ABSTRACT

Orbifold Branes in String Theory and Their Applications to Cosmology

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This dissertation contains two distinct compactification schemes of 10-dimensional string theory, as well as some of the implications of one of these schemes for string cosmology. The first half of this work begins with a brief overview of cosmology and goes through constructing and then analyzing the first model, inspired by the work of Santos and Wang. The second part consists of an attempt to construct similar models using the the popular warped conifold compactification scheme, as well as an appendix with a variant of the first model and its derivation. The work concludes with the observation that the latter attempt does not admit solutions of the same form, and that the variant model in the appendix is degenerate to previously studied KK-type models.

Orbifold Branes in String Theory and Their Applications to Cosmology

by

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER ONE

Introduction to Modern Cosmology

1.1 Three Principles

There are three main ideas that have guided the development of cosmology for almost a century. Each represented a philosophical shift away from an anthropocentric view of the universe, a privileged place in terms of position, motion, or local matter arrangement.

1.1.1 Cosmological Principle

The main assumptions of cosmology are homogeneity and isotropy. That is, that laws we observe on earth and in the solar system can be applied to the rest of the universe as well. This is also often cited as stating that the location of Earth has no special significance to the physics we observe beyond our solar system. Extending this idea to time as well as space was one motivation for Einstein's original belief in a static universe. At the largest scales the universe does appear homogeneous and isotropic, though dynamic. For this reason, and for the simplification it yields, the universe is generally modeled by the Friedmann-Robertson-Walker (FRW) metric,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right), \tag{1.1}$$

where $a(t)$ is the expansion factor, and k the spatial curvature, which takes the values $k = 0, \pm 1$. The space-times are called flat for $k = 0$, closed for $k = +1$, and open for $k = -1$, that is,

$$k = \begin{cases} 0, & \text{flat,} \\ +1, & \text{closed,} \\ -1, & \text{open.} \end{cases} \tag{1.2}$$

All of the experiments carried out so far are consistent with $k = 0$ [1].

1.1.2 Weyl's Postulate

The second assumption generally accepted is that all the matter in the universe follows geodesics in this space-time. This motion is a consequence of relativity in the absence of any non-gravitational forces that significantly affect the motion at such distances, though the dark matter halos within galaxies could hypothetically have such a long range interaction. Mathematically this means that the matter fields in the universe can be described by a perfect fluid,

$$T_{ab} = (\rho + p)u_a u_b - p g_{ab}, \; (a, b = 0, 1, 2, 3), \tag{1.3}$$

where p and ρ denote, respectively, the pressure and energy density of the fluid with its four-velocity u_a, measured by co-moving observers. It should be noted that p and ρ should be understood to be the sum of all the species,

$$p = \sum p_i, \quad \rho = \sum \rho_i. \tag{1.4}$$

In general, the equation of state of the fluid can be written as

$$p_i = w_i \rho_i, \tag{1.5}$$

where for different eras and species of matter we have different equations of state. For example, in the radiation-dominated epoch, $w_\gamma = 1/3$, while in the matter-dominated epoch, $w_m = 0$, and in the dark energy dominated epoch, $w_{DE} < -1/3$. When $w_{stiff} = 1$, it is called a stiff fluid. The cosmological constant term can be considered as a particular case of a perfect fluid with $w_\Lambda = -1$ and

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G}. \tag{1.6}$$

1.1.3 Einstein's General Relativity

General relativity is assumed to describe the evolution of the universe, except possibly in the quantum-gravity regime within a short time of the big bang. General relativity is encoded in the second-order partial differential equations [2],

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab} + \Lambda g_{ab}, \tag{1.7}$$

where R_{ab} denotes the Ricci tensor, R the Ricci scalar, and G and Λ are, respectively, the Newtonian and cosmological constants. The energy-momentum tensor T_{ab} satisfies the conservation laws of the energy and momentum,

$$\nabla^a T_{ab} = 0, \tag{1.8}$$

where ∇_a denotes the covariant derivative with respect to the metric g_{ab}.

For the FRW metric (1.1), the non-vanishing components of the Einstein tensor G_{ab} are given by

$$\begin{aligned} G_{00} &= 3\left(H^2 + \frac{k}{a^2}\right), \\ G_{ij} &= \left(2\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2}\right) g_{ij}, \ (i, j = 1, 2, 3) \end{aligned} \tag{1.9}$$

where

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}, \quad H \equiv \frac{\dot{a}}{a}, \tag{1.10}$$

and $\dot{a} \equiv da(t)/dt$. In the co-moving coordinates, we have $u_a = \delta^t_a$, and the Einstein field equations (1.7) yield,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{1}{3}\Lambda, \tag{1.11}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + p) + \frac{1}{3}\Lambda. \tag{1.12}$$

Eq.(1.11) is often referred to as the Friedmann equation. Combining these two equations, we obtain the conservation law of energy,

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{1.13}$$

which can be also obtained from the conservation laws of the energy and momentum, Eq.(1.8).

1.2 Three Observational Supports

The familiar big bang cosmology is supported primarily by three observations. The pattern of redshifts as a function of luminosity, which is interpreted to be a relation of velocity to distance, also known as Hubble's law, is the first and most dramatic evidence,

$$v = Hd, \tag{1.14}$$

where v denotes the receding velocity of a galaxy, d its distance, and H the Hubble parameter. The current value of H is $H_0 \simeq 72$ (km/s)/Mpc [1].

The 2.725 degree cosmic microwave background, and the ratios of light isotopes in space are the others [3]. The fact that the expansion of the universe is currently accelerating is a newer and more exciting finding, and is most clearly visible in the supernova redshift data collected in the past decade [4, 5, 6, 7].

The ratios of deuterium, helium 3 and 4, and lithium to hydrogen are all consistent with the big bang model using roughly the same value for the matter to radiation ratio [3]. The evolution of the universe during this epoch is fairly well accounted for without including dark energy, which after inflation, will only become important again at later times.

The standard model of modern cosmology is known as the lambda-CDM model. It fits the known expansion data to a FRW model with nonzero cosmological constant and supplements observed baryonic matter with what is known as cold dark matter [3]. The cold dark matter models most favored are hidden sector particles that behave similarly to ordinary dust, but are inert with respect to the electromagnetic field. Models including other fluids with negative pressures greater than 1/3 of the density are known as dark energy models. The cosmological constant is the special case where pressure is exactly negative of density. The evidence for dark matter consists primarily of the galactic rotation curve and anomalous gravitational lensing. Some efforts have been made to determine how dark matter might affect the measurements of planets or satellites in the solar system, but generally the effects are expected to be too weak to have been measured so far. The cosmological constant favored by experiment is extremely small, and its theoretical value is cutoff dependent, quite large in even the most conservative case. This is the well-known cosmological constant problem [8].

CHAPTER TWO

Orbifold Branes in String/M-Theory

One of the major fascinations with string theory in the last decade, especially with regards to cosmology, is the braneworlds scenario [9]. The traditional approach to compactification of ten dimensional string theory is based on Kaluza and Klein's original model, supplemented with the idea of bulk dimensions. Matter is allowed to exist in a number of 'large' dimensions greater than four, but is 'stuck' to an embedded four dimensional surface called a brane. The same mechanism used by early string theorists to explain confinement of quarks is employed on a wider scale, confining all nongravitational forces to the brane, and branes are the only 'allowed endpoints' of strings in type IIB, the theory of closed strings. Branes themselves obey a Nambu-Goto type action in the simplest case, equal to a world volume swept out times the brane tension. In the presence of fluxes this generalizes to the Dirac-Born-Infeld action. Their intrinsic metrics act as curved space-time for particles confined to the surface, as they sweep out minimal surfaces of their corresponding dimensions plus one in the ten dimensional space-time in which they are embedded. So, one can visualize branes as a literal interpretation of the rubber sheet metaphor often used to describe curved space-times in GR. The standard approach to building a braneworld model involves compactifying ten dimensions down to five or six, then using an effective theory to describe the motions of the branes themselves, from which we end up with familiar four dimensional Einstein's equations used for cosmology. The appeal of brane models is twofold: they allow for the creation of more scenarios near the testable TeV scale of current colliders, and they also address the original contents of Einstein with Kaluza and Klein's scenario, that of the 'speciality' of tying up some dimensions and not others. Also, the scales of the brane models are typically used to try to explain the hierarchy problem, avoiding the 'Plankian desert', the large range of energies separating the standard model from Planck scale physics, in which it is supposed that no new phenomena occur. So D-branes, so named because of the Dirichlet type boundary conditions imposed on strings connecting to them, are often used with non-compactified extra dimensions in string models to generate such physics. Here we use the standard compactification techniques to get an effective theory with five large dimensions, then give this bulk a reflection symmetry about two parallel co-dimension-one volumes in it, on which we put the branes. The bulk is also made periodic with respect to the normal direction to the branes. This is called a S_1/Z_2 orbifold symmetry.

2.1 $(D + d_+ + d_-)$-Dimensional Decompositions

Let us begin with the action for the NS/NS sector in $(D+d_+ +d_-)$-dimensions, $\hat{M}_N = M_D \times \mathcal{M}_{d_+} \times \mathcal{M}_{d_-}$, where $\mathcal{M}_{d_+}$ and $\mathcal{M}_{d_-}$ are d_+ and d_- dimensional spaces, respectively, and $N \equiv D + d_+ + d_-$. Then the action may be written as, [10, 11, 12],

$$\begin{aligned} \hat{S}_N &= -\frac{1}{2\kappa_N^2}\int d^N x\sqrt{|\hat{g}_N|}e^{-\hat{\Phi}} \\ &\quad \times \left\{\hat{R}_N[\hat{g}] + \left(\hat{\nabla}\hat{\Phi}\right)^2 - \frac{1}{12}\hat{H}^2\right\}, \end{aligned} \tag{2.1}$$

where $\hat{\nabla}$ is the covariant derivative with respect to $\hat{g}^{AB}$ with $A, B = 0, 1, ..., N-1$, and $\hat{\Phi}$ is the dilaton field. The NS three-form field $\hat{H}_{ABC}$ is defined as

$$\hat{H}_{ABC} = 3\partial_{[A}\hat{B}_{BC]}, \tag{2.2}$$

$$\hat{H}^2 = \hat{H}^{ABC}\hat{H}_{ABC}, \tag{2.3}$$

where the square brackets imply total antisymmetrization over all indices, and

$$\hat{B}_{CD} = -\hat{B}_{DC}, \quad \partial_A\hat{B}_{CD} \equiv \frac{\partial\hat{B}_{CD}}{\partial x^A}. \tag{2.4}$$

The first step in evaluating this action to get an effective five dimensional theory is to take a block diagonal metric ansatz given by,

$$\begin{aligned} d\hat{s}_N^2 &= \hat{g}_{AB} dx^A dx^B \\ &= \tilde{g}_{ab}(x)\, dx^a dx^b + e^{\sqrt{\frac{2}{d_+}}\psi_+(x)} h_{ij}^+(z_+)\, dz_+^i dz_+^j \\ &\quad + e^{\sqrt{\frac{2}{d_-}}\psi_-(x)} h_{pq}^-(z_-)\, dz_-^p dz_-^q, \end{aligned} \tag{2.5}$$

where $\tilde{g}_{ab}(x)$ is the metric on M_D parametrized by the coordinates x^a with $a, b, c = 0, 1, ..., D-1$, $h_{ij}^+(z_+)$ is the metric on the compact space $\mathcal{M}_{d_+}$ with coordinates z_+^i, where $i, j = D, D+1, ..., D+d_+-1$, and $h_{ij}^-(z_-)$ is the metric on the compact space $\mathcal{M}_{d_-}$ with coordinates z_-^p, where $p, q = D+d_+, D+d_++1, ..., N-1$.

In addition, we also assume the dilaton field $\hat{\Phi}$ is a function of x^a, and that the flux $\hat{B}_{CD}$ is block diagonal as well,

$$\left(\hat{B}_{CD}\right) = \begin{pmatrix} B_{ab}(x) & 0 & 0 \\ 0 & e^{\xi_+(x)} B_{ij}(z_+) & 0 \\ 0 & 0 & e^{\xi_-(x)} B_{pq}(z_-) \end{pmatrix}. \tag{2.6}$$

This considerably simplifies the action. In particular, the non-vanishing components of $\hat{H}_{ABC}$ are

$$\begin{aligned} \hat{H}_{abc} &= H_{abc} = 3\partial_{[a} B_{bc]}, \\ \hat{H}_{ijk} &= e^{\xi_+} H_{ijk} = 3e^{\xi_+}\partial_{[i} B_{jk]}, \\ \hat{H}_{pqr} &= e^{\xi_-} H_{pqr} = 3e^{\xi_-}\partial_{[p} B_{qr]}, \\ \hat{H}_{aij} &= B_{ij} e^{\xi_+} \tilde{\nabla}_a \xi_+, \\ \hat{H}_{apq} &= B_{pq} e^{\xi_-} \tilde{\nabla}_a \xi_-, \end{aligned} \tag{2.7}$$

where $\tilde{\nabla}_a$ denotes the covariant derivative with respect to $\tilde{g}^{ab}$.

The Ricci scalar for the complete metric may be broken up into the scalars of the product spaces as well as a few cross terms containing the warp factors,

$$\begin{aligned}\hat{R}_N[\hat{g}] &= \tilde{R}_D[\tilde{g}] + e^{-\sqrt{\frac{2}{d_+}}\psi_+} R_{d_+}\left[h^+\right] \\ &+e^{-\sqrt{\frac{2}{d_-}}\psi_-} R_{d_-}\left[h^-\right] \\ &-2\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b Q - \frac{(d_+ + 1)}{2}\left(\tilde{\nabla}\psi_+\right)^2 \\ &-\frac{(d_- + 1)}{2}\left(\tilde{\nabla}\psi_-\right)^2 \\ &-\sqrt{d_+ d_-}\left(\tilde{\nabla}\psi_+\right)\left(\tilde{\nabla}\psi_-\right),\end{aligned} \tag{2.8}$$

where

$$Q \equiv \sqrt{\frac{d_+}{2}}\,\psi_+ + \sqrt{\frac{d_-}{2}}\,\psi_-. \tag{2.9}$$

To further simplify the problem, we shall remove the coupling of the dilaton to the Ricci scalar and factor the volume element as well. This is done with the standard transformation from the string frame to the Einstein frame by absorbing the dilation with the conformal transformation of

$$g_{ab} = \Omega^2 \tilde{g}_{ab}, \quad \Omega = e^{\frac{Q-\hat{\Phi}}{D-2}}. \tag{2.10}$$

The Ricci scalar and Q term transform accordingly,

$$\begin{aligned}\tilde{R}_D[\tilde{g}] &= \Omega^2\left\{R_D[g] + 2(D-1)\Box\ln\Omega\right. \\ &\quad \left.-(D-2)(D-1)\left(\nabla\ln\Omega\right)^2\right\}, \\ \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b Q &= \Omega^2\left(\Box Q\right. \\ &\quad \left.-(D-2)\left(\nabla Q\right)\left(\nabla\ln\Omega\right)\right),\end{aligned} \tag{2.11}$$

where $\Box \equiv g^{ab}\nabla_a\nabla_b$, and ∇_a denotes the covariant derivative with respect to g^{ab}. Then, combining Eqs.(2.8) and (2.11), we obtain

$$\sqrt{|\hat{g}_N|}e^{-\hat{\Phi}}\left\{\hat{R}_N[\hat{g}] + \left(\hat{\nabla}\hat{\Phi}\right)^2 - \frac{1}{12}\hat{H}^2\right\}$$
$$= \sqrt{|g_D h^+ h^-|}\left\{R_D[g] + e^{-2\frac{Q-\hat{\Phi}}{D-2}}\left(e^{-\sqrt{\frac{2}{d_+}}\psi_+} R_{d_+}\right.\right.$$

$$
\begin{aligned}
&+e^{-\sqrt{\frac{2}{d_-}}\psi_-}R_{d_-} - \frac{1}{12}\hat{H}^2\Big) + \frac{2}{D-2}\Box Q \\
&-\frac{2(D-1)}{D-2}\Box\hat{\Phi} - \frac{1}{D-2}\left(\nabla\left(Q-\hat{\Phi}\right)\right)^2 \\
&-\frac{1}{2}\left(\nabla\psi_+\right)^2 - \frac{1}{2}\left(\nabla\psi_-\right)^2\Big\}.
\end{aligned} \tag{2.12}
$$

The transformed flux action becomes

$$
\begin{aligned}
\hat{H}^2 \;=\;& e^{\frac{6(Q-\hat{\Phi})}{D-2}}H^2 \\
&+3e^{\frac{2(Q-\hat{\Phi})}{D-2}}\Bigg(e^{2\left(\xi_+-\sqrt{\frac{2}{d_+}}\psi_+\right)}B_+^2\left(\nabla\xi_+\right)^2 \\
&+e^{2\left(\xi_--\sqrt{\frac{2}{d_-}}\psi_-\right)}B_-^2\left(\nabla\xi_-\right)^2\Bigg) \\
&+e^{2\xi_+-3\sqrt{\frac{2}{d_+}}\psi_+}H_+^2 \\
&+e^{2\xi_--3\sqrt{\frac{2}{d_-}}\psi_-}H_-^2,
\end{aligned} \tag{2.13}
$$

with

$$
\begin{aligned}
H^2 &= H_{abc}(x)H^{abc}(x), \\
H_+^2 &= H_{ijk}\left(z_+\right)H^{ijk}\left(z_+\right), \\
H_-^2 &= H_{pqr}\left(z_-\right)H^{pqr}\left(z_-\right), \\
B_+^2 &= B_{ij}\left(z_+\right)B^{ij}\left(z_+\right), \\
B_-^2 &= B_{pq}\left(z_-\right)B^{pq}\left(z_-\right),
\end{aligned} \tag{2.14}
$$

and

$$
g^{ab}g_{ac} = \delta^b_c, \quad h^{+ik}h^+_{ij} = \delta^k_j, \quad h^{-pq}h^-_{pr} = \delta^q_r. \tag{2.15}
$$

