ABSTRACT

Global $\widetilde{SL(2,\mathbb{R})}$ Representations of the Schrödinger Equation
with Time-dependent Potentials

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We study the representation theory of the solution space of the one-dimensional Schrödinger equation with time-dependent potentials that possess $\mathfrak{sl}_2$-symmetry. We give explicit local intertwining maps to multiplier representations and show that the study of the solution space for potentials of the form $V(t,x) = g_2(t)x^2+g_1(t)x+g_0(t)$ reduces to the study of the potential free case. We also show that the study of the time-dependent potentials of the form $V(t,x) = \lambda x^{-2}+g_2(t)x^2+g_0(t)$ reduces to the study of the potential $V(t,x) = \lambda x^{-2}$. Therefore, we study the representation theory associated to solutions of the Schrödinger equation with this potential only. The subspace of solutions for which the action globalizes is constructed via nonstandard induction outside the semisimple category.

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A Dissertation

Approved by the Department of Mathematics

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Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

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May 2012

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CHAPTER ONE

Introduction

The original prolongation algorithm of Sophus Lie, was used in the early seventies to show that the one-dimensional Schrödinger equation,

$$2iu_t + u_{xx} = 2V(t,x)u$$

has different time-independent potentials that admit non-trivial, inequivalent Lie symmetries (c.f. [4], [9]). These are

$$V_1(x) = \lambda \tag{1.1a}$$

$$V_2(x) = \lambda x \tag{1.1b}$$

$$V_3(x) = \lambda x^2 \tag{1.1c}$$

$$V_4(x) = \lambda x^{-2} \tag{1.1d}$$

$$V_5(x) = \lambda_1 x^2 + \lambda_2 x^{-2} \tag{1.1e}$$

with arbitrary constants $\lambda, \lambda_i \in \mathbb{R}$. Lie's prolongation method provides the Lie algebra of symmetry operators. If $\mathfrak{h}_3$ denotes the three-dimensional Heisenberg algebra, then the symmetry algebra is isomorphic to $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{h}_3(\mathbb{R})$ for (1.1a), (1.1b), and (1.1c) and it is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}$ for (1.1d) and (1.1e). Early in the eighties, it was shown that the time dependent potential $V(t,x) = g_2(t)x^2 + g_1(t)x + g_0(t)$ has the same symmetry Lie algebra $\mathfrak{g}$ (c.f. [12]). General time-dependent potentials $V(t,x) = \overline{V}(t,x) + g_2(t)x^2 + g_1(t)x + g_0(t)$, where $\overline{V}(t,x)$ is a function not of the form $g_2(t)x^2 + g_1(t)x + g_0(t)$, have smaller symmetry algebras. It is natural to use representation theory to study the solution space of these differential operators. However, since the resulting actions are not global, the techniques of representation theory do not always apply.

However, in the mid-nineties, it was shown in [6] that these actions could be extended to global actions of the group. This motivated the study of the representation theory of the solution space for many other differential equations (c.f. [6], [11], [10]). For instance, in 2005 M. Sepanski and R. Stanke decomposed the solution space for the 1-dimensional potential free Schrödinger equation and studied it as a global Lie group representation in [10]. Recently, they analyzed the n-dimensional case, for the potential free Schrödinger equation (c.f. [11]).

By partially compactifying $\mathbb{R}^2$, it is possible to work with a natural subspace of solutions for which the action of the symmetry group globalizes. The resulting representations are analyzed. It turns out that a change of variables reduces (1.1a)-(1.1c) to the potential free case and so are well understood [11].

However, the potential related to the inverse of the square of x, (1.1d) and the potential (1.1e) can be related to an eigenvalue problem of the potential free case for, essentially, the Casimir element. The study of the potential (1.1d) is important in the study of the motion of a dipole in a cosmic string background (c.f. [3]). This potential is also relevant in the fabrication of nanoscale atom optical devices, the study of dipole-bound anions of polar molecules, and in the study of the behavior of three-body systems in nuclear physics (c.f. [2]). Parts of this work have been published in [7].

This work is organized as follows. In Section 2.1 the realization of the symmetry group is exposed. The solution spaces are embedded in the standard induced representation spaces, the latter are exposed in this section, finally the group and the algebra actions are explicitly calculated.

In Section 3.1 the spaces and isomorphisms by which the constant potential reduces to the potential free case are shown. A very similar procedure is followed in Sections 3.2 and 3.3 to show the isomorphisms used for the linear and quadratic potentials respectively.

In Section 4 we study the inverse square potential, (1.1d). The resulting representations on the solution space are analyzed and decomposed into irreducible $\mathfrak{sl}_2$-modules. The eigenvalues that yield smooth solutions are indexed by the triangular numbers. Moreover, their direct sum inherits a joint action of $\widetilde{SL(2,\mathbb{R})}$ and the three-dimensional Heisenberg group. This representation is also decomposed into indecomposables. Finally, Section 4.5 shows how (1.1e) reduces to the same eigenvalue problem as (1.1d).

In the last section we study the time-dependent potential $V(t,x) = g_2(t)x^2 + g_1(t)x + g_0(t)$. It will be shown to reduce via change of variables to the potential free case. Moreover, the potential $V(t,x) = \lambda/x^2 + g_2(t)x^2 + g_0(t)$ will be shown to reduce to the eigenvalue problem studied in Chapter 4 via the same change of variables.

CHAPTER TWO

Preliminary Notation

2.1 The Group

Following [7], let $G_0 = SL(2,\mathbb{R})$ and let H_3 denote the three dimensional Heisenberg group with product,

$$(v_1, v_2, v_3)(v_1', v_2', v_3') = (v_1 + v_1', v_2 + v_2', v_3 + v_3' + v_1 v_2' - v_2 v_1').$$

Following the realization of the two-fold cover of G_0 in [8], define the complex upper half plane $D := \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$ and let G_0 act on D by fractional linear transformations, that is, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_0$ and $z \in D$ then

$$g.z = \frac{az + b}{cz + d}.$$

Define $\mathrm{d} : G_0 \times D \to \mathbb{C}$ by $\mathrm{d}(g, z) := cz + d$. Then there are exactly two smooth square roots of $\mathrm{d}(g, z)$ for each $g \in G_0$ and $z \in D$. The double cover can be realized as:

$$\begin{aligned} \widetilde{G_0} = \{(g, \epsilon) | g \in SL(2,\mathbb{R}) &\text{ and smooth } \epsilon : D \to \mathbb{C} \\ &\text{such that } \epsilon(z)^2 = \mathrm{d}(g, z) \text{ for } z \in D\} \end{aligned}$$

with the product defined by

$$(g_1, \epsilon_1(z))(g_2, \epsilon_2(z)) = (g_1 g_2, \epsilon_1(g_2.z)\epsilon_2(z)).$$

Finally, the symmetry group that we are interested in, is $G := \widetilde{G_0} \ltimes H_3$. Here $\widetilde{G_0}$ acts on H_3 by the standard action on the first two coordinates and leaves the third fixed.

2.2 Parabolic Subgroup and Induced Representations

As in [11], we consider the parabolic subalgebra of lower triangular matrices $\overline{\mathfrak{q}} \subset \mathfrak{sl}(2,\mathbb{R})$ with Langlands decomposition $\mathfrak{m} \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}}$. If $\exp_{\widetilde{G}} : \mathfrak{sl}(2,\mathbb{R}) \to \widetilde{G_0}$ denotes the exponential map then:

$$A := \exp_{\widetilde{G_0}}(\mathfrak{a}) = \{(\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right), z \mapsto e^{-t/2}) | t \in \mathbb{R}^{\geq 0}\}$$

$$N := \exp_{\widetilde{G_0}}(\mathfrak{n}) = \{(\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right), z \mapsto 1) | t \in \mathbb{R}\}$$

$$\overline{N} := \exp_{\widetilde{G_0}}(\overline{\mathfrak{n}}) = \{(\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right), z \mapsto \sqrt{tz+1}) | t \in \mathbb{R}\}.$$

Let $\mathfrak{k} := \{\left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right) : \theta \in \mathbb{R}\}$ then

$$K := \exp_{\widetilde{G_0}}(\mathfrak{k}) = \{(\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right), z \mapsto \sqrt{\cos\theta - z\sin\theta}) | \theta \in \mathbb{R}\},$$

where $\sqrt{\cdot}$ denotes the principal square root in $\mathbb{C}$. Writing M for the centralizer of A in K then

$$M = \{m_j := (\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^j, z \to i^{-j}) | j = 0,1,2,3\}.$$

Let $W \subset H_3$ be given by $W = \{(0,v,w) | v,w \in \mathbb{R}\} \cong \mathbb{R}^2$ and let $X := \{(x,0,0) | x \in \mathbb{R}\}$. Let us write $\mathfrak{w}$ for the Lie algebra of W. Then $\overline{P} = MA\overline{N} \ltimes W$ is the analogue of a parabolic subgroup in G corresponding to $\overline{\mathfrak{p}} := \overline{\mathfrak{q}} \ltimes \mathfrak{w}$.

For later use, we notice that an element in $g = [(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z \mapsto \epsilon(z)), (u,v,w)] \in G$ is in the image of the mapping $\overline{P} \times (N \times X) \to G$ given by $(\overline{p}, n) \mapsto \overline{p}n$, if $a \neq 0$. This induces a decomposition of such g into its $\overline{P}$ and $N \times X$ components,

$$\begin{aligned}
&[(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z \mapsto \epsilon(z)), (u,v,w)] = \\
&\qquad \left[(\left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right), z \mapsto \epsilon(z+b/a)), (0, v, w + (u + bv/a)v)\right] \\
&\qquad\qquad \cdot \left[(\left(\begin{smallmatrix} 1 & b/a \\ 0 & 1 \end{smallmatrix}\right), z \mapsto 1), (u + bv/a, 0, 0)\right].
\end{aligned}$$

On the open dense set where $a \neq 0$, let $\overline{p} : G \to \overline{P}$ and $n : G \to N \times X$ be the projections from the previous decomposition.

It is well known that the character group on A is isomorphic to the additive group $\mathbb{C}$ so any character on A can be indexed by a constant $r \in \mathbb{C}$ and defined by

$$\chi_r\big(\big(\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right), z \mapsto e^{-t/2}\big)\big) = t^r$$

for $t > 0$. A character on M is parametrized by $q \in \mathbb{Z}_4$ and defined by $\chi_q(m_j) = i^{jq}$. A character on W can be parametrized by $s \in \mathbb{C}$ and defined by,

$$\chi_s\big((0, v, w)\big) = e^{sw}.$$

Finally, any character on $\overline{P}$ that is trivial on N is parametrized by a triplet (q, r, s) where $s, r \in \mathbb{C}$ and $q \in \mathbb{Z}_4$ and defined by

$$\chi_{q,r,s}\left(((-1)^j \left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right), z \mapsto i^{-j} e^{-a/2}\sqrt{acz+1}), (0, v, w)\right) = i^{jq}|a|^r e^{sw}.$$

The representation space induced by $\chi_{q,r,s}$ will be denoted by $I(q, r, s)$ and defined by

$$I(q, r, s) := \{\phi : G \to \mathbb{C} | \phi \in C^\infty \text{ and } \phi(g\overline{p}) = \chi_{q,r,s}^{-1}(\overline{p})\phi(g) \text{ for } g \in G, \overline{p} \in \overline{P}\}$$

the G-action on $I(q, r, s)$ is given by $(g_1.\phi)(g_2) = \phi(g_1^{-1}g_2)$.

Since $H_3 = XW$ then $G = (N \times X)\overline{P}$ a.e. But $N \times X$ is isomorphic to $\mathbb{R}^2$ via $(t, x) \mapsto N_{t,x} := [((\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}), z \mapsto 1), (x, 0, 0)]$. Since a section in the induced representation is determined by its restriction to $N \times X$, this restriction induces an injection of $I(q, r, s)$ into $C^\infty(\mathbb{R}^2)$ which is identified as

$$I'(q, r, s) = \{f \in C^\infty(\mathbb{R}^2) | f(t, x) = \phi(N_{t,x}) \text{ for some } \phi \in I(q, r, s)\}.$$

This space is endowed with the corresponding action so that the map $\phi \mapsto f$ where $f(t, x) = \phi(N_{t,x})$, becomes intertwining. Thus $I(q, r, s) \cong I'(q, r, s)$ as $\widetilde{G}_0$-modules. As in the semisimple case, we will call this, the non-compact picture.

2.2.1 *The Action of G on $I'(q,r,s)$*

Fix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_0$. In order to write down the action of G on $I'(q,r,s)$ it is helpful to find an explicit way extension of $\epsilon : D \to \mathbb{C}$ to the real line. This is done in [11]. We review the construction here. Let $(g,\epsilon) \in \widetilde{G_0}$ then $\mathrm{d}(g,z) = cz + d$ and $\epsilon(z)$ can be written as

$$\epsilon(z) = |d|^{1/2} i^p \sqrt{c/dz+1}$$

for some $p \in \mathbb{Z}_4$. For $x \neq -d/c$ the limit value can be calculated,

$$\epsilon(x) = \lim_{z \to x, z \in D} \epsilon(z).$$

For such x, it follows that

$$\epsilon(x) = \begin{cases} |d|^{1/2} i^p \sqrt{c/dx+1} & \text{if } c/dx+1 > 0 \\ |d|^{1/2} i^{p+1} \sqrt{|c/dx+1|} & \text{if } c/dx+1 < 0 \text{ and } c/d > 0 \\ |d|^{1/2} i^{p-1} \sqrt{|c/dx+1|} & \text{if } c/dx+1 < 0 \text{ and } c/d < 0 \end{cases}$$

This can be extended to any real value of x.

Proposition 2.1. *Let $f \in I'(q,r,s)$, $(g,\epsilon) \in \widetilde{G_0}$, and $(u,v,w) \in H_3$. Then,*

$$((g,\epsilon).f)(t,x) = (a-ct)^{r-q/2} \epsilon(g^{-1}.(t+z)) e^{\frac{-scx^2}{a-ct}} f\left(\frac{dt-b}{a-ct}, \frac{x}{a-ct}\right) \tag{2.1a}$$

$$((u,v,w).f)(t,x) = e^{-s(uv-2vx-tv^2+w)} f(t, x-u-tv). \tag{2.1b}$$

Proof. This result is proved in a more general setting in [11]. Here we prove the action in (2.1a) for our particular case. Consider the $NMA\overline{N}$ decomposition for a given $(g,\epsilon) \in \widetilde{G_0}$,

$$(g,\epsilon) = (\left(\begin{smallmatrix} 1 & b/d \\ 0 & 1 \end{smallmatrix}\right), 1)(\mathrm{sgn}(d) I_2, i^p)(\left(\begin{smallmatrix} |d|^{-1} & 0 \\ 0 & |d| \end{smallmatrix}\right), |d|^{1/2})(\left(\begin{smallmatrix} 1 & 0 \\ c/d & 1 \end{smallmatrix}\right), \sqrt{c/dz+1}).$$

The action is computed as

$$((g,\epsilon).f)(t,x) = \phi((g,\epsilon)^{-1} N_{t,x}) = \phi\left(\left(\left(\begin{smallmatrix} d & dt-b \\ -c & a-ct \end{smallmatrix}\right), \epsilon(g^{-1}.(t+z))^{-1}\right), (x,0,0)\right).$$

We look at the $N\overline{P}$ decomposition,

$$\left[\left(\left(\begin{smallmatrix} d & dt-b \\ -c & a-ct \end{smallmatrix}\right), \epsilon(g^{-1}.(t+z))^{-1}\right), (x,0,0)\right] = \left[\left(\left(\begin{smallmatrix} 1 & \frac{dt-b}{a-ct} \\ 0 & 1 \end{smallmatrix}\right), 1\right), \left(\frac{x}{a-ct}, 0, 0\right)\right]$$
$$\cdot \left[\left(\left(\begin{smallmatrix} \frac{1}{a-ct} & 0 \\ \frac{-c}{a-ct} & a-ct \end{smallmatrix}\right), \epsilon(g^{-1}.(t+z))^{-1}\right), \left(0, \frac{-cx}{a-ct}, \frac{cx^2}{a-ct}\right)\right].$$

Using the definition of $I(q,r,s)$ and $\chi_{q,r,s}$ we obtain

$$((g,\epsilon).f)(t,x) = (a-ct)^{r-q/2}\epsilon(g^{-1}.(t+z))e^{\frac{-scx^2}{a-ct}} f\left(\frac{dt-b}{a-ct}, \frac{x}{a-ct}\right).$$

$\square$

Corollary 2.1. *The action of* $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R})$ *on* $I'(q,r,s)$ *is given by the differential operator*

$$(ct-a)x\partial_x + (ct^2 - 2at - b)\partial_t + (ra - csx^2 - rct). \tag{2.2}$$

An element $(u,v,w) \in \mathfrak{h}_3$ *acts on* $I'(q,r,s)_{\mu 1}$ *by the differential operator*

$$(tv-u)\partial_x + s(w-2vx).$$

Proof. It follows from differentiating the group actions on $I'(q,r,s)$. For instance, to compute the action of the element $\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R})$ we first use Proposition 2.1 to compute the action of the element

$$\left(\left(\begin{smallmatrix} e^{a\tau} & 0 \\ 0 & e^{-a\tau} \end{smallmatrix}\right), \epsilon\right).f(t,x) = e^{a\tau r} f\left(e^{-2a\tau}t, e^{-a\tau r}x\right)$$

Then we take $\frac{d}{d\tau}\Big|_{\tau=0}$ to obtain,

$$ra - 2at\partial_t - ax\partial_x$$

which proves the statement for this element. The rest of the generators of the algebra are computed similarly.

