ABSTRACT

Indecomposability in Inverse Limits

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Topological inverse limits play an important in the theory of dynamical systems and in continuum theory. In this dissertation, we investigate classical inverse limits of Julia sets and set-valued inverse limits of arbitrary compacta. Using the theory of Hubbard trees, the trunk of the Julia set of a postcriticallly finite polynomial is introduced. Using this trunk, a characterization of indecomposability is provided for inverse limits of post-critically finite polynomials restricted to their Julia sets.

Inverse limits with upper semicontinuous set-valued bonding maps are also examined. We provide necessary and sufficient conditions for inverse limits of upper semicontinuous functions to have the full projection property, answering a question posed by Ingram [30]. The full projection property is an important tool in the study of indecomposable inverse limits. A characterization of the full projection property for arbitrary compacta is given based solely on the dynamics of the bonding functions and a second characterization is given for the class of continuum-valued maps of trees that are residual-preserving.

Indecomposability in Inverse Limits

by

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CHAPTER ONE

Introduction

1.1 Motivation

Inverse limits play important roles in continuum theory and in the theory of dynamical systems. Regarding dynamical systems, inverse limits allow dynamical behavior to be investigated topologically. Dynamical properties of a function can often be obtained by investigating the topological properties of the inverse limit that the function generates.

In continuum theory, inverse limits often allow for very simple descriptions of topological spaces that are otherwise very complex. All continua are homeomorphic to inverse limits of compact, connected polyhedra (Theorem 2.13 in [45]), and even many rather exotic continua, such as the pseudo-arc, can be constructed using an inverse limit with a single bonding function [25].

Moreover, inverse limits provide important tools for constructing novel spaces with interesting properties. One such class of interesting spaces is the class of indecomposable continua (see Definition 1.2.1). In 1910, Brouwer discovered the first indecomposable continua when he constructed continua that are the common boundary of three regions in the plane [14]. By 1920, indecomposable continua were being studied in their own right by Mazurkiewicz, Janiszewski and Kuratowski [33, 41]. Since then, indecomposable continua have been studied extensively, both in applications and in their own right, and inverse limits have played a prominent role in their construction. Using inverse limits, infinite families of indecomposable continua can be constructed rather easily (see, for example, Definitions 1.3.3 and 1.4.3) and there are many simple conditions that imply inverse limits are indecomposable. For example, every map of the unit interval that is topologically exact generates an inverse limit that is indecomposable (see Proposition 2.2.5).

The remainder of this dissertation is organized as follows. In the remainder of Chapter 1, we provide background material. In Chapter 2, inverse limits of postcritically finite polynomials are considered. In Chapter 3, we characterize the full projection property for inverse limits of upper semincontinuous set-valued functions. In Chapter 5, we investigate particular inverse limits of S^2 known as Hagopian Spheres. Finally, Chapter 5 contains concluding remarks.

1.2 Continuum Theory and General Topology

In this section, fundamental definitions and results from continuum theory are provided.

A *continuum* is a compact, connected metrizable space. A *Hausdorff continuum* is a compact, connected Hausdorff space. A *compactum* is a compact Hausdorff space.

The class of continua is very deep and rich. Indecomposable continua provide an example of the types of complex behavior that continua can exhibit.

Definition 1.2.1. A continuum, X, is said to be *decomposable*, if there exists two proper subcontinua of X, $A, B \subset X$, such that $X = A \cup B$. Otherwise, X is said to be *indecomposable*.

The Brouwer-Janiszewski-Knaster (BJK) continuum is an example of an indecomposable continuum. The BJK continuum can be constructed by winding a ray over a Cantor set, as in Figure 1.1, and then taking the closure of the constructed space. For more details see pg. 204 of [35]. Like the BJK continuum, all indecomposable continua "wind back on themselves" in the sense that no indecomposable continuum is locally connected at any point (an easy corollary of Theorem V.48.V.2 of [35]).

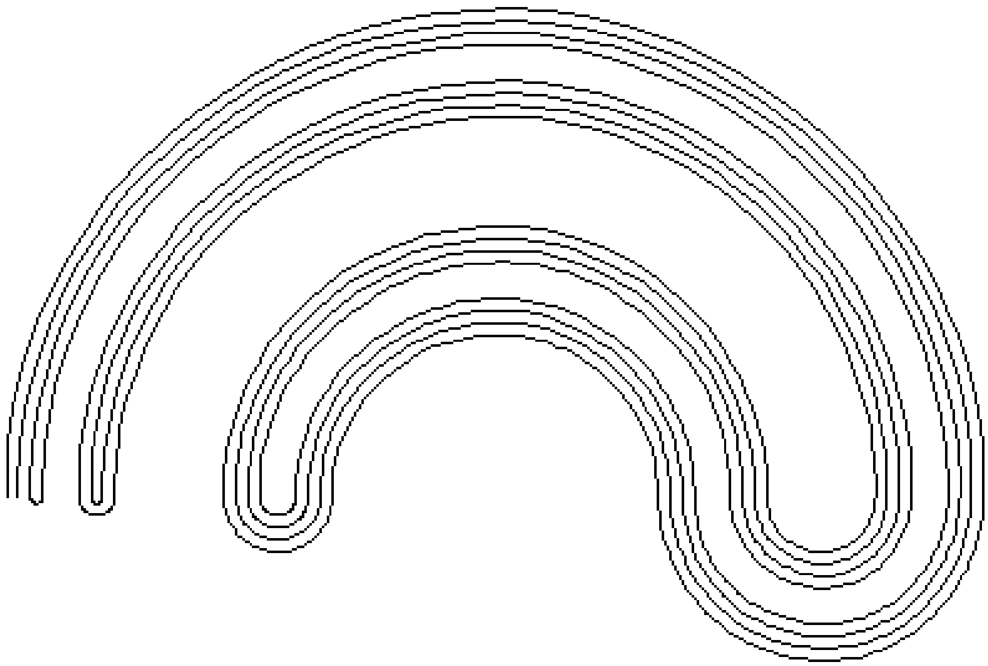

Figure 1.1. The Brouwer-Janiszewski-Knaster (BJK) continuum

The study of indecomposable continua naturally leads to the study of the subcontinua of spaces and to the study of *composants*.

Definition 1.2.2. Let X be a metric space. The *composant* of a point $x \in X$ is the set of all points p in X for which there is a proper subcontinuum of X that contains both p and x. Equivalently, the composant of x is the union of all proper subcontinua of X that contain x.

As an example, the interval $[0, 1]$ has three composants: the composant of 0 is $[0, 1)$, the composant of 1 is $(0, 1]$, and the composant of every other point is $[0, 1]$. Composants play an important role in the study of indecomposable continua, as illustrated by the following theorem.

Theorem 1.2.3. (Theorems 11.15 and 11.17 of [45]) Let X be a continuum. Then X is indecomposable iff X has at least two disjoint composants, in which case X has uncountably many composants, all of which are pairwise disjoint.

Having uncountably many composants is, in fact a characterization of indecomposable continua: all decomposable continua have either one or three composants (Theorems 11.13 of [45]). However, some indecomposable Hausdorff continua have only one or two composants [10].

The theory of dimension also plays an important role in continuum theory. Before higher dimensions can be defined, zero dimensional space must be defined.

Definition 1.2.4. [26] A topological space X is said to be *zero-dimensional* if for each point $x \in X$ there is a neighborhood basis at x consisting of open sets whose boundaries are empty.

To define higher dimensions, first point-wise dimension must be defined. The following definition is inductive in nature.

Definition 1.2.5. [26] A topological space X is said to be have *dimension* $\leq N$ *at a point* x, if there is a neighborhood basis of x of open sets whose boundaries have dimension $\leq N-1$. The space X is said to have *dimension* N *at* x if the dimension of X at x is $\leq N$, but not $\leq N-1$.

The dimension of a topological space can then be defined inductively.

Definition 1.2.6. [26] A topological space X is said to have *dimension* $\leq N$ if X has dimension $\leq N$ at each point $x \in X$. The space X is said to have *dimension* N if X has dimension $\leq N$ but not $\leq N-1$. The space X is said to be *infinite dimensional* if X does not have dimension $\leq N$ for any $N \in \mathbb{N}$.

We close this section with the definition of topological conjugacy. Topological conjugacy can be thought of as the defining equivalency on topological dynamical systems, playing the same role that homeomorphisms and isomorphisms do for topological and algebraic spaces respectively.

Definition 1.2.7. Let X and Y be topological spaces and let $f : X \to X$, $g : Y \to Y$ be continuous. Then f and g are said to be *topologically conjugate*, if there is a homeomorphism $h : Y \to X$ such that $f \circ h = h \circ g$.

1.3 Inverse Limits

Inverse limits provide an important bridge between topological dynamics and continuum theory. Under relatively mild hypotheses, they form compact, connected metric spaces, allowing the tools of continuum theory to be used to describe dynamical behavior. Moreover, even simple inverse limits can exhibit exotic topological behavior. This enables novel continua to be constructed relatively easily and it often allows known continua to be described in a simpler fashion.

Definition 1.3.1. For each $i \in \mathbb{N}$, let X_i be a compact Hausdorff space and let $f_i : X_{i+1} \to X_i$ be a continuous function. Then the *inverse limit* of the sequence $\{X_i, f_i\}$'s, $\varprojlim\{X_i, f_i\}$, is the space $\{(x_1, x_2, \ldots) \in \Pi_{i\in\mathbb{N}} X_i : f_j(x_{j+1}) = x_j$ for each $j \in \mathbb{N}\}$, viewed as subspace of the product space, $\Pi_{i\in\mathbb{N}} X_i$.

We will frequently refer to the functions in Definition 1.3.1 as the *bonding functions* of the inverse limit, and the compacta X_i as the *factor spaces* of the inverse limit. When the factor spaces are clear from context, we simply write $\varprojlim f_i$ for $\varprojlim\{X_i, f_i\}$. As mentioned above, only mild hypotheses are needed to ensure that an inverse limit is a continuum, as the next theorem demonstrates.

Theorem 1.3.2. (Theorems 1.2, 1.3, 1.5 of [29]) For each $i \in \mathbb{N}$, let X_i be a compact Hausdorff space and let $f_i : X_{i+1} \to X_i$ be a continuous function. Then $\varprojlim f_i$ is a compact Hausdorff space. Moreover, if each X_i is connected, then $\varprojlim f_i$ is connected, and if each X_i is metrizable, then $\varprojlim f_i$ is metrizable. In particular, the inverse limit of continua under continuous maps is a continuum.

The *Knaster continua* are examples of fairly exotic spaces that can be constructed from simple inverse limits.

Definition 1.3.3. A *Knaster Continuum* is a continuum homeomorphic to $\varprojlim\{[0,1], f\}$ for some open map $f : [0,1] \to [0,1]$ that is not monotone.

The previously mentioned BJK continuum is an example of a Knaster continuum, because it is homeomorphic to the inverse limit of the full tent map,

$$f(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \text{for } \frac{1}{2} < x \leq 1 \end{cases}$$

(See [45] for more details). All Knaster continua are indecomposable [48].

In 2004, Mahavier [38] investigated a generalized form of inverse limits on the unit interval, by replacing the bonding functions that exist in traditional inverse limits, with multi-valued functions satisfying certain constraints. Two years later, Ingram and Mahavier generalized this new type of inverse limit to arbitrary compacta, using the notion of upper-semicontinuous set-valued functions.

Definition 1.3.4. [32] Let X and Y be compacta and let $f : X \to 2^Y$, where 2^Y is the set of closed subsets of Y. Then f is said to be *upper semi-continuous*, if for each $x \in X$ and each open set $U \subset Y$ such that $f(x) \subset U$, there is an open set $V \subset X$ containing x, such that $f(V) \subset U$.

Figure 1.2 provides an example of a set-valued function that is upper semincontinuous and one that is not. Upper semicontinuous functions can be easily characterized in terms of their graphs.

Definition 1.3.5. Let X and Y be compacta and let $f : X \to 2^Y$. The *graph of f*, $G(f)$, is the set $\{(x, y) \in X \times Y : y \in f(x)\}$.

Theorem 1.3.6. (Theorem 1 of [32]) Let X and Y be compacta and let $f : X \to 2^Y$. Then f is upper semi-continuous iff $G(f)$ is a closed subset of $X \times Y$.

Inverse limits with upper semicontinuous functions are defined similarly to classical inverse limits.

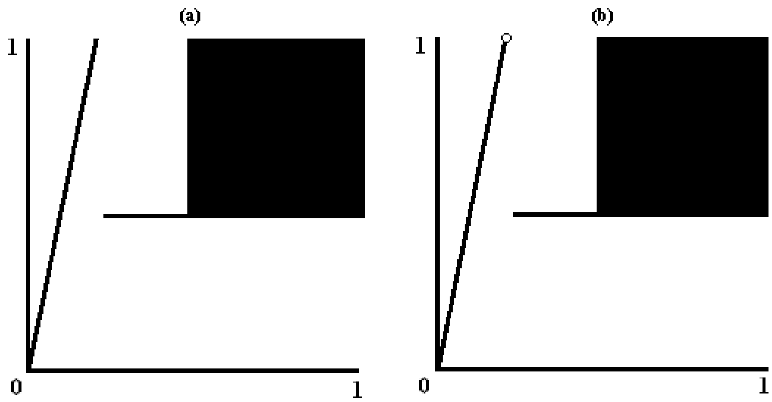

Figure 1.2: Set-Valued Functions. An example of a set-valued function that is upper semi-continuous (a) and an example of a set-valued function that is not upper semi-continuous (b).

Definition 1.3.7. [32] For each $i \in \mathbb{N}$, let X_i be a compact Hausdorff space, and let $f_i : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous set-valued function. Then the *inverse limit*, $\varprojlim\{X_i, f_i\}$, is the space $\{(x_1, x_2, \ldots) \in \Pi_{i\in\mathbb{N}} X_i : x_j \in f(x_{j+1})$ for each $j \in \mathbb{N}\}$, viewed as subspace of the product space, $\Pi_{i\in\mathbb{N}} X_i$.

While we have not defined them in this context, inverse limits can be defined over factor spaces which are indexed by directed sets other than the natural numbers. We will not work with inverse limits in this generality, but occasionally we will investigate inverse limits whose factor spaces are indexed over the integers, rather than over the natural numbers, using the following notation.

Definition 1.3.8. For each $i \in \mathbb{Z}$, let X_i be a compact Hausdorff space, and let $f_i : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous set-valued function. Then the *inverse limit*, $\varprojlim\{X_i, f_i, \mathbb{Z}\}$ is the space $\{(\ldots, x_{-1}, x_0, x_1, x_2, \ldots) \in \Pi_{i\in\mathbb{Z}} X_i : x_j \in f(x_{j+1})$ for each $j \in \mathbb{Z}\}$, viewed as subspace of the product space, $\Pi_{i\in\mathbb{Z}} X_i$.

When working with inverse limits over $\mathbb{N}$ and $\mathbb{Z}$ simultaneously, we will use the notation $\varprojlim\{X_i, f_i, \mathbb{N}\}$ for the inverse limit with directed set $\mathbb{N}$.

Like classical inverse limits, inverse limits with upper semicontinuous set-valued functions form compacta under mild hypotheses.

Theorem 1.3.9. (Theorem 3 of [32]) For each $i \in \mathbb{N}$, let X_i be a compact Hausdorff space, and let $f_i : X_{i+1} \to 2^{X_i}$ be an upper semi-continuous set-valued function. Then $\varprojlim f_i$ is a compactum. Moreover, if each X_i is metrizable then $\varprojlim f_i$ is metrizable.

Similarly, $\varprojlim\{X_i, f_i, \mathbb{Z}\}$ is a compactum if each X_i is a compactum and each f_i is upper semi-continuous, and $\varprojlim\{X_i, f_i, \mathbb{Z}\}$ is metrizable if each X_i is metrizable.

Unlike inverse limits with ordinary functions, inverse limits with set-valued functions needn't be connected, even if every factor space is connected and the graph of every bonding function is connected (see Example 1 of [32]). Establishing necessary and sufficient conditions for connectedness of a set-valued inverse limit is an open problem [31]. The following provides a useful sufficient condition for connectedness of an inverse limit.

Theorem 1.3.10. (Theorem 5 of [32]) Suppose that for each i, X_i is a Hausdorff continuum, and $f_i : X_{i+1} \to 2^{X_i}$ is upper semi-continuous, such that for each $x \in X_{i+1}$, $f(x)$ is connected. Then $\varprojlim f_i$ connected.

Definition 1.3.11. An upper semi-continuous function $f : X \to 2^Y$ is said to be *continuum-valued*, if, for each $x \in X$, $f(x)$ is a continuum.

In view of Theorem 1.3.10, continuum-valued upper semicontinuous functions on continua give rise to inverse limits that are continua.

1.4 Homogeneous Continua

In this section we consider a special class of continua: the homogeneous continua. In [51], Rogers described the classification of homogeneous continua as the most active area of research in continuum theory in the 1980s, and the topic continues be a very active area of research to this day. We borrow heavily from [37, 50, 51] in this section.

Definition 1.4.1. A continuum X is said to be *homogeneous* if, for every $x, y \in X$, there exists a homeomorphism $h : X \to X$, such that $h(x) = y$.

Informally, homogeneous continua can be thought of as continua that exhibit a high degree of uniformity. The simplest example of a homogeneous continuum is a simple closed curve. Another is the Menger curve, which we define below. A graph of the Menger curve is given in Figure 1.3.

Definition 1.4.2. [20] For a cube $C \subset [0,1] \times [0,1] \times [0,1]$ let $D(C)$ denote the decomposition of C into 27 congruent subcubes. Let $C_0 = \{[0,1] \times [0,1] \times [0,1]\}$. For $n \geq 0$, define C_{n+1} inductively by $C_{n+1} = \{C' \in D(C) : C \in C_n$ and C' meets the 1-skeleton of $C_n^*\}$, where $C_n^* = \cup C_n$. The set $M = \bigcap_{n \in \mathbb{N}} C_n^*$ is called the *Menger curve*.

R. D. Anderson [2] showed that the simple closed curve and the Menger curve are the only one-dimensional non-degenerate homogeneous continua that are locally connected. The simple closed curve can be thought of as a special member of the class of solenoids.

Definition 1.4.3. [24] Let $\mathbb{D}$ denote the unit disk in the complex plane, that is $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. Let $d \in \mathbb{N}$. Then for the function $f_d : \partial\mathbb{D} \to \partial\mathbb{D}$ given by $f_d(z) = z^d$, the inverse limit $\varprojlim f_d$ is called the *d-adic solenoid*. More generally, a topological space M is said to be a *solenoid* iff it is homeomorphic to an inverse limit of the form $\varprojlim f_{d_i}$ for some sequence $(d_i)_{i \in \mathbb{N}}$ of positive integers.

Since the factor space $\partial\mathbb{D}$ is a topological group (under complex multiplication) and for each $d \in \mathbb{N}$, f_d is a group endomorphism, solenoids are examples of group-

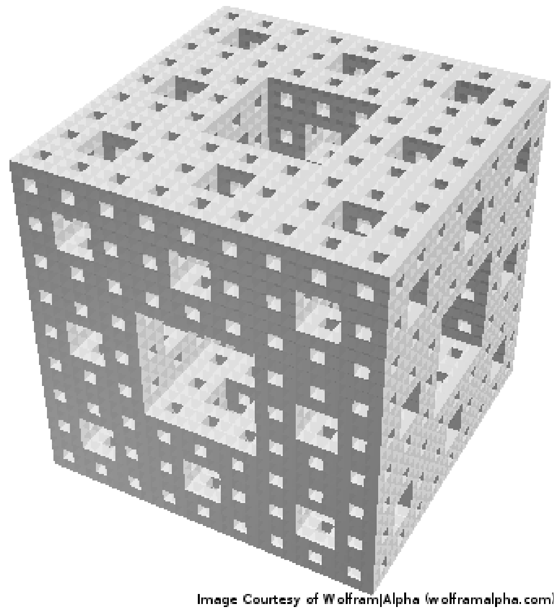

Figure 1.3. The Menger Curve

theoretic inverse limits as well as topological inverse limits, and in fact, solenoids are topological groups. It follows that all solenoids are homogeneous.

