ABSTRACT

A Combinatorial Property of Bernstein-Gelfand-Gelfand Resolutions of Unitary Highest Weight Modules

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It follows from a formula by Kostant that the difference between the highest weights of consecutive parabolic Verma modules in the Bernstein-Gelfand-Gelfand-Lepowsky resolution of the trivial representation is a single root. We show that an analogous property holds for all unitary representations of simply laced type. Specifically, the difference between consecutive highest weights is a sum of positive noncompact roots all with multiplicity one.

A Combinatorial Property of Bernstein-Gelfand-Gelfand Resolutions of Unitary Highest Weight Modules

by

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A Dissertation

Approved by the Department of Mathematics

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Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

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Accepted by the Graduate School
August 2013

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ACKNOWLEDGMENTS

This work could not have been finished without the assistance, support, and encouragement of numerous people. First and foremost, I would like to thank my advisor, Dr. Markus Hunziker, for his instruction and guidance in learning some beautiful mathematics. He is not only an excellent mathematician, but also a true gentleman; his kindness and sense of humor have been greatly appreciated. I would like to thank committee members Dr. Manfred Dugas, Dr. Mark Sepanski, Dr. Ron Stanke, and Dr. James Stamey for serving as such. I would also like to thank the Office of the President, the Graduate School, and anyone else involved with the Presidential Scholarship, without which I could not have afforded my - rather expensive - horse habit. I am very grateful to my fellow graduate students for their friendship and support, especially our first year study group: Kelly Aceves, Jordan Alexander, Curtis Graham, Jessica Stewart, and Bradley Wagner. Also a special separate thank you to those who helped cover my classes while I was gone for the Feast of Tabernacles.

I would like to thank the Waco congregation of the United Church of God for their fellowship and support, and for welcoming me into the body of Christ. Thank you to Lisa and Leighton Stewart at L S Equestrian; it put my mind at ease knowing Pirate was in good hands whenever I went away. Also thanks to Shirley Stewart for English riding lessons, allowing me a much needed break from math during the frantic weeks before the defense. I would like to thank Prof. Benjamin Donath of Buena Vista University for being my mathematical mentor before my time at Baylor, and friend since. And finally, I would like to thank my family, without whom none of this would have been possible. Thanks, praise, and glory to God the Father and our Lord Jesus Christ!

CHAPTER ONE

Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $\Phi \subseteq \mathfrak{h}^*$ denote the associated root system of $\mathfrak{g}$. Choose a positive system Φ^+. Let W be the associated Weyl group and for $w \in W$, define $\Phi_w := \Phi^+ \cap w(-\Phi^+)$. In [13], Kostant observed that $w\rho = \rho - \langle \Phi_w \rangle$ for all $w \in W$, where $\langle \Psi \rangle = \sum_{\beta \in \Psi} \beta$ for $\Psi \subseteq \Phi^+$ and $\rho = \frac{1}{2} \langle \Phi^+ \rangle$, as usual. It follows from this formula that $w \cdot \lambda = \lambda - \langle \Phi_w \rangle$ when $\lambda = 0$. Thus the highest weights of consecutive parabolic Verma modules in the BGG resolution of the irreducible $\mathfrak{g}$-module with highest weight λ differ by a single root. The goal of this work is to show that all unitary modules of simply laced type have a sum of roots as the difference between consecutive parabolic Verma modules. To that end, in Chapter Two we cover requisite background material, preparing for the main result in Chapter Three. Then in Chapter Four we detail the proof of the main result.

CHAPTER TWO

Background

2.1 Parabolic Subalgebras of Hermitian Type

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $\Phi \subseteq \mathfrak{h}^*$ denote the associated root system of $\mathfrak{g}$. Choose a simple system $\Delta \subseteq \Phi$ and enumerate the simple roots as $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ as in Bourbaki [4]. This determines a positive system Φ^+. Then $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \bigoplus_{\alpha \in -\Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ is the one-dimensional root space for each $\alpha \in \Phi$. The standard Borel subalgebra is given by

$$\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

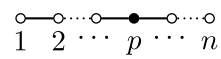
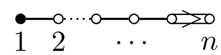
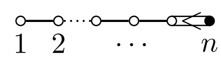
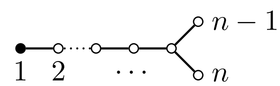
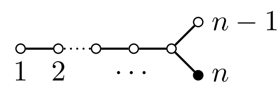
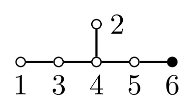
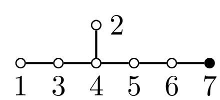
A standard parabolic subalgebra is any subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$. There is a correspondence between such parabolics $\mathfrak{p}$ and subsets of Δ as follows. Let $\mathrm{I} \subseteq \Delta$ and $\Phi_\mathrm{I} := \Phi \cap \operatorname{span}_\mathbb{Z}\mathrm{I}$. This defines a root system. Define

$$\mathfrak{m}_\mathrm{I} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_\mathrm{I}} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{u}_\mathrm{I} := \bigoplus_{\alpha \in \Phi^+ \setminus \Phi_\mathrm{I}} \mathfrak{g}_\alpha.$$

Then $\mathfrak{p}_\mathrm{I} := \mathfrak{m}_\mathrm{I} \oplus \mathfrak{u}_\mathrm{I}$ is a standard parabolic subalgebra. If $|\Delta \setminus \mathrm{I}| = 1$ we say $\mathfrak{p}_\mathrm{I}$ is a maximal parabolic subalgebra, which will be the focus for the remainder of this work. To simplify notation we drop the subscript I and write $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$. For later reference, define $\Phi(\mathfrak{u}) := \Phi^+ \setminus \Phi_\mathrm{I}$.

A maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{u}$ is of Hermitian type if $\mathfrak{u}$ is abelian, or equivalently, the unique simple root in $\Delta \setminus \mathrm{I}$ has coefficient one in the expression for highest root θ. In this setting, there is a noncompact real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$, a compact real form $\mathfrak{m}_{\mathbb{R}}$ of $\mathfrak{m}$, and an element $h_0 \in \mathfrak{h}$ such that $\mathfrak{m}_{\mathbb{R}}$ is a maximal compact subalgebra of $\mathfrak{g}_{\mathbb{R}}$ with center $\mathbb{R}ih_0$ and $\mathfrak{u} = \{x \in \mathfrak{g} \mid [h_0, x] = x\}$. The following table lists all possibilities (see [9]). The crossed node of the Dynkin diagram denotes the simple root in $\Delta \setminus \mathrm{I}$.

Table 2.1. Maximal Parabolic Subalgebras of Hermitian Type

$\mathfrak{g}_{\mathbb{R}}$	Type	Dynkin diagram
$\mathfrak{su}(p,q)$	A_n	
$\mathfrak{so}(2n-1,2)$	B_n	
$\mathfrak{sp}(n,\mathbb{R})$	C_n	
$\mathfrak{so}(2n-2,2)$	D_n	
$\mathfrak{so}^*(2n)$	D_n	
$\mathfrak{e}_{\mathrm{VI}}$	E_6	
$\mathfrak{e}_{\mathrm{VII}}$	E_7	

2.2 Diagrams of Hermitian Type

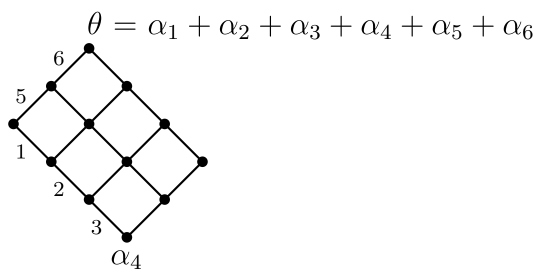
When $\mathfrak{p}$ is of Hermitian type, the posets $\Phi(\mathfrak{u})$ have Hasse diagrams which are two-dimensional, as in Jakobsen [12].

Definition 2.1. Define ${}^{\mathrm{I}}W := \{w \in W \mid \Phi_w \subseteq \Phi(\mathfrak{u})\}$.

We view the Hasse diagram of the lower order ideal Φ_w, $w \in {}^{\mathrm{I}}W$, as a subdiagram of the Hasse diagram of $\Phi(\mathfrak{u})$. We also associate a generalized Young diagram to each ideal Φ_w: draw a square for each node of the Hasse subdiagram and rotate the result clockwise so that the square corresponding to the simple root in $\Delta \setminus \mathrm{I}$ is in the top left.

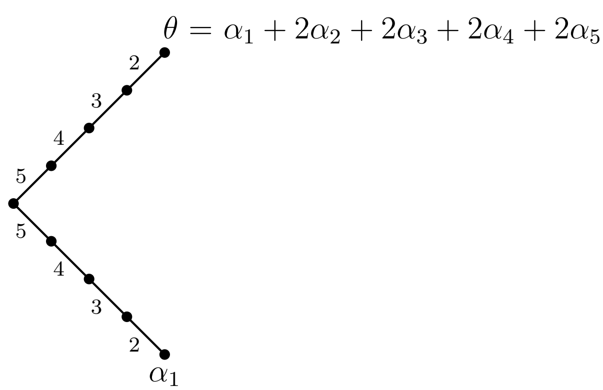
Listed below are the Hasse and Young diagrams of the longest element of ${}^{\mathrm{I}}W$ for each of the types from Table 2.1. If $\beta, \beta' \in \Phi(\mathfrak{u})$ with $\beta' - \beta = \alpha_i \in \Delta$, then the edge connecting the nodes corresponding to β and β' in the Hasse diagram is labelled by the integer i. Parallel edges have the same label. The boxes of the Young diagrams are numbered as in [9]. Note the useful fact that every Young diagram contains a fattened version of the Dynkin diagram.

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p, q)$ for $p = 4$ and $q = 3$:



4	3	2	1
5	4	3	2
6	5	4	3

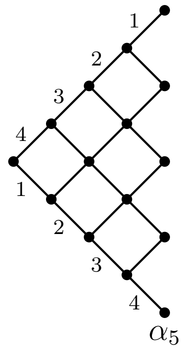
$\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(2n - 1, 2)$ for $n = 5$:



1	2	3	4	5
				4
				3
				2
				1

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$ for $n = 5$:

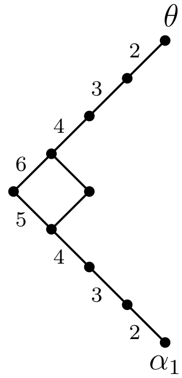
$\theta = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$



5	4	3	2	1
	5	4	3	2
		5	4	3
			5	4
				5

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(2n-2, 2)$ for $n = 6$:

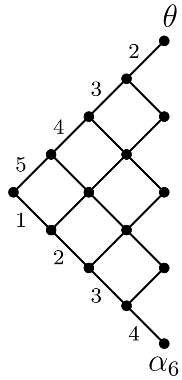
$\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$



1	2	3	4	5
			6	4
				3
				2
				1

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n)$ for $n = 6$:

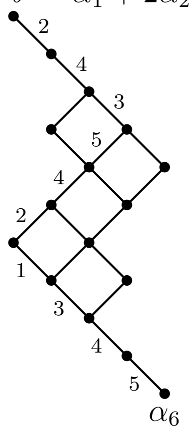
$\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$

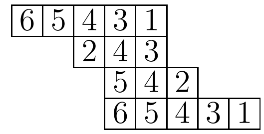


6	4	3	2	1
	5	4	3	2
		6	4	3
			5	4
				6

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{e}_{\mathrm{VI}}$:

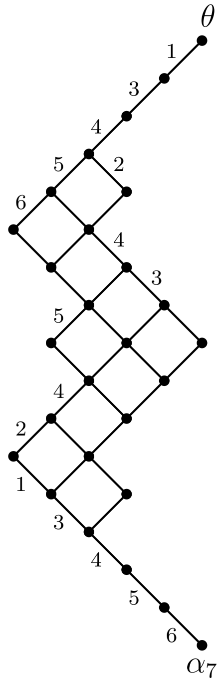
$$\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$$

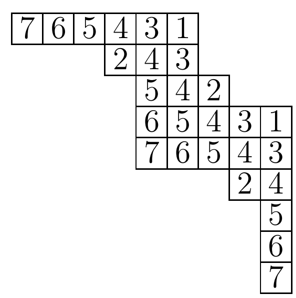




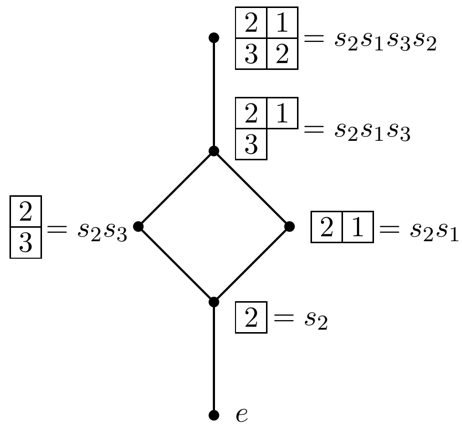
$\mathfrak{g}_{\mathbb{R}} = \mathfrak{e}_{\mathrm{VII}}$:

$$\theta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$$





Young subdiagrams provide a convenient graphical way of writing reduced expressions for elements of ${}^{\mathrm{I}}W$. For example, the reduced expressions for the elements $w \in {}^{\mathrm{I}}W$ for $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(2,2)$ are given by the following Hasse diagram:



2.3 Parabolic Category $\mathcal{O}^{\mathfrak{p}}$

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. The parabolic category $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory of the category of $U(\mathfrak{g})$-modules with objects V satisfying:

(1) V is a finitely generated $U(\mathfrak{g})$-module;

(2) V is a direct sum of finite dimensional simple modules, i.e. a semisimple $U(\mathfrak{m})$-module;

(3) $\dim U(\mathfrak{u})v < \infty$ for all $v \in V$, i.e. V is locally $\mathfrak{u}$-finite.