$\square$

If we denote the potential free Schrödinger operator by $\Box$, then

$$\Box = 2i\partial_t + \partial_x^2.$$

By equation (3.8), the standard $\mathfrak{sl}_2$-triple $\{h, e^{\pm}\}$ acts by

$$h = -x\partial_x - 2t\partial_t + r \tag{2.3}$$

$$e^+ = -\partial_t \tag{2.4}$$

$$e^- = tx\partial_x + t^2\partial_t - (sx^2 + rt) \tag{2.5}$$

on the non-compact picture. Let

$$\Omega = 1/2h^2 - h + 2e^+e^-$$

be the Casimir element in the enveloping algebra of $\mathfrak{sl}(2, \mathbb{R})$. For use in later sections, define the central element

$$\Omega' = 2\Omega - r(r+2).$$

Corollary 2.2. *On $I'(q, r, s)$, Ω acts by*

$$\Omega = \frac{1}{2}\left(4sx^2\partial_t + x^2\partial_x^2 - (1+2r)x\partial_x + r(r+2)\right).$$

In particular, for $r = -1/2$ and $s = i/2$, Ω acts by

$$\Omega = \frac{1}{2}(x^2\Box + r(r+2)).$$

Proof. A straightforward calculation using the actions of the standard $\mathfrak{sl}_2$-triple and the definition of the Casimir element gives the desired result. $\Box$

CHAPTER THREE

Time Independent Potentials

3.1 Constant Potential

3.1.1 The Algebra

We consider the constant potential case (1.1a). It is well-known that the algebra of symmetry operators is isomorphic to $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{h}_3(\mathbb{R})$, however we were unable to find a convenient reference for the explicit form of the generators, so we calculate them here. We calculate the conformal invariant differential operators according to [12]. Similar results can be obtained by using the standard prolongation algorithm.

With an eye towards future use, we begin the calculation for an arbitrary time-dependent potential $V(t,x)$. Suppose that $L = A(t,x)\partial_x + B(t,x)\partial_t + C(t,x)$ is a differential operator in $\mathfrak{g}$. Then it has to satisfy the following condition

$$[\Box - 2V(t,x), L] = \iota(t,x)(\Box - 2V(t,x)),$$

which gives

$$\begin{aligned}2BV_x + C_{xx} + 2iC_t + 2AV_t + (2iA_t + A_{xx})\partial_t + (2C_x + B_{xx} + 2iB_t)\partial_x \\ + 2A_x\partial_{xt} + 2B_x\partial_x^2 = \iota(t,x)(\Box - 2V(t,x)). \quad (3.1)\end{aligned}$$

Equating coefficients, we obtain the following system of partial differential equations

$$A_x = 0 \quad (3.2a)$$

$$B_x = \iota/2 \quad (3.2b)$$

$$AV_t + BV_x + \frac{1}{2}C_{xx} + iC_t = -\iota V \quad (3.2c)$$

$$2iA_t + A_{xx} = 2i\iota \quad (3.2d)$$

$$2C_x + B_{xx} + 2iB_t = 0. \quad (3.2e)$$

This gives the general form of the functions A, B, and C. For instance (3.2a) implies $A(t,x) = A(t)$ and together with (3.2d), gives $B(t,x) = 1/2xA_t + b(t)$ for some function $b(t)$ dependent on t. Equation (3.2e) gives $C(t,x) = -1/4iA_{tt}x^2 - ib'(t)x + c(t)$ for some function c. We substitute this in (3.2c) to obtain c. In the particular case of the constant potential, the form of $V(t,x) = V_1(x) = \lambda$ yields $A_{ttt} = 0$ which implies that

$$A(t) = c_1t^2 + c_2t + c_3.$$

We can also see that $b''(t) = 0$ thus $b(t) = c_4t + c_5$ which gives

$$B(t,x) = (c_1t + \frac{c_2}{2})x + c_4t + c_5.$$

The zero degree coefficient in (3.1) gives

$$c'(t) = i(\lambda(2c_1t + c_2) + 1/2A_{tt})$$

from which we get

$$C(t,x) = -\frac{1}{2}c_1x^2 - ic_4 - i(\lambda(c_1t^2 + c_2t) + 1/2c_1t) + c_6.$$

In summary, besides multiplying by a constant, we obtain the following conformal invariant differential operators

$$\begin{aligned} X_1 &= t^2\partial_t + tx\partial_x - \frac{1}{2}(ix^2 - t - i\lambda t^2) \\ X_2 &= t\partial_t + \frac{1}{2}x\partial_x - i\lambda t \\ X_3 &= \partial_t \\ X_4 &= t\partial_x - ixu \\ X_5 &= \partial_x. \end{aligned}$$

Considering the bracket relations between these operators and considering distinguished linear combinations, we obtain the following basis elements,

$$e^+ := -\partial_t + i\lambda \tag{3.4a}$$

$$e^- := t^2\partial_t + tx\partial_x + \frac{1}{2}(t + 2it^2\lambda - ix^2) \tag{3.4b}$$

$$h := -2t\partial_t - x\partial_x - (\frac{1}{2} + 2it\lambda) \tag{3.4c}$$

$$\xi := -\partial_x \tag{3.4d}$$

$$\psi := t\partial_x - ix \tag{3.4e}$$

$$\zeta := \frac{1}{2}i \tag{3.4f}$$

with $\{h, e^\pm\}$ being isomorphic to the standard basis of $\mathfrak{sl}(2,\mathbb{R})$ and $\{\xi, \psi, \zeta\}$ spanning an isomorphic copy of the three dimensional Heisenberg algebra, $\mathfrak{h}_3(\mathbb{R})$, with ζ as the central element. Hence, the algebra spanned is $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{h}_3(\mathbb{R})$.

3.1.2 Multiplier Representation

From the point of view of physics, it is very natural to realize the action of the symmetry group on a multiplier representation instead of a more Lie-theoretical standard induced representation. In this section we set up an intertwining operator between these two pictures as well as determine a space for which the action globalizes.

It is convenient to work first with what might be called the non-compact picture of a multiplier representation. To that end, let $\nu_\lambda : N \times X \to \mathbb{C}$ be given by

$$\nu_\lambda(N_{t,x}) = e^{i\lambda t}$$

and extend it to an open dense set of G (namely $\overline{P}(N \times X)$) by

$$\nu_\lambda(g) = \nu_\lambda(n(g)).$$

For $f \in I'(q, r, s)$ define

$$\tilde{f}(t,x) = e^{i\lambda t} f(t,x) \tag{3.5}$$

and define

$$I'(q,r,s)_{\mu_\lambda} = \{\tilde{f} \in C^\infty(\mathbb{R}^2) | \tilde{f}(t,x) = e^{i\lambda t} f(t,x) \text{ for } f \in I'(q,r,s)\}.$$

The reason for the subscript μ_λ will become evident below. This space inherits a unique G-module structure so that the map $f \to \tilde{f}$ is intertwining.

Now we turn to a standard multiplier representation on $G/\overline{P}$. We start by defining the multiplier $\mu_\lambda : \overline{P}(N \times X) \times \overline{P}(N \times X) \to \mathbb{C}$ by

$$\mu_\lambda(g_1, g_2) = \nu_\lambda(g_2^{-1} g_1)/\nu_\lambda(g_2^{-1}).$$

Let $\phi \in I(q,r,s)$ and define the map $\tilde{\phi}$ on an open dense set of $G/\overline{P}$ by

$$\tilde{\phi}(g\overline{P}) := \mu_\lambda(g^{-1}, I)^{-1} \phi(g).$$

The multiplier representation space on $G/\overline{P}$, denoted by $\mathcal{F}$, is defined as the image of this mapping, that is

$$\mathcal{F} = \{\tilde{\phi} \mid \phi \in I(q,r,s)\},$$

with the G-action given by

$$(g_1.\tilde{\phi})(g_2\overline{P}) = \mu_\lambda(g_1, g_2)\tilde{\phi}(g_1^{-1} g_2 \overline{P}).$$

We notice that the map $\phi \to \tilde{\phi}$ is an intertwining map, because the easily verified equality $\mu_\lambda(g_2^{-1} g_1, I) = \mu_\lambda(g_2^{-1}, I)\mu_\lambda(g_1, g_2)$ implies that

$$\begin{aligned}(g_1.\tilde{\phi})(g_2\overline{P}) = \mu_\lambda(g_1, g_2)\tilde{\phi}(g_1^{-1} g_2\overline{P}) &= \mu_\lambda(g_1, g_2)\mu_\lambda(g_2^{-1} g_1, I)^{-1}\phi(g_1^{-1} g_2) \\ &= \mu_\lambda(g_2^{-1}, I)^{-1}\phi(g_1^{-1} g_2) = \widetilde{g_1.\phi}(g_2\overline{P}).\end{aligned}$$

Now we have a commutative diagram,

$$\begin{array}{ccc} \phi \in I(q,r,s) & \to & f \in I'(q,r,s) \\ \downarrow & & \downarrow \\ \tilde{\phi} \in \mathcal{F} & \to & \tilde{f} \in I'(q,r,s)_{\mu_\lambda}. \end{array}$$

Let $\tilde{\phi} \in \mathcal{F}$ then

$$\tilde{\phi}(N_{t,x}\overline{P}) = \nu_\lambda(N_{-t,-x})^{-1} f(t,x) = e^{i\lambda t}\nu_\lambda(N_{-t,-x})^{-1}\tilde{f}(t,x) = \tilde{f}(t,x).$$

Hence, the bottom map is given by $\tilde{\phi} \mapsto \tilde{f}$ if $\tilde{f}(t,x) = \tilde{\phi}(N_{t,x}\overline{P})$.

3.1.3 Group Action on $I'(q,r,s)_{\mu_\lambda}$

We will use the isomorphism between $I'(q,r,s)$ and $I'(q,r,s)_{\mu_\lambda}$ to transfer the action of the group and consequently the action of the algebra to the latter space, recovering the symmetry operators calculated in Section 3.1.1.

Proposition 3.1. *Let $\tilde{f} \in I'(q,r,s)_{\mu_\lambda}$, $(g,\epsilon) \in \widetilde{G_0}$, and $(u,v,w) \in H_3$. Then,*

$$\begin{aligned} ((g,\epsilon).\tilde{f})(t,x) = (a-ct)^{r-q/2}\epsilon(g^{-1}.(t+z)) \\ \cdot e^{i\lambda\left(\frac{b+(a-d)t-c(isx^2/\lambda+t^2)}{a-ct}\right)}\tilde{f}\left(\frac{dt-b}{a-ct}, \frac{x}{a-ct}\right) \end{aligned} \tag{3.6a}$$

$$((u,v,w).\tilde{f})(t,x) = e^{-s(uv-2vx-tv^2+w)}\tilde{f}(t,x-u-tv). \tag{3.6b}$$

Proof. The proof of Equation (3.6b) is not different than (2.1b), because the multiplier is trivial when restricted to the Heisenberg group, and (3.6a) is easily obtained from (2.1a) by applying the intertwining operator (3.5) two times as follows:

$$\begin{aligned} ((g,\epsilon).\tilde{f})(t,x) &= e^{i\lambda t}((g,\epsilon).f)(t,x) = e^{i\lambda t}(a-ct)^{r-q/2}\epsilon(g^{-1}.(t+z)) \\ &\cdot e^{\frac{-scx^2}{a-ct}} f\left(\frac{dt-b}{a-ct}, \frac{x}{a-ct}\right) = e^{-i\frac{dt-b}{a-ct}t}e^{i\lambda t}(a-ct)^{r-q/2}\epsilon(g^{-1}.(t+z)) \\ &\cdot e^{\frac{-scx^2}{a-ct}}\tilde{f}\left(\frac{dt-b}{a-ct}, \frac{x}{a-ct}\right), \end{aligned} \tag{3.7}$$

which proves the proposition. $\square$

Corollary 3.1. *The action of $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R})$ on $I'(q,r,s)_{\mu_\lambda}$ is given by the differential operator*

$$(ct-a)x\partial_x + (ct^2-2at-b)\partial_t + (ra - csx^2 - rct - (2at+b-ct^2)i\lambda). \tag{3.8}$$

An element $(u,v,w) \in \mathfrak{h}_3$ *acts on* $I'(q,r,s)_{\mu_\lambda}$ *by the differential operator*

$$(tv-u)\partial_x + s(w-2vx).$$

Proof. It follows from differentiating the group actions on $I'(q,r,s)_{\mu_\lambda}$. For example, to calculate the action of be^+, we consider the action of the group element from Proposition 3.1

$$\left(\left(\begin{smallmatrix} 1 & 0 \\ c\tau & 1 \end{smallmatrix}\right), \epsilon\right).f(t,x) = (1-\tau t)^r e^{\frac{-c\tau s x^2 - i\lambda c t^2}{1-c\tau t}} f\left(\frac{t}{1-c\tau t}, \frac{x}{1-c\tau t}\right).$$

Then we take $\frac{d}{d\tau}\big|_{\tau=0}$ and we obtain

$$-crt - csx^2 - i\lambda ct^2 + ct^2\partial_t + cxt\partial_x,$$

which corresponds to the asserted action. The rest of the generators of the algebra are computed similarly. $\square$

Remark 3.1. Notice that the action of the algebra on $I'(q,r,s)_{\mu_1}$ with $r=-1/2$ and $s=i/2$ correspond with the action of the algebra of symmetry operators for the constant potential Schrödinger equation. Also, one can recover the actions on $I'(q,r,s)$ by setting $\lambda=0$.

By Equation (3.8), the standard $\mathfrak{sl}_2$-triple acts on $I'(q,r,s)_{\mu_\lambda}$ by

$$h = -x\partial_x - 2t\partial_t + (r - 2i\lambda t)$$

$$e^+ = -\partial_t - i\lambda$$

$$e^- = tx\partial_x + t^2\partial_t - (sx^2 - i\lambda t^2 + rt).$$

Using these expressions, the following corollary follows directly.

