(7) The process terminates when $w = e$.

The inverse algorithm is clear from this description.

For $m = 8$, the same algorithm works, but on 9-tuples instead of 8-tuples. Otherwise, the quotient structure is the same.

For $\widetilde{E}_6/E_6$, we again begin by sorting the given tuple, recording rightward jumps. If the tuple is sorted, but not the identity, apply $\gamma(w)$ or $\mu(w)$, whichever is positive, to the tuple, and mark the next number recorded (after sorting again) with a $+$ if s_2 was used (i.e.$\gamma(w)$ applied) or $-$ if s_0 was used (i.e. $\mu(w)$ was applied).

Remark 5.1. *In all cases, given w in $\widetilde{X}_n/X_n$ for one of the types discussed above, we may write $\lambda(w)$ for the partition associated to w by Eriksson's game.*

5.2.4 An Important Corollary of the Game

Examining the algorithms as they have been presented, Eriksson's game can be seen as producing reduced decompositions of Weyl group elements in a canonical way. This will be made explicitly clear as each type is discussed individually. One important corollary follows from this observation, and we present it here.

Corollary 5.1. *(Of Erikkson's construction). If u, v are minimal length coset representatives in one of the previously mentioned quotient groups, and $\lambda(u), \lambda(v)$ are the* *partitions associated to u and v, respectively, by Eriksson's game, then $\lambda(u) < \lambda(v)$ in the containment order on partitions implies that $u < v$.*

Proof. As $\lambda(u)$ is covered by $\lambda(v)$, then Eriksson's game provides an explicit reduced decomposition of u as a subword of v. □

This shows that the Bruhat order contains the Young order via the association of permutations and partitions. In fact, Bruhat order is strictly finer than Young order.

CHAPTER SIX

Rational Smoothness in $\widetilde{A}_n/A_n$

6.1 Combinatorics in $\widetilde{A}_n/A_n$

6.1.1 The Reduced Word Interpretation of the Game: $\widetilde{A}_n/A_n$

Recall that for $w \in \widetilde{A}_n$, $w(k+n+1) = w(k)+n+1$. By adding $n+1$ to $w(1)$ and sorting, counting the number of jumped entries *from the right side of the fundamental window,* we have really just shifted the window right by one unit, and sorted from the back. So the first sort can be realized as right multiplication by $v_{1,\lambda_1}^{-1} := s_0 s_n \cdots s_{n+2-\lambda_1}$. Similarly, the next sort shifts the window right by one unit, so the second sort can be realized by right multiplication by $v_{2,\lambda_2}^{-1} := s_1 s_0 s_n \cdots s_{n+3-\lambda_2}$. In the same way, the k^{th} sort can be realized by right multiplication by $v_{k,\lambda_k}^{-1} := s_{k-1}s_{k-2}\cdots s_{n+1+k-\lambda_k}$, where the indices are counted from the residues modulo $n+1$.

This leads to the following:

Theorem 6.1. *If $w \in \widetilde{A}_n/A_n$ is associated to partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, then w has a reduced decomposition*

$$w = v_{k,\lambda_k} \cdots v_{1,\lambda_1}.$$

Proof. That $w = v_{k,\lambda_k} \cdots v_{1,\lambda_1}$ follows from the argument given above. That the expression is reduced follows from the fact that each v_{i,λ_i} has at most n simple elements multiplied together, and by construction,for $0 \le j \le n$, s_j appears at most once in v_{i,λ_i}. This implies that each v_{i,λ_i} is reduced, and that $\ell(v_{i,\lambda_i}) = \lambda_i$. But

$$|\lambda| = \ell(w) \le \sum_{i=1}^{k} \ell(v_{i,\lambda_i}) = |\lambda|$$

implying that the decomposition for w is indeed reduced. □

We can encode this information in an array as in Chapter 3 of this thesis:

s_0	s_n	$\cdots$	s_2
s_1	s_0	$\cdots$	s_3
$\vdots$	$\vdots$	$\cdots$	$\vdots$
s_n	s_{n-1}	$\cdots$	s_1
s_0	s_n	$\cdots$	s_2
$\vdots$			$\vdots$

Example 6.1. *(Continuing Example 5.1) For $n = 4$, the block above becomes*

0	4	3	2
1	0	4	3
2	1	0	4
3	2	1	0

with length limited to four rows. Cover this with the partition $\lambda = (4, 3, 1, 1)$ to get

0	4	3	2
1	0	4	
2			
3			

which corresponds to the reduced decomposition

$$w = (s_3)(s_2)(s_4 s_0 s_1)(s_2 s_3 s_4 s_0)$$

as being the unique reduced decomposition associated to λ.

As in Chapter 3.2, we can realize elements in $\widetilde{A}_n/A_n$ by thinking of shaded subdiagrams of these large rectangles. A similar statement holds for the other types.

6.2 Small Elements

6.2.1 Definition

Definition 6.1. *We say that a minimal coset representative w of $\widetilde{A}_n/A_n$ with associated partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is small provided that $k + \lambda_1 \leq n + 1$. That is to say, w is small if the maximum row length plus maximum column length of $\lambda(v)$ is smaller than or equal to $n + 1$.*

In this section, we examine the structure of the fundamental windows of small permutations. For ease of discussion, q will denote the maximum row length, and p will denote the maximum column length for a given permutation. We will abuse notation and say that a permutation v has row length q provided the maximal row length of $\lambda(v)$ is q, and similarly for column length. Also, the partition $\lambda = (q, q, \ldots, q)$ of length p may be denoted $\lambda = (q^p)$. We call such a partition, and the minimal length coset representative associated to it, rectangular.

6.2.2 Combinatorial Structure of Small Rectangular Elements.

Proposition 6.1. *Let $v \in \widetilde{A}_n/A_n$ be associated to the partition $\lambda = (r^t)$, and suppose that $r+t = n+1$. Then $v = [a_1, a_2, \ldots, a_t, b_1, \ldots, b_r]$, where $a_i = a_{i-1}+1$ for $2 \leq i \leq t$ and $b_i = b_{i-1} + 1$ for $2 \leq i \leq r$. Furthermore, $v(1) = 1 - r$, and $v(n+1) = t + n + 1$.*

Proof. Let v, r and t as above. To explicitly construct v, we start at the window $[t+1, \ldots, t+n+1]$, which after application of the first r deletes the entry $t+n+1-r$ and replaces it with $t+n+1-r-(n+1) = r-t$ (modulo $n+1$) in position 1. The last $r-1$ elements are left untouched by this and every other application of r to the window, yielding the statement on the string of b-values, and that $v(n+1) = t+n+1$. Every other application of r takes the element in the new $t+n+1-q$ position, subtracts $n+1$ from it, and sorts it to the beginning of the window, and this happens a total of t times. As $r+t = n+1$, this implies that each of the first t elements is moved exactly once, so the minimal element in the complete notation must be

$$u(1) = t + 1 - (n+1) = t - n = t - (r + t - 1) = 1 - r,$$

and that $u(i) = 1 - r + i$ for $1 \leq i \leq t$, finishing the proposition. □

Corollary 6.1. *Let $w \in \widetilde{A}_n/A_n$ be associated to the partition $\lambda = (r^t)$. Then if a_t is as above, and $1 \leq i \leq n+1$ is defined to be the solution to the equation $a_t + 1 \equiv i$ (mod $n+1$), then s_i is a left descent of w (counting $s_{n+1} := s_0$).*

Proof. We can see explicitly from the argument in the proposition that $b_1 - a_t = n+2$, and so by construction, the unique element whose residue is one more than that of

$v(t)$ (counting a residue of 0 as being one higher than that of n) is b_1. Furthermore, the residue of $v(t)$ modulo $n+1$ is equal to the residue of $t-r$ modulo $n+1$, denoting this residue by i so s_i, multiplying v on the left, will serve to shorten v. $\square$

We can use the same argument to provide a similar result for rectangular permutations with r rows and t columns where $r+t<n+1$:

Proposition 6.2. *Let $v \in \widetilde{A}_n/A_n$ have partition $\lambda = (r^t)$, and suppose that $r+t<n+1$. Then $v = [a_1, a_2, \dots, a_t, b_1, \dots, b_s, c_1, \dots, c_r]$ where $s = n+1-r-t$, where for $1 \le i \le t$, $t+1 < j \le t+s$, and $t+s+1 < k \le t+n+1$, $a_{i-1}+1 = a_i$, $b_{j-1}+1 = b_j$, and $c_{k-1}+1 = c_k$. Furthermore, $v(1) = 1-r$, and $v(n+1) = t+n+1$.*

6.2.3 The Function $M_{i,j}(v)$.

Define for $1 \le i < j \le n+1$ a function

$$M_{i,j}(v) = \left\lfloor \frac{|v(j)-v(i)|}{n+1} \right\rfloor. \qquad (6.1)$$

Lemma 6.1. *Let $v \in \widetilde{A}_n/A_n$ be rectangular, let r be the number of columns of $\lambda(v)$ and t the number of rows of $\lambda(v)$, and suppose that $r+t \le n+1$. Then*

$$M_{i,j}(v) = \begin{cases} 1 & \text{if } 1 \le i \le t, t+s+1 \le j \le n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the above, for $1 \le i \le t$ and $t+1 \le j \le n+1$,

$$|v(i)-v(j)| \le |v(1)-v(n+1)| = 2n+1,$$

but $\lfloor \frac{2n+1}{n+1} \rfloor = 1$. Conversely, for i and j as before, $|v(i)-v(j)| \ge |b_1 - a_q| = n+2$, and $\lfloor \frac{n+2}{n+1} \rfloor = 1$, yielding the result. $\square$

Now suppose that u and v are elements in the quotient such that u is covered by v. Then there is a transposition $t = T_{ij}$ so that $ut = v$, and $u(k) = v(k)$ for $k \ne i, j$. Recall there are three requirements that must be met for u to be in the quotient: (1), $u(i) < u(i+1)$ for $1 \le i \le n$; (2) the elements $u(i)$ for $1 \le i \le n+1$ must exhaust every equivalence class of integers modulo $n+1$; and (3) $\sum_{i=1}^{n+1} u(i) = \binom{n+2}{2}$. The first requirement ensures $u(i) < u(j)$ and $v(i) < v(j)$, the second forces $u(i) \equiv v(j) \pmod{n+1}$ and $u(j) \equiv v(i) \pmod{n+1}$, and the third implies that if $u(j)+m = v(j)$, then $u(i) - m = v(i)$. If m is positive, then $M_{k,l}(u) \geq M_{k,l}(v)$ for $l > k \geq i$, for $k < l \leq j$, and for $(k,l) = (i,j)$. But this is all possible combinations of i and j, which implies that $\ell(u) \geq \ell(v)$, which contradicts our assumption that v covers u. Thus we have the following:

Lemma 6.2. *For $u, v \in \widetilde{A}_n/A_n$ such that $u \xrightarrow{t} v$, where $t = T_{ij}$, then*

$$v(i) < u(i) < u(j) < v(j). \tag{6.2}$$

These two propositions yield immediately a bijection between small elements of the quotient with the Young order on the set of partitions with row length no longer than n:

Proposition 6.3. *If $u \to v$ elements of $\widetilde{A}_n/A_n$, where the partition associated to v is $\lambda = (r^t)$, and $r + t \leq n+1$, then $M_{k,l}(u) \leq M_{k,l}(v)$ for each pair (k,l) with $1 \leq k < l \leq n+1$. Furthermore, in the case that $u \to v$, there is precisely one pair (k,l) such that $M_{k,l}(u) < M_{k,l}(v)$.*

Proof. Let $u \to v$ be as above, with transposition $t = T_{ij}$ witnessing the cover. Let $s = n + 1 - r - t$. Consider the following cases:

(1) If neither k nor l is equal to i or j, then as u and v differ only at positions i and j, then $M_{k,l}(u) = M_{k,l}(v)$.

(2) $1 \leq k < j$: $u(j) < v(j)$ implies $M_{k,j}(u) \leq M_{k,j}(v)$.

(3) $1 \leq k \leq i$: As $u(i) < u(j)$, then $M_{k,i}(u) \leq M_{k,j}(u)$.

(4) $i < l \leq n+1$: As $u(i) > v(i)$, then $M_{i,l}(u) \leq M_{i,l}(v)$.