Theorem 1.4.4. (Problems 2.8, 2.16 of [45]) Every solenoid is homogeneous. Moreover, for $d > 1$, the d-adic solenoid is indecomposable.

Solenoids, are in fact, the only homogeneous continua whose only proper subcontinua are arcs [24]. However, while solenoids are one dimensional, they are not embeddable in the plane. In fact, there are only four known homogeneous continua that are planar [37]: the point and the circle, and two more exotic continua, the pseudo-arc and the circle of pseudo-arcs which we will describe briefly.

The pseudo-arc is an example of a continuum that is *hereditarily indecomposable*, that is, it is indecomposable and all of its proper subcontinua are also indecomposable. We will not provide a constructive definition of the pseudo-arc, but, non-constructively, the pseudo-arc may be defined as the only non-degenerate hereditarily indecomposable continuum that is homeomorphic to an inverse limit of arcs [11]. Continua that are homeomorphic to inverse limits of arcs are often called *chainable*. The pseudo-arc can also be characterized as the only non-degenerate homogeneous continuum that is chainable [12].

The circle of pseudo-arcs is a decomposable continuum that can be mapped continuously onto the circle by a map f such that for every point x on the circle $f^{-1}(x)$ is a pseudo-arc [13]. While it is unknown if any other planar homogeneous continua exist beside the four we have listed, it is known that any other planar homogeneous continuum that doesn't not separate the plane must be indecomposable [34]. Moreover, any other planar homogeneous continuum that is indecomposable, whether is separates the plane or not, must also be hereditarily indecomposable [23].

Many other non-planar, one dimensional homogeneous continua have been constructed (e.g. [44]). In higher dimensions, all compact, connected manifolds are examples of decomposable homogeneous continua, as are countable products of non-degenerate locally connected homogeneous continua [50]. Rogers [49] has shown that no homogeneous continuum of dimension greater than one is hereditarily indecomposable. However, it is unknown if there are homogeneous continua of any dimension greater than one that are indecomposable [9].

In Chapter 4, we consider a class of spaces that have been proposed as possible candidates for higher dimensional indecomposable homogeneous continua.

1.5 Complex Dynamics

Questions about convergence of the Newton-Ralphson method for approximating zeros of differentiable functions, led to the development the field of complex dynamics, which attempts to describe the behavior of iterative systems of functions of complex variables.

In this section we provide some preliminary definitions and results from complex dynamics. We will restrict ourselves to dynamics of analytic functions of the Riemann sphere, which are precisely the rational functions [8]. Because the dynamics of rational functions of degree 1 are trivial and degenerate (see Problem 4-a of [43]), we will make a standing assumption that all rational functions under consideration are of degree ≥ 2.

By $\hat{\mathbb{C}}$, we will denote the Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, which is the one-point compatification of the complex plane, $\mathbb{C}$.

Given a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the Riemann sphere can be decomposed into two subsets: the region on which the dynamical behavior of f is simple and the region on which the dynamical behavior of f is complicated. Before we make the previous statement more precise, we provide a few definitions.

Definition 1.5.1. [8] Let X, Y be metric spaces and let for each $n \in \mathbb{N}$, let $f_n : X \to Y$. Then $\{f_n\}_{n \in \mathbb{N}}$ is said to *converge locally uniformly* to a function f, if, for each $x \in X$, there is an open set U containing x such that f_n converges uniformly to f on U.

Definition 1.5.2. [8] Let X, Y be metric spaces. A collection of maps $\mathcal{F} = \{f_\alpha : X \to Y | \alpha \in A\}$ is said to *normal*, if each sequence of maps $f_n \in \mathcal{F}$ contains a subsequence that converges locally uniformly.

Normal families of functions of the Riemann sphere correspond precisely with families of equicontinuous functions of the Riemann sphere, per the Arzelà-Ascoli Theorem.

Theorem 1.5.3. (Arzelà-Ascoli) [8] Let $X \subset \hat{\mathbb{C}}$ be a connected open set. Then a collection of maps $\mathcal{F} = \{f_\alpha : X \to \hat{\mathbb{C}} | \alpha \in A\}$ is normal iff it is equicontinuous.

We can now divide the Riemann sphere into the two subsets that we previously mentioned.

Definition 1.5.4. [8] Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be analytic (i.e. a rational function). The *Fatou set* of f, $F(f)$ is the domain of normality of f, that is, the maximal open set on which the family $\{f^n : n \in \mathbb{N}\}$ is normal. The *Julia set* of f, $J(f)$ is the complement of $F(f)$.

We will often simply write J for the Julia set of f when the function f is clear from context. Both J and $F(f)$ are *fully invariant*, that is $f^{-1}(J) = J$ and $f^{-1}(F(f)) = F(f)$.

By definition, the Fatou set is a portion of the Riemann sphere on which the dynamics of f are simple. It is, in fact, the only portion of the Riemann sphere on which the dynamics of f are simple. On the Julia set, the dynamics of f are very complicated. We illustrate this with several theorems, but first we provide a few definitions.

Definition 1.5.5. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. A periodic point p of f is said to be *attracting* if $f^p(U) \subset U$ for any sufficiently small neighborhood of p. If there is a neighborhood U of p such that for every $z \in U \backslash \{p\}$, there is an $n \in \mathbb{N}$, such that $f^n(p) \notin U$, then p is said to be *repelling*.

Definition 1.5.6. [19] Let X be a topological space. A function $f : X \to X$ is said to be *topologically transitive* if for every pair of open sets $U, V \subset X$, there is a $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$.

Definition 1.5.7. [19] Let X be a metric space. A function $f : X \to X$ is said to have *sensitive dependence on initial conditions* if there exists a $\delta > 0$ such that, for any $x \in X$ and any neighborhood N of x, there is a $y \in N$ and an $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.

Theorem 1.5.8. (Corollary 4.16 of [43]) There is a residual set $G \subset J(f)$ such that for every $z \in G$, $\{f^n(z) : n \in \mathbb{N}\}$ is dense in J.

Theorem 1.5.9. (Theorem 14.2 of [43]) Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. Then for any open set $U \subset \hat{\mathbb{C}}$ such that $U \cap J(f) \neq \emptyset$, there exists an $n \in \mathbb{N}$, such that $f^n(U \cap J(f)) = J(f)$.

Theorem 1.5.10. (Theorem 14.1 of [43]) Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. Then $J(f)$ is the closure of the periodic repelling points of f.

In [19], Devaney gave one of the most widely used definitions for dynamical chaos.

Definition 1.5.11. (Definition 8.5 of Part I of [19]) Let X be a topological space and let $f : X \to X$ be continuous. Then f is said to be *chaotic* on X if

(1) f has sensitive dependence on initial conditions,

(2) f is topologically transitive, and

(3) periodic points are dense in X.

From Theorems 1.5.9 and 1.5.10, it is quickly deduced that $f|_{J(f)}$ is chaotic for any rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with a non-degenerate Julia set.

We have made a distinction between attracting and repelling periodic points, but we will need to draw finer distinction between periodic points. To this end, for each periodic point p of f, we associate a complex number, λ, called the *multiplier* of p.

Definition 1.5.12. [43] For a rational function, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and a fixed point p of $f(w)$, let z be a uniformizing parameter so that $z = 0$ corresponds to the point p. Then, in local coordinates, $f(z)$ has a power series about $z = 0$ of the form $f(z) = \lambda z + \mathcal{O}(z^2)$. The complex number λ is called the *multiplier* of the fixed point. If p is a periodic point of period n, then the multiplier of p is defined to be the multiplier of p as a fixed point of the function $f^n(z)$.

The multiplier of a periodic point encapsulates the notions of attracting and repelling, as the following theorem demonstrates.

Theorem 1.5.13. (Theorem 8.1 and Lemma 8.10 of [43]) A periodic point, p, is attracting iff $|\lambda| < 1$. If $|\lambda| > 1$, then p is a repelling periodic point.

A periodic point for which $|\lambda| = 1$ is called a *neutral* periodic point. As a special case of attracting periodic points, if $\lambda = 0$, then p is said to be *superattracting.* One important property that distinguishes complex polynomials of degree two or greater from other rational functions is that ∞ is a superattracting fixed point of every polynomial [43].

At times, we will make use of Filled Julia Sets.

Definition 1.5.14. Let $K \subset \hat{\mathbb{C}}$ be compact. Then the *filling* of K, is the union of K and the bounded components of $\hat{\mathbb{C}} \backslash K$. The *Filled Julia set*, $K(f)$, is the filling of $J(f)$.

If f is a polynomial then $\infty \in F(f)$, and $J(f)$ is the boundary of the unbounded Fatou component (Theorem IV.1.1 of [16]). For polynomials, the unbounded Fatou component is precisely the basin of attraction of ∞.

Definition 1.5.15. Let X be a metric space. Let $f : X \to X$ be continuous and let $c \in X$ be periodic under f. Then the *basin of attraction* of c is $\mathcal{A}_c := \{x \in X : f^n(x) \to c \text{ as } n \to \infty\}$.

We now turn to some important topological properties of the Julia set. In regards to dimension, the Julia set of a rational function is always a one-dimensional compactum, unless it is the entire Riemann sphere. The latter case cannot occur if f is a polynomial.

Julia sets exhibit a high degree of self-similarity: locally almost every region of the Julia set “looks like” almost every other region of the Julia set. More precisely, we have the following theorem.

Theorem 1.5.16. (Problem 4-d in [43]) Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. Then there is a finite (possibly empty) set $X := \{z_0 \in J : \forall z \in \varprojlim f$ such that $f(z_1) = z_0, \exists i \in \mathbb{N}$ such that z_i is a critical point of $f\}$ such that for any $z_0 \in J(f)\backslash X$, there is a dense set $D \subset J(f)$ such that for any $z \in D$ there is a homeomorphism h from a neighborhood N of z_0 to a neighborhood N' of z such that $h(z_0) = z$.

We now turn to the components of the Fatou set, which also provide valuable information about the dynamics of the function f. We note two important results.

Theorem 1.5.17. (Problem 4-i of [43]) Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function and U be a component of $F(f)$. Then $f(U)$ is also a component of $F(f)$.

Theorem 1.5.18. (Sullivan's Nonwandering Domain Theorem) (Theorem 1 of [53]) Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function. Then every component of $F(f)$ is eventually periodic. That is, for every component U of $F(f)$, there are integers $n \geq 0$ and $m > 0$, such that $f^n(U) = f^{n+m}(U) = V$, for some component V of $F(f)$.

For arbitrary rational functions, even for polynomials, many topological properties of Julia sets are not well understood. The following are important open questions.

Question 1.5.19. ([40] and [17]) Does there exists a rational function whose Julia set is an indecomposable continuum? Does there exists a polynomial function whose Julia set is an indecomposable continuum? Does there exists a quadratic polynomial function whose Julia set is an indecomposable continuum?

It is known that Julia sets of rational functions can contain indecomposable continua. Milnor and Lei [42] constructed a rational function whose Julia set is a Sierpienski curve. Sierpienski curves, which are homeomorphic to a "face" of the Menger curve defined in Section 3, are universal one dimensional planar continua and hence contain homeomorphic copies of all one dimensional planar continua.

Because many properties of Julia sets of arbitrary polynomials and rational functions are so poorly understood, our main focus is on the set of polynomials that are post-critically finite.

Definition 1.5.20. A polynomial function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is said to be *post-critically finite*, if each critical point c of f satisfies $\{f^n(c) : n \in \mathbb{N}\}$ is finite.

Note that all critical points of post-critically finite polynomials are either *periodic* (that is, $f^n(c) = c$ for some $n \geq 0$) or *pre-periodic* (that is, c is not periodic but, $f^n(c) = f^{n+m}(c)$ for some $n, m \geq 1$). In addition, all periodic points of post-critically finite polynomials are either repelling or super-attracting (Corollary 14.5 of [43]).

In general, Julia sets of polynomials needn't be connected, but if f is a post-critically finite polynomial, then $J(f)$ is both connected and locally connected (Theorems 9.5, 19.6 and 19.7 in [43]). Moreover, the boundary of every Fatou component is locally connected, which is a special case of Theorem 19.7 in [43].

Theorem 1.5.21. Let f be a post-critically finite polynomial. Then the boundary of every component of $F(f)$ is locally connected.

In [47], Roesch and Yin proved a stronger result: for arbitrary polynomials, if F is a Fatou component that is not eventually mapped to a Siegel disk, then ∂F is a Jordan curve. Siegel disks, whose definition we omit (see [43], only occur at certain types of neutral fixed points and hence are not present in the Fatou sets of post-critically finite polynomials. For more information about Julia sets we refer the reader to [43] and [16].

1.6 Hubbard Trees

In [21], Douady and Hubbard constructed a combinatorial object, the *Hubbard Tree*, to describe the dynamics of a postcritically finite polynomial on its Julia set.

Topologically, Hubbard trees are much simpler than Julia sets. However, up to conjugacy, it is possible to recover a polynomial f from its Hubbard tree, $H(f)$, together with the dynamics of f on $H(f)$, and few other simple pieces of information related to $H(f)$ (see [21] and [46] for more details). In the remainder of this section we formally define Hubbard trees and then list some important results about them.

We will first consider the case where f is a postcritically finite polynomial with no periodic critical points. In this case, $J(f)$ is a locally connected continuum that does not contain a simple closed curve, that is a *dendrite* (see Theorem V.4.2 of [16]). It follows that $J(f)$ is uniquely arc-connected. Moreover, since dendrites are nowhere dense in the plane, it follows that $F(f) = \mathcal{A}_\infty$, so every critical point of f is an element of $J(f)$.

Definition 1.6.1. Let f be a postcritically finite polynomial such that no critical point of f is periodic. The *Hubbard Tree* of f, $H(f)$, is the minimal tree in $J(f)$ that contains all the critical points of f and their forward orbits. More formally, if $C = \{z \in \mathbb{C} : z = f^n(c)$ for some $n \geq 0$ and some $c \in \mathbb{C}$ such that $f'(c) = 0\}$, and for each $a, b \in J(f)$, $[a, b]$ denotes the unique arc in $J(f)$ that connects them, then $H(f) = \bigcup_{a,b \in C}[a, b]$.

If f has at least one periodic critical point, c, of period p, then c is a superattracting critical point of f^p, and hence is an element of $F(f^p) = F(f)$.

Theorem 1.6.2. [46] Let f be a postcritically finite polynomial and let F be a component of $F(f)$. Then there is a homeomorphism $\phi_F : F \to \mathbb{D}$ that conjugates $f|_F$ to z^d for some $d \in \mathbb{N}$.

Definition 1.6.3. Let f be a postcritically finite polynomial. If F is a bounded component of $F(f)$ then an *internal ray* of F is a ray of the form $R_\theta = \phi_F^{-1}(\{re^{i\theta} : 0 \leq r < 1\})$. If F is the unbounded component of f, then a ray of the form $R_\theta = \phi_F^{-1}(\{re^{i\theta} : 0 \leq r < 1\})$ is called an *external ray*.

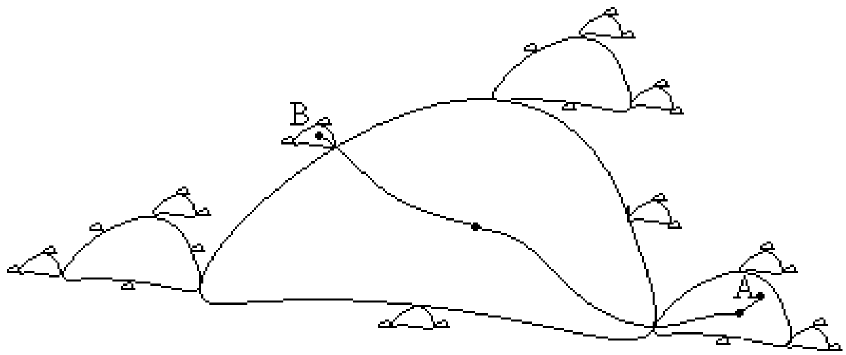

Figure 1.4. A regulated arc

Using the above definitions, we may define regulated arcs, which will allow us to define Hubbard trees. See Figure 1.4 for an example of a regulated arc.

Definition 1.6.4. Let f be a postcritically finite polynomial and let $x, y \in K(f)$. An arc $A \subset K(f)$ from x to y is said to be *regulated* if, for every bounded Fatou component F that meets A, $A \cap \overline{F}$ is a subset of the closure of the union of two internal rays of F.

Note that if f is pre-periodic, then $K(f) = J(f)$ and all arcs in $K(f)$ are regulated.

Definition 1.6.5. Let f be a postcritically finite polynomial and let C denote the set of critical points of f together with their forward orbits. Then the *Hubbard tree* of f, $H(f)$, is the union of the regulated arcs between the elements of C.

Note that if f has no periodic critical points then $K(f) = J(f)$ and the above definition reduces to our previous definition of $H(f)$.

We close this section with a few properties of Hubbard trees and regulated arcs.

Theorem 1.6.6. [46] Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a postcritically finite polynomial. Then,

(1) H(f) is forward invariant, that is, $f(H(f)) \subset H(f)$,

(2) $H(f) \cap \partial F$ is finite for every bounded Fatou component, F and

(3) if A is a regulated arc in $K(f)$ such that $\partial A \subset H(f)$, then $A \subset H(f)$.

We discuss more properties of Hubbard trees in Section 2.2.

CHAPTER TWO

Inverse Limits of Julia Sets

2.1 Introduction

Inverse limits have proven to be a valuable tool for the investigation of dynamical systems. For example, in dynamical systems of the unit interval, Barge and Martin [5, 7] established numerous relationships between the topology and dynamics of inverse limits and the dynamics of the bonding functions of those inverse limits. Inverse limits have also been used to establish relationships among dynamical properties, as in [6] by the same two authors.

However, to date, inverse limits of Julia sets have not received much attention in the literature. In [15], Cabrera investigated inverse limits of quadratic polynomials of the form $f_c(z) = z^2 + c$ with 0 periodic with special attention given to "regular leaf spaces", which can be obtained by removing finitely many points from the inverse limit, $\varprojlim\{f, \hat{\mathbb{C}}\}$. Cabrera showed that distinct values of c give distinct regular leaf spaces. To date, this is the only paper we are aware that relates directly to the study of inverse limits of Julia sets.

In this chapter, we consider the larger class of postcritically finite polynomials, but we restrict our attention to the inverse limit along the Julia set, that is $\varprojlim\{f|_J, J\}$. In the case that J is a dendrite, this relates somewhat to a paper of Baldwin [3], who gave, among other things, results on inverse limits of dendrites under functions with a single critical point using kneading sequence theory.

The main result of this chapter is the following (see Theorems 2.4.5 and 2.5.11).

Theorem 2.1.1. Let f be a postcritically finite polynomial. Then the following are equivalent:

(1) J is homeomorphic to either S^1 or $[0, 1]$.

(2) f is conjugate to either z^d or $\pm T$ for some Chebyshev polynomial, T.

(3) $\varprojlim f|_J$ is indecomposable.

(4) $\varprojlim f|_J$ is either a solenoid or Knaster continuum.

In addition we prove a number of other results concerning inverse limits of postcritically finite polynomials. Most involve the "trunk" of J, a forward-invariant subset of J that is closely related to the Hubbard tree of f. Perhaps the most interesting is the following (Theorem 2.6.3):

Theorem 2.1.2. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a postcritically finite polynomial with Julia set J with trunk T such that $J \backslash T \neq \emptyset$. Let $X = \varprojlim\{f|_J, J\}$ and let $\tilde{T} = \bigcap_{i \in \mathbb{N}} \pi_i^{-1}(T) = \varprojlim\{f|_T, T\}$. Then

(1) $X \backslash \tilde{T}$ is connected,

(2) $X \backslash \tilde{T}$ has $\mathfrak{c}$-many composants,

(3) the arc components of $X \backslash \tilde{T}$ are precisely its composants,

(4) $\mathfrak{c}$-many arc components of $X \backslash \tilde{T}$ are dense in $X \backslash \tilde{T}$ and

(5) infinitely-many arc components of $X \backslash \tilde{T}$ are not dense in $X \backslash \tilde{T}$.

The remainder of this chapter is organized as follows. We give general definitions in the remainder of this section. In Section 2 general results regarding indecomposability of inverse limits are presented. Section 3 summarizes known results concerning Hubbard trees that we will need. Sections 4 and 5 address indecomposability in inverse limits of postcritically finite polynomials without and with periodic critical points respectively. Finally in Section 6 we consider the inverse limit restricted to the trunk and also its complement.

