ABSTRACT

Spectral Analysis of the Exceptional Laguerre and Jacobi Equations

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It was believed that Bochner's characterization of all sequences of polynomials $\{p_n\}_{n=0}^{\infty}$, with $\deg p_n = n \geq 0$, that are eigenfunctions of a second-order differential equation and are orthogonal with respect to a positive Borel measure on the real line having finite moments of all orders, was the only classification result of its kind. This result has been generalized, most notably in 2009 by Gómez-Ullate, Kamran, and Milson who characterized all sequences of polynomials $\{p_n\}_{n=1}^{\infty}$, with $\deg p_n = n \geq 1$, which have the remaining properties as those polynomial systems in Bochner's result. Up to a complex linear change of variable, the only such sequences are the *exceptional* X_1-Laguerre and the X_1-Jacobi polynomials. Additionally, their result was later extended to include exceptional X_m polynomial sequences; that is sequences which omit m polynomials from the standard sequence $\{p_n\}_{n=0}^{\infty}$, but still satisfy the remaining properties as the polynomial systems from Bochner's result. In fact, there are two existing families of generalized X_m-Laguerre polynomials, Type I and Type II, and we show the existence of a Type III family. The X_1 and generalized X_m families are excellent examples on which to apply the classical Glazman, Krein, Naimark theory as it pertains to the study of spectral analysis. The full spectral analysis for each of these families of exceptional polynomials as well as the analysis for extreme parameter choices is given in this dissertation.

Spectral Analysis of the Exceptional Laguerre and Jacobi Equations

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CHAPTER ONE

Introduction

The classical orthogonal polynomial families of Laguerre, Jacobi and Hermite developed as applications in applied mathematics, probability and statistics, and physics. In particular, their origins may be traced to Legendre's work on planetary motion. Interest in these polynomials has increased in recent years as a result of activity in approximation theory and numerical analysis [4]. These families of polynomials serve as the foundational examples on which orthogonal polynomial theory is built.

Due to the well-known "Bochner" classification, among the class of all orthogonal polynomials, only the Laguerre, Jacobi, and Hermite polynomials, have a full-algebraic sequence of polynomials which satisfy second-order differential equations and are orthogonal with respect to a positive-definite inner product of the form

$$(p, q) = \int_{\mathbb{R}} p\overline{q} d\mu.$$

However, these classical examples give rise to several interesting generalizations, specifically the *exceptional* orthogonal polynomial families of Laguerre and Jacobi, which we study here.

In 2009, Gómez-Ullate, Kamran, and Milson [19] (see also [18,20–24]) amended the classification problem answered by Bochner by requiring that the associated differential equation not permit a solution of degree zero. In other words, they introduced a differential equation with a complete set of eigenvectors of deg n, $n \in \mathbb{N}$. Formally, they characterized all polynomial sequences $\{p_n\}_{n=1}^{\infty}$, with $\deg p_n = n \geq 1$, which satisfy the following conditions:

(i) there exists a second-order differential expression

$$\ell[y](x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x),$$

and a sequence of complex numbers $\{\lambda_n\}_{n=1}^{\infty}$ such that $y = p_n(x)$ is a solution of

$$\ell[y](x) = \lambda_n y(x) \quad (n \in \mathbb{N});$$

where each coefficient $a_i(x)$ $(i = 0, 1, 2)$ is a function of the independent variable x and does not depend on the degree of the polynomial eigenfunctions;

(ii) if c is any non-zero constant, $y(x) \equiv c$ is *not* a solution of $\ell[y](x) = \lambda y(x)$ for any $\lambda \in \mathbb{C}$;

(iii) there exists an interval I and a Lebesgue measurable function $w(x)$ $(x \in I)$ such that

$$\int_I p_n(x)p_m w(x)dx = K_n\delta_{n,m},$$

where $K_n > 0$ for each $n \in \mathbb{N}$;

(iv) all moments $\{\mu_n\}_{n=0}^{\infty}$ of w, defined by

$$\mu_n = \int_I x^n w(x)dx \quad (n = 0, 1, 2, \ldots),$$

exist and are finite.

Up to a complex linear change of variable, the authors in [19] show that the only solutions to this classification problem are the *exceptional* X_1-Laguerre and the X_1-Jacobi polynomials. In particular, we say that the X_1-Laguerre and X_1-Jacobi polynomial sequences are the only sequences of codimension one whose polynomial subspace is preserved under their respective second-order differential operators.

It is natural to follow the study of the X_1 exceptional polynomials by looking for exceptional polynomial systems having codimension $m \geq 2$. As in the codimension one case, the X_m exceptional polynomials are solutions to a second-order differential equation, are orthogonal with respect to a positive weight function, and the moments of all orders exist and are finite. However, it is not necessarily the case that the first m polynomials (that is, polynomials of degrees $\{0, 1, 2, \cdots, m-1\}$) must be removed to produce exceptional polynomial systems of codimension m. As m increases, we have more flexibility in which polynomials we remove from the sequence. Thus, the restrictions of the Bochner classification are further generalized.

Even though the authors in [19] introduce the notion of exceptional polynomials via Sturm-Liouville theory, the path they followed to their discovery was motivated by their interest in quantum mechanics, specifically with their intent to extend exactly solvable and quasi-exactly solvable potentials beyond the Lie algebraic setting via the Darboux transformation. Their result was the first classification result for which the subspace spanned by a specific polynomial system was preserved under its associated operator. Gómez-Ullate, Kamran, and Milson [22] were also the first to formalize the notation and definitions which provide the framework for studying exceptional polynomials.

Two decades prior to their work, however, these exceptional polynomials were being observed in the area of quantum mechanics; in particular, examples of Hermite-like polynomial systems with positive codimension had been studied in the context of supersymmetric quantum mechanics as rational modifications of the harmonic oscillator [8, 10]. Recently, more activity has been seen in the area of exceptional polynomial sequences. The impetus that led to the work of Gómez-Ullate, Kamran, and Milson in [19] came from an earlier paper [29] of Kamran, Milson, and Olver on evolution equations reducible to finite-dimensional dynamical systems. Although they set out to construct potentials that would be solvable by polynomials which fall outside the realm of the classical theory of orthogonal polynomials, it is important to note as well that the work in [19] was not originally inspired by orthogonal polynomials. To further note, their work was inspired by the paper of Post and Turbiner [40] who formulated a *generalized Bochner problem* of classifying the linear differential operators in one variable leaving invariant a given vector space of polynomials. Reformulation within the framework of one-dimensional quantum mechanics and shape invariant potentials followed [19] by various other authors; for example, see [39] and [41].

Chapter 2 of this thesis contains background material on orthogonal polynomials and associated classification results. In conjunction with introducing the well-known properties of the X_1-Laguerre and X_1-Jacobi families and the more general Type I and II X_m-Laguerre families, we introduce a new Type III X_m-Laguerre polynomial system in Chapter 3. It is the case that the exceptional Laguerre and Jacobi polynomials are excellent examples on which to apply the classical Glazman, Krein, Naimark theory as it pertains to the study of spectral analysis. Chapter 4 contains a summary of the theory of extensions of symmetric operators. The full spectral analysis of the exceptional polynomials follows in Chapter 5. The spectral analysis of the X_1-Laguerre polynomials has been published in [2]. Manuscripts containing the spectral analysis of the X_1-Jacobi polynomials and the Type I, II, and III X_m-Laguerre polynomials are in progress ([36] and [35] respectively). Chapter 6 introduces the notion of Sobolev orthogonality and left-definite theory as introduced by Littlejohn and Wellman, which is applicable for studying extreme parameter choices. Chapter 7 contains the spectral analysis for the X_1-Laguerre differential expression $\ell_1^{\alpha}[\cdot]$ for the extreme parameter choice of $\alpha = 0$. This work has been published in conjunction with the spectral analysis of the X_1-Laguerre polynomials [2]. A discussion of the X_1-Jacobi differential expression $\ell_1^{(\alpha,\beta)}[\cdot]$ for the extreme parameter choice of $\alpha = 0$ follows in Chapter 8 and is also being prepared for submission [36].

CHAPTER TWO

Orthogonal Polynomials

2.1 Introduction

All polynomials in this text are assumed to be real polynomials, that is, having real number coefficients, and to be of the real variable x. By a *polynomial system*, we mean a sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with $\deg(p_n) = n$, $n \geq 0$. Note that a polynomial system forms a basis for $\mathcal{P} := span\{1, x, x^2, \ldots\}$, the space of all real polynomials. If, for each $n \in \mathbb{N}_0$, p_n is a monic polynomial, then $\{p_n\}_{n=0}^{\infty}$ is called a *monic polynomial system.*

Let $I = (a, b) \subseteq \mathbb{R}$ be an open interval with $-\infty \leq a < b \leq \infty$. Furthermore, let $\mu(x) : I \to \mathbb{R}$ be a function of bounded variation such that the *nth*-moments, where the *nth-moment* $(n \in \mathbb{N}_0)$ is defined as

$$\int_I x^n d\mu(x)\,,$$

exist and are finite for all $n \in \mathbb{N}_0$.

A polynomial system $\{p_n\}_{n=0}^{\infty}$ is said to be *orthogonal* on I with respect to the weight function $w(x)$ if

$$\int_I p_n(x)p_m(x)dw(x) = K_n\delta_{nm} \quad (m \text{ and } n \geq 0)$$

where K_n are non-zero real constants and

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

is the Kronecker delta function. The function $w(x)$ is referred to as the *weight function* associated with the polynomial system. If each constant K_n is positive $(n \in \mathbb{N}_0)$, then the polynomial system $\{p_n(x)\}_{n=0}^{\infty}$ is a *positive definite* orthogonal polynomial system.

The well-known orthogonal polynomial systems satisfy fixed linear differential equations of spectral type; that is,

$$\ell_N[y](x) = \sum_{i=1}^{N} a_i(x) y^{(i)}(x) = \sum_{i=1}^{N} \sum_{j=0}^{i} a_{ij} x^j y^{(i)}(x) = \lambda_n y(x) \tag{2.1}$$

where a_{ij} are real constants and $\lambda_n = a_{11}n + \cdots + a_{NN}n(n-1)\cdots(n-N+1)$. We will see later that the particular orthogonal polynomial systems in which we are interested do not satisfy linear differential equations of this form.

We call an orthogonal polynomial system $\{p_n\}_{n=0}^{\infty}$ a *Bochner-Krall orthogonal polynomial system* of order N (≥ 1) if $\{p_n\}_{n=0}^{\infty}$ satisfies a differential equation of the form

$$\ell_N[p_n](x) = \lambda_n p_n(x)\,,$$

where ℓ_N has order N and is of the form (2.1), but does not satisfy any differential equation of order $< N$. It is then the case that we write $\{p_n\}_{n=0}^{\infty} \in BKS(N)$.

For $N = 2$, (2.1) simplifies to

$$\ell[y](x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = \lambda_n y(x) \tag{2.2}$$

where $a_2(x)$, $a_1(x)$, and $a_0(x)$ are real-valued functions, independent of n, and λ_n is a real constant which depends only upon n. If the orthogonal polynomial system $\{p_n(x)\}_{n=0}^{\infty} \in BKS(2)$, then the system is considered *classical*.

The second-order differential equation (2.2) is said to be in *Sturm-Liouville* or *symmetric form* if it can be rewritten as

$$-\left(p(x)y'(x)\right)' + q(x)y(x) = \lambda w(x)y\,, \tag{2.3}$$

where $p(x) > 0$, $q(x)$ and $w(x) > 0$ are continuous on the open interval I° and $p(x)$ has a continuous derivative. By multiplying both sides of (2.2) by an appropriate integrating factor,

$$\mu(x) = \frac{1}{a_2(x)} e^{-\int a_1(x)/a_2(x)\,dx}\,,$$

(2.2) may be written in symmetric form (2.3) with

$$p(x) = e^{-\int a_1(x)/a_2(x)\,dx}, \quad q(x) = \frac{a_0(x)}{a_2(x)} e^{-\int a_1(x)/a_2(x)\,dx} \quad \text{and} \quad w(x) = \mu(x).$$

In most cases, the integrating factor is the weight function $w(x)$.

Given a second-order differential equation in symmetric form, a value for λ, and additional restrictions called *boundary conditions*, the task of showing that there exists a non-trivial solution $y(x)$ is referred to as a *Sturm-Liouville problem*. We will discuss the boundary conditions on $y(x)$ in greater detail in Chapter 4. For the time being, the reader may ignore any restrictions imposed by boundary conditions.

2.2 Classification Results

This provides background material on the subject of orthogonal polynomial sequences with attention given to the classification of sequences which are solutions of the Bochner–Krall Problem.

2.2.1 Classification of the $BKS(2)$

The classification of solutions to the $BKS(2)$ problem is generally attributed to Bochner [3] in 1929 and Lesky [34] in 1962. Although this result is referred to as the “Bochner” classification, in recent years, it has become clear that the question was addressed earlier by Routh [42] in 1885; see [28, p. 509].

Bochner classified all orthogonal polynomial solutions $y(x)$ to the second-order equation (2.2). He showed that if there exist solutions to (2.2) of degree n, $n = 0$, 1, or 2, then $a_2(x)$, $a_1(x)$, and $a_0(x)$ are at most of degree 2, 1 and 0 respectively. By studying the possible location of the roots of $a_2(x)$, it was shown that up to a complex change of variable, the only solutions to (2.2) are the following:

(1) Laguerre polynomials: $\{L_n^\alpha\}_{n=0}^\infty$ with $-\alpha \notin \mathbb{N}$

(2) Jacobi polynomials: $\{P_n^{(\alpha,\beta)}\}_{n=0}^\infty$ with $-\alpha, -\beta, -(\alpha+\beta+1) \notin \mathbb{N}$

(3) Hermite polynomials: $\{H_n\}_{n=0}^{\infty}$

(4) Bessel polynomials: $\{y_n^a\}_{n=0}^{\infty}$ with $-(a+1) \notin \mathbb{N}$ (there is clear evidence that Bochner knew of the existence of the Bessel orthogonal polynomial sequence, but these polynomials were not officially discovered until 1948).

It has also been shown that the classical orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ of Laguerre, Jacobi and Hermite can be characterized, up to a complex linear change of variable, in several ways, not just as solutions to the $BKS(2)$ problem or as a result of studying the placement of the roots of $a_2(x)$. Indeed, they are the only positive-definite orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ satisfying the following four equivalent conditions:

(a) the sequence of first derivatives $\{p_n'(x)\}_{n=1}^{\infty}$ is also orthogonal with respect to a positive measure [26];

(b) the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a Rodrigues formula

$$p_n(x) = K_n^{-1}(w(x))^{-1}\frac{d^n}{dx^n}\left(\rho^n(x)w(x)\right) \quad (n \in \mathbb{N}),$$

where $K_n > 0$, $w > 0$ on some interval $I = (a, b)$, and where ρ is a polynomial of degree ≤ 2 [9, 12, 44];

(c) the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a differential recursion relation of the form

$$\pi(x)p_n'(x) = (\alpha_n x + \beta_n)p_n(x) + \gamma_n p_{n-1}(x) \quad (n \in \mathbb{N},\ p_{-1}(x) = 0)$$

where $\pi(x)$ is some polynomial and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ are real number sequences [15];

(d) there exists a second-order differential expression

$$m[y](x) = b_2(x)y''(x) + b_1(x)y'(x) + b_0(x)y(x),$$

and a sequence of complex numbers $\{\mu_n\}_{n=0}^{\infty}$ such that $y = p_n(x)$ is a solution of the differential equation

$$m[y](x) = \mu_n y(x) \quad (n \in \mathbb{N}_0).$$

We conclude this section by presenting some properties of the Laguerre and Jacobi polynomials that are relevant to the research within the dissertation. For further properties of these polynomials and the Hermite and Bessel polynomials, the reader is referred to Chihara [4], Ismail [28], and Szegö [43].

2.2.1.1 *Laguerre Polynomials*

Let $\alpha > -1$ and consider the classical Laguerre expression given by

$$\ell_\alpha[y](x) := xy''(x) + (\alpha + 1 - x)\, y'(x)\,. \tag{2.4}$$

The classical Laguerre polynomials [4] can be defined by the Rodrigues formula

$$L_n^\alpha(x) := (n!)^{-1} x^{-\alpha} e^x \frac{d^n}{dx^n}\left[x^{n+\alpha} e^{-x}\right]\,.$$

With the use of the Leibniz formula, we obtain the explicit formula

$$L_n^\alpha(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}\,.$$

Consider the weight function

$$w_\alpha(x) = x^\alpha e^{-x}\,, \tag{2.5}$$

and the associated space $L^2((0,\infty), w_\alpha(x))$. The classical Laguerre polynomials are orthogonal with respect to this weight:

$$\int_0^\infty L_m^\alpha(x) L_n^\alpha(x) w_\alpha(x)\, dx = c(\alpha, n)\delta_{mn}$$

where $c(\alpha, n) = (-1)^n \Gamma(n + \alpha + 1)$ and $\Gamma(x)$ is the gamma function. Moreover, for each $n \in \mathbb{N}_0$, they satisfy the eigenvalue equation

$$\ell_\alpha[L_n^\alpha] = \lambda_n L_n^\alpha \tag{2.6}$$

for $\lambda_n = n$. For their three term recurrence relation, see [4], and for further details about the classical Laguerre polynomials, also see [43].

2.2.1.2 *Jacobi Polynomials*

Let $\alpha, \beta > -1$ and consider the classical Jacobi expression given by

$$\ell_{\alpha,\beta}[y](x) := (x^2-1)y''(x) + ((\alpha+\beta+2)x + \alpha - \beta)\, y'(x). \tag{2.7}$$

The classical Jacobi polynomials [4] can be defined by the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) := \frac{(-2)^n}{n!(1-x)^\alpha(1+x)^\beta}\,\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right].$$

With use of the Leibniz formula we obtain the explicit formula

$$P_n^{(\alpha,\beta)}(x) = 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k}.$$

Consider the weight function

$$w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta, \tag{2.8}$$

and the associated space $L^2((0,\infty), w_\alpha(x))$. The classical Jacobi polynomials are orthogonal with respect to this weight:

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)w_{\alpha,\beta}(x)dx = c(\alpha,\beta,n)\delta_{mn},$$

where

$$c(\alpha,\beta,n) = \frac{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}$$

and $\Gamma(x)$ is the gamma function. Furthermore, for each $n \in \mathbb{N}_0$ they satisfy the eigenvalue equation

$$\ell_{\alpha,\beta}[P_n^{(\alpha,\beta)}] = \lambda_n P_n^{(\alpha,\beta)} \tag{2.9}$$

for $\lambda_n = n(n+\alpha+\beta+1)$. For their three-term recurrence relation, see [4], and for further details about the classical Jacobi polynomials, also see [43].

2.2.2 Other Classification Results

Extensions of Bochner's characterization to a *real* linear change of variable was considered by Kwon and Littlejohn [31] in 1997 and to orthogonality with respect to a Sobolev inner product by Kwon and Littlejohn [32] in 1998. Each of these extensions are a modification of Bochner's original classification result and although they are not directly related to the exceptional polynomials, the techniques used for understanding their properties and spectrum are applicable to the exceptional case considered in this dissertation.

In [31], the authors obtain new necessary and sufficient conditions on the coefficients $a_1(x)$ and $a_2(x)$ so that (2.2) has orthogonal polynomial solutions. Using this result, it is shown that if we allow for a real linear change of variable then there are exactly six orthogonal polynomial families which are solutions. Four of the six are those classified by Bochner—the Jacobi, Laguerre, Hermite and Bessel. We gain two additional sets

(1) the twisted Hermite polynomials $\{H_n\}_{n=0}^{\infty}$

(2) the twisted Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ with $-(\alpha+\beta+1)\notin\mathbb{N}$ and $\alpha=\overline{\beta}$.

By considering the $BKS(2)$ problem where the measure has been generalized to include two functions via a Sobolev inner product of the form

$$\Phi(p,q) := \sum_{j=0}^{1}\int_{\mathbb{R}} p^{(j)}q^{(j)}d\mu_j\,,$$

where μ_j $(j=0,1)$ are (possibly signed) Borel measures and $\mu_1\neq 0$, the authors [32] produce a classification result which shows that up to a complex linear change of variable, the only polynomial sequences which are a $BKS(2)$ with a Sobolev inner product are

(1) Laguerre polynomials $\{L_n^{\alpha}\}_{n=0}^{\infty}$ with $-\alpha\notin\mathbb{N}$

(2) Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ with $-\alpha, -\beta, -(\alpha+\beta+1) \notin \mathbb{N}$

(3) Hermite polynomials $\{H_n\}_{n=0}^{\infty}$

(4) Bessel polynomials $\{y_n^a\}_{n=0}^{\infty}$ with $-(a+1) \notin \mathbb{N}$

(5) Jacobi polynomials $\{P_n^{(-1,-1)}\}_{n=0}^{\infty}$

(6) Jacobi polynomials $\{P_n^{(\alpha,-1)}\}_{n=0}^{\infty}$ with $-\alpha \notin \mathbb{N}$

(7) Jacobi polynomials $\{P_n^{(-1,\beta)}\}_{n=0}^{\infty}$ with $-\beta \notin \mathbb{N}$

(8) Laguerre polynomials $\{L_n^{-1}\}_{n=0}^{\infty}$.

These solutions are variations of the classical orthogonal families given by Bochner; however, by changing the Sobolev bilinear form and thus the definition of orthogonality, we are allowed some parameter choices which were previously excluded in the Bochner classification case.

Note that in the classical Bochner classification and these particular extensions, the sequence of orthogonal polynomial solutions is a full algebraic sequence—that is, we have a polynomial solution of degree n for each $n \in \mathbb{N}_0$. By modifying the parameter N in a Bochner-Krall system and/or changing the definition of orthogonality, we change only the order of our differential expression and/or the Sobolev bilinear form. The degrees of the polynomials in Bochner-Krall polynomial systems remain the same, $n \in \mathbb{N}_0$. This will not be the case when we introduce *exceptional* orthogonal polynomials.

CHAPTER THREE

Exceptional Orthogonal Polynomials

3.1 Framework for Exceptional Orthogonal Polynomials

In the following exposition, we precisely introduce what it means to be an exceptional orthogonal polynomial system using the definitions and framework developed in [16–24].

Consider the polynomial system $P = \{p_{n_i}(x)\}_{i=0}^{\infty}$ with $\deg(p_{n_i}) = n_i$, where $n_i \in G$ and G is an increasing sequence of non-negative integers,

$$G = \{n_0, n_1, n_2, \ldots\} \subset \mathbb{N}_0 \,.$$

In addition, let $m := |\mathbb{N}_0 \setminus G| \neq 0$ and be finite. P will span a subset of the polynomial space $\mathcal{P} = span\{1, x, x^2, \ldots\}$ and $\dim(\mathcal{P} \setminus P) = m$. We say that P has *codimension* m. By construction, the polynomial system P is *degree-regular*, that is $\deg p_{n_i} < \deg p_{n_{i+1}}$ for all $i \in \mathbb{N}_0$. If the polynomials of the system share no common factor of degree ≥ 1, the system is said to be *primitive*.

For $i \geq 1$, let $U_i = span\{p_{n_0}, p_{n_1}, \ldots p_{n_{i-1}}\}$ be the span of the first i polynomials of P. Let $m_i = n_{i-1}+1-i$ be the codimension of U_i in $\mathcal{P}_{n_{i-1}} = span\{1, x, \ldots, x^{n_{i-1}}\}$. It follows that $U_1 \subset U_2 \subset U_3 \subset \cdots$ and that $m_i \leq m$ for all $i \in \mathbb{N}$. Additionally, because P is degree-regular, $m_i \leq m_{i+1}$. We say that the codimension sequence $\{m_i\}_{i=0}^{\infty}$ is *semi-stable* when P is degree-regular and m is finite (in our case, this will always be true). When the codimension sequence $\{m_i\}_{i=0}^{\infty}$ is constant with $m_i = m$ for all $i \in \mathbb{N}$, then the sequence is *stable*.

Furthermore, suppose P is a primitive system, orthogonal with respect to a weight function w on an interval I and that each p_{n_i} ($i \in \mathbb{N}_0$) in P satisfies a fixed differential equation. P is *preserved* (i.e. is an invariant subspace) under a fixed second-order differential expression $\ell[\cdot]$ if

$$\ell[p_n](x) \in \mathit{span}\, P$$

for all p_n $(n \in G)$.

If the primitive semi-stable orthogonal polynomial system P is the sequence of smallest codimension which is preserved under the second-order differential expression $\ell[\cdot]$, (2.2), and the corresponding sequence $\{\ell[p_{n_i}]\}_{i=0}^{\infty}$ is also semi-stable, then we call the polynomial system P an *exceptional orthogonal polynomial system* or abbreviated, X_m-exceptional polynomial system.

Note that if $m = |\mathbb{N}_0 \setminus G| = 0$, then $G \equiv \mathbb{N}_0$ and the exceptional polynomial system $\{p_{n_i}\}_{n_i \in G}$ has codimension zero. It will then be the case that $\{p_{n_i}\}_{n_i \in G}$ must be one of the classical orthogonal polynomial systems classified by Bochner.

Furthermore, [22] shows the following proposition using Cramer's Rule.

Proposition 3.1. *Let y_1, y_2, and y_3 be three linearly independent solutions to the second-order differential equation*

$$\ell[y_i](x) = a_2(x)y_i''(x) + a_1(x)y_i'(x) + a_0(x)y_i(x) = g_i(x) \quad \textit{for } i = 1, 2, 3,$$

where each g_i is a polynomial. Then $a_2(x)$, $a_1(x)$, and $a_0(x)$ are rational functions whose denominator contains the Wronskian

$$W(y_1, y_2, y_3) = \det \begin{vmatrix} y_0'' & y_0' & y_0 \\ y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \end{vmatrix}.$$

Let us turn to some examples [22] which illustrate the definitions above:

Example 3.1. Consider the polynomial system $P = \{1, x^2, x^3, \ldots\}$ associated with the non-polynomial expression

$$\ell[y] = y'' - \frac{2y'}{x}.$$

The expression has semi-stable codimension with $m_1 = 0$ and $m_i = 1$ for $i \geq 2$. Applying $\ell[\cdot]$ to each of the polynomials in P, we see that

$$\{\ell[1], \ell[x^2], \ell[x^3], \ell[x^4], \ldots, \ell[x^n], \ldots\} = \{0, -3, 0, 4x^2, \ldots, n(n-3)x^{n-2}, \ldots\}\,.$$

The corresponding codimension sequence is semi-stable, but not stable.

Example 3.2. In comparison, the polynomial system $P = \{x+1, x^2, x^3, \ldots\}$ has stable codimension equal to one. Additionally, this sequence is exceptional since it is preserved by the non-polynomial expression

$$\ell[y] = y'' - 2\left(1 + \frac{1}{x}\right)y' + \left(\frac{2}{x}\right)y\,,$$

and this expression does not preserve a flag of smaller codimension, namely the system $\{1, x, x^2, \ldots\}$.

Example 3.3. Let $h_k(x)$ be the degree k Hermite polynomial. The polynomial system $\{h_1, h_2, h_3, \ldots\}$ is not exceptional. The polynomial system is preserved under the expression

$$\ell[y] = y'' + xy'\,,$$

however, this expression also preserves the system $\{1, x, x^2, \ldots\}$ which has codimension zero. Hence, $\ell[\cdot]$ preserves a polynomial system codimension smaller than one.