(5) $j < l \leq n+1$: $v(i+1) = u(i+1) \leq u(j)$, so we have from Proposition 5 that $M_{i+1,l}(v) = 0$, which implies

$$n+1 > v(l) - v(i+1) > u(l) - u(i+1) \geq u(l) - u(j),$$

which implies that $M_{j,l}(u) = 0$.

The final statement is clear as $\ell(v) = \ell(u) + 1$, and $\sum_{1 \leq i < j \leq n+1} M_{i,j}(v) = \ell(v)$ and similarly for u. □

Corollary 6.2. *Let $u < v \in \widetilde{A}_n/A_n$ be such that $\lambda(v) = (r^t)$, with $r + t \leq n+1$. Then $\lambda(u) < \lambda(v)$.*

Proof. Given $u < v$, use the Chain property in the quotient to find a sequence $u = w_0 < w_1 < \cdots < w_k = v$, where $\ell(w_i) = \ell(u) + i$ and recursively apply the covering case in the proposition above. □

Corollary 6.3. *If $u \in \widetilde{A}_n/A_n$ has partition $\lambda = (r^t)$, and $r + t \leq n+1$, then $M_{i,j}(u) \in \{0,1\}$ for each $(i,j) \in [1,n] \times [2, n+1]$.*

Proof. Let u be as above, and set v to be the permutation whose partition is (r^t). But $M_{i,j}(v) \in \{0,1\}$, and by the Proposition above, $M_{i,j}(u) \leq M_{i,j}(v)$, yielding the result. □

6.2.4 The Array $\Delta^n(v)$.

Given a partition λ and an integer $n+1$ such that, for each i, the i^{th} row of λ is of length less than or equal to $n+1-i$, define $\Delta^n(\lambda)$ in the following way: Write λ as a Young diagram with λ_i squares in the i^{th} row. Fill the diagram with ones, and then extend, for each $i \leq n+1$, the i^{th} row to a length of $n+1-i$, and fill the new boxes with zeroes.

Example 6.2. *Let $n = 3$; then the tableau $\lambda = (1, 1)$ satisfies the requirements of the above, so we can form $\Delta^3(\lambda)$ to be the array*

$$\Delta^3(\lambda) = \begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & \\ 0 & & \end{array} .$$

We can abbreviate the above notation by referring to the array as $((1, 0, 0), (1, 0), (0))$.

Lemma 6.3. *Let λ be a partition of $k \leq m$ be such that in the Young diagram for λ, there are p rows, q columns, with $p + q \leq n + 1$. Then there is a unique permutation $v \in \widetilde{A}_n/A_n$ associated to λ in such a way that*

$$\Delta^n_{i,j}(\lambda) = M_{i,j}(v)$$

for all $1 \leq i < j \leq n + 1$.

Proof. Given such a partition λ, it is a consequence of Eriksson and Eriksson's derivation of Bott's formula for $\widetilde{A}_n/A_n$ that a unique permutation v exists such that $\ell(v) = k$, and this permutation is associated to λ through the numbers game used before. Write $\lambda = (x_1, \ldots, x_q)$ and construct the permutation starting in the window $[q + 1, \ldots, q + n + 1]$. The first action of the game subtracts $n + 1$ from $q + n + 1 - x_q$ and resorts the list. As $q - x_q < q + 1$, then $q - x_q$ is the first element in the new sequence, and as

$$\sum_{i=1}^{q-1} x_i \leq \sum_{i=1}^{q-1} p \leq n + 1 - q, \tag{*}$$

there are two possibilities: if $q = 1$, the game is finished, $M_{1,l}(v) = 1$ for $q+n+1-x_q \leq k \leq q + n + 1$, while $M_{i,j}(v) = 0$ for all other pairs (i, j), and so we have the array $((1, 1, \ldots, 1, 0, \ldots, 0), (0, \ldots, 0), \ldots, (0))$, with x_q ones in the first entry. If $q > 1$, then $(*)$ implies that $x_{q-1} < n$, which means that the game will not move the element $q-x_q$ again, i.e. $q-x_q$ will be an entry in the permutation. The same argument shows after the second step of the numbers game that no later step will change the values $\{q-x_q, q-x_{q-1}-1\}$, and so on with each additional step of the numbers game. Putting this in the language of the structure proposition earlier, the permutation v associated to λ will be $(a_1, \ldots, a_q, b_1, \ldots, b_s, c_1, \ldots, c_p$, where $c_i = q + n + 1 - p + i$, the elements $b_1, \ldots, b_s$ are the nondeleted subset of $[q+1, q+n+1-x_q]$, and $a_i = q - x_{q-i} - q + i$. By construction, for $1 \leq i \leq q$ and $i < j \leq q + s + i$, $M_{i,j}(v) = 0$, while for $1 \leq i \leq q$ and $q + s + i + 1 \leq j \leq n + 1$, $M_{i,j}(v) = 1$. Also, for $p + 1 \leq i \leq n$ and $i < j \leq n + 1$, $M_{i,j}(0) = 0$. As $\Delta^{n+1}(\lambda)$ is associated to λ in a unique way via Lemma 6.2.3, this gives the bijection. ☐

Theorem 6.2. *Given $n \in \mathbb{N}$, for $v \in \widetilde{A}_n/A_n$ small, then $[e, v]$ in the Bruhat order is isomorphic as a poset to $[\lambda(e), \lambda(v)]$ in the Young order.*

Proof. Let $v \in \widetilde{A}_n/A_n$ be a given rectangular element with associated partition $\lambda(v) := (r^t)$, with $r + t \leq n + 1$. Associate $\lambda(v)$ with $\Delta^n(v)$, and suppose that u is covered by v. By Proposition 6.2.3 and the definition of Δ^n, we see $\Delta^n(u)$ and $\Delta^n(v)$ are the same for all but one entry, and by summing the rows of $\Delta^n(u)$ and $\Delta^n(v)$ to get the partitions $\lambda(u)$ and $\lambda(v)$, we see that the partitions differ in exactly one place as well. Thus $\lambda(v)$ covers $\lambda(u)$. But Theorem 6.2 stated that $\lambda(u) < \lambda(v)$ implies $u < v$ in the Bruhat order. ☐

6.2.5 Complementary Partitions and Palindromicity.

Next we show that an interval in the Young order whose highest element is rectangular must be palindromic. First consider the dual cases of $\lambda = (r)$ and $\lambda' = (1, 1, \ldots, 1)$ (length t): In each case, there is exactly one partition μ covered by λ (or λ'), and μ is also rectangular. By Theorem 6.2.4, the Young order of $[\lambda(e), \lambda(v)]$ and the Bruhat order on $[e, v]$ coincide, so the Poincaré polynomial of $[e, v]$ is palindromic. This suggests a proof of the palindromicity of any rectangle, but first we need a definition.

Definition 6.2. *Given tableau $\mu = (a_1, \ldots, a_m)$ and $\lambda = (b_1, \ldots, b_n)$ where $N =$ $\max\{m, n\}$, we write*

$$\mu \oplus \lambda = (a_1 + b_N, a_2 + b_{N-1}, \ldots, a_N + b_1)$$

where $a_i = 0$ *if* $i > m$*, and* $b_i = 0$ *if* $i > n$*. We say that tableau or partitions* μ *and* λ *are complementary relative to the partition* κ provided $\mu \oplus \lambda = \kappa$.

Theorem 6.3. *If* $v \in \widetilde{A}_n/A_n$ *is a rectangular element with* r *columns and* t *rows, and* $r + t \leq n + 1$*, then the Poincaré polynomial of* v *is palindromic.*

Proof. Consider again the case $\lambda = (r)$. The first deletion yields a partition $\lambda_1 = (r - 1)$, and we can think of the deleted entry as being a partition (1), and $(r-1) \oplus (1) = \lambda$. Similarly, deletion of the next entry yields a partition $(r - 2)$, and we can append the deleted element on to (1) to yield (2), and $(r-2) \oplus (2) = \lambda$. The same pairing continues throughout. Note that deletion of an outer corner in λ_i corresponds to appending a box to the complement of λ_i relative to λ in such a way that the ammended complement remains a Young diagram, and is, in fact, complementary to the deleted λ_i relative to λ. Thus the process of deleting squares from a rectangular partition corresponds exactly to the process of building that partition up from the empty set. Thus the interval $[e, \lambda]$ is palindromic in the Young order.

In general, let λ be any rectangular partition $(r, \ldots, r)$ (t entries). For a subpartition λ' of λ (partition lying entirely within λ), define a complement μ' of λ' by $\mu'(1) = t - \lambda'(t)$. For every possible deletion of an outer corner of λ' there corresponds a place where a square can be appended to μ' in the Young diagram, and thus the posets $[\emptyset, \lambda]$ (Young partial order) and $[\lambda, \emptyset]$ (reverse Young partial order) are isomorphic. □

6.3 Large Elements

6.3.1 What Is To Be Shown

This section is concerned with proving the following theorem.

Theorem 6.4. *Let* $w \in \widetilde{A}_n/A_n$ *be large; the partition associated to* w *by the Eriksson game is of the form* $\lambda = (1^{nk})$ *or* (n^k) *if and only if* $[e, w]$ *is palindromic.*

6.3.2 The Shape of the Fundamental Window of w Large

In this section, we extend some earlier results on small elements to large elements. These extensions allow us to construct descents not corresponding to outside corners of partitions associated to large elements.

Recall the Propositions of Section 6.2.2. In those technical propositions, it was shown that small rectangular elements with columns of length t and rows of length r can be written as permutations whose fundamental window took the shape of series of consecutive integers

$$w = [a_1, \ldots, a_t, b_1, \ldots, b_s, c_1, \ldots, c_r]$$

with $s = n + 1 - r - t$. The same statement holds for large elements.

Proposition 6.4. *Let w be a large rectangular permutation associated to partition (r^t). Let $j = n+1-r$, and let k be the residue of t modulo j (using j for the zero residue). Then*

$$w = [a_1, \ldots, a_k, b_1, \ldots, b_{j-k}, c_1, \ldots, c_r]$$

where the a_i, b_i and c_i constitute strings of consecutive integers. Furthermore, the sequence

$$\{b_1, \ldots, b_{j-k}, a_1, \ldots, a_k, c_1, \ldots, c_r\}$$

is a string of consecutive integers modulo $n + 1$.

Proof. The first statement of the proposition follows exactly as in Section 6.2.2. So suppose that $j = n + 1 - r$, that $j \neq t$, and that j divides t. Then each element in the a-sequence is moved exactly t/j times in the inverse game to construct w from its partition, implying that the b-sequence is empty. Furthermore, as each element in the a sequence is moved at least twice, while no element of the c-sequence is moved in the game.

The second statement is similarly, noting that, at each step of the inverse game, the number in position $n+1-r$ is replaced with itself minus $(n+1)$ and sorted left r

positions. Thus the residue of a_1 is one larger than the residue of b_{j-k}, and the residue of c_1 is one larger than that of a_k, all residues being taken modulo $n+1$.

$\square$

This has an immediate corollary that will be useful in the sequel.

Corollary 6.4. *If w is a large rectangular element with partition $\lambda = (r^t)$, $j := n+1-r$, and $t > j$, then partitioning w as before into three segments $\{a_i, 1 \leq i \leq j\}, \{b_i, 1 \leq i \leq s\}, \{c_i, 1 \leq i \leq r\}$ of consecutive integers, and $c_1 - b_s \geq \lfloor \frac{t}{j} \rfloor$, with equality holding only in the case that j divides t.*

Proof. Denote the residue of $w(i)$ modulo $n+1$ as $\widetilde{w(i)}$. As the inverse game is played starting from the $n+1$ tuple $[r+1, r+2, \ldots, r+n+1]$ and each element in the a-segment is moved exactly t/j times, we see that $\widetilde{w(r)} = \widetilde{p+r}$, and $\widetilde{w(r+1)} = \widetilde{p+r}+1$. In particular, if u represents the results of j steps of the inverse game for the partition (r^t), the initial $r+1$ elements of u are $\{r+1-(n+1), \ldots, r+j-(n+1), r+j+1\}$. Then $u(r+1) - u(r) = n+2$. Another j steps of the game yields an element v with $v(r+1) - v(r) = (n+2) + (n+1) = 2(n+1)+1$. Continuing the process yields the inequality.