Substituting Eqs.(2.13) and (2.14) into Eq.(2.1), and then integrating by parts, we obtain the D-dimensional effective action in the Einstein frame,

$$
S_D^{(E)} = -\frac{1}{2\kappa_D^2}\int\sqrt{|g_D|}d^Dx\left(R_D[g] - \mathcal{L}_D^{(E)}\left(\phi_n,\xi_\pm\right)\right), \tag{2.16}
$$

where $\phi_n = \{\phi, \psi_\pm\}$, and

$$\kappa_D^2 \equiv \frac{\kappa_N^2}{V_{d_+} V_{d_-}}, \tag{2.17}$$

$$\begin{aligned}
V_{d_\pm} &\equiv \int \sqrt{|h^\pm|}\, d^{d_\pm} z_\pm, \\
\mathcal{L}_D^{(E)} &= \frac{1}{2} \sum_n (\nabla \phi_n)^2 + \frac{1}{12} e^{-\sqrt{\frac{8}{D-2}}\phi} H^2 \\
&\quad + \alpha_+ e^{2\xi_+ - \sqrt{\frac{8}{d_+}}\psi_+} (\nabla \xi_+)^2 \\
&\quad + \alpha_- e^{2\xi_- - \sqrt{\frac{8}{d_-}}\psi_-} (\nabla \xi_-)^2 \\
&\quad - e^{\sqrt{\frac{2}{D-2}}\phi} \left(\beta_+ e^{-\sqrt{\frac{2}{d_+}}\psi_+} \right. \\
&\quad + \beta_- e^{-\sqrt{\frac{2}{d_-}}\psi_-} - \gamma_+ e^{2\xi_+ - \sqrt{\frac{18}{d_+}}\psi_+} \\
&\quad \left. - \gamma_- e^{2\xi_- - \sqrt{\frac{18}{d_-}}\psi_-} \right),
\end{aligned} \tag{2.18}$$

and

$$\phi \equiv \sqrt{\frac{2}{D-2}} \left(\hat{\Phi} - Q \right), \tag{2.19}$$

$$\begin{aligned}
\alpha_\pm &\equiv \frac{1}{4V_{d_\pm}} \int d^{d_\pm} z_\pm \sqrt{|h^\pm|} B_\pm^2 (z_\pm), \\
\beta_\pm &\equiv \frac{1}{V_{d_\pm}} \int d^{d_\pm} z_\pm \sqrt{|h^\pm|} R_{d_\pm} (z_\pm), \\
\gamma_\pm &\equiv \frac{1}{12V_{d_\pm}} \int d^{d_\pm} z_\pm \sqrt{|h^\pm|} H_\pm^2 (z_\pm).
\end{aligned} \tag{2.20}$$

2.2 Orbifold Branes

The action for the brane can be written as,

$$\begin{aligned}
S_{D-1,m}^{(E,I)} &= -\epsilon_I \int_{M_{D-1}^{(I)}} \sqrt{\left| g_{D-1}^{(I)} \right|} V_{D-1}^{(I)} (\phi_n, \xi_\pm)\, d^{D-1} \xi_{(I)} \\
&\quad + \int_{M_{D-1}^{(I)}} d^{D-1} \xi_{(I)} \sqrt{\left| g_{D-1}^{(I)} \right|} \\
&\quad \times \mathcal{L}_{D-1,m}^{(I)} (\phi_n, \xi_\pm, \chi),
\end{aligned} \tag{2.21}$$

where $I = 1, 2$, $V_{D-1}^{(I)} (\phi_n, \xi_\pm)$ denotes the potential of the scalar fields ϕ_n on the branes, and $\xi_{(I)}^\mu$'s are the intrinsic coordinates of the branes with $\mu, \nu = 0, 1, ..., D-2$, and $\epsilon_1 = -\epsilon_2 = 1$. χ denotes collectively the matter fields. The surface of the (I)th brane is given explicitly as,

$$\Phi_I\left(x^a\right) = 0, \tag{2.22}$$

or parametrically as

$$x^a = x^a\left(\xi^{\mu}_{(I)}\right). \tag{2.23}$$

$g^{(I)}_{D-1}$ denotes the determinant of the reduced metric $g^{(I)}_{\mu\nu}$ of the I-th brane, defined as

$$g^{(I)}_{\mu\nu} \equiv g_{ab} e^{(I)a}_{(\mu)} e^{(I)b}_{(\nu)}\Big|_{M^{(I)}_{D-1}}, \tag{2.24}$$

$$e^{(I)\,a}_{(\mu)} \equiv \frac{\partial x^a}{\partial \xi^{\mu}_{(I)}}. \tag{2.25}$$

Then, the total action is given by the sum of the D-dimensional bulk action and the brane boundary actions,

$$S^{(E)}_{total} = S^{(E)}_{D} + \sum_{I=1}^{2} S^{(E,I)}_{D-1,m}. \tag{2.26}$$

The variation of the total action (2.26) with respect to the metric g^{ab} yields the field equations,

$$\begin{aligned} G^{(D)}_{ab} &= \kappa^2_D T^{(D)}_{ab} + \kappa^2_D \sum_{I=1}^{2} \mathcal{T}^{(I)}_{\mu\nu} e^{(I,\,\mu)}_{a} e^{(I,\,\nu)}_{b} \\ &\quad \times \sqrt{\left|\frac{g^{(I)}_{D-1}}{g_D}\right|}\,\delta\left(\Phi_I\right), \end{aligned} \tag{2.27}$$

where $\delta(x)$ denotes the Dirac delta function, normalized in the sense of [13], and the energy-momentum tensors $T^{(D)}_{ab}$ and $\mathcal{T}^{(I)}_{\mu\nu}$ are defined as,

$$\begin{aligned} \kappa^2_D T^{(D)}_{ab} \equiv\ & \frac{1}{2}\left(\nabla_a \phi^n\right)\left(\nabla_b \phi_n\right) \\ & + \alpha_+ e^{2\xi_+ - \sqrt{\frac{8}{d_+}}\psi_+}\left(\nabla_a \xi_+\right)\left(\nabla_b \xi_+\right) \\ & + \alpha_- e^{2\xi_- - \sqrt{\frac{8}{d_-}}\psi_-}\left(\nabla_a \xi_-\right)\left(\nabla_b \xi_-\right) \\ & + \frac{1}{4} e^{-\sqrt{\frac{8}{D-2}}\,\phi} H_{acd} H_b{}^{cd} \end{aligned}$$

$$-\frac{1}{2}g_{ab}\mathcal{L}_D^{(E)}, \tag{2.28}$$

$$\mathcal{T}_{\mu\nu}^{(I)} \equiv \mathcal{S}_{\mu\nu}^{(I)} + \tau_p^{(I)} g_{\mu\nu}^{(I)},$$

$$\mathcal{S}_{\mu\nu}^{(I)} \equiv 2\frac{\delta\mathcal{L}_{D-1,m}^{(I)}}{\delta g^{(I)\,\mu\nu}} - g_{\mu\nu}^{(I)}\mathcal{L}_{D-1,m}^{(I)}, \tag{2.29}$$

where $\phi^n = \phi_n$, and

$$\tau_p^{(I)} \equiv \epsilon_I V_{D-1}^{(I)}\left(\phi_n, \xi_\pm\right). \tag{2.30}$$

Variation of the action (2.26), with respect to ϕ, $\psi_\pm$, $\xi_\pm$ and B_{ab}, gives equations for the matter fields,

$$\begin{aligned}\Box\phi &= -\frac{1}{12}\sqrt{\frac{8}{D-2}}e^{-\sqrt{\frac{8}{D-2}}\,\phi}H^2 \\ &\quad -\sqrt{\frac{2}{D-2}}e^{\sqrt{\frac{2}{D-2}}\,\phi}\Big(\beta_+ e^{-\sqrt{\frac{2}{d_+}}\,\psi_+} \\ &\quad +\beta_- e^{-\sqrt{\frac{2}{d_-}}\,\psi_-} - \gamma_+ e^{2\xi_+ - \sqrt{\frac{18}{d_+}}\,\psi_+} \\ &\quad -\gamma_- e^{2\xi_- - \sqrt{\frac{18}{d_-}}\,\psi_-}\Big) \\ &\quad -\sum_{i=1}^{2}\left(2\kappa_D^2\epsilon_I\frac{\partial V_{D-1}^{(I)}}{\partial\phi} + \sigma_\phi^{(I)}\right) \\ &\quad \times\sqrt{\left|\frac{g_{D-1}^{(I)}}{g_D}\right|}\,\delta\left(\Phi_I\right),\end{aligned} \tag{2.31}$$

$$\begin{aligned}\Box\psi_\pm &= -\alpha_\pm\sqrt{\frac{8}{d_\pm}}e^{2\xi_\pm - \sqrt{\frac{8}{d_\pm}}\psi_\pm}\left(\nabla\xi_\pm\right)^2 \\ &\quad e^{\sqrt{\frac{2}{D-2}}\,\phi}\Bigg(\beta_\pm\sqrt{\frac{2}{d_\pm}}e^{-\sqrt{\frac{2}{d_\pm}}\psi_\pm} \\ &\quad -\gamma_\pm\sqrt{\frac{18}{d_\pm}}e^{2\xi_\pm - \sqrt{\frac{18}{d_\pm}}\psi_\pm}\Bigg) \\ &\quad -\sum_{i=1}^{2}\left(2\kappa_D^2\epsilon_I\frac{\partial V_{D-1}^{(I)}}{\partial\psi_\pm} + \sigma_{\psi_\pm}^{(I)}\right) \\ &\quad \times\sqrt{\left|\frac{g_{D-1}^{(I)}}{g_D}\right|}\,\delta\left(\Phi_I\right),\end{aligned} \tag{2.32}$$

$$
\begin{aligned}
\Box\xi_\pm &= -\left(\nabla\xi_\pm\right)^2 + \sqrt{\frac{8}{d_\pm}}\left(\nabla_a\xi_\pm\right)\left(\nabla^a\psi_\pm\right) \\
&\quad + \frac{\gamma_\pm}{\alpha_\pm} e^{\sqrt{\frac{2}{D-2}}\,\phi - \sqrt{\frac{2}{d_\pm}}\psi_\pm} - \frac{\gamma_\pm}{2\alpha_\pm} e^{\sqrt{\frac{8}{d_\pm}}\psi_\pm - 2\xi_\pm} \\
&\quad \times \sum_{I=1}^{2}\left(2\kappa_D^2\epsilon_I \frac{\partial V_{D-1}^{(I)}}{\partial\xi_\pm} + \sigma_{\xi_\pm}^{(I)}\right) \\
&\quad \times \sqrt{\left|\frac{g_{D-1}^{(I)}}{g_D}\right|}\,\delta\left(\Phi_I\right),
\end{aligned} \tag{2.33}
$$

$$
\begin{aligned}
\nabla^c H_{cab} &= \sqrt{\frac{8}{D-2}}\, H_{cab}\nabla^c\phi \\
&\quad - \sum_{i=1}^{2}\sigma_{ab}^{(I)}\sqrt{\left|\frac{g_{D-1}^{(I)}}{g_D}\right|}\,\delta\left(\Phi_I\right),
\end{aligned} \tag{2.34}
$$

$$
\begin{aligned}
\sigma_\phi^{(I)} &\equiv -2\kappa_D^2 \frac{\delta\mathcal{L}_{D-1,m}^{(I)}}{\delta\phi}, \\
\sigma_{\psi_\pm}^{(I)} &\equiv -2\kappa_D^2 \frac{\delta\mathcal{L}_{D-1,m}^{(I)}}{\delta\psi_\pm}, \\
\sigma_{\xi_\pm}^{(I)} &\equiv -2\kappa_D^2 \frac{\delta\mathcal{L}_{D-1,m}^{(I)}}{\delta\xi_\pm}, \\
\sigma_{ab}^{(I)} &\equiv -4\kappa_D^2 e^{\sqrt{\frac{8}{D-2}}\,\phi} \frac{\delta\mathcal{L}_{D-1,m}^{(I)}}{\delta B^{ab}}.
\end{aligned} \tag{2.35}
$$

Deriving the equations in the bulk involves simply dropping the delta function terms from the above equations. Those boundary terms can then be put into the Israel junction conditions to give boundary conditions on the bulk fields' solutions' normal derivatives at the brane [14, 15]. Alternatively, the Gauss-Codacci and Lanczos equations for the $(D-1)$ dimensional gravitational field equations can be used [16]. In the following, we shall follow the second approach.

Let's begin with the Gauss-Codacci equations

$$
G_{\mu\nu}^{(D-1)} = \mathcal{G}_{\mu\nu}^{(D)} + E_{\mu\nu}^{(D)} + \mathcal{F}_{\mu\nu}^{(D-1)}, \tag{2.36}
$$

with

$$\begin{aligned}
\mathcal{G}^{(D)}_{\mu\nu} &\equiv \frac{D-3}{(D-2)}\left\{G^{(D)}_{ab}e^a_{(\mu)}e^b_{(\nu)}\right.\\
&\left.-\left[G_{ab}n^a n^b + \frac{1}{D-1}G^{(D)}\right]g_{\mu\nu}\right\},\\
E^{(D)}_{\mu\nu} &\equiv C^{(D)}_{abcd}n^a e^b_{(\mu)}n^c e^d_{(\nu)},\\
\mathcal{F}^{(D-1)}_{\mu\nu} &\equiv K_{\mu\lambda}K^{\lambda}_{\nu} - KK_{\mu\nu}\\
&-\frac{1}{2}g_{\mu\nu}\left(K_{\alpha\beta}K^{\alpha\beta} - K^2\right),
\end{aligned} \tag{2.37}$$

where n^a denotes the normal vector to the brane, $G^{(D)} \equiv g^{ab}G^{(D)}_{ab}$, and $C^{(D)}_{abcd}$ the Weyl tensor. The extrinsic curvature $K_{\mu\nu}$ is defined as

$$K_{\mu\nu} \equiv e^a_{(\mu)}e^b_{(\nu)}\nabla_a n_b. \tag{2.38}$$

And the Lanczos equations read [17],

$$\left[K^{(I)}_{\mu\nu}\right]^- - g^{(I)}_{\mu\nu}\left[K^{(I)}\right]^- = -\kappa^2_D\mathcal{T}^{(I)}_{\mu\nu}, \tag{2.39}$$

where

$$\begin{aligned}
\left[K^{(I)}_{\mu\nu}\right]^- &\equiv \lim_{\Phi_I\to 0^+}K^{(I)\,+}_{\mu\nu} - \lim_{\Phi_I\to 0^-}K^{(I)\,-}_{\mu\nu},\\
\left[K^{(I)}\right]^- &\equiv g^{(I)\,\mu\nu}\left[K^{(I)}_{\mu\nu}\right]^-.
\end{aligned} \tag{2.40}$$

If the fields have reflection symmetry about the brane surfaces, then the discontinuity in the normal derivatives will simply be twice the boundary value.

Then, substituting into the extrinsic curvatures $K^{(I)}_{\mu\nu}$, the effective energy-momentum tensor $\mathcal{T}^{(I)}_{\mu\nu}$ through the Lanczos equations (2.39), and setting

$$\mathcal{S}^{(I)}_{\mu\nu} = \tau^{(I)}_{\mu\nu} + g^{(I)}_k g^{(I)}_{\mu\nu}, \tag{2.41}$$

where $g^{(I)}_k$ is a coupling constant of the I-th brane [18], we find that

$$\mathcal{T}^{(I)}_{\mu\nu} = \tau^{(I)}_{\mu\nu} + \left(g^{(I)}_k + \tau^{(I)}_p\right)g^{(I)}_{\mu\nu}. \tag{2.42}$$

Then, $G_{\mu\nu}^{(D-1)}$ given by Eq.(2.36) can be cast in the form,

$$
\begin{aligned}
G_{\mu\nu}^{(D-1)} &= \mathcal{G}_{\mu\nu}^{(D)} + E_{\mu\nu}^{(D)} + \mathcal{E}_{\mu\nu}^{(D-1)} + \kappa_D^4 \pi_{\mu\nu} \\
&\quad + \kappa_{D-1}^2 \tau_{\mu\nu} + \Lambda_{D-1} g_{\mu\nu},
\end{aligned} \tag{2.43}
$$

where we have defined

$$
\begin{aligned}
\pi_{\mu\nu} &\equiv \frac{1}{4}\Bigg\{\tau_{\mu\lambda}\tau_{\nu}^{\lambda} - \frac{1}{D-2}\tau\tau_{\mu\nu} \\
&\quad - \frac{1}{2} g_{\mu\nu}\left(\tau^{\alpha\beta}\tau_{\alpha\beta} - \frac{1}{D-2}\tau^2\right)\Bigg\}, \\
\mathcal{E}_{\mu\nu}^{(D-1)} &\equiv \frac{\kappa_D^4 (D-3)}{4(D-2)}\tau_p \\
&\quad \times \left[\tau_{\mu\nu} + \left(g_k + \frac{1}{2}\tau_p\right) g_{\mu\nu}\right],
\end{aligned} \tag{2.44}
$$

and

$$
\begin{aligned}
\kappa_{D-1}^2 &= \frac{D-3}{4(D-2)} g_k \kappa_D^4, \\
\Lambda_{D-1} &= \frac{D-3}{8(D-2)} g_k^2 \kappa_D^4.
\end{aligned} \tag{2.45}
$$

Taking the energy momentum tensor for a perfect fluid,

$$
\tau_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p g_{\mu\nu}, \tag{2.46}
$$

where u_μ is the four-velocity of the fluid, we obtain,

$$
\begin{aligned}
\pi_{\mu\nu} &= \frac{D-3}{4(D-2)}\rho \\
&\quad \times \left[(\rho + p)\, u_\mu u_\nu - \left(p + \frac{1}{2}\rho\right) g_{\mu\nu}\right].
\end{aligned} \tag{2.47}
$$

Note that in writing Eqs.(2.43)-(2.47), the super indices (I) were dropped.