Example 3.4. For $k = 2, 3, 4, \ldots$, let

$$y_{2k-1} = x^{2k-1} - (2k-1)x$$
$$y_{2k} = x^{2k} - kx^2\,.$$

The polynomial system of codimension two defined by $\{1, y_3, y_4, y_5, \ldots\}$. The associated codimension sequence $\{0, 2, 2, \ldots\}$ is semi-stable but not stable. This flag is preserved by the expressions [17]:

$$\ell_1[y] = y'' + x\left(1 - \frac{4}{x^2-1}\right)y' \,,$$
$$\ell_2[y] = xy'' - 2\left(1 + \frac{2}{x^2-1}\right)y' \quad \text{and}$$
$$\ell_3[y] = (x^2-1)y'' - 2xy' \,.$$

The system is exceptional since $\ell_1[\cdot]$ and $\ell_2[x]$ do not preserve the polynomial system of codimension zero, $\{1, x, x^2, \ldots\}$, and cannot preserve a system of codimension one since they each have more than one pole (see [22], Lemma 3.2).

3.2 X_1-Exceptional Polynomials

We now provide some properties of the X_1-Laguerre and X_1-Jacobi polynomials which are relevant to our study. Each of these codimension one polynomial systems arise from choosing $G = \{1, 2, 3, \ldots\}$. For further reading, see [19].

3.2.1 X_1-Laguerre Polynomials

We summarize some of the main properties of the X_1-Laguerre polynomials $\{L^{\alpha}_{1,n}\}_{n=1}^{\infty}$, as discussed in [19]. Unless otherwise indicated, we assume that the parameter $\alpha > 0$. The X_1-Laguerre polynomial $y = L^{\alpha}_{1,n}(x)$ $(n \in \mathbb{N})$ satisfies the second-order differential equation

$$\ell_1^{\alpha}[y](x) = \lambda_n y(x) \quad (0 < x < \infty), \tag{3.1}$$

where

$$\ell_1^{\alpha}[y](x) := -xy''(x) + \frac{(x-\alpha)(x+\alpha+1)}{(x+\alpha)}y'(x) - \frac{(x-\alpha)}{(x+\alpha)}y(x) \tag{3.2}$$

and

$$\lambda_n = n - 1 \quad (n \in \mathbb{N}). \tag{3.3}$$

Remark 3.1. Recall Proposition 3.1 which states that in the case of exceptional polynomials, at least one of the coefficients $a_1(x)$ and $a_0(x)$ in the second-order differential expression (2.2) are not polynomials, but rather rational functions. This variation is only found in the exceptional case—in the classical setting, all coefficients must be polynomial. It is remarkable that for an infinite number of polynomial solutions the denominator terms cancel, simplifying the left-hand side of (2.2) to a polynomial. In fact, the authors in [24] show that in order for a differential expression to be exceptional, at least one coefficient of a lower-order term is necessarily a non-polynomial function.

The polynomials $\{L_{1,n}^{\alpha}\}_{n=1}^{\infty}$are orthogonal on $(0,\infty)$ with respect to the weight function

$$w_1^{\alpha}(x) = \frac{x^{\alpha}e^{-x}}{(x+\alpha)^2} \quad (x \in (0,\infty)). \tag{3.4}$$

Note that the denominator of the weight function equals the square of the classical Laguerre polynomial $L_1^{\alpha-1}(-x)$.

The X_1-Laguerre polynomials form a complete orthogonal set in the Hilbert-Lebesgue space $L^2((0,\infty);w_1^{\alpha})$, defined by

$$L^2((0,\infty);w_1^{\alpha}) := \left\{ f : (0,\infty) \to \mathbb{C} \mid f \text{ is measurable and } \int_0^{\infty} |f|^2 \, w_1^{\alpha} < \infty \right\}.$$

More specifically, with

$$\|f\|_{\alpha} := \left(\int_0^{\infty} |f(x)|^2 \, w_1^{\alpha}(x)dx \right)^{1/2},$$

it is the case that

$$\left\| L_{1,n}^{\alpha} \right\|_{\alpha}^2 = \left(\frac{\alpha+n}{\alpha+n-1} \right) \frac{\Gamma(\alpha+n)}{(n-1)!} \quad (n \in \mathbb{N}).$$

The fact that $\{L_{1,n}^{\alpha}\}_{n=1}^{\infty}$ forms a complete set $L^2((0,\infty);w_1^{\alpha})$ is rather surprising since $f(x) = 1$ is in $L^2((0,\infty);w_1^{\alpha})$ but not in the span of $\{L_{1,n}^{\alpha}\}_{n=1}^{\infty}$. For the proof of completeness, see [19, Section 3].

The X_1-Laguerre polynomials can be written in terms of the classical Laguerre polynomials $\{L_{1,n}^{\alpha}\}_{n=0}^{\infty}$; specifically,

$$L_{1,n}^{\alpha}(x) = -(x+\alpha+1)L_{n-1}^{\alpha}(x) + L_{n-2}^{\alpha}(x) \quad (n \in \mathbb{N}),$$

where $L^{\alpha}_{-1}(x) := 0$. Moreover, the X_1-Laguerre polynomials satisfy a three-term recurrence relation:

$$\begin{aligned} 0 &= (n+1)\left((x+\alpha)^2(n+\alpha)-\alpha\right) L^{\alpha}_{1,n+2}(x) \\ &\quad + (n+\alpha)\left((x+\alpha)^2(x-2n-\alpha-1)+2\alpha\right) L^{\alpha}_{1,n+1}(x) \\ &\quad + (n+\alpha-1)\left((x+\alpha)^2(n+\alpha+1)-\alpha)\right) L^{\alpha}_{1,n}(x)\,, \end{aligned}$$

where $L^{\alpha}_{-1}(x) = L^{\alpha}_{0} := 0$. However in stark contrast with the classical Laguerre polynomials $\{L^{\alpha}_{n}\}_{n=0}^{\infty}$, whose roots are all contained in the interval $(0,\infty)$ when $\alpha > -1$, the X_1-Laguerre polynomials $\{L^{\alpha}_{1,n}\}_{n=1}^{\infty}$ have $n-1$ roots in $(0,\infty)$ and one zero in the interval $(-\infty, -\alpha)$ whenever $\alpha > 0$ [25].

3.2.2 X_1-Jacobi Polynomials

Let us summarize some properties of the X_1-Jacobi polynomials $\{P^{(\alpha,\beta)}_{1,n}\}_{n=0}^{\infty}$, as discussed in [19]. Consider the parameters α and β, which are assumed to satisfy the following properties (unless otherwise indicated):

$$\alpha, \beta \in (-1,\infty), \quad \alpha \neq \beta, \quad \text{and} \quad \operatorname{sgn}(\alpha) = \operatorname{sgn}(\beta). \tag{3.5}$$

In the literature, it is commonly assumed that $\operatorname{sgn}(0) = 0$; this, in conjunction with the restrictions on α and β require that neither $\alpha = 0$ nor $\beta = 0$. Later, in Chapter 8, we do consider the extreme cases when the restrictions on α and β are loosened so that $\alpha = 0$ or $\beta = 0$ is permitted. We define parameters $a, b, c \in \mathbb{R}$ by:

$$a := \frac{1}{2}(\beta-\alpha)\,, \quad b := \frac{\beta+\alpha}{\beta-\alpha}\,, \quad \text{and} \quad c := b + \frac{1}{a}\,.$$

The assumptions (3.5) imply $|b| > 1$. Indeed, suppose $|b| \leq 1$. Then either $\alpha, \beta \in (-1,0)$ or $\alpha, \beta > 0$. In the first case, $|b| = \frac{|\beta+\alpha|}{|\beta-\alpha|} \leq 1$. That is, $|\beta+\alpha| \leq |\beta-\alpha|$, which is a contradiction. The argument for $\alpha, \beta > 0$ follows similarly.

Define the weight function

$$w_1^{\alpha,\beta}(x) := \frac{(1-x)^{\alpha}(1+x)^{\beta}}{(x-b)^2} \quad (x \in (-1,1)). \tag{3.6}$$

Since $|b| > 1$, the weight function $w_1^{\alpha,\beta}$ is bounded in and a.e. positive on $[-1, 1]$. We define the norm

$$\|f\|_{\alpha,\beta} := \left(\int_{-1}^{1} |f|^2 \, w_1^{\alpha,\beta} \right)^{1/2}$$

and consider the Hilbert-Lebesgue space

$$L^2((-1,1); w_1^{\alpha,\beta}) := \left\{ f : (-1,1) \to \mathbb{C} \mid f \text{ is measurable and } \|f\|_{\alpha,\beta} < \infty \right\}.$$

The X_1-Jacobi polynomials $\{P_{1,n}^{(\alpha,\beta)}\}_{n=1}^{\infty}$ are obtained from applying the Gram-Schmidt orthogonalization process with respect to the inner product induced within $L^2((-1,1); w_1^{\alpha,\beta})$ to the sequence of vectors

$$u_1 = x - c\,, \qquad u_i = (x-b)^i \text{ for } i \geq 2.$$

Instead of choosing to normalize based on the lengths of the resulting vectors, we require

$$P_{1,n}^{(\alpha,\beta)}(1) = \frac{\alpha + n}{\beta - \alpha} \binom{\alpha + n - 2}{n - 1}. \tag{3.7}$$

In [19, Section 3], it was shown that $\{P_{1,n}^{(\alpha,\beta)}\}_{n=1}^{\infty}$ form a complete set of eigenvectors in $L^2((-1,1); w_1^{\alpha,\beta})$ for the polynomial Sturm-Liouville problem given by the second-order differential expression

$$\ell_1^{\alpha,\beta}[y](x) := (x^2-1)y''(x) + 2a\left(\frac{1-bx}{b-x}\right)\left[(x-c)y'(x) - y(x)\right], \tag{3.8}$$

that is, for each $n \in \mathbb{N}$ and all $x \in (-1,1)$:

$$\ell_1^{\alpha,\beta}\left[P_{1,n}^{(\alpha,\beta)}\right](x) = \lambda_n P_{1,n}^{(\alpha,\beta)}(x) \quad \text{ where } \quad \lambda_n = (n-1)(\alpha+\beta+n). \tag{3.9}$$

The X_1-Jacobi polynomials can be written in terms of the classical Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$; specifically, with $P_{-1}^{(\alpha,\beta)} := 0$ we have

$$P_{1,n}^{(\alpha,\beta)} = -\frac{1}{2}(x-b)P_{n-1}^{(\alpha,\beta)} + \frac{bP_{n-1}^{(\alpha,\beta)} - P_{n-2}^{(\alpha,\beta)}}{\alpha+\beta+2n-2} \qquad (n \in \mathbb{N}). \tag{3.10}$$

Moreover, it is also the case that the X_1-Jacobi polynomials satisfy a three term recurrence relation [19].

3.3 Darboux Transform

The Darboux transform is a well-known technique from quantum mechanics that is used to generate new exactly solvable potentials from known ones [20]. In relation to orthogonal polynomials, the Darboux transform allows us to obtain new second-order differential operators from known ones. We must be cautious as it might not always be the case that applying the Darboux transform results in a new, well-defined Sturm-Liouville problem. For us, the Darboux transform will provide a means of moving between the classical and exceptional polynomial settings. This overview is based on the work in [20] who describe the Darboux transform and its relation to exceptional orthogonal polynomials. For further reading regarding the Darboux transform in general, see [33].

As a result of Favard's Theorem [15], a monic polynomial system $\{p_n\}_{n=0}^{\infty}$ is a monic orthogonal polynomial system if and only if $\{p_n\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation

$$p_{n+1}(x) = (x - b_n)p_n(x) - c_n p_{n-1}(x) \quad (n \in \mathbb{N}_0 \text{ and } P_{-1}(x) = 0)\,,$$

where $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are real numbers with $c_n \neq 0$ for $n \geq 1$. The three-term recurrence relation of a monic orthogonal polynomial system induces a second-order difference operator. The Darboux transform factors this second-order difference operator into a product of two first-order difference operators. By changing the order of these two factors, we may obtain a new second-order differential operator.

Let $y = y(x)$ and $\ell[y](x) = p(x)y''(x) + q(x)y'(x) + r(x)y(x)$ be a second-order differential expression with rational coefficients. We can view $\ell[\cdot]$ as a linear operator, mapping the function $y(x)$ to another function $\ell[y](x)$.

Let

$$T(y) := \ell[y](x)$$

be a differential operator with rational coefficients. We say that T has a *rational* *factorization* if

$$T - \lambda = BA \tag{3.11}$$

where $A = A(y)$ and $B = B(y)$ are first-order operators with rational coefficients and λ is a constant. Let us write

$$A(y) = b(y' - cy)$$
$$B(y) = \hat{b}(y' - \hat{c}y)$$

where $c(x)$, $\hat{c}(x)$, $b(x)$, and $\hat{b}(x)$ are rational functions. We refer to $b(x)$ as the *factorization gauge*. Given the rational factorization (3.11) of T, define the *partner operator* $\hat{T}$ by

$$\hat{T} = AB + \lambda\,, \text{ that is,}$$
$$\hat{T}(y) = py'' + \hat{q}y' + \hat{r}y\,,$$

where

$$\hat{q} = q + p' - 2pb'/b$$
$$\hat{r} = -p(\hat{c}' + \hat{c}^2) - \hat{q}\hat{c} + \lambda\,.$$

Let

$$\Phi(x) = \exp\left(\int^x c\right)$$

be called a *factorization eigenfunction*, and note that it satisfies

$$T(\Phi) = \lambda\Phi\,.$$

In other words, the function

$$c(x) = (\log(\Phi(x)))'\,,$$

is a solution to the equation:

$$p(c' + c^2) + qc + r = \lambda\,.$$

If we consider two factorization gauges, $b_1(x)$ and $b_2(x)$, and their corresponding partner operators $\hat{T}_1(y)$ and $\hat{T}_2(y)$, then

$$\hat{T}_2 = \mu^{-1}\hat{T}_1\mu \quad \text{where } \mu(x) = b_1(x)/b_2(x)\,.$$

Thus for a given operator T, a rational factorization is determined uniquely by the factorization gauge and the factorization eigenfunction. We require our exceptional polynomial sequences to be primitive since a factorization gauge can preserve a polynomial sequence of lower codimension.

Example 3.5. This example is discussed in [22]. Consider the codimension one flag $\{x, x^2, x^3, \ldots\}$ which is not exceptional because it is not primitive. Although the subspace is preserved under the rational operator

$$\hat{T}(y) = y'' - \frac{2y'}{x} + \frac{2y}{x^2}\,,$$

there exists the partner operator $T(y) = y''$ which is related to $\hat{T}$ via the guage transformation $\hat{T}(y) = xTx^{-1}$ which preserves the standard sequence of codimension zero, $\{1, x, x^2, \ldots\}$. By requiring exceptional polynomial sequences to be primitive, we ensure that there does not exist a gauge transformation which preserves a flag of smaller codimension.

For our purposes, the relationship between the partner operators is of great importance as these links will provide us with a means of connecting the classical and exceptional cases. In fact, partner operators satisfy the following intertwining relations:

$$\hat{T}A = AT\,, \quad B\hat{T} = TB\,.$$

It is also the case that if T is exactly solvable by a sequence of polynomials and has $\{y_j\}_{j\in J\subseteq\mathbb{N}}$ as eigenpolynomials, then $\{A(y_j)\}_{j\in J}$ are eigenfunctions of the partner operator $\hat{T}$. The $\{A(y_j)\}_{j\in J}$ will also be a polynomial sequence for appropriate choices of $b(x)$. Requiring the sequence $\{y_j\}_{j\in J}$ to be primitive determines the choice of $b(x)$ up to multiplication by a scalar. The relationship between T and $\hat{T}$ also provides a correspondence between the weight functions

$$w(x) := \frac{\exp\left(\int^x q/p\right)}{p(x)}$$

and $\hat{w}(x)$, which is analogously defined, as it is elementary to show

$$\hat{w} = pw/b^2 \,.$$

A Darboux transform may be classified into one of three categories—state-deleting, state-adding, or isospectral. We call a Darboux transformation *state-deleting* if the factorization eigenfunction $\Phi(x)$ is in $L^2(I, w)$ and the formal factorizing eigenvalue λ is the minimum value of the spectrum of T. In the case of our particular Sturm-Liouville problems, the spectrum will always be bounded below. On the other hand, if the corresponding factorization eigenfunction $\hat{\Phi}(x)$ from the partner operator is in $L^2(I, \hat{w})$ and the formal factorizing eigenvalue λ is the minimum value of the spectrum of T, then we say the transformation is *state-adding*. If

$$\Phi(x) \notin L^2(I, w) \quad \text{and} \quad \hat{\Phi}(x) \notin L^2(I, \hat{w}) \,,$$

then λ must be smaller than the minimum value in the spectrum of T, and we call the corresponding Darboux transformation *isospectral*. Here we are assuming both T and $\hat{T}$ correspond to polynomial Sturm-Liouville problems. So if the transformation is state-deleting, then $\Phi = y_1$, where y_1 is the first eigenpolynomial of T. When the transformation is state-adding, $\hat{\Phi} = A(y_1)$ will be a polynomial. And lastly, the transformation is isospectral when both Φ and $\hat{\Phi}$ are non-polynomial.

Suppose that we now have a family of operators $\{T_k\}_{k\in K}$ which have polynomial eigenfunction solutions. That is,

$$T_k(y) = p_k(x)y'' + q_k(x)y' + r_k(x)y \,, \quad k \in K \,.$$

We call the operators and polynomials *shape–invariant* if the family is closed with respect to the state–deleting Darboux transformation.

We now summarize the rational factorizations associated with the Darboux transform for the classical Laguerre and Jacobi polynomials.

Recall the classical Laguerre differential expression (2.4), the corresponding weight function (2.5), and eigenvalue equation (2.6). The classical Laguerre polynomials are shape-invariant under the factorizations:

$$\ell_\alpha = B_\alpha A_\alpha$$
$$\ell_{\alpha+1} = A_\alpha B_\alpha + 1\,,$$

where

$$A_\alpha(y) = y' \quad \text{and} \tag{3.12}$$
$$B_\alpha(y) = xy' + (\alpha + 1 - x)y\,. \tag{3.13}$$

The quasi-rational eigenfunctions and eigenvalues of $\ell_\alpha[y]$ are:

$$\begin{aligned}
\Phi_1(x) &= L_m^\alpha(x), & \lambda &= -m\\
\Phi_2(x) &= x^{-\alpha} L_m^{-\alpha(x)}, & \lambda &= \alpha - m\\
\Phi_3(x) &= e^x L_m^\alpha(-x), & \lambda &= \alpha + 1 + m\\
\Phi_4(x) &= x^{-\alpha} e^x L_m^\alpha(-x), & \lambda &= m + 1
\end{aligned}$$

where $m \in \mathbb{N}_0$. The factorizations corresponding to each of these eigenfunctions have been studied, see e.g. [16]. It has been shown that Φ_1 with $m = 0$ corresponds to a state–deleting transformation and corresponds to the classical Laguerre polynomials. For $m > 0$, the eigenfunctions corresponding to Φ_1 yield singular operators, which means that no new families of orthogonal polynomials arise. The family associated with Φ_4 is state-adding and therefore, the resulting orthogonal polynomials are not of codimension m. The factorizations of Φ_2 and Φ_3 result in new orthogonal polynomials—in fact, these factorizations respectively produce the Type I and

Type II exceptional orthogonal polynomials of codimension m which are discussed at length later in Subsections 3.4.1 and 3.4.2 respectively.

Now consider the classical Jacobi differential expression (2.7) and the corresponding weight function (2.8) and eigenvalues (2.9). The classical Jacobi polynomials are shape-invariant under the factorizations:

$$\begin{aligned} \ell_{\alpha,\beta} &= B_{\alpha,\beta}A_{\alpha,\beta} \\ \ell_{\alpha+1,\beta+1} &= A_{\alpha+1,\beta+1}B_{\alpha+1,\beta+1} + \alpha + \beta + 2\,, \quad \text{where} \\ A_{\alpha,\beta}(y) &= y' \quad \text{and} \\ B_{\alpha,\beta}(y) &= (1-x^2)y' + (\beta - \alpha + (\alpha+\beta+2)x)y \\ &= (1-x)^{-\alpha}(1+x)^{-\beta}(y(1-x)^{\alpha+1}(1+x)^{\beta+1})'\,. \end{aligned}$$

The quasi-rational eigenfunctions and eigenvalues of $\ell_{\alpha,\beta}[y] = \lambda y$ are [13]:

$$\begin{aligned} &\Phi_1(x) = P_m^{(\alpha,\beta)}(x), && \lambda = -m(1+\alpha+\beta+m) \\ &\Phi_2(x) = (1-x)^{-\alpha}(1+x)^{-\beta}P_m^{(-\alpha,-\beta)}(x), && \lambda = (1+m)(\alpha+\beta-m) \\ &\Phi_3(x) = (1-x)^{-\alpha}P_m^{(-\alpha,\beta)}(x), && \lambda = (1+\beta+m)(\alpha-m) \\ &\Phi_4(x) = (1-x)^{-\beta}P_m^{(\alpha,-\beta)}(x), && \lambda = (1+\alpha+m)(\beta-m) \end{aligned}$$

where $m = 0, 1, 2, \ldots$. The factorizations corresponding to each of these factorizations have been studied in [16]. It has been shown that Φ_1 with $m = 0$ corresponds to a state-deleting transformation and underlies the shape-invariance of the classical Jacobi operator. When Φ_1 and $m > 0$, singular operators result and hence do not yield new orthogonal polynomials. The Φ_2 results in a state-adding transformation for all values $m \geq 0$. The factorizations Φ_3 and Φ_4 result in novel orthogonal polynomials, given some additional restrictions upon parameters α and β. The resulting families are related by the transformation $\alpha \longleftrightarrow \beta$ and $x \longleftrightarrow -x$. Without loss of generality, we may focus only on the Φ_3 factorization. This factorization leads to a general form of exceptional Jacobi polynomials of codimension m, $m \geq 1$. In this dissertation, we will focus only on the case $m = 1$ for the exceptional Jacobi polynomials. For further study regarding exceptional Jacobi polynomials of codimension $m \geq 1$, the reader is directed to [16], [22], and [25].

3.4 X_m-Laguerre Polynomials

There are at least three families of Laguerre-type which span a subset of $\mathcal{P}$ of codimension m. In this section, we describe the properties of the Type I, II, and III X_m-Laguerre polynomial sequences. To distinguish among the families of X_m-Laguerre polynomials, we will use a superscript "I", "II" and "III" on the polynomials, weight function, differential expression, etc. to denote the Type I, II, and III X_m-Laguerre polynomials respectively.

3.4.1 X_m^I-Laguerre Polynomials

Here we assume that the parameter $\alpha > 0$. Taking $\Phi_2(x)$ as the factorization eigenfunction and the classical Laguerre polynomial of degree m, $L_m^\alpha(-x)$, to be the factorization guage, it can be seen that the classical Laguerre expression $\ell_\alpha[\cdot]$ (2.4) may be rewritten as

$$\begin{aligned} \ell_\alpha &= B_{m,\alpha}^I A_{m,\alpha}^I + \alpha + m + 1, \qquad \text{where} \\ A_{m,\alpha}^I(y) &= L_m^\alpha(-x)y' - L_m^{\alpha+1}(-x)y \qquad \text{and} \\ B_{m,\alpha}^I(y) &= \frac{xy' + (1+\alpha)y}{L_m^\alpha(-x)}\,. \end{aligned}$$

The associated partner eigenfunction is $\hat{\Phi}(x) = x^{-1-k}$. Then, the Type I Laguerre expression $\ell_m^{I,\alpha}[\cdot]$ may be written as

$$\ell_m^{I,\alpha} = A_{\alpha-1,m}^I B_{\alpha-1,m}^I + k + m\,.$$

Computation shows that

$$\ell_m^{I,\alpha}[y] := -\ell_\alpha[y] + 2(\log L_m^{\alpha-1}(-x))'(xy' + \alpha y) - my\,. \tag{3.14}$$

By construction, the X_m^I-Laguerre polynomial $y = L_{m,n}^{I,\alpha}(x)$ $(n \in \mathbb{N}\backslash\{1, 2, \cdots, m-1\})$ satisfies the eigenvalue equation

$$\ell_m^{I,\alpha}[y] = \lambda_n y \quad (0 < x < \infty)$$

where $\lambda_n = n \quad (n \in \mathbb{N}_0)$.

The X_m^I-Laguerre polynomials $\{L_{m,n}^{I,\alpha}\}_{n=m}^{\infty}$ are orthogonal on $(0, \infty)$ with respect to the weight function

$$w_m^{I,\alpha}(x) = \frac{x^\alpha e^{-x}}{L_m^{\alpha-1}(-x)^2} \quad (x \in (0, \infty))\,. \tag{3.15}$$

The requirement $\alpha > 0$ ensures that the weight function will be positive and $L_m^{\alpha-1}(-x)$ will have no zeros in $[0, \infty)$.

In Lagrangian symmetric form, the X_m^I-Laguerre differential expression is given by

$$\ell_m^{I,\alpha}[y](x) = \frac{1}{w_m^{I,\alpha}(x)} \left(\left(\frac{x^{\alpha+1} e^{-x}}{(L_m^{\alpha-1}(-x))^2} y'(x) \right)' - \frac{2\alpha x^\alpha e^{-x} (L_m^{\alpha-1}(-x))'}{(L_m^{\alpha-1}(-x))^3} y(x) \right)\,. \tag{3.16}$$

The $(n-m)$th $(n \geq m)$ X_m^I-Laguerre polynomial can be expressed in terms of classical Laguerre polynomials by

$$L_{m,n}^{I,\alpha}(x) = L_m^\alpha(-x) L_{n-m}^\alpha(x) - L_{m-1}^\alpha(-x) L_{n-m-1}^\alpha(x) \quad \text{ for } (n \geq m)\,.$$

It may also be expressed in terms of the operators $A_{\alpha,m}^I$ and $B_{\alpha,m}^I$ as

$$L_{m,n}^{I,\alpha} = -A_{m,\alpha-1}^I (L_{m,n-m}^{I,\alpha})\,.$$

Note that $m \in \mathbb{N}_0$ and in fact, when $m = 0$, these definitions reduce to the classical Laguerre differential expression, polynomial eigenfunctions, weight function, etc.