Let $\Lambda_{\mathrm{I}}^{+} := \{\lambda \in \mathfrak{h}^* \mid (\lambda + \rho, \alpha^\vee) \in \mathbb{Z}_{>0} \quad \forall \alpha \in \Phi_{\mathrm{I}}^{+}\}$ where $(\ ,\ )$ is the nondegenerate bilinear form on $\mathfrak{h}^*$ induced from the Killing form of $\mathfrak{g}$ and $\alpha^\vee := 2\alpha/(\alpha, \alpha)$. There is a correspondence given by $\lambda \mapsto F_\lambda$ between Λ_{I}^{+} and the set of finite-dimensional simple $\mathfrak{m}$-modules, where F_λ has highest weight $\lambda \in \Lambda_{\mathrm{I}}^{+}$. We view F_λ as a $\mathfrak{p}$-module by having $\mathfrak{u}$ act trivially. The parabolic Verma module with highest weight λ is then

$$M_{\mathrm{I}}(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_\lambda \ ,$$

which is a quotient of the ordinary Verma module of highest weight λ. Let $L(\lambda)$ denote the unique simple quotient of $M_{\mathrm{I}}(\lambda)$.

The ordinary Verma module of highest weight λ and its subquotients $M_{\mathrm{I}}(\lambda)$ and $L(\lambda)$ have an infinitesimal character $\chi_\lambda : Z(U(\mathfrak{g})) \to \mathbb{C}$, i.e. a character of

$Z(U(\mathfrak{g}))$ with $z \cdot v = \chi_\lambda(z)v$ for all $z \in Z(U(\mathfrak{g}))$ and all v in the particular module. The infinitesimal block $\mathcal{O}^{\mathfrak{p}}_\lambda$ of $\mathcal{O}^{\mathfrak{p}}$ is the subcategory of modules such that $z - \chi_\lambda(z)$ acts locally nilpotently for all $z \in Z(U(\mathfrak{g}))$.

The Weyl group W acts on $\mathfrak{h}^*$ via "dot action:" $w \cdot \lambda = w(\lambda + \rho) - \rho$. By Harish-Chandra, $\chi_\lambda = \chi_\mu$ for $\lambda, \mu \in \mathfrak{h}^*$ if and only if $\lambda \in W \cdot \mu$. If $(\mu + \rho, \alpha^\vee) \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Phi^+$, $\mu \in \mathfrak{h}^*$ is said to be *antidominant*. In fact, if $\lambda \in \mathfrak{h}^*$ is integral, then there is a unique antidominant $\mu \in \mathfrak{h}^*$ with $\lambda \in W \cdot \mu$. Unless otherwise stated, μ will denote an antidominant integral element of $\mathfrak{h}^*$ throughout this work; in other words

$$(\mu + \rho, \alpha^\vee) \in \mathbb{Z}_{\leq 0} \quad \forall \alpha \in \Phi^+.$$

If $|W \cdot \mu| = |W|$, μ is called regular. This is equivalent to requiring $(\mu + \rho, \alpha^\vee) \neq 0$ for all $\alpha \in \Phi$. For μ regular, the categories $\mathcal{O}^{\mathfrak{p}}_\mu$ are Morita equivalent and we will write $\mathcal{O}^{\mathfrak{p}}_{\mathrm{reg}}$.

Let $W_{\mathrm{I}} = \langle s_\alpha \mid \alpha \in \mathrm{I} \rangle$ with w_{I} its longest element. Then for $w \in {}^{\mathrm{I}}W$, $w_{\mathrm{I}} w \cdot \mu \in \Lambda^+_{\mathrm{I}}$.

Definition 2.2. Define $\Sigma = \Sigma_\mu := \{\alpha \in \Delta \mid (\mu + \rho, \alpha^\vee) = 0\}$. Let
${}^{\mathrm{I}}W^\Sigma := \{w \in {}^{\mathrm{I}}W \mid w < ws_\alpha \text{ and } ws_\alpha \in {}^{\mathrm{I}}W \quad \forall \alpha \in \Sigma\}$.

Note that μ is regular if and only if $\Sigma = \emptyset$. There is a correspondence between ${}^{\mathrm{I}}W^\Sigma$ and the set of simple modules in $\mathcal{O}^{\mathfrak{p}}_\mu$ by sending $w \in {}^{\mathrm{I}}W^\Sigma$ to $L(w) := L(w_{\mathrm{I}} w \cdot \mu)$. The following will be important in our proof of the main result, and in its application.

Lemma 2.1. *Let $w \in {}^{\mathrm{I}}W^\Sigma$ with $\lambda = w_{\mathrm{I}} w \cdot \mu$. Then*

(1) $\Phi_w = \{\beta \in \Phi(\mathfrak{u}) \mid (\lambda + \rho, w_{\mathrm{I}}\beta^\vee) > 0\}$, *and*

(2) $\Phi_{ww_\Sigma} = \{\beta \in \Phi(\mathfrak{u}) \mid (\lambda + \rho, w_{\mathrm{I}}\beta^\vee) \geq 0\}$.

Proof. (1) Let $\beta \in \Phi(\mathfrak{u})$ such that $(\lambda + \rho, w_{\mathrm{I}}\beta^\vee) = (\mu + \rho, w^{-1}\beta^\vee) > 0$. Then, since μ is antidominant, $w^{-1}\beta \in -\Phi^+$ and hence $\beta \in \Phi_w$.

For the other inclusion, suppose that $\beta \in \Phi_w$. Then $w^{-1}\beta \in -\Phi^+$ and it follows that $(\mu + \rho, w^{-1}\beta^\vee) = (\lambda + \rho, w_{\mathrm{I}}\beta^\vee) \geq 0$. Suppose, to the contrary, that $(\mu + \rho, w^{-1}\beta^\vee) = 0$. Then $w^{-1}\beta \in -\Phi_\Sigma^+$ and hence $(ww_\Sigma)^{-1}\beta = w_\Sigma w^{-1}\beta \in \Phi_\Sigma^+$. This implies $\beta \notin \Phi_{ww_\Sigma}$. On the other hand, since $w \in {}^{\mathrm{I}}W^\Sigma$, we have $ww_\Sigma \geq w$ so that $\Phi_w \subseteq \Phi_{ww_\Sigma}$. This is a contradiction and we conclude that $(\lambda+\rho, w_{\mathrm{I}}\beta^\vee) > 0$. $\square$

Proposition 2.1. *Let $x, y \in {}^{\mathrm{I}}W^\Sigma$ such that $\Phi_x \subseteq \Phi_y$. Set $\xi := w_{\mathrm{I}}x \cdot \mu$ and $\eta := w_{\mathrm{I}}y \cdot \mu$. Then $\xi = s_{\beta_k} \cdots s_{\beta_1} \cdot \eta$, where $\{\beta_1, \ldots, \beta_k\}$ is a subset of $w_{\mathrm{I}}\Phi_y \subseteq \Phi(\mathfrak{u})$ such that*

$$\xi = \eta - m_1\beta_1 - m_2\beta_2 - \cdots - m_k\beta_k$$

where $m_i := (s_{\beta_{i-1}} \cdots s_{\beta_1}(\eta + \rho), \beta_i^\vee) > 0$ for $1 \leq i \leq k$.

Proof. Since $x \leq y$, we can write $\Phi_y = \{\beta_1, \ldots, \beta_l\}$ where the β_j are chosen such that $\{\beta_1, \ldots, \beta_j\}$ is a lower order ideal of $\Phi(\mathfrak{u})$ for all j and $\Phi_x = \{\beta_1, \ldots, \beta_k\}$. Then

$$\begin{aligned}
\eta + \rho = w_{\mathrm{I}}y(\mu + \rho) &= w_{\mathrm{I}}s_{\beta_l} \cdots s_{\beta_{k+1}}x(\mu + \rho) \\
&= (w_{\mathrm{I}}s_{\beta_l}w_{\mathrm{I}}^{-1})(w_{\mathrm{I}}s_{\beta_{l-1}}w_{\mathrm{I}}^{-1}) \cdots (w_{\mathrm{I}}s_{\beta_{k+1}}w_{\mathrm{I}}^{-1})w_{\mathrm{I}}x(\mu + \rho) \\
&= s_{w_{\mathrm{I}}\beta_l}s_{w_{\mathrm{I}}\beta_{l-1}} \cdots s_{w_{\mathrm{I}}\beta_{k+1}}(\xi + \rho).
\end{aligned}$$

Equivalently, $\xi + \rho = s_{w_{\mathrm{I}}\beta_{k+1}}s_{w_{\mathrm{I}}\beta_{k+2}} \cdots s_{w_{\mathrm{I}}\beta_l}(\eta + \rho)$. $\square$

2.4 Classification of Unitary Highest Weight Modules

If there is a $\mathfrak{g}_{\mathbb{R}}$-invariant Hermitian scalar product on the simple module $L(\lambda)$ in $\mathcal{O}^{\mathfrak{p}}$ then $L(\lambda)$ is *unitary*. We follow the classification given by [5]. See also [6]. Any highest weight λ must be an element of a cone of the form vertex λ_a plus a certain sum of fundamental weights C_a as follows. Extend the Dynkin diagrams from Table 2.1 in the usual way by adding a node for $-\theta$, where θ is the highest root. Draw a subdiagram of the extended Dynkin diagram containing the node $-\theta$ and the nodes for α_i not the crossed node such that ω_i does not occur in the highest weight λ_a. The connected component containing $-\theta$ is then the Dynkin diagram for a reduced root system Q with $-\theta$ as the crossed node. The cone C_a is given by the sum of $a_i\omega_i$, $a_i \in \mathbb{Z}_{\geq 0}$, for α_i not in the Dynkin diagram of Q and is indexed by $a = (Q, l)$. We list the cones for Types A and D. See Tables 2.2 and 2.3 for the remaining information.

For Type A, $G = \mathrm{SU}(p, q)$ and $C_a = \{\sum_{i=p'}^{n-q'} a_i\omega_i \mid a_p = -\sum_{i\neq p} a_i\}$.

For Type D, $G = \mathrm{SO}^*(2m)$, there are two possibilities for Q. If $Q = \mathrm{SO}^*(2p)$ with $3 \leq p \leq m$, then $C_a = \{\sum_{i=p}^{m} a_i\omega_i \mid a_m = -a_{m-1} - \sum_{i=p}^{m-2} 2a_i\}$. If $Q = \mathrm{SU}(1, q)$, $1 \leq q \leq m-1$, we have that $C_a = \{a_1\omega_1 + \sum_{i=q+1}^{m} a_i\omega_i \mid a_m = -a_1 - a_{m-1} - \sum_{i=q+1}^{m-2} 2a_i\}$.

For Type D, $G = \mathrm{SO}(2m-2, 2)$ and $Q = \mathrm{SU}(1, p)$, $1 \leq p \leq m-1$, we have $C_a = \{a_1\omega_1 + \sum_{i=p+1}^{m} a_i\omega_i \mid a_1 = -a_{m-1} - a_m - \sum_{i=p+1}^{m-2} 2a_i\}$.

2.5 BGG Resolutions and Kostant Modules

The construction of a complex for any simple module $L(w)$ in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{reg}}$ is directly comparable to the construction of the resolution of a finite dimensional simple module given by Lepowsky [14], and, for $\mathfrak{p} = \mathfrak{b}$, by Bernstein, Gelfand, and Gelfand [2]. If $x, y \in {}^{\mathrm{I}}W$ are connected in the Hasse diagram for ${}^{\mathrm{I}}W$ with $x \leq y$, then there is a nonzero $\mathfrak{g}$-module map $M_{\mathrm{I}}(x) \to M_{\mathrm{I}}(y)$ which lifts to the standard map between the orginary Verma modules with the same respective highest weights. It is possible to assign ± 1 to the sides of every square in W so that the product of the four is -1. This fact, together with the map $M_{\mathrm{I}}(x) \to M_{\mathrm{I}}(y)$, can be used to construct a matrix of maps $d_i : C_i \to C_{i-1}$ where $C_i := \bigoplus_{\substack{x \leq w \\ l(w)-l(x)=i}} M_{\mathrm{I}}(x)$

for $1 \leq i \leq l(w)$. Together with the canonical quotient map $d_0 : C_0 = M_{\mathrm{I}}(w) \to L(w)$ this gives the BGG complex of $L(w)$ (see [9]).