2.2 General Indecomposability Results

Before considering inverse limits of postcritically finite polynomials, we present several results on indecomposability of inverse limits, most of which we will make use of later.

Definition 2.2.1. A continuum X will be said to *decompose finely* if for every $\delta > 0$, there is a decomposition $X = A \cup B$ such that neither A nor B contains a ball of radius δ.

Definition 2.2.2. If X is a topological space, then a map $f : X \to X$ is said to be *topologically exact* if for every open set $U \subset X$ there is an $n \in \mathbb{N}$ such that $f^n(U) = X$.

Remark 2.2.3. In the previous definition, if X is compact metric, then n depends only on the size of U, that is, for every $\delta > 0$ there exists an $n \in N$ such that for any $x \in X$ and any $U \supset B_\delta(x)$, $f^n(U) = X$.

Proposition 2.2.4. Suppose X is a continuum that does not decompose finely and suppose that $f : X \to X$ is topologically exact. Then $\varprojlim\{X, f\}$ is indecomposable.

Proof. Let A, B be subcontinua of $\varprojlim\{X, f\}$ such that $A \cup B = \varprojlim\{X, f\}$. Then as X does not decompose finely, there is a $\delta > 0$ such that for each n, one of $\pi_n(A)$, $\pi_n(B)$ contains a ball of radius δ. So we may assume that $\pi_n(A)$ contains a δ-ball for infinitely many $n \in N$. Then by topological exactness, $\pi_n(A) = X$ for infinitely many $n \in N$ and hence for every $n \in N$. It follows that $A = \varprojlim\{X, f\}$. Thus $\varprojlim\{X, f\}$ is indecomposable. $\square$

Proposition 2.2.5. Let X be a continuum containing an arc A, such that $Int_X(A) \neq \emptyset$. Then X does not decompose finely and hence $\varprojlim\{X, f\}$ is indecomposable for any topologically exact map $f : X \to X$.

Proof. Let $C \subset Int_X(A)$ be an arc and let $\gamma : [0,1] \to C$ be a homeomorphism. Let $\delta > 0$ be small enough that $\gamma(0, \frac{1}{3})$, $\gamma(\frac{1}{3}, \frac{2}{3})$ and $\gamma(\frac{2}{3}, 1)$ each contain a ball of radius δ in X. Then for any decomposition $X_1 \cup X_2 = X$, one of X_1, X_2, say X_1, must contain at least one of $\gamma(0, \frac{1}{3})$, $\gamma(\frac{1}{3}, \frac{2}{3})$, $\gamma(\frac{2}{3}, 1)$. Then X_1 contains a ball of radius δ, so X does not decompose finely. □

Proposition 2.2.6. Let X be a continuum irreducible about a finite set, but not irreducible about any 2-point set. Then X does not decompose finely and hence $\varprojlim\{X, f\}$ is indecomposable for any topologically exact map $f : X \to X$.

Proof. Let $F = \{a_1, \ldots a_n\}$ be a minimal set about which X is irreducible. For each $i \leq n$, let $F_i \supset F \backslash \{a_i\}$ be a proper subcontinuum of F. Choose $\delta > 0$ such that, for each $i \leq n$, $X \backslash F_i$ contains a ball of radius δ.

Now let $A \cup B = X$ be any decomposition of X. Then at least one of A, B contains two elements of $\{a_1, \ldots, a_n\}$. Without loss of generality, suppose that $a_1, a_2 \in A$. Then $a_2 \in A \cap F_1$ and $A \cup F_1 \supset F$, so $A \cup F_1 = X$. Then $A \supset X \backslash F_1$ and hence A contains a δ-ball. □

The following is proved in a more general setting in [36].

Theorem 2.2.7. (Kuykendall) Let X be a continuum and $f : X \to X$ be a continuous surjection. Then $\varprojlim\{X, f\}$ is indecomposable iff for every $\epsilon > 0$ there is a positive integer n and three distinct points $x, y, z \in X$ such that for any continuum M containing two of them and any $w \in X$, $d(f^n(M), w) < \epsilon$.

2.3 Hubbard Trees and Related Concepts

In Chapter 1, we provided the definition of *Hubbard trees*. In this section we provide some results on Hubbard trees that we make use of in later sections.

Lemma 2.3.1. Let $x, y \in K(f)$, with A the regulated arc between them and let $\mathcal{F}_B$ denote the set of bounded Fatou components of f. Then for any other arc $B \subset K(f)$

from x to y,

(1) $A \cap (J(f) \backslash \bigcup_{F \in \mathcal{F}_B} \partial F) = B \cap (J(f) \backslash \bigcup_{F \in \mathcal{F}_B} \partial F)$ and

(2) each bounded Fatou component, F, satisfies $A \cap \partial F \subset B \cap \partial F$.

Thus $A \cap J(f) \subset B \cap J(f)$.

Proof. (1) Let B be an arc between x and y. Since $J(f) = \partial K(f)$, it follows by Proposition 6 in Chapter 2, Section 6 of [21], that there is a regulated arc A' between x and y such that $A' \cap (J(f) \backslash \bigcup_{F \in \mathcal{F}_B} \partial F) = B \cap (J(f) \backslash \bigcup_{F \in \mathcal{F}_B} \partial F)$. As the regulated arc between x and y is unique, $A = A'$.

(2) Immediate from the construction of regulated arcs. (See Proposition 6 in Chapter 2, Section 6 of [21].) $\square$

Lemma 2.3.2. (Lemma 1.8 of [46]) Let A be a regulated arc such that $Int_{K(f)}(A)$ doesn't contain any critical points. Then $f|_A$ is a homeomorphism.

2.4 Polynomials with Only Preperiodic Critical Points

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial such that every finite critical point is preperiodic. By Theorem V.4.2 in [16], $J(f)$ is a dendrite. In this section we characterize when $\varprojlim f|_J$ is indecomposable, but first we consider a more general case.

Theorem 2.4.1. Suppose that X is a dendrite and $g : X \to X$ is topologically exact. Suppose further that there is a closed, forward invariant, connected set $T \subset X$ such that:

(1) $X \backslash T$ has at least three components and

(2) for any arc $A \subset X$ with $Int_X(A) \subset X \backslash T$, $g|_A$ is injective.

Then $\varprojlim g$ is decomposable.

Proof. Let $x, y \in X$ and let $\overline{xy}$ denote the arc between them. For each n, define $\overline{xy}_n = g^n(\overline{xy})$. We claim that, for each n, $cl(\overline{xy}_n \backslash T)$ consists of 0, 1 or 2 arcs. As T is connected and X is uniquely arc-connected, $\overline{xy}_0 \cap T$ is connected, so $cl(\overline{xy}_0 \backslash T)$

has at most 2 components and the components must be arcs since $\overline{xy}_0$ is an arc. Now suppose that $cl(\overline{xy}_n \backslash T)$ consists of 0, 1 or 2 arcs. We will show that the same holds for $cl(\overline{xy}_{n+1} \backslash T)$. If $\overline{xy}_n \backslash T = \emptyset$, then $\overline{xy}_{n+1} \subset T$ since T is forward invariant. Otherwise, $cl(\overline{xy}_n \backslash T) = A \cup B$, where A and B are arcs (allowing $A = B$). If A and B are disjoint from T then $\overline{xy}_n$ is an arc disjoint from T and hence $\overline{xy}_{n+1}$ is an arc and the argument for the $n = 0$ case can be applied. So suppose that A meets T. Then $g(A)$ is an arc by (2) with at least one endpoint in T, so $cl(g(A) \backslash T)$ is either empty or it is an arc. The same holds for $g(B)$ and hence $cl(\overline{xy}_{n+1} \backslash T)$ consists of 0, 1 or 2 arcs.

Now by hypothesis, $X \backslash T$ has at least three components, but for each n, $g^n(\overline{xy}) \backslash T$ misses all but possibly two of them. Thus by Kuykendall's Theorem (2.2.7), $\varprojlim g$ is decomposable. $\square$

Lemma 2.4.2. Let $A \subset J$ be an arc such that no separating point of A is a critical point of f. Then $f|_A$ is injective.

Proof. This follows from Lemma 2.3.2. $\square$

The following is stated without proof in [1].

Lemma 2.4.3. If J is not an arc, then J has infinitely many branch points.

Proof. Let z_0 be a branch point of J. If $z_1 \in f^{-1}(z_0)$ is a critical point, then $z_1 \neq z_0$ as f has no periodic critical points. Moreover, f is at least two-to-one in a neighborhood of z_1, so the valency of z_1 is greater than the valency of z_0. Hence z_1 is also a branch point. Then as f has only finitely many critical points, it follows that there is a branch point z of f such that no iterated preimage of z is a critical point. Then by Theorem 1.5.16, J has infinitely many branch points. $\square$

Lemma 2.4.4. If J is not an arc, then $J \backslash H(f)$ has infinitely many components.

Proof. As $H(f)$ is the union of finitely many arcs and is uniquely arc-connected, it suffices to show that $H(f)$ contains infinitely many branch points of J. Suppose not. Then there exists a neighborhood U in $H(f)$ such that no point of U is a branch point of J. Then U has interior in J. For cofinitely many $z \in J$ there is a dense set $D_z \subset J$ such that for any $z' \in D_z$ there is a homeomorphism that maps a neighborhood of z to a neighborhood of z' (Theorem 1.5.16). Thus for cofinitely many $z \in J$, there is a neighborhood of z that is an arc. Thus J has only finitely many branch points, which is a contradiction to Lemma 2.4.3. $\square$

Theorem 2.4.5. Let f be a polynomial whose finite critical points are all preperiodic. Then the following are equivalent:

(1) $\varprojlim f|_J$ is indecomposable.

(2) J is an arc.

(3) (Steinmetz) f is conjugate to $\pm T_n$, where T_n is a Chebyshev polynomial.

(4) $\varprojlim f|_J$ is a Knaster continuum.

Proof. ($1 \Leftrightarrow 2$) If J is an arc, then the result follows from Proposition 2.2.5 since f is topologically exact (Corollary 14.2 of [43]). If J is not an arc then by the preceding lemmas, we may apply Theorem 2.4.1 and hence $\varprojlim f|_J$ is decomposable.

($2 \Leftrightarrow 3$) This is part of Theorem A of [52].

($3 \Rightarrow 4$) It is well-known (see for example, Theorem 1.4.1 of [8]) that $J(T_n) = [-1, 1]$. Since $T_n|_{[-1,1]}$ has exactly $n-1$ critical points, each of which maps to either 1 or -1, and $T_n(\{-1, 1\}) \subset \{-1, 1\}$, the result follows.

($4 \Rightarrow 1$) Well-known.

$\square$

2.5 Polynomials with Periodic Critical Points

Throughout this section, let $f(z)$ be a postcritically finite polynomial such that at least one finite critical point of f is periodic. Recall that the boundary of every Fatou component of f is a Jordan curve (see Theorem 1.5.21 and the following comments). In this section, we characterize when $\varprojlim f|_J$ is indecomposable.

Lemma 2.5.1. Let $A, B \subset J$ be distinct Jordan curves. Then $|A \cap B| \leq 1$.

Proof. Suppose that $|A \cap B| \geq 2$. Let C be a component of $B \backslash A$. Then $A \cup C$ is a theta-curve (that is a curve homeomorphic to the letter θ) and hence J is not the boundary of the unbounded Fatou component, a contradiction. $\square$

Definition 2.5.2. The *trunk* of f, T is defined to be $T = (H(f) \cap J(f)) \cup (\bigcup\{\partial F : F$ is a component of $F(f)$ and $F \cap H(f) \neq \emptyset\})$. (See Figure 2.1).

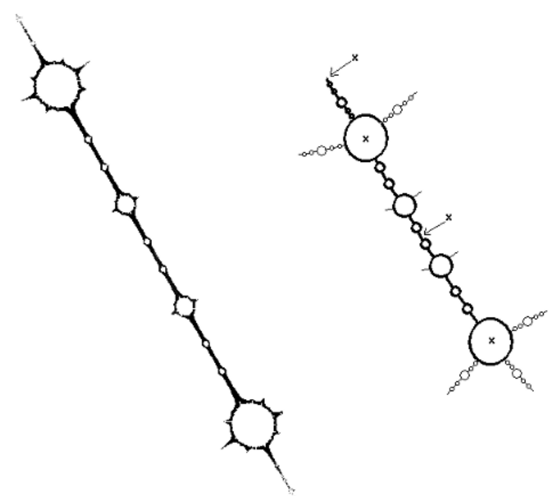

Figure 2.1: The Julia set (left) for the postcritically finite polynomial $f(z) \approx z^4 + (2.683 + 4.647i)z^3 + (-3.599 + 6.234i)z^2$ and an illustration (right) highlighting the structure of the Fatou components. The x's denote the locations of points of critical orbits and the bold region of the diagram denotes the trunk. "Lines" in the diagram indicate regions where there are very small Fatou components. (Recall that the Julia set of a rational function is the boundary of the union of the Fatou components in the grand orbit of any Fatou component containing a periodic critical point (Cor. 4.12 of [43]))

Lemma 2.5.3. T is a path-connected continuum.

Proof. To prove that T is compact, it suffices to show that $T' := cl(\bigcup\{\partial F : F$ is a component of $F(f)$ and $F \cap H(f) \neq \emptyset\}) \subset T$. Let $x \in T'$. For each $i \in \mathbb{N}$, let F_i be a bounded Fatou component of f with $F_i \cap H(f) \neq \emptyset$, and let $x_i \in \partial F_i$ such that $x_i \to x$. If there is a Fatou component F such that $F_i = F$ infinitely often, then $x \in \partial F \subset T$. Otherwise, for any $\epsilon > 0$, only finitely many Fatou components have diameter greater than ϵ (by Lemma 19.4, Theorem 19.6 and the remarks after Theorem 19.7 in [43]), so $diam(\partial F_i) \to 0$. Hence $x \in H(f)$. Since $\forall i \in \mathbb{N},\ x_i \in J$, it follows that $x \in H(f) \cap J \subset T$. Thus T is compact.

It remains only to show that T is path-connected. Let $x, y \in T$. Then there is a regulated arc $A_0 \subset K(J)$ from x to y. Let $\{F_i\}$ enumerate the bounded components of $F(f)$. For each $i \in \mathbb{N}$ such that $F_i \cap A_0 \neq \emptyset$, $|A_0 \cap \partial F_i| = 2$ (by the definition of regulated arcs), so we may choose an arc $C_i \subset \partial F_i$ such that $\partial C_i = A_0 \cap \partial F_i$. Then for each $i \in \mathbb{N}$, define an arc A_i by

$$A_i = \begin{cases} C_i \cup (A_{i-1} \backslash (A_0 \cap F_i)) & \text{if } F_i \cap A_0 \neq \emptyset \\ A_{i-1} & \text{otherwise.} \end{cases}$$

As only finitely many Fatou components have diameter greater than any ϵ, the A_i converge to some arc A containing both x and y. For $z \in A$, either $z \in A_0 \backslash \bigcup_{i \in \mathbb{N}} F_i \subset H(f) \cap J(f) \subset T$ or $z \in \partial F_i$ for some i such that $F_i \cap A_0 \neq \emptyset$, in which case $F_i \cap H(f) = \emptyset$, so $z \in T$. Thus $A \subset T$ is an arc between x and y as desired. □

Lemma 2.5.4. If F is a bounded Fatou component such that $|\partial F \cap T|$ is infinite, then $\partial F \subset T$.

Proof. Let $\mathcal{A} = \{G : G$ is a Fatou component, $G \cap H(f) \neq \emptyset$ and $\partial G \cap \partial F \neq \emptyset\}$. As $H(f) \cap \partial F$ is finite by Theorem 1.6.6 and for any bounded Fatou component $G \neq F$, we have that $|\partial G \cap \partial F| \leq 1$ by Lemma 2.5.1, it follows, by the definition of T, that $\mathcal{A}$

is infinite. Let $G, H \in \mathcal{A}$ be distinct and let $x \in \partial G \cap \partial F, y \in \partial H \cap \partial F, x_G \in G \cap H(f)$ and $x_H \in H \cap H(f)$. Then there are regulated arcs A_1, A_2 and A_3 connecting x_G to x, x to y and y to x_H respectively such that $Int(A_2) \subset F$. It follows that $A = A_1 \cup A_2 \cup A_3$ is the regulated arc from x_G to x_H. Then by Theorem 1.6.6, $A \subset H(f)$, so $\partial F \subset T$ as desired.

$\square$

Lemma 2.5.5. Let $A \subset J$ be an arc. Then $A \cap T$ is connected and furthermore, for any bounded Fatou component F, $A \cap \partial F$ is connected as well.

Proof. Identify A with the closed interval $[0, 1]$ and suppose that $(a, b) \subset J \backslash T$ with $a, b \in A \cap T$. Let $\overline{ab} \subset T$ be an arc between a and b. Then $C = \overline{ab} \cup (a, b)$ is a Jordan curve in J. Then C must enclose a bounded Fatou component F. It follows that $\partial F = C$ as otherwise $\partial F \backslash C$ would not be accessible from the unbounded Fatou component. Then $\overline{ab} \subset \partial F \cap T$ so by Lemma 2.5.4, $\partial F \subset T$. But this implies $(a, b) \subset T$, a contradiction. Thus $A \cap T$ is connected.

To prove that $A \cap \partial F$ is connected for any bounded Fatou component F, we may construct a Jordan curve C like before. Then C and ∂F are distinct Jordan curves in J with $a, b \in C \cap \partial F$, which contradicts Lemma 2.5.1.

$\square$

Remark 2.5.6. Applying Lemma 2.5.5, we may strengthen Lemma 2.5.4: If F is a bounded Fatou component such that $|\partial F \cap T| > 1$, then $\partial F \subset T$.

Lemma 2.5.7. Every component of $J \backslash T$ gets mapped homeomorphically onto its image by f.

Proof. Let C be a component of $J \backslash T$ and let $x, y \in C$. Since J contains no θ-curves, it follows from Theorem VI.52.IV.4 of [35] that J is hereditarily locally connected. Then $\overline{C}$ is a locally connected continuum and hence is also path-connected. Then by Corollary 5.4 in [22], $\overline{C}$ is locally path-connected. Let $\tilde{C}$ denote the filling of $\overline{C}$. Then $\tilde{C}$ is a locally path-connected continuum. Let A denote the regulated arc from x to y in $K(f)$. As $\tilde{C}\backslash T$ is a connected open subset of $\tilde{C}$, it follows that $\tilde{C}\backslash T$ is path-connected. Let B denote an arc in $\tilde{C}\backslash T$ from x to y. Then by Lemma 2.3.1, $A \cap T \subset (B \cap J) \cap T = \emptyset$, so $A \subset \tilde{C}\backslash T$. Thus no critical point of f is in A, so by Lemma 2.3.2, $f|_A$ is injective. Then $f(x) \neq f(y)$, so $f|_C$ is injective. $\square$

Remark 2.5.8. An argument similar to the one in the proof Lemma 2.5.7 can be used to show that, for every Fatou component F which doesn't contain any critical points, $f|_{\overline{F}}$ is a homeomorphism.

Lemma 2.5.9. $f(T) \subset T$.

Proof. As $H(f)$ and J are both forward invariant, it suffices to show that $f(\partial F) \subset T$ for every bounded Fatou component F, satisfying $F \cap H(f) \neq \emptyset$. Let F be any such Fatou component and let x, y be in the postcritical set of f such that $\overline{xy} \cap F \neq \emptyset$, where $\overline{xy}$ is the unique arc in $H(f)$ from x to y. Then $f(\overline{xy}) \subset H(f)$, so $f(F) \cap H(f) \neq \emptyset$. As F is a Fatou component, $f(\partial F) = \partial(f(F))$, so by the definition of T, $f(\partial F) \subset T$ as desired. $\square$

Lemma 2.5.10. If $F(f)$ has infinitely many connected components then so does $J\backslash T$.