Now let T be the largest multiple of j smaller than t, and let u be the element obtained by executing T steps of the inverse game on the starting position $[r+1, \ldots, r+n+1]$. Then $u = [a_1, \ldots, a_j, c_1, \ldots, c_r]$, and $c_1 - a_j = \frac{T}{j}(n+1)+1$. Applying more steps of the game causes the gap between the c-segment and the a and b segment to grow further, establishing the result. $\square$

6.3.3 *w Large, $\lambda(w)$ Rectangular: A Case*

Suppose $w \in \widetilde{A}_n/A_n$ is associated to a rectangular partition (r^t), with $r+t > n+1$. Define $j := n+1-r$, and suppose that j divides t. Let s_m be the simple reflection associated to the lower right corner element in the labelled partition (r^t). Let $w' := ws_m$. We show that w' has three left descents, which, as s_0 is covered by two elements in $\widetilde{A}_n/A_n$, implies that $[e, w]$ is not palindromic.

Begin by noting that the partition λ' associated to w' is the partition associated to w with its lower right corner deleted; thus λ' has two apparent left descents via Theorem **outside corner = right descent**, and we construct a third.

As the last simple reflection in the first row of λ has label $1-r$ modulo $n+1$, and each subsequent row ends in a simple reflection of one higher index than the previous, it is easy to see that m is the residue of $1-r+(t-1)$ modulo $n+1$.

Define the element $r := r(p,q,j,t)$ to be the reflection that deletes the simple reflection s associated with the last box in row $q-j$ in λ'. This reflection is

$$r = (s_{m+1}\cdots s_{m+p-1})\,(s_{m-1}\cdots s_{m-j+1})\,s_{m-j}\,(s_{m-1}\cdots s_{m-j+1})^{-1}\,(s_{m+1}\cdots s_{m+p-1})^{-1}$$

where all indices are considered modulo $n+1$. As r is a reflection, we would like to realize it as an affine transposition. Suppose that $m=0$; understanding r in this case will translate to every other case by translating along the Dynkin diagram. In this case,

$$r = (s_1\cdots s_{p-1})(s_n\cdots s_{n-j+2})s_{n-j+1}(s_n\cdots s_{n-j+2})^{-1}(s_1\cdots s_{p-1})^{-1}.$$

As $j = n+1-p$, we can rewrite this as

$$\begin{aligned} r &= (s_1\cdots s_{p-1})(s_n\cdots s_{p+1})s_p(s_n\cdots s_{p+1})^{-1}(s_1\cdots s_{p-1})^{-1} \\ &= (s_n\cdots s_{p+1})(s_1\cdots s_{p-1})s_p(s_1\cdots s_{p-1})^{-1}(s_n\cdots s_{p+1})^{-1} \\ &= (s_n\cdots s_{p+1})(s_p\cdots s_2)s_1(s_p\cdots s_2)^{-1}(s_n\cdots s_{p+1})^{-1}. \end{aligned}$$

This is the transposition interchanging $1+k(n+1)$ and $n+k(n+1)$ for every integer k. If $m \neq 0$, the same process yields a transposition switching $1+m+k(n+1)$ with $n+m+k(n+1)$. So let us consider the "shape" of the permutation w'.

As w' is large, and we assume that j divides t, we can bring Proposition 6.4 into play. We can write $w = [a_1,\ldots,a_j,c_1,\ldots,c_r]$, and as w is large, $c_1 - a_j \geq 2n+3$. Furthermore, the residue of a_j is $r+j$ modulo $n+1$, and the residue of c_1 is $r+j+1$.

Letting $w' := s_m w$ as before. Then it is easy to see that

$$w' = [a_1,\ldots,a_{j-1},a_j+1,c_1-1,c_2,\ldots,c_r],$$

with the residue of $w'(j)$ modulo $n+1$ equal to $r+j+1$ and the residue of $w'(j+1)$ modulo $n+1$ equal to $r+j$. Applying r as above to w' yields a second descent, and thus contradicts palindromicity.

6.3.4 w large, $\lambda(w)$ Rectangular: The Final Non-Palindromic Case

Suppose $w \in \widetilde{A}_n/A_n$ has associated partition (r^t) with $r+t > n+1$, let $j := n+1-r$ as before, and suppose that j does not divide t. We show that w covers at least two elements in $\widetilde{A}_n/A_n$, which then implies that $[e,w]$ is not palindromic.

Our argument relies on the following observation: there is nothing special about playing the Eriksson game from right to left in each permutation. In particular, we have the following reformulations of previous theorems.

Theorem 6.5. *There is a game that assigns to any permutation $w \in \widetilde{A}_n/A_n$ a labelled partition $\lambda^c(w)$ such that, if s_m is the label of an outside corner, then s_m is a right descent of w.*

Proof. The proof of this result relies on reformulating the Eriksson game to be played from the left side of each permutation, instead of from the right. As right multiplication by simple reflections corresponds to shuffling positions, playing the game from the left sorts elements rightwards, yielding increasing sequences of simple reflections. Thus w may be realized as a subdiagram of the array

0	1	$\dots$	$n-1$
n	0	$\dots$	$n-2$
$n-1$	n	$\dots$	$n-3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

All of the theorems of Sections 6.1 and 6.2 can be derived as easily from the realization of the game. □

We call the alternate Eriksson game the *left* game for the remainder of this section.

To finish this case, we show that w as above has two outside corners in its representation in the left game.

So let w satisfy the condition above; then by Proposition 6.1, the permutation representation of w has the form

$$w = [a_1, \ldots, a_k, b_1, \ldots, b_{j-k}, c_1, \ldots c_r].$$

Playing the left game on w, we see that as $a_1 \equiv b_{j-k} + 1 \pmod{n+1}$, and $b_{j-k} - a_1 < n+1$ forces the first output of the game to be j, and similarly for the first $K := \lfloor \frac{t}{j} \rfloor \cdot r$ outputs. But at the $K+1$ iteration of the reverse algorithm, the inserted entry fits between the a ad b sequences, and so appends a k to the partition. Hence the left game yields a partition $\lambda^c = (j^K, k^r)$. But this has two outside corners, and thus w has two left descents, showing $[e, w]$ can't be palindromic.

6.3.5 The Case $\lambda = (n^k)$

Now consider the partition $\lambda = (n, n) \in \widetilde{A}_n / A_n$. The element w associated to λ has reduced decomposition

$$w = \big((s_0 s_n \cdots s_2)(s_1 s_0 s_n \cdots s_3) \big)^{-1}$$

Deletion of any element of $(s_0 s_n \cdots s_2)$ yields an element outside of the quotient via a straightforward set of braid moves. Fix $a \neq 2, 3$, and suppose that s_a is deleted from the second row. Let b be the smallest nonnegative integer equivalent to $a+1$ modulo $n+1$, and let c be the same for $a+2$. Then we can write $w = ((u s_b s_a u')(v s_c s_b v'))^{-1}$. When s_a is deleted, as s_b commutes with v and u' by the construction of the diagrams, we obtain

$$\begin{aligned} w' &= ((u s_b s_c u')(v s_b v'))^{-1} \\ &= (u s_b s_c s_b u' v')^{-1} \\ &= (u s_c s_b s_c u' v')^{-1} \end{aligned}$$

which, as s_c commutes with u, is not in the quotient.

Thus, within $\widetilde{A}_n/A_n$, the only right descent of w is s_3, resulting in the partition $(n, n-1)$. Then the subword criterion combined with the canonical decomposition gives that $(n-1, n-1)$ and $(n, n-2)$ both lie below $(n, n-1)$. The same argument as above shows that no other elements do. In the same way, partition $(n, n-k)$ covers only $(n-1, n-k)$ and $(n, n-k-1)$ for $1 \leq k \leq n$. Thus partition order is Bruhat order on the interval $[e, w]$, and as w is rectangular, the interval must be palindromic.

Now we improve the previous paragraph to the case $\lambda = (n^m)$ for $m \leq n+1$, with associated word w. In this case, if a covered reflection s in the canonical decomposition for w is deleted, then we can restrict our attention to the row of the deleted box and the row above it. Then the previous section, modulo a renumbering depending on the row, details how braid relations yield a new right descent s for the word associated to these two row. But then the same argument allows us to generate a new right descent of the row word associated to the third higher row, and recursively until we see that w' with the covered s deleted is not in the quotient. Thus only uncovered reflections may be deleted, so the only descents are outside corners. Thus partition order and Bruhat order coincide, and as the partition associated to w is rectangular, this forces $[e, w]$ to be a palindromic interval.

6.3.6 The Case $\lambda = (1^{nk})$

We can apply the left Eriksson game to the permutation associated to $\lambda = (1^{nk})$ to obtain the conjugate partition $\lambda = (n^k)$, using the results of Section 6.3.4. Going back to the proof in Section 6.3.5, we see that they were obtained simply by using braid relations. As the interval $[e, w]$ does not depend on any particular decomposition of w, the result carries over to this case as well. This finishes the proof of Theorem 6.4, and the case of the affine symmetric group.

CHAPTER SEVEN

Rational Smoothness in $\widetilde{D}_n/D_n$

7.1 Combinatorics in $\widetilde{D}_n/D_n$

7.1.1 The Reduced Word Interpretation of the Game

If $w \in \widetilde{D}_n/D_n$ is such that Phase 1 is used to construct the partition associated to w, then by asumption, the first sort requires $2n-2$ descents. Simple elements s_i act by left multiplication, and sorting $w(n+1)$ into the window corresponds to left multiplication by the element

$$a_{2n-2} = s_0 s_2 s_3 \cdots s_{n-2} s_n s_{n-1} s_{n-2} \cdots s_2 s_1.$$

We generalize slightly here, and define the multiset

$$A = \{0, 2, 3, \ldots, n-2, n, n-1, n-2, \ldots, 3, 2, 1\}$$

and define a_k to be the (ordered) product of the first k elements of A for $k > 0$, $a_0 := e$. Similarly, define the following ordered multisets:

$$\begin{aligned} A' &= \{0, 2, \ldots, n-2, n-1\} \\ B &= \{1, 2, \ldots, n-2, n, n-1, n-2, \ldots, 1\} \\ B' &= \{1, 2 \ldots, n-2, n-1\}. \end{aligned}$$

We take a'_k to be the (ordered) product of the first k elements of A', and similarly define b_k and b'_k, again using the convention that a subscript of zero yields the identity element. Left multiplication by s_0 brings the elements at position $n+1$ and $n+2$ into the fundamental window at positions $n-1$ and n, respectively, and we sort the element at position $n-1$ first. Hence we count s_0 with the first sort, and s_1 with the second, i.e. use the list A to sort $w(n-1)$, and the list B for sorting $w(n)$. We use the primed lists for the sort if and only if the element being sorted will lie immediately right of the zero position, and mark such a length decrease with a dot.

The inverse of the above bijection is clear: reduce the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ to pairs $(\lambda_1, \lambda_2), \ldots, (\lambda_{k-1}, \lambda_k)$, where we allow the possibility that $\lambda_k = 0$. Then multiply the identity tuple by the element $(a_{\lambda_1} b_{\lambda_2})^{-1}(a_2 b_2)^{-1} \cdots (a_{\lceil k/2 \rceil} b_{\lceil k/2 \rceil})^{-1}$ to reconstruct the permutation associated with the partition.

7.1.2 A Simplification

It is necessary to show that the $(n-1)^\bullet$ has only one possible place in any partition λ.