Proposition 2.2. *Let $L(w)$ be a simple module in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{reg}}$. Then the sequence $0 \to C_{\ell(w)} \to \cdots \to C_1 \to C_0 \to L(w) \to 0$ is a complex. Moreover, the restriction of d_i to $M_{\mathrm{I}}(x)$ is nonzero for each $x \leq w$ with $l(x) = l(w) - i$.*

Table 2.2. Classification of Unitary Highest Weight Modules of Types A and D

G	λ_a	Q	Parameters	l
$\mathrm{SU}(p,q)$	$\omega_{p'}+\omega_{n-q'}-(n+l+1-p'-q')\omega_p$	$\mathrm{SU}(p',q')$	$1\le p'\le p$, $1\le q'\le q$	$1\le l\le \min(p',q')$
$\mathrm{SO}^*(2m)$	$\omega_2-(2m-2)\omega_m$	$\mathrm{SU}(1,1)$		1
	$\omega_1+\omega_{q+1}-(2m-q)\omega_m$	$\mathrm{SU}(1,q)$	$2\le q\le m-3$	1
	$\omega_1+\omega_{m-1}-(m+1)\omega_m$	$\mathrm{SU}(1,m-2)$		1
	$\omega_1-(m-1)\omega_m$	$\mathrm{SU}(m-1)$		1
	$\omega_p-2(m-p+l)\omega_m$	$\mathrm{SO}^*(2p)$	$3\le p\le m-2$	$1\le l\le [\frac{p}{2}]$
	$\omega_{m-1}-(1+2l)\omega_m$	$\mathrm{SO}^*(2m-2)$		$1\le l\le [\frac{m-1}{2}]$
	$-(2l-2)\omega_m$	$\mathrm{SO}^*(2m)$		$1\le l\le [\frac{m}{2}]$
$\mathrm{SO}(2m-2,2)$	$-(2m-p-1)\omega_1+\omega_{p+1}$	$\mathrm{SU}(1,p)$	$1\le p\le m-3$	1
	$-(m+1)\omega_1+\omega_{m-1}+\omega_m$	$\mathrm{SU}(1,m-2)$		1
	$-(m-1)\omega_1+\omega_m$	$\mathrm{SU}(1,m-1)$		1
	$-(m-1)\omega_1+\omega_{m-1}$	$\mathrm{SU}(1,m-1)$		1
	0	$\mathrm{SO}(2m-2,2)$		1
	$-(m-2)\omega_1$	$\mathrm{SO}(2m-2,2)$		2

Table 2.3. Classification of Unitary Highest Weight Modules of Type E

G	λ_a	Q	l
E_6	$\omega_2 - 12\omega_6$	$\mathrm{SU}(1,1)$	1
	$\omega_4 - 12\omega_6$	$\mathrm{SU}(1,2)$	1
	$\omega_3 + \omega_5 - 12\omega_6$	$\mathrm{SU}(1,3)$	1
	$\omega_3 - 9\omega_1$	$\mathrm{SU}(1,4)$	1
	$\omega_1 + \omega_5 - 10\omega_6$	$\mathrm{SU}(1,4)$	1
	$\omega_5 - 8\omega_6$	$\mathrm{SU}(1,5)$	1
	$\omega_1 - 5\omega_6$	$\mathrm{SO}(2,8)$	1
	$\omega_1 - 8\omega_6$	$\mathrm{SO}(2,8)$	2
	$-3\omega_6$	E_6	2
E_7	$\omega_1 - 18\omega_7$	$\mathrm{SU}(1,1)$	1
	$\omega_3 - 18\omega_7$	$\mathrm{SU}(1,2)$	1
	$\omega_4 - 18\omega_7$	$\mathrm{SU}(1,3)$	1
	$\omega_2 + \omega_5 - 18\omega_7$	$\mathrm{SU}(1,4)$	1
	$\omega_5 - 15\omega_7$	$\mathrm{SU}(1,5)$	1
	$\omega_2 + \omega_6 - 16\omega_7$	$\mathrm{SU}(1,5)$	1
	$\omega_2 - 13\omega_7$	$\mathrm{SU}(1,6)$	1
	$\omega_6 - 10\omega_7$	$\mathrm{SO}(2,10)$	1
	$\omega_6 - 14\omega_7$	$\mathrm{SO}(2,10)$	2
	$-4\omega_7$	E_7	2
	$-8\omega_7$	E_7	3

For example, in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{reg}}$ for $\mathfrak{su}(2,2)$ the BGG complex of the finite dimensional simple module is given by

$$0 \to M_{\mathrm{I}}(e) \to M_{\mathrm{I}}(\ydiagram{1}) \to M_{\mathrm{I}}(\ydiagram{2}) \oplus M_{\mathrm{I}}(\ydiagram{1,1}) \to M_{\mathrm{I}}(\ydiagram{2,1}) \to M_{\mathrm{I}}(\ydiagram{2,2}) \to L(\ydiagram{2,2}) \to 0.$$

By truncating this complex we obtain complexes for the other simple modules in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{reg}}$, such as

$$0 \to M_{\mathrm{I}}(e) \to M_{\mathrm{I}}(\ydiagram{1}) \to M_{\mathrm{I}}(\ydiagram{2}) \to L(\ydiagram{2}) \to 0.$$

Definition 2.3. A simple module $L(w)$ in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{reg}}$ is a *Kostant module* if

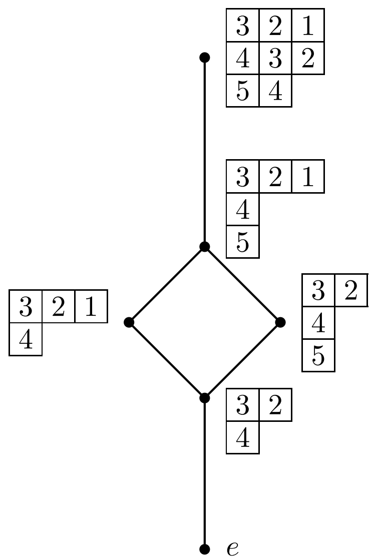
$$H^i(\mathfrak{u}, L(w)) \cong \bigoplus_{\substack{x \le w \\ l(w)-l(x)=i}} F_x$$

as an $\mathfrak{m}$-module for $i \ge 0$.

By Kostant's theorem [13], $L(w)$ is a Kostant module for w the longest element of ${}^{\mathrm{I}}W$ and by Enright [7], every unitary highest weight module is a Kostant module. Moreover, Boe and Hunziker [3] showed the following.

Proposition 2.3. *Let $L(w)$ be a simple module in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{reg}}$ for $w \in {}^{\mathrm{I}}W$. Then $L(w)$ is a Kostant module if and only if the truncated BGG complex for $L(w)$ is exact.*

In [10, 11], Enright and Shelton showed that, in particular when the Dynkin diagram is simply laced, there is an equivalence of categories given by an exact functor $\mathcal{E} : \mathcal{O}^{\mathfrak{p}'}_{\mathrm{reg}} \to \mathcal{O}^{\mathfrak{p}}_{\mu}$ which maps simple modules to simple modules, and Verma modules to Verma modules. Here $\mathfrak{p}'$ is a parabolic subalgebra of Hermitian type of a complex simple Lie algebra $\mathfrak{g}'$ of rank at most n. Thus we may extend the definition of BGG resolutions, and hence Kostant modules, to any infinitesimal block $\mathcal{O}^{\mathfrak{p}}_{\mu}$ by applying the exact functor $\mathcal{E}$. For example, consider $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(3,3)$ with $\Sigma = \{\alpha_3\}$. Then ${}^{\mathrm{I}}W^{\Sigma}$ has the following poset:



Since this is isomorphic to the poset ${}^{\mathrm{I}}W$ for $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(2,2)$, we obtain a BGG resolution

$$0 \to M_{\mathrm{I}}(e) \to M_{\mathrm{I}}(\ydiagram{2,1}) \to M_{\mathrm{I}}(\ydiagram{3,1}) \oplus M_{\mathrm{I}}(\ydiagram{2,1,1}) \to M_{\mathrm{I}}(\ydiagram{3,1,1}) \to M_{\mathrm{I}}(\ydiagram{3,3,2}) \to L(\ydiagram{3,3,2}) \to 0.$$

of the simple module $L(\ydiagram{3,3,2})$ in $\mathcal{O}^{\mathfrak{p}}_{\mu}$.

CHAPTER THREE

Theorem

Let $\lambda = 0$ be the highest weight of the trivial representation. If $M_{\mathrm{I}}(w \cdot \lambda) \to M_{\mathrm{I}}(w' \cdot \lambda)$ is a map in the BGG resolution of $L(\lambda) = L(0)$, then there is a unique root β in $\Phi_{w'} \setminus \Phi_w$. From Kostant's formula we have $w \cdot \lambda = \lambda - \langle \Phi_w \rangle$. Therefore $w \cdot \lambda = w' \cdot \lambda - \beta$. For example, in $\mathfrak{su}(2,2)$ we have the following resolution:

$$\begin{tikzcd}
 & & & M(\cdot) \arrow[dr, "\epsilon_1-\epsilon_3"] & & & & \\
0 \arrow[r] & M(\cdot) \arrow[r, "\epsilon_1-\epsilon_4"] & M(\cdot) \arrow[ur, "\epsilon_2-\epsilon_4"] \arrow[dr, "\epsilon_1-\epsilon_3"'] & \bigoplus & M(\cdot) \arrow[r, "\epsilon_2-\epsilon_3"] & M(\lambda) \arrow[r] & L(\lambda) \arrow[r] & 0 \\
 & & & M(\cdot) \arrow[ur, "\epsilon_2-\epsilon_4"'] & & & &
\end{tikzcd}$$

Here the labels on the arrows denote the root β that is subtracted moving from right to left. This pleasing property characterizes the trivial representation in the set of all finite dimesional representations. For any other representation, we must subtract multiples of roots, as in the following example for $\mathfrak{su}(2,2)$:

$$\begin{tikzcd}
 & & & M(\cdot) \arrow[dr, "2(\epsilon_1-\epsilon_3)"] & & & & \\
0 \arrow[r] & M(\cdot) \arrow[r, "\epsilon_1-\epsilon_4"] & M(\cdot) \arrow[ur, "\epsilon_2-\epsilon_4"] \arrow[dr, "2(\epsilon_1-\epsilon_3)"'] & \bigoplus & M(\cdot) \arrow[r, "\epsilon_2-\epsilon_3"] & M(\lambda) \arrow[r] & L(\lambda) \arrow[r] & 0 \\
 & & & M(\cdot) \arrow[ur, "\epsilon_2-\epsilon_4"'] & & & &
\end{tikzcd}$$

Moreover, if $\mathfrak{p}$ is of Hermitian type and $\mathfrak{g}_{\mathbb{R}}$ is noncompact, the trivial representation is the only unitarizable finite dimensional representation. The goal of this work is to show that the BGG resolution of every unitary highest weight module satisfies an analoguous property, namely that the m_i from Proposition 2.1 are equal to 1. To see this, let $f : \Phi(\mathfrak{u}) \to \Delta$ be given by $f(\beta) := \nu^{-1}\beta$ for $\nu \in {}^{\mathrm{I}}W$ such that $\Phi_\nu = \{\gamma \in \Phi(\mathfrak{u}) \mid \gamma < \beta\}$.

Lemma 3.1. *Let $x, y \in {}^{\mathrm{I}}W$ such that $\Phi_y = \Phi_x \cup \{\beta\}$ for some $\beta \in \Phi(\mathfrak{u})$. Then*

$$w_{\mathrm{I}}x \cdot \mu = w_{\mathrm{I}}y \cdot \mu + (\mu + \rho, f(\beta)^\vee) w_{\mathrm{I}}\beta.$$

Proof. Since $\Phi_y = \Phi_x \cup \{\beta\}$, we have $y = s_\beta x$. Hence

$$\begin{aligned} w_{\mathrm{I}}y(\mu+\rho) &= w_{\mathrm{I}}s_\beta x(\mu+\rho) \\ &= (w_{\mathrm{I}}s_\beta w_{\mathrm{I}}^{-1}) w_{\mathrm{I}}x(\mu+\rho) \\ &= s_{w_{\mathrm{I}}\beta} w_{\mathrm{I}}x(\mu+\rho) \\ &= w_{\mathrm{I}}x(\mu+\rho) - (w_{\mathrm{I}}x(\mu+\rho), w_{\mathrm{I}}\beta^\vee) w_{\mathrm{I}}\beta \\ &= w_{\mathrm{I}}x(\mu+\rho) - (\mu+\rho, x^{-1}\beta^\vee) w_{\mathrm{I}}\beta \\ &= w_{\mathrm{I}}x(\mu+\rho) - (\mu+\rho, f(\beta)^\vee) w_{\mathrm{I}}\beta, \end{aligned}$$

using the fact that $s_{w_{\mathrm{I}}\beta}\lambda = \lambda - (\lambda, (w_{\mathrm{I}}\beta)^\vee) w_{\mathrm{I}}\beta$ and that $f(\beta) = x^{-1}\beta$. □

Proposition 3.1. *Let $x, y \in {}^{\mathrm{I}}W$ with $x < y$. If $\eta = w_{\mathrm{I}}y \cdot \mu$ and $\xi = w_{\mathrm{I}}x \cdot \mu$, then*

$$\xi = \eta + \sum_{\beta \in \Phi_y \setminus \Phi_x} (\mu+\rho, f(\beta)^\vee) w_{\mathrm{I}}\beta.$$

Proof. There exists a sequence $x = x_0 \to x_1 \to \cdots \to x_k = y$ in ${}^{\mathrm{I}}W$. For $1 \le i \le k$, we then have $\Phi_{x_i} = \Phi_{x_{i-1}} \cup \{\beta_i\}$, where $\beta_i \in \Phi(\mathfrak{u})$. By Lemma 3.1,

$$w_{\mathrm{I}}x \cdot \mu = w_{\mathrm{I}}y \cdot \mu + \sum_{i=1}^{k} (\mu+\rho, f(\beta_i)^\vee) w_{\mathrm{I}}\beta_i.$$

This proves the proposition since $\Phi_y \setminus \Phi_x = \{\beta_1, \ldots, \beta_k\}$. □

Consequently, we are interested in the coefficients $(\mu+\rho, f(\beta_i)^\vee)$ of $w_{\mathrm{I}}\beta_i$.

