ABSTRACT

Smoothness of Polynomials With Respect to Boundary
Conditions As Solutions of Ordinary Differential Equations

Vy K. Nguyen

Director: Johnny L. Henderson, Ph.D.

In this paper, we investigate special polynomial solutions of linear ordinary differential equations as functions of certain boundary conditions. We first look at solutions of $y'' = 0$ and $y''' = 0$, and observe the smooth dependence of the solutions on the boundary conditions by computing partial derivatives. Then, we characterize these partial derivatives according to their own respective boundary conditions. We also establish some relationships among these partial derivatives. In our generalization, we consider $(n-1)^{st}$ degree polynomials as solutions of the boundary value problems of linear ordinary differential equations of n derivative. We established these partial derivatives as $(n-1)^{st}$ degree polynomials and satisify their own respective boundary conditions. Finally, we discuss possible applications of the results of this paper through the modeling of hybrid phenomena with dynamic equations on time scales, and we mention extensions of the results to nonlinear differential equations.

APPROVED BY DIRECTOR OF HONORS THESIS:

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Dr. Johnny L. Henderson, Department of Mathematics

APPROVED BY THE HONORS PROGRAM:

____

Dr. Andrew Wisely, Director

DATE: ______________

SMOOTHNESS OF POLYNOMIALS WITH RESPECT TO BOUNDARY CONDITIONS AS SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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Vy K. Nguyen

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Vy Kim Nguyen

CHAPTER ONE

Introduction

We consider special polynomial solutions of linear ordinary differential equations as functions of certain boundary conditions. That is, we consider differentiation of the polynomials with respect to boundary conditions and characterization of their derivatives as polynomials in x and as solutions of boundary value problems themselves.

There is a long history of work devoted to smoothness of solution with respect to boundary conditions for differential equation. In 1976, Peterson [17] was concerned with smoothness of solutions of $(k, n-k)$ conjugate boundary value problems for n^{th} order linear ordinary differentiation in equations, with comparison of solution results. Then in 1975, Spencer [18] extended Peterson's results for conjugate boundary value problems for nonlinear differential equations. In 1984, Henderson [9] dealt with analogues of the Spencer results in the context of right focal point boundary conditions for nonlinear ordinary differential equations. Then in 1987, Henderson [10] dealt with the n^{th} order differential equation $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$ and solutions to the variational equation $z^{(n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(x, y(x, y'(x), \ldots, y^{(n-1)}(x))z^{(i-1)}$ in order to study the differentiability with respect to boundary conditions of solutions for boundary problems for n^{th} order differential equations. He obtained results that are an analogue of a theorem due to Peano for initial value problems. In 1993, Ehme [4] studied solutions between boundary value problems for the ordinary differential equation $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$ and obtained results in which solutions of other boundary value problems are differentiated with respect to boundary conditions. In order to characterize both continuous dependence and differentiability of solutions with respect to the boundary conditions, Ehme [5] extended his studies to a system of first order functional differential equations that satisfy a set of multipoint boundary conditions. In 1996, Henderson and Ehme [6] directed their attention to the smoothness of solutions for functional boundary value problems. They specifially looked at solutions to the equations: $y'(t) = f(t, y(t), f(\varphi(t)))$, $t \in [a, b]$, and $M_y(\tau(t)) + N_y(\sigma(t)) = \Gamma(t)$, $t \in [\alpha, a]$ and discovered that under appropriate conditions, these solutions are continuously differentiable as a function of φ, τ, σ, and Γ. Working with systems of difference equations, Lawrence and Henderson [13] considered derivatives of solutions to the first order systems of difference equations that satisfy multipoint boundary conditions and confirmed the existence of these derivatives. In 2002, Lawrence [16] extended her works to study the differentiability of solutions of a nonlinear dynamic system with multipoint boundary conditions. In 2004, Henderson and Tisdell [14] investigated a three-point boundary value problem for second order ordinary differential equations to determine the boundary data smoothness for solutions. A year later, Henderson, Karna and Tisdell [12] continued the study of the second order, three-point boundary value problem in order to employ shooting methods to obtain solutions to the boundary value problem. In 2007, with attention directed to nonlocal boundary-value problems, Ehrke, Henderson, Kunkel, and Sheng [7] obtained smoothness results for solutions with respect to nonlocal boundary conditions $y(x_1) = y_1$, $y(x_2) - \sum_{i=1}^{m} r_i y(n_i) = y_2$, for second order ordinary differential equations. Continuing the study of smoothness of solutions for nonlocal boundary value problems, Henderson, Hopkins, Kim, and Lyons [11] extended that study to n^{th} order differential equations. Hopkins, Kim, Lyons, and Speer [15] extended the study of boundary data smoothness for solutions of nonlocal boundary value problems to second order equations in 2009. With this work, the study of smoothness of solutions of boundary value problems, with respect to boundary conditions, has been extended to polynomials, when viewed as solutions of linear differential equations satisfying integral boundary conditions.