3.4.2 X_m^{II}-Laguerre Polynomials

Here, we assume the parameter $\alpha > m-1$. Choosing the factorization function Φ_3 and letting $L_m^{-\alpha}(x)$ be the factorization gauge, the classical Laguerre differential expression may be written as

$$\begin{aligned} \ell_\alpha &= B^{II}_{m,\alpha}A^{II}_{m,\alpha} + \alpha - m \quad \text{where} \\ A^{II}_{m,\alpha}(y) &= xL^{-\alpha}_m(x)y' + (k-m)L^{-\alpha-1}_m(x)y \quad \text{and} \\ B^{II}_{m,\alpha}(y) &= \frac{y'-y}{L^{-\alpha}_m(x)}\,. \end{aligned}$$

The corresponding partner eigenfunction is $\hat{\Phi}(x) = e^x$. Based on this factorization, we define the X^{II}_m-Laguerre expression $\ell^{II,\alpha}_m[\cdot]$ by

$$\begin{aligned} \ell^{II,\alpha}_m[y] &= A^{II}_{m,\alpha+1}B^{II}_{m,\alpha+1}(y) + \alpha + 1 \\ &= -\ell_\alpha[y] - 2x(\log L^{-\alpha-1}_m(x))'(y-y') + my\,. \end{aligned} \tag{3.17}$$

The X^{II}_m-Laguerre polynomial $y = L^{II,\alpha}_{m,n}(x)$ where $n \in \mathbb{N} \setminus \{1, 2, \cdots, m-1\}$ satisfies the eigenvalue equation (3.17)

$$\ell^{II,\alpha}_m[y] = \lambda_n y \quad (0 < x < \infty)$$

where $\lambda_n = n \quad (n \in \mathbb{N}_0)$.

The X^{II}_m-Laguerre polynomials $\{L^{II,\alpha}_{m,n}\}^\infty_{n=m}$ are orthogonal on $(0, \infty)$ with respect to the weight function

$$w^{II,\alpha}_m(x) = \frac{x^\alpha e^{-x}}{(L^{-\alpha-1}_m(x))^2} \quad (x \in (0,\infty))\,. \tag{3.18}$$

Requiring $\alpha > m - 1$ is equivalent to $L^{-\alpha-1}_m(x)$ having no zeros in $[0, \infty)$, see [39, Proposition 4.1]. The restriction ensures that $w^{II,\alpha}_m(x)$ is a weight function that gives finite moments of all orders.

In Lagrangian symmetric form, the X^{II}_m- Laguerre differential expression is given by

$$\ell^{II,\alpha}_m[y](x) = \frac{1}{w^{II,\alpha}_m(x)}\left(\left(\frac{x^{\alpha+1}e^{-x}}{(L^{-\alpha-1}_m(x))^2}y'(x)\right)' - \frac{2x^{\alpha+1}e^{-x}\left(L^{-\alpha-1}_m(x)\right)'}{\left(L^{-\alpha-1}_m(x)\right)^3}y(x)\right) \tag{3.19}$$

where $w^{II,\alpha}_m(x)$ is the Type II weight function.

The $(n-m)$th $(n \geq m)$ X_m^{II}-Laguerre polynomial can be expressed in terms of the first order operators $A_{m,\alpha}^{II}$ and $B_{m,\alpha}^{II}$ applied to the the classical Laguerre polynomials and the function $L_m^{-\alpha}(x)$ by

$$
\begin{aligned}
L_{m,n}^{II,\alpha}(x) &= -A_{m,\alpha+1}^{II}(L_{n-m}^{\alpha+1}(x)) \\
&= xL_m^{-\alpha-1}(x)L_{n-m-1}^{\alpha+2}(x) + (m+\alpha-1)L_m^{-\alpha-2}(x)L_{n-m}^{\alpha+1}(x) \quad \text{for } (n \geq m)\,.
\end{aligned}
$$

For $m = 0$, the above definitions reduce, as in the Type I case, to the classical Laguerre polynomials; however, these polynomials will have a different normalization:

$$L_{0,n}^{II,\alpha}(x) = -(\alpha+1+n)L_n^{\alpha}(x)\,.$$

3.4.3 X_m^{III}-Laguerre Polynomials

The Type III exceptional X_m-Laguerre polynomials $\{L_{m,n}^{III,\alpha}(x)\}$ $(n = 0, m+1, m+2, m+3, \ldots)$ are a new class of Laguerre-type orthogonal polynomials [35]. Their existence does not arise from the quasi-rational eigenfunctions of the classical Laguerre polynomials, unlike in the case of the Type I and Type II exceptional Laguerre polynomials. Rather, their existence stems from the Type I exceptional Laguerre polynomials.

Consider the transformation

$$z(x) = x^{-\alpha}y(x)\,.$$

Remark 3.2. The function $z(x)$ is a well-known transformation in the case of the classical Laguerre polynomials: recall that the classical Laguerre polynomial $y = L_n^{\alpha}(x)$ satisfies the second-order differential equation

$$\ell[y](x) = -ny(x)$$

where $\ell[y](x)$ is defined in (2.4). If we consider the case of $z(x)$,

$$\ell[z](x) = x^{-\alpha}m[z](x)\,,$$

where

$$m[z](x) = xz''(x) + (1+\alpha-x)z'(x) + \alpha z(x)\,.$$

The equation $m[\cdot]$ is also a Laguerre equation and therefore, for $n \in \mathbb{N}_0$, the function $z(x) = L_n^{-\alpha}(x)$ is a solution of the second-order differential equation

$$m[z](x) = (-n+\alpha)z(x)\,.$$

Concerning the correspondence between the Type I and Type III polynomials, a calculation shows that

$$\ell_n^{I,\alpha}[z](x) = x^{-\alpha}\ell_n^{III,-\alpha}[y](x)\,,$$

where

$$\ell_n^{III,-\alpha}[y](x) := -xy'' + \left(-1+\alpha+x+2x\frac{(L_n^{\alpha-1}(-x))'}{L_n^{\alpha-1}(-x)}\right)y' + (-n-\alpha)y\,. \quad (3.20)$$

The expression $\ell_n^{III,\alpha}[\cdot]$ is called the X_m^{III}-Laguerre differential expression.

From these calculations, it is unclear whether the eigenvalue problem

$$\ell_n^{III,\alpha}[y](x) = (n-m+\alpha)y(x) \quad (3.21)$$

will have polynomial solutions and if it does, whether those polynomial solutions will be orthogonal with respect to a positive-definite weight function. To demonstrate that this eigenvalue problem has orthogonal solutions, it is helpful to turn to the technique of Darboux transformations.

Recall the intertwining operators of the classical Laguerre expression $A_{n,\alpha}^I[\cdot]$ and $B_{n,\alpha}^I[\cdot]$ defined in (3.12) and (3.13) respectively which give rise to the Type I second-order expression. There is a second rational factorization of the classical Laguerre expression $\ell_\alpha[\cdot]$, which yields the Type III second-order expression.

Let

$$A_{m,\alpha}^{III}(y) = xL_m^{-\alpha}(-x)y' - (m+1)L_{m+1}^{-\alpha-1}(-x)y$$
$$B_{m,\alpha}^{III}(y) = \frac{y'}{L_m^{-\alpha}(-x)}\,.$$

The classical Laguerre operator may also be written as

$$\ell_\alpha = B_{m,\alpha}^{III}A_{m,\alpha}^{III} + m + 1\,,$$

and the Darboux transformation associated with the above factorizations yields another representation of the Type III second-order expression

$$\ell_m^{III,\alpha} = A_{m,\alpha}^{III}B_{m,\alpha}^{III} + m + 1\,.$$

The corresponding weight for the Type III case is

$$w_{m,\alpha}^{III}(x) = \frac{x^\alpha e^{-x}}{\left(L_m^{-\alpha-1}(-x)\right)^2} \tag{3.22}$$

and in fact, it is the case that

$$w_{m,\alpha}^{III}(x) = x^{2\alpha}w_{m,-\alpha}^{I}(x)\,.$$

Thus, the Type I and Type III weight functions are related by a multiplicative factor and a relabeling of the parameter α. As a result of this relationship, the Type I and Type III Sturm-Liouville problems are related via a multiplicative operator or a gauge transformation. The orthogonality of the polynomial eigenfunctions in the Type I case is preserved under such transformations. Therefore, we are ensured that our Type III Sturm-Liouville problem will have orthogonal polynomial solutions.

It is necessary to note, however, that the restrictions upon the parameter α are not preserved under such transformations. The Type III eigenvalue equation will have orthogonal polynomial solutions if and only if $-1 < \alpha < 0$. This restriction on the parameter α ensures that the denominator of the weight function will have no zeros in $[0, \infty)$ and that the moments remain finite.

The Type III differential equation (3.21) will have a polynomial solution $y = L_{m,n}^{III,\alpha}(x)$ of degree n for $n = 0$ and for $n \geq m+1$; that is, solutions of degrees $\{1, 2, \ldots, m\}$ are missing. On the other hand, for the Type I polynomials, solutions of degrees $\{0, 1, \ldots, m-1\}$ are missing.

We may write the Type III exceptional Laguerre polynomials explicitly in terms of the classical Laguerre polynomials

$$L_{m,n}^{III,\alpha}(x) = \begin{cases} xL_{j-1}^{\alpha+2}(x)L_m^{-\alpha-1}(-x)+ & \\ \qquad (m+1)L_j^{\alpha+1}(x)L_{m+1}^{-\alpha-2}(-x), & n = m+1, m+2, \ldots \\ 1 & n = 0 \end{cases}$$

and using the Darboux transformation

$$L_{m,n}^{III,\alpha}(x) = \begin{cases} -A_{m,\alpha+1}^{III}[L_{n-m-1}^{\alpha+1}(x)], & n = m+1, m+2, \ldots \\ 1 & n = 0\,. \end{cases}$$

With respect to the Type III weight function (3.22), these polynomials are orthogonal in the Hilbert space $L^2((0,\infty); w_{m,\alpha}^{III})$ and the norms are given by

$$\int_0^\infty L_{m,n}^{III,\alpha}(x)^2 w_m^{III,\alpha}(x)dx = \begin{cases} \dfrac{n\,\Gamma(n-m+\alpha+1)}{(n-m-1)!}, & \text{if } n \geq m+1 \\ \dfrac{\Gamma(\alpha+1)}{\binom{m-\alpha-1}{m}} & \text{if } n = 0\,. \end{cases}$$

At this time it is still unknown if these polynomials form a complete orthogonal set in $L^2((0,\infty); w_m^{III,\alpha})$.

Lastly, in Lagrangian symmetric form, the X_m^{III}-Laguerre differential expression is given by

$$\ell_m^{III,\alpha}[y](x) = \frac{1}{w_m^{III,\alpha}(x)}\left(\left(\frac{x^{\alpha+1}e^{-x}}{(L_m^{-\alpha-1}(x))^2}y'(x)\right)' - \frac{2x^{-\alpha-1}e^{-x}((L_m^{-\alpha-1}(x))'}{(L_m^{-\alpha-1}(x))^3}y(x)\right). \tag{3.23}$$

CHAPTER FOUR

Self-adjoint Operators and Spectral Theory

In order to apply spectral theory to a non-self-adjoint symmetric operator, we must first find a self-adjoint extension. This procedure is neither trivial nor possible in general. The operators which arise in the exceptional Laguerre and Jacobi cases are differential operators which means that they are unbounded and thus not defined everywhere on the corresponding Hilbert space. We are required then to study self-adjoint extensions of the operator induced by their respective differential expressions. Much of this chapter is devoted to the theory of self-adjoint extensions. We will conclude this chapter with a review of spectral theory. The material for this chapter may be found in [1, 11, 27, 30, 38].

4.1 Preliminary Operator Theory

In this section H will denote a complex Hilbert space with inner product $(\cdot,\cdot)_H$. Let $T : \mathcal{D}(T) \to H$ be *densely defined*; that is, the domain of T is a dense subset of H. We now define the *Hilbert-adjoint operator* T^* of T. The domain of the Hilbert-adjoint operator $\mathcal{D}(T^*)$ consists of all $y \in H$ such that the mapping $f_y : x \mapsto (Tx, y)_H$ is continuous on $\mathcal{D}(T)$. By the Hahn-Banach theorem, f_y has a continuous extension to all of H. Therefore, by the Riesz representation theorem, there exists a unique $y^* \in H$ such that $(Tx, y)_H = (x, y^*)_H$ for all $x \in \mathcal{D}(T) = H$. We define $T^*y = y^*$.

An operator T is said to be *Hermitian* if, for all $x, y \in \mathcal{D}(T)$,

$$(Tx, y)_H = (x, Ty)_H .$$

If an operator T is both Hermitian and densely defined, then T is called *symmetric.*

If a linear operator T is defined on all of H and it also satisfies

$$(Tx, y)_H = (x, Ty)_H \text{ for all } x, y \in \mathcal{D}(T)\,,$$

then T is a bounded linear operator. Hence, the domain of an unbounded symmetric operator T is a proper subspace of H and it is of interest to study the self-adjoint extensions (if they exist) of T in H.

We will now study the relationship between the adjoint of a symmetric operator and the adjoint of its symmetric extension. An operator T is an *extension* of the operator S if the following hold:

$$\mathcal{D}(S) \subseteq \mathcal{D}(T)$$
$$S[f] = T[f], \quad f \in \mathcal{D}(S)\,.$$

If we have a densely defined operator T, then

(1) T is symmetric if and only if $T \subseteq T^*$.

(2) If S is an extension of T, then T^* is an extension of S^*, that is $T \subseteq S$ implies $S^* \subseteq T^*$.

(3) If T is symmetric, then every symmetric extension S of T satisfies

$$T \subseteq S \subseteq S^* \subseteq T^*\,.$$

A densely defined operator with with property that $T = T^*$ is said to be *self-adjoint*. In terms of the inner product, a self-adjoint operator satisfies

$$(Tx, y)_H = (x, T^*y)_H = (x, Ty)_H\,.$$

It follows from (1) that every self-adjoint operator is necessarily symmetric, but not every symmetric operator is self-adjoint. From (3), we see that the most general symmetric extension in H (in particular, the most general self-adjoint extension) of a symmetric operator T is a suitably chosen restriction of the adjoint T^* of T.

4.2 Weyl Theory—Endpoint Analysis

Weyl Theory (see [27]) applies equally well to all bounded and unbounded intervals of the real line. Let $I = (a, b)$ be an interval. Suppose that for each $k \in \{0, 1, \ldots, n\}$, $a_k : I \to \mathbb{C}$ is such that $a_k \in C^k(a, b)$. Let

$$\ell[y] = \sum_{k=0}^{n} a_k y^{(k)}, \quad y \in C^n(a, b). \tag{4.1}$$

The *Lagrange adjoint* (or *formal adjoint*) of $\ell[\cdot]$ is the differential expression

$$\ell^*[y] = \sum_{k=0}^{n} (-1)^k (\bar{a}_k y)^{(k)}.$$

The expression $\ell[\cdot]$ is said to be *formally symmetric* if $\ell[y] = \ell^*[y]$ for all $y \in C^n(a, b)$. For all formally symmetric differential operators with real-valued coefficients, we have the following theorem which determines their general form.

Theorem 4.1. *Suppose $\ell[\cdot]$ is of the form* (4.1) *with each coefficient $a_k \in C^k(a, b)$ being real-valued. If $\ell[\cdot]$ is formally symmetric, then n is necessarily even and $\ell[\cdot]$ may be written as:*

$$\ell[y] = \sum_{k=0}^{j} (-1)^k (b_k y^{(k)})^{(k)},$$

where $n = 2j$.

In 1910, Weyl [45] studied the particular case when $n = 2$. He studied solutions to the *Sturm-Liouville differential equation*,

$$D_\lambda[y](x) := -\left(p(x)y'(x)\right)' + \left(\lambda w(x) - q(x)\right) y(x) = 0 \quad (x \in I); \tag{4.2}$$

where $\lambda \in \mathbb{C}$ is fixed and p, p', q, w: $I \to \mathbb{R}$ are continuous functions with both p and w positive on I. The operator $D_\lambda[\cdot]$ is studied on the Hilbert space

$$L^2(I, w) := \left\{ f : I \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_a^b |f|^2 \, w \, dx < \infty \right\}.$$

We define the *second-order Sturm-Liouville expression* to be

$$\ell[y] := \frac{1}{w}((-py')' + qy).$$

Then, $w\ell[y]$ is formally symmetric and the equation $\ell[y] = \lambda y$ is equivalent to $D_\lambda y = 0$. We say $y(x) : I \to \mathbb{C}$ is a *solution* of (4.2) if $y(x) \in L^2(I, w)$ and $\ell_\lambda[y](x) - \lambda y = 0$ for each $x \in I$.

Recall that a general second-order differential equation in non-symmetric form (2.2) may be put into Sturm-Liouville form by multiplying by an appropriate integrating factor. This integrating factor is $w(x)$, the weight function, from classical orthogonal polynomial theory.

The following two theorems are attributed to Weyl and are useful in establishing the self-adjointness of certain Sturm-Liouville operators. For proofs, see [7] and [27].

Theorem 4.2 (Weyl's First Theorem). *Fix $x_0 \in I^\circ$ and choose $\lambda_0 \in \mathbb{C}$ such that for this value λ_0, it is the case that all solutions $\tilde{y}(x)$ of $D_{\lambda_0}[\cdot] = 0$ satisfy*

$$\int_{x_0}^{b} |\tilde{y}(x)|^2 \, w(x) \, dx < \infty \,. \tag{4.3}$$

Then, for arbitrary $\lambda \in \mathbb{C}$, it will be the case that every solution of $D_\lambda[\cdot](x) = 0$ will satisfy the inequality (4.3).

Similarly, fix $x_0 \in I^\circ$ and choose $\tilde{\lambda}_0 \in \mathbb{C}$ such that for this value $\tilde{\lambda}_0$, it is the case that all solutions $\tilde{y}(x)$ of $D_{\tilde{\lambda}_0}[\cdot] = 0$ satisfy

$$\int_{a}^{x_o} |\tilde{y}(x)|^2 \, w(x) \, dx < \infty \,. \tag{4.4}$$

Then, for arbitrary $\lambda \in \mathbb{C}$, it will be the case that every solution of $D_\lambda[\cdot](x) = 0$ will satisfy the inequality (4.4).

Note that the first statement of Weyl's first theorem studies the behavior of solutions at the right-hand endpoint of I; while the second looks at the behavior of solutions at the left-hand endpoint of the interval I.

Theorem 4.3 (Weyl's Second Theorem). *Consider the Sturm-Liouville equation (4.2) in I. Let $x_0 \in (a, b)$ and suppose $\lambda \in \mathbb{C}$ is such that* $\mathrm{Im}(\lambda) \neq 0$. *Then the following statements hold:*

(1) There exists at least one solution $\tilde{y}$ *of* $D_\lambda y = 0$ *satisfying*

$$\int_{x_o}^{b} |\tilde{y}(x)|^2 \, w(x) \, dx < \infty \, .$$

(2) Suppose there exists a solution $\hat{y}$ *of* $D_\lambda y = 0$ *such that*

$$\int_{x_0}^{b} |\hat{y}(x)|^2 \, w(x) \, dx = \infty \, .$$

Then for each solution y *of* $D_\lambda y = 0$ *satisfying*

$$\int_{x_o}^{b} |y(x)|^2 \, w(x) \, dx < \infty \, ,$$

it is the case that

$$\lim_{x \to b^-} p(x) \left(y'(x)\overline{y}(x) - y(x)\overline{y'}(x) \right) = 0 \, .$$

(3) Suppose every solution y *of* $D_\lambda y = 0$ *satisfies*

$$\int_{x_0}^{b} |y(x)|^2 w(x) \, dx < \infty \, .$$

Then there exists a circle $C_b(\lambda)$ *in the complex plane* $\mathbb{C}$ *with radius* $r_b(\lambda) > 0$ *and a linearly independent set of solutions* $\{v_1, v_2\}$ *of* $D_\lambda y = 0$ *such that if* $z \in \mathbb{C}$ *is any complex number on this circle and*

$$y_z(x) = z v_1(x) + v_2(x) \, ,$$

then

$$\lim_{x \to b^-} p(x) \left(y_z'(x)\overline{y}_z(x) - y_z(x)\overline{y'}_z(x) \right) = 0 \, .$$

Corresponding statements are true at the endpoint a *of* I.

As a result of Weyl's second theorem, we say that $x = b$ is in the *limit-circle case* with respect to λ if for this value λ,

$$\int_{x_0}^{b} |\tilde{y}(x)|^2 \, w(x) \, dx < \infty$$

for every solution $\tilde{y}(x)$ of $D_\lambda y = 0$. On the other hand, $x = b$ is in the *limit-point case* with respect to λ if for this value of λ, there exists only one linearly independent solution $\tilde{y}(x)$ of $D_\lambda y = 0$ for which

$$\int_{x_0}^{b} |\tilde{y}(x)|^2 \, w(x) \, dx = \infty \, .$$

Once again, we have similar terminology for the end point a of I.

It is obvious from the above definitions that the differential equation $\ell_\lambda[\cdot]$ is, for a or b and any value $\lambda \in \mathbb{C}$, either limit-point or limit-circle and cannot be both. Up to this point, it appears that the definitions for limit-point and limit-circle are dependent upon the endpoint a or b and λ; however, we see below a very useful result, the Weyl alternative theorem, which removes dependency on λ.

Theorem 4.4 (The Weyl Alternative). *The classification of $\ell_\lambda[\cdot]$ at an endpoint a or b as limit-point and limit-circle is independent of $\lambda \in \mathbb{C}$.*

4.3 Extensions of Symmetric Operators

Suppose $T : \mathcal{D}(T) \to H$ is a linear operator. The *graph* $\mathcal{G}(T)$ of an operator T is the subset of $H \oplus H$ consisting of all points of the form (x, Tx) with $x \in \mathcal{D}(T)$. T is said to be a *closed operator* if and only if $\mathcal{G}(T)$ is a closed subset of $H \oplus H$ in the inner product defined by

$$((x_1, x_2), (y_1, y_2)) = (x_1, y_1)_H + (x_2, y_2)_H \, .$$

A linear operator T is said to be *closable* if there exists a closed linear operator T_1 with $T \subseteq T_1$. T_1 is a *minimal closed linear extension* of T whenever S is a closed linear extension of T, we have $T_1 \subset S$. In this case, we call T_1 the *closure* of T and write $\overline{T} = T_1$. Every symmetric operator T is closable and $\overline{T}$ is a uniquely defined symmetric operator. Additionally, if T is self-adjoint, then T is closed; and more generally, if T is densely defined, then T^* is closed [11].

For the remainder of this section, assume $T : \mathcal{D}(T) \subseteq H \to H$ is symmetric ($T \subseteq T^*$) and densely defined on H. We will be citing the work of von Neumann [11] who showed the characteristics of T which are necessary for T to have self-adjoint extensions in H and furthermore, if T has self-adjoint extensions, how to characterize the extensions.

We define the *positive* and *negative deficiency spaces of* T respectively by

$$D_+ = \{f \in \mathcal{D}(T^*) \mid T^* f = if\} \quad \text{and} \quad D_- = \{f \in \mathcal{D}(T^*) \mid T^* f = -if\} \, ,$$

where $i = \sqrt{-1}$. The dimensions, denoted n_+ and n_-, respectively, are called the *positive* and *negative deficiency indices* of T. Recall that the Weyl alternative theorem (Theorem 4.4) shows that in the case of the Sturm-Liouville operator, the deficiency indices n_+ and n_- are independent of the number i.

T will have self-adjoint extensions in H if and only if $n_+ = n_-$. Furthermore if $n_+ = n_- = 0$, then T has only one self-adjoint extension, namely $\overline{T}$.

Let $H \oplus H$ be the space inherited when $\mathcal{D}(T^*)$ is identified with $\mathcal{G}(T^*)$ via the map $x \to (x, T^*x)$. If $x, y \in \mathcal{D}(T^*)$, we define the inner product $(x, y)^*$ by

$$(x, y)^* := (x, y) := (x, y)_H + (T^*x, T^*y)_H \, .$$

It can be verified that $(\mathcal{D}(T^*), (\cdot, \cdot)^*)$ is a complete Hilbert space. von Neumann [11] showed that $\mathcal{D}(\overline{T})$, D_+, and D_- are all closed, mutually orthogonal subspaces in $(\mathcal{D}(T^*), (\cdot, \cdot)^*)$. Furthermore, he gave a decomposition of the Hilbert space

$$\mathcal{D}(T^*) = \mathcal{D}(\overline{T}) \oplus D_+ \oplus D_- \, .$$

We call the above decomposition *von Neumann's formula.*

Suppose a S is a self-adjoint extension of the symmetric operator T. Recall from Section 4.1

$$T \subseteq S = S^* \subseteq T^* \, .$$

In particular, each self-adjoint extension of T is a restriction of T^*, i.e. $Sx = T^*x$ for $x \in \mathcal{D}(S)$. From this, we see that the space $D_+ \oplus D_-$ plays an important role in our search for self-adjoint extensions of the operator T. Indeed, the following theorem, known as von Neumann's classification theorem, gives us a connection between $D_+ \oplus D_-$ and the domains of our closed symmetric extensions of T.

Theorem 4.5 (von Neumann's Classification Theorem). *Suppose $T : \mathcal{D}(T) \subseteq H \to H$ is a densely defined symmetric operator and that $n_+ = n_- =: n \in \mathbb{N}$. Then T has self-adjoint extensions. Let $\{\phi_1, \phi_2, \ldots, \phi_n\}$ be a basis for D_+ and let $\{\psi_1, \psi_2, \ldots, \psi_n\}$ be a basis for D_-. Define $B_i(x) = (x, \phi_i)^*$ $(i = 1, \ldots, n)$ and $C_i(x) = (x, \psi_i)^*$ $(i = 1, 2, \ldots, n)$. Then, every self-adjoint extension S of T in H has the form*

$$Sx = T^*x$$
$$\mathcal{D}(S) = \left\{ x \in H \mid B_i - \sum_{j=1}^{n} \theta_{ij} C_j = 0 \quad (i = 1, \ldots, n) \right\}$$

where $\theta = \theta_{ij}$ is an $n \times n$ unitary matrix (i.e. $\bar{\theta}^T = \theta^{-1}$.)

We call B_i and C_i $(i = 1, \ldots, n)$ *boundary values* of T and the equations $B_i = 0$ and $C_i = 0$ $(i = 1, \ldots, n)$ the *boundary conditions* for T.

For a more detailed look, the reader is referred to [1] and [11].

4.4 Extensions of Symmetric Differential Operators

Let $I = (a, b)$ be an open interval of $\mathbb{R}$. In this section, our Hilbert space will be $L^2(a, b)$. Recall Theorem 4.1, which states that every formally symmetric differential expression $\ell[\cdot]$ of order $2n$ may be written as

$$\ell[y](x) = \sum_{k=0}^{n} (-1)^k (a_k(x) y^{(k)}(x))^{(k)} . \tag{4.5}$$

The right-hand endpoint, b is called a *regular* endpoint of $\ell[\cdot]$ if $b < \infty$ and for some $\epsilon > 0$,

$$\frac{1}{a_n}, a_0, a_1, \ldots, a_{n-1} \in L_{loc}(b - \epsilon, b) .$$

In this case, we say $\ell[\cdot]$ is *regular* at b. A similar definition holds for the left-hand endpoint a. If b is not regular, it is *singular*, and we cannot talk about the derivative at singular endpoints. The expression $\ell[\cdot]$ is said to be *regular* if both a and b are regular endpoints. Otherwise, $\ell[\cdot]$ is called *singular*.