Using the surface $\Phi_I(x) = 0$ to define the brane, we can divide the space-time into two regions, one with $\Phi_I(x) > 0$ and the other with $\Phi_I(x) < 0$ [see Fig.2.1]. Since the field equations are the second-order differential equations, so the matter fields have to be at least continuous across this surface, although in general their

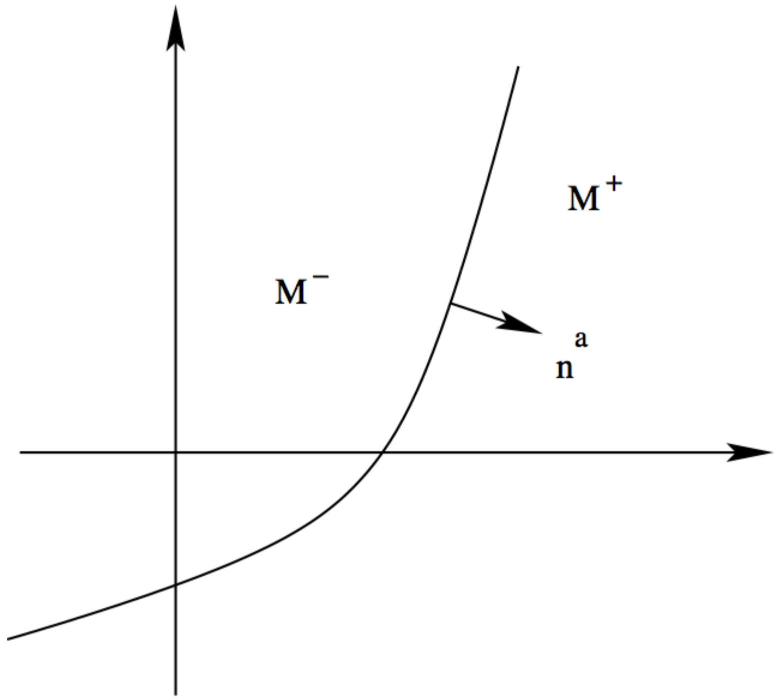

Figure 2.1: The hypersurface Σ, where $\Sigma \equiv \left\{x^A : \Phi_I(x) = 0\right\}$, divides the spacetime into two regions, $M^{\pm}$, where $M_+ \equiv \left\{x^A : \Phi_I(x) > 0\right\}$ and $M_- \equiv \left\{x^A : \Phi_I(x) < 0\right\}$.

first-order directives need not be. Introducing the Heaviside function $H(x)$, defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{2.48}$$

in the neighborhood of $\Phi_I(x) = 0$ we can write the matter fields in the form,

$$F(x) = F^+(x)H(\Phi_I) + F^-(x)\left[1 - H(\Phi_I)\right], \tag{2.49}$$

where $F \equiv \{\phi,\ \psi_\pm,\ \xi_\pm,\ B\}$, and F^+ (F^-) is defined in the region $\Phi_I > 0$ ($\Phi_I < 0$). Then, we find that

$$\begin{aligned} F_{,a}(x) &= F^+_{,a}(x)H(\Phi_I) + F^-_{,a}(x)\left[1 - H(\Phi_I)\right], \\ F_{,ab}(x) &= F^+_{,ab}(x)H(\Phi_I) + F^-_{,ab}(x)\left[1 - H(\Phi_I)\right] \\ &\quad + \left[F_{,a}\right]^- \frac{\partial \Phi_I(x)}{\partial x^b}\,\delta(\Phi_I), \end{aligned} \tag{2.50}$$

where $[F_{,a}]^-$ is defined as that in Eq.(2.40). Projecting $F_{,a}$ onto n^a and $e^a_{(\mu)}$ directions, we find

$$F_{,a} = F_{,\mu} e^{(\mu)}_a - F_{,n} n_a, \tag{2.51}$$

where

$$F_{,n} \equiv n^a F_{,a}, \quad F_{,\mu} \equiv e^a_{(\mu)} F_{,a}. \tag{2.52}$$

Then, we have

$$\begin{aligned} \left[F_{,a}\right]^- n^a &= \left[F_{,n}\right]^- , \\ \left[F_{,a}\right]^- e^a_{(\mu)} &= 0. \end{aligned} \tag{2.53}$$

Inserting Eqs.(2.51)-(2.53) into Eq.(2.50), we find

$$\begin{aligned} F_{,ab}(x) &= F^+_{,ab}(x) H\left(\Phi_I\right) + F^-_{,ab}(x)\left[1 - H\left(\Phi_I\right)\right] \\ &\quad - \left[F_{,n}\right]^- n_a n_b N_I \, \delta\left(\Phi_I\right), \end{aligned} \tag{2.54}$$

where $N_I \equiv \sqrt{|\Phi_{I,c}\Phi_I^{,c}|}$, and

$$n_a = \frac{1}{N_I} \frac{\partial \Phi_I(x)}{\partial x^a}. \tag{2.55}$$

Substituting Eq.(2.54) into Eqs.(2.31)-(2.34), we find that the matter field equations on the branes are,

$$\left[\phi^{(I)}_{,n}\right]^- = -\Psi^{(I)} \left(2\kappa_D^2 \epsilon_I \frac{\partial V^{(I)}_{D-1}}{\partial \phi} + \sigma^{(I)}_\phi\right), \tag{2.56}$$

$$\left[\psi^{(I)}_{\pm,n}\right]^- = -\Psi^{(I)} \left(2\kappa_D^2 \epsilon_I \frac{\partial V^{(I)}_{D-1}}{\partial \psi_\pm} + \sigma^{(I)}_{\psi_\pm}\right), \tag{2.57}$$

$$\begin{aligned} \left[\xi^{(I)}_{\pm,n}\right]^- &= -\frac{\Psi^{(I)}}{2\alpha_\pm} e^{\sqrt{\frac{8}{d_\pm}}\psi_\pm - 2\xi_\pm} \\ &\quad \times \sum_{I=1}^{2} \left(2\kappa_D^2 \epsilon_I \frac{\partial V^{(I)}_{D-1}}{\partial \xi_\pm} + \sigma^{(I)}_{\xi_\pm}\right), \end{aligned} \tag{2.58}$$

$$\left[H^{(I)}_{nab}\right]^- = -\Psi^{(I)} \, \sigma^{(I)}_{ab}, \tag{2.59}$$

where

$$H_{nab} \equiv H_{cab} n^c, \quad \Psi^{(I)} \equiv \frac{1}{N_I} \sqrt{\left|\frac{g^{(I)}_{D-1}}{g_D}\right|}. \tag{2.60}$$

2.3 Orbifold Branes in 5-Dimensional Spacetimes

Choosing $D = 5, d_+ = 3, d_- = 2$ we return to our metric ansatz, for the five-dimensional space-time with a three dimensional spatial space that is homogeneous, isotropic, and independent of time. It can always be written in the form [19],

$$ds_5^2 = g_{ab}dx^a dx^b = g_{MN}dx^M dx^N - e^{2\omega\left(x^M\right)}d\Sigma_k^2, \tag{2.61}$$

where $M,\ N = 0, 1$. In the conformal gauge,

$$g_{00} = g_{11}, \quad g_{01} = 0, \tag{2.62}$$

this becomes,

$$ds_5^2 = e^{2\sigma(t,y)}\left(dt^2 - dy^2\right) - e^{2\omega(t,y)}d\Sigma_k^2. \tag{2.63}$$

Note that metric (2.63) still has the gauge freedom,

$$t = f(t' + y') + g(t' - y'), \quad y = f(t' + y') - g(t' - y'), \tag{2.64}$$

where $f(t' + y')$ and $g(t' - y')$ are arbitrary functions.

From this point on we remove the flux to keep things simple. Let

$$\hat{B}_{CD} = 0, \tag{2.65}$$

so that

$$\xi_\pm = 0, \quad \alpha_\pm = 0, \quad \gamma_\pm = 0. \tag{2.66}$$

2.3.1 Field Equations Outside of Branes

Then, outside the two branes the independent equations of Eqs.(2.27) are,

$$\begin{aligned}
&\omega_{,tt} + \omega_{,t}\left(\omega_{,t} - 2\sigma_{,t}\right) + \omega_{,yy} + \omega_{,y}\left(\omega_{,y} - 2\sigma_{,y}\right) \\
&= -\frac{1}{6}\left(\phi_{,t}^2 + \phi_{,y}^2 + \psi_{+,t}^2 + \psi_{+,y}^2 + \psi_{-,t}^2 + \psi_{-,y}^2\right), \qquad (2.67) \\
&2\sigma_{,tt} + \omega_{,tt} - 3{\omega_{,t}}^2 - \left(2\sigma_{,yy} + \omega_{,yy} - 3{\omega_{,y}}^2\right) - 4ke^{2(\sigma-\omega)}
\end{aligned}$$

$$= -\frac{1}{2}\left(\phi_{,t}^2 - \phi_{,y}^2 + \psi_{+,t}^2 - \psi_{+,y}^2 + \psi_{-,t}^2 - \psi_{-,y}^2\right), \tag{2.68}$$

$$\omega_{,ty} + \omega_{,t}\omega_{,y} - (\sigma_{,t}\omega_{,y} + \sigma_{,y}\omega_{,t})$$
$$= -\frac{1}{6}\left(\phi_{,t}\phi_{,y} + \psi_{+,t}\psi_{+,y} + \psi_{-,t}\psi_{-,y}\right), \tag{2.69}$$

$$\omega_{,tt} + 3\omega_{,t}{}^2 - \left(\omega_{,yy} + 3\omega_{,y}{}^2\right) + 2ke^{2(\sigma-\omega)}$$
$$= \frac{1}{3}e^{2\sigma}V_5. \tag{2.70}$$

On the other hand, the corresponding Klein-Gordon equations take the form,

$$\phi_{,tt} + 3\phi_{,t}\omega_{,t} - (\phi_{,yy} + 3\phi_{,y}\omega_{,y})$$
$$= -\sqrt{\frac{2}{3}}e^{2\sigma}V_5, \tag{2.71}$$

$$\psi_{+,tt} + 3\psi_{+,t}\omega_{,t} - (\psi_{+,yy} + 3\psi_{,y}\omega_{,y})$$
$$= e^{2\sigma}\beta_+ e^{\sqrt{2/3}\phi - \psi_+}, \tag{2.72}$$

$$\psi_{-,tt} + 3\psi_{-,t}\omega_{,t} - (\psi_{-,yy} + 3\psi_{,y}\omega_{,y})$$
$$= \sqrt{\frac{2}{3}}e^{2\sigma}\beta_- e^{\sqrt{2/3}\phi - \psi_-}, \tag{2.73}$$

with

$$V_5 = e^{\sqrt{\frac{2}{3}}\phi}\left(\beta_+ e^{-\psi_+} + \beta_- e^{-\sqrt{\frac{2}{3}}\psi_-}\right). \tag{2.74}$$

2.3.2 Field Equations On the Branes

On each of the two branes, the metric reduces to

$$ds_5^2\Big|_{M_4^{(I)}} = g_{\mu\nu}^{(I)} d\xi_{(I)}^\mu d\xi_{(I)}^\nu = d\tau_I^2 - a^2(\tau_I)\, d\Sigma_k^2, \tag{2.75}$$

where $\xi_{(I)}^\mu \equiv \{\tau_I, r, \theta, \varphi\}$, and τ_I denotes the proper time of the I-th brane, defined by

$$\begin{aligned} d\tau_I &= e^{\sigma[t_I(\tau_I), y_I(\tau_I)]}\sqrt{1 - \left(\frac{\dot{y}_I}{\dot{t}_I}\right)^2}\, dt_I, \\ a(\tau_I) &\equiv e^{\omega[t_I(\tau_I), y_I(\tau_I)]}, \end{aligned} \tag{2.76}$$

with $\dot{y}_I \equiv dy_I/d\tau_I$, etc. For the sake of simplicity we shall drop all the indices "I", unless some specific attention is needed. The normal vector n_a and the tangential vectors $e^a_{(\mu)}$ are given, respectively, by

$$\begin{aligned} n_a &= e^{2\sigma}\left(-\dot{y}\delta^t_a + \dot{t}\delta^y_a\right), \\ n^a &= -\left(\dot{y}\delta^a_t + \dot{t}\delta^a_y\right), \\ e^a_{(\tau)} &= \dot{t}\delta^a_t + \dot{y}\delta^a_y, \quad e^a_{(r)} = \delta^a_r, \\ e^a_{(\theta)} &= \delta^a_\theta, \quad e^a_{(\varphi)} = \delta^a_\varphi. \end{aligned} \tag{2.77}$$

Proceeding to the Gauss-Codacci relations we find that

$$\begin{aligned} \mathcal{G}^{(5)}_{\mu\nu} &= \mathcal{G}^{(5)}_\tau \delta^\tau_\mu \delta^\tau_\nu - \mathcal{G}^{(5)}_\theta \delta^m_\mu \delta^n_\nu g_{mn}, \\ E^{(5)}_{\mu\nu} &= E^{(5)}\left(3\delta^\tau_\mu\delta^\tau_\nu - \delta^m_\mu\delta^n_\nu g_{mn}\right), \end{aligned} \tag{2.78}$$

where

$$\begin{aligned} \mathcal{G}^{(5)}_\tau &\equiv \frac{1}{3}e^{-2\sigma}\left(\phi^2_{,t} - \phi^2_{,y} + \psi^2_{+,t} - \psi^2_{+,y} + \psi^2_{-,t} - \psi^2_{-,y}\right) \\ &\quad -\frac{5}{24}\left[(\nabla\phi)^2 + (\nabla\psi_+)^2 + (\nabla\psi_-)^2\right] + \frac{1}{4}V_5, \\ \mathcal{G}^{(5)}_\theta &\equiv \frac{1}{3}\left[\phi^2_{,n} + \psi^2_{+,n} + \psi^2_{-,n}\right] \\ &\quad +\frac{5}{24}\left[(\nabla\phi)^2 + (\nabla\psi_+)^2 + (\nabla\psi_-)^2\right] - \frac{1}{4}V_5, \\ E^{(5)} &\equiv \frac{1}{6}e^{-2\sigma}\left[(\sigma_{,tt} - \omega_{,tt}) - (\sigma_{,yy} - \omega_{,yy})\right. \\ &\quad \left. +ke^{2(\sigma-\omega)}\right], \end{aligned} \tag{2.79}$$

with $\phi_{,n} \equiv n^a\nabla_a\phi$. The four-dimensional equations on each brane take the form,

$$\begin{aligned} H^2 + \frac{k}{a^2} &= \frac{8\pi G}{3}(\rho + \tau_p) + \frac{1}{3}\Lambda + \frac{1}{3}\mathcal{G}^{(5)}_\tau + E^{(5)} \\ &\quad + \frac{2\pi G}{3\rho_\Lambda}(\rho + \tau_p)^2, \end{aligned} \tag{2.80}$$

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p - 2\tau_p) + \frac{1}{3}\Lambda - E^{(5)} \\ &\quad -\frac{1}{6}\left(\mathcal{G}^{(5)}_\tau + 3\mathcal{G}^{(5)}_\theta\right) - \frac{2\pi G}{3\rho_\Lambda}\left[\rho(2\rho + 3p)\right. \\ &\quad \left. + (\rho + 3p - \tau_p)\tau_p\right], \end{aligned} \tag{2.81}$$

where $H \equiv \dot{a}/a,$, $\Lambda \equiv \Lambda_4$ and $G \equiv G_4$. On the other hand, from Eqs.(2.56) and (2.57), we find

$$\left[\phi^{(I)}_{,n}\right]^{-} = -\left(2\kappa_5^2\epsilon_I\frac{\partial V_4^{(I)}}{\partial\phi} + \sigma_\phi^{(I)}\right)\ \Psi, \tag{2.82}$$

$$\left[\psi^{(I)}_{\pm,n}\right]^{-} = -\left(2\kappa_5^2\epsilon_I\frac{\partial V_4^{(I)}}{\partial\psi_\pm} + \sigma_{\psi_\pm}^{(I)}\right)\ \Psi. \tag{2.83}$$

CHAPTER THREE

Radion Stability

In the two-brane models, one important question is the stability of branes. In this section, we shall address this issue.

3.1 The Spacetime Background with Poincaré Symmetry

The five dimensional static metric with a four dimensional Poincaré symmetry is given by Eq.(2.63) with $k = 0$ and $\sigma(y) = \omega(y)$, that is,

$$ds_5^2 = e^{2\sigma(y)} \left(\eta_{\mu\nu} dx^\mu dx^\nu - dy^2\right). \tag{3.1}$$

Our specific solutions are given by,

$$\begin{aligned}
\sigma(y) &= \frac{1}{3} ln \left(\frac{|y| + y_0}{L}\right), \\
\phi(y) &= c_1 ln \left(\frac{|y| + y_0}{L}\right) + \phi_0, \\
\psi_+(y) &= c_2 ln \left(\frac{|y| + y_0}{L}\right) + \psi_+^0, \\
\psi_-(y) &= \sqrt{\frac{3}{2}} c_2 \ln \left(\frac{|y| + y_0}{L}\right) + \psi_-^0,
\end{aligned} \tag{3.2}$$

where $c_1, y_0,\ L,\ \ \sigma_0,\ \phi_0,$ and ψ_+^0 are all arbitrary constants, and

$$\begin{aligned}
16 &= 6c_1^2 + 15c_2^2, \\
\psi_-^0 &= \sqrt{\frac{3}{2}} \left(\psi_+^0 - \ln\left(\frac{-\beta_+}{\beta_-}\right)\right).
\end{aligned} \tag{3.3}$$

The function $|y|$ is defined as that given in Fig.3.1.