Proposition 7.1. $(n-1)^\bullet$ *follows all occurrences of* $n-1$ *with no dot.*

Proof. To see this, suppose otherwise. Then the following subword is contained in w:

$$s_{n-1}s_{n-2}\cdots s_1 s_{n-1}s_{n-2}\cdots s_2 s_0 = s_{n-2}s_{n-1}s_{n-3}\cdots s_1 s_2 s_0 s_1.$$

If this occurrence of $n-1$ is terminal in λ, this shows that w is not in the quotient. On the other hand, if this occurence of $n-1$ is not terminal, we have still to consider the suffix $(a_{\lambda_k} b_{\lambda_{k+1}})^{-1} \cdots (a_{\lambda_{k+m}} b_{\lambda_{k+m+1}})^{-1}$ where $m+1 < n-1$. This suffix is a reduced word that is a product of reduced words of the form

$$s_{\lambda_k - 1}\cdots s_2 s_1 s_{\lambda_k}\cdots s_2 s_0.$$

But we see that multiplication of the above by s_1 yields an element with a right descent of s_3. A similar argument for later elements yields descents s_{2m+1} for some integer m, which must at some point commute with all elements in the reduced word associated to the pair $(\lambda_k, \lambda_{k+1})$, forcing a right descent other than s_0, yielding a contradiction to the fact that $w \in \widetilde{D}_n/D_n$. □

7.1.3 Order, Partitions, and Reduced Decompositions

The set of all partitions of the above form can be ordered in the following way: order the numbers $\{1, 2, \ldots, n-2, n-1, (n-1)^\bullet, n, \ldots, 2n-2\}$ with $a < b$ provided $a < b$ in the integer order, with the assumption that n covers $n-1$ and $(n-1)^\bullet$, $n-1$ covers both $n-2$ and $(n-1)^\bullet$, and $(n-1)^\bullet$ covers $n-2$. Then we say that

$$\lambda(u) = (\lambda_1, \ldots, \lambda_j) < (\mu_1, \ldots, \mu_k) = \lambda(w)$$

provided:

(1) $j \leq k$,

(2) $\sum_{i=1}^{j} \lambda_i < \sum_{i=1}^{k} \mu_i$ (as integers), and

(3) for $1 \leq i \leq j$, $\lambda_i \leq \mu_i$.

Note that these rules mirror the selection of the order in the sets A, A', B, and B' from before, and the sorts used in the game above. Consider the diagram

0	2	3	$\cdots$	$n-2$	a	b	$n-2$	$\ldots$	2	0
1	2	3	$\cdots$	$n-2$	a	b	$n-2$	$\ldots$	2	1
0	2	3	$\cdots$	$n-2$	a	b	$n-2$	$\ldots$	2	0
1	2	3	$\cdots$	$n-2$	a	b	$n-2$	$\ldots$	2	1

Here we write

$$\{a, b\} = \{n-1, n\}, \text{ with } a = \begin{cases} n-1 & \text{if } n-1 \text{ is dotted} \\ n & \text{otherwise} \end{cases}$$

as the commutation rules mean that we are free to allow either $a = 0, b = 1$ or $a = 1, b = 0$ with no difference. The block diagram makes clear how the algorithm associates a partition λ to an element w. Cover the above table with the partition λ by matching upper left corner to upper left corner: then the reduced decomposition of w is read from the masked portion of the above diagram right to left and bottom to top.

Example 7.1. *The partition* $(8, 4^\bullet, 2, 1)$ *corresponding to* $w = (-2, 5, 7, 8, 21)$ *in* $\widetilde{D}_5/D_5$ *placed on the above diagram masks the boxed elements as seen below:*

0	2	3	a	b	3	2	0
1	2	3	a	b	3	2	1
0	2	3	a	b	3	2	0
1	2	3	a	b	3	2	1

with length limited to four rows. Cover this with with the partition $\lambda = (8, 4^{\bullet}, 2, 1)$ to get

0	2	3	4	5	3	2	0
1	2	3	4				
0	2						
1							

giving a reduced decomposition

$$w = (s_1)(s_2s_0)(s_4s_3s_2s_1)(s_0s_2s_3s_5s_4s_3s_2s_0)$$

associated to λ.

7.2 Small Elements

7.2.1 Definition

Definition 7.1. *We define a small element of $\widetilde{D}_n/D_n$ to be an element whose reduced decompositions form a connected proper subgraph of the Dynkin diagram for $\widetilde{D}_n$.*

Remark 7.1. *This condition is easily seen to hold for all small elements of $\widetilde{A}_n/A_n$, and in fact, the converse is true.*

Whereas the small elements were easy to describe combinatorially in Type A, they are more subtle in Type D, and so we introduce functions similar to those we use in the earlier sections.

Our immediate goal is to prove that, as in $\widetilde{A}_n/A_n$, there is a distinguished class of permutations whose Bruhat order intervals are isomorphic to the partition intervals described above.

7.2.2 $M_{i,j}$ Functions.

Recall that the characterization of $\widetilde{D}_n/D_n$ includes the requirement that for w a minimal coset representative, we have that

$$w(-2) < w(1) < w(2) < \cdots < w(n).$$

As $w(-2) = -w(2) < w(2)$, we must have $w(2) > 0$, and as $w(-2) = -w(2) < w(1)$, we know that $w(1) + w(2) = w(1) - w(-2)$, so that if $w(1) < 0$, then $w(1) + w(2) > 0$

so $\mathrm{nsp}(w) = 0$. Furthermore, this characterization ensures that there are no inversions in w, so the double sums are the only contributing entries. Thus, it is clear that we can reduce the two double sums into a single function

$$M_{i,j}(w) = \begin{cases} \left\lfloor \frac{v(j)-v(i)}{2n+1} \right\rfloor & i < j \\ \left\lfloor \frac{v(j)+v(i)}{2n+1} \right\rfloor & i > j \\ 0 & \text{otherwise.} \end{cases}$$

7.2.3 Divided Diagrams

We will arrange the data from the $M_{i,j}$ functions as a diagram as follows:

$$\begin{array}{cccc|cccc}
M_{n,1}(w) & \cdots & M_{n,n-2}(w) & M_{n,n-1}(w) & M_{n-1,n}(w) & M_{n-2,n}(W) & \cdots & M_{1,n}(w) \\
 & \ddots & & \vdots & \vdots & & & \\
 & & M_{3,1}(w) & M_{3,2}(w) & M_{2,3}(w) & M_{1,3}(w) & & \\
 & & & M_{2,1}(w) & M_{1,2}(w) & & &
\end{array}$$

For $M_{i,j}(w) \neq 0$ with $i > j$ (left-hand side of the bar), we require that either $M_{i+1,j}(w) \neq 0$, or $M_{i,j-1}(w) \neq 0$. Similarly, for $i < j$ (right hand side of the bar), we require that either $M_{i+1,j}(w) \neq 0$ or $M_{i,j-1}(w) \neq 0$. The immediate question arises whether this realization in a bijective map to the set of all such divided diagrams, but this is implied in [39], treating each element w as v^{-1}, and such diagrams comprise a faithful representation of $\widetilde{D}_n/D_n$, not just the small elements.

Denote the divided diagram of the element w by $D(w)$. We place a partial order structure on the set of divided diagrams satisfying the above rules by saying that $D(u) < D(v)$ provided for each $1 \leq i, j \leq n$, $M_{i,j}(u) \leq M_{i,j}(v)$. Viewing each divided diagram as an element in $\mathbb{N}^{n(n-1)}$, this is just the product order.

7.3 Palindromicity in Type D

7.3.1 Combinatorial Structure of Certain Permutations

Observations tell us that for $\widetilde{D}_n/D_n$, the palindromic elements have associated partitions

$$(\emptyset), (1), (2), \ldots, (n-1), ((n-1)^\bullet), (2,1), (3,2,1) \ldots,$$
$$(n-1, n-2, \ldots, 1), ((n-1)^\bullet, n-2, \ldots, 1), (2n-2).$$

Note that in the usual order on Young's lattice, there are three maximal elements in the above collection: $(2n-2)$, $((n-1)^\bullet, n-2, \ldots, 1)$, and $(n-1, n-2, \ldots, 1)$. We call such elements triangles. It is nteresting to note certain details about the combinatorial structure of triangular elements, as well as details in their divided diagrams. The following lemma is immediate from the definitions:

Lemma 7.1. *For $w \in \widetilde{D}_n/D_n$, with associated partition*

$$\pi(w) \in \{(2n-2), ((n-1)^\bullet, n-2, \ldots, 1), (n-1, n-2, \ldots, 1)\},$$

$M_{i,j}(w) \in \{0,1\}$ *for* $1 \leq i, j \leq n$.

Proof. For $\pi(w) = (2n-2)$, we have from the algorithm that

$$w = [-1, 2, 3, \ldots, n-1, 3n+2]$$

and it is immediate that the maximal $M_{i,j}(w) = \left\lfloor \frac{4n+1}{2n+1} \right\rfloor = 1$. Similarly, if $\pi(w) = ((n-1)^\bullet, n-2, \ldots, 1)$, we have that

$$(n \text{ even}) \;:\; w = [n+1, n+2, \ldots, 2n-1, 2n]$$
$$(n \text{ odd}) \;:\; w = [n, n+2, \ldots, 2n-1, 2n]$$

with maximal $M_{i,j}(w) = \left\lfloor \frac{4n-1}{2n+1} \right\rfloor = 1$. Again, if $\pi(w) = (n-1, n-2, \ldots, 1)$, we see

$$(n \text{ even}) \;:\; w = [-n-1, n+2, \ldots, 2n-1, 2n+2]$$
$$(n \text{ odd}) \;:\; w = [-n, n+2, \ldots, 2n-1, 2n+2]$$

which has maximal $M_{i,j}(w) = \left\lfloor \frac{4n+1}{2n+1} \right\rfloor = 1$. ∎

7.3.2 Small and Binary Elements in Type D.

If an element w is such that $\pi(w) < \lambda$ for any of the three shapes listed in the lemma, we say that w is *small.* We say that $w \in \widetilde{D}_n/D_n$ is *binary* provided $D(w)$ is composed solely of zeroes and ones. Not all binary elements are small, for example, in $\widetilde{D}_n/D_n$, the element $w = (-1, 4, 6, 11) = (4, 1)$. The goal is to prove that for small elements, we have the poset isomorphisms

$$\begin{aligned}\{\text{small elements of } \widetilde{D}_n/D_n\}_{D-\text{order}} &\overset{\simeq}{\leftrightarrow} \{\text{small elements of } \widetilde{D}_n/D_n\}_{P-\text{order}} \\ &\overset{\simeq}{\leftrightarrow} \{\text{small elements of } \widetilde{D}_n/D_n\}_{\text{Bruhat order}}.\end{aligned}$$

Theorem 7.1. *Let w be binary, and suppose $s \in D_L(w)$ is simple. Then $D(sw)$ differs from $D(w)$ in exactly one position.*

Proof. We know that at least one entry is altered, as $\sum_{i,j} M_{i,j}(w) = \ell(w)$, and left multiplication of w by a simple element s results in an element of length $\ell(w) \pm 1$. So let w be binary and consider first the case $s := s_i$, for $1 \leq i \leq n-1$, so $s = (i, i+1)$ (written as a transposition). Then sw (in complete notation) is identical to w except in positions a and b, where $w(a) \equiv \pm i \pmod{2n+1}$ and $w(b) \equiv \pm(i+1) \pmod{2n+1}$. Then $sw(a) \equiv \pm(i+1) \pmod{2n+1}$ and $sw(b) \equiv \pm i \pmod{2n+1}$, where the signs remain as they were in the case of w. Consider $a < c \leq n$, $c \neq b$: then

$$M_{c,a}(sw) = \left\lfloor \frac{sw(a) + sw(c)}{2n+1} \right\rfloor = \left\lfloor \frac{w(a) + 1 + w(c)}{2n+1} \right\rfloor.$$

The difference in the numerators is only one. By the definition of the representation, $w(a)+w(c) \in [1+k(2n+1), 2n-1+k(2n+1)]$ for some integer k, and $sw(a)+sw(c) \in [1+k'(2n+1), 2n-1+k'(2n+1)]$ for some integer k'. If $k \neq k'$, this says that either $w(a)+w(c) = k'(2n+1)$, or $sw(a)+w(c) = (k+1)(2n+1)$, both of which are absurd. A similar contradiction occurs if $1 \leq c < a$, $c \neq b$. Thus $M_{c,a}(sw) = M_{c,a}(w)$ and $M_{a,c}(w) = M_{a,c}(sw)$. The same argument shows $M_{c,b}(sw) = M_{c,b}(w)$ and $M_{b,c}(w) = M_{b,c}(sw)$. Thus we must have that $M_{a,b}(sw) < M_{a,b}(w)$, which, as $M_{i,j}(w) \geq 0$ for all pairs (i, j), implies $0 = M_{a,b}(sw) < M_{a,b}(w) = 1$, and sw is binary.