The maximal operator T_1 generated by the expression

$$\ell[\cdot] : \mathcal{D}(\ell) \subseteq L^2(I) \to L^2(I)$$

of the form (4.5) is defined by

$$\begin{aligned} T_1[y] &= \ell[y] \\ \Delta := \mathcal{D}(T_1) &= \left\{ y : I \to \mathbb{C} \mid y^{(k)} \in AC_{loc}(I), k = 0, 1, \ldots, 2n-1; y, \ell[y] \in L^2(I) \right\} . \end{aligned}$$

The term 'maximal' comes from the fact that Δ is the largest subspace of functions in $L^2(I)$ for which T_1 maps back into $L^2(I)$. It is not difficult to see that Δ is dense in $L^2(I)$; consequently, the adjoint T_0 of T_1 exists as a densely defined operator in the space $L^2(I)$.

Before we define T_0, let us turn to Green's Formula. If $y, z \in \mathcal{D}(T_1)$ and $[\alpha, \beta]$ is a compact subinterval of I, then

$$\int_\alpha^\beta \ell[y](x)\bar{z}(x)\,dx - \int_\alpha^\beta y(x)\ell[\bar{z}(x)]dx = [y, z](x)\Big|_\alpha^\beta \tag{4.6}$$

where $[\cdot, \cdot]$ is the sesquilinear form given by

$$[y, z] = \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{k+j} \left[\left(a_k \bar{z}^{(k)}\right)^{(k-j)} y^{(j-1)} - \left(a_j y^{(k)}\right)^{(k-j)} \bar{z}^{(j-1)} \right] . \tag{4.7}$$

For two arbitrary elements $y, z \in \mathcal{D}(T_1)$, the limits

$$\lim_{x \to a} [y, z](x) \quad \text{and} \quad \lim_{x \to b} [y, z](x)$$

will exist and be finite [38].

The *preminimal operator* T_0' is defined to be

$$T_0'[y] = \ell[y]$$
$$\mathcal{D}(T_0') = \{y \in \mathcal{D}(T_1) \mid y \text{ has compact support in } I\}\,.$$

It can be shown that T_0' is Hermitian in $L^2(I)$. Hence, there is a symmetric closure of T_0' in $L^2(I)$. Let

$$T_0 := \overline{T}_0'\,.$$

From [1] or [38], this *minimal operator* $T_0 : \mathcal{D}(T_0) \subset L^2(I) \to L^2(I)$ is defined in the following manner

$$T_0 f = \ell_\alpha[f]$$
$$f \in \mathcal{D}(T_0) : = \{f \in \Delta \mid [f, g] \mid_{x=a}^{x=b} = 0 \text{ for all } g \in \Delta\}.$$

By construction we have the following relationship between the maximal operator T_1 and the minimal operator T_0

$$T_0^* = T_1 \quad \text{and} \quad T_1^* = T_0\,.$$

Therefore, in order to find self-adjoint operators generated by $\ell[\cdot]$, we must look at either extensions of T_0 or restrictions of T_1. The minimal operator T_0 is a closed, symmetric operator in $L^2(I)$; which implies that T_0 necessarily has equal deficiency indices m, where m is an integer satisfying $0 \leq m \leq 2n$. Therefore, from the general Stone-von Neumann theory (see Section 4.3) of self-adjoint extensions of symmetric operators, T_0 has self-adjoint extensions. We are now prepared for our main theorem.

Theorem 4.6 (Glazman-Krein-Naimark Theorem). *Let T be a self-adjoint extension of the minimal operator T_0 with equal deficiency indices $m = n_+ = n_-$. Then there exist $g_1, g_2, \ldots, g_m \in \mathcal{D}(T)$ which are linearly independent modulo $\mathcal{D}(T_0)$ such that*

$$T[y] := \ell[y]$$

$$\mathcal{D}(T) = \{y \in \Delta \mid [y, g_i] \mid_a^b = 0, i = 1, 2, \ldots, m\}.$$

Conversely, if $\{g_1, g_2, \ldots, g_m\} \subset \mathcal{D}(T)$ *are linearly independent modulo* $\mathcal{D}(T_0)$ *and*

$$[g_j, g_i](b) - [g_j, g_i](a) = 0 \quad \textit{for all} \quad i, j = 1, 2, \ldots, m\,,$$

then the subspace

$$\{y \in \Delta \mid [y, g_i] \mid_a^b = 0, i = 1, 2, \ldots, m\}$$

is the domain of a self-adjoint extension of T_0*.*

Each g_i $(i = 1, 2, \ldots, m)$ is called a *boundary function* and the requirement that $[y, g_i] \mid_a^b = 0$ $(i = 1, 2, \ldots, m)$ for all $y \in \Delta$ is referred to as the (Glazman) *symmetry conditions*. Obviously, for the deficiency indices $m = n_+ = n_-$, m independent boundary conditions must be satisfied in order to obtain a self-adjoint operator.

4.5 The Method of Frobenius—Finding Linearly Independent Solutions

In this section, we shall describe the Method of Frobenius, which is a technique for finding n linearly independent solutions, in the form of generalized power series, for certain types of ordinary differential equations. This method can be used to determine whether a Sturm-Liouville expression $\ell_\lambda[\cdot]$ is limit-point or limit-circle.

Consider the differential equation (2.2) given by

$$\ell[y](z) := a_2(z)y''(z) + a_1(z)y'(z) + a_0(z)y(z) = 0 \tag{4.8}$$

where each a_k $(k = 0, 1, 2)$ is analytic in some open neighborhood $N(a)$ of $a \in \mathbb{C}$. We say that $z = a$ is a *regular point* of $\ell[\cdot]$ if $a_2(a) \neq 0$. The point $z = a$ is a *regular singular point* of $\ell[\cdot]$ if

$$\frac{a_k(z)}{a_2(z)} \text{ has a pole of order } \leq 2 - k \quad (k = 0, 1)$$

otherwise, $z = a$ is called an *irregular singular point* of $\ell[\cdot]$.

If $z = a$ is a regular singular point of $\ell[\cdot]$, then

$$P_k(z) = \frac{(z-a)^{2-k} a_k(z)}{a_2(z)} \quad (k = 0, 1)$$

is analytic in $N(a)$. Then (4.8) may be rewritten as

$$\ell[y](z) = \frac{a_2(z)}{(z-a)^2} \left[(z-a)^2 y''(z) + (z-a) P_1(z) y'(z) + P_0(z) y(z) \right] .$$

The expression

$$(z-a)^2 y''(z) + (z-a) P_1(z) y'(z) + P_0(z) y(z) \tag{4.9}$$

is called the *canonical form* of $\ell[y](z)$.

The Method of Frobenius shows that (4.9) always has a solution of the form

$$y_1(z) = \sum_{k=0}^{\infty} a_k (z-a)^{k+r_1}$$

for some value $r_1 \in \mathbb{C}$. In fact, $r = r_1$ is a root of the *indicial equation*, which is given by

$$r(r-1) + r P_1(a) + P_0(a) = 0 \, . \tag{4.10}$$

The following is a precise statement of the Method of Frobenius for second-order differential equations:

Theorem 4.7. *Consider the second-order differential equation* (4.10)*, and suppose* $r = r_1$ *and* $r = r_2$ *are the roots of the indicial equation* (4.10) *with* $Re(r_1) \geq Re(r_2)$.

(1) If $r_1 - r_2 \notin \mathbb{N}_0$*, then (4.10) has a basis of solutions* $\{y_1(x), y_2(x)\}$ *(without loss of generality) of the form:*

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1)(x-a)^{n+r_1} \quad (a_0(r_1) \neq 0) \, ,$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n(r_2)(x-a)^{n+r_2} \quad (b_0(r_1) \neq 0) \, ;$$

(2) If $r_1 = r_2$, then (4.10) has a basis of solutions $\{y_1(x), y_2(x)\}$ of the form:

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1)(x-a)^{n+r_1} \quad (a_0(r_1) \neq 0)\,,$$
$$y_2(x) = \ln(x-a)y_1(x) + \sum_{n=0}^{\infty} b_n(r_2)(x-a)^{n-r_2}\,;$$

(3) If $r_1 - r_2 \in \mathbb{N}$, then (4.10) has a basis of solutions $\{y_1(x), y_2(x)\}$ of the form:

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1)(x-a)^{n+r_1} \ (a_0(r_1) \neq 0)\,,$$
$$y_2(x) = k\ln(x-a)y_1(x) + \sum_{n=0}^{\infty} b_n(r_2)(x-a)^{n+r_2} \ (b_0(r_1) \neq 0\,,\ k \in \mathbb{C})\ .$$

Remark 4.1. If the roots of the indicial equation involve a parameter (as we will see later in the case of the exceptional polynomials), then one has to consider all possible values of the parameter to determine the form of the basis of solutions.

The Frobenius Method provides us a means of finding linearly independent solutions, which we can to then use to determine whether an expression is limit-point or limit-circle at the endpoints and consequently, we can find the deficiency indices of the expression.

4.6 Spectral Theory

In this section, let $X \neq \{0\}$ be a complex normed space and $T : \mathcal{D}(T) \subseteq X \to X$ be a linear self-adjoint operator. With T, we associate the operator

$$T_\lambda = T - \lambda I$$

where $\lambda \in \mathbb{C}$ and I is the identity operator on $\mathcal{D}(T)$. If T_λ has an inverse, then we denote the inverse as $R_\lambda(T)$, that is,

$$R_\lambda(T) = T_\lambda^{-1}\,.$$

We call $R_\lambda(T)$ the *resolvent operator of* T or the *resolvent of* T since it helps to solve the equation $T_\lambda x = y$. Studying properties of $R_\lambda(T)$ helps us to better understand the original operator T. Many properties of $R_\lambda(T)$ and T_λ are dependent upon the choice of λ.

We say that λ is a *regular value* of T if each of the following occur:

(1) The resolvent operator $R_\lambda(T)$ exists, that is, T_λ is one-to-one;

(2) $R_\lambda(T)$ is bounded; and

(3) $\mathcal{D}(R_\lambda(T))$ is dense in X.

The collection of all regular values λ is called the *resolvent set* and is denoted $\rho(T)$. The complement of the resolvent set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T, and a value $\lambda \in \sigma(T)$ is called a *spectral value* of T. The spectrum may be written as three disjoint sets: the point spectrum, continuous spectrum, and residual spectrum. The *point spectrum* $\sigma_p(T)$ is the set of all points for which T_λ is not one-to-one, in other words, $R_\lambda(T)$ does not exist. Note that $\sigma_p(T)$ is the set of eigenvalues of T since $\lambda \in \sigma_p(T)$ implies that

$$T_\lambda x = (T - \lambda I)x = 0 \text{ for some } 0 \neq x \in \mathcal{D}(T).$$

The *continuous spectrum* $\sigma_c(T)$ consists of all values $\lambda \in \sigma(T)$ for which $R_\lambda(T)$ exists and $\mathcal{D}(R_\lambda(T))$ is dense in X, but $R_\lambda(T)$ is unbounded. Lastly, the *residual spectrum* $\sigma_r(T)$ is composed of all $\lambda \in \sigma(T)$ such that $R_\lambda(T)$ exists, but $\mathcal{D}(R_\lambda(T))$ is not dense in X. It is the case that each of these sets are mutually disjoint and their union is equal to the spectrum. Together,

$$\mathbb{C} = \rho(T) \cup \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

CHAPTER FIVE

Spectral Analysis of the Exceptional Polynomials

In this chapter, we will see how spectral theory may be applied to an important class of problems called boundary value problems. The second-order differential equations of the exceptional Laguerre and Jacobi polynomials are important examples illustrating Stone-von Neumann theory and Glazman-Krein-Naimark theory of differential operators. Here we focus on the X_1-Laguerre and X_1-Jacobi and then turn our attention to the more general Type I, II, and III X_m-Laguerre settings.

5.1 Spectral Analysis of the X_1-Laguerre Polynomials

The spectral analysis of the X_1-Laguerre polynomials was completed for this thesis and appears in [2].

In Lagrangian symmetric form, the X_1-Laguerre differential expression (3.2) is given by

$$\ell_1^\alpha[y](x) = \frac{1}{w_1^\alpha(x)}\left(-\left(\frac{x^{\alpha+1}e^{-x}}{(x+\alpha)^2}y'(x)\right)' - \frac{x^\alpha e^{-x}(x-\alpha)}{(x+\alpha)^3}y(x)\right) \quad (x>0), \tag{5.1}$$

where $w_1^\alpha(x)$ is the X_1-Laguerre weight defined in (3.4).

The maximal domain associated with $\ell_1^\alpha[\cdot]$ in the Hilbert space $L^2((0,\infty); w_1^\alpha)$ is defined to be

$$\Delta := \left\{f : (0,\infty) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(0,\infty); f, \ell_1^\alpha[f] \in L^2((0,\infty); w_1^\alpha)\right\}.$$

The associated maximal operator

$$T_1 : \mathcal{D}(T_1) \subset L^2((0,\infty); w_1^\alpha) \to L^2((0,\infty); w_1^\alpha),$$

is defined by

$$T_1 f = \ell_1^\alpha[f] \tag{5.2}$$

$$f \in \mathcal{D}(T_1) : = \Delta.$$

An application of integration by parts shows that for $f, g \in \Delta$, Green's formula can be written as

$$\int_0^\infty \ell_1^\alpha[f](x)\overline{g}(x)w_1^\alpha(x)dx = [f,g](x) \mid_{x=0}^{x=\infty} + \int_0^\infty f(x)\ell_1^\alpha[\overline{g}](x)w_1^\alpha(x)dx,$$

where $[\cdot,\cdot](\cdot)$ is the sesquilinear form defined by

$$[f,g](x) := \frac{x^{\alpha+1}e^{-x}}{(x+\alpha)^2}\left(f(x)\overline{g}'(x) - f'(x)\overline{g}(x)\right) \quad (0 < x < \infty), \tag{5.3}$$

and where

$$[f,g](x) \mid_{x=0}^{x=\infty} := [f,g](\infty) - [f,g](0).$$

By definition of Δ (and the classical Hölder's inequality), notice that the limits

$$[f,g](0) := \lim_{x\to 0^+}[f,g](x) \quad \text{and} \quad [f,g](\infty) := \lim_{x\to\infty}[f,g](x)$$

exist and are finite for each $f, g \in \Delta$.

The minimal operator $T_0 : \mathcal{D}(T_0) \subset L^2((0,\infty); w_1^\alpha) \to L^2((0,\infty); w_1^\alpha)$ is defined by

$$T_0 f = \ell_1^\alpha[f] \tag{5.4}$$

$$f \in \mathcal{D}(T_0) : = \{f \in \Delta \mid [f,g] \mid_{x=0}^{x=\infty} = 0 \text{ for all } g \in \Delta\}.$$

The minimal operator T_0 is a closed, symmetric operator in $L^2((0,\infty); w_1^\alpha)$; furthermore, because the coefficients of $\ell_1^\alpha[\cdot]$ are real, T_0 necessarily has equal deficiency indices m, where m is an integer satisfying $0 \le m \le 2$. Therefore, from the general Stone-von Neumann [11] theory of self-adjoint extensions of symmetric operators, T_0 has self-adjoint extensions. We seek to find the self-adjoint extension T in $L^2((0,\infty); w_1^\alpha)$, generated by $\ell_1^\alpha[\cdot]$, which has the X_1-Laguerre polynomials $\{L_{1,n}^\alpha\}_{n=1}^\infty$ as eigenfunctions. In order to compute the deficiency indices, it is necessary to study the behavior of solutions of $\ell_1^\alpha[y] = 0$ near each of the singular endpoints $x = 0$ and $x = \infty$.

5.1.1 *Endpoint Behavior Analysis*

The endpoint $x = 0$ is, in the sense of Frobenius, a regular singular endpoint of the X_1-Laguerre equation (3.1), for any value of $\lambda \in \mathbb{C}$. The Frobenius indicial equation (4.10) at $x = 0$ is

$$r(r + \alpha) = 0.$$

Consequently, two linearly independent solutions of $\ell_1^\alpha[y] = 0$ will behave asymptotically like

$$z_1(x) := 1 \quad \text{and} \quad z_2(x) := x^{-\alpha} \quad (0 < x \leq 1)$$

Now, for any $\alpha > 0$,

$$\int_0^1 |z_1(x)|^2 \, w_1^\alpha(x) dx < \infty;$$

however,

$$\int_0^1 |z_2(x)|^2 \, w_1^\alpha(x) dx < \infty$$

only when $0 < \alpha < 1$. In the vernacular of the Weyl limit-point/limit-circle analysis (see Section 4.2), this Frobenius analysis shows that the X_1-Laguerre differential expression is in the limit-circle case at $x = 0$ when $0 < \alpha < 1$ and in the limit-point case when $\alpha \geq 1$.

The analysis at the endpoint $x = \infty$ is more complicated since $x = \infty$ is an irregular singular endpoint of the X_1-Laguerre differential expression; consequently, another asymptotic method must be employed. Fortunately, we are able to explicitly solve the differential equation $\ell_1^\alpha[y](x) = 0$ for a basis $\{y_1(x), y_2(x)\}$ of solutions. Indeed,

$$y_1(x) = x + \alpha + 1 \quad (x > 0)$$

and, for fixed but arbitrary $x_0 > 0$,

$$y_2(x) = (x + \alpha + 1) \int_{x_0}^x \frac{e^t \, (t + \alpha)^2}{t^{\alpha+1} \, (t + \alpha + 1)^2} \, dt \quad (x > 0) \tag{5.5}$$

are independent solutions to $\ell_1^\alpha[y](x) = 0$. In fact, $y_1(x) = L_{1,1}^\alpha(x)$ is the X_1-Laguerre polynomial of degree 1, while $y_2(x)$ is obtained by the well-known reduction of order method. It is straightforward to see that

$$\int_0^\infty |y_1(x)|^2 \, w_1^\alpha(x)dx < \infty. \tag{5.6}$$

However,

Lemma 5.1. *For any $\alpha > 0$, the solution $y_2(x)$, defined in (5.5), satisfies*

$$\int_1^\infty |y_2(x)|^2 \, w_1^\alpha(x)dx = \infty; \tag{5.7}$$

that is to say, $y_2 \notin L^2((1,\infty); w_1^\alpha)$.

Proof. To begin, we note that, for $x_0 > 1$,

$$\begin{aligned}\int_{x_0}^x \frac{e^t\,(t+\alpha)^2}{t^{\alpha+1}\,(t+\alpha+1)^2}\,dt &\geq \frac{(x_0+\alpha)^2}{(x_0+\alpha+1)^2}\int_{x_0}^x \frac{e^t}{t^{\alpha+1}}dt \\ &\geq \frac{(x_0+\alpha)^2}{(x_0+\alpha+1)^2}\int_{x_0}^x e^{t/2}dt \text{ for large enough } x_0 > 0 \\ &\geq Ae^{x/2} \quad (x \geq x_1 \geq x_0)\end{aligned}$$

for large enough $x_1 \geq x_0$ and where A is some positive constant. It follows that

$$\begin{aligned}|y_2(x)|^2 &= (x+\alpha+1)^2\left(\int_{x_0}^x \frac{e^t\,(t+\alpha)^2}{t^{\alpha+1}\,(t+\alpha+1)^2}\,dt\right)^2 \\ &\geq A^2(x+\alpha+1)^2 e^x \text{ for } x \geq x_1.\end{aligned}$$

Hence, for any $\alpha > 0$, we see that

$$\begin{aligned}\int_1^\infty |y_2(x)|^2\, w_1^\alpha(x)dx &\geq \int_{x_1}^\infty |y_2(x)|^2\, w_1^\alpha(x)dx \\ &\geq A^2\int_{x_1}^\infty \frac{x^\alpha(x+\alpha+1)^2}{(x+\alpha)^2}dx \\ &\geq A^2\int_{x_1}^\infty x^\alpha dx = \infty.\end{aligned}$$

$\square$

To summarize,

Theorem 5.1. *For $\alpha > 0$, let $\ell_1^\alpha[\cdot]$ be the X_1-Laguerre differential expression, defined in* (3.2) *or* (5.1)*, on the interval* $(0, \infty)$.

(a) *$\ell_1^\alpha[\cdot]$ is in the limit-point case at $x = 0$ when $\alpha \geq 1$ and is in the limit-circle case at $x = 0$ when $0 < \alpha < 1$;*

(b) *$\ell_1^\alpha[\cdot]$ is in the limit-point case at $x = \infty$ for any choice of $\alpha > 0$.*

Consequently,

Theorem 5.2. *Let T_0 be the minimal operator in $L^2((0,\infty); w_1^\alpha)$, defined in* (5.4)*, generated by the X_1-Laguerre differential expression $\ell_1^\alpha[\cdot]$.*

(a) *If $0 < \alpha < 1$, the deficiency index of T_0 is $(1, 1)$;*

(b) *If $\alpha \geq 1$, the deficiency index of T_0 is $(0, 0)$.*

5.1.2 Spectral Analysis for $\alpha \geq 1$

From Theorem 5.2(b), the next result follows from the general theory of self-adjoint extensions of symmetric operators and from Chapter 4, in particular the completeness of the X_1-Laguerre polynomials in $L^2((0,\infty); w_1^\alpha)$ and the fact that $\lambda_n \to \infty$ as $n \to \infty$, where λ_n is defined in (3.3).

Theorem 5.3. *For $\alpha \geq 1$, the maximal and minimal operators coincide. In this case, $T_0 = T_1$ is self-adjoint and is the only self-adjoint extension of the minimal operator T_0 in $L^2((0,\infty); w_1^\alpha)$. Furthermore, in this case, the X_1-Laguerre polynomials $\{L_{1,n}^\alpha\}_{n=1}^\infty$ are eigenfunctions of T_0; the spectrum $\sigma(T_0)$ consists only of eigenvalues and is given by*

$$\sigma(T_0) = \mathbb{N}_0.$$

Hence, for $\alpha \geq 1$, no boundary condition restrictions of the maximal domain are needed to generate a self-adjoint extension of the minimal operator T_0.

5.1.3 *Spectral Analysis for* $0 < \alpha < 1$

From Theorem 5.2, the general Glazman-Krein-Naimark theory implies that one appropriate boundary condition at $x = 0$ is needed in order to obtain a self-adjoint extension of the minimal operator T_0. Necessarily, this boundary condition takes the form

$$[f, g_0](0) = 0 \quad (f \in \Delta),$$

for some appropriately chosen $g_0 \in \Delta \setminus \mathcal{D}(T_0)$, where $[\cdot, \cdot](\cdot)$ is the sesquilinear form defined in (5.3). We show that $g_0(x) = 1$ is such an appropriate function which generates the self-adjoint extension T having the X_1-Laguerre polynomials $\{L^{\alpha}_{1,n}\}_{n=1}^{\infty}$ as eigenfunctions. Notice that, for any $\alpha > 0$, the function $1 \in \Delta$.

The function $y(x) = x^{-\alpha} \in L^2((0,\infty); w_1^{\alpha})$ if and only if $0 < \alpha < 1$. Remarkably, a calculation shows that

$$\ell_1^{\alpha}[x^{-\alpha}] = -(\alpha + 1)x^{-\alpha};$$

consequently, $x^{-\alpha} \in \Delta$ when $0 < \alpha < 1$. Moreover, from (5.3), we find that

$$[x^{-\alpha}, 1](0) = \alpha \lim_{x \to 0^+} \frac{e^{-x}}{(x+\alpha)^2} = \frac{1}{\alpha} \neq 0.$$

Hence $1 \notin \mathcal{D}(T_0)$. Therefore, from the Glazman-Krein-Naimark theorem (4.6), we obtain the following result.

Theorem 5.4. *Suppose* $0 < \alpha < 1$. *The operator*

$$T : \mathcal{D}(T) \subset L^2((0,\infty); w_1^{\alpha}) \to L^2((0,\infty); w_1^{\alpha}),$$

defined by

$$Tf = \ell_1^{\alpha}[f]$$

$$f \in \mathcal{D}(T) := \{f \in \Delta \mid [f, 1](0) = 0\},$$

is self-adjoint in $L^2((0,\infty);w_1^\alpha)$ and has the X_1-Laguerre polynomials $\{L_{1,n}^\alpha\}_{n=1}^\infty$ as eigenfunctions. Moreover, the spectrum of T consists only of eigenvalues and is given by

$$\sigma(T)=\mathbb{N}_0.$$

5.2 Spectral Analysis of the X_1-Jacobi Polynomials

The spectral analysis of the X_1-Jacobi polynomials was completed for this thesis and may be found in [36].

In Lagrangian symmetric form, the X_1-Jacobi differential expression (3.8) is given by

$$\begin{aligned}\ell_{\alpha,\beta}[y](x) &= \frac{1}{w_1^{\alpha,\beta}(x)}\left(-\left(p_{\alpha,\beta}(x)y'(x)\right)'+q_{\alpha,\beta}(x)y(x)\right), \text{ where}\\ p_{\alpha,\beta}(x) &:= \frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{(x-b)^2}, \text{ and}\\ q_{\alpha,\beta}(x) &:= 2a\left(\frac{1-bx}{b-x}\right)(x-c)\frac{(1-x)^\alpha(1+x)^\beta}{(x-b)^2}\end{aligned}$$

where $w_1^{\alpha,\beta}(x)$ is the X_1-Jacobi weight function (3.6).

The maximal domain associated with the differential expression $\ell_1^{\alpha,\beta}[\cdot]$ in the Hilbert space $L^2((-1,1);w_1^{\alpha,\beta})$ is

$$\Delta := \left\{f:(-1,1)\to\mathbb{C} \mid f,f'\in AC_{\text{loc}}(-1,1); f,\ell_1^{\alpha,\beta}[f]\in L^2((-1,1);w_1^{\alpha,\beta})\right\}. \quad (5.8)$$

The associated maximal operator

$$S_1:\mathcal{D}(S_1)\subset L^2((-1,1);w_1^{\alpha,\beta})\to L^2((-1,1);w_1^{\alpha,\beta}),$$

is defined by

$$S_1 f=\ell_1^{\alpha,\beta}[f] \quad (5.9)$$

$$f\in\mathcal{D}(S_1):=\Delta.$$

For $f,g\in\Delta$, Green's formula can be written as

$$\int_{-1}^1 \ell_1^{\alpha,\beta}[f](x)\overline{g}(x)w_1^{\alpha,\beta}(x)dx=[f,g](x)\mid_{x=-1}^{x=1}+\int_{-1}^1 f(x)\ell_1^{\alpha,\beta}[\overline{g}](x)w_1^{\alpha,\beta}(x)dx, \quad (5.10)$$

where $[\cdot,\cdot](\cdot)$ is the sesquilinear form defined by

$$[f,g](x) := \frac{(1-x)^{\alpha+1}(1+x)^{\beta+1}}{(x-b)^2}\left(f(x)\overline{g}'(x) - f'(x)\overline{g}(x)\right) \quad (-1 < x < 1), \quad (5.11)$$

and

$$[f,g](x)\mid_{x=-1}^{x=1} := [f,g](1) - [f,g](-1).$$

By definition of Δ (and the classical Hölder's inequality), notice that the limits

$$[f,g](-1) := \lim_{x\to -1^+}[f,g](x) \quad \text{and} \quad [f,g](1) := \lim_{x\to 1^-}[f,g](x)$$

exist and are finite for each $f, g \in \Delta$.