Theorem 3.1. *Let $L(w) \neq M_{\mathrm{I}}(w)$ be a unitary highest weight module in $\mathcal{O}^{\mathfrak{p}}_\mu$. If Φ has only one root length, i.e. if Φ is of Type A, D, or E, then*

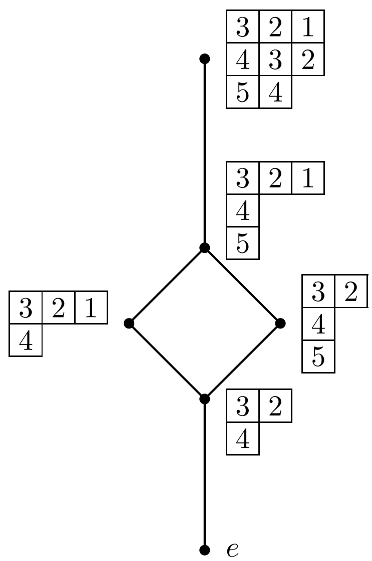
$$(\mu+\rho, \alpha^\vee) \in \{0, -1\} \quad \textit{for all } \alpha \in \operatorname{supp}(w),$$

where $\operatorname{supp}(w) = \{\alpha_i \in \Delta \mid s_{\alpha_i}$ occurs in the reduced expression for $w\}$.

The result then follows directly from this theorem: to obtain the highest weights of consecutive parabolic Verma modules we subtract possibly multiple roots, but never multiples of a root. For example, a Wallach representation of $\mathfrak{su}(3,3)$ with $\Sigma = \{\alpha_3\}$ gives us the following resolution:

$$\begin{tikzcd}
 & & & M(\cdot) \arrow[dr, "\epsilon_1-\epsilon_4"] & & & & \\
0 \arrow[r] & M(\cdot) \arrow[r, "\epsilon_1-\epsilon_5", "\epsilon_2-\epsilon_6"'] & M(\cdot) \arrow[ur, "\epsilon_3-\epsilon_6"] \arrow[dr, "\epsilon_1-\epsilon_4"'] & \bigoplus & M(\cdot) \arrow[r, "\epsilon_2-\epsilon_4", "\epsilon_3-\epsilon_5"'] & M(\lambda) \arrow[r] & L(\lambda) \arrow[r] & 0 \\
 & & & M(\cdot) \arrow[ur, "\epsilon_3-\epsilon_6"'] & & & &
\end{tikzcd}$$

A quick method of determining the roots to subtract is to look at the Hasse diagram of the poset ${}^{\mathrm{I}}W^{\Sigma}$. For this particular example we have:



The roots to subtract are then the corresponding entries of $w_{\mathrm{I}}\Phi(\mathfrak{u})$ for the boxes removed going from top to bottom. For example, starting from the top Young diagram, we must remove two boxes, namely the 2 and 4 on the lower right. These correspond to $\epsilon_3 - \epsilon_5$ and $\epsilon_2 - \epsilon_4$.

We distinguish the proof of this our main result with its own chapter.

CHAPTER FOUR

Proof

We proceed by cases corresponding to Tables 2.2 and 2.3, and write roots and weights using coordinates with respect to the standard bases given by ϵ_i's. The general procedure will be as follows. Starting with a general highest weight from 2.4, we evaluate $(\lambda+\rho,\beta^\vee)$ for all $\beta\in\Phi(\mathfrak{u})$. Then we arrange these values in the shape of a Young diagram by labeling each node of the poset diagram for $\Phi(\mathfrak{u})$ with the corresponding $(\lambda+\rho,\beta^\vee)$, taking the involution w_{I} into account. By Lemma 2.1, we are only concerned with the positive entries in order to determine $\mathrm{supp}(w)$. In fact, it will suffice to look at the entries corresponding to the Dynkin diagram within the Young diagram, which is what we will list in each case below. Finally, we determine $\mu+\rho$ from $\lambda+\rho$ by antidominance and calculate $(\mu+\rho,\alpha^\vee)$ for $\alpha\in$ $\mathrm{supp}(w)$.

4.1 Type A: $\mathfrak{g}_{\mathbb{R}}=\mathfrak{su}(p,q)$

From 2.4 we have that a general weight is of the form:

$$\lambda=(\underbrace{p'+q'+1-n-l,\dots,p'+q'+1-n-l}_{p'},$$

$$\underbrace{p'+q'-n-l-a_{p'},\dots,p'+q'-n-l-a_{p'}-\cdots-a_{p-1}}_{p-p'},$$

$$\underbrace{1+a_{p+1}+\cdots+a_{n-q'},\ 1+a_{p+2}+\cdots+a_{n-q'},\dots,1+a_{n-q'}}_{q-q'},$$

$$\underbrace{0,\dots,0}_{q'})$$

Since in this type $\rho=(n-1,\ n-2,\dots,1,\ 0)$, we then have

$$\lambda+\rho=(\underbrace{p'+q'-l,\ p'+q'-l-1,\ p'+q'-l-2,\dots,p'+q'-l-(p'-1)}_{p'},$$

$$\underbrace{q'-l-1-a_{p'},\ q'-l-2-a_{p'}-a_{p'+1},\dots,q'-l-(p-p')-a_{p'}-\cdots-a_{p-1}}_{p-p'},$$

$$\underbrace{q+a_{p+1}+\cdots+a_{n-q'},\ q-1+a_{p+2}+\cdots+a_{n-q'},\dots,q'+1+a_{n-q'}}_{q-q'},$$

$$\underbrace{q'-1,\ q'-2,\dots,1,\ 0}_{q'})$$

For this type the relevant entries are the first row: (for $\beta=\epsilon_1-\epsilon_n,\ \epsilon_2-\epsilon_n,\dots,\epsilon_p-\epsilon_n$)

$$\underbrace{p'+q'-l,\ p'+q'-l-1,\ p'+q'-l-2,\ \cdots\ ,\ q'-l+1}_{p'},\ q'-l-1-a_{p'},$$

$$q'-l-2-a_{p'}-a_{p'+1},\ \cdots\ ,\ p'+q'-l-p-a_{p'}-\cdots-a_{p-1}$$

and the first column: $(\beta=\epsilon_1-\epsilon_n,\ \epsilon_1-\epsilon_{n-1},\dots,\epsilon_1-\epsilon_{n-q+1})$

$$q'\begin{cases} p'+q'-l \\ p'+q'-l-1 \\ \vdots \\ p'-l+2 \\ p'-l+1 \end{cases}$$

$$p'-l-1-a_{n-q'}$$

$$\vdots$$

$$p'+q'-l-q+1-a_{p+2}-\cdots-a_{n-q'}$$

$$p'+q'-l-q-a_{p+1}-\cdots-a_{n-q'}$$

We consider two cases for the smallest and largest subdiagrams. An arbitrary subdiagram then follows from recursion in the following way. Begin with the smallest subdiagram, i.e. for $a_{n-q'}\geq p'-l-1$ and $a_{p'}\geq q'-l-1$. Then add one box to the diagram at a time, noting how this changes $\mu+\rho$, until we reach the largest subdiagram, i.e. for all $a_i=0$. For example, consider $p=q=6$, $p'=5$, $q'=2$, and $l=1$. Then the largest subdiagram is

$$\begin{array}{ccccc}
6 & 5 & 4 & 3 & 2 \\
5 & 4 & 3 & 2 & 1 \\
3-a_{10} & 2-a_{10} & 1-a_{10} & & \\
2-a_9-a_{10} & 1-a_9-a_{10} & & & \\
1-a_8-a_9-a_{10} & & & &
\end{array}$$

while $\lambda+\rho = (6,\ 5,\ 4,\ 3,\ 2,\ -a_5,\ 6+a_7+a_8+a_9+a_{10},\ 5+a_8+a_9+a_{10},\ 4+a_9+a_{10},\ 3+a_{10},\ 1,\ 0)$. The smallest subdiagram, for $a_{10} \geq 3$, is a 2x5 rectangle giving $\mathrm{supp}(w) = \{\alpha_2, \ldots, \alpha_7\}$ and $\mu+\rho = (-a_5,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 3+a_{10},\ 4+a_9+a_{10},\ 5+a_8+a_9+a_{10},\ 6+a_7+a_8+a_9+a_{10})$. Adding a box to the lower left means $a_{10} = 2$ and the $3+a_{10}$ entry shifts one slot left in $\mu+\rho$. Continue adding boxes on the lower left; the number of boxes in the third row gives the value of a_{10}, in the fourth the value of a_9, and so on. Each added box shifts entries from the right tail of $\mu+\rho$ left one slot until we arrive at $\mu+\rho = (-a_5,\ 0,\ 1,\ 2,\ 3,\ 3+a_{10},\ 4,\ 4+a_9+a_{10},\ 5,\ 5+a_8+a_9+a_{10},\ 6,\ 6+a_7+a_8+a_9+a_{10})$ corresponding to $a_8 = a_9 = a_{10} = 0$.

4.1.1 Case 1: $a_{n-q'} \geq p'-l-1$ and $a_{p'} \geq q'-l-1$

By antidominance we have

$$\begin{aligned}
\mu+\rho = (&\underbrace{q'-l-(p-p')-a_{p'}-\cdots-a_{p-1}, \ldots, q'-l-1-a_{p'}}_{p-p'}, \\
&\underbrace{0,\ 1, \ldots, q'-l}_{q'-l+1},\ \underbrace{q'-l+1,\ q'-l+1, \ldots, q'-1,\ q'-1}_{2(l-1)}, \\
&\underbrace{q',\ q'+1, \ldots, p'+q'-l-1,\ p'+q'-l}_{p'-l+1}, \\
&\underbrace{q'+1+a_{n-q'}, \ldots, q+a_{p+1}+\cdots+a_{n-q'}}_{q-q'})
\end{aligned}$$

This case corresponds to the smallest possible subdiagram, which has q' entries in the first column and p' in the first row. Thus

$$\mathrm{supp}(w) = \{\alpha_{p-p'+1},\ \alpha_{p-p'+2}, \ldots, \alpha_{p-1},\ \alpha_p,\ \alpha_{p+1}, \ldots, \alpha_{p+q'-1}\}.$$

Hence $(\mu+\rho,\alpha_i^\vee) = \underbrace{-1,\ldots,-1}_{q'-l+1},\ \underbrace{0,\ -1,\ 0,\ldots,-1,\ 0,\ -1}_{2(l-1)},\ \underbrace{-1,\ldots,-1}_{p'-l}$
for $i = p-p'+1,\ldots,p+q'-1$.

4.1.2 Case 2: all $a_i = 0$

Here we are in the same situation as Case 1 if $p'-l-1 \le 0$ or $q'-l-1 \le 0$, so suppose each is strictly greater than zero. This case corresponds to the largest possible subdiagram above, which has $p'+q'-l-1$ entries in the first column and in the first row. However, $\mu+\rho$ depends on two additional relations. They are independent, but for sake of brevity we combine them into two subcases.

4.1.2.1 *Subcase 1: $q'-l \le p-p'$ and $p'-l \le q-q'$.* There must be some $1 < i < p-p'$ and $q' < j < n-p-1$ such that $-a_{p'}-\cdots-a_{p-i}$ and $a_{n-j}+\cdots+a_{n-q'}$ have constant terms 0 and $p'+q'-l$, respectively. Then

$$\mu+\rho = (\underbrace{q'-l-(p-p')-a_{p'}-\cdots-a_{p-1},\ldots,-1-a_{p'}-\cdots-a_{p-i+1}}_{p-p'-q'+l},$$

$$\underbrace{-a_{p'}-\cdots-a_{p-i},\ 0,\ 1-a_{p'}-\cdots-a_{p-i-1},\ 1,\ldots,q'-l-1-a_{p'},\ q'-l-1}_{2(q'-l)},$$

$$\underbrace{q'-l,\ q'-l+1,\ q'-l+1,\ q'-l+2,\ q'-l+2,\ldots,q'-1,\ q'-1,\ q'}_{2l},$$

$$\underbrace{q'+1,\ q'+1+a_{n-q'},\ldots,p'+q'-l,\ p'+q'-l+a_{n-j}+\cdots+a_{n-q'}}_{2(p'-l)},$$

$$\underbrace{p'+q'-l+1+a_{n-j-1}+\cdots+a_{n-q'},\ldots,q+a_{p+1}+\cdots+a_{n-q'}}_{q-q'-p'+l}).$$

Since $\mathrm{supp}(w) = \{\alpha_{p-p'-q'+l+2},\ \alpha_{p-p'-q'+l+3},\ldots,\alpha_{p+p'+q'-l-2}\}$, we have

$$(\mu+\rho,\alpha_i^\vee) = \underbrace{-1,\ 0,\ -1,\dots,0,\ -1}_{2(q'-l)-1},\ -1,\ \underbrace{0,\ -1,\ 0,\dots,-1,\ 0,\ -1}_{2(l-1)},\ -1,$$
$$\underbrace{0,\ -1,\ 0,\dots,-1,\ 0,\ -1}_{2(p'-l-1)}$$
for $i = p-p'-q'+l+2,\dots,p+p'+q'-l-2$.