Proof. Let F be a Fatou component that contains a critical point of f. As $T\backslash\partial F$ only meets finitely many components of $J\backslash\partial F$, it suffices to show that ∂F contains infinitely many separating points of J. Suppose not. Then there is a point $x \in \partial F$ and a neighborhood U of x in J such that no point of $U \cap \partial F$ is a separating point of J. As J is locally connected, we may assume that U is connected. Then $f^k(U) = J$ for sufficiently large k, but $f^k(\partial F) = \partial(f^k(F)) \subsetneq J$, so $U\backslash\partial F \neq \emptyset$. Then $U \cap J$ contains a simple triod S that has one endpoint, z, in $U\backslash\partial F$ and two endpoints in ∂F. Let y denote the branch point of this triod and let A denote the arc from z to

y in S. Then as $y \in U$ is not a separating point of J, there is an arc $B \subset J \backslash \{y\}$ from z to a point w of $\partial F \backslash \{y\}$, with $B \cap \partial F = \{w\}$. Then $A \cup B$ contains an arc C such that $C \cap \partial F = \{y, w\}$, a contradiction to Lemma 2.5.5. $\square$

Theorem 2.5.11. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a postcritically finite polynomial such that at least one finite critical point of f, c, is periodic. Then the following are equivalent:

(1) $\varprojlim f|_J$ is indecomposable

(2) $\varprojlim f|_J$ is a d-adic solenoid for some $d \geq 2$

(3) J is homeomorphic to S^1

(4) the Fatou set of f has exactly 2 components

(5) c is the only critical point of f and $f(c) = c$

(6) f is topologically conjugate to $g(z) = z^d$ for some $d \geq 2$.

Proof. ($6 \Rightarrow 3$) and ($3 \Rightarrow 4$) are well-known. We will show ($2 \Rightarrow 1 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 2$).

($2 \Rightarrow 1$) Well-known.

($1 \Rightarrow 4$) Suppose that $F(f)$ does not have exactly 2 components. As at least one finite critical point of f is in $F(f)$, $F(f)$ cannot be connected. It follows by Theorem IV.1.2 of [16], that $F(f)$ must have infinitely many components. Let U_1, U_2, U_3 be distinct components of $J \backslash T$ as guaranteed by Lemma 2.5.10 and let $x, y \in J$. Let $\overline{xy}$ be an arc between x and y in J. Then $\overline{xy} \backslash T$ consists of 0, 1, or 2 arcs, by Lemma 2.5.5. Proceeding by induction, define $\overline{xy}_n := f^n(\overline{xy})$ and suppose that $cl(\overline{xy}_n \backslash T)$ consists of at most two components, A and B, both of which are arcs. Then by Lemma 2.5.9, $f(\overline{xy}_n \cap T) \subset T$. $A \backslash T$ is a topological ray, so $f(A \backslash T)$ is as well, by Lemma 2.5.7. It follows that $f(A)$ is an arc with at least one endpoint in T, so by Lemma 2.5.5, $cl(f(A) \backslash T)$ is either empty or it is an arc. The same holds for $f(B)$. Thus, by induction, for every n, $\overline{xy}_n$ misses one of U_1, U_2, U_3. Therefore by Kuykendall's Theorem (2.2.7), $\varprojlim f|_J$ is decomposable.

($4 \Rightarrow 5$) Let F_c and F_∞ denote the components of the Fatou set containing c and ∞ respectively. Then it follows from our hypothesis and Corollary 4.12 of [43] that F_c and F_∞ are the basins of attraction of c and ∞ respectively. So if $c' \neq c$ were another a point of a periodic critical orbit, then it would be a superattracting fixed point under some iterate of f and hence would belong to a Fatou component distinct from F_c and F_∞, which is a contradiction. Thus we have that c is a fixed critical point and there are no other periodic critical points.

Now suppose that c' is a preperiodic critical point in J. Then at least one external ray lands at $f(c')$, so at least 2 external rays land at c' since f is at least 2-to-1 in a neighborhood of c' (see Lemma 18.1 in [43]). Then c' separates J (Lemma 17.5 of [43]), so $J \backslash \partial F_c \neq \emptyset$, a contradiction to $J = \partial F_c$ (see Corollary 4.12 of [43]).

Now suppose that c_1 is a preperiodic critical point in $F(f)$. Since $F(f)$ has only 2 components and $f^n(c_1) \not\to \infty$, it follows that c_1 is an element of F_c, which is the basin of attraction of c. Let m_1 denote the least positive integer such that $f^{m_1}(c_1) = c$. Let $[c, f^{m_1-1}(c_1)]$ denote the regulated arc between c and $f^{m-1}(c_1)$. Then $[c, f^{m_1-1}(c_1)]$ does not get mapped injectively by f, so it must contain a critical point c_2 in its interior. As with c_1, $f^{m_2}(c_2) = c$ for some least integer m_2. Proceeding by induction, for each $i \in \mathbb{N}$, let $c_{i+1} \in Int([c, f^{m_i-1}(c_i)])$ be a critical point of f and let m_{i+1} denote the least positive integer such that $f^{m_{i+1}}(c_{i+1}) = c$ so that for each $i \in \mathbb{N}$, we have $f^{m_i-1}([c, c_i]) \supset [c, c_{i+1}]$. As f only has finitely many critical points, there is a least integer k such that $c_k = c_i$ for some $i \neq k$. Let r denote the least integer greater than k such that $c_k = c_r$, and let $\mu = \Sigma_{i=k}^{r-1}(m_i - 1)$. Then $f^\mu([c, c_k]) \supset [c, c_r]$ and $f^\mu(c) = f^\mu(c_k) = c$ so, for some $x \in [c, c_k]$, $f^\mu(x) = c_k$. It follows that $[x, c_k]$ contains a fixed point of f^μ, call it y. Then $y \in J(f^\mu) = J(f)$, so $c_k \notin F_c$, a contradiction.

Thus f has a single finite critical point and that critical point is fixed.

(5 $\Rightarrow$ 6) The only such polynomials are of the form $f(z) = a(z-c)^d + c$, which are topologically conjugate to $g(z) = z^d$.

(6 $\Rightarrow$ 2) It is well known that $J(g) = \partial\mathbb{D}$. So $f|_J$ is topologically conjugate to a d-fold cover of S^1, and hence $\varprojlim f|_J$ is a d-adic solenoid. $\square$

2.6 Structure of Inverse Limits of PCF Polynomials

In this section we extend the definition of trunk to the entire collection of postcritically finite polynomials. The inverse limit of the trunk plays a role in inverse limits of Julia sets similar to that played by the core of a tent map in inverse limits of intervals (see [28]). We will show that $\varprojlim f|_J$ contains at least one indecomposable subcontinuum (Proposition 2.6.4 and Theorem 2.6.7) and that any indecomposable subcontinuum of $\varprojlim f|_J$ must intersect $\varprojlim f|_T$ (follows from Theorem 2.6.3).

Definition 2.6.1. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a postcritically finite polynomial. If f has at least one periodic critical point then the *trunk* of f was given by Definition 2.5.2. If all critical points of f are preperiodic we will now define the *trunk* of f by $T = H(f)$.

Lemma 2.6.2. $f^{-n}(T)$ meets only finitely many components of $J\backslash T$.

Proof. $f^{-n}(H(f))$ is a tree whose endpoints are among the orbits of $f^{-n}(\{f^m(c) : c \in \mathbb{C},\ m \in \mathbb{N}_0 \text{ and } f'(c) = 0\})$ (Corollary 2.2 of [46]). The result is clear if $T = H(f)$. Otherwise $f^{-n}(T) = f^{-n}((H(f) \cap J) \cup \bigcup\{\partial F : F \text{ is a Fatou component and } F \cap H(f) \neq \emptyset\}) = (f^{-n}(H(f)) \cap J) \cup \bigcup\{\partial F : F \text{ is a Fatou component and } F \cap f^{-n}(H(f)) \neq \emptyset\}$ and the result follows. $\square$

Theorem 2.6.3. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a postcritically finite polynomial with Julia set J with trunk T such that $J\backslash T \neq \emptyset$. Let $X = \varprojlim\{J, f|_J\}$ and let $\tilde{T} = \bigcap_{i\in\mathbb{N}} \pi_i^{-1}(T) = \varprojlim\{T, f|_T\}$. Then

(1) $X\backslash\tilde{T}$ is connected,

(2) $X\backslash\tilde{T}$ has $\mathfrak{c}$-many composants,

(3) the arc components of $X\backslash\tilde{T}$ are precisely its composants,

(4) $\mathfrak{c}$-many arc components of $X\backslash\tilde{T}$ are dense in $X\backslash\tilde{T}$ and

(5) infinitely-many arc components of $X\backslash\tilde{T}$ are not dense in $X\backslash\tilde{T}$.

Proof. (1) follows immediately from (4), so we only prove (2) - (5). Let C denote the set of components of $J\backslash T$.

Let $\Gamma = \{\eta \cap X : \eta \in C^{\mathbb{N}} \text{ and } \forall i \in \mathbb{N},\ f(\eta_{i+1}) \supset \eta_i\}$. Define $\hat{\sigma} : \Gamma \to \Gamma$ by $\hat{\sigma}(\eta \cap X) = (\eta_2 \times \eta_3 \times \eta_4 \times \cdots) \cap X$. For each $n \in \mathbb{N}$ and each $\gamma \in \Gamma$, define $A^n_\gamma = X \cap (f^{n-1}(\gamma_n) \times f^{n-2}(\gamma_n) \times \cdots \times f(\gamma_n) \times \gamma_n \times \gamma_{n+1} \times \cdots) = \{x \in X : x_i \in \gamma_i\ \forall i \geq n\}$.

For each $\gamma \in \Gamma$, $cl(\gamma_1) \simeq \varprojlim\ (f|_{cl(\gamma_{i+1})}, cl(\gamma_i)) = cl(\gamma)$, since each $f|_{cl(\gamma_{i+1})}$ is a homeomorphism. Then since T is forward-invariant, $\gamma \simeq \gamma_1$ which is an open, connected subset of the locally arc-connected space J, and hence is arc-connected. Then for each $\gamma \in \Gamma$, $U_\gamma := \bigcup_{n\in\mathbb{N}} A^n_\gamma$ is the nested union of arc-connected sets and hence is arc-connected. Note that $X\backslash\tilde{T} = \bigcup_{\gamma\in\Gamma} U_\gamma$. We will show that each U_γ is both a composant and an arc-component of $X\backslash\tilde{T}$.

Suppose $U_\gamma \neq U_\lambda$. It follows that $\gamma_i \neq \lambda_i$ for infinitely many i. Let $x \in U_\gamma$ and $y \in U_\lambda$, and choose $N \in \mathbb{N}$ large enough that $x \in A^N_\gamma$ and $y \in A^N_\lambda$. Suppose that $\overline{xy} \subset X$ is a continuum containing x and y. Then we have for infinitely many $i > N$, that $\gamma_i \neq \lambda_i$ and $\pi_i(\overline{xy})$ is a connected set that meets both γ_i and λ_i. For such i, $\pi_i(\overline{xy}) \cap T \neq \emptyset$. Then for every $i \in \mathbb{N}$, $\pi_i(\overline{xy}) \cap T \neq \emptyset$ since T is forward-invariant. For each $i \in \mathbb{N}$, let $z^i \in \overline{xy}$ such that $\pi_i(z^i) \in T$. Since T is forward-invariant, it follows that $\pi_k(z^i) \in T$ for all $k < i$. Let z^{i_j} be a subsequence that converges to some $z \in \overline{xy}$. Since T is closed, it follows that $z \in \tilde{T}$. Thus $\overline{xy} \cap \tilde{T} \neq \emptyset$, and hence U_γ and U_λ are in distinct composants of $X\backslash\tilde{T}$. So the arc components (and composants) of $X\backslash\tilde{T}$ are precisely the distinct U_γ's. This proves (3).

Now define an equivalence relation $\sim$ on Γ by $\gamma \sim \lambda$ iff $\gamma_i = \lambda_i$ cofinitely. Then as C is countable, for each $\gamma \in \Gamma$, the equivalence class $[\gamma]$ is countable. However, for each $\eta \in C$ there are least two distinct elements $\eta', \eta'' \in C$ such that $f(\eta'), f(\eta'') \supset \eta$, so $|\Gamma| = \mathfrak{c}$. Thus $|\{U_\gamma : \gamma \in \Gamma\}| = |\Gamma/\sim| = \mathfrak{c}$, which proves (2).

For each $n \in \mathbb{N}_0$ let $T_n = \{\eta \in C : f^n(\eta) \cap T = \emptyset$ and $f^{n+1}(\eta) \cap T \neq \emptyset\}$. By Lemma 2.6.2, T_n is finite for each n. As f is topologically exact, T_n must also be nonempty for each n. So for each $n \in \mathbb{N}$, let $\eta^n \in T_n$. Since $f^{n-1}(\eta^n) \in T_1$ for each n, there must be an $\eta_1 \in T_1$ such that $f^{n-1}(\eta^n) = \eta_1$ for infinitely many n. Then for every $n \in \mathbb{N}$, $\exists \lambda \in T_n$ such that $f^{n-1}(\lambda) = \eta_1$. By induction, for each $i > 1$ we may choose an $\eta_i \in T_i$ such that (a) $f(\eta_i) = \eta_{i-1}$ and (b) for all $n > i$, $\exists \lambda \in T_n$ such that $f^{n-i}(\lambda) = \eta_i$. Then $(\eta_1 \times \eta_2 \times \cdots) \cap X \in \Gamma$ and $f(\eta_n) \cap T = \emptyset$ for all $n \in \mathbb{N}$. Let $A = \{U_{\hat{\sigma}^i(\eta_1 \times \eta_2 \times \cdots)} : i \in \mathbb{N}\}$. Note that A is infinite.

Claim: U_λ is dense in $X \backslash T$ iff $f(\lambda_n) \cap T \neq \emptyset$ for infinitely many $n \in \mathbb{N}$.

If $f(\lambda_n) \cap T \neq \emptyset$ for only finitely-many n then choose N large enough that $f(\lambda_n) \cap T = \emptyset$ for all $n \geq N$. Then $\pi_N^{-1}(J \backslash (T \cup \lambda_N))$ is an open set that doesn't intersect U_γ.

On the other hand, if $f(\lambda_n) \cap T \neq \emptyset$ for infinitely many $n \in \mathbb{N}$, then for each such n, $\lambda_n \in T_0$. Since T_0 is finite and each element of T_0 is open in $J \backslash T$, there must be a $p \in \mathbb{N}$ such that $f^p(\mu) = J$ for each $\mu \in T_0$. Then for all $x \in X \backslash \tilde{T}$ and all $n \in \mathbb{N}$, there exists a $y \in U_\lambda$ such that $\pi_i(x) = \pi_i(y)$ for all $i < n$. Thus U_λ is dense.

By the claim, A is an infinite set of distinct components of $X \backslash \tilde{T}$ which are not dense. This proves (5).

Now let $\eta \in T_0$ and let λ, λ' be elements of C that meet $f^{-1}(\eta)$. Choose $p \in \mathbb{N}$ large enough that $f^p(\eta) = J$. Then for each $x \in 2^\omega$, there is a $\gamma \in \Gamma$ such that for all $n \in \mathbb{N}$, $\gamma_{n(p+1)} = \eta$ and $\gamma_{n(p+1)+1} = \begin{cases} \lambda & \text{if } x_n = 0 \\ \lambda' & \text{if } x_n = 1. \end{cases}$

This proves (4). $\square$

We now restrict our attention to the case where f has no periodic critical points and $J \backslash T \neq \emptyset$. Let $\mathcal{A}$ denote the set of vertices of T, that is, $\mathcal{A} = \{A \subset T : A$

is an arc and $A \cap V = \partial A\}$ where V is the union of the set of points in the orbits of the critical points of f and the set of branch points of $H(f)$.

Proposition 2.6.4. Let f be a postcritically finite polynomial with no periodic critical points. Then there exists a positive integer N such that $g(x) := f^N(x)$ satisfies $g(I) = g^2(I)$ for every $I \in \mathcal{A}$. Moreover, for every $I \in \mathcal{A}' = \{I \in \mathcal{A} : \forall I' \in \mathcal{A},\ g(I) \subset g(I')$ or $g(I) \cap g(I')$ is finite$\}$, the inverse limit $\varprojlim f|_{g(I)}$ is indecomposable.

Proof. For each $I \in \mathcal{A}$, let k_I and p_I denote the preperiod and period of I respectively; that is, the least positive integers, k_I and p_I such that $f^{k_I}(I) = f^{k_I+p_I}(I)$. Let N be any positive integer such that $LCM_{I \in \mathcal{A}}(p_I)$ divides N and $N \geq k_I,\ \forall I \in \mathcal{A}$. It follows that for all $I \in \mathcal{A}$, $g(I) := f^N(I) = g^2(I)$, as desired.

We will now show that $\varprojlim g|_{g(I)}$ is indecomposable. It follows from Theorem A of [46] that there are at least two distinct arcs, $I_1, I_2 \in \mathcal{A}' \cap 2^{g(I)}$. Then $g(I_1) \subset g(g(I)) = g(I)$, so $g(I_1) = g(I_2) = g(I)$, as $I \in \mathcal{A}'$.

Now suppose that $\varprojlim g|_{g(I)} = A \cup B$ where A, B are subcontinua of $\varprojlim g|_{g(I)}$. Let $n \in \mathbb{N}$. Then as $g(I)$ is uniquely arc-connected, one of $\pi_{n+1}(A), \pi_{n+1}(B)$ contains I_i for some $i \in \{1, 2\}$. Without loss of generality, suppose that $\pi_{n+1}(A) \supset I_1$. Then $\pi_n(A) \supset g(I_1) = g(I)$. As n was arbitrary, $\varprojlim g|_{g(I)}$ is indecomposable. □

Corollary 2.6.5. Let f and g be as in Proposition 2.6.4. Let $\sim$ denote the equivalence relation on $\mathcal{A}'$ given by $I_1 \sim I_2$ iff $g(I_1) = g(I_2)$. Then $\varprojlim f|_T$ contains at least $|\mathcal{A}'/\sim|$ distinct indecomposable subcontinua, each two of which meet at most at a single point.

We now consider the case where f has at least one periodic critical point.

Lemma 2.6.6. Let f be a PCF polynomial with at least one periodic critical point. Suppose that the Fatou set of f has at least one component F which (1) doesn't contain any point of any critical orbit and (2) satisfies $\partial F \subset T$. Then $\varprojlim f|_T$ is decomposable.

Proof. By Sullivan's Non-wandering Domain Theorem (Theorem 16.4 of [43]), every Fatou component is either periodic or preperiodic. Thus either every Fatou component F satisfying hypotheses (1) and (2) is periodic or we may assume that F is strictly preperiodic. By Proposition 1.6.6, $H(f) \cap \partial F$ is finite, so $\partial F \backslash H(f)$ is the union of disjoint open arcs, $A_1, \ldots, A_m$. As $\partial F \subset T$ and F doesn't contain any points of critical orbits, T must separate ∂F, so $m \geq 2$. Let G denote a Fatou component that contains a periodic critical point. Choose ϵ small enough so that
(i) $\forall w \in \partial G$, $\partial G \backslash \{w\}$ contains an ϵ-ball in T,
(ii) $\forall w \in \partial F$, $\partial F \backslash \{w\}$ contains an ϵ-ball in T,
(iii) $\forall i \leq m$, A_i contains an ϵ-ball in T and
(iv) if F is periodic with period p then $\forall n \leq p, \forall i \leq m$, $f^n(A_i)$ contains an ϵ-ball in T.

Let $\{a, b, c\}$ be any three points of T. By Kuykendall's Theorem (2.2.7), it suffices to show that there is a continuum $\overline{ab}$ between a and b such that for every $n \in \mathbb{N}$, $f^n(\overline{ab})$ misses an ϵ-ball in T.

If $a, b \in \partial F$ then we may choose an arc $\overline{ab} \subset \partial F$. Then if F is strictly preperiodic, for all $n \in \mathbb{N}$ we have that $|f^n(\overline{ab}) \cap \partial F| \leq 1$ by Lemma 2.5.1. On the other hand, if F is periodic then $|f^n(\overline{ab}) \cap \partial G| \leq 1$ by Lemma 2.5.1. In either case, for every $n \in \mathbb{N}$, $f^n(\overline{ab})$ misses an ϵ-ball in T as desired.