It is not difficult to see that Δ is dense in $L^2((-1,1); w_1^{\alpha,\beta})$; consequently, the adjoint S_0 of S_1 exists as a densely defined operator in $L^2((-1,1); w_1^{\alpha,\beta})$. For obvious reasons, S_0 is called the minimal operator associated with $\ell_1^{\alpha,\beta}[\cdot]$. From [1] or [38], this minimal operator

$$S_0 : \mathcal{D}(S_0) \subset L^2((-1,1); w_1^{\alpha,\beta}) \to L^2((-1,1); w_1^{\alpha,\beta})$$

is defined by

$$S_0 f = \ell_1^{\alpha,\beta}[f] \quad (5.12)$$
$$f \in \mathcal{D}(S_0) := \{f \in \Delta \mid [f,g]\mid_{x=-1}^{x=1} = 0 \text{ for all } g \in \Delta\}.$$

The minimal operator S_0 is a closed, symmetric operator in $L^2((-1,1); w_1^{\alpha,\beta})$; furthermore, because the coefficients of $\ell_1^{\alpha,\beta}[\cdot]$ are real, S_0 necessarily has equal deficiency indices m, where m is an integer satisfying $0 \le m \le 2$. Therefore, from the general Stone-von Neumann [11] theory of self-adjoint extensions of symmetric operators, S_0 has self-adjoint extensions. We seek to find the self-adjoint extension S in $L^2((-1,1); w_1^{\alpha,\beta})$, generated by $\ell_1^{\alpha,\beta}[\cdot]$, which has the X_1-Jacobi polynomials $\{P_{1,n}^{(\alpha,\beta)}\}_{n=1}^{\infty}$ as eigenfunctions. In order to compute the deficiency indices, it is necessary to study the behavior of solutions of $\ell_1^{\alpha,\beta}[y] = 0$ near each of the singular endpoints $x = -1$ and $x = 1$.

5.2.1 Endpoint Behavior Analysis

As the following result states, the behavior at the endpoint $x = 1$, or $x = -1$, depends only on the parameter α or β, respectively.

First, we apply the Frobenius analysis to the endpoint $x = 1$. Multiplying the differential expression, we obtain

$$\left(\frac{x-1}{x+1}\right)\left(\ell_1^{\alpha,\beta}[y] - \lambda y\right) = (x-1)^2 y''(x) + p(x)(x-1)y'(x) + q(x)y(x)$$

with

$$p(x) = \frac{2a(1-bx)(x-c)}{(b-x)(x+1)} \quad \text{and} \quad q(x) = \left(\frac{1-x}{x+1}\right)\left[\lambda + \frac{2a(1-bx)}{(b-x)}\right].$$

This yields the indicial equation (4.10):

$$0 = r(r-1) + rp(1) + q(1) = r(r-1-a(1-c)).$$

Using the definitions of a, b and c in terms of the parameters α and β, we see that

$$a(1-c) = -\alpha - 1.$$

Therefore, two linearly independent solutions of $\ell_1^{\alpha,\beta}[y] = 0$ asymptotically (near $x = 1$, e.g. on the interval $(0,1)$) behave like

$$z_1(x) = 1 \quad \text{and} \quad z_2(x) = (x-1)^{-\alpha}.$$

We have

$$\int_0^1 |z_1(x)|^2 w_1^{\alpha,\beta}(x)dx < \infty$$

for all feasible values for α and β (recall assumptions (3.5)).

However,

$$\int_0^1 |z_2(x)|^2 w_1^{\alpha,\beta}(x)dx < \infty,$$

if and only if $\alpha < 1$, independent of the value of β.

Second, we see that the endpoint $x = -1$ works out by analogy: By applying the Frobenius method to the expression $\left(\frac{x+1}{x-1}\right)\left(\ell_1^{\alpha,\beta}[y] - \lambda y\right)$, we obtain the indicial equation

$$0 = r(r + 1 + a(1 + c)).$$

And the two linearly independent solutions (near $x = -1$)

$$y_1(x) = 1 \quad \text{and} \quad y_2(x) = (x + 1)^{-\beta}.$$

To summarize,

Theorem 5.5. *For the parameters α, β recall the restrictions* (3.5)*, let $\ell_{\alpha,\beta}[\cdot]$ be the X_1-Jacobi differential expression given by 3.8 on the interval* $(-1, 1)$*.*

(a) At the endpoint $x = 1$, the expression $\ell_1^{\alpha,\beta}[\cdot]$ is limit-point for $\alpha \geq 1$ and limit-circle in case $\alpha < 1$.

(b) At the endpoint $x = -1$, the expression $\ell_1^{\alpha,\beta}[\cdot]$ is limit-point for $\beta \geq 1$ and limit-circle in case $\beta < 1$.

Let us mention an immediate consequence of Theorem 5.5, which tells us the number of boundary conditions we need to choose in order to achieve self-adjointness.

Corollary 5.1. *Consider the minimal operator S_0 in $L^2((-1, 1); w_1^{\alpha,\beta})$ as defined in* (5.12)*, generated by the X_1-Jacobi differential expression $\ell_1^{\alpha,\beta}$.*

(a) For $\alpha, \beta \geq 1$, the minimal operator S_0 has deficiency index $(0, 0)$*.*

(b) For $\alpha \geq 1$, and $\beta < 1$, the minimal operator S_0 has deficiency index $(1, 1)$*. The same is true for $\alpha < 1$ and $\beta \geq 1$.*

(c) For $\alpha, \beta < 1$, the minimal operator S_0 has deficiency index $(2, 2)$*.*

5.2.2 *Spectral Analysis*

We are now ready to prove the main results of this section. Theorem 5.6 gives the appropriate domain for our self-adjoint operator S and Theorem 5.7 indicates the eigenvalues and eigenfunctions for S.

Theorem 5.6. *The domain $\mathcal{D}(S)$ consists of functions from the maximal domain*

$$f \in \Delta = \left\{ f : (0, \infty) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); f, \ell_1^{\alpha,\beta}[f] \in L^2((-1,1); w_1^{\alpha,\beta}) \right\}$$

such that the following condition(s) are satisfied:

- *Case 1: For $\alpha, \beta \geq 1$, no extra condition is necessary.*
- *Case 2: For $\alpha \geq 1$ and $\beta < 1$,*

$$\lim_{x \to -1^+} (1+x)^{\beta+1} f'(x) = 0 \,.$$

- *Case 3: For $\alpha < 1$ and $\beta \geq 1$,*

$$\lim_{x \to 1^-} (1-x)^{\alpha+1} f'(x) = 0 \,.$$

- *Case 4: For $\alpha, \beta < 1$,*

$$\lim_{x \to 1^-} (1-x)^{\alpha+1} f'(x) = \lim_{x \to -1^+} (1+x)^{\beta+1} f'(x) = 0 \,.$$

Proof. If the parameters satisfy $\alpha, \beta \geq 1$, then it follows trivially from part (a) of Corollary 5.1 that the minimal operator S_0 given by (5.12) coincides with the maximal operator S_1 given by (5.9); i.e. we have $S_0 = S_1$, including their domains $\Delta = \mathcal{D}(S_0)$.

Let $\alpha < 1$ and $\beta \geq 1$. By part (b) of Corollary 5.1, the deficiency index equals $(1,1)$. In particular, $\mathcal{D}(S_0)$ is a subspace of Δ of codimension two. Let us define a corresponding self-adjoint operator S via reducing the maximal domain Δ.

We will reduce the maximal domain by imposing a suitable condition invoking the sesquilinear form $[\cdot,\cdot](\cdot)$ given by (5.11). By Theorem 5.5 the restriction will be at $x=1$. First note that $h(x)=(1-x)^{-\alpha}\in\Delta$, since

$$\ell_1^{\alpha,\beta}[h]=\mathcal{O}((1-x)^{-\alpha}) \qquad \text{(near } x=1\text{)},$$

which implies $\ell_1^{\alpha,\beta}[h]\in L^2((-1,1);w_1^{\alpha,\beta})$ as $\alpha<1$. Further, the constant function satisfies $1\in\Delta$, while the sesquilinear form

$$[h,1]|_{x=-1}^{x=1}=[h,1](1)=-\frac{2^{\beta+1}}{(1-b)^2}\neq 0.$$

For the parameters, $\alpha\geq 1$ and $\beta<1$ the corresponding statement can be proved by analogy. If $\alpha,\beta<1$, we combine the above. □

Theorem 5.7. *The X_1-Jacobi polynomials $\left\{P_{1,n}^{(\alpha,\beta)}\right\}_{n=1}^{\infty}$ form a complete set of eigenfunctions for the self-adjoint operators defined in the previous theorem. Moreover, the spectrum of S is pure point and consists of the set of eigenvalues*

$$\lambda_n=(n-1)(\alpha+\beta+n) \text{ and } n\in\mathbb{N}$$

corresponding to the eigenfunctions $\left\{P_{1,n}^{(\alpha,\beta)}\right\}_{n=1}^{\infty}$.

In order to perform the spectral analysis for the self-adjoint operator in Theorem 5.6 and prove Theorem 5.7 we require the following lemma.

Lemma 5.2. *We have $P_{1,n}^{(\alpha,\beta)}(x)\in\mathcal{D}(S)$.*

Proof. Observe that for each $n\in\mathbb{N}$, $P_{1,n}^{(\alpha,\beta)}(x)$ are polynomials by definition. Therefore, $P_{1,n}^{(\alpha,\beta)}(x)\in\Delta$, where the maximal domain Δ is given by (5.8). For fixed $n\in\mathbb{N}$, $P_{1,n}^{(\alpha,\beta)}(x)$ is bounded at the endpoints; that is, $\left|P_{1,n}^{(\alpha,\beta)}(1)\right|<\infty$ and $\left|P_{1,n}^{(\alpha,\beta)}(-1)\right|<\infty$. Therefore, we obtain

$$\left[P_{1,n}^{(\alpha,\beta)},1\right]=\mathcal{O}\left((1-x)^{\alpha+1}(1+x)^{\beta+1}\right)$$

near $x=1$ and $x=-1$. Thus we have $\left[P_{1,n}^{(\alpha,\beta)},1\right](1)=\left[P_{1,n}^{(\alpha,\beta)},1\right](-1)=0$ and it follows that $P_{1,n}^{(\alpha,\beta)}(x)\in\mathcal{D}(S)$ for all feasible α and β. □

Proof of Theorem 5.7. Lemma 5.2 implies Theorem 5.7. □

5.3 *Spectral Analysis of the X_m^I-Laguerre Polynomials*

The spectral analysis of the X_m^I-Laguerre polynomials was completed for this thesis and may be found in [35].

Recall that the Type I differential expression (3.14) with its associated weight function (3.15) may be written in symmetric form (3.16). The maximal domain associated with $\ell_m^{I,\alpha}[\cdot]$ in the Hilbert space $L^2\left((0,\infty);w_m^{I,\alpha}\right)$ is defined to be

$$\Delta := \left\{f:(0,\infty)\to\mathbb{C} \mid f,f'\in AC_{\text{loc}}(0,\infty); f,\ell_m^{I,\alpha}[f]\in L^2((0,\infty);w_m^{I,\alpha})\right\}.$$

The associated maximal operator

$$T_1^I:\mathcal{D}(T_1^I)\subset L^2((0,\infty);w_m^{I,\alpha})\to L^2((0,\infty);w_m^{I,\alpha}),$$

is defined to be

$$T_1^I f = \ell_m^{I,\alpha}[f] \tag{5.13}$$
$$f\in\mathcal{D}(T_1^I) := \Delta.$$

For $f,g\in\Delta$, Green's formula can be written as

$$\int_0^\infty \ell_m^{I,\alpha}[f](x)\bar{g}(x)w_m^{I,\alpha}(x)dx = [f,g](x)\big|_{x=0}^{x=\infty} + \int_0^\infty f(x)\ell_m^{\alpha}[\bar{g}](x)w_m^{I,\alpha}(x)dx\,,$$

where $[\cdot,\cdot](\cdot)$ is the sesquilinear form defined by

$$[f,g](x) := \frac{x^{\alpha+1}e^{-x}}{(L_m^{\alpha-1}(-x))^2}(f(x)\bar{g}'(x)-f'(x)\bar{g}(x)) \quad (0<x<\infty)$$

and where

$$[f,g](x)\,|_{x=0}^{x=\infty} := [f,g](\infty)-[f,g](0)\,.$$

The adjoint of the maximal operator in $L^2\left((0,\infty);w_m^{I,\alpha}\right)$ is the minimal operator, $T_0^I:\mathcal{D}(T_0)\subset L^2((0,\infty);w_m^{I,\alpha})\to L^2((0,\infty);w_m^{I,\alpha})$. The minimal operator is defined by

$$T_0^I f = \ell_m^{I,\alpha}[f] \tag{5.14}$$
$$f \in \mathcal{D}(T_0^I) := \{f \in \Delta \mid [f,g] \mid_{x=0}^{x=\infty} = 0 \text{ for all } g \in \Delta\}.$$

We seek to find the self-adjoint extension T^I in $L^2\left((0,\infty); w_m^{I,\alpha}\right)$ generated by $\ell_\alpha^I[\cdot]$, which has the X_m^I-Laguerre polynomials $\left\{L_{m,n}^{I,\alpha}\right\}_{n=m}^{\infty}$ as eigenfunctions. To do this, we first need to study the behavior of solutions near the singular endpoints $x = 0$ and $x = \infty$ to determine the deficiency indices; and secondly, we will determine the number of (if any) and appropriate boundary conditions.

5.3.1 Endpoint Analysis for the X_m^I-Laguerre Expression

Our main goal is to show the following theorem:

Theorem 5.8. *For $\alpha > 0$, let $\ell_m^{I,\alpha}[\cdot]$ be the X_m^I-Laguerre differential expression on the interval $(0,\infty)$.*

(a) $\ell_m^{I,\alpha}[\cdot]$ is in the limit-circle case at $x = 0$ when $0 < \alpha < 1$ and is in the limit-point case at $x = 0$ when $\alpha \geq 1$.

(b) $\ell_m^{I,\alpha}[\cdot]$ is in the limit-point case at $x = \infty$ for any choice of $\alpha > 0$.

Proof. The endpoint $x = 0$ is, in the sense of Frobenius, a regular singular endpoint of the X_m^I-Laguerre expression $\ell_m^{I,\alpha}[y] = \lambda y$ for any value $\lambda \in \mathbb{C}$. The Frobenius indicial equation at $x = 0$ is

$$r(r+\alpha) = 0\,.$$

Consequently, two linearly independent solutions of $\ell_m^{I,\alpha}[y] - \lambda y = 0$ will behave asymptotically like

$$z_1(x) := 1 \quad \text{ and } \quad z_2(x) := x^{-\alpha}$$

near $x = 0$.

Fix $\epsilon > 0$. Now, for any $\alpha > 0$,

$$\int_0^{\epsilon} |z_1(x)|^2 \, w_m^{I,\alpha}(x)\, dx = \int_0^{\epsilon} \frac{x^{\alpha} e^{-x}}{(L_m^{\alpha-1}(-x))^2}\, dx < \infty\,.$$

The integral is finite near $x = 0$ since, $x^{\alpha} = 0$, $e^{-x} = 1$ and $L_m^{\alpha-1}(0) \neq 0$ for all $\alpha > 0$. $L_m^{\alpha-1}(0) \neq 0$ follows from the identity for evaluating the classical Laguerre polynomials at 0,

$$L_n^{\alpha}(0) = \frac{(\alpha+1)_n}{n!}\,,$$

where

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1) & \text{if } n \geq 0 \\ 1 & \text{if } n = 0\,. \end{cases}$$

However,

$$\int_0^{\epsilon} |z_2(x)|^2 \, w_m^{I,\alpha}(x)\, dx = \int_0^{\epsilon} \frac{x^{-\alpha} e^{-x}}{(L_m^{\alpha-1}(-x))^2}\, dx < \infty$$

if and only if $0 < \alpha < 1$.

In the language of the Weyl limit-point and limit-circle analysis, this Frobenius analysis shows that the X_m^I-Laguerre differential expression is in the limit-circle case at $x = 0$ when $0 < \alpha < 1$ and is in the limit-point case at $x = 0$ when $\alpha \geq 1$.

Since $x = \infty$ is an irregular singular endpoint of the X_m^I-Laguerre differential expression, the analysis is more complicated than at the endpoint $x = 0$. As a result, another asymptotic method must be employed. Fortunately, we are able to explicitly solve the differential equation

$$\ell_m^{I,\alpha}[y](x) - \lambda y(x) = 0$$

for a basis $\{y_1(x), y_2(x)\}$ of solutions. Let $y_1(x) = L_{m,m}^{I,\alpha}(x)$, the X_m^I-Laguerre polynomial of degree m, i.e.

$$y_1(x) = L_{m,m}^{I,\alpha}(x) = L_m^{\alpha}(-x) \quad (x > 0)\,.$$

Using the well-known reduction of order method, we obtain $y_2(x)$. For fixed, but arbitrary $x_0 > 0$,

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{t^{-(\alpha+1)} e^t (L_m^{\alpha-1}(-t))^2}{y_1(t)^2} \, dt \, .$$

Note that $y_1(x)$ and $y_2(x)$ are linearly independent solutions to $\ell_m^{I,\alpha}[y](x) - \lambda y(x) = 0$.

Since $y_1(x)$ is a solution to the X_m^I-Laguerre expression, by construction, it is in the Hilbert space $L^2((\epsilon, \infty), w_m^{I,\alpha})$. However in Lemma 5.3 below, we see that $y_2(x) \notin L^2((\epsilon, \infty), w_m^{I,\alpha})$. □

Lemma 5.3. *For any $\alpha > 0$ and a fixed $\epsilon > 0$, the solution $y_2(x)$, defined above, satisfies*

$$\int_{\epsilon}^{\infty} |y_2(x)|^2 \, w_m^{I,\alpha} \, dx = \infty \, ;$$

that is to say, $y_2 \notin L^2((\epsilon, \infty), w_m^{I,\alpha})$.

Proof. To begin, we note that

$$\lim_{x \to \infty} \frac{L_m^{\alpha-1}(-x)}{y_1(x)^2} = 1 \, .$$

For $x_0 > \epsilon$,

$$\begin{aligned} \int_{\epsilon}^{x} \frac{e^t}{t^{\alpha+1}} \left(\frac{L_m^{\alpha-1}(-t)}{y_1(t)} \right)^2 dt &\geq \left(\frac{L_m^{\alpha-1}(-x_0)}{y_1(x_0)} \right)^2 \int_{x_0}^{x} \frac{e^t}{t^{\alpha+1}} \, dt \\ &\geq A \int_{x_1}^{x} e^{t/2} \, dt \quad \text{(for some } x_1 \text{ with } x \geq x_1 \geq x_0) \, . \end{aligned}$$

The final inequality is based on the following observation: there exists $x_1 > 0$ such that for $t \geq x_1$,

$$t^{\alpha+1} \leq e^{t/2}$$

thus

$$\frac{e^t}{t^{\alpha+1}} \geq \frac{e^t}{e^{t/2}} = e^{t/2} \quad (t \geq x_1) \, .$$

Therefore, for large enough $x_1 \geq x_0$ and where A is some positive constant, it follows that

$$\begin{aligned} |y_2(x)|^2 &= y_1(x)^2 \left(\int_\epsilon^x \frac{e^t}{t^{\alpha+1}} \left(\frac{L_m^{\alpha-1}(-t)}{y_1(t)} \right)^2 dt \right)^2 \\ &= A^2 y_1(x)^2 e^x \quad \text{for } x \geq x_1 . \end{aligned}$$

Hence, for any $\alpha > 0$,

$$\begin{aligned} \int_\epsilon^\infty |y(t)|^2 \, w_m^{I,\alpha}(t)\, dt &\geq \int_{x_1}^\infty |y_2(t)|^2 \, w_m^{I,\alpha}(t)\, dt \\ &\geq A^2 \int_{x_1}^\infty \frac{t^\alpha y_1(t)^2}{L_m^{\alpha-1}(-t)}\, dt \\ &\geq \int_{x_1}^\infty t^\alpha \, dt = \infty . \end{aligned}$$

The last inequality follows since $y_1(x) = L_m^\alpha(-x)$ has no positive zeros. □

As a result,

Theorem 5.9. *Let T_0^I be the minimal operator in $L^2\left((0,\infty); w_m^{I,\alpha}\right)$ which is generated by the X_m^I-Laguerre differential expression $\ell_m^{I,\alpha}[\cdot]$.*

(a) If $0 < \alpha < 1$, the deficiency index of T_0^I is $(1,1)$;

(b) If $\alpha \geq 1$, the deficiency index of T_0^I is $(0,0)$.

5.3.2 Spectral Analysis for $0 < \alpha < 1$

When $0 < \alpha < 1$, there are infinitely many self-adjoint extensions. In order to obtain a self-adjoint extension of the minimal operator T_0^I which has the X_m^I-Laguerre polynomials $\left\{L_{m,n}^{I,\alpha}\right\}_{n=m}^\infty$ as eigenfunctions, we are required to impose one boundary condition, $g_0 \in \Delta \setminus \mathcal{D}(T_0^I)$ such that

$$[f, g_0](0) = 0 \quad (f \in \Delta) .$$

We claim $g_0 = 1$ is an appropriate choice.

Note that the function $y(x) = x^{-\alpha} \in L^2\left((0,\infty); w_m^{I,\alpha}\right)$ if and only if $0 < \alpha < 1$. The calculation,

$$\ell_m^{I,\alpha}[x^{-\alpha}] = -\alpha x^{-\alpha}\,,$$

shows that $x^{-\alpha} \in \Delta$ for $0 < \alpha < 1$. Additionally,

$$[x^{-\alpha}, 1](0) = \alpha \lim_{x\to 0^+} \frac{e^{-x}}{(L_m^{\alpha-1}(-x))^2} \neq 0\,.$$

Hence, $g_0 \neq 1 \notin \mathcal{D}(T_0^I)$. Therefore, we obtain the following result.

Theorem 5.10. *Suppose* $0 < \alpha < 1$. *The operator*

$$T^I : \mathcal{D}(T^I) \subset L^2\left((0,\infty); w_m^{I,\alpha}\right) \to L^2\left((0,\infty); w_m^{I,\alpha}\right)\,,$$

defined by

$$T^I f = \ell_m^{I,\alpha}[f]$$
$$f \in \mathcal{D}(T^I) : = \{f \in \Delta \mid [f,1](0) = 0\}\,,$$

is self-adjoint in $L^2\left((0,\infty); w_m^{I,\alpha}\right)$ *and has the* X_m^I*-Laguerre polynomials* $\left\{L_{m,n}^{I,\alpha}\right\}_{n=m}^{\infty}$ *as eigenfunctions. Moreover, the spectrum of* T^I *consists only of eigenvalues and is given by*

$$\sigma(T^I) = \mathbb{N}_0\,.$$

5.3.3 Spectral Analysis for $\alpha \geq 1$

As a consequence of part (b) of Theorem 5.9, for $\alpha \geq 1$,

Theorem 5.11. *For* $\alpha \geq 1$, *the maximal and minimal operators coincide; that is,* $T_0^I = T_1^I$. *This is the unique self-adjoint operator in* $L^2\left((0,\infty); w_m^{I,\alpha}\right)$ *generated by* $\ell_m^{I,\alpha}[\cdot]$. *Furthermore, in this case, the* X_m^I*-Laguerre polynomials* $\left\{L_{m,n}^{I,\alpha}\right\}_{n=m}^{\infty}$ *are eigenfunctions of* T_0^I*; the spectrum* $\sigma(T_0^I)$ *consists only of eigenvalues and is given by the following*

$$\sigma(T_0^I) = \mathbb{N}_0\,.$$

Therefore, for $\alpha \geq 1$, no boundary condition restrictions of the maximal domain are needed to generate a self-adjoint extension of the minimal operator T_0^I.

5.4 Spectral Analysis of the X_m^{II}-Laguerre Polynomials

The spectral analysis of the X_m^{II}-Laguerre polynomials was completed for this thesis and may be found in [35].

Recall that the Type II differential expression (3.17) with its associated weight function (3.18) may be written in symmetric form (3.19). The maximal domain associated with $\ell_m^{II,\alpha}[\cdot]$ in the Hilbert space $L^2\left((0,\infty); w_m^{II,\alpha}\right)$ is defined to be

$$\Delta := \left\{ f : (0,\infty) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(0,\infty); f, \ell_m^{II,\alpha}[f] \in L^2((0,\infty); w_m^{II,\alpha}) \right\}.$$

The associated maximal operator

$$T_1^{II} : \mathcal{D}(T_1^{II}) \subset L^2((0,\infty); w_m^{II,\alpha}) \to L^2((0,\infty); w_m^{II,\alpha}),$$

is defined by

$$T_1^{II} f = \ell_m^{II,\alpha}[f] \tag{5.15}$$
$$f \in \mathcal{D}(T_1^{II}) := \Delta.$$

For $f, g \in \Delta$, Green's formula can be written as

$$\int_0^\infty \ell_m^{II,\alpha}[f](x)\bar{g}(x) w_m^{II,\alpha}(x)dx = [f,g](x)\Big|_{x=0}^{x=\infty} + \int_0^\infty f(x)\ell_\alpha[\bar{g}](x) w_m^{II,\alpha}(x)dx\,,$$

where $[\cdot,\cdot](\cdot)$ is the sesquilinear form defined by

$$[f,g](x) := \frac{x^{\alpha+1}e^{-x}}{(L_m^{-\alpha-1}(x))^2}(f(x)\bar{g}'(x) - f'(x)\bar{g}(x)) \quad (0 < x < \infty)$$

and where

$$[f,g](x)\,|_{x=0}^{x=\infty} := [f,g](\infty) - [f,g](0)\,.$$

The adjoint of the maximal operator in $L^2\left((0,\infty); w_m^{II,\alpha}\right)$ is the minimal operator, $T_0^{II} : \mathcal{D}(T_0^{II}) \subset L^2((0,\infty); w_m^{II,\alpha}) \to L^2((0,\infty); w_m^{II,\alpha})$. The minimal operator is defined by

$$T_0^{II} f = \ell_m^{II,\alpha}[f] \tag{5.16}$$
$$f \in \mathcal{D}(T_0^{II}) : = \{f \in \Delta \mid [f,g] \mid_{x=0}^{x=\infty} = 0 \text{ for all } g \in \Delta\}.$$

In the same manner as in the X_m^I-Laguerre case, we seek to find the self-adjoint extension T^{II} in $L^2\left((0,\infty); w_m^{II,\alpha}\right)$ generated by $\ell_m^{II,\alpha}[\cdot]$, which has the X_m^{II}-Laguerre polynomials $\left\{L_{m,n}^{II,\alpha}\right\}_{n=m}^{\infty}$ as eigenfunctions.

5.4.1 Endpoint Analysis for the X_m^{II}-Laguerre Expression

We will now show the following theorem for the X_m^{II}-Laguerre expression:

Theorem 5.12. *For $\alpha > m-1$ let $\ell_m^{II,\alpha}[\cdot]$ be the X_m^{II}-Laguerre differential expression on the interval $(0,\infty)$.*

(a) $\ell_m^{II,\alpha}[\cdot]$ is in the limit-point case at $x = 0$ when $\alpha \geq m-1$.

(b) $\ell_m^{II,\alpha}[\cdot]$ is in the limit-point case at $x = \infty$ for any choice of $\alpha > 0$.

Proof. We will provide a sketch of the proof, as many of the details are similar to those of the X_m^I-Laguerre case.