4.1.2.2 *Subcase 2: $q'-l > p-p'$ and $p'-l > q-q'$.* Here the entries containing a_i's are entirely contained within the constant entries:

$$\mu+\rho = (\underbrace{0,\ 1,\dots,p'+q'-l-p-1}_{p'+q'-l-p},$$
$$\underbrace{q'-l-(p-p')-a_{p'}-\cdots-a_{p-1},\ p'+q'-l-p,\dots,q'-l-1-a_{p'},\ q'-l-1}_{2(p-p')},$$
$$\underbrace{q'-l,\ q'-l+1,\ q'-l+1,\ q'-l+2,\ q'-l+2,\dots,q'-1,\ q'-1,\ q'}_{2l},$$
$$\underbrace{q'+1,\ q'+1+a_{n-q'},\dots,q,\ q+a_{p+1}+\cdots+a_{n-q'}}_{2(q-q')},\ \underbrace{q+1,\ q+2,\dots,p'+q'-l}_{p'+q'-l-q}).$$

By the hypothesis of this subcase, $p'+q'-l > p$ and $p'+q'-l > q$ so the subdiagram is actually the full Young diagram for $\mathrm{SU}(p,q)$. That is $\mathrm{supp}(w) = \{\alpha_1,\ \alpha_2,\dots,\alpha_{n-1}\}$. Thus

$$(\mu+\rho,\alpha_i^\vee) = (\underbrace{-1,\dots,-1}_{p'+q'-l-p},\ \underbrace{0,\ -1,\ 0,\dots,-1,\ 0,\ -1}_{2(p-p')},$$
$$\underbrace{-1,\ 0,\ -1,\ 0,\dots,-1,\ 0,\ -1,\ -1}_{2l},$$
$$\underbrace{0,\ -1,\ 0,\dots,-1,\ 0,\ -1}_{2(q-q')},\ \underbrace{-1,\dots,-1}_{p'+q'-l-q-1})$$

for $i = 1,\dots,n-1$.

4.2 Type D: $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2m)$

Recall that for this type, the Weyl group includes action by an even number of sign changes. This will cause some inconvience in determining $\mu+\rho$ for the $Q = \mathrm{SU}(1,q)$ cases. However, for $Q = \mathrm{SO}^*(2p)$, $\lambda+\rho$ always contains a zero term, allowing us to negate entries with impunity.

4.2.1 *Case 1: $Q = \mathrm{SO}^*(2p)$, $3 \le p \le m-2$*

From Table 2.2 we have

$$\lambda = (\underbrace{1-(m-p+l), \ldots, 1-(m-p+l)}_{p},$$

$$\underbrace{-(m-p+l)-a_p, \ldots, -(m-p+l)-a_p-\cdots-a_{m-1}}_{m-p})$$

and since ρ is the same as for Type A, it follows that

$$\lambda+\rho = (\underbrace{p-l,\ p-l-1,\ p-l-2, \ldots, 1-l}_{p},$$

$$\underbrace{-l-1-a_p,\ -l-2-a_p-a_{p+1}, \ldots, -(m-p+l)-a_p-\cdots-a_{m-1}}_{m-p}).$$

For Type D the relevant entries of the Young diagram are the first row, that is for $\beta = \epsilon_1+\epsilon_2,\ \epsilon_1+\epsilon_3, \ldots, \epsilon_1+\epsilon_n$:

$$2p-2l-1,\, 2p-2l-2,\ \cdots\ ,\ p-2l+1,\ p-2l-1-a_p,\ p-2l-2-a_p-a_{p+1},$$

$$\cdots\ ,\ 2p-2l-m-a_p-\cdots-a_{m-1}$$

and the first entry of the second row: $2p-2l-3$ (for $\beta = \epsilon_2+\epsilon_3$). We follow cases similar to those of Type A.

4.2.1.1 *Subcase 1: $a_p \ge p-2l-1$ OR $p-2l-1 \le 0$.* In this case by antidominance we have

$$\mu+\rho = (\underbrace{-(m-p+l)-a_p-\cdots-a_{m-1}, \ldots, -l-2-a_p-a_{p+1},\ -l-1-a_p}_{m-p},$$

$$\underbrace{-(p-l),\ -(p-l-1), \ldots, -l}_{p-2l+1},$$

$$\underbrace{-l+1,\ -l+1,\ -l+2,\ -l+2, \ldots, -1,\ -1}_{2(l-1)},\ 0)$$

By hypothesis, we have $p-1$ positive entries in the above first row. Also, since $l \le [\frac{p}{2}]$, $2p-2l-3 > 0$. Hence $\mathrm{supp}(w) = \{\alpha_{m-p+1},\ \alpha_{m-p+2}, \ldots, \alpha_{m-2},\ \alpha_{m-1},\ \alpha_m\}$ and so $(\mu+\rho, \alpha_i^\vee) = \underbrace{-1, \ldots, -1}_{p-2l+1},\ \underbrace{0,\ -1,\ 0, \ldots, -1,\ 0,\ -1}_{2(l-1)},\ -1$ for $i = m-p+1, \ldots, m$.

4.2.1.2 *Subcase 2: all a_i's 0 and $2p-2l-1<m$.* Here

$$\mu+\rho=(\underbrace{-(m-p+l)-a_p-\cdots-a_{m-1},\ldots,-1-(p-l)-a_p-\cdots-a_{2p-2l}}_{m-2p+2l},$$
$$\underbrace{-(p-l)-a_p-\cdots-a_{2p-2l-1},\ -(p-l),\ldots,-l-1}_{2(p-2l)},$$
$$-\,l,\ \underbrace{-l+1,\ -l+1,\ -l+2,\ -l+2,\ldots,-1,\ -1}_{2(l-1)},\ 0)$$

Assuming that $p-2l-1>0$ (otherwise we have the same situation as Subcase 1), there are $2p-2l-2$ positive entries in the first row. Thus

$$\operatorname{supp}(w)=\{\alpha_{m-2p+2l+2},\ \alpha_{m-2p+2l+3},\ldots,\alpha_{m-p+1},\ \alpha_{m-p+2},\ldots,\alpha_{m-2},\ \alpha_{m-1},\ \alpha_m\}.$$

It follows that

$$(\mu+\rho,\alpha_i^\vee)=\underbrace{-1,\ 0,\ -1,\ldots,0,\ -1,\ -1}_{2(p-2l)},\ \underbrace{0,\ -1,\ 0,\ldots,-1,\ 0,\ -1}_{2(l-1)},\ -1$$

for $i=m-2p+2l+2,\ldots,m$.

4.2.1.3 *Subcase 3: all a_i's 0 and $2p-2l-1\geq m$.* In this case

$$\mu+\rho=(\underbrace{-(p-l),\ -(p-l-1),\ldots,-(m-p+l+1)}_{2p-2l-m},$$
$$\underbrace{-(m-p+l)-a_p-\cdots-a_{m-1},\ -(m-p+l),\ldots,-l-1-a_p,\ -l-1}_{2(m-p)},$$
$$-\,l,\ \underbrace{-l+1,\ -l+1,\ -l+2,\ -l+2,\ldots,-1,\ -1}_{2(l-1)},\ 0).$$

Moreover, the entire first row above is positive, i.e.

$$\operatorname{supp}(w)=\{\alpha_1,\ \alpha_2,\ldots,\alpha_{m-2},\ \alpha_{m-1},\ \alpha_m\}.$$

Therefore

$$(\mu+\rho,\alpha_i^\vee)=\underbrace{-1,\ldots,-1}_{2p-2l-m},\ \underbrace{0,\ -1,\ 0,\ldots,-1,\ 0,\ -1}_{2(m-p)},\ -1,$$
$$\underbrace{0,\ -1,\ 0,\ldots,-1,\ 0,\ -1}_{2(l-1)},\ -1$$

for $i=1,\ldots,m$.

4.2.2 Case 2: $Q = \mathrm{SO}^*(2p)$, $p = m-1$

By 2.4 we have that $\lambda = (\underbrace{-l, \ldots, -l}_{m-1},\ -1-l-a_{m-1})$. Thus

$$\lambda + \rho = (m-l-1,\ m-l-2, \ldots, 2-l,\ 1-l,\ -1-l-a_{m-1})$$

and so the entries in the first row are

$$2m-2l-3,\ 2m-2l-4,\ \cdots,\ m-2l+1,\ m-2l,\ m-2l-2-a_{m-1}$$

with $2m-2l-5$ in the first entry of the second row.

4.2.2.1 *Subcase 1: $a_{m-1} \geq m-2l-2$ OR $m-2l-2 \leq 0$.* Here we have $\mathrm{supp}(w) = \{\alpha_2,\ \alpha_3, \ldots, \alpha_{m-2},\ \alpha_{m-1},\ \alpha_m\}$ and

$$\mu + \rho = (\,-1-l-a_{m-1},\ \underbrace{-(m-l-1),\ -(m-l-2), \ldots, -l}_{m-2l},$$

$$\underbrace{-l+1,\ -l+1,\ -l+2,\ -l+2, \ldots, -1,\ -1}_{2(l-1)},\ 0).$$

Hence $(\mu+\rho, \alpha_i^\vee) = \underbrace{-1, \ldots, -1}_{m-2l},\ \underbrace{0,\ -1,\ 0, \ldots, -1,\ 0,\ -1}_{2(l-1)},\ -1$ for $i = 2, \ldots, m$.

4.2.2.2 *Subcase 2: $a_{m-1} = 0$ and $m-2l-2 > 0$.* In this case $\mathrm{supp}(w) = \{\alpha_1,\ \alpha_2, \ldots, \alpha_{m-2},\ \alpha_{m-1},\ \alpha_m\}$ and

$$\mu + \rho = (\underbrace{-(m-l-1),\ -(m-l-2), \ldots, -(l+2)}_{m-2l-2},$$

$$-1-l-a_{m-1},\ -l-1,\ -l,\ \underbrace{-l+1,\ -l+1,\ -l+2,\ -l+2, \ldots, -1,\ -1}_{2(l-1)},\ 0).$$

Therefore $(\mu+\rho, \alpha_i^\vee) = \underbrace{-1, \ldots, -1}_{m-2l-2},\ 0,\ -1,\ -1,\ \underbrace{0,\ -1,\ 0, \ldots, -1,\ 0,\ -1}_{2(l-1)},\ -1$ for $i = 1, \ldots, m$.

4.2.3 *Case 3:* $Q = \mathrm{SO}^*(2p)$, $p = m$

Observe that by 2.4 we have $\lambda = (-l+1, \ldots, -l+1)$ and so

$$\lambda + \rho = (m-l,\ m-l-1, \ldots, -l+2,\ -l+1)$$

giving us a first row in the diagram of

$$2m-2l-1,\ 2m-2l-2,\ \cdots\ ,\ m-2l+2,\ m-2l+1$$

and $2m-2l-3$ as the first entry in the second row. Thus

$\mathrm{supp}(w) = \{\alpha_1,\ \alpha_2, \ldots, \alpha_{m-2},\ \alpha_{m-1},\ \alpha_m\}$ and

$$(\mu+\rho, \alpha_i^\vee) = \underbrace{-1, \ldots, -1}_{m-2l+1},\ \underbrace{0,\ -1,\ 0, \ldots, -1,\ 0,\ -1}_{2(l-1)},\ -1$$

for $i = 1, \ldots, m$.

4.2.4 *Case 4:* $Q = \mathrm{SU}(1,q)$, $q = 1$

By 2.4

$$\lambda = (2-m+\frac{1}{2}a_1,\ 2-m-\frac{1}{2}a_1,\ 1-m-\frac{1}{2}a_1-a_2, \ldots, 1-m-\frac{1}{2}a_1-a_2-\cdots-a_{m-1}).$$

Thus

$$\begin{aligned}\lambda+\rho = (1+\frac{1}{2}a_1,\ -\frac{1}{2}a_1,\ -2-\frac{1}{2}a_1-a_2,\ -3-\frac{1}{2}a_1-a_2-a_3, \ldots,\\ 1-m-\frac{1}{2}a_1-a_2-\cdots-a_{m-1})\end{aligned}$$

yielding a first row of

$$1,\ -1-a_2,\ -2-a_2-a_3,\ \cdots\ ,\ 3-m-a_2-\cdots-a_{m-2},\ 2-m-a_2-\cdots-a_{m-1}$$

and first entry in the second row of $-2-a_1-a_2$. Clearly $\mathrm{supp}(w) = \{\alpha_m\}$. Now in this case

$$\begin{aligned}\mu+\rho = (1-m-\frac{1}{2}a_1-a_2-\cdots-a_{m-1},\ 2-m-\frac{1}{2}a_1-a_2-\cdots-a_{m-2}, \ldots,\\ -2-\frac{1}{2}a_1-a_2,\ -1-\frac{1}{2}a_1,\ \frac{1}{2}a_1)\end{aligned}$$

so $(\mu+\rho, \alpha_i^\vee) = -1$ for $i = m$.