Now consider the case where $a \notin \partial F$. Then we may choose an arc $A \subset T$ containing a and b such that $A \cap A_l = \emptyset$ for some $l \leq m$ and such that for all Fatou components U with $A \cap \partial U \neq \emptyset$, we have $\partial(A \cap \partial U) \subset H(f)$.

If every Fatou component that satisfies (1) and (2) is periodic then let p denote the period of F so that $f^p(F) = F$. Then if T' denotes the filling of T, F is the only pre-image of itself under $f^p|_{T'}$, as otherwise there would be a preperiodic Fatou component that satisfies (1) and (2). By Remark 2.5.8, for each i, $f^i|_{\partial F}$ is a homeomorphism. Then $f^p|_{\partial F}$ permutes $\partial F \cap H(f)$ and so as it is a homeomorphism, it must permute the set $\{A_1, \ldots, A_m\}$. So for every $n \in \mathbb{N}$, $\exists k \leq m$ such that $f^{np}(A) \cap A_k = \emptyset$. Then for all $n \in \mathbb{N}$, $\exists k \leq m$ such that $f^n(A) \cap f^{n \bmod p}(A_k) = \emptyset$. Thus for every $n \in \mathbb{N}$, $f^n(\overline{ab})$ misses an ϵ-ball in T as desired.

It remains only to consider the case where F is strictly preperiodic. In this case, for each bounded Fatou component U with $\partial U \subset T$, we will define an arc A_U such that for all $n \in \mathbb{N}$, $f^n(A_U) \cap A_l = \emptyset$. If $f^n(A \cap \partial U) \cap A_l = \emptyset$ for all $n \in \mathbb{N}$ then let $A_U = A \cap \partial U$. Otherwise $f^N(U) = F$ for precisely one $N \in \mathbb{N}$ as F is strictly preperiodic. In this case, since F doesn't contain any points of critical orbits, neither does U, $f(U)$, $\ldots$, $f^N(U) = F$, so $f^N|_{\partial U}$ is a homeomorphism (Remark 2.5.8). Then $f^N(A \cap \partial U) \supset A_l$, so we define $A_U = cl(\partial U \backslash A)$. Thus in either case, we have for all $n \in \mathbb{N}$, that

$$f^n(A_U) \cap A_l = \emptyset. \tag{2.1}$$

Now let $\{U_1, U_2, \ldots\}$ enumerate $\{U \in \text{Fatou Components} : |A \cap \partial U| > 1\}$, which may or may not be finite. Let $A_1 = (A \backslash \partial U_1) \cup A_{U_1}$ and by induction define

$$A_{i+1} = \begin{cases} (A_i \backslash \partial U_{i+1}) \cup A_{U_i} & \text{if } i+1 < |\{U_1, U_2, \ldots\}| \\ A_i & \text{otherwise.} \end{cases}$$

Then $\overline{ab} = \lim A_i$ is a subcontinuum of T containing a and b. For any $x \in \overline{ab}$, we have that $x \in A_{U_i}$ for some i, and hence, by (2.1), for all $n \in \mathbb{N}$, $f^n(x) \notin A_l$. Thus $f^n(\overline{ab})$ misses an ϵ-ball as desired. $\square$

For each $x \in \hat{\mathbb{C}}$, let $\omega(x) = \overline{\{f^n(x) : n \in \mathbb{N}_0\}}$.

Theorem 2.6.7. Let f be a PCF polynomial with at least one periodic critical point and let $C = \bigcup \omega(c)$ where the union is taken over the set of periodic critical points. Then either:

(1) $\varprojlim f|_T$ is a solenoid, or

(2) $\varprojlim f|_T$ is a decomposable continuum containing at least $|C|$ distinct solenoids, no two of which meet at more than a single point.

Proof. Distinct points c_1, c_2 of C are superattracting fixed points of sufficiently high iterates of f and hence lie in distinct bounded Fatou components (see Cor 4.12 of [43]). It follows that $\varprojlim f|_T$ contains $|C|$ distinct solenoids, $S_1, \ldots S_{|C|}$, no two of which can meet at more than a single point. We now show that $\varprojlim f|_T$ is decomposable whenever it is not a solenoid.

If there is some bounded Fatou component F with $\partial F \subset T$ such that F does not contain any point of any critical orbit, then $\varprojlim f|_T$ is decomposable by Lemma 2.6.6. Otherwise every bounded Fatou component F with $\partial F \subset T$, contains a point of a critical orbit.

Let $T' = \bigcap_{n \in \mathbb{N}} f^n(T)$ so that $f(T') = T'$ and $\varprojlim f|_T = \varprojlim f|_{T'}$. Let $A_1, \ldots, A_m$ denote the closures of the components (if there are any) of $(J \cap H(f) \cap T') \setminus \bigcup \{\partial F : F$ is a bounded complementary domain of $T'\}$. Note that each A_i has interior in T'. Let $F_1, \ldots, F_M$ denote the bounded complementary domains of T', which are also Fatou components of f. As $f(H(f) \cap T) \subset H(f) \cap T$ and f has at least one periodic critical point, it follows that $M \geq 1$. If $m + M = 1$ then $\varprojlim f|_T$ is a solenoid by Theorem 2.5.11. Otherwise, $\mathcal{S} := \{A_1, \ldots, A_m, \partial F_1, \ldots \partial F_M\}$ has at least two elements.

Now, for each i, j, $|A_i \cap A_j| \leq 1$ by definition, and $|\partial F_i \cap \partial F_j| \leq 1$ by Lemma 2.5.1. By Theorem 1.6.6, $A_i \cap \partial F_j$ is finite, so by Lemma 2.5.1, $|A_i \cap \partial F_j| \leq 1$. Thus for all $D, D' \in \mathcal{S}$, if $D \neq D'$ then $|D \cap D'| \leq 1$. For each x in T', let $V(x)$ denote the number of elements of $\mathcal{S}$ that contain x. (This is the "valency" of x, in some sense.) For each A_i, $f(A_i)$ is a subset of $H(f)$, so for each F_j, we have that $f(A_i) \cap \partial F_j$ is finite. Since $f(A_i)$ is also a connected subset of J, $|f(A_i) \cap \partial F_j| \leq 1$. Then, since $\bigcup_{j \leq M} \partial F_j$ is fully invariant under $f|_{T'}$, it follows that, for each A_i, $f(A_i) \subset A_k$ for some A_k. Then as $\mathcal{S}$ is finite and $f|_{T'}$ is surjective, it follows that f permutes the elements of $\mathcal{S}$. It follows for every $x \in T'$, that $V(f(x)) \geq V(x)$ and hence that $V(f(x)) = V(x)$, since $\mathcal{S}$ is finite.

We will say that an element $S \in \mathcal{S}$ separates T' if $\cup(\mathcal{S}\backslash\{S\})$ is disconnected. Choose $L \in \mathcal{S}$ such that L does not separate T'. Then there is a sequence $\{L_i \in \mathcal{S}\}_{i\in\mathbb{N}}$ with $L_1 = L$ such that $f(L_{i+1}) = L_i$ for each $i \in \mathbb{N}$. It follows that $A = \varprojlim\{f, L_i\}$ is a continuum. Now if $D \in \mathcal{S}$ such that D separates T', then there are distinct points $x_1, x_2 \in D$ with $V(x_1), V(x_2) > 1$. Then $f(x_1), f(x_2) \in f(D)$ and $f(x_1) \neq f(x_2)$ since $V(x_1) = V(f(x_1))$, so $f(D)$ separates T'. Thus the pre-image of a non-separating element of $\mathcal{S}$ is a non-separating element of $\mathcal{S}$. Thus $K_i = \bigcup \mathcal{S}\backslash\{L_i\}$ is a continuum for each i. Since $f(K_{i+1}) = K_i$, it follows that $B = \varprojlim\{f, K_i\}$ is a continuum. Then $A \cup B$ is a decomposition of $\varprojlim f|_T$. □

Remark 2.6.8. In Theorem 2.6.7, even if $|C| = 1$, $\varprojlim f|_T$ need not be a solenoid: for the postcritically finite polynomial $f(z) = 4(-1-\sqrt{2})^{2/3}z^2 + 4(-1-\sqrt{2})^{1/3}z^3 + z^4$, it can be shown that $|C| = 1$ but $\varprojlim f|_T$ is still decomposable. (See Figure 2.1.)

Remark 2.6.9. In the proof of Theorem 2.6.7, a case was considered where $J \cap H(f)$ possibly had interior $H(f)$. However, we conjecture that $J \cap H(f)$ is nowhere dense in $H(f)$ for any postcritically finite polynomial with a periodic critical point.

CHAPTER THREE

Inverse Limits of Upper Semi-Continuous Functions

3.1 Motivation

For each $i \in \mathbb{N}$, let X_i be a compact Hausdorff space and let $f_i : X_{i+1} \to 2^{X_i}$ be a surjective, upper semicontinuous set-valued function so that $\varprojlim f_i$ is a compactum. We will define $f^i_j(x) = \pi_j(\pi_i^{-1}(x))$. Note that if $j < i$ then $f^i_j(x) = f_j \circ \cdots \circ f_{i-1}(x)$ and if $j > i$ then $f^i_j(x) = (f_{j-1})^{-1} \circ \cdots \circ (f_i)^{-1}(x)$. We will denote $\{(x_j, x_{j+1}) : \exists z \in \varprojlim f_i$ such that $z_j = x_j$ and $z_{j+1} = x_{j+1}\}$ by $\pi_{j,j+1}(\varprojlim f_i)$. If $U_i \subset X_i$ for each $i \leq n$, we will denote $\{x \in \varprojlim f_i : x_i \in U_i \ \forall i \leq n\}$ by $\overbrace{U_1 \times \cdots \times U_n}$. In general we will not distinguish between an upper semicontinuous function $f : X \to 2^Y$, and its graph $G(f) = \{(x, y) \in X \times Y : y \in f(x)\}$. Throughout this chapter, we will use the notation $|x|$ exclusively for cardinality. Following Ingram [31], if $f : X \to 2^Y$ is *single-valued* at every point, that is $|f(x)| = 1$ for all $x \in X$, then we will call f a *mapping.*

To motivate our study, consider the following result from the study of classical inverse limits.

Proposition 3.1.1. Suppose that for each $i \in \mathbb{N}$, X_i is a continuum and $f_i : X_{i+1} \to X_i$ is a continuous surjective mapping. If K is a proper closed subset of $\varprojlim f_i$, then $\pi_j(K) \neq X_j$ for cofinitely many $j \in \mathbb{N}$.

The above proposition provides an important tool for establishing the indecomposability of inverse limits. In particular, if it can be shown that all subcontinua H, K of $\varprojlim f_i$ such that $H \cup K = \varprojlim f_i$ satisfy $\pi_j(H) = X_j$ or $\pi_j(K) = X_j$ infinitely often, then it follows that $\varprojlim f$ is indecomposable.

In general, Proposition 3.1.1 does not hold if the bonding functions are allowed to be upper-semicontinuous set-valued functions, as the following example illustrates.

Example 3.1.2. [31] Let $f : [0,1] \to 2^{[0,1]}$ be given by $f(x) = [0,1]$ for all $x \in [0,1]$. Then $\varprojlim f = [0,1]^\omega$ and $K = \{(x, x, \ldots) \in [0,1]^\omega\}$ satisfies $\pi_i(K) = [0,1]$ for all $i \in \mathbb{N}$.

A more interesting example is provided in Example 3.3. In light of the above, we define the following.

Definition 3.1.3. (Compare [31]) Let D be either $\mathbb{N}$ or $\mathbb{Z}$. A set $K \subset \varprojlim\{X_i, f_i, D\}$ will be said to have *full projections in infinitely many coordinates* if $\pi_i(K) = X_i$ for infinitely many $i \in D$. We will say that $\varprojlim f_i$ has the *full projection property* if there is no proper closed subset of $\varprojlim f_i$ that has full projections in infinitely many coordinates.

Definition 3.1.4. (Compare [31]) Let D be either $\mathbb{N}$ or $\mathbb{Z}$. We will say that a connected inverse limit, $\varprojlim f_i$, has the *continuum full projection property* if there is no proper subcontinuum of $\varprojlim f_i$ that has full projections in infinitely many coordinates.

In [31], Ingram asked for conditions on upper semicontinuous functions that imply that $\varprojlim f_i$ has the full projection property and the continuum full projection property. He proved the following theorem, which makes use of the full projection property:

Theorem 3.1.5. [31] For each $i \in N$, let $n_i \in \mathbb{N}$, let T_i be a simple n_i-od (that is, a cone over a n_i-point discrete set) and let $f_i : T_{i+1} \to 2^{T_i}$ be upper semicontinuous such that there are disjoint open sets $U, V \subset T_{i+1}$ such that $f|_{U \cup V}$ is single-valued and $\overline{f_i(U)} = \overline{f_i(V)} = T_i$. If $\varprojlim f$ is connected and $\varprojlim f$ has the continuum full projection property then $\varprojlim f$ is indecomposable.

Varagona [55] showed that some upper semicontinuous functions from $[0,1]$ to $[0,1]$ whose graphs are $\sin(1/x)$ curves have indecomposable inverse limits. Along the way, he provided a condition for upper semicontinuous functions of the unit interval to have the full projection property:

Lemma 3.1.6. (Lemma 3.1 of [55]) Let $f : [0,1] \to 2^{[0,1]}$ be an upper semicontinuous function with the property that $\varprojlim f$ is a continuum. Suppose that, for some $A \subset [0,1]$, $f|_{[0,1]\setminus A}$ is a function, $f([0,1]\setminus A) = [0,1]$, and $P = \{(p_1, p_2, \ldots) \in \varprojlim f : p_i \notin A$ for all $i\}$ is a dense subset of $\varprojlim f$. Then whenever H is a proper closed subset of $\varprojlim f$, there exists some positive integer N such that if $n \geq N$, $\pi_n(H) \neq [0,1]$.

Cornelieus [18] gave a necessary and sufficient condition for an inverse limit to have the full projection property, but in order to apply his characterization, one needs detailed knowledge of the inverse limit. In the remainder of this chapter, we provide more conditions that are necessary and/or sufficient to imply the full projection property. In the next section, we provide necessary conditions, sufficient conditions and also necessary and sufficient conditions (Theorems 3.2.6 and 3.2.10) for inverse limits of upper semicontinuous functions of arbitrary compacta to have the full projection property. Then in Section 3, we restrict our attention to residual-preserving continuum-valued maps of graphs. We provide a necessary and sufficient condition for residual-preserving continuum-valued maps of trees to have the full projection property (Theorem 3.3.17). Our focus in both sections is on conditions that don't rely upon knowledge of the inverse limit, but depend only upon the dynamics of the bonding functions. We then conclude this chapter with some examples in Section 4.

3.2 Full Projections in Inverse Limits of Compact Hausdorff Spaces

For this section, we will make a standing assumption that all upper semicontinuous functions are surjective.

Definition 3.2.1. An inverse limit, $\varprojlim f_i$, will be said to have the *weak full projection property* if there is no proper closed subset K of $\varprojlim f_i$, such that $\pi_i(K) = X_i$ for all $i \in \mathbb{N}$.

Definition 3.2.2. Let $f : X \to 2^Y$ be upper semicontinuous. f is said to be *irreducible with respect to domain* if no closed subgraph of $G(f_i)$ has full domain, that is, $\pi_1(H) \neq X$ for every closed set $H \subsetneq G(f_i)$.

Lemma 3.2.3. $\varprojlim f_i$ has the full projection property iff $\varprojlim f_i$ has the weak full projection property and for each $i \in \mathbb{N}$, f_i is irreducible with respect to domain.

Proof. By contrapositive. Clearly if $\varprojlim f_i$ does not have the weak full projection property, then it does not have the full projection property. So suppose that for some $i \in \mathbb{N}$, there is an $H \subsetneq G(f_i)$, such that H is closed and the domain of H is X_{i+1}. Let $K = \pi_{i,i+1}^{-1}(H)$. Then $K_{i+1} = X_{i+1}$ and it follows that $K_j = X_j$ for all $j > i$. Then K is a proper closed subset of $\varprojlim f_i$ with full projections in infinitely many coordinates. Thus $\varprojlim f_i$ does not have the full projection property.

Now suppose that $\varprojlim f_i$ has the weak full projection property and for each $i \in \mathbb{N}$, f_i is irreducible with respect to domain. Suppose that $K \subset \varprojlim f_i$ has full projections in infinitely many coordinates. Let $n \in N$ such that $\pi_{n+1}(K) = X_{n+1}$. Then $\pi_{n+1,n}(K)$ is a closed subgraph of $G(f_n)$ with full domain, so $\pi_{n+1,n}(K) = G(f_n)$. Hence, $\pi_n(K) = X_n$. It follows inductively that $\pi_i(K) = X_i$ for all $i \in \mathbb{N}$. By hypothesis, $\varprojlim f_i$ has the weak full projection property, so it follows that $K = \varprojlim f_i$. Thus $\varprojlim f_i$ has the full projection property. $\square$

Lemma 3.2.4. $\varprojlim\{X_i, f_i, \mathbb{Z}\}$ has the full projection property iff $\varprojlim\{X_i, f_i, \mathbb{Z}\}$ has the weak full projection property, and for each $i \in \mathbb{N}$, no subgraph of $G(f_i)$ has full domain and no subgraph of $G(f_i)$ has full range.

Proof. Clearly if $\varprojlim\{X_i, f_i, \mathbb{Z}\}$ does not have the weak full projection property, then it does not have the full projection property. Suppose that for some $i \in \mathbb{Z}$, there is an $H \subsetneq G(f_i)$, such that H is closed and either the domain of H is X_{i+1} or the range of H is X_i. Let $K = \pi_{i,i+1}^{-1}(H)$. If the domain of H is X_{i+1} then $K_j = X_j$ for all $j > i$. Otherwise, the range of H is X_i and $K_j = X_j$ for all $j < i$. Either way, K is a proper closed subset of $\varprojlim f_i$ with full projections in infinitely many coordinates. Thus $\varprojlim\{X_i, f_i, \mathbb{Z}\}$ does not have the full projection property.

To prove the converse, suppose that $K \subset \varprojlim\{X_i, f_i, \mathbb{Z}\}$ has full projections in infinitely many coordinates. Let $n \in N$ such that $\pi_n(K) = X_n$. Then $\pi_{n+1,n}(K)$ is a closed subgraph of $G(f_n)$ with full range. Then by hypothesis, $\pi_{n+1}(K) = X_{n+1}$. By induction, it follows that $\pi_i(K) = X_i$ for all $i \geq n$. Similarly, $\pi_{n,n-1}(K)$ is a closed subgraph with full domain, so $\pi_{n-1}(K) = X_{n-1}$ and by a similar argument $\pi_i(K) = X_i$ for all $i \leq n$. By hypothesis, $\varprojlim\{X_i, f_i, \mathbb{Z}\}$ has the weak full projection property, so it follows that $K = \varprojlim\{X_i, f_i, \mathbb{Z}\}$. Thus $\varprojlim\{X_i, f_i, \mathbb{Z}\}$ has the full projection property. $\square$

Lemma 3.2.5. Suppose that $U = \overbrace{U_1 \times \cdots \times U_n}$ is a basic open set in $\varprojlim f_i$ such that $\pi_k(\varprojlim f_i \backslash U) \neq X_k$. Then there is a $j \leq n$ such that $\pi_j(\varprojlim f_i \backslash U) \neq X_j$.

Proof. If $k \leq n$ we are done, so suppose that $k > n$. Let $K = \varprojlim f_i \backslash U$. Let $z \in X_k \backslash K$. Then for all $i \leq n$, $f_i^k(z) \subset U_i$. Let $w \in f_n^k(z)$. Then for all $x \in \varprojlim f_i$ such that $x_n = w$, $x_i \in U_i$ for all $i \leq n$. Thus $\pi_n(\varprojlim f_i \backslash U) \subset X_n \backslash \{w\}$. $\square$

Theorem 3.2.6. $\varprojlim f_i$ has the weak full projection property iff for all $n \in \mathbb{N}$, for all $x \in \varprojlim f_i$ and for every basic open neighborhood $U_1 \times \cdots \times U_n$ of $(x_1, \ldots, x_n)$, there is a positive integer $i \leq n$ and a $y \in X_i$ such that $f_j^i(y) \subset U_j$ for all $j \leq n$.