The endpoint $x = 0$ is, in the sense of Frobenius, a regular singular endpoint of the X_m^{II}-Laguerre expression $\ell_m^{II,\alpha}[y] = \lambda y$ for any value $\lambda \in \mathbb{C}$. The Frobenius indicial equation at $x = 0$ is the same as above and two linearly independent solutions of $\ell_m^{II,\alpha}[y] - \lambda y = 0$ behave asymptotically like

$$z_1(x) := 1 \quad \text{and } z_2(x) := x^{-\alpha}$$

near $x = 0$.

Recall that the restriction $\alpha > m-1$ is equivalent to $L_m^{-\alpha-1}(x)$ having no zeros in $[0,\infty)$. Therefore, for any $\alpha > m-1$ and $\epsilon > 0$,

$$\int_0^{\epsilon} |z_1(x)|^2 w_m^{II,\alpha}(x)\, dx = \int_0^{\epsilon} \frac{x^{\alpha}e^{-x}}{(L_m^{-\alpha-1}(x))^2}\, dx < \infty\,,$$

however,

$$\int_0^{\epsilon} |z_2(x)|^2 \, w_m^{II,\alpha}(x)\, dx = \int_0^{\epsilon} \frac{x^{-\alpha}e^{-x}}{(L_m^{-\alpha-1}(x))^2}\, dx = \infty$$

for $\alpha > m - 1 \geq 1$.

Thus, the X_m^{II}-Laguerre differential expression will be in the limit-point case for $\alpha > m - 1$.

The endpoint $x = \infty$ is an irregular singular endpoint, but we are able to explicitly solve the differential equation

$$\ell_m^{II,\alpha}[y](x) - \lambda y(x) = 0$$

for a basis $\{y_1(x), y_2(x)\}$ of solutions. Two linearly independent solutions are the X_m^{II}-Laguerre polynomial of degree m,

$$y_1(x) = L_{m,m}^{II,\alpha}(x) = (m - \alpha - 1)L_m^{-\alpha-2}\,,$$

and for fixed but arbitrary $x_0 > \epsilon$,

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{t^{-(\alpha+1)}e^t(L_m^{-\alpha-1}(t))^2}{y_1(x)^2}\, dt\,.$$

Since $y(x)$ is a solution of $\ell_m^{II,\alpha}[y](x) - \lambda y(x) = 0$, by construction $y_1(x) = L^2((0,\infty), w_m^{II,\alpha})$. However, arguing in the same manner as Lemma 5.3 for our newly constructed $y_2(x)$, it can be shown that $y_2 \notin L^2((0,\infty), w_m^{II,\alpha})$. Thus, we are in the limit-point case at $x = \infty$ for $\alpha > m - 1$. $\square$

Consequently,

Theorem 5.13. *Let T_0^{II} be the minimal operator in $L^2\left((0,\infty); w_m^{II,\alpha}\right)$ which is generated by the X_m^{II}-Laguerre differential expression $\ell_m^{II,\alpha}[\cdot]$. Then for $\alpha > m - 1$, the deficiency index of T_0^{II} is $(0,0)$.*

5.4.2 Spectral Analysis for $\alpha > m-1$

Theorem 5.14. *For $\alpha \geq m-1$, the maximal and minimal operators coincide; that is, $T_0^{II} = T_1^{II}$. This is the unique self-adjoint operator in $L^2\left((0,\infty); w_m^{II,\alpha}\right)$ generated by $\ell_m^{II,\alpha}[\cdot]$. Furthermore, in this case, the X_m^{II}-Laguerre polynomials $\left\{L_{m,n}^{II,\alpha}\right\}_{n=m}^{\infty}$ are eigenfunctions of T_0^{II}; the spectrum $\sigma(T_0^{II})$ consists only of eigenvalues and is given by the following*

$$\sigma(T_0^{II}) = \mathbb{N}_0 \, .$$

Therefore, for $\alpha \geq m-1$, no boundary condition restrictions of the maximal domain are needed to generate a self-adjoint extension of the minimal operator T_0^{II}.

5.5 Spectral Analysis of the X_m^{III}-Laguerre Polynomials

The spectral analysis of the X_m^{III}-Laguerre polynomials was completed for this thesis and also may be found in [35].

Recall that the Type III differential expression (3.20) with its associated weight function (3.22) may be written in symmetric form (3.23). The maximal domain associated with $\ell_m^{III,\alpha}[\cdot]$ in the Hilbert space $L^2\left((0,\infty); w_m^{III,\alpha}\right)$ is defined to be

$$\Delta := \left\{ f : (0,\infty) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(0,\infty); f, \ell_m^{III,\alpha}[f] \in L^2((0,\infty); w_m^{III,\alpha}) \right\}.$$

The associated maximal operator

$$T_1^{III} : \mathcal{D}(T_1^{III}) \subset L^2((0,\infty); w_m^{III,\alpha}) \to L^2((0,\infty); w_m^{III,\alpha}),$$

is defined to be

$$\begin{aligned} T_1^{III} f &= \ell_m^{III,\alpha}[f] \\ f \in \mathcal{D}(T_1^{III}) &: = \Delta. \end{aligned} \tag{5.17}$$

For $f, g \in \Delta$, Green's formula can be written as

$$\int_0^\infty \ell_m^{III,\alpha}[f](x)\overline{g}(x) w_m^{III,\alpha}(x)dx = [f,g](x)\Big|_{x=0}^{x=\infty} + \int_0^\infty f(x)\ell_\alpha[\overline{g}](x) w_m^{III,\alpha}(x)dx \, ,$$

where $[\cdot,\cdot](\cdot)$ is the sesquilinear form defined by

$$[f,g]\,(x) := \frac{x^{\alpha+1}e^{-x}}{(L_m^{-\alpha-1}(-x))^2}(f(x)\bar{g}'(x) - f'(x)\bar{g}(x)) \quad (0 < x < \infty)$$

and where

$$[f,g]\,(x)\mid_{x=0}^{x=\infty} := [f,g]\,(\infty) - [f,g]\,(0)\,.$$

The adjoint of the maximal operator in $L^2\left((0,\infty); w_m^{III,\alpha}\right)$ is the minimal operator, $T_0^{III} : \mathcal{D}(T_0^{III}) \subset L^2((0,\infty); w_m^{III,\alpha}) \to L^2((0,\infty); w_m^{III,\alpha})$. The minimal operator is defined by

$$T_0^{III} f = \ell_m^{III,\alpha}[f] \tag{5.18}$$
$$f \in \mathcal{D}(T_0^{III}) : = \{f \in \Delta \mid [f,g]\mid_{x=0}^{x=\infty} = 0 \text{ for all } g \in \Delta\}.$$

In the same manner as in the X_m^I and X_m^{II}-Laguerre cases, we seek to find the self-adjoint extension T^I in $L^2\left((0,\infty); w_m^{III,\alpha}\right)$ generated by $\ell_m^{III,\alpha}[\cdot]$, which has the X_m^{III}-Laguerre polynomials $\left\{L_{m,n}^{III,\alpha}\right\}_{n=m}^{\infty}$ as eigenfunctions.

5.5.1 Endpoint Analysis for the X_m^{III}-Laguerre Expression

Theorem 5.15. *For $-1 < \alpha < 0$, let $\ell_m^{III,\alpha}[\cdot]$ be the X_m^{III}-Laguerre differential expression on the interval $(0,\infty)$. Then $\ell_m^{III,\alpha}[\cdot]$ is in the limit-circle case at $x = 0$ when $-1 < \alpha < 0$.*

Proof. The endpoint $x = 0$ is, in the sense of Frobenius, a regular singular endpoint of the X_m^{III}-Laguerre expression $\ell_m^{III,\alpha}[y] = \lambda y$ for any value $\lambda \in \mathbb{C}$. The Frobenius indicial equation at $x = 0$ is

$$r(r+\alpha) = 0\,.$$

Consequently, two linearly independent solutions of $\ell_m^{III,\alpha}[y] - \lambda y = 0$ will behave asymptotically like

$$z_1(x) := 1 \quad \text{and} \quad z_2(x) := x^{-\alpha}$$

near $x = 0$. Looking at the behavior of these solutions near $x = 0$, we see that the X_m^{III}-Laguerre expression is in the limit-circle case when $-1 < \alpha < 0$.

For the analysis at the irregular singular endpoint, $x = \infty$, we obtain two linearly independent solutions using the Reduction of Order method. Solving the differential equation, $\ell_m^{III,\alpha}[y](x) - \lambda y(x) = 0$, we have a basis of solutions $\{y_1(x), y_2(x)\}$, where

$$y_1(x) = 1$$

is the solution of degree zero and

$$y_2(x) = \int_{x_0}^{x} t^{-(\alpha+1)} e^t (L_m^{-\alpha-1}(-t))^2 \, dt \, .$$

In the same spirit as the Type I and Type II cases, we see that $y_1(x) \in L^2((\epsilon, \infty), w_m^{III,\alpha})$, but $y_2(x) \notin L^2((\epsilon, \infty), w_m^{III,\alpha})$. □

Consequently,

Theorem 5.16. *Let T_0^{III} be the minimal operator in $L^2((0,\infty), w_m^{III,\alpha})$ which is generated by the X_m^{III}-Laguerre differential expression $\ell_m^{III,\alpha}[\cdot]$. For $-1 < \alpha < 0$, the deficiency index of T_0^{III} is $(1,1)$.*

5.5.2 Spectral Analysis

For $-1 < \alpha < 0$, we must impose one boundary condition in order to obtain a self-adjoint extension T^{III} of the minimal operator T_0^{III}. Note that the function $y(x) = x^{-\alpha}$ if and only if $-1 < \alpha < 0$. The calculation

$$\ell_m^{III,\alpha}[x^{-\alpha}] = x^{-\alpha} \left(\frac{2\alpha L_{m-1}^{-\alpha}(-x) + (1+\alpha+m) L_m^{-\alpha-1}(-x)}{L_m^{-\alpha-1}(-x)} \right)$$

shows that $x^{-\alpha} \in \Delta$ for $0 < \alpha < 1$. Furthermore,

$$[x^{-\alpha}, 1](0) = \alpha \lim_{x \to 0} \frac{e^{-x}}{L_m^{-\alpha-1}(-x)^2} \neq 0 \, .$$

Theorem 5.17. *Suppose* $-1 < \alpha < 0$. *The operator*

$$T^{III} : D(T^{III}) \subset L^2((0,\infty), w_m^{III,\alpha}) \to L^2((0,\infty), w_m^{III,\alpha})\,,$$

defined by

$$T^{III} f = \ell_m^{III,\alpha}[f]$$
$$f \in \mathcal{D}(T^{III}) := \{f \in \Delta \mid [f,1](0) = 0\}\,,$$

is self-adjoint in $L^2((0,\infty), w_m^{III,\alpha})$ *and has* $\left\{L_{m,n}^{III,\alpha}\right\}_{n=m}^{\infty}$, *the* X_m^{III}*-Laguerre polynomials, as eigenfunctions. Moreover, the spectrum of* T^{III} *consists only of eigenvalues and is given by*

$$\sigma(T^{III}) = \{1 + m - n\}\ , \quad n \in \{0, m+1, m+2, \ldots\}\ .$$

CHAPTER SIX

Sobolev Orthogonality in the Left-definite Setting

6.1 Introduction

The motivation for left-definite theory stems from the study of differential equations of the form

$$\ell[y](x) = \lambda w(x) y(x) \quad (x \in I) \tag{6.1}$$

where $\ell[\cdot]$ is a Lagrangian symmetric differential expression of order $2n$; $I = (a, b)$ is an open interval of $\mathbb{R}$; and $w(x) > 0$ on I. Recall (4.5) which states that $\ell[\cdot]$ may be given by

$$\ell[y](x) := \sum_{j=0}^{n} (-1)^j (a_j(x) y^{(j)}(x))^{(j)},$$

where each coefficient $a_j(x)$ is positive and infinitely differentiable on I. Up to this point, we have studied one particular setting—the Hilbert space $L^2(I; w)$ with associated inner product

$$(f, g) = \int_a^b f(x)\overline{g}(x) w(x)\, dx$$

—in our spectral analysis. Notice that $w(x)$ appears on the *right-hand* side of (6.1). Therefore, it is natural to refer to $L^2(I; w)$ as the classic *right-definite* spectral setting for $w^{-1}\ell[\cdot]$.

An alternative to viewing $\ell[\cdot]$ in the right-definite setting is to study $\ell[\cdot]$ within the framework of left-definite theory (see Section 6.2). We will see that left-definite theory plays a central role in studying the "extreme" cases of the X_1-Laguerre and X_1-Jacobi polynomials. Another useful tool in our analysis will be the Chisholm-Everitt Inequality, see Theorem 6.4. We will also extend our results in the extreme case to include a full-algebraic sequence of polynomials as eigenfunctions by redefining orthogonality and using Sobolev spaces (see Section 6.4).

6.2 Left-Definite Theory

The following material is borrowed from Littlejohn-Wellman [37]. Let V denote a vector space and suppose that $(\cdot,\cdot)$ is an inner product with norm $\|\cdot\|$ generated from $(\cdot,\cdot)$ such that $H=(V,(\cdot,\cdot))$ is a Hilbert space. For the expression $\ell[\cdot]$ and functions f,g from the maximal domain, we have Green's formula (4.6). A related formula is Dirichlet's formula given by

$$\int_a^b \ell[f](x)\overline{g}(x)\,dt=\sum_{j=0}^{n}\int_a^b f^{(j)}\overline{g}^{(j)}(t)\,dt+\{f,g\}(x)\Big|_{x=a}^{x=b}$$

where f,g are again from the maximal domain and $\{\cdot,\cdot\}$ is another bilinear form related to (4.7). Suppose that there exists $T:\mathcal{D}(T)\subseteq H\to H$ which is a self-adjoint extension of the minimal operator associated with $\ell[\cdot]$. Then

$$(Tf,g)=\int_a^b \ell[f](x)\overline{g}(x)\,dx=\sum_{j=0}^{n}\int_a^b a_j(x)f^{(j)}(x)\overline{g}^{(j)}(x)\,dx\quad(f,g\in\mathcal{D}(T))\,. \tag{6.2}$$

This implies that for $f,g\in\mathcal{D}(T)$, the Dirichlet form $\{f,g\}(x)\big|_{x=a}^{x=b}$ is zero. Furthermore, for the remainder of the chapter, we will suppose that T is a self-adjoint operator that is bounded below by a positive number $k>0$, i.e.,

$$(Af,f)\geq k(f,f)\qquad(f\in\mathcal{D}(A))\,,$$

In addition, $\ell[\cdot]$ generates a Sobolev Space H_1 with the *Dirichlet inner product*,

$$(f,g)_1:=\sum_{j=0}^{n}\int_a^b b_j(t)f^{(j)}(t)\overline{g}^{(t)}\,dt\quad(f,g\in\mathcal{H}_1)\,. \tag{6.3}$$

Note that from (6.2) and (6.3), we have

$$(Tf,g)=(f,g)_1\quad(f,g\in\mathcal{D}(T))\,.$$

Let $H_1=(V_1,(\cdot,\cdot)_1)$, where V_1 is a subspace of V and $(\cdot,\cdot)_1$ is an inner product on V_1. Since the inner product $(\cdot,\cdot)_1$ is generated from the left-hand side of (6.1), we refer to H_1 as the *left-definite setting* for $w^{-1}\ell[\cdot]$ and call H_1 the *left-definite* Hilbert space associated with $w^{-1}\ell[\cdot]$ and the pair (H,T). In other words, H_1 is the left-definite Hilbert space if each of the following conditions hold:

(a) H_1 is a Hilbert space,

(b) $\mathcal{D}(T)$ is a subspace of V_1,

(c) $\mathcal{D}(T)$ is dense in H_1,

(d) $(f,f)_1 \geq k(f,f) \qquad (f \in V_1)$,

(e) $(f,g)_1 = (Af,g) \qquad (f \in \mathcal{D}(A), g \in V_1)$.

H_1 may also be defined as the closure of $\mathcal{D}(T)$ in the topology generated by the norm $\|\cdot\|_1 = (\cdot,\cdot)_1^{1/2}$. In [37] and Theorem 6.1, we see that there is actually a continuum of left-definite spaces associated with T; therefore, H_1 is really the *first* left-definite space associated with T.

Theorem 6.1. *Suppose T is a self-adjoint operator in the Hilbert space $H = (V, (\cdot,\cdot))$ that is bounded below by kI, where $k > 0$. Let $r > 0$. Define $H_r = (V_r, (\cdot,\cdot)_r)$ by*

$$V_r = \mathcal{D}(T^{r/2}),$$

and

$$(f,g)_r = (T^{r/2}f, A^{r/2}g) \qquad (f,g \in V_r).$$

Then H_r is an r^{th} left-definite space associated with the pair (H,T) in the sense of the above definition. Moreover, suppose $H_r = (V_r, (\cdot,\cdot)_r)$ and $H_r' = (V_r', (\cdot,\cdot)_r')$ are r^{th} left-definite spaces associated with the pair (H,T). Then $V_r = V_r'$ and $(f,g)_r = (f,g)_r'$ for all $f,g \in V_r = V_r'$; i.e. $H_r = H_r'$. Consequently, $H_r = (V_r, (\cdot,\cdot)_r)$, as defined above, is the unique r^{th} left-definite Hilbert space associated with (H,T).

Let $r > 0$ and suppose H_r is the r^{th} left-definite space associated with (H,T). If there exists a self-adjoint operator $T_r : H_r \to H_r$ that is a restriction of T, i.e.,

$$T_r f = Tf$$

$$f \in \mathcal{D}(T_r) \subset \mathcal{D}(T),$$

then we call such an operator an r^{th} *left-definite operator associated with* (H,T). The following theorem shows that associated with the unique r^{th} left-definite Hilbert space is a unique operator.

Theorem 6.2. *Suppose T is a self-adjoint operator in a Hilbert space H that is bounded below by kI for some $k > 0$. For $r > 0$, let $H_r = (V_r, (\cdot,\cdot)_r)$ be the r^{th} left-definite space associated with (H,T). Then there exists a unique left-definite operator T_r in H_r associated with (H,T). More specifically, if there exists a self-adjoint operator $\tilde{T}_r : H_r \to H_r$ such that $\tilde{T}_r f = Af$ for all $f \in \mathcal{D}(\tilde{T}_r) \subset \mathcal{D}(T)$, then $T_r = \tilde{T}_r$. Furthermore,*

$$\mathcal{D}(T_r) = V_{r+2},$$

and T_r is bounded below by kI in H_r.

By studying the left-definite operators and spaces associated with a self-adjoint, bounded below operator T, we are able to glean information about the spectrum of the original operator T.

Theorem 6.3. *For each $r > 0$, let T_r denote the r^{th} left-definite operator associated with the self-adjoint operator T that is bounded below by kI where $k > 0$. Then*

(a) *the point spectra of T and T_r coincide; i.e. $\sigma_p(T_r) = \sigma_p(T)$;*

(b) *the continuous spectra of T and T_r coincide; i.e. $\sigma_c(T_r) = \sigma_c(T)$;*

(c) *the resolvent sets of T and T_r coincide; i.e. $\rho(T) = \rho(T_r)$.*

6.3 Chisholm-Everitt Inequality

The following theorem, referred to as the Chisholm-Everitt Inequality, is extremely useful [5] as it is a remarkably helpful tool in understanding general properties of certain operator domains. In 1999, the result was extended to the general case of conjugate indices p and q ($p, q \geq 1$) in [6].

Theorem 6.4 (Chisholm-Everitt Inequality). *Let $I = (a, b)$ where $-\infty \le a < b \le \infty$ and let ω be a positive Lebesgue measurable function on I. Let $\varphi, \psi : I \to \mathbb{C}$ satisfy:*

(a) $\varphi, \psi \in L^2_{loc}(I; \omega)$

(b) There exists a $c \in (a, b)$ *such that* $\varphi \in L^2((a, c]; \omega)$ *and* $\psi \in L^2([c, b); \omega)$

(c) For all $[\alpha, \beta] \subset I$,

$$\int_a^\alpha |\varphi|^2 \omega dx > 0 \text{ and } \int_\beta^b |\psi|^2 \omega dx > 0.$$

Define $A, B : L^2(I; \omega) \to L^2_{loc}(I; \omega)$ *by*

$$(Af)(x) = \varphi(x) \int_x^b \psi(t) f(t) \omega(t) dt,$$
$$(Bf)(x) = \psi(x) \int_a^x \varphi(t) f(t) \omega(t) dt, \text{ and}$$
$$K(x) := \left(\int_a^x \varphi^2(t) \omega(t) dt \right)^{\frac{1}{2}} \left(\int_x^b \psi^2(t) \omega(t) dt) \right)^{\frac{1}{2}}$$

with $K := \sup_{x \in I} K(x)$. *Then, a necessary and sufficient condition for both* A *and* B *to be bounded linear operators is* $K < \infty$. *Furthermore, in this case,* $||Af|| \le 2K||f||$ *and* $||Bf|| \le 2K||f||$.

6.4 Sobolev Spaces

The area of Sobolev orthogonal polynomials deals with the study of sequences of polynomials which are orthogonal to quasi-definite symmetric bilinear forms of the type

$$(p, q)_N = \sum_{k=0}^N \int_{\mathbb{R}} p^{(k)}(x) q^{(k)}(x) \, d\mu_k$$

where each (signed) Borel measure μ_k, $0 \le k \le N$, has finite moments on the real line $\mathbb{R}$. When $N = 0$, we are in the case of classical orthogonal polynomial theory; however, for $N > 0$, the algebraic and analytic theory of the polynomials is quite different. For example, in general, Sobolev orthogonal polynomials do not satisfy a three-term recurrence relation.

CHAPTER SEVEN

Extreme Case of the X_1-Laguerre Expression

The spectral analysis for extreme parameter choices in the case of the X_1-Laguerre polynomials was completed for this thesis and may be found in [2].

The X_1-Laguerre weight function $w_1^\alpha(x)$, defined in (3.4), is defined for the parameter $\alpha > 0$. When $\alpha = 0$, this weight function reduces to $w_1^0(x) = x^{-2}e^{-x}$. Consequently, $\alpha = 0$ in the X_1-Laguerre case corresponds to $\alpha = -2$ in the ordinary Laguerre case. In fact, it is the case that

$$\ell_{-2}[\cdot] = \ell_1^0[\cdot]\,; \quad w_{-2}(x) = w_1^0(x); \quad L_n^{-2}(x) = L_{1,n}^0(x) \quad etc.$$

In this situation, the X_1-Laguerre equation (3.1) reduces to the ordinary Laguerre differential equation

$$\ell_0[y](x) := -xy''(x) + (x+1)y'(x) - y(x) = \lambda y(x) \quad (0 < x < \infty). \tag{7.1}$$

For each $n \in \mathbb{N}_0$, $y(x) = L_n^{-2}(x)$, the Laguerre polynomial with $\alpha = -2$, is a solution of $\ell^{-2}[y](x) = (n-1)y(x)$. The Laguerre polynomials $\{L_n^{-2}\}_{n=2}^\infty$ (of degree ≥ 2) form a complete orthogonal set in the Hilbert space $L^2((0,\infty); w_{-2})$; moreover, for $i = 0, 1$, the polynomials $L_i^{-2} \notin L^2((0,\infty); w_{-2})$. The spectral theory for this Laguerre case, in $L^2((0,\infty); w_{-2})$, was established in [14]; specifically, the authors in [14] develop the self-adjoint operator in $L^2((0,\infty); w_{-2})$, generated by the Laguerre expression (7.1), which has the Laguerre polynomials $\{L_n^{-2}\}_{n=2}^\infty$ as eigenfunctions. These authors also discuss the spectral theory of (7.1) which has the *entire* sequence of Laguerre polynomials $\{L_n^{-2}\}_{n=0}^\infty$ as eigenfunctions. The analysis, in this case, is in the Hilbert-Sobolev space $(S, (\cdot,\cdot))$, where

$$S = \{f : [0,\infty) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}[0,\infty), f'' \in L^2((0,\infty); e^{-x})\},$$

and where $(\cdot,\cdot)$ is the positive-definite inner product defined by

$$(f,g) = 3f(0)\overline{g}(0) - f'(0)\overline{g}(0) - f(0)\overline{g}'(0) + \int_0^\infty f''(x)\overline{g}''(x)e^{-x}dx.$$

The authors find, and discuss, the self-adjoint operator in $(S,(\cdot,\cdot))$ having the Laguerre polynomials $\{L_n^{-2}\}_{n=0}^\infty$ as a complete orthogonal set of eigenfunctions. The key to establishing this analysis of (7.1) in $(S,(\cdot,\cdot))$ is the general left-definite spectral theory developed by Littlejohn and Wellman in [37].

CHAPTER EIGHT

Extreme Case for the X_1-Exceptional Jacobi Expression

The spectral analysis for extreme parameter choices in the case of the X_1-Jacobi polynomials was completed for this thesis and may be found in [36].

8.1 Introduction

The case when $b = \pm 1$, that is when $\alpha = 0$ or $\beta = 0$, is an "extreme" case not considered by Gómez-Ullate, Kamran and Milson in [19]. Without loss of generality, consider $\alpha = 0$ and hence, $b = 1$; symmetry provides a similar argument for $\beta = 0$ and $b = -1$. In this chapter, we will loosen the restrictions on α and β. With $\alpha = 0$ and $\beta \neq 0$, it is the case that

$$a = \frac{\beta}{2}, \quad b = 1, \quad \text{and} \quad c = 1 + \frac{2}{\beta}.$$

The second-order differential expression (3.8) for the X_1-Jacobi polynomials simplifies to

$$\ell_1^{0,\beta}[y](x) = (x^2 - 1)y''(x) + \beta\left(\left(x - 1 - \frac{2}{\beta}\right)y'(x) - y(x)\right), \tag{8.1}$$

and the weight function (3.6) reduces to

$$w_1^{0,\beta}(x) = (1 - x)^{-2}(1 + x)^{\beta}. \tag{8.2}$$

Recall the classical Jacobi expression (2.7), which is defined for $\alpha, \beta > -1$. If we loosen the restrictions on α in the classical case, allowing $\alpha \in \mathbb{R}$, notice that the X_1-Jacobi differential equation (8.1) is equal the classical differential equation (2.7) when $\alpha = -2$. That is, $\alpha = 0$ in the exceptional case corresponds to $\alpha = -2$ in the classical case:

$$\ell_{-2,\beta} = \ell_1^{0,\beta}, \quad P_n^{(-2,\beta)}(x) = P_{1,n}^{(0,\beta)}(x), \quad w_{-2,\beta} = w_1^{0,\beta}, \quad \text{etc.},$$

and similarly for the eigenvalues $\lambda_n = n(n+\beta-1)$ (where we did not explicitly keep track of the parameters α and β). Therefore, the results provided in Sections 8.2.1 and 8.3 are not subject to the generalized Bochner classification by Gómez-Ullate, Kamran and Milson [19].

In Section 8.2.1 we study this extreme case in the right-definite setting by defining a self-adjoint operator in $L^2((-1,1);w_{-2,\beta})$ similarly to the exceptional X_1-Laguerre case; by also dropping $P_1^{(-2,\beta)}$.