4.2.5 Case 5: $Q = \mathrm{SU}(1,q)$, $2 \leq q \leq m-3$

By 2.4

$$\lambda = (2-m+\frac{1}{2}q+\frac{1}{2}a_1,\ \underbrace{1-m+\frac{1}{2}q-\frac{1}{2}a_1,\ldots,1-m+\frac{1}{2}q-\frac{1}{2}a_1,}_{q}$$
$$\underbrace{-m+\frac{1}{2}q-\frac{1}{2}a_1-a_{q+1},\ldots,-m+\frac{1}{2}q-\frac{1}{2}a_1-a_{q+1}-\cdots-a_{m-1}}_{m-q-1})$$

and so

$$\lambda+\rho = (\underbrace{1+\frac{1}{2}q+\frac{1}{2}a_1,\ -1+\frac{1}{2}q-\frac{1}{2}a_1,\ -2+\frac{1}{2}q-\frac{1}{2}a_1,\ldots,-\frac{1}{2}q-\frac{1}{2}a_1,}_{q+1}$$
$$\underbrace{-2-\frac{1}{2}q-\frac{1}{2}a_1-a_{q+1},\ldots,-m+\frac{1}{2}q-\frac{1}{2}a_1-a_{q+1}-\cdots-a_{m-1}}_{m-q-1}).$$

Thus the first row and first entry of the second row are

$$q,\ q-1,\ q-2,\ \cdots,\ 1,\ -1-a_{q+1},\ -2-a_{q+1}-a_{q+2},\ \cdots,$$
$$1-m+q-a_{q+1}-\cdots-a_{m-1}$$

and $q-3-a_1$, respectively.

4.2.5.1 *Subcase 1: $a_1 \geq q-3$ OR $q=2$.* For this case $\mathrm{supp}(w) = \{\alpha_{m-q}, \alpha_{m-q+1},\ldots,\alpha_{m-2},\ \alpha_m\}$. By antidominance

$$\mu+\rho =$$
$$(-m+\frac{1}{2}q-\frac{1}{2}a_1-a_{q+1}-\cdots-a_{m-1},\ 1-m+\frac{1}{2}q-\frac{1}{2}a_1-a_{q+1}-\cdots-a_{m-2},$$
$$\ldots,-\frac{1}{2}q-2-\frac{1}{2}a_1-a_{q+1},\ -\frac{1}{2}q-1-\frac{1}{2}a_1,\ -\frac{1}{2}q-\frac{1}{2}a_1,\ -\frac{1}{2}q+1-\frac{1}{2}a_1,\ldots,$$
$$-2+\frac{1}{2}q-\frac{1}{2}a_1,\ -\frac{1}{2}q+1+\frac{1}{2}a_1).$$

Therefore $(\mu+\rho,\alpha_i^\vee) = -1,\ldots,-1$ for $i = m-q,\ldots,m-2,\ m$.

4.2.5.2 *Subcase 2: $a_1 = 0$ and $q > 2$.* Here $\mathrm{supp}(w) = \{\alpha_{m-q},\ \alpha_{m-q+1}, \ldots,\ \alpha_{m-2},\ \alpha_{m-1},\ \alpha_m\}$, however $\mu+\rho$ depends on the parity of q. If q is even we have

$$\begin{aligned}
&\mu+\rho = \\
&\Big(-m+\frac{1}{2}q-\frac{1}{2}a_1-a_{q+1}-\cdots-a_{m-1},\ 1-m+\frac{1}{2}q-\frac{1}{2}a_1-a_{q+1}-\cdots-a_{m-2},\\
&\ \ldots,-\frac{1}{2}q-2-\frac{1}{2}a_1-a_{q+1},\ -\frac{1}{2}q-1-\frac{1}{2}a_1,\ -\frac{1}{2}q-\frac{1}{2}a_1,\ -\frac{1}{2}q+1-\frac{1}{2}a_1,\\
&\ -\frac{1}{2}q+1+\frac{1}{2}a_1,\ -\frac{1}{2}q+2-\frac{1}{2}a_1,\ldots,-1-\frac{1}{2}a_1,\ -1+\frac{1}{2}a_1,\ \frac{1}{2}a_1\Big).
\end{aligned}$$

If q is odd then

$$\begin{aligned}
&\mu+\rho = \\
&\Big(-m+\frac{1}{2}q-\frac{1}{2}a_1-a_{q+1}-\cdots-a_{m-1},\ 1-m+\frac{1}{2}q-\frac{1}{2}a_1-a_{q+1}-\cdots-a_{m-2},\\
&\ \ldots,-\frac{1}{2}q-2-\frac{1}{2}a_1-a_{q+1},\ -\frac{1}{2}q-1-\frac{1}{2}a_1,\ -\frac{1}{2}q-\frac{1}{2}a_1,\ -\frac{1}{2}q+1-\frac{1}{2}a_1,\\
&\ -\frac{1}{2}q+1+\frac{1}{2}a_1,\ -\frac{1}{2}q+2-\frac{1}{2}a_1,\ldots,-\frac{3}{2}-\frac{1}{2}a_1,\ -\frac{3}{2}+\frac{1}{2}a_1,\ -\frac{1}{2}-\frac{1}{2}a_1,\\
&\ \pm\Big(\frac{1}{2}-\frac{1}{2}a_1\Big)\Big)
\end{aligned}$$

where the sign on the last entry depends on the parity of k, $q = 2k+1$. Hence $(\mu+\rho, \alpha_i^\vee) = -1,\ -1,\ 0,\ -1,\ldots,0,\ -1,\ -1$ or $(\mu+\rho,\alpha_i^\vee) = -1,\ -1,\ 0,\ -1,\ldots,\ 0,\ -1,\ -1,\ 0$ (with the last two entries switched for the negative final entry) for $i = m-q,\ m-q+1,\ldots,m-1,\ m$.

4.2.6 Case 6: $Q = \mathrm{SU}(1,q)$, $q = m-2$

From 2.4

$$\lambda = \Big(1-\frac{1}{2}m+\frac{1}{2}a_1,\ -\frac{1}{2}m-\frac{1}{2}a_1,\ -\frac{1}{2}m-\frac{1}{2}a_1,\ldots,-\frac{1}{2}m-\frac{1}{2}a_1,\ -1-\frac{1}{2}m-\frac{1}{2}a_1-a_{m-1}\Big)$$

and

$$\begin{aligned}
\lambda+\rho = \Big(&\frac{1}{2}m+\frac{1}{2}a_1,\ \frac{1}{2}m-2-\frac{1}{2}a_1,\ \frac{1}{2}m-3-\frac{1}{2}a_1,\ldots,1-\frac{1}{2}m-\frac{1}{2}a_1,\\
&-1-\frac{1}{2}m-\frac{1}{2}a_1-a_{m-1}\Big).
\end{aligned}$$

Thus we have $m-2,\ m-3,\ m-4,\ \cdots,\ 1,\ -1-a_{m-1}$ as the first row and $m-5-a_1$ as the first entry in the second row.

4.2.6.1 *Subcase 1: $a_1 \geq m-5$ OR $m \leq 5$.* Here $\mathrm{supp}(w) = \{\alpha_2,\ \alpha_3, \ldots, \alpha_{m-2},\ \alpha_m\}$ and

$$\begin{aligned}\mu+\rho = (&-\frac{1}{2}m-1-\frac{1}{2}a_1-a_{m-1},\ -\frac{1}{2}m-\frac{1}{2}a_1,\ -\frac{1}{2}m+1-\frac{1}{2}a_1, \ldots,\\ &\frac{1}{2}m-3-\frac{1}{2}a_1,\ -\frac{1}{2}m+2+\frac{1}{2}a_1).\end{aligned}$$

It follows that $(\mu+\rho, \alpha_i^\vee) = -1, \ldots, -1$ for $i = 2,\ 3, \ldots, m-2,\ m$.

4.2.6.2 *Subcase 2: $a_1 = 0$ and $m > 5$.* For this case $\mathrm{supp}(w) = \{\alpha_2,\ \alpha_3, \ldots, \alpha_{m-2},\ \alpha_{m-1},\ \alpha_m\}$, however $\mu+\rho$ depends on the parity of m. If m is even we have

$$\begin{aligned}\mu+\rho = (&-\frac{1}{2}m-1-\frac{1}{2}a_1-a_{m-1},\ -\frac{1}{2}m-\frac{1}{2}a_1,\ -\frac{1}{2}m+1-\frac{1}{2}a_1,\\ &-\frac{1}{2}m+2-\frac{1}{2}a_1,\ -\frac{1}{2}m+2+\frac{1}{2}a_1, \ldots, -1-\frac{1}{2}a_1,\ -1+\frac{1}{2}a_1,\ \frac{1}{2}a_1).\end{aligned}$$

If m is odd then

$$\begin{aligned}\mu+\rho = (&-\frac{1}{2}m-1-\frac{1}{2}a_1-a_{m-1},\ -\frac{1}{2}m-\frac{1}{2}a_1,\ -\frac{1}{2}m+1-\frac{1}{2}a_1,\\ &-\frac{1}{2}m+2-\frac{1}{2}a_1,\ -\frac{1}{2}m+2+\frac{1}{2}a_1, \ldots, -\frac{3}{2}-\frac{1}{2}a_1,\ -\frac{3}{2}+\frac{1}{2}a_1,\\ &-\frac{1}{2}-\frac{1}{2}a_1,\ \pm(\frac{1}{2}-\frac{1}{2}a_1))\end{aligned}$$

where the sign on the last entry depends on the parity of the number of entries in $\lambda+\rho$ with positive constant term. Therefore $(\mu+\rho, \alpha_i^\vee) = -1,\ -1,\ 0,\ -1, \ldots, 0,\ -1,\ -1$ or $(\mu+\rho, \alpha_i^\vee) = -1,\ -1,\ 0,\ -1, \ldots, 0,\ -1,\ -1,\ 0$ (where the last two entries are switched for the negative final entry) for $i = 2,\ 3, \ldots, m-1,\ m$.

4.2.7 Case 7: $Q = \mathrm{SU}(1,q)$, $q = m-1$

For this case, by 2.4

$$\lambda = (\frac{3}{2}-\frac{1}{2}m+\frac{1}{2}a_1,\ \frac{1}{2}-\frac{1}{2}m-\frac{1}{2}a_1,\ \frac{1}{2}-\frac{1}{2}m-\frac{1}{2}a_1, \ldots, \frac{1}{2}-\frac{1}{2}m-\frac{1}{2}a_1).$$

Thus

$$\lambda+\rho=(\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}a_1,\ \frac{1}{2}m-\frac{3}{2}-\frac{1}{2}a_1,\ \frac{1}{2}m-\frac{5}{2}-\frac{1}{2}a_1,\dots,\frac{3}{2}-\frac{1}{2}m-\frac{1}{2}a_1,\\ \frac{1}{2}-\frac{1}{2}m-\frac{1}{2}a_1)$$

resulting in a first row of $m-1,\ m-2,\ m-3,\ \cdots,\ 2,\ 1$ and $m-4-a_1$ as the first entry in the second row.

4.2.7.1 *Subcase 1: $a_1 \geq m-4$ OR $m=4$.* In this case $\operatorname{supp}(w)=\{\alpha_1,\ \alpha_2,\dots,\alpha_{m-2},\ \alpha_m\}$. Now

$$\mu+\rho=(-\frac{1}{2}m-\frac{1}{2}-\frac{1}{2}a_1,\ -\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}a_1,\ -\frac{1}{2}m+\frac{3}{2}-\frac{1}{2}a_1,\dots,\\ \frac{1}{2}m-\frac{5}{2}-\frac{1}{2}a_1,\ -\frac{1}{2}m+\frac{3}{2}+\frac{1}{2}a_1)$$

and so $(\mu+\rho,\alpha_i^\vee)=-1,\dots,-1$ for $i=1,\ 2,\dots,m-2,\ m$.