These are the solutions to the equations outside the branes Eqs.(2.67)-(2.72). Boundary conditions must still be enforced using the on brane equations Eqs.(2.80)-(2.81) and Eqs.(2.82)-(2.83). The normal vector $n_{(I)}^a$ to the I-th brane is given by

$$n_{(I)}^a = -\epsilon_y^{(I)} e^{-\sigma(y_I)} \delta_y^a, \tag{3.4}$$

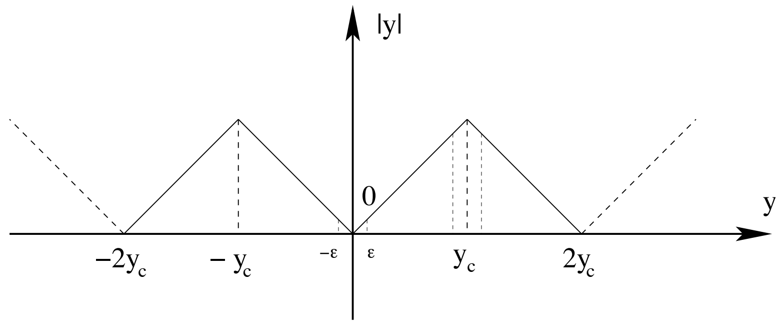

Figure 3.1: The function $|y|$ defined in Eq.(3.2).

and

$$\begin{aligned}\dot{t} &= e^{-\sigma(y_I)}, \;\; \dot{y} = 0,\\ \mathcal{G}^{(5)}_{\tau} &= -\mathcal{G}^{(5)}_{\theta} = -\frac{2}{9L^2}\left(\frac{L}{y_I + y_0}\right)^{\frac{8}{3}},\end{aligned} \tag{3.5}$$

where $y_1 = y_c > 0$ and $y_2 = 0$. Inserting the above into Eqs.(2.80) and (2.81), and $H = 0$, the equations are satisfied for $\tau^{(I)}_{\mu\nu} = 0$, provided that the tension $\tau^{(I)}_p$ defined by Eq.(2.30) satisfies the relation,

$$\left(\tau^{(I)}_{(\phi,\psi_\pm)} + 2\rho_\Lambda{}^{(I)}\right)^2 = \frac{\rho^{(I)}_\Lambda}{9\pi G_4 L^2}\left(\frac{L}{y_I + y_0}\right)^{8/3}, \tag{3.6}$$

where $\rho^{(I)}_\Lambda$ denotes the corresponding energy density of the effective cosmological constant on the I-th brane, defined as $\rho^{(I)}_\Lambda = \Lambda^{(I)}/(8\pi G)$. From Eqs.(2.82) and (2.83), we find that

$$\frac{\partial V^{(I)}_4}{\partial \phi} = \frac{c_1 \epsilon_I}{\kappa^2_5(y_I + y_0)}, \tag{3.7}$$

$$\frac{\partial V^{(I)}_4}{\partial \psi_+} = \frac{c_2 \epsilon_I}{\kappa^2_5(y_I + y_0)}, \tag{3.8}$$

$$\frac{\partial V^{(I)}_4}{\partial \psi_-} = -\frac{\sqrt{3}c_2 \epsilon_I}{\sqrt{2}\kappa^2_5(y_I + y_0)}. \tag{3.9}$$

To study the radion stability, it is convenient to introduce the proper distance Y, defined by

$$\mathcal{D}(t) = \int_{y_2}^{y_1} e^{\sigma(t,y)} dy. \tag{3.10}$$

where $y_2(t_2) < y < y_1(t_1)$, and $y = y_I(t_I)$ denote the locations of the two branes. For our case this gives

$$Y = \frac{3L}{4}\left(\left(\frac{y_I + y_0}{L}\right)^{4/3} - \left(\frac{y_0}{L}\right)^{4/3}\right). \quad (3.11)$$

Then, in terms of Y, the static solution (3.1) can be written as

$$ds_5^2 = e^{-2A(Y)}\eta_{\mu\nu}dx^\mu dx^\nu - dY^2, \quad (3.12)$$

with

$$\begin{aligned} A(Y) &= -\frac{1}{4}\ln\left(\frac{4\left(|Y| + Y_0\right)}{3L}\right), \\ \phi(Y) &= \frac{3}{4}c_1\ln\left(\frac{4\left(|Y| + Y_0\right)}{3L}\right) + \phi_0, \\ \psi_+(Y) &= \frac{3}{4}c_2\ln\left(\frac{4\left(|Y| + Y_0\right)}{3L}\right) + \psi_+^0, \\ \psi_-(Y) &= \sqrt{\frac{27}{32}}\,c_2\ln\left(\frac{4\left(|Y| + Y_0\right)}{3L}\right) + \psi_-^0, \end{aligned} \quad (3.13)$$

where

$$Y_0 = \frac{3L}{4}\left(\frac{y_0}{L}\right)^{4/3}. \quad (3.14)$$

In comparison to the Randal-Sundrum setup, this metric has $\sqrt{Y}$ warp factor, instead of e^{-Y}. In order to get a feel for this space, consider null rays propagating normal to the brane. These will obey

$$Y^5 = \frac{3L}{4}\left(t_0 \pm \frac{5}{4}t\right)^4. \quad (3.15)$$

So while we do not have the Cauchy problem of incoming signals arriving from infinity in finite time, we we still find it useful to use two branes to compactify this bulk space. The monotonic nature of the warp factor then implies that, at least for static branes, one will have positive and the other negative tension.

Following [20],we consider a massive scalar test field Φ with the actions,

$$\begin{aligned} S_b &= \int d^4x \int_0^{Y_c} dY \sqrt{-g_5}\left((\nabla\Phi)^2 - M^2\Phi^2\right), \\ S_I &= -\alpha_I \int_{M_4^{(I)}} d^4x \sqrt{-g_4^{(I)}}\left(\Phi^2 - v_I^2\right)^2, \end{aligned} \tag{3.16}$$

where α_I and v_I are real constants. In the background of Eq.(3.12), the field Φ satisfies the Klein-Gordon equation

$$\Phi'' - 4A'\Phi' - M^2\Phi = \sum_{I=1}^{2} 2\alpha_I \Phi\left(\Phi^2 - v_I^2\right)\delta(Y - Y_I). \tag{3.17}$$

Using a pill box integration over the the I-th brane, we find that

$$\left.\frac{d\Phi(Y)}{dY}\right|_{Y_I-\epsilon}^{Y_I+\epsilon} = 2\alpha_I \Phi_I\left(\Phi_I^2 - v_I^2\right), \tag{3.18}$$

where $\Phi_I \equiv \Phi(Y_I)$. Since

$$\begin{aligned} \lim_{Y\to Y_c^+}\frac{d\Phi(Y)}{dY} &= -\lim_{Y\to Y_c^-}\frac{d\Phi(Y)}{dY} \equiv -\Phi'(Y_c), \\ \lim_{Y\to 0^-}\frac{d\Phi(Y)}{dY} &= -\lim_{Y\to 0^+}\frac{d\Phi(Y)}{dY} \equiv -\Phi'(0), \end{aligned} \tag{3.19}$$

the conditions (3.18) can be written as,

$$\Phi'(Y_c) = -\alpha_1 \Phi_1\left(\Phi_1^2 - v_1^2\right), \tag{3.20}$$

$$\Phi'(0) = \alpha_2 \Phi_2\left(\Phi_2^2 - v_2^2\right). \tag{3.21}$$

Inserting the above solution back to the actions (3.16), and integrating over Y, we get the effective potential for the radion Y_c,

$$\begin{aligned} V_\Phi(Y_c) &\equiv -\int_{0+\epsilon}^{Y_c-\epsilon} dY\sqrt{-g_5}\left((\nabla\Phi)^2 - M^2\Phi^2\right) \\ &\quad + \sum_{I=1}^{2}\alpha_I \int_{Y_I-\epsilon}^{Y_I+\epsilon} dY\sqrt{-g_4^{(I)}}\left(\Phi^2 - v_I^2\right)^2 \\ &\quad\quad \times\delta(Y - Y_I) \\ &= \left. e^{-4A(Y)}\Phi(Y)\Phi'(Y)\right|_0^{Y_c} \\ &\quad + \sum_{I=1}^{2}\alpha_I\left(\Phi_I^2 - v_I^2\right)^2 e^{-4A(Y_I)}. \end{aligned} \tag{3.22}$$

For our solution given by Eq.(3.13) and Eq.(3.17) in the region $0 < Y < Y_c$, we find that

$$\frac{d^2\Phi}{dz^2} + \frac{1}{z}\frac{d\Phi}{dz} - \Phi = 0, \tag{3.23}$$

where $z \equiv M(Y + Y_0)$. Eq.(3.23) has the general solution,

$$\Phi = aI_0(z) + bK_0(z), \tag{3.24}$$

where $I_0(z)$ and $K_0(z)$ denote the modified Bessel function of the first and second kind, respectively [21]. In the limit that α_I's are very large [20], Eqs.(3.20) and (3.21) show that there are solutions only when $\Phi(0) \simeq v_2$ and $\Phi(Y_c) \simeq v_1$, that is,

$$v_1 \simeq aI_0^c + bK_0^c, \tag{3.25}$$

$$v_2 \simeq aI_0^0 + bK_0^0, \tag{3.26}$$

where

$$\begin{aligned} &z_c = M(Y_c + Y_0), \quad z_0 = MY_0, \\ &I_0^i \equiv I_0(z_i), \quad K_0^i \equiv K_0(z_i). \end{aligned} \tag{3.27}$$

Eqs.(3.25) and (3.26) have the solution,

$$\begin{aligned} a &\simeq \frac{1}{\Delta}\left(v_1K_0^0 - v_2K_0^c\right), \\ b &\simeq \frac{1}{\Delta}\left(v_2I_0^c - v_1I_0^0\right), \end{aligned} \tag{3.28}$$

where

$$\Delta \equiv I_0^cK_0^0 - I_0^0K_0^c. \tag{3.29}$$

Inserting Eqs.(3.24) and (3.28) into Eq.(3.22), we find that

$$\begin{aligned} V_\Phi(Y_c) \simeq \; & \frac{4}{3L\Delta}\left\{v_1z_c\left[v_1\left(I_0^0K_1^c + I_1^cK_0^0\right)\right.\right. \\ & \left. -v_2\left(I_0^cK_1^c + I_1^cK_0^c\right)\right] \\ & +v_2z_0\left[v_2\left(I_0^cK_1^0 + I_1^0K_0^c\right)\right. \\ & \left.\left. -v_1\left(I_0^0K_1^0 + I_1^0K_0^0\right)\right]\right\}. \end{aligned} \tag{3.30}$$

When $z_0 \gg 1$, we have $z_c = z_0 + MY_c \gg 1$. Then, we find

$$\begin{aligned} I_0(z) &\simeq I_1(z) \simeq \sqrt{\frac{1}{2\pi z}} e^z, \\ K_0(z) &\simeq K_1(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z}. \end{aligned} \tag{3.31}$$

Substituting them into Eq.(3.30), we get

$$V_\Phi(Y_c) \simeq \frac{4z_0}{3L\sinh(MY_c)} \left\{ \left({v_1}^2 + {v_2}^2 \right) \cosh(MY_c) - 2v_1 v_2 \right\}, \tag{3.32}$$

which has a minimum at

$$Y_c^{min.} = \frac{1}{M} \cosh^{-1} \left(\frac{{v_1}^2 + {v_2}^2}{2v_1 v_2} \right), \tag{3.33}$$

where

$$\begin{aligned} \left. \frac{\partial^2 V_\Phi(Y_c)}{\partial {Y_c}^2} \right|_{Y_c = Y_c^{min.}} &\simeq \left(\frac{16 z_0 M^2}{3L} \right) \frac{(v_1 v_2)^2}{|v_1^2 - v_2^2|} > 0, \\ V_\Phi(Y_c) &\simeq \begin{cases} \infty, & Y_c = 0, \\ \infty, & Y_c = \infty. \end{cases} \end{aligned} \tag{3.34}$$

This shows the potential always has a minimum at a finite and non-zero value of Y_c. As a result, the radion is stabilized.

To calculate the corresponding radion mass, we need to know the precise relation between Y_c and the radion scalar φ. Following [19, 20], we find that

$$\begin{aligned} \varphi &= \left(\frac{12}{\kappa_5^2} \int_0^{Y_c} e^{-2A} dY \right)^{1/2} = \sqrt{6LM_5^3} \\ &\quad \times \left\{ \left(\frac{4(Y_c + Y_0)}{3L} \right)^{3/2} - \left(\frac{4Y_0}{3L} \right)^{3/2} \right\}^{1/2}. \end{aligned} \tag{3.35}$$

So,

$$\begin{aligned} m_\varphi^2 &\equiv \left. \frac{\partial^2 V_\Phi(Y_c)}{2\partial \varphi^2} \right|_{Y_c = Y_c^{min.}} = \frac{M^2}{M_5^3} \left(\frac{16 Y_0}{27L} \right)^{1/2} \\ &\quad \times \frac{(v_1 v_2)^2}{|v_1^2 - v_2^2|} \cosh^{-1} \left(\frac{{v_1}^2 + {v_2}^2}{2 v_1 v_2} \right), \end{aligned} \tag{3.36}$$

where $M_5^3 = M_{10}^8 V_{d_+} V_{d_-}$.

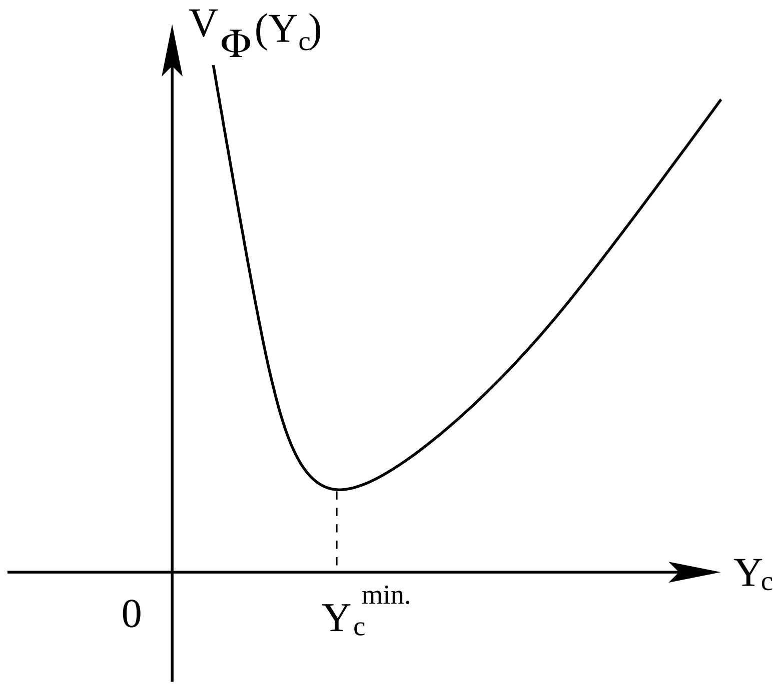

Figure 3.2: The potential $V_\Phi(Y_c)$ defined by Eq.(3.32) for $M \gg 1/Y_0$.

When $z \ll 1$, we find

$$\begin{aligned} I_0(z) &\simeq 1, \quad I_1(z) \simeq \frac{z}{4}, \\ K_0(z) &\simeq -\ln(z), \quad K_1(z) \simeq \frac{1}{z}. \end{aligned} \tag{3.37}$$

Then, Eq.(3.30) reduces to

$$\begin{aligned} V_\Phi(Y_c) \simeq& \frac{v_1 - v_2}{3LY_c} \left\{ (v_1 - v_2) \left(4 - z_0^2 \ln(z_0)\right) Y_0 \right. \\ & \left. + z_0^2 \left(v_2 - 2v_1 \ln(z_0)\right) Y_c \right\}, \end{aligned} \tag{3.38}$$

for $Y_c \ll Y_0$. This potential has no minimum, as shown in Fig. 3.3, leading to an unstable radion for $M \ll 1/Y_0$.

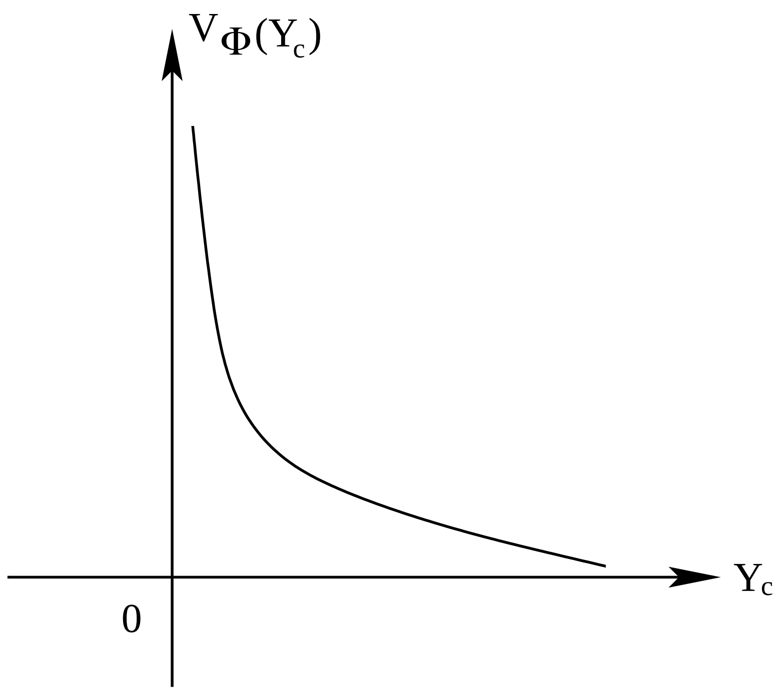

Figure 3.3: The potential $V_\Phi\,(Y_c)$ defined in Eq.(3.38) for $M \ll 1/Y_0$.

CHAPTER FOUR

Localization of Gravity and 4D Effective Newtonian Potential

One of the most important aspects to any model is where it connects to experiment. The most important is to reproduce the newtownian gravity limit. In models with five large dimensions, care must be taken to verify that the inverse square law is still approximately obeyed by masses on the brane. In simple compactification schemes the use of gauss's law gives the correct limit when the distance is large compared to the size of the extra dimension. This is the source of some of the most stringent experimental constraint on the size of extra dimensions. In the single brane Randal-Sundrum type models the limit is accomplished by a more subtle effect, that of the warped nature of the extra dimension itself, namely the convergent nature of the ADS warp factor. In these orbifold models this issue remains important. To study the localization of gravity and the four-dimensional effective gravitational potential, in this section let us consider small fluctuations h_{ab} of the five dimensional static metric with a four dimensional Poincaré symmetry, given by Eqs.(3.1) in its conformally flat form.

4.1 Tensor Perturbations and the KK Towers

Since such tensor perturbations are not coupled with scalar ones [22], without loss of generality we can set the perturbations of the scalar fields to zero, i.e., $\delta\phi_n = 0$. We choose the gauge [23, 24]

$$h_{ay} = 0, \quad h^{\lambda}_{\lambda} = 0 = \partial^{\lambda} h_{\mu\lambda}. \tag{4.1}$$

Then, it can be shown that [25]

$$\begin{aligned}\delta G^{(5)}_{ab} &= -\frac{1}{2}\Box_5 h_{ab} - \frac{3}{2}\left\{(\partial_c\sigma)\left(\partial^c h_{ab}\right)\right. \\ &\quad \left. -2\left[\Box_5\sigma + (\partial_c\sigma)\left(\partial^c\sigma\right)\right] h_{ab}\right\},\end{aligned}$$

$$\begin{aligned}
\kappa_5^2 \delta T_{ab}^{(5)} &= -\frac{1}{4} h_{ab} \left(\sum_{n=1} \left(\nabla \phi_n\right)^2 - 2V_5 \right), \\
\delta T_{\mu\nu}^{(4)} &= \left(\tau_p + \lambda\right) h_{\mu\nu}, \qquad (4.2)
\end{aligned}$$

where $\Box_5 \equiv \eta^{ab}\partial_a\partial_b$ and $(\partial_c\sigma)(\partial^c h_{ab}) \equiv \eta^{cd}(\partial_c\sigma)(\partial_d h_{ab})$, with η^{ab} being the five-dimensional Minkowski metric. Substituting the above expressions into the gravitational field equations (2.27) with $D = 5$, we find that in the present case there is only one independent equation, given by

$$\Box_5 \tilde{h}_{\mu\nu} + \frac{3}{2}\left(\sigma'' + \frac{3}{2}\sigma'^2\right)\tilde{h}_{\mu\nu} = 0, \qquad (4.3)$$

where $h_{\mu\nu} \equiv e^{-3\sigma/2}\tilde{h}_{\mu\nu}$. Setting

$$\begin{aligned}
&\tilde{h}_{\mu\nu}(x,y) = \hat{h}_{\mu\nu}(x)\psi(y), \\
&\Box_5 = \Box_4 - \nabla_y^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu - \partial_y^2, \\
&\Box_4 \hat{h}_{\mu\nu}(x) = -m^2 \hat{h}_{\mu\nu}(x), \qquad (4.4)
\end{aligned}$$

we find that Eq.(4.3) takes the form of the Schrödinger equation,

$$\left(-\nabla_y^2 + V\right)\psi = m^2\psi, \qquad (4.5)$$

where

$$\begin{aligned}
V &\equiv \frac{3}{2}\left(\sigma'' + \frac{3}{2}\sigma'^2\right) \\
&= -\frac{1}{4\left(|y| + y_0\right)^2} + \frac{\delta(y)}{y_0} - \frac{\delta\left(y - y_c\right)}{y_c + y_0}. \qquad (4.6)
\end{aligned}$$

From the above expression we can see clearly that the potential has a delta-function well at $y = y_c$, which is responsible for the localization of the graviton on this brane. In contrast, the potential has a delta-function barrier at $y = 0$, which makes the gravity delocalized on the $y = 0$ brane. Fig. 4.1 shows the potential schematically.