For $i = n$, $s_i = (1, 2, \ldots, n-2, n+1, n+2)$. Our assumptions are still that $sw < w$, and that w is binary. Take a and b as the positions of difference between w and sw. If $M_{a,b}(sw) > M_{a,b}(w)$, then there exist pairs (a_i, b_i), $i \in \{1, 2\}$, such that $M_{a_i,b_i}(sw) < M_{a_i,b_i}(w)$, since the length difference is one. Suppose $sw(a) = w(a) + 3$, and let $1 \leq c < a$. Then

$$M_{c,a}(w) = \left\lfloor \frac{sw(a) - sw(c)}{2n+1} \right\rfloor = \left\lfloor \frac{w(a) + 3 - w(c)}{2n+1} \right\rfloor$$

which is different from $M_{c,a}(w)$ if and only if $w(c) \equiv \pm 1, \pm 2 \pm 3 \pmod{2n+1}$. The first four cases would imply $c = a$ or b, and the final two cases would imply that $w(a) + w(c)$ is a multiple of $2n+1$, all of which are absurd. The same arguments show $M_{c,d}(w) = M_{c,d}(sw)$ and $M_{d,c}(sw) = M_{d,c}(w)$ for $d \in \{a, b\}$. Thus the only change can occur with $M_{a,b}(sw)$, which must be a decrease by one due to our assumptions of w binary. The same proof establishes the result for $s = s_0$. □

Corollary 7.1. *If w is binary, and s is a simple left descent for w, then sw is binary.*

Proof. Only one position is changed, and as

$$\sum_{i,j} M_{i,j}(sw) = \ell(sw) < \ell(w) = \sum_{i,j} M_{i,j}(w),$$

this change must be a decrease. As each entry is nonnegative, the decrease must be a change from 1 to 0. □

Corollary 7.2. *All small elements are binary.*

Proof. The "top" small elements are binary via Lemma 7.3.1. Every other small element is below one of these top elements by definition, and by the above, must also be binary. □

We remark some notation for binary divided diagrams to make the next section easier. If $D(w)$ is binary, write $D(w) = (\mu_1, \mu_2, \ldots, \mu_k | \nu_1, \ldots, \nu_k)$, where μ_i is the number of ones in the i^{th} row of the left side of the divided diagram (counted from the top), and ν_i is the number of ones in the $n - i^{\text{th}}$ row of right side of $D(w)$ (again, from the top). Note that for binary elements, due to the requirements that columns be weakly decreasing (read top to bottom) and rows be weakly decreasing (read from the bar to the outside), this assignment is unambiguous.

7.3.3 From Partitions to Divided Diagrams

Our goal is to find an injection F from small partitions to binary divided diagrams. The map F is as follows: let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of a small element $w \in \widetilde{D}_n/D_n$. If λ_1 is marked with a dot, or if $\lambda_1 < n-1$, define $F(\lambda)$ to be the binary divided diagram $(\lambda_1, \lambda_2, \ldots, \lambda_k|0, \ldots, 0)$. If $\lambda_1 = n-1$, define $\nu_i = 1$ if $\lambda_i = n-i$ and 0 otherwise. Then we have

$$F(\lambda) = (\mu_1, \mu_2, \ldots, \mu_k|\nu_1, \ldots, \nu_k).$$

The final case is $n \leq \lambda_1 \leq 2n-2$, in which we take $\mu_1 = n-1$ and $\nu_1 = \lambda_1 - \mu_1$, yielding the diagram $F(\lambda) = (n-1|\lambda_1 - n + 1)$.

Recall the previous definitions of the elements a_i, b_i, a_i'; for future reference, we note the following ($n-i$ deleted in each tuple):

Table 7.1. a_i Parameters For Binary Divided Diagrams

i	a_i^{-1} $(a_i'^{-1}$ for $(n-1)^\bullet)$	$D(a_i^{-1})$
$i < n-1:$	$[1, 2, \ldots, n-1, n+1, n+1+i]$	$(\lambda_1|0) = F((i))$
$i = n-1:$	$[-2, 3, \ldots, n-1, n+1, 2n+2]$	$(n-2|1) = F((i))$
$i = (n-1)^\bullet:$	$[2, 3, \ldots, n-1, n+1, 2n]$	$(n-1|0) = F((i))$
$i = n:$	$[-1, 3, \ldots, n-1, n+1, 2n+3]$	$(n-1|1) = F((i))$
$2n-1 > i > n$	$[-1, 2, \ldots, n-1, n+1, n+3+i]$	$(n-1|i-n+1) = F((i))$

This already shows that F maps partitions to the appropriate binary divided diagrams in the case $\lambda = (m|0)$, $1 \leq m \leq 2n-2$. A similar computation for other small partitions will need a few additional technical details: the elements b_i^{-1}, we have

(1) $b_i^{-1} = [1, 2, \ldots, \widehat{n-i}, \ldots, n, n-i]$, $i < n-1$:

(2) $b_i^{-1} = [-2, 3, 4, \ldots, n, -1]$, $i = n-1$:

Note that we can have no occurence of b'_{n-1} in a small element, as the second entry in a small partition is always smaller than $n-1$. Define the permutation

$$t_i = \begin{cases} a_{\lambda_i}^{-1} & i \equiv 1 \pmod 2, i \neq n-1 \\ {a'_{\lambda_i}}^{-1} & i \equiv 1 \pmod 2, i = (n-1)^{\bullet} \\ b_{\lambda_i}^{-1} & i \equiv 0 \pmod 2 \end{cases} \cdot$$

In view of the original algorithm based on sorts by simple elements, we then have that the permutation associated to λ is

$$w = t_k t_{k-1} \cdots t_1.$$

Now consider the cases $\lambda = (\lambda_1, \ldots, \lambda_k)$, k even, and $\lambda_1 \leq n-1$. Permutation multiplication gives us that

$$w = [1, 2, \ldots, \widehat{n-\lambda_1}, \ldots, \widehat{n-\lambda_2} \ldots, \widehat{n-\lambda_k} \ldots, n, n+\lambda_k+1, n_1+\lambda_{k-1} \cdots n+1+\lambda_1].$$

Computing the left side of the divided diagram of this permutation gives a first entry of k plus the number of unskipped entries between $n-\lambda_1$ and $n-1$ (inclusive), and in general, an i^{th} entry of $k+1-i$ plus the number of unskipped entries between $n-\lambda_i$ and $n-1$ (inclusive). This follows from the fact that the t_i deletes the entry w_i at position $n-\lambda_i$ and replaces it with $2n+1-w_i$ in the last or next to last position (depending on parity of i). An entry right of the deleted position is larger than the deleted element w_i, so adding it to $2n+1+w_i$ yields an integer between $2n+1$ and $2(2n+1)$. Upon dividing by $2n+1$ and taking the floor, this yields a one in the divided diagram. Conversely, for an undisturbed entry left of w_i, the magnitude is small, so the sum has magnitude smaller than $2n+1$, yielding a zero in the divided diagram. For the right side, no entries are entered, as every permutation entry is positive, with a maximum gap between them of $n+\lambda_1 < 2n+1$.

Analogously, we see that for k odd and $\lambda_1 < n-1$,

$$w = [1, \ldots, \widehat{n-\lambda_1}, \ldots \widehat{n-\lambda_2} \ldots, \widehat{n-\lambda_k} \ldots, \widehat{n}, n+1, n+1+\lambda_k, \ldots, n+1+\lambda_1]$$

and that the i^{th} row of the left side of the divided diagram has $k+g_i$ ones, where g_i is the number of unmoved elements in the interval $[n-\lambda_i, n]$, with no entries on the right side. The final note to make for these two cases is that $\lambda_i = k + h_i$, where h_i is the number of skipped entries right of the i^{th} position in λ as compared to $(n-1, n-2, \ldots, 1)$, showing that $D(w) = F(\lambda)$ as desired.

The next case to consider is k even, $\lambda_1 = (n-1)^{\bullet}$: for this, we again construct w associated to λ and deduce the form of $D(w)$. We have

$$w = [1, \ldots, \widehat{n - \lambda_1}, \ldots, \widehat{n - \lambda_k}, \ldots, n, n+1+\lambda_1, n+1+\lambda_2, \ldots, n+1+\lambda_k].$$

Again writing $h_i(\lambda) := h_i$ for the number of entries in w from the interval $(n - \lambda_i, n)$, we have $\mu_i = k + 1 - i + h_i$, and $\nu_i = 0$, yielding $F(\lambda)$ as before. Similarly, for k odd, $\lambda_1 = n - 1$, we have

$$w = [\widehat{n - \lambda_1}, 2, \ldots, \widehat{n - \lambda_k}, \ldots, \hat{n}, n+1, n+1+\lambda_1, n+1+\lambda_2, \ldots, n+1+\lambda_k].$$

Calculating the divided diagram for such a permutation, we see that $\mu_i = k+1-i+h_i = \lambda_i$, so $F(\lambda) = D(\pi^{-1}(\lambda))$.

For the same calculation in the case $\lambda_1 = n-1$, k even, we get a slightly different form:

$$\pi^{-1}(\lambda) = [\text{first entry negative}, \widehat{n - \lambda_1}, \ldots, \widehat{n - \lambda_k}, n, n+1+\lambda_k, n+1+\lambda_{k-1}, \ldots, n+3+\lambda_1].$$

The first entry negative comment means that the first row in the divided diagram has one fewer entry on the right, as the first entry a must be smaller than or equal to -2, and so

$$n + 3 + \lambda_1 + a = n + 3 + n - 1 + a = 2n + 2 + a < 2n + 1,$$

but on the other hand

$$n + 3 + \lambda_1 - a = 2n + 2 - a > 2n + 1$$

implying that $\nu_1 = 1$. Similarly, for each i such that the $|a| > \lambda_i$, we see $\nu_i = 1$, and $\mu_i = k - i + h_i$. A similar argument holds for k odd. Thus we have shown the following:

Theorem 7.2. *The map F takes partitions to divided diagrams so that*

$$F(\lambda) = D(\pi^{-1}(\lambda)).$$

An inverse to F immediately suggests itself: given a divided diagram

$$D(\lambda) = (\mu_1, \dots, \mu_k | \nu_1, \dots, \nu_k),$$

for $i > 1$, define $\lambda_i = \mu_i + \nu_i$, and if $\mu_1 = n-1$, $\nu_1 = 0$, then $\lambda_1 = (n-1)^\bullet$. Then take $\lambda = (\lambda_1, \dots, \lambda_k)$. Denote this map by G.

Theorem 7.3. *$F(G(D(w))) = D(w)$ and $G(F(\lambda)) = \lambda$ for small elements w and small partitions λ.*

Proof. Let w be small, thus binary, by Corollary 7.2. Write $D(w) = (\mu_1, \dots, \mu_k | \nu_1 \dots, \nu_k)$, so $G(D(w)) = (\mu_1 + \nu_1, \dots, \mu_k + \nu_k)$. We have three cases:

(1) If $\mu_2 + \nu_2 = 0$, so $D(w) = (\mu_1 | \nu_1)$ with $\mu_1 < n-1$ and $\nu_1 = 0$, or $\mu_1 = n-1$ and $\nu_1 = \lambda_1 - \mu_1$. Then $G(D(w)) = (\lambda_1)$, and $F(G(D(w))) = (\mu_1 | 0)$ is $\mu_1 < n-1$ or $F(G(D(w))) = \mu_1 | \nu_1)$ if $\mu_1 = n-1$ by the definition of F. Thus $F(G(D(w))) = D(w)$. Similarly, if $\lambda = (\lambda_1)$, then $F(\lambda) = (\mu_1 | \nu_1)$ with $\nu_1 = 0$ unless $\lambda_1 > n-1$, in which case $\nu_1 = \lambda_1 - n + 1$, and $\mu_1 = n-1$. In either case, applying G to the resulting tuple shows that $G(F(\lambda)) = \lambda$.

(2) If $\mu_2 > 0$ and $\nu_1 = 0$, then $G(D(w)) = (\mu_1^\bullet, \mu_2, \dots, \mu_k)$, and $F(\mu_1^\bullet, \mu_2, \dots, \mu_k) = (\mu_1, \dots, \mu_k | 0)$ so F is a left inverse for G. Similarly, if $\lambda = ((n-1)^\bullet, \lambda_2, \dots, \lambda_k)$, then $F(\lambda) = (n-1, \lambda_2, \dots, \lambda_k | 0)$ by the definition of F, and G applies to this tuple again yields λ by construction.