Proof. Suppose that for all $n \in \mathbb{N}$, for all $x \in \varprojlim f_i$ and for every basic open neighborhood $U_1 \times \cdots \times U_n$ of $(x_1, \ldots, x_n)$, there is a positive integer $i \leq n$ and a $y \in X_i$ such that $f_j^i(y) \subset U_j$ for all $j \leq n$. Let $K \subsetneq \varprojlim f_i$ be closed. Then for some $n \in N$ and for each $i \leq n$ there is an open set $U_i \subset X_i$ such that $U = \overbrace{U_1 \times \cdots \times U_n} \neq \emptyset$ and $K \subset \varprojlim f_i \backslash U$. Then by hypothesis, there is an $i \leq n$ and a $z \in X_i$ such that for all $y \in \varprojlim f_i$ with $y_i = z$, we have that $y \in U$. Then $z \notin \pi_i(K)$. As K was arbitrary, $\varprojlim f_i$ has the weak full projection property.

Now suppose that $\varprojlim f_i$ has the weak full projection property. Let $x \in \varprojlim f_i$ and let $U = \overbrace{U_1 \times \cdots \times U_n}$ be a basic open neighborhood of x. Then $K = \varprojlim f_i \backslash U$ does not have full projections in all coordinates, so by Lemma 3.2.5, there is an $i \leq n$ such that $U_i \backslash \pi_i(K) \neq \emptyset$. Let $y \in U_i \backslash \pi_i(K)$. Then for all $z \in \varprojlim f_i$ such that $z_i = y$, $z_j \in U_j$ for all $j \leq n$. Thus $f_j^i(y) \subset U_j$ for all $j \leq n$, as desired. $\square$

Observe that Theorem 3.2.6 proves and generalizes Proprosition 3.1.1. We also have some novel corollaries.

Corollary 3.2.7. If $f : X \to X$ is a mapping, then $\varprojlim f^{-1}$ has the weak full projection property.

Proof. As f is a mapping, $\varprojlim f$ has the weak full projection property, and hence satisfies the conclusion of Theorem 3.2.6. Then $\varprojlim f^{-1}$ also satisfies the conclusion, and hence the hypothesis as well. $\square$

By $f|_{A \times B}$, we will denote the upper semicontinuous function whose graph is $G(f) \cap (A \times B)$. For iterates of f and closed sets $K \subset \varprojlim f_i$, we will use $f_j^i|_K(x)$ to denote $\pi_j((\pi_i|_K)^{-1}(x))$.

In [31], Ingram asked for conditions on bonding functions under which closed subsets of an inverse limit are the inverse limits of their projections. We provide conditions with the following two corollaries.

Corollary 3.2.8. Suppose $K \subset \varprojlim f_i$ is closed. If for every basic open set $U = \overbrace{U_1 \times \cdots \times U_n}$ that meets $\varprojlim f_i|_{K_i \times K_{i+1}}$, there is a $y \in K$ and an $i < n$ such that $f_j^i|_K(y_i) \subset U_j$ for all $j \leq n$, then $K = \varprojlim f_i|_{K_i \times K_{i+1}}$

Proof. By Theorem 3.2.6, $\varprojlim f_i|_{K_i \times K_{i+1}}$ has the weak full projection property. Then since $K \subset \varprojlim f_i|_{K_i \times K_{i+1}}$ has full projections in each coordinate, $K = \varprojlim f_i|_{K_i \times K_{i+1}}$. $\square$

Corollary 3.2.9. Every closed subset of $\varprojlim f_i$ is the inverse limit of its projections iff for every closed set $K \subset \varprojlim f_i$ and for every basic open set $U = \overbrace{U_1 \times \cdots \times U_n}$ that meets $\varprojlim f_i|_{K_i \times K_{i+1}}$, there is a $y \in K$ and an $i < n$ such that $f_j^i|_K(y_i) \subset U_j$ for all $j \leq n$.

Proof. If there is a closed subset K, for which the no such y exists, then by Theorem 3.2.6, there is a closed subset K' of K with the same projections as K in all coordinates. Then K' is not the inverse limit of its projections. The converse follows follows from Corollary 3.2.8. $\square$

The following is an altered version of Theorem 3.2.6 which we make use of later.

Theorem 3.2.10. For each $i \in \mathbb{N}$, let X_i be a compactum and let $f_i : X_{i+1} \to 2^{X_i}$ be upper semicontinuous. Then $\varprojlim f_i$ has the weak full projection property iff, for all $n \in \mathbb{N}$, for all $x \in \varprojlim f_i$ and for every basic open neighborhood $U_1 \times \cdots \times U_n$ of $(x_1, \ldots, x_n)$, there is a positive integer $i \leq n$ and a $y \in X_i$ such that $f_j^i(y) \subset \pi_j(\overbrace{U_1 \times \cdots \times U_n})$ for all $j \leq n$.

Proof. This follows immediately from Theorem 3.2.6 as $U_j \supset \pi_j(\overbrace{U_1 \times \cdots \times U_n})$ for each $j \leq n$.

Let $n \in \mathbb{N}$ and let $x \in \varprojlim f_i$. Let $U_1 \times \cdots \times U_n$ be a basic open neighborhood of $(x_1, \ldots, x_n)$. By Theorem 3.2.6, there is a positive integer $i \leq n$ and a $y \in X_i$ such that $f_j^i(y) \subset U_j$ for all $j \leq n$. It follows that for each $j \leq n$, $f_j^i(y) \subset \pi_j(\overbrace{U_1 \times \cdots \times U_n})$. $\square$

Theorem 3.2.11. Let $f : X \to 2^X$. Suppose that for each $x \in X$, and each open set V that meets $f^{-1}(x)$, there is an open set U containing x such that for every $z \in U$ there is a $w \in V$ such that $f(w) = \{z\}$. Then $\varprojlim f$ has the weak full projection property.

Proof. Let $x \in \varprojlim f_i$ and let $U_1 \times \cdots \times U_n$ be an open neighborhood of $(x_1, \ldots, x_n)$. Let $W_n = U_n$. Now suppose that $0 \leq m < n-1$ and for each $i \leq m$, W_{n-i} has been defined such that W_{n-i} is open, $x_{n-i} \in W_{n-i} \subset U_{n-i}$ and if $i \geq 1$ then, for all $z \in W_{n-i}$, there is a $w \in W_{n-i+1}$ such that $f(w) = \{z\}$. Then by hypothesis, there is an open set U containing x_{n-m-1} such that for all $z \in U$ there is a $w \in W_{n-m}$ such that $f(w) = \{z\}$. Then we may define $W_{n-m-1} = U \cap U_{n-m-1}$. Since $x_{n-m-1} \in U \cap U_{n-m-1}$, W_{n-m-1} is non-empty. By induction, we may define W_i for each $i \in \{1, \ldots, n\}$.

Now let $y_1 \in W_1$. Then there is a $y_2 \in W_2$ such that $f(y_2) = \{y_1\}$. Proceeding by induction, for each $1 < i \leq n$, by the construction of W_i, there is a $y_i \in W_i$ such that $f(y_i) = \{y_{i-1}\}$. Then for each $i < n$, $f^i(y_n) = \{y_{n-i}\} \subset W_i \subset U_i$. Therefore, by Theorem 3.2.6, $\varprojlim f$ has the weak full projection property. □

Proposition 3.2.12. $\varprojlim\{X, f, \mathbb{N}\}$ has the weak full projection property iff $\varprojlim\{X, f, \mathbb{Z}\}$ has the weak full projection property.

Proof. Suppose that $\varprojlim\{X, f, \mathbb{N}\}$ has the weak full projection property. Let $U = \overbrace{U_{-n} \times \cdots \times U_n}$ be a basic open set in $\varprojlim\{X, f, \mathbb{Z}\}$. Then $U' := \{(x_1, x_2, \ldots) \in \varprojlim\{X, f, \mathbb{N}\} : \exists(\ldots, y_{-1}, y_0, y_1, \ldots) \in \varprojlim\{X, f, \mathbb{Z}\}$ such that $x_i = y_{i-n-1}$ for all $1 \leq i \leq 2n+1\}$ is open in $\varprojlim\{X, f, \mathbb{N}\}$. Then by Lemma 3.2.5, $\varprojlim\{X, f, \mathbb{N}\} \backslash U'$ does not have full projection in some coordinate $i \leq n$. It follows that $\varprojlim\{X, f, \mathbb{Z}\} \backslash U$ does not have full projection in the $(i-n-1)^{st}$ coordinate. Thus $\varprojlim\{X, f, \mathbb{Z}\}$ has the weak full projection property.

Now suppose that $\varprojlim\{X, f, \mathbb{N}\}$ does not have the weak full projection property. Then there is a basic open set $U = \overbrace{U_1 \times \cdots \times U_n}$ of $\varprojlim\{X, f, \mathbb{N}\}$ and such that $\varprojlim\{X, f, \mathbb{N}\} \backslash U$ has full projections in all coordinates. Let $U' = \{(\ldots, x_{-1}, x_0, x_1, \ldots) \in \varprojlim\{X, f, \mathbb{Z}\} : (x_1, x_2, \ldots) \in U\}$. Then $\varprojlim\{X, f, \mathbb{Z}\} \backslash U'$ has full projections in all positive coordinates. Also, as $\pi_1(\varprojlim\{X, f, \mathbb{N}\} \backslash U) = X$, it follows, from the surjectivity of f, that $\varprojlim\{X, f, \mathbb{Z}\}\backslash U'$ has full projections in all non-positive coordinates. Thus $\varprojlim\{X, f, \mathbb{Z}\}$ does not have the weak full projection property. $\square$

Definition 3.2.13. Let X be a compact Hausdorff space. An upper semicontinuous function $f : X \to 2^X$ is said to be *re-expanding* if there is an open set $U \subset X$ such that $f^n(U) = \{x\}$ for some $x \in X$ and some $n \in N$ and $f^m(x)$ is non-degenerate for some $m \in \mathbb{N}$.

Proposition 3.2.14. Suppose that X is a compact Hausdorff space with no isolated points, and let $f : X \to 2^X$ be upper semicontinuous. If f is re-expanding, then $\varprojlim f$ does not have the weak full projection property.

Proof. Let U, n, m, x be as given in Definition 3.2.13. As X has no isolated points, let $U_{n+m+1} \subsetneq U$ be non-empty open. Let U_1 be any open set such that $f^m(x) \cap U_1 \neq \emptyset$ and $f^m(x)\backslash U_1 \neq \emptyset$. Consider the non-empty open set $W = \overbrace{U_1 \times X_2 \times \cdots \times X_{n+m} \times U_{n+m+1}}$, where $X_i := X$ for all i. Let $y \in W$. Then $y_{m+1} = x$, so, for all $i \geq m+1$, $f_1^i(y_i)$ meets $X\backslash U_1$, by definition of U_1. Similarly, for all $i \leq m$, $f_m^i(y_i) \supset \{x\}$, so $f_{n+m+1}^i(y_i) \cap (U\backslash U_{n+m+1}) \neq \emptyset$. Therefore, by Theorem 3.2.6, $\varprojlim f$ does not have the weak full projection property.

$\square$

3.3 *Continuum-Valued Maps of Graphs*

As in the previous section, we will assume that all upper semi-continuous functions are surjective.

Establishing necessary and sufficient conditions for an inverse limit of upper semi-continuous functions to be connected is currently an open problem (Problems 2.2 and 2.3 of [31]). One significant sufficient condition is the following.

Theorem 3.3.1. (Theorem [31]) Let X be a continuum and $f : X \to 2^X$ be upper-semicontinuous. If f is the union of continuum-valued upper semicontinuous functions, at least one of which intersects all the others and is surjective, then $\varprojlim f$ is connected.

However, if such an f is the union of more than one continuum-valued function, then it will not have the full projection property, since it will necessarily fail to be irreducible with respect to domain. Thus, for the purpose of constructing indecomposable continua using the full projection property, only the subclass of continuum-valued upper semicontinuous functions needs to be considered. In this section, we restrict our attention to continuum-valued maps of finite graphs. We will freely make use of the following two propositions.

Proposition 3.3.2. (Theorem 4 of [32]) Let $f : X \to 2^Y$ be upper semi-continuous and continuum-valued. Then $G(f)$ is connected.

Proposition 3.3.3. Let G be a finite graph and Y a compactum. Let $f : G \to 2^Y$ be upper semicontinuous and continuum-valued. Then for any connected set $C \subset G$, $f(C)$ is connected.

Proof. Let $C_n \subset C$ be a sequence of continua such that $C_i \subset C_{i+1}$ for each i, and $C = \bigcup_{n\in\mathbb{N}} C_n$. It follows from Proposition 3.3.2, that each $f(C_n)$ is connected, since $f|_{C_n}$ is an upper semicontinous, continuum-valued function. Then $f(C) = \bigcup_{n\in\mathbb{N}} f(C_n)$ is the union of connected sets containing a common point, and hence is connected. $\square$

Definition 3.3.4. An upper semicontinuous function $f : X \to 2^X$ is *residual-preserving*, if, for each open set $U \subset X$ and each residual set $R \subset U$, $f(R)$ is residual in $f(U)^\circ$.

Before examining residual preserving maps, we provide an example of a class of maps that are irreducible with respect to domain but not residual preserving.

Example 3.1. Let C denote the Cantor middle-thirds set. Let $\{A_n\}_{n\in\mathbb{N}}$ enumerate the maximal open intervals in $[0, 1]\backslash C$. For each $n \in N$, let $a_n \in [0, 1]$. Then $G = (C \times [0, 1]) \cup \bigcup_{n\in\mathbb{N}}(A_n \times \{a_n\})$ is the graph of an upper-semicontinuous function

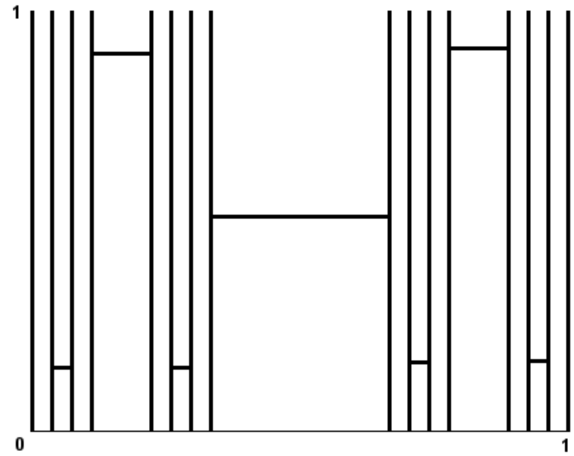

Figure 3.1: The graph of an upper semicontinuous function that is irreducible with respect to domain but is not residual preserving.

$f : [0,1] \to 2^{[0,1]}$. A graph of one such G is given in Figure 3.1. The a_n's may be chosen so that f is irreducible with respect to domain. Since $[0,1] \backslash C$ is dense and open, but $f([0,1]\backslash C) = \{a_n : n \in \mathbb{N}\}$ is countable, f is not residual preserving. The a_n's may be chosen so that f is re-expanding, in which case, by Theorem 3.2.14, $\varprojlim f$ does not have the full projection property. Alternatively, if the a_n's are chosen such that f is not re-expanding, it is not difficult to show that $\varprojlim f$ has the full projection property, using Theorem 3.2.6.

Lemma 3.3.5. Let G be a finite graph. Suppose that $f : G \to 2^G$ is continuum-valued such that $G(f)$ is irreducible with respect to domain. Then $S = \{x \in G : f(x)$ is single-valued $\}$ is residual.

Proof. Let d be the arc-length metric on G and let D be a countable dense subset of G that contains the endpoints of G. For each $n \in \mathbb{N}$, let $A_n = \{x \in G : diam(f(x)) \geq \frac{1}{n}\}$. Note that each A_n is closed. Suppose, for the sake of contradiction, that A_n has interior in G for some $n \in \mathbb{N}$. Let $U \subset A_n$ be open in G and for each $y \in G$ let $T_y = \{x \in U : y \in f(x)\}$. As f is upper semicontinuous, each T_y is closed in U. Since $U = \bigcup_{y \in D} T_y$, it follows, by the Baire Category Theorem, that there is a $q \in D$ such that T_q has interior in U. Let $A = \{y \in G : d(q,y) = 1/(2n)\}$. Since G is a graph, A is finite. Also, for each $x \in T_q$, $diam(f(x)) \geq \frac{1}{n}$ as $T_q \subset A_n$, so $x \in \bigcup_{a \in A} T_a$. Then $T_q^\circ \subset \bigcup_{a \in A} T_a$, so by the Baire Category Theorem, there is an $a \in A$ such that T_a has interior in T_q°. Let $V \subset T_q \cap T_a$ be open in G. Let $B_1, \ldots B_m$ enumerate the arcs in G that connect a and q. For each $v \in V$, we have that $q, a \in f(v)$, so $B_j \subset f(v)$ for some B_j. Then $V \subset \bigcup_{i \leq m} f^{-1}(B_i^\circ)$, so, by the Baire Category Theorem, there is a $k \leq n$ such that $f^{-1}(B_k^\circ)$ has interior in V. Let $W \subset f^{-1}(B_k^\circ) \cap V$ be non-empty open in G. Then $G(f) \backslash (W \times B_k^\circ)$ is a proper closed subset of $G(f)$ with full domain, which contradicts our hypothesis. Thus each A_n is nowhere dense.

Then by the Baire Category Theorem, $\bigcup_{n \in \mathbb{N}} A_n$ is nowhere dense in G. Since $G = S \cup (\bigcup_{n \in \mathbb{N}} A_n)$, it follows that S is residual. □

Corollary 3.3.6. Suppose that $f : G \to 2^G$ is continuum-valued such that $G(f)$ is irreducible with respect to domain. Then for every open neighborhood U of every point $x \in G$, and for every open neighborhood V of every $y \in f(x)$, there is a $u \in U$ and a $v \in V$ such that $f(u) = \{v\}$.

Proof. Let $K = \{(x,y) \in G(f) : f(x) = \{y\}\}$. Then by Lemma 3.3.5, $\pi_1(K)$ is dense in G. Then $\pi_1(\overline{K}) = G$, so by hypothesis, $\overline{K} = G(f)$. The result follows. □

Definition 3.3.7. If $f : X \to 2^X$ is upper semicontinuous such that $f^{-1}(x)$ is totally disconnected for each $x \in X$, then f is said to be a *light map*.

Note that for continuum-valued maps on graphs, $f : G \to 2^G$ is light iff for every open set $U \subset G$, $f(U)$ is not a singleton. We will provide a characterization of when inverse limits of residual-preserving maps on trees have the full projection property, but first we provide a simpler sufficient condition for maps on graphs that is often easier to verify.

Theorem 3.3.8. Let G be a finite graph and $f : G \to 2^G$ be a light, residual-preserving, continuum-valued, upper-semicontinuous map such that for every $x \in G$, every $y \in f(x)$, every connected open neighborhood U of x, and every component C of $U \backslash \{x\}$, $y \in \overline{f(C)}$. Then $\varprojlim f$ has the full projection property iff f is irreducible with respect to domain.

Proof. If f is not irreducible with respect to domain, then by Lemma 3.2.3, f does not have the full projection property.

Now suppose that f is irreducible with respect to domain. By Lemma 3.2.3, it suffices to show that f has the weak full projection property.

Let $U = \overbrace{U_1 \times \cdots \times U_n}$ be a non-empty open set and let $x \in U$. Let C_n be a component of $U_n \backslash \{x_n\}$. Now suppose that C_{i+1} has been defined, such that C_{i+1} is a component of $V \backslash \{x_{i+1}\}$ for some connected open set $V \subset U_{i+1}$. Then as f is light and continuum-valued, $f(C_i)$ is a non-degenerate continuum, and hence has interior in G. Moreover, since $x_i \in \overline{f(C_{i+1})}$ by hypothesis, it follows that $f(C_i) \cap U_i$ has interior in U_i. Then there is a connected open set V_i containing x_i, such that $f(C_{i+1}) \cap U_i$ contains a component, C_i of $V_i \backslash \{x_i\}$. Then by induction, we have, for each $i < n$, an open set $C_i \subset U_i$, such that $f(C_{i+1}) \supset C_i$.