Integration of Eq.(4.5) in the neighborhood of $y = 0$ and $y = y_c$ yields, respectively, the boundary conditions,

$$\lim_{y \to y_c^-} \psi'(y) = \frac{1}{2\left(y_c + y_0\right)} \lim_{y \to y_c^-} \psi(y), \qquad (4.7)$$

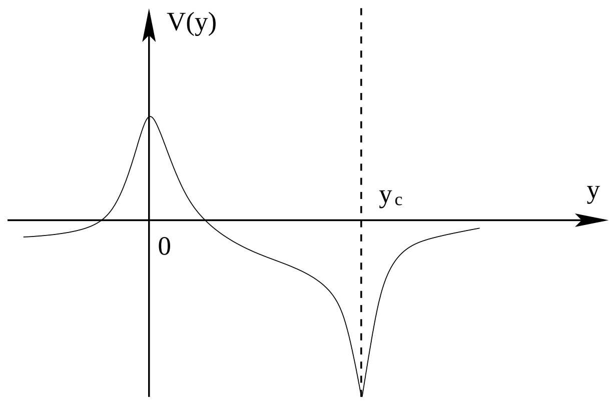

Figure 4.1: The potential defined by Eq.(4.6).

$$\lim_{y \to 0^+} \psi'(y) \;\; = \;\; \frac{1}{2y_0} \lim_{y \to 0^+} \psi(y). \tag{4.8}$$

Note that in writing the above equations we had used the Z_2 symmetry of the wave function ψ.

Introducing the operators,

$$Q \equiv \nabla_y - \frac{3}{2}\sigma', \quad Q^\dagger \equiv -\nabla_y - \frac{3}{2}\sigma', \tag{4.9}$$

Eq.(4.5) can be written in the form of a supersymmetric quantum mechanics problem,

$$Q^\dagger \cdot Q\psi = m^2\psi, \tag{4.10}$$

which, together with the boundary conditions (4.7) and (4.8), guarantees that the operator $Q^\dagger \cdot Q$ is Hermitian [26, 19]. Then, by the usual theorems from Quantum Mechanics [27], we can see that all eigenvalues m^2 are non-negative, and their corresponding wave functions $\psi_n(y)$ are orthogonal to each other and form a complete basis. Therefore, the background in the current setup is gravitationally stable.

4.1.1 Zero Mode

The four-dimensional gravity is given by the existence of the normalizable zero mode, for which the corresponding wavefunction is given by

$$\psi_0(y) = N_0 \left(|y| + y_0\right)^{1/2}, \tag{4.11}$$

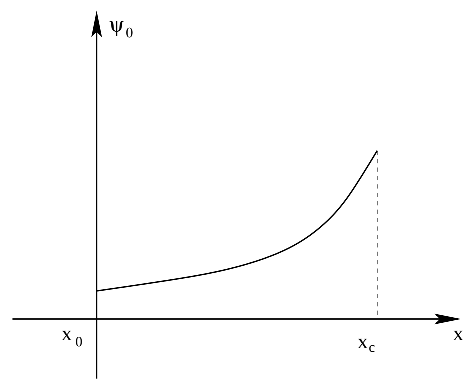

Figure 4.2: The wavefunction $\psi_0(y)$ defined by Eq.(4.11) for the zero mode.

where N_0 is the normalization factor, defined as

$$N_0 = \sqrt{\frac{2}{y_c\,(y_c + 2y_0)}}. \tag{4.12}$$

Eq.(4.11) shows clearly that the wavefunction is increasing as y increases from 0 to y_c [see Fig. 4.2]. Therefore, the gravity is indeed localized near the $y = y_c$ brane.

4.1.2 Non-Zero Modes

In order to have localized four-dimensional gravity, we require that the corrections to the Newtonian law from the non-zero modes, the KK modes, of Eq.(4.5), be very small, so that they will not contradict observations. When $m \neq 0$, it can be shown that Eq.(4.5) has the general solution,

$$\psi = x^{1/2}\left(cJ_0(x) + dY_0(x)\right), \tag{4.13}$$

where $x \equiv m\,(y + y_0)$, and $J_0(x)$ and $Y_0(x)$ are the Bessel functions of the first and second kind, respectively [21]. The integration constants c and d are determined from the boundary conditions, Eqs.(4.7) and (4.8), which can now be cast in the form,

$$\begin{pmatrix} J_1(x_c) & Y_1(x_c) \\ J_1(x_0) & Y_1(x_0) \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 0, \tag{4.14}$$

where $x_0 \equiv my_0$ and $x_c \equiv x_0 + my_c$. Clearly, there are no trivial solutions only when

$$\Delta\,(x_0, x_c) \;\equiv\; J_1(x_c)\,Y_1(x_0) - J_1(x_0)\,Y_1(x_c) = 0. \tag{4.15}$$

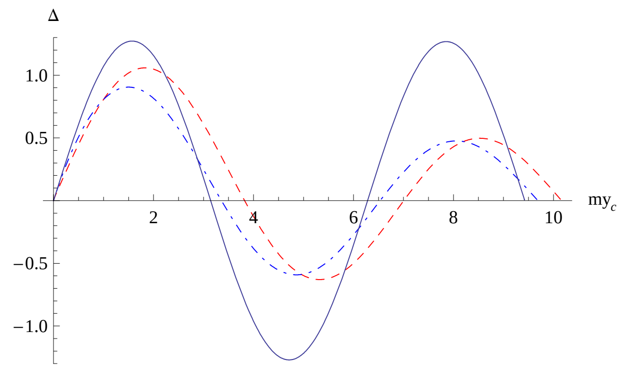

Figure 4.3: The re-scaled function of Δ defined by Eq.(4.15), where the dashed, dot-dashed and solid lines are, respectively, for $\Delta\left(x_0 = 0.01\right)/35$, $\Delta\left(x_0 = 1.0\right)/0.5$ and $\Delta\left(x_0 = 1000\right)/0.005$.

Table 4.1: The first three modes m_n $(n = 1, 2, 3)$ for $x_0 = 0.01,\ 1.0,\ 1000$, respectively.

x_0	$m_1 y_c$	$m_2 y_c$	$m_3 y_c$
0.01	3.82	7.01	10.16
1.0	3.36	6.53	9.69
1000	3.14	6.28	9.42

Fig. 4.3 shows the function $\Delta\left(x_0, m y_c\right)$ for $x_0 = 0.01,\ 1.0,\ 1000$, respectively. From this figure, we find that the spectrum of the gravitational KK towers is discrete, and weakly depends on the specific values of x_0.

Table I shows the first three modes m_n $(n = 1, 2, 3)$ for $x_0 = 0.01,\ 1.0,\ 1000$, from which it can be seen that to find m_n, it is sufficient to consider only the case $x_0 \gg 1$. When $x_0 \gg 1$, we find that $x_c = x_0 + m y_c \gg 1$ and [21]

$$\begin{aligned} J_1(x) &\simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{3}{4}\pi\right), \\ Y_1(x) &\simeq \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{3}{4}\pi\right). \end{aligned} \tag{4.16}$$

Inserting the above expressions into Eq.(4.15), we obtain

$$\Delta = \frac{2}{\pi\sqrt{x_0 x_c}} \sin\left(m y_c\right), \tag{4.17}$$

with roots given by

$$m_n = \frac{n\pi}{y_c}, \quad (n = 1, 2, ...). \tag{4.18}$$

In particular, we have

$$\begin{aligned} m_1 &\simeq 3.14 \times \left(\frac{10^{-19}\ \mathrm{m}}{y_c}\right) \mathrm{TeV} \\ &\simeq \begin{cases} 1\ \mathrm{TeV}, & y_c \simeq 10^{-19}\ \mathrm{m}, \\ 10^{-2}\ \mathrm{eV}, & y_c \simeq 10^{-5}\ \mathrm{m}, \\ 10^{-4}\ \mathrm{eV}, & y_c \simeq 10^{-3}\ \mathrm{m}. \end{cases} \end{aligned} \tag{4.19}$$

It should be noted that the mass m_n calculated above is measured by the observer with the metric $\eta_{\mu\nu}$. However, since the warped factor $e^{\sigma(y)}$ is not one at $y = y_c$, the physical mass on the visible brane should be given by [23]

$$m_n^{obs} = e^{-\sigma(y_c)} m_n = \left(\frac{y_c + y_0}{L}\right)^{1/3} m_n. \tag{4.20}$$

Without introducing any new hierarchy, we expect that $\left[(y_c + y_0)/L\right]^{1/3} \simeq \mathcal{O}(1)$. As a result, we have

$$m_n^{obs} = \left(\frac{y_c + y_0}{L}\right)^{1/5} m_n \simeq m_n. \tag{4.21}$$

For each m_n that satisfies Eq.(4.15), the wavefunction $\psi_n(y)$ is given by

$$\psi_n(y) = N_n x_n^{1/2} \left(\frac{J_0(x_n)}{J_1(x_{0,n})} - \frac{Y_0(x_n)}{Y_1(x_{0,n})}\right), \tag{4.22}$$

where

$$\begin{aligned} x_{0,n} &\equiv m_n y_0 \simeq n\pi \left(\frac{y_0}{y_c}\right), \\ x_n &\equiv m_n (y_0 + y) \simeq n\pi \left(\frac{y_0 + y}{y_c}\right). \end{aligned} \tag{4.23}$$

The normalization factor $N_n [\equiv N_n(m_n, y_c)]$ is determined by the condition,

$$\int_0^{y_c} |\psi_n(y)|^2 \, dy = 1. \tag{4.24}$$

Figs. 4.4, 4.5 and 4.6 show $\psi_1(y)$, $\psi_2(y)$ and $\psi_3(y)$ for $x_{0,1} = 100,\ 102,\ 104$, respectively.

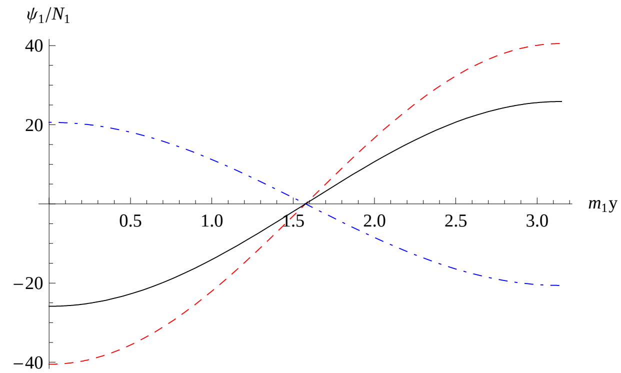

Figure 4.4: The wavefunction, $\psi_1(y)$, defined by Eq.(4.22) vs $m_1 y$ where $y \in [0, y_c]$. The dashed, dot-dashed and solid lines are, respectively, for $x_{0,1} = 100,\ 102,\ 104$.

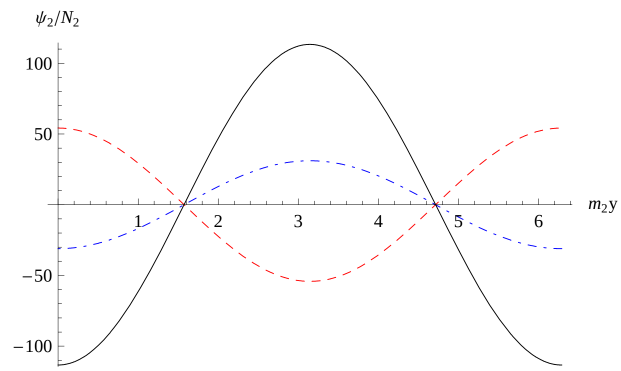

Figure 4.5: The wavefunction, $\psi_2(y)$, defined by Eq.(4.22), vs $m_2 y$ where $y \in [0, y_c]$. The dashed, dot-dashed and solid lines are, respectively, for $x_{0,1} = 100,\ 102,\ 104$.

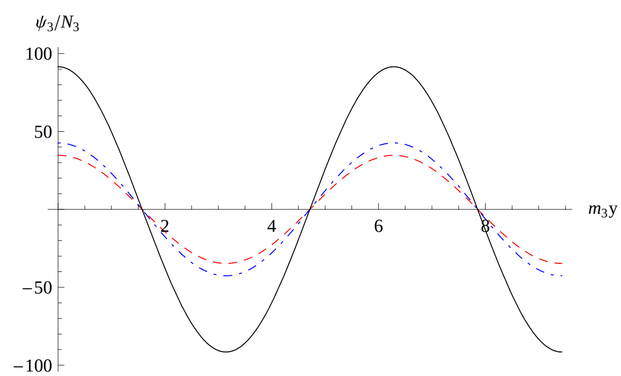

Figure 4.6: The wavefunction, $\psi_3(y)$, defined by Eq.(4.22), vs $m_3 y$ where $y \in [0, y_c]$. The dashed, dot-dashed and solid lines are, respectively, for $x_{0,1} = 100,\ 102,\ 104$.

To calculate the four-dimensional effective Newtonian potential and its corrections, we consider two point-like sources of masses M_1 and M_2, located on the brane at $y = y_c$. Then, the discrete eigenfunction $\psi_n(z)$ of mass m_n has a Yukawa correction to the four dimensional gravitational potential between the two particles [28, 25],

$$U(r) = G_4 \frac{M_1 M_2}{r} + \frac{M_1 M_2}{M_5^3 r} \sum_{n=1}^{\infty} e^{-m_n r} \left|\psi_n\left(y_c\right)\right|^2, \tag{4.25}$$

where $\psi_n\left(y_c\right)$ is given by Eq.(4.22), with

$$x_{c,n} \equiv m_n\left(y_c + y_0\right) \simeq \frac{n\pi y_0}{y_c} + n\pi. \tag{4.26}$$

When $x_{0,1} = m_1 y_0 \gg 1$, we find that

$$\begin{aligned} N_n &\simeq \frac{\cos\left(2m_n y_0\right)}{\sqrt{2n\pi y_0}}, \\ \psi_n\left(y_c\right) &\simeq (-1)^{n+1}\sqrt{\frac{2}{y_c}}. \end{aligned} \tag{4.27}$$

Then, we obtain,

$$\left|\psi_n\left(y_c\right)\right|^2 \simeq 2M_{pl}\left(\frac{l_{pl}}{y_c}\right). \tag{4.28}$$

Clearly, by properly choosing y_c, the corrections of the four dimensional Newtonian potential due to the high order gravitational KK modes are negligible.

CHAPTER FIVE

Cosmological Model in the String/M-Theory

On each of the two branes, the metric reduces to

$$ds_5^2\Big|_{M_4^{(I)}} = g^{(I)}_{\mu\nu} d\xi^{\mu}_{(I)} d\xi^{\nu}_{(I)} = d\tau_I^2 - a^2\left(\tau_I\right) d\Sigma_k^2, \tag{5.1}$$

where $\xi^{\mu}_{(I)} \equiv \{\tau_I, r, \theta, \varphi\}$, and τ_I denotes the proper time of the I-th brane, defined by

$$\begin{aligned} d\tau_I &= e^{\sigma[t_I(\tau_I), y_I(\tau_I)]}\sqrt{1 - \left(\frac{\dot{y}_I}{\dot{t}_I}\right)^2}\, dt_I, \\ a\left(\tau_I\right) &\equiv e^{\omega[t_I(\tau_I), y_I(\tau_I)]}, \end{aligned} \tag{5.2}$$

with $\dot{y}_I \equiv dy_I/d\tau_I$, etc. For the sake of simplicity and without of causing any confusion, from now on we shall drop all the indices "I", unless some specific attention is needed. Then, the normal vector n_a and the tangential vectors $e^a_{(\mu)}$ are given, respectively, by

$$\begin{aligned} n_a &= e^{2\sigma}\left(-\dot{y}\delta^t_a + \dot{t}\delta^y_a\right), \\ n^a &= -\left(\dot{y}\delta^a_t + \dot{t}\delta^a_y\right), \\ e^a_{(\tau)} &= \dot{t}\delta^a_t + \dot{y}\delta^a_y, \quad e^a_{(r)} = \delta^a_r, \\ e^a_{(\theta)} &= \delta^a_\theta, \quad e^a_{(\varphi)} = \delta^a_\varphi. \end{aligned} \tag{5.3}$$

Thus, we find that

$$\begin{aligned} \mathcal{G}^{(5)}_{\mu\nu} &= \mathcal{G}^{(5)}_{\tau}\delta^{\tau}_{\mu}\delta^{\tau}_{\nu} - \mathcal{G}^{(5)}_{\theta}\delta^{m}_{\mu}\delta^{n}_{\nu} g_{mn}, \\ E^{(5)}_{\mu\nu} &= E^{(5)}\left(3\delta^{\tau}_{\mu}\delta^{\tau}_{\nu} - \delta^{m}_{\mu}\delta^{n}_{\nu} g_{mn}\right), \end{aligned} \tag{5.4}$$

where $m,\ n = r,\ \theta,\ \varphi$, Then, it can be shown that the four-dimensional field equations on each of the two branes take the form,

$$H^2 + \frac{k}{a^2} = \frac{2\pi G}{3\rho_\Lambda}\left(\rho + \tau_\phi + 2\rho_\Lambda\right)^2 + \frac{1}{3}\mathcal{G}^{(5)}_{\tau} + E^{(5)}, \tag{5.5}$$

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{2\pi G}{3}\left(\rho+3p-2\rho_\Lambda-\tau_\phi\right)\left(\rho+2\rho_\Lambda+\tau_\phi\right) \\ &\quad -E^{(5)}-\frac{1}{6}\left(\mathcal{G}_\tau^{(5)}+3\mathcal{G}_\theta^{(5)}\right), \end{aligned} \tag{5.6}$$

where $H\equiv \dot{a}/a,,\ \Lambda\equiv\Lambda_4,\ G\equiv G_4$ and $\rho_\Lambda=\Lambda/(8\pi G)$.