Corollary 3.2. *On* $I'(q,r,s)_{\mu_\lambda}$, Ω *acts by*

$$\Omega = \frac{1}{2}\left(4sx^2\partial_t + x^2\partial_x^2 - (1+2r)x\partial_x + r(r+2) + 4is\lambda x^2\right).$$

In particular, for $s=i/2$ *and* $r=-1/2$, *it acts by* $\Omega = \frac{1}{2}(x^2(\Box - 2\lambda) + r(r+2))$.

An immediate consequence of this corollary is that for the special parameters, $r = -1/2$ and $s = i/2$, we have

$$\ker \Omega' = \ker(\Box - 2\lambda)$$

in $I'(q,r,s)_{\mu_\lambda}$. In the setting of $I(q,r,s)$ and $I'(q,r,s)$ this kernel has been decomposed into its K-types and the representation theory of this space has been worked in more generality (c.f. [11]). Under the isomorphism (3.5) all these results can be transfered to the case of $\mathcal{F}$ and $I'(q,r,s)_{\mu_\lambda}$.

3.2 Linear Potential

We now turn to the case where $V_2(x) = \lambda x$. Proceeding as in Section 3.1.1 one can calculate the algebra generators substituting $V(t,x) = V_2(x)$. In this case, the algebra of symmetry operators is spanned by

$$e^+ := -\partial_t + t\lambda\partial_x + \frac{1}{2}\lambda(t^2\lambda - 2x)i \tag{3.9a}$$

$$e^- := t^2\partial_t + t(x - \frac{t^2\lambda}{2})\partial_x - \frac{1}{8}(4it + t^4\lambda^2 - 12t^2\lambda x + 4x^2)i \tag{3.9b}$$

$$h := -2t\partial_t + \frac{1}{2}(3t^2\lambda - 2x)\partial_x + \frac{1}{2}(t^3\lambda^2 - 6t\lambda x + i)i \tag{3.9c}$$

$$\xi := -\partial_x - i\lambda t \tag{3.9d}$$

$$\psi := t\partial_x - (x - \frac{\lambda t^2}{2})i \tag{3.9e}$$

$$\zeta := \frac{1}{2}i. \tag{3.9f}$$

The algebra spanned is $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{h}_3(\mathbb{R})$ therefore, we will continue to use the group G.

3.2.1 Multiplier Representation

As in Section 3.1.2 we need to define the appropriate multiplier representation space. We start by defining a change of coordinates map $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$(t,x) \mapsto (t, x + \frac{\lambda t^2}{2}).$$

Now define $\nu_\lambda : N \times X \to \mathbb{C}$ by

$$\nu_\lambda(N_{t,x}) = e^{-\frac{1}{3}it^3\lambda^2 + i\lambda tx}$$

and extend it to a map on an open dense subset of G by $\nu_\lambda(g) = \nu_\lambda(n(g))$. Notice that ν_λ was defined differently in Section 3.1. However, we will use the same notation because these two maps will play the same role in each respective case and there is no risk of confusion. This will be done in the next cases without further comment. For $f \in I'(q, r, s)$ define

$$\tilde{f}(t, x) = e^{\frac{1}{6}i\lambda^2 t^3 + it\lambda x} f\left(t, x + \frac{\lambda t^2}{2}\right). \tag{3.10}$$

The space that would be called the non-compact picture of the multiplier representation is

$$I'(q, r, s)_{\mu_\lambda} = \{\tilde{f}(t, x) \in C^\infty(\mathbb{R}^2) | f(t, x) \in I'(q, r, s)\}$$

with the G-action that makes the map $f \to \tilde{f}$ an intertwining map.

To define the multiplier representation, we let $\mu_\lambda : \overline{P}(N \times X) \times \overline{P}(N \times X) \to \mathbb{C}$ by $\mu_\lambda(g_1, g_2) = \nu_\lambda(g_2^{-1} g_1)/\nu_\lambda(g_2^{-1})$, and for $\phi \in I(q, r, s)$ we define

$$\tilde{\phi}(g\overline{P}) = \mu_\lambda(g^{-1}, I)^{-1} \phi(g)$$

on an open dense set of $G/\overline{P}$. As before, the multiplier representation space, $I(q, r, s)_{\mu_\lambda}$, is defined as the image of this mapping. Since

$$\begin{aligned} \tilde{\phi}(N_{t, x + \frac{\lambda t^2}{2}} \overline{P}) = \nu_\lambda(N_{-t, -x - \frac{\lambda t^2}{2}}) \phi(N_{t, x + \frac{\lambda t^2}{2}}) \\ = \nu_\lambda(N_{-t, -x - \frac{\lambda t^2}{2}}) f(t, x + \frac{\lambda t^2}{2}) = \tilde{f}(t, x) \end{aligned}$$

then the intertwining map from $\tilde{\phi} \to \tilde{f}$ is given by

$$\tilde{f}(t, x) = \tilde{\phi}(N_{t, x + \frac{\lambda t^2}{2}} \overline{P}).$$

We then have the commutative diagram

$$\begin{array}{ccc} \phi \in I(q,r,s) & \to & f \in I'(q,r,s) \\ \downarrow & & \downarrow \\ \tilde{\phi} \in \mathcal{F} & \to & \tilde{f} \in I'(q,r,s)_{\mu_\lambda}. \end{array}$$

3.2.2 Group Actions on $I(q,r,s)_{\mu_\lambda}$

We can translate the action of G on $I'(q,r,s)$ to an action on $I'(q,r,s)_{\mu_\lambda}$. We record this in the following

Proposition 3.2. *Let $\tilde{f} \in I'(q,r,s)_{\mu_\lambda}$ and $(u,v,w) \in H_3$ then*

$$((u,v,w).\tilde{f})(t,x) = e^{-i\lambda t(u+tv)-s(uv-2vx-tv^2+w)}\tilde{f}(t,x-u-tv).$$

Let $(g,\epsilon) \in \widetilde{G_0}$ and define $\Gamma := \frac{x+\lambda t^2/2}{a-ct} - \frac{\lambda(dt-b)^2}{2(a-ct)^2}$ and $\Phi := \frac{dt-b}{a-ct}$ then

$$\begin{aligned} ((g,\epsilon).\tilde{f})(t,x) = (a-ct)^{r-q/2}\epsilon(g^{-1}.(t+z))e^{i\lambda^2 t(\frac{1}{6}t^2-x)} \\ \cdot e^{i\lambda^2\Phi(-\frac{1}{6}\Phi^2+\Gamma)}e^{\frac{-sc(x+\lambda t^2/2)^2}{a-ct}}\tilde{f}(\Phi,\Gamma). \end{aligned}$$

Proof. This is straightforward calculation using (3.10) on (2.1a) and on (2.1b). For $\widetilde{G_0}$, apply Equation (3.10), then Equation (2.1a) on the second line, and then Equation (3.10) on the third line:

$$\begin{aligned} ((g,\epsilon).\tilde{f})(t,x) &= e^{\frac{1}{6}i\lambda^2 t^3+it\lambda x}((g,\epsilon).f)\left(t, x+\frac{\lambda t^2}{2}\right) \\ &= e^{\frac{1}{6}i\lambda^2 t^3+it\lambda x}(a-ct)^{r-q/2}\epsilon(g^{-1}.(t+z))e^{\frac{-sc(x+\lambda t^2/2)^2}{a-ct}} f\left(\frac{dt-b}{a-ct}, \frac{x+\lambda t^2/2}{a-ct}\right) \\ &= (a-ct)^{r-q/2}\epsilon(g^{-1}.(t+z))e^{i\lambda^2 t(\frac{1}{6}t^2-x)}e^{i\lambda^2\Phi(-\frac{1}{6}\Phi^2+\Gamma)}e^{\frac{-scx^2}{a-ct}}\tilde{f}(\Phi,\Gamma). \end{aligned}$$

The calculation for H_3 is similar and omitted. $\square$

Corollary 3.3. *The action of* $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R})$ *on* $I'(q,r,s)_{\mu_\lambda}$ *is given by the differential operator*

$$(b\lambda t + a\lambda t^2 + (x - \frac{\lambda t^2}{2})(ct-a))\partial_x + (ct^2 - 2at - b)\partial_t + \Bigg(ra - cs\left(x + \frac{\lambda t^2}{2}\right)^2 \\ -rct - i\lambda(at\left(\frac{\lambda t^2}{2} - 3x\right) + b\left(x - \frac{\lambda t^2}{2}\right) - 2ct^2x)\Bigg). \quad (3.11)$$

An element $(u,v,w) \in \mathfrak{h}_3$ *acts on* $I'(q,r,s)_{\mu_\lambda}$ *by the differential operator*

$$(tv - u)\partial_x + [2v(sx + \frac{\lambda t^2(i+s)}{2}) - itu\lambda + sw].$$

Proof. This follows directly from differentiating the actions in Proposition 3.2. For example we compute the action of ce^-. Using Proposition 3.2 we calculate,

$$(\left(\begin{smallmatrix} 1 & 0 \\ c\tau & 1 \end{smallmatrix}\right), \epsilon).f(t,x) = (1-c\tau t)^r e^{i\lambda^2 t(\frac{1}{6}t^2 - x)} \cdot e^{i\lambda^2\Phi(-\frac{1}{6}\Phi^2+\Gamma)} e^{\frac{-sc\tau(x+\lambda t^2/2)^2}{1-c\tau t}} f\big(\Phi,\Gamma\big)$$

with $\Phi = \frac{t}{1-\tau t}$ and $\Gamma = \frac{x+\lambda^2t^2/2}{1-c\tau t} - \frac{\lambda^2 t^2}{2(1-c\tau t)^2}$. The coefficient that multiplies ∂_x is

$$\left.\frac{d\Gamma}{d\tau}\right|_{\tau=0} = ct(x + \lambda^2t^2/2) - c\lambda^2t^3 = ct(x - \lambda^2t^2/2).$$

The coefficient multiplying ∂_t is

$$\left.\frac{d\Phi}{d\tau}\right|_{\tau=0} = ct^2.$$

Finally, the multiplication term of the differential operator corresponds to the derivative of the terms that multiply $\tilde{f}$ in the group action, for which we obtain

$$\begin{aligned} &- rct - \frac{i\lambda^2(t^2-2x)}{2}\left.\frac{d\Phi}{d\tau}\right|_{\tau=0} + i\lambda^2\Phi\left.\frac{d\Gamma}{d\tau}\right|_{\tau=0} - sc(x+\frac{\lambda t^2}{2})^2 \\ &\qquad = -rct - \frac{i\lambda^2(t^2-2x)}{2}ct^2 + i\lambda^2ct^2(x - \lambda^2t^2/2) - sc(x+\frac{\lambda t^2}{2})^2 \\ &\qquad\qquad = -rct + 2i\lambda^2ct^2x - sc(x+\frac{\lambda t^2}{2})^2. \quad (3.12) \end{aligned}$$

The other generators are calculated similarly. □

From Corollary 3.3, the standard $\mathfrak{sl}_2$-triple acts on $I'(q,r,s)_{\mu_\lambda}$ by

$$e^+ := -\partial_t + t\lambda\partial_x + \frac{1}{2}\lambda(t^2\lambda - 2x)i$$
$$e^- := t^2\partial_t + t(x - \frac{t^2\lambda}{2})\partial_x - rt + 2i\lambda^2 t^2 x - s(x + \frac{\lambda t^2}{2})^2$$
$$h := -2t\partial_t + \frac{1}{2}(3t^2\lambda - 2x)\partial_x + \frac{1}{2}(t^3\lambda^2 - 6t\lambda x) + r.$$

Corollary 3.4. *Let $s = i/2$ and $r = -1/2$. Then, the Casimir element, Ω, acts on $I'(q,r,s)_{\mu_\lambda}$ by $\Omega = 1/2((x + \frac{\lambda t^2}{2})^2(\Box - 2\lambda x)) + r(r+2))$. In particular,*

$$\ker \Omega' = \ker(\Box - 2\lambda x)$$

in $I'(q,r,s)_{\mu_\lambda}$.

This means that the space of solutions to the Schrödinger equation with linear potential in $I'(q,r,s)_{\mu_\lambda}$ is isomorphic to the space of solutions $\ker \Omega' \subset I'(q,r,s)$. The latter has been studied in more generality in [11].

3.3 Harmonic Oscillator

For the potential $V_3(x)$ we consider $-\lambda^2/8$ instead of λ, allowing λ to be real or purely imaginary, in order to simplify the calculations. When λ is real the equation represents the repulsive harmonic oscillator, if it is imaginary, the equation corresponds to the attractive harmonic oscillator. The symmetry operators in both cases correspond to each other through $\lambda \mapsto i\lambda$, so there is no loss of generality in considering just one of these two cases.

Substituting $V(t,x) = -1/8\lambda^2 x^2$ in equation (3.1) the generators of the algebra can be computed as in Section 3.1.1. The algebra of symmetry operators is spanned by

$$e^+ := -\frac{1}{\lambda}(1 + \cosh\lambda t)\partial_t - \frac{1}{2}x\sinh\lambda t\partial_x + \frac{1}{4}(i\lambda x^2\cosh\lambda t - \sinh\lambda t) \tag{3.13a}$$
$$e^- := -\frac{1}{\lambda}(1 - \cosh\lambda t)\partial_t + \frac{1}{2}x\sinh\lambda t\partial_x - \frac{1}{4}(i\lambda x^2\cosh\lambda t - \sinh\lambda t) \tag{3.13b}$$

$$h := -\frac{2}{\lambda}\sinh\lambda t\partial_t - x\cosh\lambda t\partial_x + \frac{1}{2}(\lambda x^2\sinh\lambda t - \cosh\lambda t) \tag{3.13c}$$

$$\xi := -\cosh\frac{\lambda t}{2}\partial_x + \frac{1}{2}i\lambda x\sinh\frac{\lambda t}{2} \tag{3.13d}$$

$$\psi := -\sinh\frac{\lambda t}{2}\partial_x + \frac{1}{2}i\lambda x\cosh\frac{\lambda t}{2} \tag{3.13e}$$

$$\zeta := -\frac{1}{2}i\lambda. \tag{3.13f}$$

Similar to Section 3.1, the algebra is isomorphic to $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{h}_3(\mathbb{R})$.

3.3.1 Multiplier Representation

To construct the required multiplier representation space for this case, we will follow the same procedure presented in the previous cases. Here, the change of coordinates is $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ and it is given by

$$\gamma(t,x) = \left(\tanh\frac{\lambda t}{2}, x\,\mathrm{sech}\,\frac{\lambda t}{2}\right).$$

This induces a map on N.