(3) If $\mu_2 > 0$ and $\nu_1 > 0$, then $G(D(w)) = (\mu_1 + \nu_1, \mu_2 + \nu_2, \dots, \mu_k + \nu_k) = \lambda'$. As $\nu_1 > 0$, $\mu_1 \geq n-2$. If $\mu_1 \geq n-1$, then smallness implies that $\mu_1 = 0$, contradicting our case assumption. Thus $\mu_1 = n-2$. Similarly, if $\nu_i = 1$, then $\mu_i = n - i$, and as $\lambda_i = \mu_i + \nu_i$, this forces the sequence of μ_i to be strongly descreasing. Then $G(D(w))$ is a partition with parts in $\{1, \dots, n-1\}$ and no

repetition. Hence F is defined on such a partition, and by the definition of F, $F(G(D(w))) = (\mu_1, \ldots, \mu_k|\nu_1, \ldots, \nu_k)$. For the other composition, letting $\lambda = (n-1, \ldots, \lambda_k)$, use the rule for F to define $\nu_i = 1$ iff $\lambda_i = n-i$, forcing $\mu_i = n-i-1$. Thus $F(\lambda) = (n-2, \lambda_2 - \mu_2 \ldots, \lambda_k - \mu_k|1, \mu_2, \ldots, \mu_k)$, and $GF(\lambda) = \lambda$ as desired.

Thus in each case, $GF(\lambda) = \lambda$ and $FG(D(w)) = D(w)$. □

7.3.4 Palindromicity in $\widetilde{D}_n/D_n$

Lemma 7.2. *For small elements, the order on divided diagrams is isomorphic to the order on partitions.*

Proof. Suppose that $D(w) < D(w')$; for each i, we have $\mu_i \leq \mu_i'$ and $\nu_i \leq \nu_i'$. But $\lambda_i = \mu_i + \nu_i$, so this implies that $G(D(w)) < G(D(w'))$. If we specify that $D(w) < D(w')$ is a covering relation, then there is exactly one position (i,j) such that $M_{i,j}(w) < M_{i,j}(w')$, implying that there is exactly one value k such that $\lambda_k < \lambda_k'$, so $\pi(w) < \pi(w')$ is likewise a covering relation.

Conversely, if $\pi(w) = \lambda < \pi(w') = \lambda'$ is a covering relation of small elements, there can be at most one point of difference between λ and λ', say in the i^{th} row. If $\lambda_1' < n-1$, or if $\lambda_1' = (n-1)^\bullet$, then

$$D(w) = (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_k|0, \ldots, 0) < (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_k|0, \ldots, 0) = D(w')$$

by the definition of F. If $\lambda' = n-1$, then either $\lambda_i = n-i-1$ or $\lambda_i < n-i-1$. If $\lambda_i = n-i-1$, so $\lambda_i' = n-i$, then

$$\begin{aligned} D(w) &= (\lambda_1 - 1, \ldots, \lambda_{i-1} = n-i, n-i-1, \ldots, \lambda_k|1, \ldots, \nu_i - 1 = 1, \nu_i = 0, \ldots, 0) \\ &\quad < (\lambda_1 - 1, \ldots, n-i, n-i-1, \ldots, \lambda_k|1, \ldots, \nu_i = 1, 0, \ldots, 0) = D(w'). \end{aligned}$$

If $\lambda_i < n-i-1$, then $\mu_i < n-i-1, \nu_i = 0$, $\mu_i' = \mu_i + 1, \nu_i' = 0$. Finally, if $\lambda_1 > n-1$, then $\lambda_2 = 0$, and we have $D(w) = (n-1|\lambda_1 - n + 1)$, $D(w') = (n-1|\lambda_1' - n + 1)$, so $D(w) < D(w')$ again. □

We can use the divided diagrams to characterize those binary elements w such that the interval $[e, w]$ is palindromic. Recall that we say a divided diagram a triangle provided it is of the form $(k-1, k-2, \ldots, 1|0, \ldots, 0)$ or $(k-2, k-3, \ldots, 1, 0|1, 1, \ldots, 1)$ for $1 \leq k \leq n-1$, and we call a divided diagram a row provided it is of the form $(k|0)$ or $(n-1|k)$ for $0 \leq k \leq n-1$. Similarly, we call a partition λ a triangle (row) if $\lambda = F(D(w))$, where $D(w)$ is a triangle (row).

Proposition 7.2. *If w, w' are small elements, $w < w'$ implies $\lambda(w) < \lambda(w')$.*

Proof. $w < w'$ implies w is a subword of any spelling of w'. Thus we can write w' by inserting a simple element into a reduced decomposition for w. We have characterized reduced decompositions of binary elements using partitions, so we have a canonical way of writing $w = t_k t_{k-1} \cdots t_1$. If $\pi(w)$ is a row, then the inserted simple either cancels an element of the word $w = t_1$ (contradicting that $w < w'$), or the inserted simple can be moved to the first position of t_1 either yielding a word t_1' or $t_2 t_1$, where t_2 is composed solely of the inserted element. In either case $\pi(w) < \pi(w')$. Assume that $\pi(w)$ is not a row. An appended simple element can be moved to the front of one of the subwords t_i, forcing $\pi(w) < \pi(w')$. □

Corollary 7.3. *The partition and Bruhat orders on small elements are isomorphic.*

Proof. Use Proposition 7.2, and Corollary 5.1 □

Theorem 7.4. *A small element w has $[e, w]$ palindromic if and only if $\pi(w)$ is a triangle or a row.*

Proof. Let w be small (thus binary). Then $\pi(w)$ is a triangle (row) if and only if $D(w)$ is a triangle (row). Row elements clearly cap palindromic intervals via the theorem above. Suppose $D(w)$ is a triangle $(k, k-1, \ldots, 1|0)$, $k \leq n-1$, $k \neq (n-1)^\bullet$. A subdiagram of $D(w)$ is any divided diagram δ smaller than $D(w)$ in the diagram order. Define the complement of δ with respect to $D(w)$ to be the array

$$\begin{aligned} D^C(w) \quad = \quad & \big((0, \ldots, 0, M_{k,1}, M_{k,2}, \cdots, M_{k,k-1}), \\ & (0, \ldots, 0, M_{k,2}, \ldots, M_{3,2}), \ldots, (0, \ldots, 0, M_{k,k-1})\big). \end{aligned}$$

Given a small element w, for each w' covered by w, there is a distinct partition for w', and thus a distinct complement partition $D^C(w')$. The descending order in divided diagrams is the same as the ascending order in complementary divided diagrams, and furthermore, the complementary diagrams at level j in $[\emptyset, D^C(w)]$ are precisely the divided diagrams of length $\frac{1}{2}k(k+1)+1-j$ in $[\emptyset, D(w)]$. Thus the poset $[e, D(w)]$ is palindromic. The result is proved for divided diagrams of the form $(k-1, k-2, \ldots, 1, 0|1, 1, \ldots, 1)$ in an analagous way. $\square$

Theorem 7.5. *If w is a large element, then $[e, w]$ is not palindromic in the Bruhat order.*

Proof. Take $w \in \widetilde{D}_n/D_n$ large, with partition $\pi(w) = \lambda \neq (m)^k$. As w is large, there are at least two entries in λ, and one of them exceeds $n-1$. The longest row can be shortened (regardless of any other row), and the shortest row can be shortened as well. Thus there are at least two partitions smaller than $\pi(w)$, and as partition coverings induce Bruhat coverings, the interval can't be palindromic. In the case that $\lambda = (m)^k$ for $m \neq n$ and $k \geq 3$, we have

$$(m)^k > (m^{k-1}, m-1) > (m^{k-2}, m-1) \text{ and } (m^{k-1}, m-2)$$

contradicting the possibility of palindromicity. For the case $k = 2$, we have the similar

$$(m, m) > (m, m-1) > (m-1, m-1) \text{ and } (m, m-2)$$

again contradicting palindromicity.

The final case is for the partition (m^k), $m = n$. The reduced decomposition for $\pi((m^k))$ has for its prefix the subword

$$s_{n-1}s_n s_{n-2} \cdots s_2 s_j$$

where $j = 0$ if k is even and $j = 1$ if k is odd. This element has left descents s_{n-1} and s_n (due to the fact that these elements commute), again implying that (m^k) has too many descents for $\pi(m^k))$ to be palindromic in the Bruhat order. $\square$

CHAPTER EIGHT

Rational Smoothness in $\widetilde{E}_n/E_n$

8.1 $\widetilde{E}_7/E_7$

8.1.1 Reinterpretation of the Eriksson game.

Recall the sorting game of Section 5.2.3. Note that the first sort corresponds to right multiplication by $s_0 s_1 \cdots s_{d_1-1}$, and adding γ corresponds to multiplication by s_7. Thus d_1 correspond to multiplying by the product

$$t_{d_1} = s_{d_1-1} \cdots s_1 s_0$$

It is clear that each s_i involved acts to shorten w by sorting the sequence. I claim further that right multiplication by s_7 at this point is length decreasing. To see this, note that right multiplication by any other simple element must be length increasing (because the sequence is sorted), so if $w \neq e$, then it must have some simple decent, and this must be s_7. Furthermore, the sorts in step (4) of $w(m)$ leftwards correspond to multiplication by $s_\alpha s_{\alpha+1} \cdots s_{\alpha+d_m-1}$, where $\alpha = m-2$, and as they act to further sort the sequence, they must be length decreasing. This has the added benefit of explaining why the algorithm must terminate

Note that the entry following a $^+$ may exceed the length of the $^+$ entry, but may not exceed 4. Thus the process yields a sequence $d = (d_1, \ldots, d_k)$ where $d_1 \in \{1, \ldots, 7\}$ (possibly marked with $^+$), and for $i > 2$, $d_i \leq 4$, and the d_i aggregate to chains of length 4 or less ended with a $^{+1}$ element. The sequence λ is referred to as the partition of the permutation.

To see that the game assigns a unique partition to each permutation, note again that each box in the partition corresponds to a unique simple element (via the sorting algorithm, or the definition of s_7), and the partition then corresponds to a reduced decomposition of the element (establishing the result described in **??**) for $\tilde{E}_7/E_7$). Furthermore, define a total ordering on the set of simple elements by $s_i < s_j$

provided $i < j$. As reduced decompositions are unique, so must also be the partition. Thus the minimal reduced decomposition for an element w may be obtained by filling $\pi(w)$ in the following way:

0	1	2	3	4	5	6					
			7	3	2	1	0				
				4	3	2	1				
				5	4	3	2				
				6	5	4	3				
							7	3	2	1	0
								4	3	2	1
								5	4	3	2
								6	5	4	3

and repeating the final four lines of the shape above as needed.

8.1.2 Palindromicity in $\widetilde{E}_7/E_7$

It is immediate that the poset for the quotient $\widetilde{E}_7/E_7$ has, for its first four ranks, the following structure:

$$e \text{---} s_0 \text{---} s_1 s_0 \text{---} s_2 s_1 s_0 \text{---} s_3 s_2 s_1 s_0 \text{---} \cdots .$$

It is easy to see then that the element of lengths smaller than 6 are clearly palindromic.