Theorem 8.1. *The differential expression*

$$T_1^{-2,\beta} f = \ell_{-2,\beta}[f]$$

defined on $\Delta \subset L^2((-1,1);w_{-2,\beta})$ *together with the domain*

$$f \in \mathcal{D}(T_1^{-2,\beta}) := \begin{cases} \left\{ f \in \Delta \mid \lim_{x\to -1}(1+x)^{\beta+1} f'(x) = 0 \right\} & \beta \in (-1,1) \\ \Delta & \beta \geq 1 \end{cases}$$

yields a self-adjoint operator.

We ultimately show the following result:

Theorem 8.2. *The self-adjoint operator of the previous theorem has discrete spectrum with eigenvalues* $\lambda_n = (n-1)(\alpha+\beta+n)$, $n \geq 2$, *corresponding to the eigenfunctions* $\left\{ P_n^{(-2,\beta)}(x) \right\}_{n=2}^{\infty}$.

It is noteworthy that the completeness proofs of [19] cannot be extended to show the completeness of $\left\{ P_{1,n}^{(0,\beta)}(x) \right\}_{n=2}^{\infty}$ in $L^2((-1,1);w_{-2,\beta})$. Therefore, we turn to the framework of left-definite theory. In Section 8.3 we study within the structure of left-definite theory via defining a Hilbert space based on a certain Sobolev inner product (see Subsection 8.3.1). As this construction is too involved, we now only outline the general idea and defer presenting precise results until later: We define the self-adjoint operator via the direct sum of two operators. One acts on a two dimensional space, and the other can be defined as the second left-definite operator. In the process of proving the hypotheses of left-definite theory, we apply several times the useful Chisholm-Everitt Inequality, see Theorem 6.4. In particular, the use of this result is required to prove that the operator $\ell_{-2,\beta}$ is positive (see Theorem 8.6). In return, left-definite theory immediately provides the domain as well as the spectrum. In the end, see Theorem 8.9 below, we obtain a self-adjoint operator which has the classical Jacobi polynomials (defined in Subsubsection 2.2.1.2) as eigenfunctions and describe its spectral analysis.

8.2 Right-definite Setting

8.2.1 First Approach to the Extreme Case $b = \pm 1$

We consider the Sturm-Liouville problem given by the second-order differential equation

$$\ell_{-2,\beta}[y](x) \quad (x \in (-1,1)),$$

where we have the non-classical Jacobi operator

$$\ell_{-2,\beta}[y](x) := (x^2-1)y''(x) + (\beta x - \beta - 2)y'(x), \tag{8.3}$$

or, in symmetric form:

$$\ell_{-2,\beta}[y] = -\frac{(x-1)^2}{(1+x)^\beta}\left(\frac{(1+x)^{\beta+1}}{(1-x)}\ y'\right)'.$$

While we have the classical eigenvalue equation

$$\ell_{-2,\beta}[P_n^{(-2,\beta)}](x) = \lambda_n P_n^{(-2,\beta)}(x)$$

for all non-negative integers n, it is interesting to observe that for $n = 0$ and $n = 1$:

$$P_n^{(-2,\beta)}(x) \notin L^2((-1,1); w^{-2,\beta}(x)).$$

Therefore, we must restrict our sequence of polynomials to $\{P_n^{(-2,\beta)}(x)\}$, $n \geq 2$.

Remark 8.1. In restricting our sequence of polynomials to $\left\{P_n^{(-2,\beta)}(x)\right\}_{n=2}^{\infty}$, we can now permit the case $\beta = 0$ in this section, although this is not allowed in the X_1-Jacobi theory. The linear polynomial "$P_{1,1}^{(0,0)}(x) = P_1^{(-2,0)}(x)$" is not defined due to degeneracy of the binomial coefficients. And, in fact, it can be shown that there is no linear polynomial that satisfies the eigenvalue equation when $\beta = 0$. Therefore, for the remainder of this section β may take on any value in the interval $(-1, \infty]$.

A simple computation, which we omit, shows a useful relationship between the Jacobi polynomials for parameters $\alpha = 2$ and $\alpha = -2$.

Lemma 8.1. *For $n \geq 2$:*

$$P_n^{(-2,\beta)}(x) = \frac{(n+\beta)(n+\beta-1)}{4n(n-1)}(1-x)^2 P_{n-2}^{(2,\beta)}(x).$$

In particular, we have $P_n^{(-2,\beta)}(x) \in L^2((-1,1); w_{-2,\beta})$.

The following statement about completeness is, in spirit, related to completeness results in [19,21]. However, we consider the case $\alpha = 0$, which cannot be studied using their techniques.

Lemma 8.2. *The set $\left\{P_n^{(-2,\beta)}\right\}_{n=2}^{\infty}$ is complete in $L^2((-1,1); w_{-2,\beta})$. Equivalently,*

$$\textit{span}\,\{p \in \mathcal{P} \mid p \text{ is a polynomial of } \deg \geq 2 \text{ with } p(1) = p'(1) = 0\}$$

is dense in $L^2((-1,1); w_{-2,\beta})$.

Proof. The equivalence of the two statements is immediate; we will prove the second statement. Let $\varepsilon > 0$ and $f \in L^2((-1,1); w_{-2,\beta})$. Then

$$\frac{f(x)}{(1-x)^2} \in L^2((-1,1); w_{2,\beta}).$$

Note that in this latter space, $L^2((-1,1); w_{2,\beta})$, the polynomials $\mathcal{P}$ are dense by classical Jacobi theory. So there exists $p \in \mathcal{P}$ such that

$$\varepsilon^2 > \left\|\frac{f(x)}{(1-x)^2} - p(x)\right\|_{\alpha=2}^2 = \int_{-1}^{1} \left|\frac{f(x)}{(1-x)^2} - p(x)\right|^2 w_{2,\beta}(x)\, dx.$$

Define $q(x) = p(x)(1-x)^2$. Note that $q(1) = q'(1) = 0$. Then,

$$\|f(x) - q(x)\|^2_{\alpha=-2} = \int_{-1}^{1} |f(x) - q(x)|^2 \, w_{-2,\beta}(x) \, dx$$
$$= \int_{-1}^{1} \left| \frac{f(x)}{(1-x)^2} - p(x) \right|^2 w_{2,\beta}(x) \, dx = \left\| \frac{f(x)}{(1-x)^2} - p(x) \right\|^2_{\alpha=2} < \varepsilon^2 \,,$$

proving the desired result. □

8.2.2 Maximal and Minimal Operators

The maximal domain associated with the expression $\ell_{-2,\beta}[\cdot]$ in the Hilbert space $L^2((-1,1); w_{-2,\beta})$ is:

$$\Delta := \left\{ f : (0,\infty) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); f, \ell_{-2,\beta}[f] \in L^2((-1,1); w_{-2,\beta}) \right\}.$$

The associated maximal operator

$$T_1 : \mathcal{D}(T_1) \subset L^2((-1,1); w_{-2,\beta}) \to L^2((-1,1); w_{-2,\beta}).$$

is defined by

$$T_1 f = \ell_{-2,\beta}[f]$$
$$f \in \mathcal{D}(T_1) := \Delta.$$

For $f, g \in \Delta$, the sesquilinear form used in Green's formula is now defined by

$$[f,g](x) := (1-x)^{-1}(1+x)^{\beta+1} \left(f(x)\bar{g}'(x) - f'(x)\bar{g}(x) \right) \quad (-1 < x < 1)\,. \qquad (8.4)$$

We define the minimal operator

$$T_0 : \mathcal{D}(T_0) \subset L^2((-1,1); w_{-2,\beta}) \to L^2((-1,1); w_{-2,\beta})$$

associated with $\ell_{-2,\beta}[\cdot]$ to be

$$T_0 f = \ell_{-2,\beta}[f]$$
$$f \in \mathcal{D}(T_0) := \left\{ f \in \Delta \big[f,g\big]_{x=-1}^{x=1} = 0 \text{ for all } g \in \Delta \right\}.$$

8.2.3 Endpoint Behavior Analysis

The behavior at $x = 1$ is independent of the parameter β, much unlike the behavior at $x = -1$:

Theorem 8.3. *For the parameters $\alpha = -2$ and $\beta > -1$, let $\ell_{-2,\beta}[\cdot]$ be the classical Jacobi expression given by* (8.3) *on the interval* $(-1, 1)$.

(a) At the endpoint $x = 1$, the expression $\ell_{-2,\beta}[\cdot]$ is limit-point.

(b) At the endpoint $x = -1$, the expression $\ell_{-2,\beta}[\cdot]$ is limit-circle in case $\beta \in (-1, 1)$ and limit-point for $\beta \geq 1$.

As an immediate consequence of Theorem 8.3, we have the following theorem, which tells us the number of boundary conditions we need to choose to achieve self-adjointness of the operator.

Corollary 8.1. *Consider the minimal operator T_0 in $L^2((-1, 1); w_{-2,\beta})$ as defined in Subsection 8.2.2, generated by the X_1-Jacobi differential expression $\ell_{-2,\beta}[\cdot]$.*

(a) For $\beta \in (-1, 1)$, the minimal operator has deficiency index $(1, 1)$.

(b) For $\beta \geq 1$, the minimal operator has deficiency index $(0, 0)$.

Proof. First, we apply the Frobenius analysis to the endpoint $x = 1$. With the differential expression (8.3), we obtain

$$\left(\frac{x-1}{x+1}\right)(\ell_{-2,\beta}[y] - \lambda y) = (x-1)^2 y'' + \left(\frac{x-1}{x+1}\right)(\beta x - \beta - 2)y' - \lambda\left(\frac{x-1}{x+1}\right)y$$
$$= 0\,.$$

This yields the indicial equation (4.10):

$$r(r-1) - r = r(r-2) = 0\,.$$

Therefore, two linearly independent solutions of $\ell_{-2,\beta}[y] = 0$ will asymptotically (near $x = 1$, e.g. on the interval (0,1)) behave like

$$z_1(x) = 1 \quad \text{and} \quad z_2(x) = (x-1)^2.$$

We have, for any $\beta > -1$

$$\int_0^1 |z_1(x)|^2 \, w_{-2,\beta}(x) \, dx = \infty \qquad \text{and} \qquad \int_0^1 |z_2(x)|^2 \, w_{-2,\beta}(x) \, dx < \infty \, .$$

Second, we see that the endpoint $x = -1$ works out similarly. By applying the Frobenius method to the expression

$$\left(\frac{x+1}{x-1}\right) (\ell_{-2,\beta}[y](x) - \lambda y) \ ,$$

we obtain the indicial equation (4.10):

$$r(r-1) + r\,(\beta+1) = r(r+\beta) = 0 \, .$$

For $\beta \neq 0$, the two linearly independent solutions of $\ell_{-2,\beta}[y] = 0$ will asymptotically (near $x = -1$, e.g. on the interval (-1,0)) behave like

$$y_1(x) = 1 \quad \text{and} \quad y_2(x) = (1+x)^{-\beta}.$$

Note that although

$$y_1(x) = 1 \notin L^2((-1,1); w_{-2,\beta}(x)) \, ,$$

it is in $L^2((-1,0); w_{-2,\beta})$, that is:

$$\int_{-1}^0 |y_1(x)|^2 \, w_{-2,\beta}(x) \, dx < \infty$$

for all possible values of β. On the other hand, we have

$$\int_{-1}^0 |y_2(x)|^2 \, w_{-2,\beta}(x) \, dx < \infty$$

if and only if $\beta \in (-1,1)$.

For $\alpha = \beta = 0$, the two functions y_1 and y_2 are identical and therefore, not linearly independent. However, in this case, one can easily verify that the function

$$y_2(x) = 4\ln(1+x) - x$$

satisfies

$$\begin{aligned}\ell_{0,0}[y_2](x) &= -(x-1)^2\left(\frac{1+x}{1-x}\left(\frac{4}{1+x}-1\right)\right)' \\ &= -(x-1)^2\left(\frac{4}{1-x}-(1+x)\right)' \longrightarrow 0\end{aligned}$$

as $x \to -1^+$. Furthermore, we obtain $y_2 \in L^2((-1,0); w_{-2,0})$, since the integral

$$\begin{aligned}\int_{-1}^{0} \ln^2(1+x)dx &= \lim_{\delta\to -1^+}\left[(1+x)\ln^2(1+x) - 2(1+x)\ln(1+x) + 2(1+x)\right]_\delta^0 \\ &= 2 \end{aligned} \tag{8.5}$$

by L'Hôpital's rule. This yields the second statement. □

8.2.4 Spectral Analysis

Using the notation of classical Jacobi theory, Theorem 8.1 can be re-written as follows.

Theorem 8.4. *The differential expression*

$$T_{-2,\beta} f = \ell_{-2,\beta}[f]$$

defined on the maximal domain $\Delta \subset L^2((-1,1); w_{-2,\beta})$ *together with the*

$$f \in \mathcal{D}(T_{-2,\beta}) := \begin{cases} \left\{f \in \Delta \mid \lim_{x\to -1}(1+x)^{\beta+1} f'(x) = 0\right\} & \beta \in (-1,1) \\ \Delta & \beta \geq 1 \end{cases}$$

yields a self-adjoint operator.

Proof. Consider

$$\psi(x) := P_2^{(-2,\beta)}(x) = -\frac{\beta+2}{4}(x-1)^2\,.$$

The case $\beta \geq 1$ follows immediately from part (b) of Corollary 8.1.

By Corollary 8.1, the deficiency index is (1,1) for $\beta \in (-1, 1)$. By Theorem 8.3, the restriction will be at $x = -1$.

Observe that

$$\ell_{-2,\beta}[\psi] = \mathcal{O}((x-1)^2)$$

is in $L^2((-1,1); w_{-2,\beta})$ near $x = 1$ and $x = -1$ which implies

$$\ell_{-2,\beta}[\psi] \in L^2((-1,1); w_{-2,\beta})\,.$$

Therefore, we have $\psi \in \Delta$.

It remains to find a function $g \in \Delta$ with $[g, \psi](-1) \neq 0$.

For $\beta \neq 0$, consider the function

$$g(x) = \frac{1}{w_{-2,\beta}(x)} = (1+x)^{-\beta}(x-1)^2\,.$$

We have

$$\ell_{-2,\beta}[g] = \mathcal{O}((1+x)^{-\beta}(x-1)^2).$$

Since $\beta < 1$, we obtain $\ell_{-2,\beta}[g] \in L^2((-1,1); w_{-2,\beta})$, and it follows that $g \in \Delta$.

By a computation, we obtain

$$[g, \psi]_{x=-1}^{x=1} = [g, \psi](-1) = -8\beta \neq 0.$$

For $\beta = 0$, we use $g(x) = (x-1)^2 \ln(1+x)$. Again, computation shows that

$$\ell_{-2,\beta}[g] = \mathcal{O}((x-1)^2 \ln(1+x))$$

near $x = 1$ and $x = -1$. In virtue of equation (8.5) the integral

$$\int_{-1}^{0} \ln^2(1+x)dx = 2$$

and we have $g \in \Delta$.

We observe:

$$[g, \psi]_{x=-1}^{x=1} = [g, \psi](-1) = -8 \neq 0.$$

Finally, in the case $\beta \in (-1,1)$, the boundary condition $[f,\psi](-1)=0$ can be written as

$$\lim_{x\to -1}(1+x)^{\beta+1}f'(x)=0\,.$$

□

Now we are able to prove our main theorem concerning the spectrum in the extreme parameter case of the X_1-Jacobi polynomials.

Proof of Theorem 8.2. This result follows immediately from Theorem 8.1 in combination with Lemma 8.2 and the proof of Lemma 5.2 adapted to the situation where $\alpha = 0$ (this adaption is not difficult). □

8.3 Left-definite Setting

Now, let us study $\left\{P_n^{(-2,\beta)}\right\}_{n=0}^{\infty}$ in the left-definite setting. In particular, we seek a certain Sobolev space, $(S,\Phi(\cdot,\cdot))$, such that $\left\{P_n^{(-2,\beta)}\right\}_{n=0}^{\infty}$ forms a complete orthogonal set. Notice that we seek a space in which the polynomials of degree zero and one are included. Further, we study the linear differential operator $\ell_{-2,\beta}[\cdot]$ in this Sobolev space. The objective is then to determine the self-adjoint operator T in S, which is generated by $\ell_{-2,\beta}[\cdot]$ and has $\left\{P_n^{(-2,\beta)}\right\}_{n=0}^{\infty}$ as eigenfunctions. Finally, we find its spectrum.

8.3.1 The Sobolev Inner Product and its Associated Space

Define

$$\begin{aligned}\Phi(p,q) := A[\beta p(1)\bar{q}(1) + 2(p'(1)\bar{q}(1) + p(1) + \bar{q}'(1))] \\ + Bp'(1)\bar{q}'(1) + \int_{-1}^{1} p''(x)q''(x)(1+x)^{\beta+2}\,dx\,,\end{aligned}$$

where $AB > 0$ and $\frac{-4A}{\beta} + B > 0$. In what follows we let

$$A = \frac{1}{\beta} \quad \text{and} \quad B = 1 + \frac{4}{\beta^2}\,.$$

Proposition 8.1. *Expression $\Phi(p,q)$ forms an inner product.*

Proof. We concentrate on showing that $\Phi(p,p) \geq 0$, as the other properties required for $\Phi(p,q)$ to be an inner product follow trivially.

With the choices for A and B we have

$$\Phi(p,q) = \left(p(1) + \frac{2}{\beta}p'(1)\right)\left(\bar{q}(1) + \frac{2}{\beta}\bar{q}'(1)\right) + p'(1)\bar{q}'(1) + \int_{-1}^{1} p''(x)q''(x)(1+x)^{\beta+2}\,dx\,.$$

In particular, $\Phi(p,p)$ is the sum of squared terms. □

Proposition 8.2. *The polynomial sequence $\left\{P_n^{(-2,\beta)}\right\}_{n=0}^{\infty}$ is orthogonal with respect to the inner product Φ.*

Proof. Clearly we have $\Phi(P_0^{(-2,\beta)}, P_0^{(-2,\beta)}) = \Phi(P_1^{(-2,\beta)}, P_1^{(-2,\beta)}) = 1\,.$

Note that for $n > 1$, $P_n^{(-2,\beta)}(x)$ has a double root at $x = 1$. This, in conjunction with $\left(P_0^{(-2,\beta)}\right)''(x) = \left(P_1^{(-2,\beta)}\right)''(x) = 0$ gives

$$\Phi(P_0^{(-2,\beta)}, P_n^{(-2,\beta)}) = 0 \quad \text{for } n \geq 1$$
$$\Phi(P_1^{(-2,\beta)}, P_n^{(-2,\beta)}) = 0 \quad \text{for } n \geq 2\,.$$

Lastly, we need to show that for $m,n \geq 2$, $\Phi(P_n^{(-2,\beta)}, P_m^{(-2,\beta)}) = k_n\delta_{nm}$ where $k_n \neq 0$. We recall again that for $n > 1$, $P_n^{(-2,\beta)}(x)$ has a double root at $x = 1$, and thus, we only need to look at the integral part of Φ:

$$\int_{-1}^{1} \left(P_m^{(-2,\beta)}\right)''(x)\left(P_n^{(-2,\beta)}\right)''(x)(1+x)^{\beta+2}\,dx.$$

We see that

$$(1-x)^{\alpha}(1+x)^{\beta}\left(P_n^{(-2,\beta)}\right) = \frac{(n+\beta-1)(n+\beta)}{4}P_n^{(0,\beta+2)}\,.$$

Exploiting the orthogonality properties of the classical Jacobi polynomials, it is obvious that $\left(P_m^{(-2,\beta)}\right)''$ and $\left(P_n^{(-2,\beta)}\right)''$ are orthogonal with respect to $(1+x)^{\beta+2}$. □

Let us define the set S by

$$S := \left\{ f : (-1,1] \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1];\ f'' \in L^2((-1,1);(1+x)^{\beta+2}) \right\} .$$

Proposition 8.3. *The Sobolev space $(S, \Phi(\cdot,\cdot))$ is a Hilbert space.*

Proof. We need to show the completeness of $(S, \Phi(\cdot,\cdot))$. Suppose $\{f_n\} \subseteq S$ is a Cauchy sequence. We need to find $f \in S$ such that $f_n \to f$ in S. Notice that we have

$$\begin{aligned} \|f_n - f_m\|_{\Phi}^2 &= \Phi(f_n - f_m, f_n - f_m) \\ &= \left(f_n(1) - f_m(1) + \frac{2}{\beta}\left(f_n(1) - f_m(1)\right) \right)^2 \\ &\quad + \left(f_n'(1) - f_m'(1)\right)^2 + \int_{-1}^{1} \left(f_n''(x) - f_m''(x)\right)^2 (1+x)^{\beta+2}\, dx \,. \end{aligned}$$

From this expression, we glean that sequence $\{f_n''\}_{n=0}^{\infty}$ is Cauchy in the space $L^2\left((-1,1);(1+x)^{\beta+2}\right)$, and that the sequences $\{f_n(1)\}_{n=0}^{\infty}$ as well as $\{f_n'(1)\}_{n=0}^{\infty}$ are Cauchy in $\mathbb{C}$.

Therefore, by the completeness of $L^2\left((-1,1);(1+x)^{\beta+2}\right)$ and $\mathbb{C}$, there exists a function $g(x) \in L^2((-1,1;\ (1+x)^{\beta+2})$ and scalars $a, b \in \mathbb{C}$ such that $\{f_n''\}_{n=0}^{\infty}$ converges to $g(x)$ in $L^2((-1,1;\ (1+x)^{\beta+2})$, $\{f_n'(1)\}_{n=0}^{\infty}$ converges to a, and $\{f_n(1)\}_{n=0}^{\infty}$ converges to b.

Now, the function $f : (-1,1] \to \mathbb{C}$ given by

$$f(x) := ax + (b-a) + \int_x^1 \int_t^1 g(u)\, du\, dt$$

satisfies $f(1) = b$, $f'(1) = a$ and $f(x) \in S$. □

Proposition 8.4. *The set $\mathcal{P}$ of polynomials is dense in $(S, \Phi(\cdot,\cdot))$.*

Proof. Let $f \in S$. Then, $f'' \in L^2((-1,1);(1+x)^{\beta+2})$. Since $\mathcal{P}$ is dense in the space $L^2((-1,1);\ (1+x)^{\beta+2})$, there exists $p \in \mathcal{P}$ such that

$$\int_{-1}^{1} |f''(x) - p(x)|^2 (1+\beta)^{\beta+2}\, dx < \epsilon^2.$$

With the polynomial

$$q(x) := f'(1)x + (f(1) - f'(1)) + \int_x^1 \int_t^1 p(u)\, du\, dt$$

we have $q(1) = f(1)$, $q'(1) = f'(1)$, $q''(1) = f''(1)$, as well as

$$\|f - q\|_\Phi^2 < \epsilon^2 .$$

Since $p \in \mathcal{P}$ approximates f in $(S, \Phi(\cdot,\cdot))$, the polynomials $\mathcal{P}$ are dense in S. $\square$

Summing up, we have shown:

Theorem 8.5. *The set of Jacobi polynomials* $\left\{P_n^{(-2,\beta)}\right\}_{n=0}^{\infty}$ *forms an orthogonal basis (i.e. a complete orthogonal set) for* S.

Let us write S as a direct sum of two spaces:

$$S = S_1 \oplus S_2$$

where

$$S_1 := span\left\{P_0^{(-2,\beta)}, P_1^{(-2,\beta)}\right\} = \{f \in S \mid f''(x) = 0 \text{ a.e. } x \in (-1,1]\} \text{ , and}$$
$$S_2 := span\left\{P_n^{(-2,\beta)}\right\}_{n=2}^{\infty} = \{f \in S \mid f(1) = f'(1) = 0\} \text{ .}$$

In order to construct the self-adjoint operator T in S, we consider two self-adjoint operators generated by $\ell_{-2,\beta}[\cdot]$: T_1 and T_2 will be the respective self-adjoint operators in S_1 and S_2. Since T_1 acts on a two dimensional space, it's definition and spectral analysis is straightforward. However, the definition of T_2 is rather involved. Subsections 8.3.3 through 8.3.5 are devoted to proving the hypotheses required for left-definite theory (see Subsection 6.2).

Then, $T = T_1 \oplus T_2$ yields the desired self-adjoint operator on S.

8.3.2 Constructing T_1 and its Spectral Analysis

Define

$$T_1 : \mathcal{D}(T_1) \subset S_1 \to S_1$$
$$T_1 f := \ell_{-2,\beta}[\cdot] \text{ for } f \in \mathcal{D}(T_1) \subset S_1.$$

If T_1 is symmetric, then it is self-adjoint. Indeed, S_1 is a two dimensional (finite) space.

Proposition 8.5. *Operator T_1 is symmetric in $(S_1, \Phi(\cdot,\cdot))$.*

Proof. Observe that for $f \in S_1$,

$$\ell_{-2,\beta}[f] = (\beta x - \beta - 2)f'(x) \text{ a.e. } \ x \in (-1,1]$$

since $f''(x) = 0$ a.e. $x \in (-1,1]$ and that $T_1 f''(x) = 0$. Then, we obtain

$$\begin{aligned}\Phi(T_1 f, g) &= -2f'(1)\bar{g}(1) + \frac{2}{\beta}\left[\beta f'(1)\bar{g}(1) - f'(1)\bar{g}'(1)\right] + \left(1 + \frac{4}{\beta}\right)\beta f'(1)\bar{g}'(1) \\ &= \beta f'(1)\bar{g}'(1) \\ &= \Phi(f, T_1 g).\end{aligned}$$

Therefore, T_1 is symmetric in $(S_1, \Phi(\cdot,\cdot))$. □

We know that $P_0^{(-2,\beta)}(x)$ is an eigenfunction corresponding to $\lambda_0 = 0$; and similarly, $P_1^{(-2,\beta)}(x)$ is an eigenfunction corresponding to $\lambda_1 = \beta$. Therefore, the sequence $\left\{P_j^{(-2,\beta)}\right\}_{j=0,1}$ is a complete set of eigenfunctions for T_1 and $\sigma(T_1) = \{0, \beta\}$.

8.3.3 The Positivity of the Differential Expression $\ell_{-2,\beta}[\,\cdot\,]$

In order to apply left-definite theory it is necessary to show the following theorem:

Theorem 8.6. *The operator associated induced by $\ell_{-2,\beta}[\,\cdot\,]$ is a positive operator.*

Before proving Theorem 8.6, we prove a series of useful lemmas concerning properties of functions from the maximal domain. Let us first turn our attention to the behavior at $x = -1$.

Lemma 8.3. *Let* $f \in \mathcal{D}(T)$. *Then* $\lim_{x\to -1}(1+x)^{\beta+1}f'(x) = 0$.

Proof. Let $f \in \mathcal{D}(T)$. For $-1 < \beta < 1$ the lemma is trivial, since the conclusion coincides with the boundary conditions (see Theorem 8.4).