We define $\nu_\lambda : N \times X \to \mathbb{C}$ by

$$\nu_\lambda(N_{\gamma(t,x)}) = e^{-\frac{i\lambda}{4}x^2\tanh\frac{\lambda t}{2}}(\mathrm{sech}\,\frac{\lambda t}{2})^{-1/2}$$

and we extend it to a map on an open dense set of G by $\nu_\lambda(g) = \nu_\lambda(n(g))$. Let $f \in I'(q,r,s)$ and define a map $f \mapsto \tilde{f}$ by

$$\tilde{f}(t,x) = e^{-\frac{i\lambda}{4}x^2\tanh\frac{\lambda t}{2}}(\mathrm{sech}\,\frac{\lambda t}{2})^{-1/2} f\left(\tanh\frac{\lambda t}{2}, x\,\mathrm{sech}\,\frac{\lambda t}{2}\right). \tag{3.14}$$

We define the space $I'(q,r,s)_{\mu_\lambda}$ as the image of this map. For use in the following section we record the inverse of this map,

$$f(t,x) = e^{\frac{i\lambda t x^2\cosh(\lambda t/2)^2}{4}}(1-t^2)^{-1/4}\tilde{f}\left(\frac{2}{\lambda}\,\mathrm{arctanh}\,t, x\cosh\frac{\lambda t}{2}\right) \tag{3.15}$$

for $t \in (-1,1)$. Notice that $\tilde{f} \in I'(q,r,s)_{\mu_\lambda}$ determines the map $f \in I'(q,r,s)$ only on $(-1,1) \times \mathbb{R}$. Therefore, this map gives an isomorphism between $I'(q,r,s)_{\mu_\lambda}$ and

$$I'(q,r,s)_{(-1,1)\times\mathbb{R}} := \{f \in C^\infty((-1,1)\times\mathbb{R}) \mid f(t,x)$$
$$= e^{\frac{i\lambda t x^2 \cosh(\lambda t/2)^2}{4}}(1-t^2)^{-1/4}\tilde{f}\left(\frac{2}{\lambda}\operatorname{arctanh} t, x\cosh\frac{\lambda t}{2}\right) \text{ for } f \in I'(q,r,s)_{\mu_\lambda}\}.$$

Clearly there exists a restriction map $I'(q,r,s) \to I'(q,r,s)_{(-1,1)\times\mathbb{R}}$.

In order to construct the multiplier representation space, let

$$\mu_\lambda : \overline{P}(N\times X)\times\overline{P}(N\times X) \to \mathbb{C}$$

by

$$\mu_\lambda(g_1,g_2) = \nu_\lambda(g_2^{-1}g_1)/\nu_\lambda(g_2^{-1}).$$

For $\phi \in I(q,r,s)$ define on an open dense set of $G/\overline{P}$ the intertwining map $\tilde{\phi}(g\overline{P}) = \mu_\lambda(g^{-1},I)^{-1}\phi(g)$ and regard the multiplier representation

$$I(q,r,s)_{\mu_\lambda}$$

as the image of this mapping. Finally we use the fact that

$$\tilde{f}(t,x) = \mu_\lambda(\gamma(N_{t,x})^{-1},I)f(\gamma(t,x))$$

to write the intertwining map from $I(q,r,s)_{\mu_\lambda}$ to $I'(q,r,s)_{\mu_\lambda}$ explicitly as

$$\tilde{\phi} \mapsto \tilde{f} \text{ whenever } \tilde{\phi}(N_{\gamma(t,x)}\overline{P}) = \tilde{f}(t,x).$$

We then have the commutative diagram

$$\begin{array}{ccc} \phi \in I(q,r,s) & \to & f \in I'(q,r,s) \\ \downarrow & & \downarrow \\ \tilde{\phi} \in \mathcal{F} & \to & \tilde{f} \in I'(q,r,s)_{\mu_\lambda}. \end{array}$$

where the bottom and right maps are isomorphisms.

3.3.2 Group Actions on $I'(q,r,s)_{\mu_\lambda}$

The group acts on $I'(q,r,s)_{(-1,1)\times\mathbb{R}}$ in the same way as on $I'(q,r,s)$, however the action is not a global one. In this Section, we use the isomorphism in (3.14) and

(3.15) to transfer the action of the group on $I'(q,r,s)_{(-1,1)\times\mathbb{R}}$ to a local action on $I'(q,r,s)_{\mu_\lambda}$.

Proposition 3.3. *Let* $\tilde{f} \in I'(q,r,s)_{\mu_\lambda}$ *and* $(u,v,w) \in H_3$. *Then*

$$((u,v,w).\tilde{f})(t,x) = e^{\frac{i\lambda}{4}\tanh\frac{\lambda t}{2}(2x-u\cosh\frac{\lambda t}{2}-v\sinh\frac{\lambda t}{2})(u\cosh\frac{\lambda t}{2}+v\sinh\frac{\lambda t}{2})}$$
$$\cdot e^{-s(uv-2vx\,\mathrm{sech}\,\frac{\lambda t}{2}-v^2\tanh\frac{\lambda t}{2}+w)}\tilde{f}\Big(t, x-u\cosh\frac{\lambda t}{2}-v\sinh\frac{\lambda t}{2}\Big). \tag{3.16}$$

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ *and let* $(g,\epsilon) \in \widetilde{G_0}$. *Define* $\Xi := \frac{x\,\mathrm{sech}\,\frac{\lambda t}{2}}{a-c\tanh\frac{\lambda t}{2}}$ *and* $\Psi = \frac{d\tanh\frac{\lambda t}{2}-b}{a-c\tanh\frac{\lambda t}{2}}$. *Then the action of* (g,ϵ) *on* $I'(q,r,s)_{\mu_\lambda}$ *is given by*

$$((g,\epsilon).f)(t,x) = \big(\cosh(\operatorname{arctanh}\Psi)\,\mathrm{sech}\,\frac{\lambda t}{2}\big)^{1/2}(a-c\tanh\frac{\lambda t}{2})^{r-q/2}$$
$$\cdot\,\epsilon(g^{-1}.(\tanh\frac{\lambda t}{2}+z))e^{\frac{i\lambda}{4}(x^2\tanh\frac{\lambda t}{2}-\Xi^2\tanh\frac{\lambda\Psi}{2})-scx\Xi\,\mathrm{sech}\,\frac{\lambda t}{2}}$$
$$\cdot\,\tilde{f}\left(\frac{2}{\lambda}\operatorname{arctanh}\Psi, \Xi\cosh\Big(\frac{\lambda}{2}\Psi\Big)\right). \tag{3.17}$$

Proof. This follows from applying the isomorphism (3.14) to the actions (2.1a) and (2.1b) calculated in Proposition 2.1. We will show the calculation of (3.16):

$$\begin{aligned}((u,v,w).\tilde{f})(t,x) &= e^{-\frac{i\lambda}{4}x^2\tanh\frac{\lambda t}{2}}(\mathrm{sech}\,\frac{\lambda t}{2})^{-1/2} \\ &\qquad\cdot((u,v,w).f)\left(\tanh\frac{\lambda t}{2}, x\,\mathrm{sech}\,\frac{\lambda t}{2}\right) \\ &= e^{-\frac{i\lambda}{4}x^2\tanh\frac{\lambda t}{2}}(\mathrm{sech}\,\frac{\lambda t}{2})^{-1/2}e^{-s(uv-2vx\,\mathrm{sech}\,\frac{\lambda t}{2}-\tanh\frac{\lambda t}{2}v^2+w)} \\ &\qquad\cdot f(\tanh\frac{\lambda t}{2}, x\,\mathrm{sech}\,\frac{\lambda t}{2}-u-\tanh\frac{\lambda t}{2}v).\end{aligned}$$

Using (3.15) we transform f into $\tilde{f}$ and obtain the asserted result. The same procedure is used to compute the action of elements in $\widetilde{G_0}$. □

The action of the group is just local, because (3.17) is not defined for every $(g,\epsilon) \in \widetilde{G_0}$, only for group elements such that $\frac{d\tanh\frac{\lambda t}{2}-b}{a-c\tanh\frac{\lambda t}{2}} \in (-1,1)$.

Corollary 3.5. *Let* $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R})$*, then it acts, on* $I'(q,r,s)_{\mu_\lambda}$*, by the differential operator*

$$\begin{aligned} &-\frac{1}{2}(2ax\cosh\lambda t + bx\sinh\lambda t - cx\sinh\lambda t)\partial_x - \frac{1}{\lambda}(2a\sinh\lambda t - b(1+\cosh\lambda t) \\ &\qquad + c(1-\cosh\lambda t))\partial_t + \frac{1}{2}a(1+2r-\cosh\lambda t + i\lambda x^2\sinh\lambda t) \\ &\quad - \frac{b}{4}(-i\lambda x^2\cosh\lambda t + \sinh\lambda t) + c(-sx^2 - r\tanh\frac{\lambda t}{2} + \frac{1}{4}(-ix^2(2\lambda + 4is \\ &\qquad\qquad + \lambda\cosh\frac{\lambda t}{2}) + \sinh\frac{\lambda t}{2})\tanh^2\frac{\lambda t}{2}). \qquad (3.18) \end{aligned}$$

An element $(u,v,w) \in \mathfrak{h}_3$ *acts, on* $I'(q,r,s)_{\mu_\lambda}$*, by the differential operator*

$$(v\sinh\frac{\lambda t}{2} - u\cosh\frac{\lambda t}{2})\partial_x + \frac{1}{2}i\lambda ux\sinh\frac{\lambda t}{2} - \frac{1}{4}ivx(-\lambda - 8is + \lambda\cosh\lambda t)\operatorname{sech}\frac{\lambda t}{2}.$$

Proof. These actions follow from differentiating the actions in Proposition 3.3. For this case we will calculate the action of be^+. Let $\Xi = x\operatorname{sech}\frac{\lambda t}{2}$ and $\Psi = \tanh\frac{\lambda t}{2} - \tau b$ then the coefficient that multiplies ∂_t is

$$\frac{d}{d\tau}(\frac{2}{\lambda}\operatorname{arctanh}(\tanh\frac{\lambda t}{2} - b\tau))\Big|_{\tau=0} = -\frac{2b}{\lambda}\cosh^2\frac{\lambda t}{2}.$$

The coefficient that multiplies ∂_x is

$$\frac{d}{d\tau}(x\operatorname{sech}(\frac{\lambda t}{2})\operatorname{arctanh}(\tanh\frac{\lambda t}{2} - b\tau))\Big|_{\tau=0} = -\frac{2b}{\lambda}\sinh\frac{\lambda t}{2}.$$

Finally, the zero order term in the differential operator correspondent to be^+ is

$$\begin{aligned} &\frac{d}{d\tau}(\big(\cosh(\operatorname{arctanh}\Psi)\operatorname{sech}\frac{\lambda t}{2}\big)^{1/2} e^{\frac{i\lambda}{4}(x^2\tanh\frac{\lambda t}{2} - \Xi^2\tanh\frac{\lambda\Psi}{2}) - scx\Xi\operatorname{sech}\frac{\lambda t}{2})}\Big|_{\tau=0} \\ &\qquad\qquad\qquad = -\frac{b}{2}\sinh\lambda t + \frac{b\lambda x^2}{4}i\cosh\lambda t. \end{aligned}$$

The calculation for the other generators of the algebra follows the same procedure.

□

Notice that for $r = -1/2$ and $s = i\lambda/4$, the actions in the corollary correspond with the ones in (3.13).

Corollary 3.6. *Let $s = i\lambda/4$ and $r = -1/2$. Then the Casimir element, Ω, acts on $I(q,r,s)_{\mu_\lambda}$ by $\Omega = \frac{1}{2}(x^2(\Box + \lambda^2 x^2/4) + r(r+2))$. In particular,*

$$\ker \Omega' = \ker(\Box + \lambda^2 x^2/4)$$

in $I'(q,r,s)_{\mu_\lambda}$.

Proof. By (3.18), the $\mathfrak{sl}_2$-triple acts by

$$\begin{aligned} h &= -x\cosh(\lambda t)\partial_x - \frac{2}{\lambda}\sinh(\lambda t)\partial_t + \frac{1}{2}(1 + 2r - \cosh(\lambda t) + i\lambda x^2 \sinh(\lambda t)) \\ e^+ &= -\frac{1}{2}x\sinh(\lambda t)\partial_x - \frac{1}{\lambda}(1+\cosh(\lambda t))\partial_t + \frac{1}{4}(-\sinh(\lambda t) + i\lambda x^2\cosh(\lambda t)) \\ e^- &= \frac{1}{2}x\sinh(\lambda t)\partial_x - \frac{1}{\lambda}(1-\cosh(\lambda t))\partial_t - sx^2 - r\tanh\frac{\lambda t}{2} \\ &\quad + \frac{1}{4}(-ix^2(2\lambda + 4is + \lambda\cosh\frac{\lambda t}{2}) + \sinh\frac{\lambda t}{2})\tanh^2\frac{\lambda t}{2}. \end{aligned}$$

Using these expressions, the result is obtained by a straightforward calculation. $\Box$

Let S be the image of $\ker \Omega'_{(-1,1)\times\mathbb{R}}$ under the map (3.14). Then we have the following diagram:

$$\begin{array}{ccccc} I'(q,r,s) & \to & I'(q,r,s)_{(-1,1)\times\mathbb{R}} & \to & I'(q,r,s)_{\mu_\lambda} \\ \uparrow & & \uparrow & & \uparrow \\ \ker\Omega' & \to & \ker\Omega'_{(-1,1)\times\mathbb{R}} & \to & S \subset \ker\Omega'_{\mu_\lambda} \end{array}$$

where the vertical maps are given by inclusion and the map $\ker\Omega' \to \ker\Omega'_{(-1,1)\times\mathbb{R}}$ is given by restriction. The latter map is surjective and since the elements in $\ker\Omega'$ are analytic functions, it is also injective. The study of this solution space reduces to the study of $\ker\Omega' \subset I'(q,r,s)$ which has already been studied in [11]. Notice that the right two columns of the diagram are just local representations of the group and the spaces on the left column are global representations of the group.

CHAPTER FOUR

Eigenvalue Problem of the Potential Free Schrödinger Equation

As seen, the representation theory for nearly all the potentials possessing $\mathfrak{sl}_2$-symmetry, reduces to the potential free case, therefore it is well understood (c.f. [11]). Here we study the remaining case where $V(t,x) = \lambda x^{-2}$. Interestingly, it will reduce to an eigenvalue problem of Ω'. To see this, recall from Section 2.1 the $\mathfrak{sl}_2$ part of the symmetry algebra acts by

$$e^+ := -\partial_t \tag{4.1a}$$

$$e^- := t^2\partial_t + tx\partial_x - \frac{1}{2}(x^2 + it)i \tag{4.1b}$$

$$h := -2t\partial_t + x\partial_x - \frac{1}{2} \tag{4.1c}$$

on $I'(q,r,s)$. In Corollary 2.2 we showed that the Casimir element Ω, acts by

$$\Omega = \frac{1}{2}\left(4sx^2\partial_t + x^2\partial_x^2 - (1+2r)x\partial_x + r(r+2)\right)$$

and the central element Ω' acts by the differential operator

$$\Omega' = x^2(4s\partial_t + \partial_x^2 - (1+2r)/x\partial_x)$$

on $I'(q,r,s)$. The subspace $\ker(\Omega' - 2\lambda)$ in $I'(q,r,s)$ is the same as $\ker(\Box - 2\lambda/x^2)$ in $I'(q,r,s)$.

We begin the study with a result on the invariance under $\widetilde{G_0}$ and the subgroup $\{(0,0,w)|w \in \mathbb{R}\} \subset H_3$.

Proposition 4.1. *The subspace* $\ker(\Omega' - 2\lambda)$ *in* $I'(q,r,s)$ *is invariant under the action of* $\widetilde{G_0}$ *and under the action of the subgroup* $\{(0,0,w)|w \in \mathbb{R}\}$ *of* H_3*. The space is not left invariant by the complement of* $\{(0,0,w)|w \in \mathbb{R}\}$ *in* H_3*.*

Proof. Since Ω is in the center of the enveloping algebra of $\mathfrak{g}$, the $\widetilde{G_0}$-invariance is clear. Let $(u,v,w) \in H_3$. Using the action on Corollary 2.1 we can calculate

$$\begin{aligned}
[\Box - 2\lambda/x^2, (u,v,w)] &= [2i\partial_t + \partial_x^2 - 2\lambda/x^2, (tv-u)\partial_x + i/2(w-2vx)] \\
&= [2i\partial_t, (tv-u)\partial_x] + [\partial_x^2, i/2(w-2vx)] \\
&\quad - 2[\lambda/x^2, (tv-u)\partial_x] \\
&= 2iv\partial_x - 2iv\partial_x + \frac{4\lambda(tv-u)}{x^3} = \frac{4\lambda(tv-u)}{x^3}.
\end{aligned}$$

This is zero for every $t \in \mathbb{R}$ only if $u = 0$ and $v = 0$, this proves the invariance under $\{(0,0,w)|w \in \mathbb{R}\}$ of H_3. $\square$

Though, all of H_3 does not leave $\ker(\Omega' - 2\lambda)$ invariant, it will play an important role in linking together different $\widetilde{G}_0$-invariant kernels.