4.2.7.2 *Subcase 2: $a_1=0$ and $m>4$.* Here $\operatorname{supp}(w)=\{\alpha_1,\ \alpha_2,\dots,\alpha_{m-2},\ \alpha_{m-1},\ \alpha_m\}$, and again $\mu+\rho$ depends on the parity of m. If m is odd we have

$$\mu+\rho=(-\frac{1}{2}m-\frac{1}{2}-\frac{1}{2}a_1,\ -\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}a_1,\ -\frac{1}{2}m+\frac{3}{2}-\frac{1}{2}a_1,\ -\frac{1}{2}m+\frac{3}{2}+\frac{1}{2}a_1,\\ -\frac{1}{2}m+\frac{5}{2}-\frac{1}{2}a_1,\dots,-1-\frac{1}{2}a_1,\ -1+\frac{1}{2}a_1,\ \frac{1}{2}a_1).$$

If m is even then

$$\mu+\rho=(-\frac{1}{2}m-\frac{1}{2}-\frac{1}{2}a_1,\ -\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}a_1,\ -\frac{1}{2}m+\frac{3}{2}-\frac{1}{2}a_1,\ -\frac{1}{2}m+\frac{3}{2}+\frac{1}{2}a_1,\\ -\frac{1}{2}m+\frac{5}{2}-\frac{1}{2}a_1,\dots,-\frac{3}{2}-\frac{1}{2}a_1,\ -\frac{3}{2}+\frac{1}{2}a_1,\ -\frac{1}{2}-\frac{1}{2}a_1,\ \pm(\frac{1}{2}-\frac{1}{2}a_1))$$

where again the sign on the last entry depends on the parity of the number of entries in $\lambda+\rho$ with positive constant term. Therefore $(\mu+\rho,\alpha_i^\vee)=-1,\ -1,\ 0,\ -1,\dots,0,\ -1,\ -1$ or $(\mu+\rho,\alpha_i^\vee)=-1,\ -1,\ 0,\ -1,\dots,0,\ -1,\ -1,\ 0$ (where the last two entries are switched for the negative final entry) for $i=1,\ 2,\dots,m-1,\ m$.

4.3 Type D: $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(2m-2, 2)$

For this case $\mathrm{supp}(w)$ depends on the parity of m because the longest element in the Weyl group for D_{m-1} changes the sign of $m-1$ for m odd but not for m even. See Table 1 of [1]. However, this will only be an issue for $Q = \mathrm{SU}(1, m-1)$. In the remaining cases α_{m-1} and α_m are either both included or both excluded from $\mathrm{supp}(w)$. Thus we ignore the distinction and, without loss of generality, only write the row of the Young diagram for m even when $q < m-1$.

4.3.1 Case 1: $Q = \mathrm{SU}(1, q)$, $q = m-1$ with $\lambda_a = -(m-1)\omega_1 + \omega_{m-1}$

By 2.4 we have $\lambda = (\frac{3}{2} - m - \frac{1}{2}a_{m-1},\ \frac{1}{2} + \frac{1}{2}a_{m-1}, \ldots, \frac{1}{2} + \frac{1}{2}a_{m-1},\ -\frac{1}{2} - \frac{1}{2}a_{m-1})$ and so $\lambda + \rho = (\frac{1}{2} - \frac{1}{2}a_{m-1},\ m - \frac{3}{2} + \frac{1}{2}a_{m-1}, \ldots, \frac{3}{2} + \frac{1}{2}a_{m-1},\ -\frac{1}{2} - \frac{1}{2}a_{m-1})$.

4.3.1.1 *Subcase 1: m even.* Then $\mu + \rho = (\frac{3}{2} - m - \frac{1}{2}a_{m-1}, \ldots, -\frac{3}{2} - \frac{1}{2}a_{m-1},$ $-\frac{1}{2} - \frac{1}{2}a_{m-1},\ \frac{1}{2} - \frac{1}{2}a_{m-1})$. Though the Young diagram is different, as in the previous section the relevant entries are the first row: $m-1,\ m-2, \ldots, 2,\ 1$ and the first entry of the second row: $-a_{m-1}$. Thus $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_{m-1}\}$ and we have $(\mu + \rho, \alpha_i^\vee) = -1, \ldots, -1$ for $i = 1, \ldots, m-1$.

4.3.1.2 *Subcase 2: m is odd.* Here the first entry of the second row and last entry of the first row are interchanged so that $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_{m-2},\ \alpha_m\}$. Also $\mu + \rho = (\frac{3}{2} - m - \frac{1}{2}a_{m-1}, \ldots, -\frac{3}{2} - \frac{1}{2}a_{m-1},\ -\frac{1}{2} - \frac{1}{2}a_{m-1},\ -\frac{1}{2} + \frac{1}{2}a_{m-1})$ so that $(\mu + \rho, \alpha_i^\vee) = -1, \ldots, -1,\ -1$ for $i = 1, \ldots, m-2,\ m$.

4.3.2 Case 2: $Q = \mathrm{SU}(1, q)$, $q = m-1$ with $\lambda_a = -(m-1)\omega_1 + \omega_m$

In this case $\lambda = (\frac{3}{2} - m - \frac{1}{2}a_m,\ \frac{1}{2} + \frac{1}{2}a_m, \ldots, \frac{1}{2} + \frac{1}{2}a_m)$. Then
$\lambda + \rho = (\frac{1}{2} - \frac{1}{2}a_m,\ m - \frac{3}{2} + \frac{1}{2}a_m, \ldots, \frac{1}{2} + \frac{1}{2}a_m)$.

4.3.2.1 *Subcase 1: m is even.* Here we have a first row of $m-1,\ m-2, \ldots,$ $2,\ -a_m$ with 1 in the first entry of the second row. Thus

$\operatorname{supp}(w) = \{\alpha_1, \ldots, \alpha_{m-2},\ \alpha_m\}$. Since m is even we have
$\mu + \rho = (\frac{3}{2} - m - \frac{1}{2}a_m,\ \frac{5}{2} - m - \frac{1}{2}a_m, \ldots, -\frac{1}{2} - \frac{1}{2}a_m,\ -\frac{1}{2} + \frac{1}{2}a_m)$. Therefore
$(\mu + \rho, \alpha_i^\vee) = -1, \ldots, -1,\ -1$ for $i = 1, \ldots, m-2,\ m$.

4.3.2.2 *Subcase 2: m is odd.* Again the last entry of the first row and first entry of the second row are interchanged so that $\operatorname{supp}(w) = \{\alpha_1, \ldots, \alpha_{m-1}\}$. Moreover, $\mu + \rho = (\frac{3}{2} - m - \frac{1}{2}a_m,\ \frac{5}{2} - m - \frac{1}{2}a_m, \ldots, -\frac{1}{2} - \frac{1}{2}a_m,\ \frac{1}{2} - \frac{1}{2}a_m)$ and so
$(\mu + \rho, \alpha_i^\vee) = -1, \ldots, -1$ for $i = 1, \ldots, m-1$.

4.3.3 Case 3: $Q = \mathrm{SU}(1, q), q = m - 2$

In this case we have

$$\lambda = (-m - \frac{1}{2}a_{m-1} - \frac{1}{2}a_m,\ 1 + \frac{1}{2}a_{m-1} + \frac{1}{2}a_m,\ \ldots\ ,\ 1 + \frac{1}{2}a_{m-1} + \frac{1}{2}a_m,\ -\frac{1}{2}a_{m-1} + \frac{1}{2}a_m).$$

Therefore

$$\begin{aligned}\lambda + \rho = (&-1 - \frac{1}{2}a_{m-1} - \frac{1}{2}a_m,\ m - 1 + \frac{1}{2}a_{m-1} + \frac{1}{2}a_m,\ \ldots\ ,\ 2 + \frac{1}{2}a_{m-1} + \frac{1}{2}a_m,\\ &-\frac{1}{2}a_{m-1} + \frac{1}{2}a_m)\end{aligned}$$

giving a first row of $m-2,\ m-3, \ldots, 1,\ -1 - a_m$ with $-1 - a_{m-1}$ as the first entry of the second row. Hence $\operatorname{supp}(w) = \{\alpha_1, \ldots, \alpha_{m-2}\}$. By antidominance

$$\begin{aligned}\mu + \rho = (&1 - m - \frac{1}{2}a_{m-1},\ 2 - m - \frac{1}{2} - \frac{1}{2}a_m,\ \ldots\ ,\ -2 - \frac{1}{2}a_{m-1} - \frac{1}{2}a_m,\\ &-1 - \frac{1}{2}a_{m-1} - \frac{1}{2}a_m,\ \pm(-\frac{1}{2}a_{m-1} + \frac{1}{2}a_m))\end{aligned}$$

where the sign on the final entry depends on the parity of m. Thus we have
$(\mu + \rho, \alpha_i^\vee) = -1, \ldots, -1$ for $i = 1, \ldots, m-2$.

4.3.4 Case 4: $Q = \mathrm{SU}(1, p),\ 1 \leq p \leq m - 3$

Here by 2.4 we have

$$\lambda = (p+2-2m-a_{p+1}-\cdots-a_{m-2}-\frac{1}{2}a_{m-1}-\frac{1}{2}a_m,$$
$$\underbrace{1+a_{p+1}+\cdots+a_{m-2}+\frac{1}{2}a_{m-1}+\frac{1}{2}a_m,\ldots,1+a_{p+1}+\cdots+a_{m-2}+\frac{1}{2}a_{m-1}+\frac{1}{2}a_m,}_{p}$$
$$a_{p+2}+\cdots+a_{m-2}+\frac{1}{2}a_{m-1}+\frac{1}{2}a_m,\ldots,\frac{1}{2}a_{m-1}+\frac{1}{2}a_m,\ -\frac{1}{2}a_{m-1}+\frac{1}{2}a_m)$$

and so

$$\begin{aligned}\lambda+\rho = (&p+1-m-a_{p+1}-\cdots-a_{m-2}-\frac{1}{2}a_{m-1}-\frac{1}{2}a_m,\\ &m-1+a_{p+1}+\cdots+a_{m-2}+\frac{1}{2}a_{m-1}+\frac{1}{2}a_m,\ldots,\\ &m-p+a_{p+1}+\cdots+a_{m-2}+\frac{1}{2}a_{m-1}+\frac{1}{2}a_m,\\ &m-p-2+a_{p+2}+\cdots+a_{m-2}+\frac{1}{2}a_{m-1}+\frac{1}{2}a_m,\ldots,1+\frac{1}{2}a_{m-1}+\frac{1}{2}a_m,\\ &-\frac{1}{2}a_{m-1}+\frac{1}{2}a_m).\end{aligned}$$

Here we have

$$p,\ p-1,\ldots,1,\ -1-a_{p+1},\ldots,2+p-m-a_{p+1}-\cdots-a_{m-2},$$
$$1+p-m-a_{p+1}-\cdots-a_{m-2}-a_m$$

in the first row and $1+p-m-a_{p+1}-\cdots-a_{m-1}$ as the first entry of the second row. Thus $\mathrm{supp}(w)=\{\alpha_1,\ldots,\alpha_p\}$. Now

$$\begin{aligned}\mu+\rho = (&1-m-a_{p+1}-\cdots-a_{m-2}-\frac{1}{2}a_{m-1}-\frac{1}{2}a_m,\ \ldots\ ,\\ &p-m-a_{p+1}-\cdots-a_{m-2}-\frac{1}{2}a_{m-1}-\frac{1}{2}a_m,\\ &1+p-m-a_{p+1}-\cdots-a_{m-2}-\frac{1}{2}a_{m-1}-\frac{1}{2}a_m,\\ &2+p-m-a_{p+2}-\cdots-a_{m-2}-\frac{1}{2}a_{m-1}-\frac{1}{2}a_m,\ \ldots\ ,\\ &-1-\frac{1}{2}a_{m-1}-\frac{1}{2}a_m,\ \pm(-\frac{1}{2}a_{m-1}+\frac{1}{2}a_m)),\end{aligned}$$

where the sign on the last entry depends on the parity of m. Therefore $(\mu+\rho,\alpha_i^\vee)=-1,\ldots,-1$ for $i=1,\ldots,p$.

4.3.5 *Case 5:* $Q = \mathrm{SO}(2m-2,2)$ *with* $\lambda_a = -(m-2)\omega_1$

Here we have $\lambda + \rho = (1,\ m-2,\ m-3, \ldots, 1,\ 0)$ and so $\mu + \rho = (2-m,$ $3-m, \ldots, -2,\ -1,\ -1,\ 0)$. Now the first row is $m-1,\ m-2, \ldots, 1$ with 1 in the first box of the second row. Thus $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_m\}$ and we have $(\mu + \rho, \alpha_i^\vee) = -1, \ldots, -1,\ 0,\ -1,\ -1$ for $i = 1, \ldots, m$.

4.4 *Type E:* $\mathfrak{g}_{\mathbb{R}} = \mathfrak{e}_{VI}$

For Type E much of the work was already done by Enright and Hunziker in [8]. Namely, the highest weight λ is given in terms of the fundamental weights and the numbers $(\lambda + \rho, \alpha^\vee) \geq 0$ are arranged in the poset diagram. By inspection, w_{I} is the following map:

$\alpha_1 \mapsto -\alpha_1$

$\alpha_2 \mapsto -\alpha_5$

$\alpha_3 \mapsto -\alpha_3$

$\alpha_4 \mapsto -\alpha_4$

$\alpha_5 \mapsto -\alpha_2$

$\alpha_6 \mapsto \theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.$

Thus we can write w_{I} as a matrix in the Δ basis:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Writing λ as a column vector in the ω basis, we may evaluate $w_{\mathrm{I}}\lambda$ via $c\,(w_{\mathrm{I}})_\Delta\, c^{-1}\,(\lambda)_\omega$, where c is the Cartan matrix. Retrieving $w \in {}^{\mathrm{I}}W$ from the poset in [8], we then calculate $(\mu + \rho, \alpha^\vee)$ directly for each case in Table 2.3.