Suppose $\beta \geq 1$. Then the differential expression $\ell_{-2,\beta}[\cdot]$ is limit-point at $x = -1$. According to Weyl Theory

$$\lim_{x\to -1}(1+x)^{\beta+1}\left(f'(x)\bar{g}(x) - f(x)\bar{g}'(x)\right) = 0$$

holds for all $g \in \Delta$. And, in particular true for $g \in C^2[-1,1]$ given by

$$g(x) = \begin{cases} 1 & -1 \leq x \leq -\frac{1}{2} \\ 16x^3 + 12x^2 & -\frac{1}{2} < x \leq 0 \\ 0 & 0 < x \leq 1 \end{cases}.$$

With this choice of g we have

$$0 = \lim_{x\to -1}(1+x)^{\beta+1}\left(f'(x)\bar{g}(x) - f(x)\bar{g}'(x)\right) = \lim_{x\to -1}(1+x)^{\beta+1}f'(x)\,,$$

proving the lemma. □

Lemma 8.4. *For* $f, g \in \mathcal{D}(T)$, *we have* $\lim_{x\to -1^+}(1-x)^{-1}(1+x)^{\beta+1}f'(x)\bar{g}(x) = 0\,.$

Proof. Assume f and g are both real valued. Applying integration by parts, reversing the roles of x and y, and allowing y to be zero yields

$$\begin{aligned}\int_0^x (1-t)^{-1}(1+t)^{\beta+1}f'(t)g'(t)\,dt = -f(0)g(0) &+ (1-x)^{-1}(1+x)^{\beta+1}f'(x)g(x) \\ &+ \int_0^x \ell[f](t)g(t)(1-t)^{-2}(1+t)^{\beta}\,dt \\ &- k\int_0^x f(t)g(t)(1-t)^{-2}(1+t)^{\beta}\,dt\,. \qquad (8.6)\end{aligned}$$

Since each of the integral terms are finite as $x \to -1^+$, we see that

$$c := \lim_{x\to -1^+} (1-x)^{-1}(1+x)^{\beta+1} f'(x) g(x)$$

exists and is also finite. Without loss of generality, suppose $c > 0$ Then, there exists $x^* \in (-1, 0)$ such that

$$(1-x)^{-1}(1+x)^{\beta+1} f'(x) g(x) \geq \frac{c}{2}$$

on $(-1, x^*]$. Without loss of generality, assume that $f'(x) > 0$ and $g(x) > 0$. (The case when $f'(x) < 0$ and $g(x) < 0$ can be dealt with in analogy.)

Hence,

$$g(x) \geq \frac{c}{2}(1-x)^{-1}(1+x)^{\beta+1} f'(x) \quad \text{ on } \ (-1, x^*]\,,$$

and so,

$$\left|\left((1-x)^{-1}(1+x)^{\beta+1} f'(x)\right)'\right| g(x) \geq \frac{c\left|\left((1-x)^{-1}(1+x)^{\beta+1} f'(x)\right)'\right|}{2\left((1-x)^{-1}(1+x)^{\beta+1} f'(x)\right)}.$$

Integrating yields

$$\begin{aligned}
\infty &> \int_{-1}^{1} |\ell[f](t)|\, g(t)(1-t)^{-2}(1+t)^{\beta}\, dt \\
&= \int_{-1}^{1} \left|\left((1-t)^{-1}(1+t)^{\beta+1} f'(t)\right)'\right| g(t)\, dt \\
&\geq \frac{c}{2} \int_{x}^{x^*} \frac{\left|\left((1-t)^{-1}(1+t)^{\beta+1} f'(t)\right)'\right|}{(1-t)^{-1}(1+t)^{\beta+1} f'(t)}\, dt \\
&\geq \frac{c}{2} \left| \int_{x}^{x^*} \frac{\left((1-t)^{-1}(1+t)^{\beta+1} f'(t)\right)'}{(1-t)^{-1}(1+t)^{\beta+1} f'(t)}\, dt \right| \\
&= \frac{c}{2} \left| \ln\left((1-t)^{-1}(1+t)^{\beta+1} f'(t)\right) \right|_{x}^{x^*} \\
&= \infty\,.
\end{aligned}$$

Hence, we have a contradiction and we obtain the claim of the lemma. □

Now, we will turn our attention to the more difficult case—the behavior of functions from the maximal domain at the end point $x = 1$.

Lemma 8.5. *If $f \in \Delta$, then*

(a) $f' \in L^2(0,1)$,

(b) $f \in AC_{loc}(-1,1]$,

(c) $f(1) = 0$, *and*

(d) $(1-x)^{-1} f' \in L^2(0,1)$.

Proof. (a) Let $f \in \Delta$. Then we have $\ell_{-2,\beta}[f] \in L^2((0,1); w_{-2,\beta})$, or equivalently we obtain $(1-x)(1+x)^{-\beta/2}\left((1-x)^{-1}(1+x)^{\beta+1} f'(x)\right)' \in L^2(0,1)$.

For $0 \le x < 1$,

$$f'(x) = (1-x)(1+x)^{-\beta-1} f'(0) + (1-x)(1+x)^{-\beta-1} \left(\int_0^x \frac{(1-t)^{-1}(1+t)^{\beta/2}}{(1-t)^{-1}(1+t)^{\beta/2}} \left((1-t)^{-1}(1+t)^{\beta+1} f'(t)\right)' dt\right)$$

We apply the Chisholm-Everitt Inequality with $(a,b) = (0,1)$.

$$\psi(x) = (1-x)(1+x)^{-\beta-1} \quad \text{and} \quad \varphi(x) = (1-x)^{-1}(1+x)^{\beta/2}.$$

Note that

$$\left(\int_0^x \varphi^2(t)dt\right)\left(\int_x^1 \psi^2(t)dt\right) < \infty.$$

The Chisholm-Everitt Inequality implies $f'(x) \in L^2(0,1)$.

(b) Note that from part (a), since $f'(x) \in L^2(0,1)$, $f'(x)$ also lies in $L^1(0,1)$. For $0 \le x < 1$, we can write

$$f(x) = f(0) + \int_0^x f'(t)\,dt.$$

Then, $\lim_{x\to 1^-} f(x)$ exists and is finite. We define $f(1) := \lim_{x\to 1^-} f(x)$. Therefore, $f(x) \in AC_{loc}(-1,1]$.

(c) Suppose $f(1) \neq 0$. Without loss of generality, assume $f(x)$ is real valued and $f(1) = c > 0$. This implies that there exists $x^* \in (0,1)$ such that $f(x) \geq \frac{c}{2}$ for all $x \in [x^*, 1]$. Then,

$$\begin{aligned}\infty > \int_{-1}^{1} |f|^2 (1-x)^{-2}(1+x)^{\beta}\, dx \geq \int_{x^*}^{1} |f|^2 (1-x)^{-2}(1+x)^{\beta}\, dx \\ \geq \left(\frac{c}{2}\right)^2 \int_{x^*}^{1} (1-x)^{-2}(1+x)^{\beta}\, dx = \infty\,,\end{aligned}$$

which is a contradiction. Therefore, $f(1) = 0$.

(d) To show $(1-x)^{-1} f' \in L^2(0,1)$, we again apply the Chisholm-Everitt Inequality as in the proof of part (a).

□

Lemma 8.6. *For* $f, g \in \Delta$, *we have* $\lim_{x\to 1^-} (1-x)^{-1}(1+x)^{\beta+1} f'(x) g(x) = 0\,.$

Proof. Without loss of generality, assume that $f(x)$ and $g(x)$ are real valued. We may write our Jacobi expression in Lagrangian symmetric form—that is,

$$\begin{aligned}\ell_{-2,\beta}[f](x) = \frac{1}{(1-x)^{-2}(1+x)^{\beta}} \\ \left[-\left((1-x)^{-1}(1+x)^{\beta+1} f'(x)\right)' + k(1-x)^{-2}(1+x)^{\beta+1} f(x)\right]\end{aligned}$$

where k is a constant.

Then for $-1 < a < b < 1$,

$$\begin{aligned}\int_a^b \ell_{-2,\beta}[f](x)\bar{g}(x)(1-x)^{-2}(1+x)^{\beta}\, dx - k\int_a^b f(x)\bar{g}(x)(1-x)^{-2}(1+x)^{\beta}\, dx \\ = -\int_a^b \left((1-x)^{-1}(1+x)^{\beta+1} f'(x)\right)' \bar{g}(x)\, dx\,.\end{aligned}$$

Recall expression (8.6). From Lemma 8.4, $(1-x)^{-1} f'(x)$ and $g'(x)$ are in $L^2(0,1)$. Therefore, $(1-x)^{-1} f'(x) g'(x)$ is in $L^1(0,1)$. This, in conjunction with $(1+x)^{\beta+1}$ being bounded on $(0,1)$, implies that $(1-x)^{-1}(1+x)^{\beta+1} f'(x) g'(x)$ is in

$L^1(0,1)$. As $x \to 1^-$, each of the integrals in (8.6) will be finite. In other words, $\ell_{-2,\beta}[\cdot]$ is strong limit-point at $x = 1$ for $f \in \Delta$. Hence,

$$c := \lim_{x \to 1^-} (1-x)^{-1}(1+x)^{\beta+1} f'(x) g(x)$$

exists and is finite.

Suppose that $c > 0$. Then there exists $x^* \in [0,1)$ such that $f'(x) > 0$, $g(x) > 0$ and

$$(1-x)^{-1}(1+x)^{\beta+1} \geq \frac{c}{2g(x)}$$

for all $x \in [x^*, 1)$. We then have

$$\begin{aligned}
\infty &> \int_0^1 (1-x)^{-1}(1+x)^{\beta+1} f'(x)\,|g'(x)|\; dx \\
&\geq \int_{x^*}^1 (1-x)^{-1}(1+x)^{\beta+1} f'(x)\,|g'(x)|\; dx \\
&\geq \frac{c}{2} \int_{x*}^1 \frac{|g'(x)|}{g(x)}\, dx \\
&\geq \frac{c}{2} \left| \int_{x^*}^1 \frac{g'(x)}{g(x)}\, dx \right| \\
&= \frac{c}{2} \left| \ln(g(x)) \Big|_{x^*}^1 \right| \\
&= \infty\,.
\end{aligned}$$

Thus, we have a contradiction and the lemma is proved. □

Now, we are ready to prove that the operator induced by $\ell_{-2,\beta}[\,\cdot\,]$ is positive.

Proof of Theorem 8.6. We need to show that for some $k > 0$ we have $(\ell_{-2,\beta}[f], f) \geq k(f,f)$. Integration by parts yields

$$\begin{aligned}
(\ell_{-2,\beta}[f], f) &= \int_{-1}^1 (\ell_{-2,\beta}[f])(x) f(x) (1-x)^{-2}(1+x)^{\beta}\, dx \\
&= -(1-x)^{-1}(1+x)^{\beta+1} f'(x) f(x) \Big|_{-1}^1 \\
&\quad + \int_{-1}^1 (1-x)^{-1}(1+x)^{\beta+1} \left(f'(x)\right)^2 + k(1-x)^{-2}(1+x)^{\beta} \left(f(x)\right)^2\, dx\,.
\end{aligned}$$

As the terms on the last line automatically satisfy the desired bound and Lemmas 8.4 through Lemma 8.6 explore in greater detail the properties of functions in the maximal domain, it follows that $(\ell_{-2,\beta}[f], f)$ is a positive operator. □

8.3.4 Comparing the Hilbert Spaces $(S_2, \Phi(\cdot,\cdot))$ *and* $(V_2, (\cdot,\cdot)_2)$

Recall that the second left-definite space associated with the operator is

$$V_2 = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); \\ f^{(j)} \in L^2((-1,1); (1-x)^{j-2}(1+x)^{\beta+j}),\ j = 0,1,2\big\}\,.$$

Theorem 8.7. *As sets,* $S_2 = V_2$*.*

Proof. Let $f \in V_2 = \mathcal{D}(T)$. We've already shown that for $f \in \mathcal{D}(T)$, $f, f' \in AC_{loc}(-1,1]$, $f'' \in L^2\left((-1,1);(1+x)^{\beta+2}\right)$ and $f(1) = f'(1) = 0$. Therefore, $f \in S_2$.

Now, to show $S_2 \subseteq V_2$: we let $f \in S_2$. In particular, this implies that $f'' \in L^2\left((-1,1);(1+x)^{\beta+2}\right)$. Therefore, f satisfies the condition for $j = 2$ of functions in V_2. We also need to show the condition for $j = 0$ and $j = 1$.

For $j = 1$, we need to show that $f' \in L^2\left((-1,1);(1-x)(1+x)^{\beta+1}\right)$. Note that

$$f'(x) = -\int_x^1 f''(t)\,dt$$

and

$$(1-x)^{1/2}(1+x)^{(\beta+1)/2} f'(x) = -(1-x)^{1/2}(1+x)^{(\beta+1)/2}\int_x^1 f''(t)\,dt\,.$$

To show $(1-x)^{1/2}(1+x)^{(\beta+1)/2} f'(x)$ is in $L^2(-1,1)$, we apply the Chisholm-Everitt Inequality with

$$\varphi(x) = 1 \quad \text{and} \quad \psi(x) = -(1-x)^{1/2}(1+x)^{(\beta+1)/2}\,.$$

Then $\varphi(x)$ is L^2 near 1 and since $\beta > -1$ and $\psi(x)$ is L^2 near -1. We also see that

$$\int_x^1 \varphi^2(t)\,dt \cdot \int_{-1}^x \psi^2(t)\,dt \approx (1-x)\frac{(1+t)^{\beta+2}}{\beta+2}\bigg|_{-1}^x$$

is bounded. Thus by the Chisholm-Everitt Inequality, we have

$$(1-x)^{1/2}(1+x)^{(\beta+1)/2}f'(x) \in L^2(-1,1)\,.$$

For $j=0$, we need to show that $(1+x)(1-x)^{\beta/2}f(x)$ is in $L^2(-1,1)$. Similarly to the $j=1$ case, we see that

$$f(x) = -\int_x^1 f'(t)\,dt$$

and

$$(1+x)(1-x)^{\beta/2}f(x) = -(1+x)(1-x)^{\beta/2}\int_x^1 f'(t)\,dt\,.$$

Applying the Chisholm-Everitt Inequality once again with the choices

$$\varphi(x) = 1 \quad \text{and} \quad \psi(x) = -(1+x)(1-x)^{\beta/2}\,,$$

we see that

$$(1+x)(1-x)^{\beta/2}f(x) \in L^2(-1,1)\,.$$

□

As a result of Theorem 8.7, we can view the vector space $V_2 = S_2$ with two different inner products, $(V_2, (\cdot,\cdot)_2)$ and $(V_2, \Phi(\cdot,\cdot))$. In fact, both of these spaces are Hilbert spaces.

Proposition 8.6. *The norms $\|\cdot\|_2$ and $\|\cdot\|_\Phi$ are equivalent.*

Proof. Let $f \in S_2 = V_2$. Then,

$$\|f\|_\Phi^2 = \Phi(f,f) = \int_{-1}^1 |f''(x)|^2 (1+x)^{\beta+2}\,dx\,.$$

On the other hand,

$$\begin{aligned}\|f\|_2^2 &= (f,f)_2 \\ &= \int_{-1}^{1} \Big[(1+x)^{\beta+2}\left|f''(x)\right|^2 + (2k+\beta)\left|f'(x)\right|^2(1-x)^{-1}(1+x)^{\beta+1} \\ &\qquad\qquad +k^2\left|f(x)\right|^2(1-x)^{-2}(1+x)^{\beta+2}\Big]\,dx \\ &= \int_{-1}^{1}\left|f''(x)\right|^2(1+x)^{\beta+2}\,dx = \|f\|_\Phi^2 .\end{aligned}$$

By the Open Mapping Theorem [30, p. 286] $\|f\|_\Phi = \|f\|_2$; that is, the norms are equivalent. □

8.3.5 Constructing the Self-adjoint Operator T_2

Define B_2 to be the second left-definite operator associated with

$$\left(T, L^2((-1,1);(1-x)^{-2}(1+x)^{\beta})\right) ;$$

that is, $B_2 : (V_2,(\cdot,\cdot)) \to (V_2,(\cdot,\cdot))$ and $B_2 f = \ell_{-2,\beta}[f]$ where $f \in V_4$. V_4 is the fourth left-definite space defined by

$$\begin{aligned}V_4 := \{f : (-1,1) \to \mathbb{C} \mid f, f', f'', f''' \in AC_{loc}(-1,1); \\ f^{(j)} \in L^2((-1,1);(1-x)^{j-2}(1+x)^{\beta+j}),\ j = 0,1,2,3,4\} .\end{aligned}$$

Then operator B_2 is self-adjoint in the Hilbert Space $(V_2,(\cdot,\cdot)_2)$ and has $\left\{P_n^{(-2,\beta)}\right\}_{n=2}^{\infty}$ as a complete set of eigenvalues. We want to show that B_2 is self-adjoint in $(V_2,\Phi(\cdot,\cdot)) = (S_2,\Phi(\cdot,\cdot))$.

Theorem 8.8. *The following statements are true:*

(a) B_2 is densely defined in $(V_2,\Phi(\cdot,\cdot))$.

(b) B_2 is Hermitian; that is, $\Phi(B_2 f, g) = \Phi(f, B_2 g)$ for all $f,g \in \mathcal{D}(B_2) = V_4$.

(c) B_2 is self-adjoint in $(S_2,\Phi(\cdot,\cdot))$ and has $\left\{P_n^{(-2,\beta)}\right\}_{n=2}^{\infty}$ as eigenfunctions.

Proof. Part (a) follows directly since $\left\{P_n^{(-2,\beta)}\right\}_{n=2}^{\infty}$ is complete in S_2. The proof of part (b) is rather technical and we see that the result will follow from Lemmas 8.7-8.12 below. Part (c) follows from parts (a) and (b), as well as [27]. □

For $f, g \in \mathcal{D}(B_2) = V_4$ we have $B_2 f, B_2 g \in S_2$. To prove part (b), we need to show that

$$\Phi(B_2 f, g) - \Phi(f, B_2 g) = 0. \tag{8.7}$$

Canceling several terms and then integrating by parts we obtain

$$\begin{aligned}\Phi(B_2 f, g) - \Phi(f, B_2 g) &= \int_{-1}^{1} (B_2 f)''(x) g''(x) - f''(x)(B_2 g)''(x) dx \\ &= (1-x)(1+x)^{\beta+3} \left[f'''(x) \mid g''(x) - f''(x)\bar{g}'''(x) \right] \Big|_{-1}^{1}.\end{aligned}$$

In what follows, we prove that the latter expression is in fact 'strong limit-point' meaning that, individually, the evaluations at the left-hand and right-hand endpoints equal zero. We note that both of these evaluations exist are finite, since

$$\int_0^x (B_2 f)''(t) g''(t) - f''(t)(B_2 g)''(t) dt$$

is finite for all $x \in [0, 1]$ (and similarly at the other endpoint).

First consider the endpoint $x = 1$.

Lemma 8.7. *For all* $f \in V_4$ *we have* $\lim_{x\to 1}(1-x)(1+x)^{\beta+3} f'''(x) = 0$.

Proof. First, we will construct a real valued function $g(x) \in V_4$ such that g'' is 0 near -1, and 1 near 1. Take

$$g(x) = \begin{cases} 0 & -1 \le x \le 0 \\ p(x) & 0 < x \le 1/2 \\ \frac{1}{2}(x-1)^2 & 1/2 < x \le 1 \end{cases}$$

where $p(x) = -704x^7 + 1208x^6 - 704x^5 + 140x^4$ ensures that $g(x) \in C^4[-1, 1] \subset V_4$.

Then, the evaluation at the right endpoint of equation (8.7) simplifies to

$$\lim_{x\to 1}(1-x)(1+x)^{\beta+3}\left[f'''(x)\bar{g}''(x)-f''(x)\bar{g}'''(x)\right]=\lim_{x\to 1}(1-x)f'''(x)$$

and it is this latter limit that we desire to show is equal to zero. We know that this limit exists and is finite, so without loss of generality suppose that

$$c:=\lim_{x\to 1}(1-x)f'''(x)>0\,.$$

There exists $x^* \in (0,1)$ such that for all $x \in [x^*,1)$ we have

$$f'''(x)\geq \frac{c}{2(1-x)}\,.$$

On the other hand recall that $f'''(x) \in L^2((-1,1);(1-x)(1+x)^{\beta+3})$. Therefore,

$$\infty > \int_{x^*}^{1}\left|f'''(x)\right|^2(1-x)(1+x)^{\beta+3}\,dx \geq \frac{c^2}{4}\int_{x^*}^{1}\frac{(1+x)^{\beta+3}}{(1-x)}\,dx=\infty\,,$$

which is a contradiction. Hence $c=0$. □

Lemma 8.8. *Let $f \in V_4$. Then $f''' \in L^2[0,1]$.*

Proof. Since $f \in V_4$, we have

$$f^{(4)} \in L^2((-1,1);(1-x)^2(1+x)^{\beta+4})\,,$$

or equivalently we obtain

$$(1-x)(1+x)^{(\beta+4)/2}f^{(4)} \in L^2(-1,1)\,.$$

In order to apply the Chisholm-Everitt Inequality, we rewrite $f'''(x)$ as

$$f'''(x)=f'''(0)+\int_0^x \frac{(1-t)(1+t)^{(\beta+4)/2}}{(1-t)(1+t)^{(\beta+4)/2}}f^{(4)}(t)\,dt\,.$$

Applying the Chisholm-Everitt Inequality with

$$\varphi(x)=\frac{1}{(1-x)(1+x)^{(\beta+4)/2}} \quad \text{and} \quad \psi(x)=1\,.$$

Noting that $\varphi(x)$ is L^2 near 0, $\psi(x)$ is L^2 near 1, and so we have

$$\left(\int_0^x \varphi^2(t)\,dt\right)\left(\int_x^1 \psi^2(t)\right)\,dt<\infty$$

on $[0,1]$. From the Chisholm-Everitt Inequality, we see that $f''' \in L^2[0,1]$. □

In particular, Lemma 8.8 shows that $f \in V_4$ implies $f,\, f',\, f'' \in AC[0,1]$. Lemma 8.8 can also be modified to show that $f''' \in L^2_{loc}(-1,1]$ and hence,

$$f,\, f',\, f'' \in AC(-1,1]\,.$$

Lemma 8.9. *Limit*

$$\lim_{x\to 1}(1-x)(1+x)^{\beta+3}\left[f'''(x)\bar{g}''(x) - f''(x)\bar{g}'''(x)\right] = 0$$

for all $f, g \in V_4$.

Proof. See Lemmas 8.7 and 8.8. □

The endpoint $x = -1$ follows in analogy:

Lemma 8.10. *For all* $f \in V_4$ *we have* $\lim_{x\to -1}(1-x)(1+x)^{\beta+3}f'''(x) = 0$.

Proof. First, we will construct a real valued function $g(x) \in V_4$ such that g'' is 1 near -1, and 0 near 1. Let

$$g(x) = \begin{cases} 0 & 0 \le x \le 1 \\ p(x) & -1/2 < x < 0 \\ \frac{1}{2}(1+x)^2 & -1 < x \le -1/2 \end{cases}$$

where $p(x) = 644x^7 + \frac{2221}{2}x^6 + \frac{1303}{2}x^5 + \frac{1045}{8}x^4$ ensures that $g(x) \in C^4[-1,1] \subset V_4$.

Then the evaluation at the left endpoint equation (8.7) simplifies to

$$\lim_{x\to -1}(1-x)(1+x)^{\beta+3}\left[f'''(x)\bar{g}''(x) - f''(x)\bar{g}'''(x)\right] = \lim_{x\to -1}(1+x)^{\beta+3}f'''(x)$$

and it is this latter limit that we desire to show is equal to zero. We know that this limit exists and is finite, so without loss of generality suppose that

$$c := \lim_{x\to -1}(1+x)^{\beta+3}f'''(x) > 0\,.$$

There exists $x^* \in (-1,0)$ such that for all $x \in (-1,x^*]$ we have

$$f'''(x) \ge \frac{c}{2(1+x)^{\beta+3}}\,.$$

On the other hand recall that

$$f'''(x) \in L^2((-1,1);(1-x)(1+x)^{\beta+3})\,.$$

Therefore,

$$\infty > \int_{-1}^{x^*} |f'''(x)|^2\,(1-x)(1+x)^{\beta+3}\,dx \geq \frac{c^2}{4}\int_{-1}^{x^*}\frac{(1-x)}{(1+x)^{\beta+3}}\,dx = \infty\,,$$

which is a contradiction. Hence $c = 0$. □

Lemma 8.11. *Let $f \in V_4$. Then $f''' \in L^2[-1,0]$.*

Proof. Since $f \in V_4$, we have

$$f^{(4)} \in L^2((-1,1);(1-x)^2(1+x)^{\beta+4})\,,$$

or equivalently we obtain

$$(1-x)(1+x)^{(\beta+4)/2} f^{(4)} \in L^2(-1,1)\,.$$

In order to apply the Chisholm-Everitt Inequality, we rewrite

$$f'''(x) = f'''(0) - \int_x^0 \frac{(1-t)(1+t)^{(\beta+4)/2}}{(1-t)(1+t)^{(\beta+4)/2}} f^{(4)}(t)\,dt\,.$$

Applying the Chisholm-Everitt Inequality with

$$\psi(x) = \frac{1}{(1-x)(1+x)^{(\beta+4)/2}} \quad \text{and} \quad \varphi(x) = 1\,.$$

Noting that $\varphi(x)$ is L^2 near -1, $\psi(x)$ is L^2 near 0, and so we have

$$\left(\int_{-1}^{x}\varphi^2(t)\,dt\right)\left(\int_x^0 \psi^2(t)\,dt\right) < \infty$$

on $[0,1]$. From the Chisholm-Everitt Inequality, we see that $f''' \in L^2[-1,0]$. □

In particular, Lemma 8.11 shows that $f \in V_4$ implies $f,\ f',\ f'' \in AC[-1,0]$. Together with Lemma 8.8 we have $f''' \in L^2_{loc}[-1,1]$ and hence, $f,\ f',\ f'' \in AC[-1,1]$.

Lemma 8.12. *The limit*

$$\lim_{x\to -1}(1-x)(1+x)^{\beta+3}\left[f'''(x)\bar{g}''(x)-f''(x)\bar{g}'''(x)\right]=0$$

for all functions $f, g \in V_4$.

Proof. See Lemmas 8.10 and 8.11. ☐

This completes the proof of (8.7) and therefore of Theorem 8.8 part (b).

We define the self-adjoint operator T_2 in S_2 to be $T_2 = B_2$.

8.3.6 Conclusion

Recall that the primary goal is to find the self-adjoint operator T in the Sobolev space S which is generated by $\ell_{-2,\beta}[\cdot]$ and has $\left\{P_n^{(-2,\beta)}\right\}_{n=0}^{\infty}$ as eigenfunctions. We have shown the following:

(a) $S = S_1 \oplus S_2$ (recall Subsection 8.3.1).

(b) There exists a self-adjoint operator T_1 in S_1 (recall Subsection 8.3.2).

(c) There exists a self-adjoint operator T_2 in S_2 (recall Subsection 8.3.5).

Additionally, we have proven the our main result, which is the following theorem, for our operator in the Sobolev space setting.

Theorem 8.9. $T = T_1 \oplus T_2$ *is self-adjoint operator in the Sobolev space* S *which is generated by* $\ell_{-2,\beta}[\cdot]$ *and has* $\left\{P_n^{(-2,\beta)}\right\}_{n=0}^{\infty}$ *as eigenfunctions.*

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