4.1 The Compact Picture

The group $\widetilde{G_0}$ has Iwasawa decomposition $\widetilde{G_0} = KA\overline{N}$ and the product induces a diffeomorphism $G \cong (K \times X) \times (A\overline{N} \ltimes W)$. Since $(A\overline{N} \ltimes W) \subset \overline{P}$, an element $\phi \in I(q,r,s)$ is completely determined by its restriction to $K \times X$. Moreover, since $(K \times X) \cap \overline{P} = M$ we have that the restriction of $\phi \in I(q,r,s)$ (which we will still denote by ϕ) satisfies $\phi(gm) = \chi_{q,r,s}(m)\phi(g)$ for $g \in K \times X$ and $m \in M$.

There exists an isomorphism between $K \times X$ and $S^1 \times \mathbb{R}$ given by $(\theta, y) \mapsto [(g_\theta, \epsilon_\theta), (y,0,0)]$ and it can be shown that this map is 4π-periodic with respect to θ. Thus we can identify $\phi \in I(q,r,s)$ with a map $F : S^1 \times \mathbb{R} \to \mathbb{C}$, $\phi \mapsto F$ iff $\phi([(g_\theta, \epsilon_\theta), (y,0,0)]) = F(\theta, y)$. Then $F \in C^\infty(S^1 \times \mathbb{R})$ and $F(\theta + 4\pi, y) = F(\theta, y)$.

The function F inherits from ϕ additional "‘parity"’ identities. By the definition, $\epsilon_{\theta+\pi j}(z)^2 = \cos(\theta + \pi j) - z\sin(\theta + \pi j) = (-1)^j \epsilon_\theta(z)$. We then get

$$\begin{aligned}
F(\theta + \pi j, (-1)^j y) &= \phi([(g_{\theta+\pi j}, \epsilon_{\theta+\pi j}), ((-1)^j y, 0, 0)]) = \phi([(g_\theta, \epsilon_\theta), (y,0,0)]m_j) \\
&= \chi_{q,r,s}(m_j)^{-1}\phi([(g_\theta, \epsilon_\theta), (y,0,0)]) = i^{-jq}F(\theta, y).
\end{aligned}$$

Define

$$I''(q,r,s) = \{F \in C^\infty(\mathbb{R}^2) | F(\theta + 4\pi, y) = F(\theta, y)$$
$$\text{and } F(\theta + j\pi, (-1)^j y) = i^{-jq} F(\theta, y)\}. \quad (4.2)$$

Then the map $\phi \mapsto F$ is a vector space isomorphism between $I(q,r,s)$ and $I''(q,r,s)$. The space $I''(q,r,s)$ inherits a unique G-module structure, so that this map becomes an intertwining map. We call this the compact picture, as in the semisimple case, though $K \times X$ is not compact here.

In turn, the isomorphism T induces an isomorphism between $I'(q,r,s)$ and $I''(q,r,s)$ which we will write out explicitly. We begin with the following decomposition:

$$[(g_\theta, \epsilon_\theta), (y, 0, 0)] = [((\begin{smallmatrix} 1 & \tan\theta \\ 0 & 1 \end{smallmatrix}), z \mapsto 1), (y \sec\theta, 0, 0)]$$
$$\cdot [(\begin{pmatrix} 1/\cos\theta & 0 \\ -\sin\theta & \cos\theta \end{pmatrix}, \epsilon_\theta), (0, -y\tan\theta, y^2 \tan\theta)].$$

Since $F(\theta, y) = \phi([(g_\theta, \epsilon_\theta), (y, 0, 0)])$ then

$$F(\theta, y) = \chi_{q,r,s}([(\begin{pmatrix} 1/\cos\theta & 0 \\ -\sin\theta & \cos\theta \end{pmatrix}, \epsilon_\theta), (0, -y\tan\theta, y^2\tan\theta)])^{-1}$$
$$\cdot \phi([((\begin{smallmatrix} 1 & \tan\theta \\ 0 & 1 \end{smallmatrix}), z \mapsto 1), (y\sec\theta, 0, 0)]) = (\cos\theta)^{-r} e^{-sy^2\tan\theta} f(\tan\theta, y\sec\theta) \quad (4.3)$$

for $f \in I(q,r,s)$ and $\theta \in (-\pi/2, \pi/2)$. Since $F \in I''(q,r,s)$, this expression can be extended smoothly to any $\theta \in \mathbb{R}$ by using the fact that $F(\theta + j\pi, y) = i^{-jq} F(\theta, (-1)^j y)$ and continuity to get to the integer multiples of $\pi/2$. Then we define the isomorphism $T : I'(q,r,s) \to I''(q,r,s)$ by $T(f) = F$. The inverse to this map can be calculated and it is:

$$f(t, x) = (1 + t^2)^{-r/2} e^{\frac{stx^2}{1+t^2}} F(\arctan t, x(1 + t^2)^{-1/2}). \quad (4.4)$$

Under this isomorphism, via the chain rule, we obtain

$$\partial_t \leftrightarrow \frac{1}{2}(-y\sin 2\theta \partial_y + \cos^2\theta \partial_\theta + 2sy^2 \cos 2\theta - 1/2r \sin 2\theta) \quad (4.5a)$$

$$\partial_x \leftrightarrow 2sy\sin\theta + \cos\theta\partial_y. \tag{4.5b}$$

This will enable us to transfer the actions of the algebra from the non-compact picture, $I'(q,r,s)$, to the compact picture, $I''(q,r,s)$.

Define a standard basis of $\mathfrak{sl}_2(\mathbb{C})$ given by

$$\kappa = i(e^- - e^+)$$

and

$$\eta^\pm = 1/2(h \pm i(e^+ + e^-)).$$

Applying Equations (4.5) to the action in Corollary 2.1, it can be shown that the $\mathfrak{sl}_2$-triple just defined acts on $I''(q,r,s)$ by the differential operators

$$\kappa = i\partial_\theta \tag{4.6}$$

$$\eta^\pm = \frac{1}{2}e^{\mp 2i\theta}\left(y\partial_y \mp i\partial_\theta - (1/2 \pm 2isy^2)\right). \tag{4.7}$$

Proposition 4.2. *If Ω'' denotes the differential operator by which the central element Ω' acts on $I''(q,r,s)$ then*

$$\Omega'' = y^2\left(4s\partial_\theta + 4s^2y^2 + \partial_y^2 + \frac{1+2r}{y}\partial_y\right).$$

Proof. Under the isomorphism $I'(q,r,s) \cong I''(q,r,s)$ we obtain the following expressions:

$$4sx^2\partial_t \mapsto y^2(-4sy\tan\theta\partial_y + 4s\partial_\theta + 8s^2y^2 - 4s^2y^2\sec^2\theta + 2sr\tan\theta)$$

$$x^2\partial_x^2 \mapsto y^2(4s^2y^2\tan^2\theta + \partial_y^2 + 4sy\tan\theta\partial_y + 2s\tan\theta)$$

$$(1+2r)x\partial_x \mapsto (1+2r)y^2(2s\tan\theta + 1/y\partial_y).$$

Adding them we get

$$\frac{1}{y^2}\Omega'' = 4s\partial_\theta + 4s^2y^2 + \partial_y^2 + \frac{1+2r}{y}\partial_y.$$

□

4.2 K-types

Lemma 4.1. *Let $(g_{\theta'}, \epsilon_{\theta'}) \in K$ and $F \in I''(q,r,s)$ then $(g_{\theta'}, \epsilon_{\theta'}).F(\theta,y) = F(\theta-\theta',y)$*

Proof.

$$(g_{\theta'}, \epsilon_{\theta'}).F(\theta,y) = \phi([(g_{\theta'}, \epsilon_{\theta'})^{-1}(g_\theta, \epsilon_\theta),(y,0,0)])$$
$$= \phi([(g_{\theta-\theta'}, \epsilon_{\theta-\theta'}),(y,0,0)]) = F(\theta-\theta',y)$$

□

There exists an isomorphism $K \cong S^1$ given by $(g_\theta, \epsilon_\theta) \mapsto e^{i\theta/2}$. Therefore, the characters on K are of the form $\chi_m^K(g_\theta, \epsilon_\theta) = e^{-im\theta/2}$ for $m \in \mathbb{Z}$. Using Lemma 4.1, a weight vector $F_m \in I''(q,r,s)$ of weight $\frac{m}{2}$, for the action of K, satisfies $(g_{\theta'}, \epsilon_{\theta'}).F_m(\theta,y) = F_m(\theta-\theta',y) = e^{-im\theta'/2}F_m(\theta,y)$. Setting $\theta = 0$ and $\theta' = -\theta$ we obtain $F_m(\theta,y) = e^{-im\theta/2}F_m(0,y)$. Let $\tilde{F}_m(y) := F_m(0,y)$ so that a weight vector is of the form

$$F_m(\theta,y) = e^{-im\theta/2}\tilde{F}_m(y).$$

Lemma 4.2. *Fix $m \in \mathbb{Z}$ and $\tilde{F} \in C^\infty(\mathbb{R})$. Then $F(\theta,y) = e^{-im\theta/2}\tilde{F}(y)$ is annihilated by $\Omega'' - 2\lambda$ if and only if $\tilde{F}(y)$ is annihilated by the differential operator*

$$\mathcal{D} = y^2\partial_y^2 - (2\lambda - my^2 + y^4)$$

Proof. Explicitly calculating the action of $\Omega'' - 2\lambda$ on $F(\theta,y) = e^{-im\theta/2}\tilde{F}(y)$, one obtains that $(\Omega'' - 2\lambda)F = e^{-im\theta/2}\mathcal{D}\tilde{F}$. □

Proposition 4.3. *There exist a K-finite vector of weight $\frac{m}{2}$ in $\ker(\Omega''-2\lambda) \subset I''(q,r,s)$ iff*

$$l = \frac{1}{2}(1+\sqrt{1+8\lambda}) \tag{4.8}$$

is a positive integer (equivalently, $\lambda = l(l-1)/2$ for $l \in \mathbb{Z}^{>0}$) and $m \equiv 2l+q \mod 4$. In this case, if $\lambda \neq 0$, there exists a unique (up to scalar multiples) weight vector of *weight* $\frac{m}{2}$ *in* $\ker(\Omega'' - 2\lambda) \subset I''(q, r, s)$ *and it is given by*

$$F_m(\theta, y) = e^{-im\theta/2} e^{-y^2/2} y^l \, {}_1F_1\left(\frac{1 + 2l - m}{4}, l + \frac{1}{2}, y^2\right) \tag{4.9}$$

where ${}_1F_1$ *are the congruent hypergeometric functions of the first kind.*

Proof. By Lemma 4.2, for F_m to be in $\ker(\Omega'' - 2\lambda) \subset I''(q, r, s)$, it is necessary that $\mathcal{D}\tilde{F}_m = 0$. Because of the form of $\mathcal{D}$, it respects the decomposition of $\tilde{F}_m$ in terms of its even and odd components. Moreover, each of the components is determined by its value on $\mathbb{R}^+$. Working first with $y \geq 0$ we can write $\tilde{F}_m(y) = e^{-y^2/2} H(y^2)$ for some smooth function H. Then, the condition $\mathcal{D}\tilde{F}_m = 0$ is equivalent to

$$\left(4y^4 \partial_y^2 + (2y^2 - 4y^4)\partial_y + ((m-1)y^2 - 2\lambda)\right) H(y^2) = 0. \tag{4.10}$$

Following [5], the Frobenius method for this equation yields a solution spanned by two linearly independent solutions. The indicial roots for this equation are

$$l_1 = \frac{1}{2}(1 - \sqrt{1 + 8\lambda})$$

and

$$l_2 = \frac{1}{2}(1 + \sqrt{1 + 8\lambda}).$$

Then, the first linearly independent solution is of the form

$$H_1(y^2) = y^{l_2}(1 + \sum_{j=1}^{\infty} c_j(l_2) y^{2j})$$

for some $c_j(l_2) \in \mathbb{R}$. This function extends to a smooth function on $\mathbb{R}$ only if $l_2 \in \mathbb{Z}^{\geq 0}$ iff $\lambda = 0$ or λ is a triangular number (i.e., $\lambda = k(k-1)/2$ for some $k \in \mathbb{Z}^{>1}$).

If $\lambda \neq 0$, the difference between the indicial roots is an odd integer (i.e., $\sqrt{1 + 8\lambda}$), and the second solution is of the form

$$H_2(y^2) = aH_1(y^2) \ln |y^2| + y^{l_1}(1 + \sum_{j=1}^{\infty} c_j(l_1) y^{2j}) \tag{4.11}$$

for some $a, c_j(l_1) \in \mathbb{R}$. Since $l_1 < 0$, H_2 is not continuous at $y = 0$.

Let $l = l_2$ and write $\tilde{F}_m(y) = e^{-y^2/2} y^l L(y^2)$. Applying the differential operator $\mathcal{D}$ to a function of the form $e^{-y^2/2} y^l L(y^2)$ we obtain the differential equation

$$4y^2 L''(y^2) + 2(1 + 2l - 2y^2)L'(y^2) - (1 + 2l - m)L(y^2) = 0. \tag{4.12}$$

Recall the confluent hypergeometric differential equation is

$$(z\partial_z^2 + (b - z)\partial_z - a)_1F_1(a, b, z) = 0$$

(c.f. [1]). This equation has well known solutions in the form of confluent hypergeometric functions of the first and second kind. However, the smoothness condition required by being in $I''(q, r, s)$ shows that the unique solution to (4.12) corresponds to a multiple of the confluent hypergeometric function of the first kind. We may therefore take $L(y^2) = {}_1F_1\left(\frac{1+2l-m}{4}, l + \frac{1}{2}, y^2\right)$.

Finally, a simple calculation using the required parity condition on elements in $I''(q, r, s)$ from Equation (4.2) applied to $F_m(\theta, y)$ reduces to

$$e^{-im\pi j/2}(-1)^{jl} = i^{-jq}$$

which is equivalent to $m - 2l \equiv q \mod 4$.

So far, we have established the theorem for non-negative values of y. Extend $\tilde{F}$ to $\mathbb{R}$ by $\tilde{F}_m(y) = e^{-y^2/2} y^l \, {}_1F_1\left(\frac{1+2l-m}{4}, l + \frac{1}{2}, y^2\right)$ which is even or odd depending on the parity of l. Since $\mathcal{D}\tilde{F}_m(y) = 0$ for $y \geq 0$ and $\mathcal{D}$ is even, $\mathcal{D}\tilde{F}_m(y) = 0$ for $y \in \mathbb{R}$. Moreover, F_m is manifestly smooth and the unique extension to all $\mathbb{R}$.

If $\lambda = 0$ then $l = 0$ or $l = 1$, which corresponds to the potential free case and again, it is known that there exists a unique solution for each l. The solutions correspond to the even ($l = 0$) and the odd ($l = 1$) solutions found there (c.f., [11]). □

Notice that we have set up a correspondence between the set of eigenvalues with non-empty eigenspace in $I''(q, r, s)$ and $\mathbb{Z}^{\geq 0}$ via $\lambda = l(l - 1)/2$. This correspondence will be one-to-one (except for $\lambda = 0$ where it is two-to-one). For $\lambda \neq 0$

the corresponding parameter l can be recovered by $l = \frac{1}{2}(1 + \sqrt{1+8\lambda})$. For the potential free case, $\lambda = 0$, we have associated the parameters $l = 0$ and $l = 1$.