CHAPTER TWO

Basic Investigative Work

In this chapter, we are interested in investigating solutions of $y'' = 0$, (which are straight lines), depending on boundary conditions, and then we consider how these lines depend smoothly upon these boundary conditions. In particular, we begin with the solution of $y'' = 0$ that satisfies that conditions: $y(a) = A$ and $\int_a^b y(x)\,dx = B$. From $y'' = \frac{\partial^2 y}{\partial x^2} = 0$, we can write the solution as

$$y(x) = m(x-a) + k,$$

and then we can perform the following operations to find k:

$$\begin{aligned} y(a) &= A \\ y(a) &= (a-a)m + k \\ &= 0 + k \\ y(a) &= k = A. \end{aligned}$$

Since we found that $k = A$, we can substitute k into the equation for $y(x)$, and the result is

$$y(x) = m(x-a) + A.$$

Now, we need to solve for m by using the second given condition: $B = \int_a^b y(x)\,dx$. Therefore, we will perform the following operations:

$$\begin{aligned} B &= \int_a^b y(x)\,dx \\ &= \int_a^b [m(x-a) + A]\,dx \\ &= \frac{m(x-a)^2}{2} + Ax\Big|_a^b \\ &= \frac{m(b-a)^2}{2} + A(b-a). \end{aligned}$$

Now, we need to solve for m, and we obtain:

$$\begin{aligned}\frac{m(b-a)^2}{2} &= B - A(b-a)\\ m &= \frac{2[B-A(b-a)]}{(b-a)^2}.\end{aligned}$$

So now that we have the variables of k and m in terms x, a, b, A, and B, we can designate our solution as

$$\begin{aligned}y(x,a,b,A,B) &= \frac{2[B-A(b-a)]}{(b-a)^2}(x-a)+A\\ &= [2B(b-a)^{-2} - 2A(b-a)^{-1}](x-a)+A\end{aligned}$$

Note: We have written the equation in the expanded form for the ease to compute several partial derivatives.

With our designated solution, we can compute the partial derivatives of interest that characterize the smooth dependence of $y(x,a,b,A,B)$ on the variables a, b, A, and B. So we let

$$r_1(x) = \frac{\partial y}{\partial A}(x) \quad r_2(x) = \frac{\partial y}{\partial B}(x) \quad z_1(x) = \frac{\partial y}{\partial a}(x) \quad z_2(x) = \frac{\partial y}{\partial b}(x).$$

Now, we want to find the following and discover any relationships among these terms:

$r_1(x) = \frac{\partial y}{\partial A}(x)$	$r_2(x) = \frac{\partial y}{\partial B}(x)$	$z_1(x) = \frac{\partial y}{\partial a}(x)$	$z_2(x) = \frac{\partial y}{\partial b}(x)$
$r_1''(x) = \frac{\partial^2 r_1}{\partial x^2}$	$r_2''(x) = \frac{\partial^2 r_2}{\partial x^2}$	$z_1''(x) = \frac{\partial^2 z_1}{\partial x^2}$	$z_2''(x) = \frac{\partial^2 z_2}{\partial x^2}$
$r_1(a)$	$r_2(a)$	$z_1(a)$	$z_2(a)$
$\int_a^b r_1(x)\,dx$	$\int_a^b r_2(x)\,dx$	$\int_a^b z_1(x)\,dx$	$\int_a^b z_2(x)\,dx$

Let us first observe the smooth dependence of $y(x,a,b,A,B)$ on the variable A.

$$\begin{aligned}r_1(x) &= \frac{\partial y}{\partial A}(x)\\ &= (x-a)[-2(b-a)^{-1}]+1\\ &= -2(x-a)(b-a)^{-1}+1.\end{aligned}$$

Then

$$r_1''(x) = \frac{\partial^2 r_1}{\partial x^2} = 0.$$

Next

$$r_1(a) = -2(b-a)^{-1}(a-a) + 1 = 1,$$

and

$$\begin{aligned}
\int_a^b r_1(x)\,dx &= \int_a^b [-2(b-a)^{-1}(x-a)+1]\,dx \\
&= \left[\frac{-2(b-a)^{-1}(x-a)^2}{2} + x\right]\Bigg|_a^b \\
&= \frac{-2(b-a)^{-1}(b-a)^2}{2} + b - \frac{-2(b-a)^{-1}(a-a)^2}{2} - a \\
&= \frac{-(b-a)^2}{(b-a)} + (b-a) \\
&= -(b-a) + (b-a) \\
&= 0.
\end{aligned}$$

Next, let us observe the smooth dependence of $y(x, a, b, A, B)$ on the variable B.

$$\begin{aligned}
r_2(x) &= \frac{\partial y}{\partial B}(x) \\
&= (x-a)[2(b-a)^{-2}] \\
&= 2(x-a)(b-a)^{-2}.
\end{aligned}$$