Theorem 8.1. *The palindromic elements of $\widetilde{E}_7/E_7$ are*

$$e, s_0, s_1 s_0, s_2 s_1 s_0, s_3 s_2 s_1 s_0, s_4 s_3 s_2 s_1 s_0, s_7 s_3 s_2 s_1 s_0,$$

$$s_5 s_4 s_3 s_2 s_1 s_0, s_6 s_5 s_4 s_3 s_2 s_1 s_0, (s_0 s_1 s_2 s_3)(s_7 s_4 s_3 s_2 s_1 s_0),$$

$$(s_0 s_1 s_2 s_3)(s_7 s_4 s_5 s_3 s_4 s_2 s_3)(s_7 s_1 s_2 s_3 s_4 s_0 s_1 s_2 s_3)(s_7 s_5 s_4 s_3 s_2 s_1 s_0).$$

We can certainly see by hand that these are only palindromic elemenets of length smaller than 28. The proof that the list is indeed complete will occupy the next several sections as an elementary case study using the canonical lexicographically minimal reduced decompositions derived above. For the duration of this section $w \in \widetilde{E}_7/E_7$ is associated to a generalized partition $\lambda = (\lambda^1, \ldots, \lambda^k)$ with $\lambda^i < (4, 4, 4, 4)$ for $i > 1$, and with the last entry marked with $^+$ for $i < k$. We begin by reducing the scope of our investigation. This section makes heavy use of the Descent Lemma 2.5.3 and Braid Lemma 2.5.4

Proposition 8.1. *If $\pi(w) = \lambda$, and $k > 1$, then a necessary condition for $[e, w]$ to be palindromic is for $\lambda^k = (4)$ or $\lambda^k = (1, 1, 1, 1)$.*

Proof. Use the subword property and the Descent Lemma above. □

Lemma 8.1. *Given w and $\lambda = \pi(w)$, if λ^i is not rectangular, then there are at least two left descents of the word associated to λ^i.*

Proof. Clear from the definitions of the λ^i blocks and the Braid Lemma. □

8.1.3 Allowable Partitions

To prove that the list of palindromic elements is indeed complete, we start by considering the allowable partitions whose first entry is 6^+ or smaller. For brevity, we drop the s in reduced decompositions, so $3 := s_3$ etc.

Lemma 8.2. *The only partition with first entry 4^+ is (4^+).*

Proof. Note that by the Descent Lemma and its Corollary, above, that if $\lambda = (4^+, l_2, \ldots, l_m)$, then $(4^+, 1)$ must be associated to an element in the quotient. But $(4^+, 1)$ is associated to the word

$$37(3210) = (7)(37)(210) = (7)(3210)(7) \notin \widetilde{E}_7/E_7.$$

□

Lemma 8.3. *The only partitions with first entry 5^+ are $(5^+), (5^+, 1), (5^+, 2), (5^+, 3)$ and $(5^+, 4)$*

Proof. Let $1 \leq \mu \leq 4$, and let $u_\mu = 2 \cdots (4 - \mu)$ be such that $(3u_\mu)$ is the row-word associated to the μ part of the partition $\lambda = (5^+, \mu, 1)$. Then

$$\pi^{-1}(\lambda) = (4)(u_\mu^{-1}3)(743210) = u_\mu^{-1}(43)(47)(3210) = u_\mu^{-1}(34)(73)(210)(7) \notin \widetilde{E}_7/E_7.$$

Then by the Descent Lemma, any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 = 5^+$ and with $k \geq 3$ yields a reduced decomposition not lying in the quotient, and by the uniqueness of minimal reduced decompositions, this is a contradiction. Thus $k \leq 2$. □

Theorem 8.2. *If $w \in \widetilde{E}_7/E_7$, and the first entry in $\lambda(w)$ is 6^+, then $\lambda(w)$ lies inside a partition*

$$\lambda^* = (6^+, 4, 4^+, 2, 2, 2, 2^+, 4, 4^+, \ldots).$$

Proof. We use the node game to show that every partition of the form λ^* is indeed associated to a word in $\widetilde{E}_7/E_7$, and then show that the existence of an entry outside of the given framework forces the associated element to be either not lexicographically minimal or to lie outside the quotient. So let w be such that $\lambda(w)_1 = 6^+$, and consider the node game on the lexicographically minimal reduced decomposition of w. The game reads entries from left to right, so we construct the output of the node game on arbitrary entries for subwords of the form $s_1s_2s_3s_4s_0s_1s_2s_3s_7$ (associated to $(4, 4^+)$) and on $s_5s_6s_4s_5s_3s_4s_2s_3$ (associated to $(2, 2, 2, 2^+)$). Let u_4 denote the word associate to $(4, 4^+)$, and u_2 the word associated to $(2, 2, 2, 2^+)$. Then playing the node game $u_2u_4s_7$ on the initial position (a, b, c, d, e, f, g, h) yields a final position

$$g(u_4u_2s_7) = (g, d + e + f, c, b, a, b + 2c + 3d + 2e + 2f + g + 2h,$$
$$e, -a - 2b - 2c - 2d - 2e - 2f - g - h).$$

The starting position for the node game is usually $(-1, -1, \ldots, -1)$, so Equation (1) above shows that after one iteration of u_2u_4, the eighth entry is positive while every other entry is negative. Furthermore, iteration of the u_2u_4 word yield a new terminal position as a function interms of the entries appearing after the first iteration, and thus the orbit of terminal position under this action can be explicitly calculated. Furthermore, the absolute value of the fourth position remains larger than the absolute value of the eigth position after two iterations, and because the recursive polynomial for the fourth position calls itself three times while calling the final position only twice, this pattern is guaranteed to continue, and the fourth position remains negative. All other positions are sums of known negatives, and are thus negative. This then forces the eigth entry to be positive through all iterations of the action of u_2u_4. Let $V = (A, B, C, D, E, F, G, H)$ represent the terminal vector after all application of u_2u_4. Then the vector associated to w is

$$\begin{aligned} u_6 \cdot V &= s_0s_1s_2s_3s_4s_5 \cdot V \\ &= (-A - B - C - D - E - F, A, B, C, D, E, F + G, D + E + F + H). \end{aligned}$$

As $|D| > |H|$, and entries $A, B, \ldots, G$ were negative while H was positive, the above vector shows $w \in \widetilde{E}_7/E_7$. To see the result for a word of the form

$$\lambda = (6^+, 4, 4^+, \ldots, 4, 4^+, 2, 2, 2, 2^+)$$

with k iterations of the word $(4,^+, 2, 2, 2, 2^+)$, consider the word

$$= (6^+, 4, 4^+, \ldots, 4, 4^+, 2, 2, 2, 2^+, 4, 4^+),$$

which, by the above belongs to the quotient, and use the Descent Lemma 1 to peel off the final $(4, 4^+)$ word.

To prove the final statement, we suppose first that w has the form

$$\lambda = (6^+, 4, 4^+, \ldots, 4^+, 3) \text{ or } (6^+, 4, 4^+, \ldots, 2^+, 4, 4, 1),$$

as any more general case can be reduced to one of those two forms using the Descent Lemma. So suppose that the given word has the form $(6^+, 4, 4^+, \ldots, 4^+, 3)$. By the Braid Lemma, it is sufficient to show that s_1 is not a left descent of the word $w' \sim (6^+, 4, 6^+, \ldots, 6^+, 2)$. To see this, we begin by noting that the reduced word

$$(123)(7)(12340123)(7)(56453423)(7)(12340123) = u(123)$$

for some reduced word u. A simple calculation shows that s_4 is a right descent of $(6^+, 4, 4^+, 3)$, s_2 is a right descent of $(6^+, 4, 4^+, 2, 2, 2, 2^+, 4, 4^+, 3)$, and s_3 is a right descent of

$$(6^+, 4, 4^+, 2, 2, 2, 2^+, 4, 4^+, 2, 2, 2, 2^+, 4, 4^+, 3).$$

Then strong induction finishes the result for the case $(6^+, 4, 4^+, \ldots, 4^+, 3)$. A similar calculation shows that s_2 is a right descent of $(6^+, 1, 1, 1)$, s_3 is a right descent of $(6^+, 4, 4^+, 2, 2, 2, 2^+, 1, 1, 1)$, and s_4 is a right descent for

$$(6^+, 4, 4^+, 2, 2, 2, 2^+, 4, 4^+, 2, 2, 2, 2^+, 1, 1, 1),$$

and strong induction again finishes the theorem. □

Lemma 8.4. *(1) The only partitions with first entry* 6^+ *lying above* $(6^+, 4, 4^+, 2, \ldots, 2^+, 4, 3^+)$ *are* $(6^+, 4, 4^+, 2, \ldots, 2^+, 3, 3^+, \mu)$ *where* $\mu = (1), (1, 1), (1, 1, 1),$ *or* $(1, 1, 1, 1)$.

(2) The only partitions with first entry 6^+ *lying above* $(6^+, 4, 4^+, 2, \ldots, 2^+, 4, 4^+)$ *are* $(6^+, 4, 4^+, 2, \ldots, 2^+, 4, 4^+, \mu)$ *where* $\mu = (1),\ (1, 1),\ (1, 1, 1),$ *or* $(1, 1, 1, 1)$.

(3) The only partitions with first entry 6^+ *lying above* $(6^+, 4, 4^+, 2, \ldots, 4^+, 2, 2, 2^+)$ *are* $(6^+, 4, 4^+, 2, \ldots, 4^+, 2, 2, 2^+, \mu)$ *where* $\mu = (1), (2), (3),$ *or* (4).

(4) The only partitions with first entry 6^+ *lying above* $(6^+, 4, 4^+, 2, \ldots, 4^+, 2, 2, 2, 4^+)$ *are* $(6^+, 4, 4^+, 2, \ldots, 4^+, 2, 2, 2, 4^+, \mu)$ *where* $\mu = (1), (2), (3),$ *or* (4).

Proof. Calculation exactly as above. □

This finishes classifying all allowable partitions whose first entry is 6^+. Better, it gives us the following results immediately:

Lemma 8.5. *Given* $w \in \widetilde{E}_7/E_7$, *with* $\lambda = (6^+, \dots, \lambda^k)$,

(1) *For* $\lambda^k = (1,1,1,1)$, λ', *obtained by converting the final* 4 *appearing in the partition for* w *to a* 3 *and leaving every other position alone, corresponds to an element* $w' < w$, *with* $\ell(w') = \ell(w) - 1$.

(2) *For* $\lambda^k = (4)$, λ'', *obtained by converting the final* 2 *appearing in the partition for* w *to a* 1 *and leaving every other position alone, corresponds to an element* $w' < w$, *with* $\ell(w') = \ell(w) - 1$.

Proof. (1) Note that $w \in \widetilde{E}_7/E_7$ implies by the corollary to the Descent Lemma that $\mu' = (6^+, 4, 4^+, \dots, 4, 4^+)$ is an element of the quotient, and by Lemma 8.4 (2), λ' corresponds to a minimal reduced expression for an element of the quotient. But w' is a subword of w formed by deleting the left-most occurrence of s_1, so w' is covered by w.

(2) As above, using Lemma 8.4 (4).

□

Corollary 8.1. *There are no palindromic elements* w *associated to partition* $(6^+, \dots)$ *of length greater than 25.*

Proof. The previous Lemma provides a left descent other than the final simple element in the lexicographically reduced expression, forcing the coefficients of t^1 and $t^{\ell(w)-1}$ in the Poincaré polynomial for $[e, w]$ to differ. □

The case $\lambda^1 = 7^+$ is more difficult. Note that the form $(7^+, \dots, k^+, 4^+, \dots)$ can never appear, as such an element is not lexicographically minimal. To see this, note that in the subword associated to the entry 4^+ is $(s_3 s_2 s_1 s_0 s_7)^{-1}$. In conjunction with the occurence of s_0 ending the previous line, then we have

$$s_7 s_0 s_1 s_2 s_3 s_7 = s_0 s_1 s_2 s_7 s_3 s_7 = s_0 s_1 s_2 s_3 s_7 s_3$$

and the third form above precedes the first form in lexicographic order. But then the decomposition of w using the third form precedes the decomposition of w using the first form, giving the result. This also implies that we will never see something of the form $(7^+,\dots,k^+,m^+,\dots)$ with $m<4$. Similarly, no partition can be of the form $(\cdots,k^+,4,4^+,4)$, $(\cdots,k^+,4,4^+,1,1,1,1)$, $(\cdots,k^+,3,3^+,4)$ or $(\cdots,k^+,3^+,1,1,1,1)$. To see this, observe that the word associated to $(k^+,4,4^+,4)$ is (writing i for s_i)

$$\begin{aligned}(0123)(7)(1234)(0123)(7)\cdot(\text{suffix}) &= (2)(03)(12)(7)(34)(012)(747)\cdot(\text{suffix})\\ &= (2)(03)(12)(7)(34)(0123)(7)(4)\cdot(\text{suffix})\end{aligned}$$

and the last of these forms is the lexicographically minimal one, and it is not identified with the partition sequence $(k^+,4,4^+,4)$. The other statements are proved in exactly the same way.