Using Eqs.(5.5) and (5.6), the conservation law can be written as

$$\dot{\rho}+\dot{\tau_\phi}+3H(\rho+p)=\frac{1}{\Delta}[\frac{1}{3}\dot{\mathcal{G}}_\tau+\dot{E^{(5)}}+H(\mathcal{G}_\tau^{(5)}+\mathcal{G}_\theta^{(5)}+4E^{(5)})] \tag{5.7}$$

where

$$\Delta=\frac{4\pi G}{3}(2\rho_\Lambda+\rho+\tau_\phi). \tag{5.8}$$

5.1 Particular Case

Based on the string or Horava-Witten hetereotic M-theory, we have several particular cases of the cosmological models. In the case of the branes in the $M_D\times M_{d^+}\times M_{d^-}$ Compactification of type II string on S^1/Z_2 [29], we find the Friedmann (2.80) equation and conservation equation (5.8) give

$$H^2=\frac{2\pi G}{3\rho_\Lambda}\left(\rho+\tau_\phi+2\rho_\Lambda\right)^2-\frac{1}{9L^2a^8}, \tag{5.9}$$

$$\dot{\rho}+\dot{\tau_\phi}+3H(\rho+p)=-4H(2\rho_\Lambda+\rho+\tau_\phi), \tag{5.10}$$

5.2 General Case

Next we will solve the kind of the Friedmann equation and conservation equation in more general case,

$$H^2=\frac{2\pi G}{3\rho_\Lambda}\left(\rho+\tau_\phi+2\rho_\Lambda\right)^2-\rho_a, \tag{5.11}$$

$$\dot{\rho}+\dot{\tau_\phi}+3H(\rho+p)=-\alpha H(2\rho_\Lambda+\rho+\tau_\phi). \tag{5.12}$$

To solve the equations Eq.(5.11) and Eq.(5.12), we assume the interaction between the matter fields and the potential field can be written as

$$\dot{\rho}+3H(\rho+p)=Q(a)H, \tag{5.13}$$

and that the state equation is given by,

$$p = \omega\rho. \tag{5.14}$$

Then the solution for ρ is given by

$$\rho = a^{-3(1+\omega)} \int Q a^{2+3\omega} + c_0 a^{-3(1+\omega) da} \tag{5.15}$$

On the other side, the potential τ_ϕ satisfies

$$\dot{\tau_\phi} = -\alpha H(2\rho_\Lambda + \rho + \tau_\phi) - QH, \tag{5.16}$$

which has the solution,

$$\tau_\phi = -2\rho_\Lambda - \frac{\alpha}{a^\alpha} \int a^{\alpha-1} \rho da - \frac{1}{a} \int a^{\alpha-1} Q da + \frac{b_0}{a^\alpha}, \tag{5.17}$$

with b_0 constant.

If we expand the interaction term $Q(a)$ as a polynomial series and consider the different components of the matter fields, we have

$$Q(a) = \sum_i Q_i(a) = \sum_{p=0,-1,-2...} Q_p^i a^p. \tag{5.18}$$

where $i = m, \gamma$ for matter and radiation. Substituting Eq.(5.18) into Eq.(5.15) and ignore the logarithm term, we have

$$\rho = \sum_i \rho_i \tag{5.19}$$

with

$$\rho_i = \sum_{p \neq -3(1+\omega_i)} \frac{Q_p^i}{p + 3 + 3\omega_i} a^p + c_o^i a^{-3(1+\omega_i)} \tag{5.20}$$

Substitute the Equations (5.18) and (5.20) into Eq.(5.17) and ignoring the logarithm term, we have

$$\begin{aligned}\tau_\phi &= -2\rho_\Lambda - \sum_i \sum_{p\neq -3(1+\omega_i), p\neq-\alpha} \frac{\alpha Q_p^i}{(p+3+3\omega_i)(p+\alpha)} a^p \\ &\quad - \sum_i \sum_{p\neq-\alpha} \frac{Q_p^i}{p+\alpha} a^p - \sum_{i,\alpha\neq 3(1+\omega_i)} \frac{\alpha c_0^i}{\alpha - 3(1+\omega_i)} a^{-3(1+\omega_i)} + b_0 a^{-\alpha}, \quad (5.21)\end{aligned}$$

where b_0, c_0^i are constants. In Eqs(5.20),(5.21), to eliminate the logarithm term, we have to set $Q^i_{-\alpha} = 0$, $Q^i_{-3(1+\omega_i)} = 0$ and $c_0^i = 0$ if $\alpha = 3(1+\omega_i)$.

Now we substitute the expressions ρ and τ_ϕ into Eq.(5.11) to get the Friedmann equation

$$\begin{aligned}H^2 &= \frac{2\pi G}{3\rho_\Lambda} \Bigg\{ \sum_i \sum_p \frac{3(1+\omega_i)}{(p+3+3\omega_i)(p+\alpha)} Q_p^i a^p \\ &\quad - \sum_i \frac{3(1+\omega_i)}{\alpha - 3(1+\omega_i)} c_0^i a^{-3(1+\omega_i)} + b_0 a^{-\alpha} \Bigg\}^2 + \rho_a. \quad (5.22)\end{aligned}$$

Combining the coefficient of polynomial, we obtain,

$$H^2 = \frac{2\pi G}{3\rho_\Lambda} \left(\sum_p q_p a^p \right)^2 + \rho_a, \quad (5.23)$$

where q_p are arbitrary constants since Q_p^i, c_0^i, b_0 are arbitrary. We can select the special values for q_p to construct the polynomial to satisfy the cosmological observations. The values of the remaining terms in the infinite series can be obtained by using the Friedmann equations to recursively determine them.

CHAPTER SIX

Effective Action of Type IIB on Warped Conifold

We start with the standard Type IIB action,

$$S_{\rm IIB} = \tfrac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-\hat{g}}\left\{\hat{R} - \tfrac{\partial_M\tau\partial^M\bar{\tau}}{2({\rm Im}\tau)^2} - \tfrac{G_3\cdot\overline{G}_3}{12{\rm Im}\tau} - \tfrac{\tilde{F}_5^2}{4\times 5!}\right\} + \tfrac{1}{8\pi i\kappa_{10}^2}\int \tfrac{C_4\wedge G_3\wedge\overline{G}_3}{{\rm Im}\tau} + S_{\rm loc} \qquad (6.1)$$

where $\hat{g}_{AB}$ is the ten dimensional metric in the Einstein frame. As our ansatz we take the block diagonal metric,

$$\hat{g}_{AB}dz^Adz^B = \tilde{g}_{\mu\nu}(x)\; dx^\mu dx^\nu + \bar{g}_{ij}(x,y)\;\; dy^i dy^j, \qquad (6.2)$$

where $x^\mu, x^\nu \ldots$ denote the coordinates on our non-compact five dimensionnal space-time, while $y^i, y^j, \ldots$ denote coordinates on the base $T^{1,1}$ of the conifold. $\tilde{g}(x)$ is assumed to be independent of the internal coordinates, while $\bar{g}(x,y)$ depends both on the internal and the external coordinates. In particular, the explicit form of $\bar{g}$ is given by

$$\bar{g} = h_1(g^5)^2 + h_2[(g^4)^2 + (g^3)^2] + h_3[(g^2)^2 + (g^1)^2]. \qquad (6.3)$$

In the above the factors h_1, h_2, h_3 all depend on x^μ and reflect the fluctuations of the various components of the internal geometry. On the other hand, the orthogonal basis $\{g^1, \ldots, g^5\}$ only depends on the internal coordinates and is related to the vielbeins $\{E^1, \ldots, E^5\}$ via

$$E^i = \frac{1}{\sqrt{6}}g^i, \quad i = 1, \ldots, 4 \qquad (6.4)$$

$$E^5 = \frac{1}{3}g^5. \qquad (6.5)$$

The vielbeins themselves are given by

$$E^1 = \frac{1}{2\sqrt{3}}\left(-\sin\theta_1 d\phi_1 - \cos\psi\sin\theta_2 d\phi_2 + \sin\psi d\theta_2\right) \qquad (6.6)$$

$$E^2 = \frac{1}{2\sqrt{3}}\left(d\theta_1 - \sin\psi\sin\theta_2 d\phi_2 - \cos\psi d\theta_2\right) \tag{6.7}$$

$$E^3 = \frac{1}{2\sqrt{3}}\left(-\sin\theta_1 d\phi_1 + \cos\psi\sin\theta_2 d\phi_2 - \sin\psi d\theta_2\right) \tag{6.8}$$

$$E^4 = \frac{1}{2\sqrt{3}}\left(d\theta_1 + \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2\right) \tag{6.9}$$

$$E^5 = \frac{1}{3}\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right). \tag{6.10}$$

In this way we get the relationship between the 10D Ricci scalar $\hat{R}$, and the Ricci scalars $\tilde{R}$ and $\bar{R}$ for $\tilde{g}$ and $\bar{g}$ respectively, to be,

$$\hat{R} = \bar{R} + \tilde{R} - \frac{1}{4}\partial_\mu\xi\partial^\mu\xi + \frac{1}{2}\partial_\mu\tilde{g}_{ij}\partial^\mu\tilde{g}^{ij} + \frac{1}{4}\tilde{g}^{ij}\tilde{g}^{kl}\partial^\mu\tilde{g}_{jl}\partial_\mu\tilde{g}_{ik}, \tag{6.11}$$

where $\xi = \ln\bar{g}$. For the $\bar{g}_{ij}$ that we have chosen this becomes

$$\hat{R} = \bar{R} + \tilde{R} - \frac{(\nabla h_1)^2}{h_1^2} - \frac{2(\nabla h_2)^2}{h_2^2} - \frac{2(\nabla h_3)^2}{h_3^2} - \frac{1}{2}\left(\frac{\nabla h_1}{h_1} + 2\frac{\nabla h_2}{h_2} + 2\frac{\nabla h_3}{h_3}\right)^2, \tag{6.12}$$

with

$$\bar{R} = \frac{48h_1(h_2+h_3) - 9(h_2-h_3)^2 - 16{h_1}^2}{4h_1h_2h_3}. \tag{6.13}$$

Finally, a conformal transformation of $\tilde{g}(x)_{\mu\nu} \equiv \Omega^{-2}(x)g(x)_{\mu\nu}$ gives the minimally coupled gravity part

$$\sqrt{\hat{g}}\tilde{R} = \sqrt{\bar{g}}\sqrt{\tilde{g}}\Omega^2 R(g) = \sqrt{g}\sqrt{\bar{g}}\Omega^{-3}R(g) = \sqrt{g}R(g), \tag{6.14}$$

where

$$\Omega = h_1^{\frac{1}{6}}h_2^{\frac{1}{3}}h_3^{\frac{1}{3}}, \tag{6.15}$$

so that

$$\sqrt{\hat{g}}\hat{R} = \sqrt{g}\left(R(g) + \Omega^{-2}\bar{R} - \frac{(\nabla h_1)^2}{h_1^2} - \frac{2(\nabla h_2)^2}{h_2^2} - \frac{2(\nabla h_3)^2}{h_3^2} - 30(\nabla\ln\Omega)^2 + 8\nabla^2\ln\Omega\right), \tag{6.16}$$

which is diagonalized by setting

$$\ln h_1 = w - \frac{20}{11}u, \tag{6.17}$$

$$\ln h_2 = u + v, \tag{6.18}$$

$$\ln h_3 = u - v, \tag{6.19}$$

$$\ln \Omega = \frac{1}{6}w + \frac{4}{11}u, \tag{6.20}$$

Thus we obtain:

$$\sqrt{\hat{g}}\hat{R} = \sqrt{g}\left(R(g) + e^{-\frac{w}{3}-\frac{8}{11}u}\bar{R} - \frac{11}{6}(\nabla w)^2 - 4(\nabla v)^2 - \frac{900}{121}(\nabla u)^2\right). \tag{6.21}$$

Now we choose the form for the fluxes, with functions K,L,P,Q, and f of external co-ordinates, and ω_n as then harmonic n forms.

$$B_2 = K(g^1 \wedge g^2) + L(g^3 \wedge g^4), \tag{6.22}$$

$$F_3 = \frac{M\alpha'}{2}\left(g^3 \wedge g^4 \wedge g^5(1-f) + g^1 \wedge g^2 \wedge g^5 f + df \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\right), \tag{6.23}$$

$$F_5 = Q\omega_5 + dP \wedge \omega_2 \wedge \omega_2 + Q' \wedge dx^5 + \star_5 dP' \wedge g_5, \tag{6.24}$$

which, upon differentiating and applying the connection on g^i we get,

$$H_3 = dK \wedge g^1 \wedge g^2 + dL \wedge g^3 \wedge g^4 + \frac{L-K}{2}(g^2 \wedge g^4 - g^1 \wedge g^3) \wedge g^5, \tag{6.25}$$

for our H flux. In calculating the norm of the flux, the conformal factor Ω will leave kinetic terms unchanged but will result in a factor Ω^{-2} multiplying terms with all internal indices as happened to the $\bar{R}$ term above.

$$(H_3)^2 = 6\left\{(\nabla K)^2 e^{-2u-2v} + (\nabla L)^2 e^{-2u+2v} + \frac{(L-K)^2}{4}e^{-\frac{w}{3}+\frac{14}{11}u}\right\}, \tag{6.26}$$

and for F_3 as well :

$$(F_3)^2 = 9M^2\alpha'^2\left[(1-f)^2 e^{-\frac{4}{3}w-2v-\frac{10}{11}u} + f^2 e^{-\frac{4}{3}w+2v-\frac{10}{11}u} + 2(\nabla f)^2 e^{-2u}\right]. \tag{6.27}$$

Lastly we take F_5 to be a combination of the volume forms of the internal and external spaces, ω_5 and dx^5, respectively, as well as an adding the derivative of a scalar P times the 4 form, and its dual.

$$F_5 = Q\omega_5 + dP \wedge \omega_2 \wedge \omega_2 + Q' \wedge dx^5 + \star_5 dP' \wedge g_5 \tag{6.28}$$

Which has a norm in the conformal frame of

$$(F_5)^2 = (5!)\left(Q^2\Omega^{-2} + Q'^2\Omega^8 + (\nabla P)^2 e^{-4u} + (\nabla P')^2\Omega^6 e^{-w+\frac{20}{11}u}\right) \tag{6.29}$$

Also, the Chern-Simons term yields,

$$F_5 \wedge F_3 \wedge B_2 = \frac{M\alpha'}{2}\Omega^{-2}Q'(K(1-f)+Lf)dx^5 \wedge \omega_5 \tag{6.30}$$

giving an effective action,

$$\begin{aligned}S_{\rm IIB} = \frac{1}{2\kappa_{10}^2}\int d^5x\sqrt{g}\; R_5 - \frac{1}{2}(\nabla\Phi)^2 - \frac{11}{6}(\nabla w)^2 - 4(\nabla v)^2 - (\frac{30\nabla u}{11})^2 \\ \left[24e^{-u}\cosh v - 9e^{\frac{42}{11}u-w}\sinh^2 v - 4e^{w-\frac{42}{11}u}\right]e^{-\frac{1}{3}w-\frac{8}{11}u} \\ -3e^{\Phi-2u}\left(e^{2v}(\nabla K)^2 + e^{-2v}(\nabla L)^2 + \frac{1}{4}e^{-\frac{1}{3}w+\frac{36}{11}u}(L-K)^2\right) \\ -\frac{3}{4}e^{-\Phi}M^2\alpha'^2\left[(1-f)^2e^{-\frac{4}{3}w-2v-\frac{10}{11}u} + f^2e^{-\frac{4}{3}w+2v-\frac{10}{11}u} + 2(\nabla f)^2e^{-2u}\right] \\ -\frac{Q^2}{4}e^{-\frac{1}{3}w-\frac{8}{11}u} - \frac{Q'^2}{4}e^{\frac{4}{3}w+\frac{32}{11}u} - (\nabla P)^2e^{-4u} - (\nabla P')^2e^{4u} \\ +\frac{M\alpha'Q'}{16\pi}e^{-\frac{1}{3}w-\frac{8}{11}u-\Phi}(K(1-f)+Lf)\end{aligned} \tag{6.31}$$

where Φ is the dilaton. The equations of motion are then

$$\begin{aligned}\nabla^2\Phi = 3e^{\Phi-2u}\left(e^{2v}(\nabla K)^2 + e^{-2v}(\nabla L)^2 + \frac{1}{4}e^{-\frac{1}{3}w+\frac{36}{11}u}(L-K)^2\right) \\ +\frac{M\alpha'Q'}{16\pi}e^{-\frac{1}{3}w-\frac{8}{11}u-\Phi}(K(1-f)+Lf) \\ -\frac{3}{4}e^{-\Phi}M^2\alpha'^2\left[(1-f)^2e^{-\frac{4}{3}w-2v-\frac{10}{11}u} + f^2e^{-\frac{4}{3}w+2v-\frac{10}{11}u} + 2(\nabla f)^2e^{-2u}\right]\end{aligned} \tag{6.32}$$

$$\begin{aligned}11\nabla^2 w = 8e^{\frac{2}{3}w-\frac{50}{11}u} + 24e^{-\frac{1}{3}w-\frac{19}{11}u}\cosh v - 36e^{-\frac{4}{3}w+\frac{34}{11}u}\sinh^2 v \\ -\frac{Q^2}{4}e^{-\frac{1}{3}w-\frac{8}{11}u} + Q'^2e^{\frac{4}{3}w+\frac{32}{11}u} + \frac{M\alpha'Q'}{16\pi}(K(1-f)+Lf)e^{-\frac{1}{3}w-\frac{8}{11}u-\Phi} \\ +\frac{3}{4}e^{\Phi-\frac{1}{3}w+\frac{14}{11}u}(L-K)^2 - 3M^2\alpha'^2e^{-\Phi-\frac{4}{3}w-\frac{10}{11}u}\left[(1-f)^2e^{-2v} + f^2e^{2v}\right]\end{aligned} \tag{6.33}$$

$$\begin{aligned}4\nabla^2 v = 9e^{-\frac{4}{3}w+\frac{34}{11}u}\sinh 2v - 12e^{-\frac{1}{3}w-\frac{19}{11}u}\sinh v \\ -\frac{3}{2}M^2\alpha'^2e^{-\Phi-\frac{4}{3}w-\frac{10}{11}u}\left[(1-f)^2e^{-2v} - f^2e^{2v}\right] \\ +3e^{\Phi-2u}\left(e^{2v}(\nabla K)^2 - e^{-2v}(\nabla L)^2\right)\end{aligned} \tag{6.34}$$