4.4.1 *Case 1:* $Q = \mathrm{SU}(1,1)$

Here the highest weight is given by

$$\lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + a_4\omega_4 + a_5\omega_5 + (-a_1 - 2a_2 - 2a_3 - 3a_4 - 2a_5 - 10)\omega_6,$$

where $a_2 \geq 1$ and $a_i \geq 0$ for $i \neq 2$. Then $\mathrm{supp}(w) = \{\alpha_6\}$ and $(\mu + \rho, \alpha_6^\vee) = -1$ as desired.

4.4.2 *Case 2:* $Q = \mathrm{SU}(1,2)$

Observe that $\lambda = a\omega_1 + b\omega_3 + c\omega_4 + d\omega_5 + (-a - 2b - 3c - 2d - 9)\omega_6$ with $c \geq 1$ and $a, b, d \geq 0$, and $\mathrm{supp}(w) = \{\alpha_5,\ \alpha_6\}$. Hence $(\mu + \rho, \alpha_i^\vee) = -1,\ -1$ for $i = 5,\ 6$.

4.4.3 *Case 3:* $Q = \mathrm{SU}(1,3)$

In this case $\lambda = a\omega_1 + b\omega_3 + c\omega_5 + (-a - 2b - 2a - 8)\omega_6$ where $b,\ c \geq 1$ and $a \geq 0$. Thus $\mathrm{supp}(w) = \{\alpha_4,\ \alpha_5,\ \alpha_6\}$ and $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1$ for $i = 4,\ 5,\ 6$.

4.4.4 *Case 4:* $Q = \mathrm{SU}(1,4)$ *with* $\lambda_a = \omega_3 - 9\omega_6$

Here $\lambda = a\omega_1 + b\omega_3 + (-a - 2b - 7)\omega_6$ with $b \geq 1$ and $a \geq 0$. In this case, $\mathrm{supp}(w)$ depends on b. If $b > 1$, then $\mathrm{supp}(w) = \{\alpha_2,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$ and $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ -1$ for $i = 2,\ 4,\ 5,\ 6$. If $b = 1$, then $\mathrm{supp}(w) = \{\alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$ and $(\mu + \rho, \alpha_i^\vee) = -1,\ 0,\ -1,\ -1,\ -1$ for $i = 2,\ 3,\ 4,\ 5,\ 6$.

4.4.5 *Case 5:* $Q = \mathrm{SU}(1,4)$ *with* $\lambda_a = \omega_1 + \omega_5 - 10\omega_6$

We have $\lambda = a\omega_1 + b\omega_5 + (-a - 2b - 7)\omega_6$ where $a,\ b \geq 1$. If $b > 1$, then $\mathrm{supp}(w) = \{\alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$ and $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ -1$ for $i = 3,\ 4,\ 5,\ 6$. If $b = 1$, then $\mathrm{supp}(w) = \{\alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$ and so $(\mu + \rho, \alpha_i^\vee) = 0,\ -1, -1,\ -1,\ -1$ for $i = 2,\ 3,\ 4,\ 5,\ 6$.

4.4.6 *Case 6:* $Q = \mathrm{SU}(1,5)$

Here $\lambda = a\omega_5 + (-2a - 6)\omega_6$ with $a \geq 1$. If $a > 2$, then $\mathrm{supp}(w) = \{\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$ and $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ -1,\ -1$ for $i = 1,\ 3,\ 4,\ 5,\ 6$.

If $a \leq 2$, then $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_6\}$. When $a = 2$ we have $(\mu + \rho, \alpha_i^\vee) = -1,\ 0,\ -1,\ -1,\ -1, -1$ for $i = 1, \ldots, 6$. When $a = 1$, $(\mu+\rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ 0,\ -1,\ -1$ for $i = 1, \ldots, 6$.

4.4.7 *Case 7: $Q = \mathrm{SO}(2,8)$ with $\lambda_a = \omega_1 - 5\omega_6$*

Note that $\lambda = a\omega_1 + (-a-7)\omega_6$, $a \geq 1$. Then $\mathrm{supp}(w) = \{\alpha_2, \ldots, \alpha_6\}$, and $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ 0,\ -1,\ -1$ for $i = 2, \ldots, 6$.

4.4.8 *Case 8: $Q = \mathrm{SO}(2,8)$ with $\lambda_a = \omega_1 - 8\omega_6$*

For this case, $\lambda = a\omega_1 + (-a-4)\omega_6$ where $a \geq 1$. If $a > 3$, then $\mathrm{supp}(w) = \{\alpha_2, \ldots, \alpha_6\}$ and $(\mu+\rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ -1,\ -1$ for $i = 2, \ldots, 6$. If $a \leq 3$, then $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_6\}$. When $a = 3$, $(\mu + \rho, \alpha_i^\vee) = 0,\ -1,\ -1,\ -1,\ -1,\ -1$ for $i = 1, \ldots, 6$. For $a = 2$, $(\mu+\rho, \alpha_i^\vee) = -1,\ -1,\ 0,\ -1,\ -1,\ -1$ for $i = 1, \ldots, 6$. And when $a = 1$, $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ 0,\ -1,\ -1$ for $i = 1, \ldots, 6$.

4.4.9 *Case 9: $Q = E_6$*

Here we have $\lambda = -3\omega_6$ and $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_6\}$. Thus $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ 0,\ -1,\ -1$ for $i = 1, \ldots, 6$.

4.5 *Type E: $\mathfrak{g}_\mathbb{R} = \mathfrak{e}_{VII}$*

We follow the same procedure as in 4.4. Here w_I does the following:

$\alpha_1 \mapsto -\alpha_6$

$\alpha_2 \mapsto -\alpha_2$

$\alpha_3 \mapsto -\alpha_5$

$\alpha_4 \mapsto -\alpha_4$

$\alpha_5 \mapsto -\alpha_3$

$\alpha_6 \mapsto -\alpha_1$

$\alpha_7 \mapsto \theta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$.

4.5.1 Case 1: $Q = \mathrm{SU}(1,1)$

By [8] we have

$$\lambda = a_1\omega_1 + \cdots + a_6\omega_6 + (-2a_1 - 2a_2 - 3a_3 - 4a_4 - 3a_5 - 2a_6 - 16)\omega_7,$$

where $a_1 \geq 1$ and $a_i \geq 0$ for $i \neq 1$. Then $\mathrm{supp}(w) = \{\alpha_7\}$ and $(\mu + \rho, \alpha_7^\vee) = -1$.

4.5.2 Case 2: $Q = \mathrm{SU}(1,2)$

In this case $\lambda = a_2\omega_2 + \cdots + a_6\omega_6 + (-2a_2 - 3a_3 - 4a_4 - 3a_5 - 2a_6 - 16)\omega_7$ with $a_3 \geq 1$ and $a_i \geq 0$ for $i \neq 3$. Now $\mathrm{supp}(w) = \{\alpha_6,\ \alpha_7\}$ and $(\mu+\rho, \alpha_i^\vee) = 0,\ -1$ for $i = 6,\ 7$.

4.5.3 Case 3: $Q = \mathrm{SU}(1,3)$

Here $\lambda = a\omega_2 + b\omega_4 + c\omega_5 + d\omega_6 + (-2a - 4b - 3c - 2d - 14)\omega_7$, $b \geq 1$ and $a, c, d \geq 0$. Then $\mathrm{supp}(w) = \{\alpha_5,\ \alpha_6,\ \alpha_7\}$ and we have $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1$ for $i = 5,\ 6,\ 7$.

4.5.4 Case 4: $Q = \mathrm{SU}(1,4)$

We have $\lambda = a\omega_2 + b\omega_5 + c\omega_6 + (-2a - 3b - 2c - 13)\omega_7$ where $a, b \geq 1$ and $c \geq 0$. It follows that $\mathrm{supp}(w) = \{\alpha_4, \ldots, \alpha_7\}$ and $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ -1$ for $i = 4, \ldots, 7$.

4.5.5 Case 5: $Q = \mathrm{SU}(1,5)$ *with* $\lambda_a = \omega_2 + \omega_6 - 16\omega_7$

Observe that $\lambda = a\omega_2 + b\omega_6 + (-2a - 2b - 12)\omega_7$, $a, b \geq 1$. If $a > 1$, then $\mathrm{supp}(w) = \{\alpha_3, \ldots, \alpha_7\}$ and $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ -1,\ -1$ for $i = 3, \ldots, 7$. If $a = 1$, then $\mathrm{supp}(w) = \{\alpha_2, \ldots, \alpha_7\}$ and $(\mu + \rho, \alpha_i^\vee) = 0,\ -1,\ -1,\ -1,\ -1,\ -1$ for $i = 2, \ldots, 7$.

4.5.6 Case 6: $Q = \mathrm{SU}(1,5)$ *with* $\lambda_a = \omega_5 - 15\omega_7$

Here we have $\lambda = a\omega_5 + b\omega_6 + (-3a - 2b - 12)\omega_7$ with $a \geq 1$ and $b \geq 0$. If $a > 1$, then $\mathrm{supp}(w) = \{\alpha_2,\ \alpha_4, \ldots, \alpha_7\}$ and $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ -1,\ -1$

for $i = 2,\ 4, \ldots, 7$. If $a = 1$, then $\mathrm{supp}(w) = \{\alpha_2, \ldots, \alpha_7\}$ and $(\mu + \rho, \alpha_i^\vee) = -1,\ 0,\ -1,\ -1,\ -1,\ -1$ for $i = 2, \ldots, 7$.

4.5.7 Case 7: $Q = \mathrm{SU}(1, 6)$

For this case $\lambda = a\omega_2 + (-2a - 11)\omega_7$ where $a \geq 1$. When $a > 2$ we have $\mathrm{supp}(w) = \{\alpha_1,\ \alpha_3, \ldots, \alpha_7\}$ and $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ -1,\ -1,\ -1$ for $i = 1,\ 3, \ldots, 7$. If $a \leq 2$, $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_7\}$. When $a = 2$, $(\mu + \rho, \alpha_i^\vee) = -1,\ 0,\ -1,\ -1,\ -1,\ -1,\ -1$ for $i = 1, \ldots, 7$. When $a = 1$, $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ 0,\ -1,\ -1,\ -1$ for $i = 1, \ldots, 7$.

4.5.8 Case 8: $Q = \mathrm{SO}(2, 10)$ *with* $\lambda_a = \omega_6 - 14\omega_7$

Note that $\lambda = a\omega_6 + (-2a - 12)\omega_7$, $a \geq 1$, and $\mathrm{supp}(w) = \{\alpha_2, \ldots, \alpha_7\}$. Then $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ 0,\ -1,\ -1,\ -1$ for $i = 2, \ldots, 7$.

4.5.9 Case 9: $Q = \mathrm{SO}(2, 10)$ *with* $\lambda_a = \omega_6 - 10\omega_7$

We have $\lambda = a\omega_6 + (-2a - 8)\omega_7$ with $a \geq 1$. If $a > 4$, then $\mathrm{supp}(w) = \{\alpha_2, \ldots, \alpha_7\}$ and $(\mu+\rho, \alpha_i^\vee) = -1, \ldots, -1$ for $i = 2, \ldots, 7$. If $a \leq 4$, then $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_7\}$. When $a = 4$, $(\mu + \rho, \alpha_i^\vee) = 0,\ -1, \ldots, -1$ for $i = 1, \ldots, 7$. When $a = 3$, $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ 0,\ -1, \ldots, -1$ for $i = 1, \ldots, 7$. When $a = 2$, $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ 0,\ -1,\ -1,\ -1$ for $i = 1, \ldots, 7$. And when $a = 1$, $(\mu + \rho, \alpha_i^\vee) = -1,\ 0,\ -1,\ -1,\ 0,\ -1,\ -1$ for $i = 1, \ldots, 7$.

4.5.10 Case 10: $Q = E_7$ *with* $l = 3$

In this case $\lambda = -8\omega_7$, $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_7\}$, and $(\mu+\rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ 0,\ -1,\ 0,\ -1$ for $i = 1, \ldots, 7$.

4.5.11 Case 11: $Q = E_7$ *with* $l = 2$

Here $\lambda = -4\omega_7$, $\mathrm{supp}(w) = \{\alpha_1, \ldots, \alpha_7\}$, and we have $(\mu + \rho, \alpha_i^\vee) = -1,\ -1,\ -1,\ 0,\ -1,\ -1,\ -1$ for $i = 1, \ldots, 7$. $\square$

CHAPTER FIVE

Conclusion

We have in fact demonstrated more than strictly necessary to prove the result, showing specifically for which $\alpha \in \mathrm{supp}(w)$ we have $(\mu + \rho, \alpha^\vee) = 0$ in each case. This is a benefit of writing the proof in this manner. It would of course be desirable to have a proof free of coordinates. Such a proof is not immediately apparent. This new property, namely the difference between the highest weights of consecutive parabolic Verma modules in the BGG resolution for $L(\lambda)$ being a sum of roots, can be defined for nonunitary modules. The original motivation behind this work was to find a sufficient condition for a Kostant module to be unitary. Unfortunately, there are Kostant modules with this new property that are not unitary. Thus our result implies there is a class of modules nestled between the unitary modules and Kostant modules which satsify this condition on the BGG resolution. For future work, it would be interesting to determine when the result is true for Types B and C. We would also like to write out the BGG resolutions for rank 2 explicitly.

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