As a preliminary note, we see by straightforward computation that $(7^+,2,2^+,1)$ is an illegal partition, so there are no partitions of the form $(7^+,2,2^+,\mu)$, and similarly, the only partitions $(7^+,2,2,2^+,\mu)$ are of the form $\mu\in\{(1),(2),(3),(4)\}$. Thus every legal partition of length greater than 19 must have at least a single occurrence of 3 or 4 in a nonterminal position.

Proposition 8.2. *There are no palindromic elements w such that $\lambda(w)=(7^+,\dots)$.*

Proof. As before, the final block λ^k in the partition for a palindromic element must be either (4) or (1, 1, 1, 1). We can check by hand to see that if the word associated to $\lambda^{k-1'}$ end in any simple other than s_4 in the case $\lambda^k=(4)$, or s_2 in the case $\lambda^k=(1,1,1,1)$, then w has two left weak descents, so in particular, w cannot be palindromic.

So suppose that w contradicts the statement of the proposition, and that $\lambda^k(w)=(4)$. For all but two possible λ^{k-1}, it is sufficient to declare a deletion position δ (counted from the right of the word u), and show that the set ρ of right descent pairs of the deleted word is equal to the set ρ_0 of right descent pairs previous to the deletion. Here, u is the word associated to the generalized partition $(\lambda^{k-1},\lambda^k)$.

Table 8.1. Penultimate Blocks with Right Descent Pairs

λ^{k-1}	u	δ	ρ_0	ρ
$(4,4,2^+)$	(0123)(7)(45)(1234)(0123)	8	(23, 43)	(23, 43)
$(4,3,2^+)$	(0123)(7)(45)(234)(0123)	8	(23, 43)	(23, 43)
$(3,3,2^+)$	(0123)(7)(45)(234)(123)	8	(23, 43)	(23, 43)
$(4,2,2^+)$	(0123)(7)(45)(34)(0123)	10	(23, 43)	(23, 43)
$(3,2,2^+)$	(0123)(7)(45)(34)(123)	10	(23, 43)	(23, 43)
$(4,4,4,3^+)$	(0123)(7)(456)(2345)(1234)(0123)	9	(23, 43)	(23, 43)
$(4,4,3,3^+)$	(0123)(7)(456)(345)(1234)(0123)	12	(23, 43)	(23, 43)
$(4,3,3,3^+)$	(0123)(7)(456)(345)(234)(0123)	15	(23, 43)	(23, 43)

Note that the cases $\lambda^{k-1} = (3,3,3,3^+)$ and $(2,2,2^+)$ are not included on this list. They must be dealt with in a more subtle manner. First, if $k = 3$, so there is no block between λ^1 and λ^{k-1}, then if $\lambda^{k-1} = (2,2,2^+)$, w has s_6 as a weak left descent, in addition to s_0, and thus is not palindromic, while if $\lambda^{k-1} = (3,3,3,3^+)$, the associated word is not reduced, so the partition is itself illegal. Thus we can assume that $k > 3$. Let u be associated to the generalized partition $(\lambda^2, \ldots, \lambda^k)$, and v be associated with λ^1. As we are assuming that the partition $(\lambda^1, \ldots, \lambda^k)$ is legal, the right descent pairs for u and the left descent pairs for v satisfy the conditions of the braid theorem. In particular, none of the simples s_5, s_6, or s_7 can be elements of right descent pairs for u. Let $v' = vs_6$, corresponding to the reduction $\lambda^1 \mapsto \lambda^{1'} = (6^+)$. Then for λ' to be an illegal partition, we must have s_0 as an entry in a right descent pair of u, or s_5 as the first such entry. The s_7 is precluded by the previous argument, and if s_5 is the first entry, then λ was illegal, as this allows s_6 to be a right descent for w, contradicting that $w \in \widetilde{E}_7/E_7$.

Now suppose $\lambda^k = (1,1,1,1)$. The possible choices for λ^{k-1} are then $(4,3^+)$, $(3,3^+)$, and $(4,4,4^+)$. Deleting the s_0 at position 4 yields the set of right descent pairs $\{32, 34\}$, which was the set right descent pairs for u. Similarly, for $\lambda^{k-1} = (4,4,4^+)$, let u be associated to $(\lambda^{k-1}, \lambda^k)$; then deletion of the s_2 at position 12 (again, counted from the right) does not change the right descent pairs of u, and thus by the Braid Lemma, corresponds to a descent for w associated to λ. Finally, for $\lambda^{k-1} = (3,3^+)$, we have an involved set of reductions. By testing each simple element, we see that the only possible left descent of $(\lambda^1, \dots, \lambda^{k-2})$ must be s_1, and so we can assume that $\lambda^{k-2} = (4, 4^+)$. In the same way, we see that the predecessor for λ^{k-2} must end in s_5 or s_6, forcing λ^{k-3} to be either 6^+, 7^+, or $(2, 2, 2, 2^+)$. We have discussed the first case, the third case will lead to a recursion of the previous two steps, and so we can assume that we are in the second case. Thus

$$\lambda = (7^+, 4, 4^+, 2, 2, 2, 2^+, \dots, 2, 2, 2, 2^+, 3, 3^+, 1, 1, 1, 1)$$

and deletion of the 13^{th} element, an s_5, preserves the conditions of the Braid Lemma for $u \sim (3, 3^+, 1, 1, 1, 1)$ and $v \sim (7^+, 4, 4^+, \dots, 2, 2, 2, 2^+$. Thus w has two descents, and thus is not palindromic. This finishes the proposition. ☐

Proof. (of Theorem 8.1) All cases have been exhausted for $\ell(w) > 20$, and the remaining possibilities may be checked by hand. ☐

8.2 $\widetilde{E}_8/E_8$

Define the game as before, yielding a shape of the form

0	1	2	3	4	5	6	7				
					9	5	6	7			
						4	5	6			
						3	4	5			
						2	3	4			
						1	2	3			
						0	1	2			
								9	5	6	7
											⋮

Theorem 8.3. *The following list comprises the full set of palindromic elements of* $\tilde{E}_8/E_8$*:*

$$e, s_8, s_8s_7, s_8s_7s_6, s_8s_7s_6s_5, s_8s_7s_6s_5s_4, s_8s_7s_6s_5s_4s_3, s_8s_7s_6s_5s_4s_3s_2$$

$$s_8s_7s_6s_5s_4s_3s_9, s_8s_7s_6s_5s_4s_3s_2s_1, s_8s_7s_6s_5s_4s_3s_2s_9s_3s_4s_5s_6s_7s_8$$

Proof. Each partition is a subform of the above, and as before, we break partitions into blocks corresponding to subwords of the 6×3 blocks in the form above. The blocks of a partition λ are enumerated $\lambda^1, \dots, \lambda^k$, and the minimal reduced decomposition for $\lambda = \pi(w)$ is found by reading overlaying the shape of λ onto the above form, and reading the covered elements in reverse order (right to left and bottom to top). If p is a position in the final block with no element below or right of p, then p constitutes a left weak descent, and hence palindromic elements containing more than three blocks must end in $(1,1,1,1,1,1)$. The commutation relations tells us then that $\lambda^{k-1} = (3, 3^+)$, and hence that the only choices for λ^{k-2} are $(2,2,2,2^+)$ and $(3,3,3,3,3^+)$. If $\lambda^{k-2} = (2,2,2,2^+)$, deletion of the s_6 at position 3 allows commutation of the s_7 and s_8 into the λ^{k-2} block, yielding

$$\cdots, 2, 2, 2, 2, 1, 1^+, 3, 3^+, 1, 1, 1$$

for the λ-tail. Assuming that the original partition was legal, this is as well via the braid lemma, and so $w = \pi^{-1}(\lambda)$ is not palindromic. Similarly, if $\lambda^{k-2} = (3,3,3,3,3^+)$, deletion of the s_7 at position 2 allows commutation of the s_8 into the λ^{k-2} block, yielding

$$\cdots, 3, 3, 3, 3, 3, 1^+, 3, 3^+, 1, 1, 1, 1$$

for the λ-tail. Again, this does not alter the right descent pairs of the tail, and thus by the braid lemma, corresponds to a legal partition supposing the original partition was legal. Hence all palindromic elements of $\widetilde{E}_8/E_8$ have two or fewer blocks. Checking elements directly, we find the full list of such palindromic elements is as above. □

8.3 $\widetilde{E}_6/E_6$

It appears as though the game may be different with $n = 6$, but in fact it is easier.

Letting the crossed node be s_1 for notational purposes, then we can recursively construct the permutation representation of the group. As before, the smallest elements of the group can have their lexicographically minimal decomposition read of by sorting the smallest element that appears, and counting the leftwards jump. This occurs for those elements whose support is contained by the set $\{s_1, \dots, s_5\}$. Larger elements are sorted as follows: if w is sorted, then either $\gamma(w)$ or $\mu(w)$ is positive (possibly both). Then s_6 or s_7, respectively, are descents for the given word. Multiply by the requisite permutation. If the word is still sorted, repeat. Otherwise, sort w_3 into the string $\{w_4, w_5, w_6\}$ counting jumps, and similarly sort w_2 into $\{w_3, w_4, w_5\}$ and w_1 into $\{w_2, w_3, w_4\}$, if necessary. Each time, the number of jumps is recorded, while s_6 and s_7 are recorded as $1^+, 1^-$ respectively. In this way, as before, the outside corners contained in the terminal 3×3 block are all descents, while any 1^- not followed by a 1^+ is also a descent.

Using this fact, and the fact that the first 4 ranks of $\tilde{E}_6/E_6$ are $\{e\}$,$\{s_1\}$, $\{s_2s_1\}$, $\{s_3s_2s_1\}$, and $\{s_4s_3s_2s_1, s_6s_3s_2s_1\}$, we see that the tail of the partition associated to a palindrome w must be of the form $(1^+, 1, 1, 1)$ or $(1^+, 3)$. If 1^- does not precede the 1^+, the possible tails then are $s_1s_2s_3s_6s_4$ and $s_5s_4s_3s_6s_2$. In the first case, the only block forms that end in s_4 are s_3s_4 and $s_3s_4s_5s_2s_3s_4$. In the first subcase,we obtain a tail of the form $s_1s_2s_3s_6s_4s_3s_6$ which has s_4 as a left descent, and so can be eliminated. In the second subcase, we have the tail

$$s_1s_2s_3s_6s_4s_3s_2s_5s_4s_3s_6 = s_1s_2s_3s_4s_5s_6s_3s_2s_4s_3s_6.$$

Now if no s_7 precedes the first s_6, the tail of the Poincaré series is $2, 2, 2, 2, 1, 1, 1, 1$. Building from the identity, the only such heads are of the form $s_1s_2s_3s_4s_5s_6s_3\cdots$, with no s_7, or similarly switching s_5 with s_7 and s_4 with s_6, and there are only finitely many such elements: the interval $[e, w_0]$ in E_6/D_5. The second case is exactly the same.

The other possible tails are of the form $s_1s_2s_3s_6s_7$ and $s_5s_4s_3s_6s_7$. Computing possible entries to precede the s_6, only s_2 is possible in the second case. Again analyzing blocks, the penultimate block in w with a palindromic Poincaré polynomial having this type of tail must be of the form $s_2s_3s_6$ or $s_2s_3s_4s_1s_2s_3s_6$. Together, this shows that the word $s_5s_4s_3s_6s_7s_2s_3$ either lies in blocks $s_5s_4s_3s_6s_7s_2s_3s_6$ or $s_5s_4s_3s_6s_7s_2s_3s_4s_1s_2s_3s_6$, and computing the possible heads for such words, we have a defect in palindromicity at rank $\ell(w) - 8$ or $\ell(w) - 12$ respectively. The second case is exactly the same.

Thus we have classified the palidromic elements of $\widetilde{E_6}/E_6$:

Theorem 8.4. *The full list of palindromic elements for $\widetilde{E_6}/E_6$ is*

$$e, s_1, s_2s_1, s_3s_2s_1, s_4s_3s_2s_1, s_6s_3s_2s_1, s_5s_4s_3s_2s_1, s_1s_2s_3s_6s_4s_3s_2s_1,$$

$$s_5s_4s_3s_2s_1s_6s_3s_2s_4s_3s_6s_5s_4s_3s_2s_1, s_7s_6s_3s_2s_1s_4s_3s_2s_6s_3s_4s_7s_6s_3s_2s_1$$

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