Let $S_n = C_n \cap S$, where $S = \{x \in G : f(x) \text{ is single-valued }\}$. It follows from Lemma 3.3.5 that S_n is residual in C_n. By induction, for each $i < n$, let $S_i = f(C_{i+1} \cap S)$. Since each C_i is open and f is residual-preserving, it follows that S_i is residual in C_i for each $i < n$.

Now let $y_1 \in S_1$. Then by induction, for each $i < n$, there is a $y_{i+1} \in S_{i+1}$ such that $f(y_{i+1}) = \{y_i\}$, since $S_i \subset S$. Then $y_n \in U_n$ and for all $i < n$, $f^i(y_n) = \{y_{n-i}\} \subset U_{n-i}$. Therefore by Theorem 3.2.6, $\varprojlim f$ has the weak full projection property. □

Lemma 3.3.9. Let G be a finite graph and let $f : G \to G$ be a continuum-valued map that is irreducible with respect to domain. Then for all $x \in G$, $\{y \in f(x) : f^{-1}(y) = \{x\}\}$ is finite.

Proof. Let $U \subset G$ be an open neighborhood of x that is small enough that $U \backslash \{x\}$ consists of disjoint open arcs $A_1, \ldots, A_n$, such that $\overline{A_i}$ is an arc, for each $i \leq n$.

Then, for each $i \leq n$, $f(A_i)$ is connected, as f is continuum-valued. Moreover, as f is irreducible with respect to domain, $(\bigcup_{i \leq n} f(A_i)) \cap f(x)$ is dense in $f(x)$. Then $\bigcup_{i \leq n} \overline{f(A_i)} = \overline{\bigcup_{i \leq n} f(A_i)} \supset f(x)$. But for each $i \leq n$, $\overline{f(A_i)} \backslash f(A_i)$ is finite, since G is a graph and $f(A_i)$ is connected. Thus $f(x) \backslash f(U \backslash \{x\})$ is finite. □

Lemma 3.3.10. Let G be a finite graph and let $f : G \to G$ be a continuum-valued map that is irreducible with respect to domain. If there are open sets $U_1, \ldots U_n \subset G$ such that $\pi_n(\overbrace{U_1 \times \cdots \times U_n}) = \{y\}$ for some $y \in G$ then $\varprojlim f$ does not have the weak full projection property.

Proof. Let $U = \overbrace{U_1 \times \cdots \times U_n}$ and let $x \in U$ so that $x_n = y$. If, for every $i \leq n$, $\pi_i(U)$ is finite, then, for every $i \leq n$, $\pi_i(\varprojlim f \backslash U) \supset \overline{G \backslash \pi_i(U)} = G$, so $\pi_i(\varprojlim f \backslash U) = G$ for every $i \leq n$. Then by Lemma 3.2.5, we are done.

So suppose that there is an $i < n$ such that $\pi_i(U)$ is infinite, and let j denote the maximal such i. We may assume for each $i \in \{j+1, \ldots, n\}$, that $\pi_i(U) = \{x_i\}$, as otherwise, we can redefine U_i so that it does not contain the finitely many elements of $\pi_i(U)$ other than x_i.

Let $Z = \{z \in \pi_j(U) : |f^{-1}(z)| = 1\}$. By Lemma 3.3.9, Z is finite, since $\pi_j(U) \subset f(x_{j+1})$. Thus $\pi_j(U) \backslash Z \neq \emptyset$. Redefine $U_j := U_j \backslash Z$, so that U is still open, but now $|f^{-1}(z)| > 1$ for all $z \in \pi_j(U)$ and hence $f^{-1}(z) \backslash U_{j+1} \neq \emptyset$, for all $z \in \pi_j(U)$. Let $w \in U$. Then, for all $i \leq j$, either $f^i_j(w_i) \subset \pi_j(U)$, in which case $f^i_{j+1}(w_i) \backslash \pi_{j+1}(U) \neq \emptyset$, or $f^i_j(w_i) \backslash \pi_j(U) \neq \emptyset$. Moreover, for all $i \in \{j+1, \ldots, n\}$, $\pi_i(\varprojlim f \backslash U) = G$, in particular, $w_i \in \pi_i(\varprojlim f \backslash U)$, so there is a $k \leq n$ such that $f^i_k(w_i) \backslash U_i \neq \emptyset$. Therefore, by Theorem 3.2.10, $\varprojlim f$ does not have the weak full projection property. □

Definition 3.3.11. For an upper semicontinuous function $f : X \to 2^X$ we define $f^{*n}(x) = \{(w_1, \dots, w_n) \in X^n : w_n \in f(x) \text{ and } w_i \in f(w_{i+1})\ \forall i < n\}$.

Definition 3.3.12. Let G be a finite graph and $f : G \to 2^G$ be upper semicontinuous. Let $x \in X$, $(w_1, \dots, w_{n-1}) \in f^{*(n-1)}(x)$ and for each $i \leq n$ let $U_i \subset G$ be connected and open such $U_1 \times \ldots \times U_n$ is a neighborhood of $(w_1, \dots, w_{n-1}, x)$. For each component C of $U_n \backslash \{x\}$, the *trail of C with respect to* $U_1 \times \cdots \times U_n$ is the collection $A_1(C), \dots, A_n(C)$ where $A_n(C) = C$ and $A_i(C) = f(A_{i+1}(C)) \cap U_i$ for each $i < n$.

Remark 3.3.13. Note that, in Definition 3.3.12, if G is a tree, then each $A_i(C)$ is connected.

Definition 3.3.14. Let G be a finite graph and let $f : G \to 2^G$ be an upper semicontinuous function. For an $x \in G$, an $n \in \mathbb{N}$, a $(w_1, w_2, \dots w_{n-1}) \in f^{*(n-1)}(x)$, and an open set $U_1 \times \cdots \times U_n$ containing $(w_1, \dots, w_{n-1}, x)$, we will say that a component C of $U_n \backslash \{x\}$ *shadows* $(w_1, \dots, w_{n-1}, x)$, if $w_i \in \overline{A_i(C)}$ for each $i < n$ and $w_i \in \overline{f(A_{i+1}(C)^\circ)}$ for each $i < n$ such that $A_{i+1}(C)^\circ \neq \emptyset$.

Definition 3.3.15. Let G be a finite graph. An upper semicontinuous function $f : G \to 2^G$ will be said to have the *component shadowing property* if for every $x \in X$, every $n \in \mathbb{N}$, every $(w_1, w_2, \dots w_{n-1}) \in f^{*(n-1)}(x)$, and every open set $U_1 \times \cdots \times U_n$ containing $(w_1, \dots, w_{n-1}, x)$ such that each U_i is connected, there is a component C of $U_n \backslash \{x\}$ that shadows $(w_1, \dots, w_{n-1}, x)$.

Examples of functions which fail to satisfy for the component shadowing property are given in Figure 3.2.

Lemma 3.3.16. Let G be a finite graph and let $f : G \to 2^G$ be an upper semicontinuous function with the component shadowing property. Let $x \in G$ and let $n \in \mathbb{N}$. Then for any point $(w_1, \dots, w_{n-1}) \in f^{*(n-1)}(x)$, there is an open set $V_1 \times \cdots \times V_n$ and a component C of $V_n \backslash \{x\}$ such that C shadows $(w_1, \dots, w_{n-1}) \in f^{*(n-1)}(x)$ with

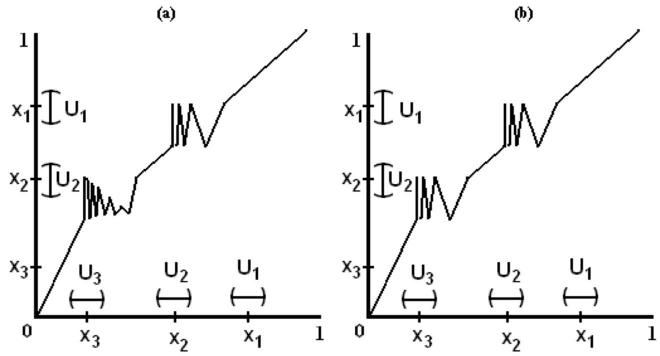

Figure 3.2: Two upper semicontinuous functions that don't have the component shadowing property. The function in graph (a) fails to satisfy $x_i \in \overline{A_i(C)}$ for each $i < 3$ and each component C of $U_3 \backslash \{x_3\}$. The function in graph (b) satisfies that condition for a component C of $U_3 \backslash \{x_3\}$, but fails to satisfy $x_i \in \overline{f(A_{i+1}(C)^\circ)}$ for each $i < 3$ such that $A_{i+1}(C)^\circ \neq \emptyset$.

respect to any open set $W_1 \times \cdots \times W_n$ such that $W_n = V_n$ and W_i is a connected open subset of V_i for each $i < n$.

Proof. Let $\mathcal{C}_n$ denote the collection of components of $U_n \backslash \{x\}$ that shadow $(w_1, \ldots, w_{n-1}) \in f^{*(n-1)}(x)$ with respect to $U_1 \times \cdots \times U_n$. Let $V_n = U_n$. Now suppose that $V_n, \ldots, V_{i+1}$ have been defined. If there exists an open set $D \subset U_i$ such that $A_{i+1}(C)^\circ = \emptyset$ with respect to $U_1 \times \cdots \times U_{i-1} \times D \times V_{i+1} \times \cdots \times V_n$, then let $V_i = D$, for some such D. Otherwise, let $V_i = U_i$.

Let $\{U_{n-1}^j : j \in \mathbb{N}\}$ enumerate a countable local basis at w_{n-1} such that each U_{n-1}^j is a connected open subset of V_{n-1} with $U_{n-1}^{j+1} \subset U_{n-1}^j$. As $\mathcal{C}_n$ is finite, there exists at least one component $C \in \mathcal{C}_n$ such that C shadows $(w_1, \ldots, w_{n-1}) \in f^{*(n-1)}(x)$ with respect to $V_1 \times \cdots \times V_{n-2} \times U_{n-1}^j \times V_n$ for infinitely many j. Let $\mathcal{C}_{n-1} = \{C \in \mathcal{C}_n : C \text{ shadows } V_1 \times \cdots U_{n-1}^j \times V_n \text{ for infinitely many } j\}$. By induction, for each $i < n$, define $\mathcal{C}_i = \{C \in \mathcal{C}_{i+1} : C \text{ shadows } V_1 \times \cdots U_i^j \times V_{i+1} \times \cdots \times V_n$ for infinitely many $j\}$.

Let $C \in \mathcal{C}_1$. For each $i \leq n$, let $W_i \subset V_i$ be a connected open set, with $w_i \in W_i$. Then as $C \in \mathcal{C}_1$, for each $i < n$ there is an open set $V_i' \subset W_i$ such that C

shadows $(w_1, \dots, w_{n-1}, x)$ with respect to $V_1' \times \cdots \times V_{n-1}' \times U_n$. Hence, C shadows $W_1 \times \cdots \times W_{n-1} \times U_n$.

$\square$

Theorem 3.3.17. Let T be a finite tree and let $f : T \to T$ be a continuum-valued, residual-preserving map. Then $\varprojlim f$ has the full projection property iff f has the component shadowing property, is irreducible with respect to domain, and is not re-expanding.

Proof. Suppose that f has the component shadowing property, is irreducible with respect to domain, and is not re-expanding. Let $n \in \mathbb{N}$ and let $x \in \varprojlim f$. Let U_n be a connect open neighborhood of x_n. Let $U_1 \times \cdots \times U_n$ denote a neighborhood of $(x_1, \dots, x_n)$ of the type that is guaranteed by Lemma 3.3.16, and let C denote the corresponding component of $U_n \backslash \{x_n\}$.

Let $S_n = C \cap S$, where $S = \{y \in X : f(y) \text{ is single-valued}\}$. By Lemma 3.3.5, S is residual in T, so S_n is residual in $C^\circ = C = A_n$. Let $V_n = U_n$. We proceed by induction.

Suppose that $V_{i+1} \subset U_{i_1}$ is a connected open neighborhood of x_i and S_{i+1} is a residual set in $A_{i+1}(C)$ with respect to $U_1 \times \cdots \times U_i \times V_{i+1} \times \cdots \times V_n$, and suppose that $A_{i+2}(C)$ is a non-degenerate connected set. If $A_{i+1}(C)^\circ = \emptyset$, then $A_{i+1}(C) = \{x_{i+1}\}$ and $i + 1 < n$ since C is a non-degenerate connected set. Then for any $y_{i+2} \in S_{i+2}$, $f^j(y_{i+2}) = \{x_{i+2-j}\}$, for each $j \in \{1, \dots, i+1\}$, as f is not re-expanding. Similarly, for each $n \geq k > i + 1$, there is a $y_k \in S_k$ such that $f(y_k) = \{y_{k-1}\}$. Then by Theorem 3.2.6 and Lemma 3.2.3, $\varprojlim f$ has the full projection property.

So we may suppose that $A_{i+1}(C)^\circ \neq \emptyset$. Then as T is a tree, $A_{i+1}(C)^\circ$ has only finitely many components. Thus, by the component shadowing property, there is a component B of $A_{i+1}(C)^\circ$ such that either

(1) $f(B) = \{x_i\}$, or

(2) $f(B)$ is a non-degenerate connected set and $x \in \overline{f(B)}$.

In the former case, let $y_{i+1} \in B \cap S_{i+1}$ and proceed as before. In the latter case, as B is open and $f(B)$ has interior in T, $f(S_{i+1} \cap B)$ is residual in $f(B)^{\circ}$, since f is residual-preserving. Then, since $f(B)$ is connected, $f(S_{i+1})$ is residual in $f(B)$. Let $V_i \subset U_i$ be a connected open neighborhood of x_i such that $f(B)$ contains a component of $U_i \backslash \{x_i\}$. Then $f(S_{i+1} \cap B)$ is residual in $f(B)$, so $S_i := f(S_{i+1}) \cap S \cap V_i$ is residual in $A_i(C)$ for the open set $U_1 \times \cdots \times U_{i-1} \times V_i \times \cdots \times V_n$.

Proceeding by induction, S_1 is residual in $A_1(C)$ for the open set $V_1 \times \cdots \times V_{n-1} \times U_n$. Define $V_n = U_n$. By induction, choose for each $i < n$, a $y_{i+1} \in S_{i+1} \cap V_{i+1}$ such that $f(y_{i+1}) = \{y_i\}$. Then for all $j < n$, $f^j(y_n) = \{y_{n-j}\} \subset V_{n-j} \subset U_{n-j}$, so by Theorem 3.2.6 and Lemma 3.2.3, $\varprojlim f$ has the full projection property.

To prove the converse, consider its contrapositive. In light of Lemma 3.2.3 and Proposition 3.2.14, suppose that $\varprojlim f$ is irreducible with respect to domain, is not re-expanding, and does not have the component shadowing property. We consider two separate cases.

First suppose that there is an $x \in T$, an $n \in \mathbb{N}$, a $(w_1, w_2, \dots w_{n-1}) \in f^{*(n-1)}(x)$, and an open set $U_1 \times \cdots \times U_n$ containing $(w_1, \dots, w_{n-1}, x)$, such that each U_i is connected and for each component C of $U_n \backslash \{x\}$ there is an $i < n$ such that $w_i \notin \overline{A_i(C)}$. Let $V_n = U_n$. For each $i < n$, let $\mathcal{C}_i$ denote the (possibly empty) set of components C of $U_n \backslash \{x\}$ such that $w_i \notin \overline{A_i(C)}$. For each $i < n$, let V_i denote the component of $U_i \backslash \overline{\bigcup_{C \in \mathcal{C}_i} A_i(C)}$ that contains w_i, so that $V := \overbrace{V_1 \times \cdots \times V_n}$ is non-empty. Then, for each $z \in U_n \backslash \{x\}$, there is an $i < n$ such that $z \notin V_i$, so $\pi_n(V) = \{x\}$. Thus, by Lemma 3.3.10, $\varprojlim f$ does not have the weak full projection property.

It remains only to consider the case where there is an $x \in X$, $n \in \mathbb{N}$, $(w_1, w_2, \dots w_{n-1}) \in f^{*(n-1)}(x)$, an open set $U_1 \times \cdots \times U_n$ containing $(w_1, \dots, w_{n-1}, x)$ such that each U_i is connected, and a non-empty collection $\mathcal{C}$ of components of $U_n \backslash \{x\}$ such that for each $C \in \mathcal{C}$, $w_i \in \overline{A_i(C)}$ for each $i < n$ and there is an $i_C < n$, such that $w_{i_C} \notin \overline{f(A_{i_C+1}(C)^\circ)}$ and $A_{i_C+1}(C)^\circ \neq \emptyset$. In this case, $w_{i_C} \in \overline{f(A_{i_C+1}(C))} \backslash \overline{f(A_{i_C+1}(C)^\circ)}$. Let V_{i_C} be a connected open subset of $U_{i_C} \backslash \overline{f(A_{i_C+1}(C)^\circ)}$ that contains w_{i_C}. Then for all $y \in U_1 \times \cdots \times V_{i_C} \times \cdots U_{n-1} \times C$, $y_{i_c} \notin \overline{f(A_{i_C+1}(C)^\circ)}$, but $y_{i_C} \in \overline{f(A_{i_C+1}(C))}$. By the upper semicontinuity of f, $\overline{f(A_{i_C+1}(C))} \subset f(\overline{A_{i_C+1}(C)})$, so $y_{i_c+1} \in \partial A_{i_C+1}(C)$. Thus $\pi_{i+1}(U_1 \times \cdots \times V_{i_C} \times \cdots U_{n-1} \times C) \subset \partial A_{i_C+1}(C)$ is finite.

If, for each $C \in \mathcal{C}$, $U_1 \times \cdots \times V_{i_C} \times \cdots U_{n-1} \times C$ is empty, then let $W_i = U_i \cap \bigcap \{V_{i_C} : C \in \mathcal{C} \text{ and } i_C = i\}$ for each $i < n$. Then for all $C \in \mathcal{C}$, $W_1 \times \cdots \times W_{n-1} \times C = \emptyset$. By an argument similar to the previous case, for each component C of $U_n \backslash \{x\}$ such that $C \notin \mathcal{C}$, $W_1 \times \cdots \times W_{n-1} \times C$ is also empty. Thus $\pi_n(W_1 \times \cdots \times W_{n-1} \times U_n) = \{x\}$. It follows by Lemma 3.3.10, that $\varprojlim f$ does not have the weak full projection property.

Similarly, if there is a $C \in \mathcal{C}$, such that $\pi_n(U_1 \times \cdots \times V_{i_C} \times \cdots U_{n-1} \times C)$ is non-empty, but finite, then, by Lemma 3.3.10, $\varprojlim f$ does not have the weak full projection property.

Otherwise, let $C \in \mathcal{C}$ and let $D \subset U_{i_C+1}$ be connected such that $\pi_{i_C+1}(U_1 \times \cdots \times V_{i_C} \times D \times \cdots \times U_n) = \{y_C\}$ for some $y_C \in V_{i_C}$. Then $f(y_C) \cap (T \backslash \overline{f(A_{i_C+1}(C)^\circ)} \neq \emptyset$. However, $A_{i_C+1}(C)^\circ \cup \{y_C\}$ is connected, so $f(A_{i_C+1}(C)^\circ \cup \{y_C\})$ is connected. Hence $f(y_C)$ meets $\overline{f(A_{i_C+1}(C)^\circ)}$. Then for all $z \in \overbrace{U_1 \times \cdots \times V_{i_C} \times D \times \cdots U_n}$, $z_{i_C+1} = y_C$. So $f(z_{i_C+1}) \cap (T \backslash V_{i_C}) \neq \emptyset$, and $|f_n^{i_C+1}(z_{i_C+1})| > 1$. Therefore, by Theorem 3.2.6, $\varprojlim f$ does not have the weak full projection property.

□

3.4 Examples

In this section, we construct examples that illustrate results on the full projection property from the previous sections in this chapter.

Example 3.4.1. Let C denote the Cantor middle-thirds set and let $\mathcal{I}$ denote the set of components of $[0,1]\backslash C$. For each $I=(a,b)\in\mathcal{I}$, let A_I denote the line segment in $[0,1]\times[0,1]$ from $(a,0)$ to $(b,1)$. Let f denote the upper semicontinuous function whose the graph is $G(f)=(C\times[0,1])\cup\bigcup_{I\in\mathcal{I}}A_I$. The graph of f is given in Figure 3.3.