To use in the following section we record the following properties of the congruent hypergeometric function (c.f. [1])

$$\frac{d^n}{dz^n}\,{}_1F_1(a,b,z) = \frac{(a)_n}{(b)_n}\,{}_1F_1(a+n,b+n,z) \tag{4.13a}$$

$$b\,{}_1F_1(a,b,z) - b\,{}_1F_1(a-1,b,z) - z\,{}_1F_1(a,b+1,z) = 0 \tag{4.13b}$$

$$\begin{aligned} b(1-b+z)\,{}_1F_1(a,b,z) + b(b-1)\,{}_1F_1(a-1,b-1,z)& \\ - az\,{}_1F_1(a+1,b+1,z) &= 0 \end{aligned} \tag{4.13c}$$

$$\begin{aligned} (a-1+z)\,{}_1F_1(a,b,z) + (b-a)\,{}_1F_1(a-1,b,z)& \\ (1-b)\,{}_1F_1(a,b-1,z) &= 0 \end{aligned} \tag{4.13d}$$

$$(a-b+1)\,{}_1F_1(a,b,z) - a\,{}_1F_1(a+1,b,z) + (b-1)\,{}_1F_1(a,b-1,z) = 0 \tag{4.13e}$$

4.3 Structure of $\ker(\Omega'' - 2\lambda) \subset I''(q,r,s)$

In this section we will study the structure of $\ker(\Omega''-2\lambda)_K$ as an $\mathfrak{sl}_2$-module.

Proposition 4.4. *With* $l = \frac{1}{2}(1+\sqrt{1+8\lambda})$ *as in Proposition 4.3 and* $m \equiv 2l + q \mod 4$*, let*

$$\Psi_{m,l}(\theta,y) = e^{-im\theta/2}e^{-y^2/2}y^l\,{}_1F_1\left(\frac{1+2l-m}{4}, l+\frac{1}{2}, y^2\right).$$

The $\mathfrak{sl}_2$*-triple* $\{\kappa, \eta^{\pm}\}$ *acts on* $\Psi_{m,l}$ *by*

$$\kappa.\Psi_{m,l} = \frac{m}{2}\Psi_{m,l} \tag{4.14}$$

$$\eta^{\pm}.\Psi_{m,l} = -\frac{2l+1\pm m}{4}\Psi_{m\pm4,l} \tag{4.15}$$

Lowest weight vectors occur if $m \equiv 2l+1 \mod 4$ *and the lowest weight vector is of the form*

$$e^{-\frac{1}{2}(2l+1)i\theta}e^{-\frac{y^2}{2}}y^l.$$

Highest weight vectors occur if $m \equiv -2l-1 \mod 4$ *and the highest weight vector is of the form*

$$e^{\frac{1}{2}(2l+1)i\theta}e^{\frac{y^2}{2}}y^l.$$

Proof. In Equations (4.6) and (4.7), we wrote down the action of the $\mathfrak{sl}_2$-triple $\{\kappa, \eta^{\pm}\}$. The stated action of κ follows by inspection. Directly applying the differential operator η^+ gives

$$\begin{aligned}\eta^+.\Psi_{m,l}(\theta, y) &= e^{-i(m\pm 4)\theta/2}e^{-y^2/2}y^l\frac{-1-2l+m}{4(1+2l)}((1+2l)\\ &\cdot {}_1F_1\Big(\frac{1+2l-m}{4}, l+\frac{1}{2}, y^2\Big) + 2y^2\,{}_1F_1\Big(\frac{5+2l-m}{4}, l+\frac{3}{2}, y^2\Big)).\end{aligned}$$

Applying (4.13c) with $a = \frac{1+2l-m}{4}$ and $b = l+\frac{1}{2}$ to the action of η^+ we obtain

$$\begin{aligned}\eta^+.\Psi_{m,l}(\theta, y) &= -\frac{1}{4}e^{-i(m\pm 4)\theta/2}e^{-y^2/2}y^l((4l-2)\\ &\cdot {}_1F_1\Big(\frac{-3+2l-m}{4}, l-\frac{1}{2}, y^2\Big) - (3-2l+m)\,{}_1F_1\Big(\frac{1+2l-m}{4}, l+\frac{1}{2}, y^2\Big)).\end{aligned}$$

Using (4.13b) we obtain the desired result. For η^-, we similarly apply (4.13b) with $a = \frac{5+2l-m}{4}$ and $b = l+\frac{1}{2}$ to obtain the desired result.

The assertion about the highest and lowest weights follow from the action of $\eta^{\pm}$ as differential operators; since the weight vectors that are annihilated by each of these are the ones correspondent to the weights $\mp(2l+1)$ respectively. Directly evaluating and observing that ${}_1F_1(a, a, z) = e^z$ and ${}_1F_1(0, b, z) = 1$, the given expressions are obtained. □

Definition 4.1. Let $H_l = \ker(\Omega'' - 2\lambda)_K$ denote the K-finite vectors in $\ker(\Omega''-2\lambda) \subset I''(q, r, s)$. For $k \in \mathbb{Z}^{\geq 0}$ define

$$H_k = \operatorname{span}_{\mathbb{C}}\{\Psi_{m,k} : m \equiv 2k+q \mod 4 \text{ for } m \in \mathbb{Z}\}. \tag{4.16}$$

For $q \equiv 1 \mod 4$ and $k \in \mathbb{Z}^{\geq 0}$ define

$$H_k^+ = \operatorname{span}_{\mathbb{C}}\{\Psi_{m,k} : m \geq 2k+1 \text{ and } m \equiv 2k+1 \mod 4 \text{ for } m \in \mathbb{Z}\}. \tag{4.17}$$

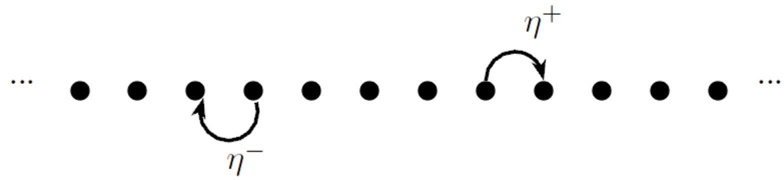

Figure 4.1. A basis of H_l is represented when q is even.

For $q \equiv -1 \mod 4$ and $k \in \mathbb{Z}^{\geq 0}$ define

$$H_k^- = \operatorname{span}\{\Psi_{m,k} : m \leq -(2k+1) \text{ and } m \equiv -(2k+1) \mod 4 \text{ for } m \in \mathbb{Z}\}. \quad (4.18)$$

Lemma 4.3. *If $q \equiv -1 \mod 4$, then H_l^- is the unique irreducible $\mathfrak{sl}_2$-submodule of H_l. If $q \equiv 1 \mod 4$, then H_l^+ is the unique irreducible $\mathfrak{sl}_2$-submodule of H_l.*

Proof. From Equation (4.15), follows irreducibility when $\pm(2l+1) \neq m$ for any $m \in \mathbb{Z}$, this occurs when $q \in 2\mathbb{Z}$. We can have $2l+1 = m$ for some $m \cong 2l+q \mod 4$ iff $q \cong 1 \mod 4$ and a lowest weight occurs in this case. Similarly, $q \cong -1 \mod 4$ implies that a highest weight occurs. Since the highest and lowest weight cannot occur in the same representation, the action Equation (4.15) implies that H_l^+ (resp. H_l^-) is clearly the unique irreducible submodule of H_l. □

Theorem 4.1. *Given $q \in \mathbb{Z}_4$ and $l = \frac{1}{2}(1+\sqrt{1+8\lambda})$, then as $\mathfrak{sl}_2$-modules:*

(1) If $q \equiv 0 \mod 4$ or $q \equiv 2 \mod 4$ then $H_l = \ker(\Omega'' - 2\lambda)_K$ is irreducible as an $\mathfrak{sl}_2$-module.

(2) If $q \equiv -1 \mod 4$, then H_l^- is the only irreducible submodule and the composition series for $\ker(\Omega'' - 2\lambda)_K$ is given by

$$0 \subset H_l^- \subset H_l.$$

(3) If $q \equiv 1 \mod 4$, then H_l^+ is the only irreducible submodule and the composition series of $\ker(\Omega'' - 2\lambda)_K$ is given by $\mathfrak{sl}_2$-submodule

$$0 \subset H_l^+ \subset H_l.$$

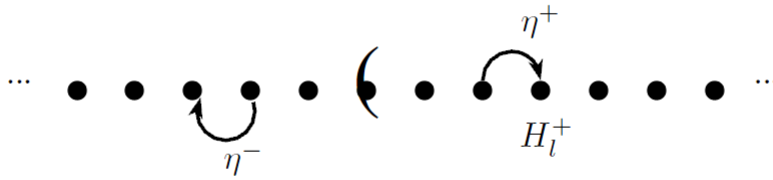

Figure 4.2. A basis of H_l is represented when $q = 1$.

Proof. Follows from Lemma 4.3. □

Figure 4.1 shows a graphic representation of a basis of H_l. Each dot represents the $\mathbb{C}$-span of the K-finite vector parametrized by m and k. Up to non-zero multiples, η^+ (resp. η^-) moves each dot to the right (resp. to the left) and κ leaves each dot fixed. Figure 4.2 shows a graphic representation of a lowest weight module. The K-finite vectors to the right of the parenthesis form H_l^+.

4.4 Heisenberg Action and Connections with Other Kernels

In this section we will examine the action of the Heisenberg algebra. This will allow us to join all the (non-zero) eigenspaces together in one representation.

Recall that the element $(u, v, 0) \in \mathfrak{h}_3(\mathbb{R})$ acts on $I'(q, r, s)$ by $(tv - u)\partial_x - 2svx$ so, under the isomorphism (4.3), the elements

$$E_{\mp} := (1, \pm i, 0) \in \mathfrak{h}_3(\mathbb{C})$$

act by the differential operators

$$\mp e^{\pm i\theta}(\pm\partial_y - 2isy)$$

Proposition 4.5. *Let $m \in \mathbb{Z}$ and $k \in \mathbb{Z}^{\geq 0}$. Then,*

$$E^-.\Psi_{m,k} = \frac{(1 + 2k - m)(k - 1)}{(2k - 1)(2k + 1)}\Psi_{m-2,k+1} - k\Psi_{m-2,k-1}$$

and

$$E^+.\Psi_{m,k} = \frac{(1 + 2k + m)(k - 1)}{2(2k - 1)}\Psi_{m+2,k+1} - k\Psi_{m+2,k-1}.$$

Proof. Combining (4.13b) and (4.13e) with $a+1$ instead of a, one obtains

$$ {}_1F_1(a,b,z) = {}_1F_1(a,b-1,z) - \frac{az}{b(b-1)}\, {}_1F_1(a+1,b+1,z). \tag{4.19} $$

Using Equation (4.13e) with $b+1$ in place of b and combining it with (4.13c), one obtains

$$ {}_1F_1(a,b,z) = -\, {}_1F_1(a-1,b-1,z) + \frac{b-a}{b-1} z\, {}_1F_1(a,b+1,z) \tag{4.20} $$

Using Equation (4.13a) we can compute the action of $E^{\pm}$ directly. Let $a = \frac{1+2k-m}{4}$ and $b = k + 1/2$. Then it is straightforward to see

$$ E^{-}.\Psi_{m,k} = -\frac{1}{2} e^{-i(m+2)\theta/2 - y^2/2} y^{k-1} \big((-1+2b)\, {}_1F_1(a,b,y^2) \\ + 4ay^2/b\, {}_1F_1(a+1,b+1,y^2) \big). $$

Applying (4.19), one gets the first equation.

A similar calculation using Equation (4.13a) shows

$$ E^{+}.\Psi_{m,k} = \frac{1}{2} e^{-i(m-2)\theta/2 - y^2/2} y^{k-1} \big((1-2b+4y^2)\, {}_1F_1(a,b,y^2) \\ - 4ay^2/b\, {}_1F_1(a+1,b+1,y^2) \big) $$

An application of (4.13c) gives

$$ E^{+}.\Psi_{m,k} = -\frac{1}{2} e^{-i(m-2)\theta/2 - y^2/2} y^{k-1} \big(4(b-1)\, {}_1F_1(a-1,b-1,y^2) \\ + (3-2b)\, {}_1F_1(a,b,y^2) \big) $$

and substituting in the expression in (4.20) gives the desired result. □

From Equation (4.8), it follows that if the eigenvalue λ corresponds to the parameter l, then $\lambda + l + 1$ corresponds to the parameter $l+1$ and $\lambda - l$ corresponds to the parameter $l-1$.

The following corollary follows immediately from the previous proposition. It will be useful in seeing that the action of the Heisenberg algebra preserves the structure of highest and lowest weight submodules in H.

Corollary 4.1. *If $m = 2k+1$, the action of $E^{\pm}$ on the lowest weight is given by*

$$E^{-}.\psi_{2k+1,k} = -k\Psi_{2k-1,k-1}$$

and

$$E^{+}.\psi_{2k+1,k} = \frac{(2k+1)(k-1)}{2k-1}\Psi_{2k+3,k+1} - k\Psi_{2k+3,k-1}.$$

If $m = -(2k+1)$, the action of $E^{\pm}$ on the lowest weight is given by

$$E^{-}.\psi_{-(2k+1),k} = \frac{2(k-1)}{2k-1}\Psi_{-2k-3,k+1} - k\Psi_{-2k-3,k-1}$$

and

$$E^{+}.\psi_{-(2k+1),k} = -k\Psi_{-2k+1,k-1}.$$

Proof. This follows directly from the previous proposition. □

We now will show how Proposition 4.5 and Corollary 4.1 imply that the action of $\mathfrak{h}_3$ ties together the kernels indexed by k, in a $\mathfrak{g}$-module. Recall, the cases where $k = 0$ and $k = 1$ correspond to the potential free case.

Definition 4.2. Let

$$H = \bigoplus_{l \in \mathbb{Z}^{\geq 0}} H_l. \tag{4.21}$$

Whenever the spaces are defined, let

$$H^{\pm} = \bigoplus_{l \in \mathbb{Z}^{\geq 2}} H_l^{\pm}. \tag{4.22}$$

Theorem 4.2. *Let $q \in \mathbb{Z}_4$ and $k \in \mathbb{Z}^{\geq 0}$. With respect to the action of $\mathfrak{g}$:*

(1) If $q = 0$ or $q = 2$, the composition series of H is

$$0 \subset H_0 \oplus H_1 \subset H$$

where each vertical strip corresponds to an irreducible $\mathfrak{sl}_2$-representation.

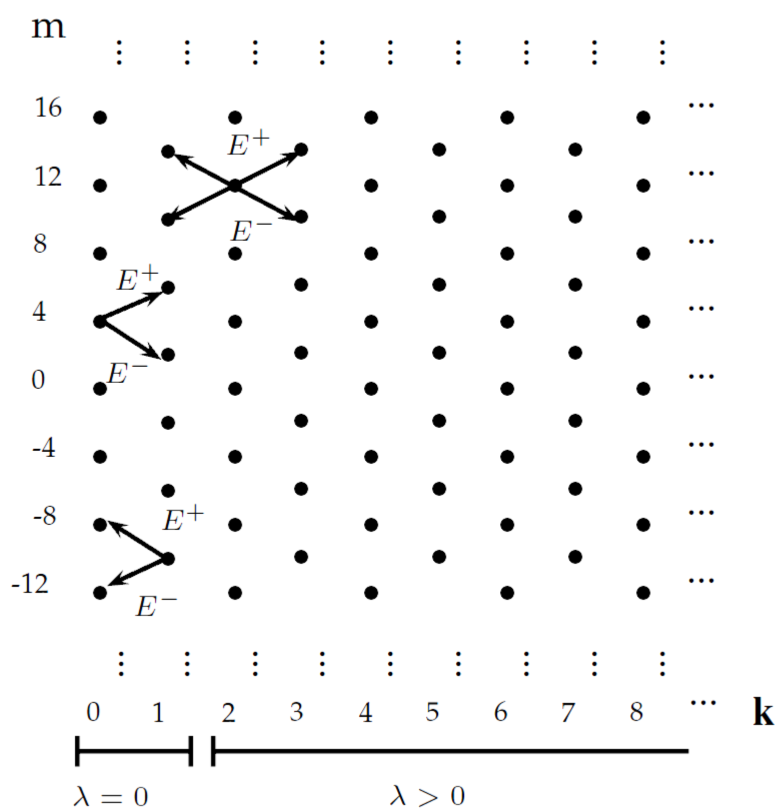

Figure 4.3. Graphical representation of the composition series when q is even.