(6.35)

$$
\begin{aligned}
&\frac{900}{11}\nabla^2 u = 153e^{-\frac{4}{3}w+\frac{34}{11}u}\sinh^2 v - 100e^{\frac{2}{3}w-\frac{50}{11}u} + 228e^{\frac{-1}{3}w-\frac{19}{11}u}\cosh v \\
&-33e^{\Phi-2u}\left(e^{2v}(\nabla K)^2 + e^{-2v}(\nabla L)^2\right) + \frac{21}{2}e^{\Phi-\frac{1}{3}w+\frac{14}{11}u}(L-K)^2 \\
&-\frac{3}{4}e^{-\Phi}M^2\alpha'^2\left[10(1-f)^2e^{-\frac{4}{3}w-2v-\frac{10}{11}u} + 10f^2e^{-\frac{4}{3}w+2v-\frac{10}{11}u} + 44(\nabla f)^2e^{-2u}\right] \\
&-2Q^2e^{-\frac{1}{3}w-\frac{8}{11}u} + 8Q'^2e^{\frac{4}{3}w+\frac{32}{11}u} - 22(\nabla P)^2e^{-4u} + 22(\nabla P')^2e^{4u} \\
&+\frac{M\alpha'Q'}{2\pi}(K(1-f)+Lf)e^{-\frac{1}{3}w-\frac{8}{11}u-\Phi} \qquad (6.36) \\
&\nabla^2 K = \nabla K\nabla(2u-2v-\Phi) + \frac{K-L}{4}e^{-\frac{1}{3}w+\frac{14}{11}u-2v} \\
&-\frac{M\alpha'Q'(1-f)}{96\pi}e^{-\frac{1}{3}w-2v-\frac{10}{11}u-2\Phi} \qquad (6.37) \\
&*\nabla^2 L = \nabla L\nabla(2u+2v-\Phi) + \frac{L-K}{4}e^{-\frac{1}{3}w+\frac{14}{11}u+2v} \\
&-\frac{M\alpha'Q'f}{96\pi}e^{-\frac{1}{3}w+2v+\frac{14}{11}u-2\Phi} \qquad (6.38) \\
&\nabla^2 f = 2\nabla f\nabla u + f\sinh 2v e^{-\frac{4}{3}w+\frac{12}{11}u} + \frac{Q'(K-L)}{48\pi M\alpha'}e^{-\frac{1}{3}w+\frac{14}{11}u} \qquad (6.39) \\
&0 = \nabla\left(e^{4u}\nabla P' + e^{-4u}\nabla P\right), \qquad (6.40)
\end{aligned}
$$

with the energy momentum tensor,

$$
\begin{aligned}
&T_{\mu\nu} = \frac{1}{2}(\nabla_\mu\Phi)(\nabla_\nu\Phi) + \frac{11}{6}(\nabla_\mu w)(\nabla_\nu w) + 4(\nabla_\mu v)(\nabla_\nu v) \\
&+\frac{900}{121}(\nabla_\mu u)(\nabla_\nu u) + 3e^{\Phi-2u}\left(e^{2v}(\nabla_\mu K)(\nabla_\nu K) + e^{-2v}(\nabla_\mu L)(\nabla_\nu L)\right) \\
&+\frac{3M\alpha'}{2}e^{-\Phi-2u}(\nabla_\mu f)(\nabla_\nu f) + e^{-4u}(\nabla_\mu P)(\nabla_\nu P) + e^{4u}(\nabla_\mu P')(\nabla_\nu P') - \frac{1}{2}g_{\mu\nu}L_5 \qquad (6.41)
\end{aligned}
$$

Finding closed form nontrivial solutions to these equations is an open problem. The following sections explore the possibility of solutions similar to those of the previous chapters, with a negative result.

6.1 F_5=0 Case

To simplify we can take

$$
K - L = P = Q = Q' = v = 0 \qquad (6.42)
$$

this eliminates the 5-flux and Chern-Simons terms, adding a symmetry between g_1, g_2 and g_3, g_4. Now,

$$\nabla^2 f = 2\nabla f \nabla u \tag{6.43}$$

$$\nabla^2 K = \nabla K \nabla(2u - \Phi) \tag{6.44}$$

$$\nabla^2 \Phi = 6e^{\Phi-2u}(\nabla K)^2 \tag{6.45}$$

$$\frac{900}{11}\nabla^2 u = -100e^{\frac{2}{3}w-\frac{50}{11}u} + 228e^{\frac{-1}{3}w-\frac{19}{11}u} - 66e^{\Phi-2u}(\nabla K)^2$$
$$-\frac{3}{4}e^{-\Phi}M^2\alpha'^2\left[10(1-f)^2e^{-\frac{4}{3}w-\frac{10}{11}u} + 10f^2e^{-\frac{4}{3}w-\frac{10}{11}u} + 44(\nabla f)^2e^{-2u}\right] \tag{6.46}$$

$$11\nabla^2 w = 8e^{\frac{2}{3}w-\frac{50}{11}u} + 24e^{-\frac{1}{3}w-\frac{19}{11}u} - 3M^2\alpha'^2e^{-\Phi-\frac{4}{3}w-\frac{10}{11}u}\left[(1-f)^2 + f^2\right] \tag{6.47}$$

if we further take

$$f = const. \tag{6.48}$$

$$M^2\alpha'^2(1 - 2f + 2f^2) = m' \tag{6.49}$$

then

$$\nabla(e^{\Phi-2u}\nabla K) = \nabla\cdot\beta_K = 0 \tag{6.50}$$

$$\nabla^2\Phi = 6e^{\Phi-2u}(\nabla K)^2 = 6e^{2u-\Phi}\beta_K^2 \tag{6.51}$$

$$\frac{900}{11}\nabla^2 u = -120e^{\frac{2}{3}w-\frac{50}{11}u} + 168e^{\frac{-1}{3}w-\frac{19}{11}u} - 11\nabla^2\Phi + \frac{110}{4}\nabla^2 w \tag{6.52}$$

$$11\nabla^2 w = 8e^{\frac{2}{3}w-\frac{50}{11}u} + 24e^{-\frac{1}{3}w-\frac{19}{11}u} - 3m'e^{-\Phi-\frac{4}{3}w-\frac{10}{11}u} \tag{6.53}$$

and

$$V_5 = 24e^{-\frac{1}{3}w-\frac{19}{11}u} - 4e^{\frac{2}{3}w-\frac{50}{11}u} - 6e^{\Phi-2u}(\nabla K)^2 - \frac{3m'}{4}e^{-\Phi-\frac{4}{3}w-\frac{10}{11}u} \tag{6.54}$$

$$T_{\mu\nu} = \frac{1}{2}(\nabla_\mu\Phi)(\nabla_\nu\Phi) + \frac{11}{6}(\nabla_\mu w)(\nabla_\nu w) + \frac{900}{121}(\nabla_\mu u)(\nabla_\nu u) + 6e^{\Phi-2u}(\nabla_\mu K)(\nabla_\nu K) - \frac{1}{2}g_{\mu\nu}L_5 \tag{6.55}$$

With a conformally flat metric

$$g_{\mu\nu} = e^{-2A}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2 \tag{6.56}$$

$$\Box = (4A_y - \partial_y)\partial_y \tag{6.57}$$

where all fields are functions of y only, then

$$\nabla K = e^{2u-\Phi}\beta_K = \beta_0 e^{4A-\Phi+2u}\hat{y} \tag{6.58}$$

$$4A_y\Phi_y - \Phi_{yy} = -6(\beta_0)^2 e^{8A-\Phi+2u} \tag{6.59}$$

$$\begin{aligned}\frac{900}{11}(4A_y u_y - u_{yy}) &= -100e^{\frac{2}{3}w-\frac{50}{11}u} + 228e^{\frac{-1}{3}w-\frac{19}{11}u} \\ &+66(\beta_0)^2 e^{8A-\Phi+2u} - \frac{15}{2}m' e^{-\Phi-\frac{4}{3}w-\frac{10}{11}u}\end{aligned} \tag{6.60}$$

$$11(4A_y w_y - w_{yy}) = 8e^{\frac{2}{3}w-\frac{50}{11}u} + 24e^{-\frac{1}{3}w-\frac{19}{11}u} - 3m'e^{-\Phi-\frac{4}{3}w-\frac{10}{11}u} \tag{6.61}$$

$$\begin{aligned}T_{yy} &= \frac{3}{4}\Phi_y^2 + \frac{11}{4}w_y^2 + \frac{1350}{121}u_y^2 + 9\beta_0^2 e^{8A-\Phi+2u} \\ &+12e^{-\frac{1}{3}w-\frac{19}{11}u} - 2e^{\frac{2}{3}w-\frac{50}{11}u} - \frac{3m'}{8}e^{-\Phi-\frac{4}{3}w-\frac{10}{11}u}\end{aligned} \tag{6.62}$$

$$\begin{aligned}T_{\mu\nu} = -\eta_{\mu\nu}e^{-2A}[&\frac{1}{4}\Phi_y^2 + \frac{11}{12}w_y^2 + \frac{450}{121}u_y^2 + 3\beta_0^2 e^{8A-\Phi+2u} \\ &+12e^{-\frac{1}{3}w-\frac{19}{11}u} - 2e^{\frac{2}{3}w-\frac{50}{11}u} - \frac{3m'}{8}e^{-\Phi-\frac{4}{3}w-\frac{10}{11}u}]\end{aligned} \tag{6.63}$$

$$R_{yy} = -4(A_y - \partial_y)A_y \tag{6.64}$$

$$R_{\mu\nu} = \eta_{\mu\nu}e^{-2A}(4A_y - \partial_y)A_y \tag{6.65}$$

now trying

$$A = a\ln(y) + a_0 \tag{6.66}$$

$$\Phi = c_0\ln(y) + \phi_0 \tag{6.67}$$

$$u = c_1\ln(y) + u_0 \tag{6.68}$$

$$w = c_2\ln(y) + w_0 \tag{6.69}$$

we have first

$$8a - c_0 + 2c_1 = -2 \tag{6.70}$$

$$4ac_0 + c_0 = -6(\beta_0)^2 e^{8a_0-\Phi_0+2u_0} \tag{6.71}$$

$$c_0 + c_2 = \frac{9}{11}c_1 \tag{6.72}$$

Thus three cases emerge from how the terms of the u equation cancel.

6.1.1 Case of Like Terms Non-quadratic in y

$$-2 \neq \frac{2}{3}c_2 - \frac{50}{11}c_1 \quad (6.73)$$

$$c_2 = \frac{31}{11}c_1 \quad (6.74)$$

which leads to

$$c_0 = -2c_1 \quad (6.75)$$

$$8a + 4c_1 = -2 \quad (6.76)$$

$$900(4a+1)c_1 = -121(4a+1)c_0 \quad (6.77)$$

Clearly in case 1, the system is overdetermined, unless the H flux is set to zero.

6.1.2 Case of Unlike Terms

$$-2 = \frac{2}{3}c_2 - \frac{50}{11}c_1 \quad (6.78)$$

$$c_2 \neq \frac{31}{11}c_1 \quad (6.79)$$

$$(6.80)$$

leading to

$$456e^{-\frac{1}{3}w_0-\frac{19}{11}u_0} = 15m'e^{-\phi_0-\frac{4}{3}w_0-\frac{10}{11}u_0} \quad (6.81)$$

$$24e^{-\frac{1}{3}w_0-\frac{19}{11}u_0} = 3m'e^{-\phi_0-\frac{4}{3}w_0-\frac{10}{11}u_0} \quad (6.82)$$

In this case the m' term cannot cancel the $e^{-\frac{w}{3}}$ in both the u and w equations.

6.1.3 Case of Like Terms with Powers of y^{-2}

$$-2 = \frac{2}{3}c_2 - \frac{50}{11}c_1 \quad (6.83)$$

$$c_2 = \frac{31}{11}c_1 \quad (6.84)$$

this leads to all terms on the right hand side having the same power of y,

$$c_0 = -2c_1 \tag{6.85}$$

$$8a + 4c_1 = -2 \tag{6.86}$$

$$\begin{aligned} 900(4a+1)c_1 &= -121(4a+1)c_0 - 1100e^{\frac{2}{3}w_0 - \frac{50}{11}u_0} \\ &+2508e^{-\frac{1}{3}w_0 - \frac{19}{11}u_0} - \frac{165m'}{2}e^{-\phi_0 - \frac{4}{3}w_0 - \frac{10}{11}u_0} \end{aligned} \tag{6.87}$$

$$\begin{aligned} 11(4a+1)c_2 &= 8e^{\frac{2}{3}w_0 - \frac{50}{11}u_0} + 24e^{-\frac{1}{3}w_0 - \frac{19}{11}u_0} \\ &-3m'e^{-\phi_0 - \frac{4}{3}w_0 - \frac{10}{11}u_0} \end{aligned} \tag{6.88}$$

so the field equations allow a solution, but all log coefficients have already been determined, and we have not considered the R_{yy} and $R_{\mu\nu}$ equations.

$$a = -\frac{5}{8} \tag{6.89}$$

$$c_0 = -\frac{3}{2} \tag{6.90}$$

$$c_1 = \frac{3}{4} \tag{6.91}$$

$$c_2 = \frac{93}{44} \tag{6.92}$$

Now,

$$V_0 \equiv +12e^{-\frac{1}{3}w_0 - \frac{19}{11}u_0} - 2e^{\frac{2}{3}w_0 - \frac{50}{11}u_0} - \frac{3m'}{8}e^{-\phi_0 - \frac{4}{3}w_0 - \frac{10}{11}u)} \tag{6.93}$$

$$T_{yy} = \frac{3}{4}(c_0)^2 + \frac{11}{4}(c_2)^2 + \frac{1350}{121}(c_1)^2 - \frac{3}{2}(4a+1)c_0 + V_0 \tag{6.94}$$

$$= -4(a+1)a + \frac{1}{2}R \tag{6.95}$$

$$-\frac{T_{\mu\nu}}{\eta_{\mu\nu}e^{-2A}} = \frac{1}{4}(c_0)^2 + \frac{11}{12}(c_2)^2 + \frac{450}{121}(c_1)^2 - \frac{1}{2}(4a+1)c_0 + V_0 \tag{6.96}$$

$$= -(4a+1)a + \frac{1}{2}R \tag{6.97}$$

$$\frac{1}{2}(\frac{3}{2})^2 + \frac{11}{6}(\frac{93}{44})^2 + \frac{900}{121}(\frac{3}{4})^2 + (1 - \frac{20}{8})(\frac{3}{2}) \neq \frac{15}{8} \tag{6.98}$$

ruling out solutions of this form for nonzero H flux.

6.2 $H = 0$ *Case*

Interestingly, the F_3 term may be absorbed into Φ, if we take $\beta_0 = 0$, letting Φ become divergenceless. There are two cases to consider.

6.2.1 *Case of Constant* Φ

now,

$$c_0 = 0 \tag{6.99}$$

$$c_2 = \frac{9}{11}c_1 \tag{6.100}$$

and either

$$c_2 = \frac{31}{11}c_1 \tag{6.101}$$

or

$$-2 = \frac{2}{3}c_2 - \frac{50}{11}c_1 \tag{6.102}$$

The first results in all fields going to zero, while the second is overdetermined when we consider the w_{yy} equation.

6.2.2 *Case of Variable* Φ

Without H, the Φ equation can also be satisfied by,

$$a = -\frac{1}{4} \tag{6.103}$$

now, from the divergence of u

$$w_0 = \frac{31}{11}u_0 + \ln(7/5) \tag{6.104}$$

and from the divergence of w,

$$\ln(8\frac{7}{5} + 24) + \phi_0 + w_0 = \ln(3m') + \frac{9}{11}u_0 \tag{6.105}$$

and demanding that the powers of y on the right hand sides match,

$$c_0 = -2c_1 \tag{6.106}$$

$$c_2 = \frac{31}{11}c_1 \tag{6.107}$$

with two Einstein equations left, and with c_1 and u_0 still free,

$$R_{\mu\nu} = -\frac{1}{3}\eta_{\mu\nu}V_0 y^k = 0 \tag{6.108}$$

$$R_{yy} = \frac{3}{4}y^{-2} = y^{-2}[\frac{1}{2}(c_0)^2 + \frac{900}{121}(c_1)^2 + \frac{11}{6}(c_2)^2] - \frac{1}{3}V_0 y^k \tag{6.109}$$

unfortunately for V_0 to vanish requires

$$e^{-\frac{1}{3}w_0 - \frac{19}{11}u_0} = 0 \tag{6.110}$$

ruling out a Kasner type solution for nonzero F_3 flux.