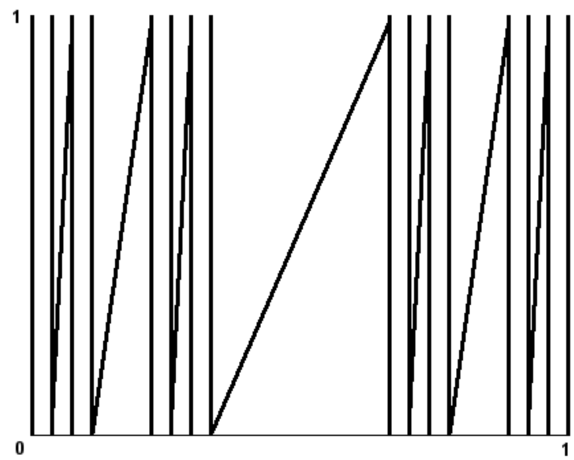

Figure 3.3. The function described in Example 3.4.1.

Since $S=\{x\in[0,1]:|f(x)|=1\}=[0,1]\backslash C$ is dense in $[0,1]$ and $G(f|_S)$ is dense in $G(f)$, it follows that f is irreducible with respect domain. It is not difficult to show that f is residual preserving. Note that for every point x at which f is multi-valued and for every connected open set $U\subset[0,1]$ that contains x, there is a component D of $U\backslash\{x\}$ such that $f(D\cap S)=[0,1]$. From this observation, it is not difficult to show that f has the component shadowing property. Hence, by Theorem 3.3.17, $\varprojlim f$ has the full projection property. Furthermore by Theorem 3.1.5, $\varprojlim f$ is indecomposable. We can generalize this example by replacing each of the line segments A_I with arbitrary single-valued functions of the form $g:I\to[0,1]$ such that $\{g(a),g(b)\}=\{0,1\}$.

Example 3.4.2. For each $n\in\mathbb{N}$, let $A_{2n}\subset[0,1]\times[0,1]$ denote the line segment from $(\frac{2^{2n-1}+1}{2^{2n}},\frac{1}{2})$ to $(\frac{2^{2n}+1}{2^{2n+1}},\frac{1}{4})$, and let $A_{2n+1}\subset[0,1]\times[0,1]$ denote the line segment

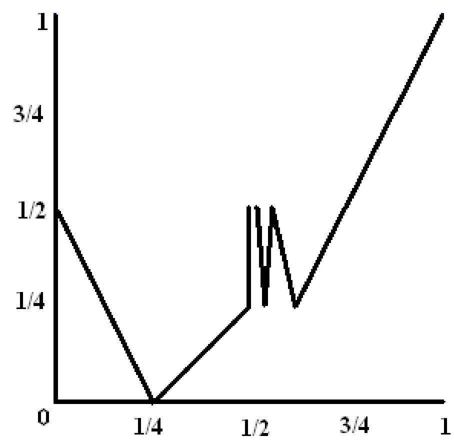

Figure 3.4. The function described in Example 3.4.2.

from $(\frac{2^{2n}+1}{2^{2n+1}}, \frac{1}{4})$ to $(\frac{2^{2n+1}+1}{2^{2n+2}}, \frac{1}{2})$, so that $S = (\frac{1}{2} \times [\frac{1}{4}, \frac{1}{2}]) \cup \bigcup_{n \geq 2} A_n$ is homeomorphic to a "$\sin(\frac{1}{x})$ curve." Let B_1 denote the line segment from $(0, \frac{1}{2})$ to $(\frac{1}{4}, 0)$ and let B_2 denote the line segment from $(\frac{1}{4}, 0)$ to $(\frac{1}{2}, \frac{1}{4})$. Let B_3 denote the line segment from $(\frac{3}{4}, \frac{1}{2})$ to $(1, 1)$. Let f be the upper-semicontinuous function whose graph is $G(f) = B_1 \cup B_2 \cup S \cup B_3$. The graph of f is given in Figure 3.4.

Note that f is irreducible with respect to domain. Since $f|_{[0,1]\setminus\{\frac{1}{2}\}}$ is residual-preserving, it is not difficult to show that f is also residual preserving. Let $x \in \pi_1^{-1}(\frac{1}{2}) \cap \pi_2^{-1}(\frac{1}{2}) \cap \pi_3^{-1}(0)$. Let $U = [0, \frac{1}{8})$. Then $U \setminus \{x_3\}$ has only one component, $C = (0, \frac{1}{8})$. But $f^2(C) \subset (0, 1/4)$, so $x_1 = \frac{1}{2} \notin \overline{f^2(C)}$. Thus f does not have the component shadowing property. Therefore by Theorem 3.3.17, $\varprojlim f$ does not have the full projection property.

CHAPTER FOUR

Hagopian Spheres

4.1 Motivation

In [54], van Mill constructed an infinite dimensional Hausdorff continuum that is indecomposable and homogeneous. However, in metric continua, the following question is still open: does there exist an indecomposable, homogeneous continuum of dimension greater than one? The question dates back at least as far as 1983 [50]

Bellamy [9] considered inverse limits of spheres using branch coverings as possible candidates for indecomposable homogeneous continua of dimension two. He dubbed these inverse limits *Hagopian Spheres* in honor of C. L. Hagopian, who suggested the idea. Bellamy focused primarily on *standard Hagopian Spheres*, which are inverse limits using maps of the following form.

Definition 4.1.1. Let $N \in \mathbb{N}$. We will call a map $f : S^2 \to S^2$, a *spin map*, if it is of the form $f = R^{-1} \circ D \circ R$, where R is an isometry of S^2 and D is given in spherical coordinates by $D(\theta, \phi) = (\theta, N\phi)$, where θ is the elevation and ϕ is the azimuthal angle. The natural number N will be called the *degree* of the map. The points $N_p(f), S_p(f) := R(\pi, 0), R(-\pi, 0)$ will be called the *north pole* and *south pole* of f, respectively.

Recall that all isometries of S^2 are rotations, inversions, or a product of the two [39]. Bellamy showed that any inverse limit with spin maps of odd degree is decomposable and posed the question as to whether any inverse limit of spin maps could be indecomposable. In this chapter, we answer that question in the affirmative, showing that for any even number N, there is a collection of spin maps f_i of degree N such that $\varprojlim f_i$ is indecomposable.

Lemma 4.2.1. Let $M \subset S^2$ be a continuum and let f be a spin map. If $f^{-1}(M)$ is connected then either M separates the north pole of f from the south pole of f or $M \cap \{N_p(f), S_p(f)\} \neq \emptyset$.

Proof. Suppose that M does not separate $N_p(f)$ from $S_p(f)$ and suppose that $M \cap \{N_p(f), S_p(f)\} = \emptyset$. Then as $S^2 \backslash M$ is open, there is an arc $A \subset S^2 \backslash M$ from $N_p(f)$ to $S_p(f)$. Then $f^{-1}(A)$ contains a simple closed curve that separates $f^{-1}(M)$.

□

Definition 4.2.2. [45] Let X, Y be topological spaces. A continuous function $f : X \to Y$ is said to be *confluent*, if for each continuum $M \subset Y$, each component of $f^{-1}(M)$ is mapped onto M by f.

Note that spin maps are confluent and compositions of confluent maps are confluent.

Lemma 4.2.3. Let $X = \varprojlim\{X_i, f_i\}$ where each f_i is confluent. Then X is indecomposable iff for all $i \in \mathbb{N}$ and for each proper decomposition $X_i = A_i \cup B_i$, either $(f_i^j)^{-1}(A_i)$ or $(f_i^j)^{-1}(B_i)$ is disconnected for some $j > i$.

Proof. Suppose that $X = A \cup B$ is a proper decomposition of X, and suppose that for each $i \in \mathbb{N}$, there exists a $j_i > i$ such that at least one of $(f_i^{j_i})^{-1}(A_i)$, $(f_i^{j_i})^{-1}(B_i)$ is disconnected. Without loss of generality, suppose that for infinitely many i, $(f_i^{j_i})^{-1}(A_i)$ is disconnected. Then for each such i there exists at least two components M_1, M_2 of $(f_i^{j_i})^{-1}(A_i)$, and, as $f_i^{j_i}$ is confluent, each gets mapped onto A_i. Then A_{j_i} meets only one of M_1, M_2, so without loss of generality, suppose that $M_2 \subset B_{j_i}$. Then $f_i^{j_i}(B_{j_i}) \supset f_i^{j_i}(M_2) = A_i$, so $B_i = X_i$. As i was arbitrarily large, it follows that $B = X$, which contradicts the definition of B.

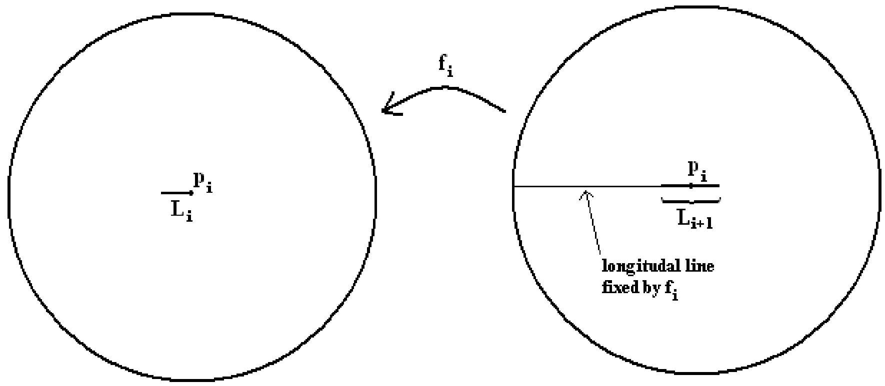

Figure 4.1. The spin map f_i from Lemma 4.2.4

Now suppose there exists an $i \in \mathbb{N}$ and some decomposition $X_i = A_i \cup B_i$, such that, for each $j > i$, both $(f_i^j)^{-1}(A_i)$ and $(f_i^j)^{-1}(B_i)$ are connected. Then $\pi_i^{-1}(A_i)$ and $\pi_i^{-1}(B_i)$ are proper subcontinua of X such that $X = \pi_i^{-1}(A_i) \cup \pi_i^{-1}(B_i)$. $\square$

Lemma 4.2.4. Let $x \in S^2$ and $\epsilon > 0$. Then for any $N \in 2\mathbb{N}$, there exist spin maps $f_1, \ldots, f_n$ of degree N such that $f_n^{-1} \circ f_{n-1}^{-1} \circ \cdots \circ f_1^{-1}(B_\epsilon(x))$ contains a point z and the antipodal point of z.

Proof. We may assume that $\epsilon < \pi$. Let p_1 be a point of distance $\frac{2}{3}\epsilon$ from x. Let f_1 be a spin map of degree N with $N_p(f_1) = p_1$ that fixes x. Then, since N is even, $f_1^{-1}(B_\epsilon(p_1))$ contains a line segment L_1 of length $\frac{4}{3}\epsilon$ with p_1 as its midpoint. Proceeding by induction, for each $i > 1$ such that L_{i-1} does not contain antipodal points, let p_i be an endpoint of L_{i-1} and let f_i be a spin map of degree N with $N_p(f_i) = p_i$ that fixes L_{i-1}. Then $L_i = f_i^{-1}(L_{i-1})$ is a line segment of length $\frac{2^i}{3}\epsilon$ (see Figure 4.1). Then for some i, the length of L_i is greater than π and hence contains a point z and its antipodal point. As $L_i \subset f_i^{-1} \circ f_{n-1}^{-1} \circ \cdots \circ f_1^{-1}(B_\epsilon(x))$, the result follows. $\square$

Theorem 4.2.5. For any $N \in 2\mathbb{N}$, there exists a collection of spin maps, $\{f_i\}$, of degree N such that $\varprojlim\{S^2, f_i\}$ is indecomposable.

Proof. For each $i \in \mathbb{N}$, let $X_i = S^2$ and let $\{B_{\epsilon_j}(x_j) : j \in \mathbb{N}\}$ denote a basis of S^2. Let $\{a_j \in \mathbb{N} \times \mathbb{N} : j \in \mathbb{N}\}$ denote an enumeration of $\mathbb{N} \times \mathbb{N}$ such that $b_i := \pi_1(a_i) > i$ for all $i > 1$ and $b_1 := \pi_1(a_1) = 1$. Then by Lemma 4.2.4, there exist spin maps $f_{b_1}, \ldots, f_{n_1 - 1}$ of degree N such that $f_{n_1-1}^{-1} \circ \cdots \circ f_{b_1}^{-1}(B_{\epsilon_1}(x_{a_1}))$ contains a point z and its antipodal point. Let f_{n_1} denote a spin map of degree N with $N_p(f_{n_1}) = z$. Then by Lemma 4.2.1, $f_{n_1}^{-1} \circ \cdots \circ f_{b_1}^{-1}(B_{\epsilon_{a_1}}(x_{a_1}))$ separates $f_{n_1}^{-1} \circ \cdots \circ f_{b_1}^{-1}(X_{b_1} \backslash B_{\epsilon_{a_1}}(x_{a_1}))$.

Proceeding by induction, suppose that $f_{n_i - 1}$ has been defined. Then $f_{n_{i-1}}^{-1} \circ \cdots \circ f_{b_i}^{-1}(B_{\epsilon_{a_i}}(x_{a_i}))$ contains an open ball, so there exist spin maps $f_{n_{i-1}+1}, \ldots, f_{n_i - 1}$ of degree N such that $f_{n_i-1}^{-1} \circ \cdots f_{b_i}^{-1}(B_{\epsilon_{a_i}}(x_{a_i}))$ contains a point z_i and its antipodal point. As before, let f_{n_i} denote a spin map of degree N with $N_p(f_{n_i}) = z_i$. Then by Lemma 4.2.1, $f_{n_i}^{-1} \circ \cdots \circ f_{b_i}^{-1}(B_{\epsilon_{a_i}}(x_{a_i}))$ separates $f_{n_i}^{-1} \circ \cdots \circ f_{b_i}^{-1}(X_{b_i} \backslash B_{\epsilon_{a_i}}(x_{a_i}))$.

Then for any i and any proper subcontinuum $M \subset X_i = S^2$, there exists an $x_{a_j} \in X_i$ such that $M \subset S^2 \backslash B_{\epsilon_{a_j}}(x_{a_j})$. Then for some $n > i$, $(f_i^n)^{-1}(M)$ is disconnected, so by Proposition 4.2.3, $\varprojlim\{S^2, f_i\}$ is indecomposable. $\square$

We note that in the construction in Theorem 4.2.5, the spin maps do not have to all be of the same degree. In fact, any function f_i may be replaced with a spin map of any even degree. Morever, after the construction of maps f_1 through f_{n_1}, any finite number of confluent maps (not just spin maps) may inserted into the inverse limit, provided that f_{n_1+1} and subsequent maps are adjusted accordingly. The same may occur after each f_{n_i} has been defined. With appropriate bonding maps inserted between f_{n_i} and f_{n_i+1} for each i, it is hoped that a homogeneous inverse limit may be constructed. However, it remains unknown if any inverse limit of spin maps is homogeneous.

Question 4.2.6. Does there exist a collection of spin maps such that $\varprojlim f_i$ is homogeneous? Does there exist a collection of spin maps such that $\varprojlim f_i$ is indecomposable and homogeneous?

CHAPTER FIVE

Conclusion

5.1 Summary

Topological inverse limits play an important role in the theory of dynamical systems and in continuum theory. In this dissertation, we investigate classical inverse limits of Julia sets and set-valued inverse limits of arbitrary compacta.

Using the theory of Hubbard trees, the trunk of a Julia set was introduced. Examination of the trunk led to a characterization of indecomposability in inverse limits of post-critically finite polynomials restricted to their Julia sets. The trunk was also observed to behave similarly to the core of a tent map. Results for inverse limits of Julia sets that are similar in nature to Bennett's Theorem for inverse limits of intervals were given. It was also shown that inverse limits of Julia sets of postcritically finite polynomials are never hereditarily decomposable. However all indecomposable subcontinua of the inverse limit of a post-critically finite polynomial must intersect the inverse limit of the trunk.

Inverse limits with upper semicontinuous set-valued bonding maps have just begun to be studied. We provided necessary and sufficient conditions for inverse limits of upper semicontinuous functions to have the full projection property, answering a question posed by Ingram [31]. In Theorem 3.2.6 such a characterization for inverse limits of arbitrary compacta was given based solely on the dynamics of the bonding functions. In Theorem 3.3.17, a characterization was given for the class of continuum-valued maps of trees that are residual-preserving. As corollaries, a condition was established for when a closed subset of an inverse limit is the inverse limit of its projections and a necessary and sufficient condition for all closed subsets of an inverse limit to have that property was provided. A number of other necessary conditions and sufficient conditions were provided in various contexts. As a corollary, a necessary and sufficient condition was given for every closed subset of an inverse limit to be the inverse limit of its projections.

Finally, we answered a question of Bellamy on whether or not inverse limits of spin maps could be indecomposable. We complimented his result showing that odd degree spin maps could not produce indecomposable inverse limits by showing that even degree spin maps could. It remains an open question as to whether or not any such inverse limit is homogeneous. An affirmative answer to that question would solve a major open problem in the classification of homogeneous continua.

5.2 Open Problems

The following are open questions for further research.

Question: Are there postcritically finite polynomials $f, g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that are not topologically conjugate such that $\varprojlim f|_{J(f)}$ is homeomorphic to $\varprojlim g_{J(g)}$?

In general, questions of about homeomorphisms of inverse limits are difficult. For example, the Ingram Conjecture asked if distinct tent maps of the interval necessarily gave rise to distinct inverse limits. The conjecture dates back to at least the mid nineties [27], was only recently proven in 2009 [4], despite considerable attention. However, while post-critically finite polynomials are in some ways more complex than tent maps (e.g., postcritically finite polynomials can have more than one critical point), this question still may be tractable, as distinct (i.e. not topologically conjugate) polynomials generally have distinct Julia sets, and when they do not (e.g. $f(z) = z^2$ and $f(z) = z^3$), their inverse limits are often easily distinguished. Moreover, the theory of Hubbard trees provide a powerful tool to study this question.

Question: Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial that is not post-critically finite. What conditions imply that $\varprojlim f|_{J(f)}$ is indecomposable? What conditions imply that $\varprojlim f|_{J(f)}$ is decomposable?

The same questions could be posed for non-polynomial rational functions. An interesting special case would be the rational functions for which $J(f) = \hat{\mathbb{C}}$.

One of the major uses of the full projection property is to prove that an inverse limit is indecomposable. For this purpose, a weaker property, the continuum full projection property is sufficient. Because connectedness in inverse limits of upper semi-continuous functions is not well understood, the continuum full projection property appears to be a difficult property to characterize at present. However, establishing relationships between the full projection property and the continuum full projection property seems to be a plausible way to develop necessary conditions and sufficient conditions for an inverse limit to have the continuum full projection property.

Definition 5.2.1. Let $f : X \to 2^X$ be upper semicontinuous. An upper semicontinuous function $g : X \to 2^X$ is said to be a *skeleton* of f, if g is irreducible with respect to domain and $G(g) \subset G(f)$.

Question: Let $f : X \to 2^X$ be upper semicontinuous and suppose that the inverse limit of every skeleton of f has the full projection property. What conditions on f and X ensure that $\varprojlim f$ has the continuum full projection property? Conversely, if the inverse limit of some skeleton of f does not have the full projection property, what conditions on f and X ensure that $\varprojlim f$ does not have the continuum full projection property?

In [31], Ingram constructed an inverse limit with the continuum full projection property and showed that it was indecomposable. Using results from this dissertation, it is not difficult to show that the inverse limit of the only skeleton of the bonding function for that inverse limit has the full projection property. Currently, we are not aware of any inverse limits for which the bonding function f has the continuum full projection property and some skeleton of f does not have the full projection property.

Question: [50] Is there an indecomposable homogeneous continuum of dimension greater than 1?

If there are indecomposable homogeneous continua of dimension greater than one, then perhaps an affirmative answer to the following question would be a significant step towards constructing one.

Question: Is there a homogeneous Hagopian sphere? Is there a standard Hagopian sphere that is homogeneous?

The class of indecomposable standard Hagopian spheres constructed in this dissertation is plausibly large enough to intersect the set of homogeneous standard Hagopian spheres, if the latter class is non-empty.

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