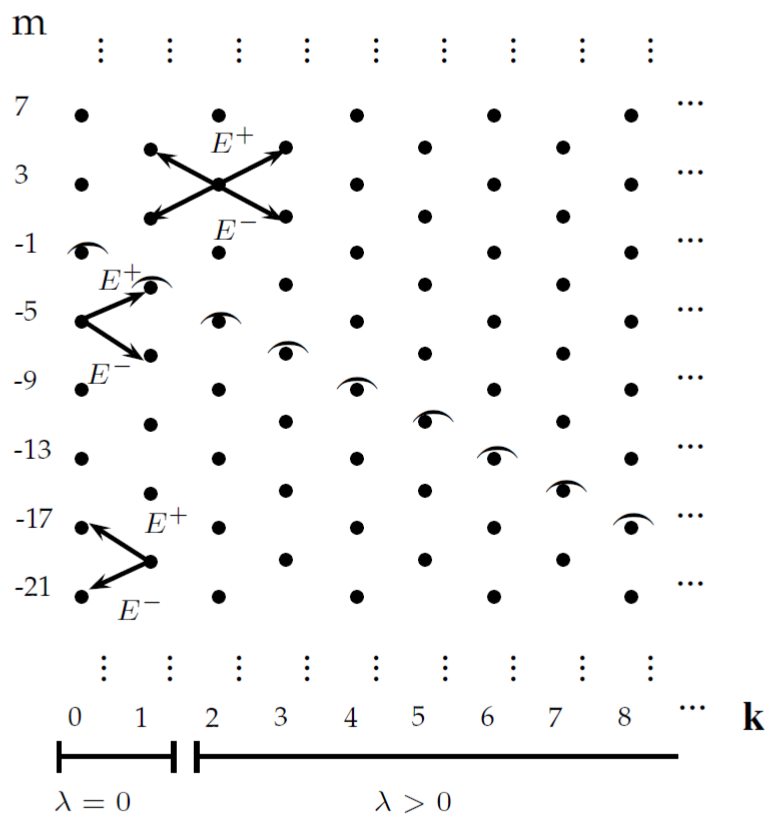

Figure 4.4. Graphical representation of the composition series when $q = 3$.

(2) If $q \equiv -1 \mod 4$*, then the composition series of* $\mathfrak{g}$*-submodules of* H *is as follows*

$$0 \subset H_0^- \oplus H_1^- \subset H_0 \oplus H_1 \subset H_0 \oplus H_1 \oplus H^- \subset H.$$

(3) If $q \equiv 1 \mod 4$*, then the composition series of* $\mathfrak{g}$*-submodules of* H *is*

$$0 \subset H_0^+ \oplus H_1^+ \subset H_0 \oplus H_1 \subset H_0 \oplus H_1 \oplus H^+ \subset H.$$

Proof. Let $q \equiv 0 \mod 4$ or $q \equiv 2 \mod 4$. Proposition 4.5 shows that the action of $E^{\pm}$ sends elements in H_0 only to H_1 and the action of $E^{\pm}$ sends elements in H_1 only to H_0. Under this assumption on q, each H_k is irreducible under the $\mathfrak{sl}_2$ action, thus $H_0 \oplus H_1$ is irreducible under the $\mathfrak{g}$ action. Now we look at the quotient $H/(H_0 \oplus H_1)$. Let $\pi : H \to H/(H_0 \oplus H_1)$ be the natural projection. Let $\overline{H}_k$ denote the image of H_k under π, then the image of H under π can be decomposed as a direct sum as

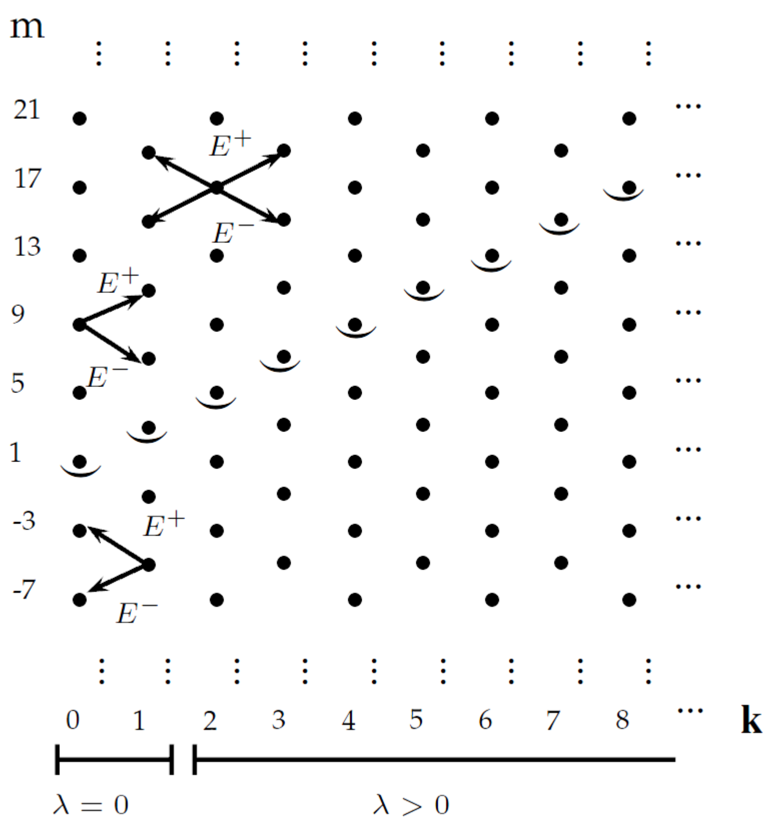

Figure 4.5. Graphical representation of the composition series when $q = 3$.

$\overline{H} = \bigoplus_j \overline{H}_{k_j}$ as an $\mathfrak{sl}_2$-module. Proposition 4.5 implies that $E^{\pm}.\overline{H}_{k_j}$ has a non-zero component in $\overline{H}_{k_j-1}$ and in $\overline{H}_{k_j+1}$, for $k_j \geq 2$.

Since the $\overline{H}_{k_j-1}$ and $\overline{H}_{k_j+1}$ are inequivalent $\mathfrak{sl}_2$-representations, then $E^{\pm}.\overline{H}_{k_j}$ generates $\overline{H}_{k_j-1} \oplus \overline{H}_{k_j+1}$ under the action of $\mathfrak{sl}(2,\mathbb{R})$. Irreducibility follows easily from this.

The proofs of (2) and (3) are essentially identical, therefore we will only look at the proof of (3). Irreducibility of $H_0^+ \oplus H_1^+$ under $\mathfrak{g}$ follows from irreducibility under $\mathfrak{sl}_2$ and from the actions on the lowest weights described in Corollary 4.1.

Next we look at the quotient $(H_0 \oplus H_1)/(H_0^+ \oplus H_1^+)$. For $j \in \{0,1\}$, write $\overline{H}_j$ for the image of H_j under the natural projection $H_0 \oplus H_1 \to (H_0 \oplus H_1)/(H_0^+ \oplus H_1^+)$. Then, $\overline{H}_0$ gets sent to $\overline{H}_1$ and $\overline{H}_1$ gets sent to $\overline{H}_0$ by the action of the Heisenberg algebra. This, together with irreducibility of $\overline{H}_0$ and $\overline{H}_1$ under $\mathfrak{sl}_2$, gives irreducibility under $\mathfrak{g}$.

Finally, we look at the quotient $(H_0 \oplus H_1 \oplus H^+)/(H_0 \oplus H_1)$. Write $\bigoplus_{k\geq 0} \overline{H}^+ k$ for the image of $H_0 \oplus H_1 \oplus H^+$ under the natural projection. The Heisenberg algebra acts as before, and $E^{\pm}.\overline{H}^+_{k_j}$ has a component in $\overline{H}^+_{k_j-1}$ and in $\overline{H}^+_{k_j+1}$, for $k_j \geq 2$. Hence any non-zero element in $\overline{H}_k$ for $k \geq 2$ generates the whole space. $\square$

4.5 An Eigenvalue Problem for the Quadratic Case

Corollary 3.6 states that Ω acts on $I'(q,r,s)_{\mu_\lambda}$ by $\Omega = \frac{1}{2}(x^2(\Box + \lambda_1^2 x^2/4) + r(r+2))$. Replacing $\lambda_1^2/4$ by $-2\lambda_1$ on obtains that $\Omega' - 2\lambda_2$ acts by

$$\Omega' - 2\lambda_2 = x^2(\Box - 2\lambda_1 x^2) - 2\lambda_2 = x^2(\Box - 2(\lambda_1 x^2 + \lambda_2/x^2)).$$

Then, the solution space of the Schödinger equation with the potential $V_5(x) = \lambda_1 x^2 + \lambda_2/x^2$, as a local G-representation, is equivalent to studying a subspace of the kernel of $\Omega' - 2\lambda_2$ in $I'(q,r,s)_{\mu_\lambda}$. To construct this subspace we let S_{λ_2} be the image of $\ker(\Omega' - 2\lambda_2) \subset I'(q,r,s)_{(-1,1)\times\mathbb{R}}$ under the map (3.14) as in Section 3.3.2.

Then we have the following diagram:

$$\begin{array}{ccccc} I'(q,r,s) & \to & I'(q,r,s)_{(-1,1)\times\mathbb{R}} & \to & I'(q,r,s)_{\mu_\lambda} \\ \uparrow & & \uparrow & & \uparrow \\ \ker(\Omega' - 2\lambda_2) & \to & \ker(\Omega' - 2\lambda_2)_{(-1,1)\times\mathbb{R}} & \to & S \subset \ker(\Omega' - 2\lambda_2)_{\mu_\lambda}. \end{array}$$

The study of this solution space, S_{λ_2}, as a local representation reduces to the study of $\ker(\Omega' - 2\lambda_2) \subset I'(q,r,s)$ which was done in Section 4.

CHAPTER FIVE

Time-Dependent Potentials

It has been shown in [12] that the only time-dependent potentials that preserve the complete $\mathfrak{sl}_2$-symmetry are potentials of the form $V(t,x) = g_2(t)x^2 + g_1(t)x + g_0(t) + \lambda/x^2$ with $\lambda g_1(t) = 0$. If $\lambda = 0$ the symmetry Lie algebra is isomorphic to $\mathfrak{g} = \mathfrak{sl}_2 \ltimes \mathfrak{h}_3$ and it is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}$ otherwise.

We show that when $\lambda = 0$, there exists a local intertwining isomorphism between the solution space of the potential free Schrödinger equation and the solution space of the Schrödinger equation with this time-dependent potential in $I(q,r,s)$. When $\lambda \neq 0$ we can reduce this problem to the eigenvalue problem studied in Section 4 under the same change of variables. Therefore, this completes the study of all non-trivial time-dependent and independent potentials that preserve at least the $\mathfrak{sl}_2$-symmetry.

By substituting $V(t,x) = g_2(t)x^2 + g_1(t)x + g_0(t)$ in (3.1) and equating the coefficients of the powers of x to find the generators of symmetry algebra, one gets the following system of ordinary differential equations:

$$A''' + 8g_2A' + 4g_2'A = 0 \tag{5.1a}$$

$$b'' + 2g_2b = -\frac{3}{2}g_1A' - g_1'A \tag{5.1b}$$

$$c' = \frac{1}{4}A'' + i(g_0A' + g_0'A) + ibg_1. \tag{5.1c}$$

It was shown in [12] that two real, linearly independent, nontrivial solutions of $b'' + 2g_2b = 0$, χ_1 and χ_2, can be used to write three linearly independent nontrivial solutions to equation (5.1a). Moreover, they can be chosen in such way that the Wronskian $W(\chi_1,\chi_2) = 1$. These three solutions are defined then by $\varphi_j = \chi_j^2$ for $j \in \{1,2\}$ and $\varphi_3 = 2\chi_1\chi_2$.

The symmetry algebra can be realized as first order differential operators. In that case, the $\mathfrak{h}_3$ part corresponds to the span of:

$$\xi = -\chi_1 \partial_x + i\chi_1' x - i\mathcal{C}_1 \tag{5.2a}$$

$$\psi = \chi_2 \partial_x - i\chi_2' x + i\mathcal{C}_2 \tag{5.2b}$$

$$\zeta = \frac{1}{2} i \tag{5.2c}$$

where $\mathcal{C}_j(t) = \int_0^t \chi_j g_1$ for $j \in \{1, 2\}$. Let

$$\mathcal{A}_l = -\chi_l \mathcal{C}_l$$

for $l \in \{1, 2\}$ and let

$$\mathcal{A}_3 = -(\chi_1 \mathcal{C}_2 + \chi_2 \mathcal{C}_1).$$

Similarly, the $\mathfrak{sl}(2, \mathbb{R})$ part is generated by the differential operators:

$$L_j = (-1)^{j+1} (\varphi_j \partial_t + (\frac{1}{2} \varphi_j' x + \mathcal{A}_j) \partial_x + \mathcal{B}_j) \tag{5.3}$$

for $1 \leq j \leq 3$, where

$$\mathcal{B}_j = -\frac{1}{4} i \varphi_j'' x^2 - i \mathcal{A}_j' x + \frac{1}{4} \varphi_j' + i g_0 \varphi_j + i \mathcal{D}_j,$$

$\mathcal{D}_l = -\frac{1}{2} \mathcal{C}_l^2$ for $l \in \{1, 2\}$ and $\mathcal{D}_3 = -\mathcal{C}_1 \mathcal{C}_2$. The following bracket relations hold: $[L_3, L_1] = -2L_1$, $[L_3, L_2] = 2L_2$, and $[L_2, L_1] = L_3$. Note that, in order to have a standard $\mathfrak{sl}_2$-triple, our definition of the L_2 operator differs in sign with respect to the definition in [12].

In order to define the appropriate multiplier representation space we start by defining a change of variables $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\gamma(t, x) := \left(\int_0^t \frac{1}{\chi_2^2} \ , \ \frac{1}{\chi_2(t)} x + \int_0^t \frac{\mathcal{C}_2}{\chi_2^2} \right).$$

In the following, we assume that the required integrability conditions are satisfied.

We define $\nu : N \times X \to \mathbb{C}$ by

$$\nu(N_{\gamma(t,x)}) = e^{\int_0^t \frac{\mathcal{B}_2(u)}{\chi_2^2(u)} + \left(\frac{1}{\chi_2^2(u)} \left(\frac{1}{2} \varphi_2'(u) x + \mathcal{A}_2(u) \right) \right)^2 du}.$$

Extend ν to a map to an open dense subset of G by $\nu(g) = \nu(n(g))$. Let $f \in I'(q,r,s)$ and define the map $f \mapsto \tilde{f}$ by

$$\tilde{f}(t,x) = e^{\int_0^t \frac{\mathcal{B}_2(u)}{\chi_2^2(u)} + \left(\frac{1}{\chi_2^2(u)}\left(\frac{1}{2}\varphi_2'(u)x + \mathcal{A}_2(u)\right)\right)^2 du} f(\gamma(t,x)). \tag{5.4}$$

The space $I'(q,r,s)_\mu$ is defined as the image of $I'(q,r,s)$ under this map, and is given the structure of a G-module that makes the map intertwining. On this space the action of the Lie algebra corresponds to the differential operators in Equation (5.2) and Equation (5.3).