CHAPTER SEVEN

Conclusions and Discussions

In this dissertation, we have systematically studied orbifold branes in the framework of string/M-Theory. In particular, in Chapter I, we have given a brief introduction to "standard modern cosmology". In Chapter II, we have a treatment of compactification into a 3-way product space with a similar H flux. The action is transformed to from the string to Einstein frames and written as an effective theory with 4 scalars joining the dilaton in the bulk. Two of them come from the warp factors of the two smaller product spaces, two from the H-flux, and one from the transformation of the dilaton and warp factors. In Chapter III, we have the a simple solution to the equations of the previous sections possessing Poincare symmetry. The orbifold symmetry imposed on the bulk scalar fields keeps the two branes at the symmetry planes of the bulk at a stable proper distance from each other. It is important not to confuse the massive scalar test field Φ of section 3.B with either the dilaton, radion, or the linear combination of dilaton and warp factors ϕ from chapter 2. The important quantity is Y_c, the inter-brane proper distance. It aquires an effective potential from the repulsive forces between branes generated by other scalars, as well as the gravitational attraction. The bulk Kasner spacetime is static, and the traditional Kasner instability in this model would manifest as a tendency for the branes to collapse toward the metric singularity at $Y = 0$. That singularity is removed by cutting and applying the orbifold symmetry, at two Y-planes. If the position of the planes along the Y direction is stabilized then the singularity cannot 'appear', and the total volume modulus of the finite bulk direction and the 2 and 3 dimensional internal spaces, traditionally referred to as the radion, remains stable. From the solutions for the fields $\psi_{\pm}$ we can see that the individual volume moduli of $d_{\pm}$ will also remain bounded as long as the argument of the log remains positive. This also is guaranteed be stabilizing the inter-brane distance, since it is that distance that measures the 'size' of the sawtooth pieces of our $|Y|$ co-ordinate. In Chapter IV, we have verified that the corrections to ordinary 4-dimensional gravity on the brane are small, by considering the mectric fluctuations near the branes. This is essentially the same approach to finding the corrections to gravity from Randall-Sundrum models, though our bulk background differs somewhat from those scenarios. The main difference between models here and standard Randall-Sundrum models is that the bulk is not an anti-de-sitter space, but an effective Kasner type space with three scalar fields arising from compactification, similar to the earlier Havora-Witten models. The boundary conditions on the branes for those scalar fields determine our moduli masses and radion mass. Our zero mode corresponding to 4D gravity on the brane is a generic result for the geometry of two branes. Gravity in the bulk is suppressed less severely than in RS1, and needs the second brane for the zero mode to be normalizble. Gravity could be thought of as not so much bound to the TeV end of the orbifold as repelled from the plank end. In Chapter V, we have focused on the cosmology of our primary model. The Friedmann equations for a perfect fluid and their solution given an arbitrary interaction term proportional to the Hubble factor, are worked out. In Chapter VI we have the compactification of $T_{1,1}$, but without solutions. In particular, it may be that solutions similar to chapter three fail because $T_{1,1}$ does not satisfy the constraint on the curvatures of the $d_{\pm}$ manifolds required by our other solutions of the D, d_{-}, d_{+} general compactification, even though the topology is $S_2 X S_3$. There are also two Appendices A and B. In Appendix A, we have a few steps to show the derivation of the solution in chapter 3. In Appendix B, we have an almost exactly analogous derivation of the chapter 3 solutions but for 6 bulk dimensions. The same constraint on the potential appears here as in appendix A. The scalar potential vanishes, leaving 3 free fields constrained by orbifold boundary conditions. In concluding this dissertation, a few comments are in order. First, the Goldberger-Wise mechanism can result in both stability of inter-brane distance and of a reasonable Yukawa correction to gravity from a TeV scale first excited state of the KK tower. However, in fitting to cosmological expansion rate, the inverse eighth power term in the Friedmann equations generated by the modular fields of the compactification does not completely replace the need for dark energy, but merely moves the source of it over to a brane-bulk interaction term, that may not be sufficiently justified from first principles. While it has for a while been the case that theory has had a playground at scales too small for the experiments of the time, there is also a rich history of the other end of the scale, in the predictions of both Neptune and Pluto, for instance. Brane world models are in many ways the analog of the predictions of very high energy particles in previous decades, granting justification for exotic equations of state for dark fluids that are needed by the data as we are able to measure higher derivatives of the Hubble parameter.

APPENDICES

APPENDIX A

Derivation Summary

The expressions

$$\begin{aligned}
L_5 &= \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}(\nabla\psi_+)^2 + \frac{1}{2}(\nabla\psi_-)^2 - V_5 && \text{(A.1)}\\
V_5 &= e^{\sqrt{\frac{2}{3}}\phi}\left(\beta_+ e^{-\sqrt{\frac{2}{3}}\psi_+} + \beta_- e^{-\psi_-}\right) && \text{(A.2)}\\
T_{ab} &= \frac{1}{2}\nabla_a\phi\nabla_b\phi + \frac{1}{2}\nabla_a\psi_+\nabla_b\psi_+ + \frac{1}{2}\nabla\psi_-\nabla_b\psi_- - \frac{1}{2}g_{ab}L_5 && \text{(A.3)}\\
R_{ab} &= T_{ab} - \frac{1}{D-2}g_{ab}T && \text{(A.4)}\\
&= \frac{1}{2}\nabla_a\phi\nabla_b\phi + \frac{1}{2}\nabla_a\psi_+\nabla_b\psi_+ + \frac{1}{2}\nabla\psi_-\nabla_b\psi_- - \frac{1}{3}g_{ab}V_5 && \text{(A.5)}
\end{aligned}$$

lead to matter equations

$$\begin{aligned}
\Box\phi &= \sqrt{\frac{2}{3}}V_5 && \text{(A.6)}\\
\Box\psi_+ &= \sqrt{\frac{2}{3}}\beta_+ e^{\sqrt{\frac{2}{3}}(\phi-\psi_+)} && \text{(A.7)}\\
\Box\psi_- &= \beta_- e^{\sqrt{\frac{2}{3}}\phi-\psi_-}. && \text{(A.8)}
\end{aligned}$$

The space-time metric takes the form

$$ds^2 = e^{-2A}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2 \qquad \text{(A.9)}$$

with its Ricci tensor being given by

$$\begin{aligned}
R_{\mu\nu} &= \eta_{\mu\nu}e^{-2A}(4A_y - \partial_y)A_y = -\frac{1}{3}\eta_{\mu\nu}V_5 && \text{(A.10)}\\
R_{yy} &= -4(A_y - \partial_y)A_y = \frac{1}{2}\left(\phi_y^2 + \psi_{+y}^2 + \psi_{-y}^2\right) - \frac{1}{3}V_5 && \text{(A.11)}\\
\Box &= (4A_y - \partial_y)\partial_y. && \text{(A.12)}
\end{aligned}$$

Notice that

$$\begin{aligned}
\Box A &= -\frac{1}{3}V_5 && \text{(A.13)}\\
\frac{1}{2}\left(\phi_y^2 + \psi_{+y}^2 + \psi_{-y}^2 - 24A_y^2\right) &= \frac{5}{3}V_5. && \text{(A.14)}
\end{aligned}$$

Now defining,

$$\begin{aligned}
\Box Q &= 0 && \text{(A.15)}\\
\psi_+ &= \sqrt{6}B + c_3 Q + \psi_{+0} && \text{(A.16)}
\end{aligned}$$

we find that for V_5 nonzero,

$$\begin{aligned}
A &= \frac{1}{4}\ln Q_y + \alpha && \text{(A.17)}\\
\phi &= -\sqrt{6}A + c_1 Q + \phi_0 && \text{(A.18)}\\
\psi_- &= -3A - 3B + c_2 Q + \psi_{-0} && \text{(A.19)}
\end{aligned}$$

and

$$\begin{aligned}
V_5 &= \beta'_- Q_y^{1/4} e^{3B+(\sqrt{2/3}c_1 - c_2)Q} + \beta'_+ Q_y^{-1/2} e^{-2B+\sqrt{2/3}(c_1-c_3)Q} && \text{(A.20)}\\
\beta'_- &= \beta_- e^{\sqrt{2/3}\phi_0 - \psi_{-0}} && \text{(A.21)}\\
\beta'_+ &= \beta_+ e^{\sqrt{2/3}(\phi_0 - \psi_{+0})} && \text{(A.22)}
\end{aligned}$$

Then taking advantage of the similarity of the ψ_+ and ψ_- matter equations, we find that

$$\sqrt{2/3}\psi_+ = \psi_- + c_4 \qquad \text{(A.23)}$$

which leads to

$$\begin{aligned}
B &= -\frac{3}{5}A && \text{(A.24)}\\
c_3 &= \sqrt{3/2}c_2 && \text{(A.25)}\\
c_4 &= \sqrt{2/3}\psi_{+0} - \psi_{-0} && \text{(A.26)}\\
\psi_{-0} &= \sqrt{2/3}\ln\left(\frac{2\beta_+}{3\beta_-}\right)\psi_{+0} && \text{(A.27)}\\
V_5 &= v_0 e^{(\sqrt{2/3}c_1 - c_2)Q - 4A/5} && \text{(A.28)}\\
\psi_- &= -\frac{6}{5}A + c_2 Q + \psi_{-0} && \text{(A.29)}\\
\psi_+ &= -\frac{3\sqrt{6}}{5}A + c_3 Q + \psi_{+0} && \text{(A.30)}\\
v_0 &= \beta'_- + \beta'_+. && \text{(A.31)}
\end{aligned}$$

This implicitly requires the constants $\beta_\pm$(and therefore $\beta'_\pm$ as well,) to be the same sign. The remaining equations are then

$$\Box Q = 0 \tag{A.32}$$

$$\Box A = -\frac{1}{3}v_0 e^{(\sqrt{2/3}c_1 - c_2)Q - 4A/5} \tag{A.33}$$

$$5A_{yy} = (\frac{1}{2}c_1^2 + \frac{5}{4}c_2^2)Q_y^2 - (2\sqrt{6}c_1 + 6c_2)Q_y A_y + \frac{64}{5}A_y^2 \tag{A.34}$$

If we assume $\partial_Q V_5 = 0$, then

$$\sqrt{2/3}c_1 = c_2 \tag{A.35}$$

$$Q_y = e^{4A} \tag{A.36}$$

$$A_{yy} = 4A_y^2 + \frac{1}{3}v_0 e^{-4A/5} \tag{A.37}$$

$$-\frac{36}{5}A_y^2 - 12c_2 e^{4A} A_y + 2c_2^2 e^{8A} - \frac{5}{3}v_0 e^{-4A/5} = 0 \tag{A.38}$$

but the last two equations are inconsistent, so V_5 must depend on Q, or be zero. Assuming

$$A = \alpha \ln y + \alpha_0 \tag{A.39}$$

$$\psi_+ = c_+ \ln y + m_+ \tag{A.40}$$

$$\psi_- = c_- \ln y + m_- \tag{A.41}$$

$$\phi = c_0 \ln y + m_0 \tag{A.42}$$

we find the constraints

$$(4\alpha + 1)c_+ y^{-2} = \sqrt{\frac{2}{3}}\beta_+ e^{\sqrt{\frac{2}{3}}((c_0 - c_+)\ln y + m_0 - m_+)} \tag{A.43}$$

$$(4\alpha + 1)c_- y^{-2} = \beta_- e^{\sqrt{(\frac{2}{3}c_0 - c_-)\ln y + \sqrt{\frac{2}{3}}m_0 - m_-)}} \tag{A.44}$$

$$(4\alpha + 1)c_0 y^{-2} = \sqrt{\frac{2}{3}}V_5 \tag{A.45}$$

$$(4\alpha + 1)\alpha y^{-2} = -\frac{1}{3}V_5 \tag{A.46}$$

$$-4\alpha(\alpha + 1)y^{-2} = \frac{1}{2}(c_0^2 + c_+^2 + c_-^2) - \frac{1}{3}V_5 \tag{A.47}$$

If $V_5 = 0$ and the space is not flat, then $\alpha = -\frac{1}{4}$, and the first two equations cannot be satisfied unless $\beta_\pm = 0$. If V_5 is nonzero then,

$$c_0 = -\sqrt{6}\alpha \quad (A.48)$$

$$c_- = 2 - 2\alpha \quad (A.49)$$

$$c_+ = \sqrt{6}(1-\alpha) \quad (A.50)$$

$$0 = 16\alpha^2 - 7\alpha + 5 \quad (A.51)$$

which again has no real roots for α. This leaves the $\beta_\pm = 0$ case, which is a Kasner type space. Gauss' theorem egregium will constrain the topolog of d_- to be T_2 but not d_+. Thus, finally we obtain

$$ds^2_{(5)} = dy^2 + \sqrt{y}(\eta_{\mu\nu}dx^\mu dx^\nu) \quad (A.52)$$

and

$$\frac{3}{2} = c_+^2 + c_-^2 + c_0^2. \quad (A.53)$$

Introducing a scalar test field into the bulk, $\Box_{(5)}F = m^2F$, we find that

$$\Box_{(5)} = g^{ab}\nabla_a\nabla_b = g^{ab}(\partial_a\partial_b - \Gamma^c_{ab}\partial_c) \quad (A.54)$$

$$= \frac{1}{\sqrt{y}}\Box_{(4)} + \frac{1}{y}\partial_y + \partial_y\partial_y \quad (A.55)$$

$$F = H(x)J(y) \quad (A.56)$$

$$m^2F = \frac{J\Box_{(4)}H}{\sqrt{y}} + HJ_{yy} + \frac{H}{y}J_y \quad (A.57)$$

$$\Box_{(4)}H = H\left(J'' + y^{-1}J' - m^2\right)\sqrt{y}, \quad (A.58)$$

from which we see that F will have massless modes in four dimensions, when J makes right hand side factor zero. J must still obey the orbifold boundary conditions as well. The solutions for nonzero m are the modified Bessel functions of chapter 3.

APPENDIX B

Another Choice of Dimension

A similar result can be obtained by compactifying for $D = 6$, $d_- = d_+ = 2$

$$
\begin{aligned}
L_6 &= \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}(\nabla\psi_+)^2 + \frac{1}{2}(\nabla\psi_-)^2 - V_6 && \text{(B.1)}\\
V_6 &= e^{\phi/\sqrt{2}}\left(\beta_+ e^{-\psi_+} + \beta_- e^{-\psi_-}\right) && \text{(B.2)}\\
T_{ab} &= \frac{1}{2}\nabla_a\phi\nabla_b\phi + \frac{1}{2}\nabla_a\psi_+\nabla_b\psi_+ + \frac{1}{2}\nabla\psi_-\nabla_b\psi_- - \frac{1}{2}g_{ab}L_6 && \text{(B.3)}\\
R_{ab} &= T_{ab} - \frac{1}{D-2}g_{ab}T && \text{(B.4)}\\
&= \frac{1}{2}\nabla_a\phi\nabla_b\phi + \frac{1}{2}\nabla_a\psi_+\nabla_b\psi_+ + \frac{1}{2}\nabla\psi_-\nabla_b\psi_- + \frac{1}{4}g_{ab}V_6 && \text{(B.5)}\\
\Box\phi &= -\frac{1}{\sqrt{2}}V_6 && \text{(B.6)}\\
\Box\psi_\pm &= \beta_\pm e^{(\phi/\sqrt{2})-\psi_\pm}. && \text{(B.7)}\\
& && \text{(B.8)}
\end{aligned}
$$

Now assume a metric of the form

$$ds^2 = dy^2 + e^{2A}d\theta^2 + e^{2B}\eta_{\mu\nu}dx^\mu dx^\nu \qquad \text{(B.9)}$$

where all functions are of y only. Then, we find

$$
\begin{aligned}
R_{\theta\theta} &= e^{2A}\Box A && \text{(B.10)}\\
R_{\mu\nu} &= \eta_{\mu\nu}e^{2B}\Box B && \text{(B.11)}\\
R_{yy} &= -A'' - 4B'' - (A')^2 - 4(B')^2 && \text{(B.12)}\\
\Box &= -(A' + 4B' + \partial_y)\partial_y && \text{(B.13)}
\end{aligned}
$$

Then, substituting into T_{ab}, we obtain

$$
\begin{aligned}
R_{\theta\theta} &= \frac{1}{4}e^{2A}V_6 && \text{(B.14)}\\
R_{\mu\nu} &= \frac{1}{4}e^{2B}V_6\eta_{\mu\nu} && \text{(B.15)}\\
R_{yy} &= \frac{1}{2}\left((\phi')^2 + (\psi'_+)^2 + (\psi'_-)^2 + \frac{1}{2}V_6\right), && \text{(B.16)}
\end{aligned}
$$

from which immediately we see that

$$\Box(A-B)=0. \tag{B.17}$$

Redefining

$$\begin{aligned} P &= A+4B && \text{(B.18)}\\ Q &= A-B && \text{(B.19)}\\ \Box &= -(P'+\partial_y)\partial_y && \text{(B.20)}\\ \Box P &= \frac{5}{4}V_6 && \text{(B.21)}\\ \Box Q &= 0 && \text{(B.22)} \end{aligned}$$

and then comparing with the equations for ϕ and $\psi_\pm$, we get

$$\begin{aligned} \phi &= \frac{-\sqrt{8}}{5}P+c_1\sqrt{2}Q && \text{(B.23)}\\ \psi_+ + \psi_- &= \frac{4}{5}P+2c_2Q+c_3 && \text{(B.24)} \end{aligned}$$

The equation for Q can be integrated to give

$$P = -\ln Q' + c_0. \tag{B.25}$$

And the similarity of the $\psi_\pm$ equations begs

$$\psi_+ + \ln\beta_- = \psi_- + \ln\beta_+, \tag{B.26}$$

where now

$$\begin{aligned} \psi_\pm &= \frac{2}{5}P+c_2Q+c_\pm && \text{(B.27)}\\ V_6 &= 2\beta_- e^{(c_1-c_2)Q-4P/5} && \text{(B.28)} \end{aligned}$$

Thus, finally we get

$$\begin{aligned} R_{yy} &= \Box P+\frac{4}{5}(P'-Q')(P'+Q')\\ &= \frac{8}{25}P'^2+\frac{4}{5}P'Q'((c_2-c_1)+(c_1^2+c_2^2)Q'^2+\frac{1}{4}V_6 && \text{(B.29)} \end{aligned}$$

which gives

$$V_6 = (c_1^2 + c_2^2 + 1)Q'^2 + (c_2 - c_1)P'Q' - \frac{3}{5}P'^2, \tag{B.30}$$

so that

$$P' = -\frac{10}{3}e^{c_0 - P}\left((c_1 - c_2) \pm \sqrt{(c_1 - c_2)^2 + 12(c_1^2 + c_2^2 + 1 - 2\beta_- w)/5}\right), \tag{B.31}$$

$$w = e^{-2c_0 + (c_1 - c_2)Q + 6P/5}. \tag{B.32}$$

In order for the solutions to be consistent, $\beta_\pm = 0$. Then,

$$P = \ln y + p_0 \tag{B.33}$$

$$Q = \alpha \ln y + q_0 \tag{B.34}$$

$$\frac{3}{5} = \alpha(c_1 - c_2) + \alpha^2(c_1^2 + c_2^2 + 1) \tag{B.35}$$

or, most generally

$$\phi = c_\phi \ln y + \phi_0 \tag{B.36}$$

$$\psi_\pm = c_\pm \ln y + \psi_{\pm 0} \tag{B.37}$$

$$A = (1 - 4\gamma)\ln y \tag{B.38}$$

$$B = \gamma \ln y \tag{B.39}$$

$$m^2 = c_\phi^2 + c_+^2 + c_-^2 \tag{B.40}$$

$$\gamma = \frac{1}{5}\left(1 \pm \sqrt{1 - 5m^2/8}\right) \tag{B.41}$$

It should be noted that as a result of Gauss' theorem egregium, the topology of the system must be $M_5 x T_5$, and is thus purely degenerate to already explored toroidal models for this choice of $d_\pm$.

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