Next we construct the multiplier representation. We start by defining the multiplier

$$\mu(g_1, g_2) = \nu(g_2^{-1} g_1)\nu(g_2^{-1})^{-1}.$$

For $\phi \in I(q,r,s)$ define, on an open dense set of $G/\overline{P}$, the map

$$\tilde{\phi}(g\overline{P}) = \mu(g^{-1}, I)^{-1}\phi(g).$$

As before, we define $I(q,r,s)_\mu$ as the image of $I(q,r,s)$ under the map $\phi \to \tilde{\phi}$.

Finally, the intertwining map from $I(q,r,s)_\mu$ to $I'(q,r,s)_\mu$ is given by

$$\tilde{\phi} \mapsto \tilde{f} \text{ whenever } \tilde{f}(t,x) = \tilde{\phi}(N_{\gamma(t,x)}\overline{P}).$$

5.1 Group Action on $I'(q,r,s)_\mu$

In this section we calculate the local action of G on $I'(q,r,s)_\mu$ and we show that the study of the solution space for this general potential reduces to the study of the kernel of the differential operator Ω' as in the potential free case. For notational convenience, define

$$\rho(t,x) = \nu(N_{\gamma(t,x)}).$$

Proposition 5.1. *Fix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_0$ and let $(g,\epsilon) \in G$. Define $\Theta(t) = \int_0^t \frac{1}{\chi_2^2}$ and $\Xi(t,x) = \frac{1}{\chi_2(t)}x + \int_0^t \frac{\mathcal{C}_2}{\chi_2^2}$. Then*

$$((g,\epsilon).\tilde{f})(t,x) = \frac{\rho(t,x)}{\rho\circ\gamma^{-1}\left(\frac{d\Theta-b}{a-c\Theta},\frac{\Xi}{a-c\Theta}\right)}(a-c\Theta)^{r-q/2}$$

$$\epsilon(g^{-1}.(\Theta+z))e^{\frac{-sc\Xi^2}{a-c\Theta}}\tilde{f}\circ\gamma^{-1}\left(\frac{d\Theta-b}{a-c\Theta},\frac{\Xi}{a-c\Theta}\right). \quad (5.5)$$

For $(u,v,w)\in H_3(\mathbb{R})$

$$((u,v,w).\tilde{f})(t,x) = \frac{\rho(t,x)}{\rho\circ\gamma^{-1}(\Theta,\Xi-u-v\Theta)}e^{-s(uv-2v\Xi-v^2\Theta+w)}$$

$$\tilde{f}\circ\gamma^{-1}(\Theta,\Xi-u-v\Theta). \quad (5.6)$$

Proof. This proposition follows directly from using the isomorphism determined by Equation (5.4) on the actions computed in Proposition 2.1. □

We will show that by differentiating these actions we recover the generators of the algebra of symmetry operators found by Truax in [12]. This reduces the study of the solution space of the time-dependent potentials in $I'(q,r,s)_\mu$ to the study of the solution space of the potential free Schrödinger equation in $I'(q,r,s)$ studied in [11]. We start with some useful calculations

Lemma 5.1. *The functions* $\chi_1(t)$ *and* $\chi_2(t)$ *satisfy*

(1)
$$\chi_1 = \chi_2\int_0^t \frac{1}{\chi_2^2},$$

(2)
$$\mathcal{A}_1 = \left(\int_0^t \frac{1}{\chi_2^2}\right)^2\mathcal{A}_2 - \chi_2\left(\int_0^t\frac{1}{\chi_2^2}\right)\int_0^t\frac{\mathcal{C}_2}{\chi_2^2},$$

(3)
$$\varphi_1' = \left(\int_0^t\frac{1}{\chi_2^2}\right)^2\varphi_2' + 2\int_0^t\frac{1}{\chi_2^2}.$$

Proof. Since $\chi_1\chi_2' - \chi_2\chi_1' = 1$, follows that $\chi_2\int_0^t\frac{1}{\chi_2^2} = \chi_2\int_0^t\frac{\chi_1\chi_2'-\chi_2\chi_1}{\chi_2^2} = \chi_1$. To prove the second statement, one uses part one in the definition of $\mathcal{A}_j$ and integration by parts. To prove the third statement, one uses part one and the definition of φ_j. □

Corollary 5.1. *For $r = -1/2$ and $s = i/2$ the standard $\mathfrak{sl}_2$-basis $\{h, e^+, e^-\}$ acts on $I'(q,r,s)_\mu$ by the differential operators $\{L_3, L_2, L_1\}$ respectively. An element $(u,v,w) \in H_3(\mathbb{R})$ acts on the same space, by*

$$(u\chi_2 - v\chi_1)\partial_x - i(u\chi_2' + v\chi_1')x + i(u\mathcal{C}_2 + v\mathcal{C}_1 - sw)$$

Proof. All these calculations are similar. We only provide the details for the action of e^- since it is more involved. Let $\gamma^{-1}(t,x) = (\Theta^{-1}(t), \Psi(t,x))$. Using (5.5), we obtain

$$\begin{aligned}(((\begin{smallmatrix}1&0\\c&1\end{smallmatrix}), \epsilon).\tilde{f})(t,x) = & \frac{\rho(t,x)}{\rho\left(\Theta^{-1}\left(\frac{\Theta}{1-c\Theta}\right), \Psi\left(\frac{\Xi}{1-c\Theta}\right)\right)}(1-c\Theta)^{r-q/2} \\ & \epsilon(g^{-1}.(\Theta+z))e^{\frac{-sc\Xi^2}{1-c\Theta}}\tilde{f}\left(\Theta^{-1}\left(\frac{\Theta}{1-c\Theta}\right), \Psi\left(\frac{\Xi}{1-c\Theta}\right)\right).\end{aligned}$$

We next take $\frac{d}{dc}\Big|_{c=0}$ to obtain the action of e^-. For the coefficient of ∂_t we obtain

$$-\Theta^2(t)\left.\frac{\partial\Theta^{-1}}{dt}\right|_{t=\Theta} = \chi_2(t)^2\left(\int_0^t \frac{1}{\chi_2^2}\right)^2 = \chi_1^2(t) = \varphi_1(t).$$

The first equality above follows from differentiating, the second equality from Lemma 5.1, and the third equality from the definition of φ_1.

For the coefficient of ∂_x we obtain

$$\begin{aligned}\Theta(t)\Xi(t,x)\frac{\partial\Psi}{\partial x} + \Theta(t)^2\frac{\partial\Psi}{\partial t} = \Theta(t)\Xi(t,x)\chi_2 + \Theta(t)^2(\frac{1}{2}\varphi_2' x + \mathcal{A}_2) = \\ x\Theta + \frac{1}{2}x(\varphi_1' - 2\Theta) + \mathcal{A}_1 = \frac{1}{2}\varphi_1' x + \mathcal{A}_1.\end{aligned}$$

The first equality above follows from differentiating, the second from computing the partial derivatives of the inverse function, and the third by Lemma 5.1.

Lastly, we compute the multiplication term. To this end, we start by computing the following expression

$$\mathcal{B}_2\Theta^2 - \frac{i}{2}\Xi^2 - r\Theta = \mathcal{B}_1 + \frac{ix^2\varphi_2'\Theta}{\chi_2^2} + ix(\Theta\frac{\mathcal{C}_2}{\chi_2} - \chi_2'\Theta\int_0^t \frac{\mathcal{C}_2}{\chi_2^2}) - i\Theta\mathcal{C}_2\int_0^t \frac{\mathcal{C}_2}{\chi_2^2}.$$

Since the multiplication term is given by $\Theta^2\mathcal{B}_2 - \frac{i}{2}\Xi^2 - r\Theta + \frac{1}{\rho(t,x)}\frac{\partial\rho}{\partial x}\frac{\partial\Psi}{\partial x}\Xi(t,x)\Theta = \Theta^2\mathcal{B}_2 + i\Theta(-\chi_2'x + \mathcal{C}_2)(1/\chi_2 x + \int \mathcal{C}_2/\chi_2^2) = \mathcal{B}_1$, the operator corresponding to e^- is L_1. $\square$

To conclude this section, we will compute the action of Ω. To begin we determine the action of the Casimir element for the $\mathfrak{sl}_2$-triple on $I'(q,r,s)_\mu$ with the special parameters $s = i/2$ and $r = -1/2$. From the definition of $\Omega = \frac{1}{2}h^2 + h + 2e^-e^+$, notice that the second order terms can come only from $\frac{1}{2}h^2 + 2e^-e^+$. The coefficient of ∂_t^2 is $\frac{1}{2}\varphi_3^2 - 2\varphi_1\varphi_2 = 0$. The coefficient of $\partial_t\partial_x$ is

$$\varphi_3(1/2\varphi_3'x + \mathcal{A}_3) - 2(\varphi_1(1/2\varphi_2'x + \mathcal{A}_2) + \varphi_2(1/2\varphi_1'x + \mathcal{A}_1)) = 0.$$

The coefficient of ∂_x^2 is

$$\frac{1}{2}\left(\frac{1}{2}\varphi_3'x + \mathcal{A}_3\right)^2 - 2\left(\frac{1}{2}\varphi_1'x + \mathcal{A}_1\right)\left(\frac{1}{2}\varphi_2'x + \mathcal{A}_2\right) = \frac{1}{2}(x - \chi_1\mathcal{C}_2 + \chi_2\mathcal{C}_1)^2.$$

The coefficient of ∂_t is equal to

$$\mathcal{B}_3\varphi_3 - 2(\mathcal{B}_2\varphi_1 + \mathcal{B}_1\varphi_2 + \varphi_1\varphi_2') + \varphi_3 = i(x - \chi_1\mathcal{C}_2 + \chi_2\mathcal{C}_1)^2.$$

The coefficient of ∂_x can be shown to equal zero. Finally, the multiplication term is

$$\begin{aligned}\frac{1}{2}\mathcal{B}_3^2 - i\frac{1}{2}\left(\frac{1}{2}\varphi_3'x + \mathcal{A}_3\right)\left(\frac{1}{2}\varphi_3''x + \mathcal{A}_3'\right) + \mathcal{B}_3 - 2\,\bigg(\mathcal{B}_1\mathcal{B}_2 \\ -i\left(\frac{1}{2}\varphi_2''x + \mathcal{A}_2'\right)(1/2\varphi_1'x + \mathcal{A}_1) + \varphi_1\frac{\partial\mathcal{B}_2}{\partial t}\bigg).\end{aligned}$$

Using the fact that $\chi_j'' = 2g_2(t)\chi_j$ for $j \in \{1,2\}$, the coefficient of x^4 is equal to

$$\begin{aligned}1/8\varphi_1''\varphi_2'' - 1/32(\varphi_3'')^2 &= 1/8(-(\chi_1\chi_2'')^2 + 2\chi_2\chi_2\chi_1''\chi_2'' - (\chi_2\chi_1'')^2 \\ &\quad - 4\chi_1'\chi_2''(\chi_1\chi_2' - \chi_2\chi_1') - 4\chi_2'\chi_1''(\chi_2\chi_1' - \chi_1\chi_2')) \\ &= -g_2(t)(\chi_1'\chi_2 - \chi_1\chi_2')^2 = -g_2(t).\end{aligned}$$

Similarly, it can be shown that the coefficient of x^3 is

$$-1/8\mathcal{A}_3'\varphi_3'' + 1/2(\mathcal{A}_1'\varphi_2'' + \mathcal{A}_2'\varphi_1'') = -g_1(t) + 2(\chi_1\mathcal{C}_2 - \chi_2\mathcal{C}_1)g_2(t).$$

The coefficient of x^2 is

$$-g_0(t) + 2(\chi_1\mathcal{C}_2 - \chi_2\mathcal{C}_1)g_1(t).$$

The coefficient of x is

$$-g_0(t)(\chi_1\mathcal{C}_2 - \chi_2\mathcal{C}_1) + 2(\chi_1\mathcal{C}_2 - \chi_2\mathcal{C}_1)g_1(t)$$

and finally, the multiplication term is

$$(\chi_1\mathcal{C}_2 - \chi_2\mathcal{C}_1)^2 g_0(t) - 3/8.$$

Putting all these together we obtain the following corollary.

Corollary 5.2. *For the parameters $r = -1/2$ and $s = i/2$, the Casimir element acts on $I(q,r,s)_\mu$ by*

$$\Omega = \frac{1}{2}\left[(x - \chi_1\mathcal{C}_2 + \chi_2\mathcal{C}_1)^2(\Box - 2(g_2(t)x^2 + g_1(t)x + g_0(t))) - 3/4\right].$$

In particular,

$$\ker \Omega' = \ker\left(\Box - 2(g_2(t)x^2 + g_1(t)x + g_0(t))\right)$$

in $I'(q,r,s)_\mu$.

If $\lambda \neq 0$ then $g_1(t) \equiv 0$ and Ω' acts by $\Omega' = x^2(\Box - 2(g_2(t)x^2 + g_0(t)))$. Thus

$$\ker(\Omega' - 2\lambda) = \ker\left(\Box - 2(g_2(t)x^2 + g_0(t) + \lambda/x^2\right)$$

in $I'(q,r,s)_\mu$.

BIBLIOGRAPHY

[1] Abramowitz, Milton and Stegun, Irene A., *Handbook of mathematical functions with formulas, graphs, and mathematical tables* Volume 55 of *National Bureau of Standards Applied Mathematics Series*, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] Bawin, M. and Coon, S. A., *Singular inverse square potential, limit cycles, and self-adjoint extensions*, Phys. Rev. A, 67 (2003), 042712.

[3] Bouaziz, Djamil and Bawin, Michel, *Singular inverse square potential in arbitrary dimensions with a minimal length: Application to the motion of a dipole in a cosmic string background*, Phys. Rev. A, 78 (2008), 032110.

[4] Boyer, Charles P., *The maximal 'kinematical' invariance group for an arbitrary potential*, Helv. Phys. Acta, 47 (1974), 589–605.

[5] Coddington, Earl, *An introduction to ordinary differential equations*, Prentice-Hall Mathematics Series, Prentice-Hall Inc.,Englewood Cliffs, N.J., 1961.

[6] Craddock, Mark, *The symmetry groups of linear partial differential equations and representation theory. I*, J. Differential Equations, 116 (1995), 202–247.

[7] Franco, Jose A., *Global* $\widetilde{SL(2,R)}$ *representations of the Schrdinger equation with singular potential*, Cent. Eur. J. Math., 2012, 10(3), 927-941.

[8] Kashiwara, M. and Vergne, M., *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math., 44 (1978), 1–47.

[9] Niederer, U., *The maximal kinematical invariance group of the free Schrödinger equation*, Helv. Phys. Acta, 45(5) (1972/73), 802–810.

[10] Sepanski, Mark R. and Stanke, Ronald J., *On global* $\mathrm{SL}(2,\mathbb{R})$ *symmetries of differential operators*, J. Funct. Anal., 224 (2005), 1–21.

[11] Sepanski, Mark R. and Stanke, Ronald J., *Global Lie symmetries of the heat and Schrödinger equation*, J. Lie Theory, 20 (2010), 543–580.

[12] Truax, D. Rodney, *Symmetry of time-dependent Schrödinger equations. I. A classification of time-dependent potentials by their maximal kinematical algebras*, J. Math. Phys., 22(9) (1981), 1959–1964.