Then,

$$r_2''(x) = \frac{\partial^2 r_2}{\partial x^2} = 0,$$

$$r_2(a) = 2(a-a)(b-a)^{-2} = 0,$$

and

$$\begin{aligned}\int_a^b r_2(x)\,dx &= \int_a^b [2(x-a)(b-a)^{-2}]\,dx \\ &= [(b-a)^{-2}(x-a)^2]\Big|_a^b \\ &= (b-a)^{-2}(b-a)^2 - (b-a)^{-2}(a-a)^2 \\ &= \frac{(b-a)^2}{(b-a)^2} \\ &= 1.\end{aligned}$$

We notice that we can represent the boundary conditions for the partial derivatives with respect to A and B in a matrix,

$$\begin{bmatrix} r_1(a) & r_2(a) \\ \\ \int_a^b r_1(x)\,dx & \int_a^b r_2(x)\,dx \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Continuing, let us observe the smooth dependence of $y(x, a, b, A, B)$ on the variable a. Let us begin by looking at

$$\begin{aligned}z_1(x) &= \frac{\partial y}{\partial a}(x) \\ &= [2B(-2)(b-a)^{-3}(-1) - 2A(-1)(b-a)^{-2}(-1)](x-a)+ \\ &\quad [2B(b-a)^{-2} - 2A(b-a)^{-1}](-1) \\ &= [4B(b-a)^{-3} - 2A(b-a)^{-2}](x-a) - [2B(b-a)^{-2} - 2A(b-a)^{-1}].\end{aligned}$$

Next,

$$z_1''(x) = \frac{\partial^2 z_1}{\partial x^2} = 0,$$

and

$$
\begin{aligned}
z_1(a) &= 2A(b-a)^{-1}[1-0] + 2B(b-a)^{-2}[2(b-a)^{-1}(0)-1] \\
&= 2A(b-a)^{-1} - 2B(b-a)^{-2} \\
&= -\left. y'(x)\right|_{x=a} \\
&= -y'(a).
\end{aligned}
$$

And we also look at

$$
\begin{aligned}
\int_a^b z_1(x)\,dx &= \int_a^b [-2B(b-a)^{-2} + 2A(b-a)^{-1} + 4B(b-a)^{-3}(x-a) \\
&\quad - 2A(b-a)^{-2}(x-a)]\,dx \\
&= \Bigg[-2B(b-a)^{-2}x + 2A(b-a)^{-1}x + \frac{4B(b-a)^{-3}(x-a)^2}{2} \\
&\quad - \frac{2A(b-a)^{-2}(x-a)^2}{2}\Bigg]\Bigg|_a^b \\
&= -2B(b-a)^{-2}b + 2A(b-a)^{-1}b + 2B(b-a)^{-3}(b-a)^2 \\
&\quad - A(b-a)^{-2}(b-a)^2 - 2B(b-a)^{-2}a - 2A(b-a)^{-1}a \\
&\quad + 2B(b-a)^{-3}(a-a)^2 + A(b-a)^{-2}(a-a)^2 \\
&= -2B(b-a)^{-2}(b-a) + 2A(b-a)^{-1}(b-a) + \frac{2B}{b-a} - A \\
&= 2A - A \\
&= A \\
&= y(a).
\end{aligned}
$$

Finally, let us observe the smooth dependence of $y(x, a, b, A, B)$ on the variable b.

$$
\begin{aligned}
z_2(x) &= \frac{\partial y}{\partial b} \\
&= (x-a)[2B(-2)(b-a)^{-3}(1) - 2A(-1)(b-a)^{-2}(1)] \\
&= 2(x-a)(b-a)^{-2}[-2B(b-a)^{-1} + A].
\end{aligned}
$$

Again,

$$z_2''(x) = \frac{\partial^2 z_2}{\partial x^2} = 0,$$

while

$$z_2(a) = 2(a-a)(b-a)^{-2}[-2B(b-a)^{-1} + A] = 0,$$

and

$$\begin{aligned}
\int_a^b z_2(a) &= \int_a^b [2(x-a)(b-a)^{-2}[-2B(b-a)^{-1} + A]\, dx \\
&= (x-a)^2(b-a)^{-2}[-2B(b-a)^{-1} + A]\Big|_a^b \\
&= (b-a)^2(b-a)^{-2}[-2B(b-a)^{-1} + A] \\
&= -2B(b-a)^{-1} + A \\
&= -y(b).
\end{aligned}$$

Now, that we have completed investigating the smooth dependence of $y(x, a, b, A, B)$ on the variables a, b, A, and B, a summary is provided below of our results.

$r_1''(x) = \frac{\partial^2 r_1}{\partial x^2} = 0$	$r_2''(x) = \frac{\partial^2 r_2}{\partial x^2} = 0$	$z_1''(x) = \frac{\partial^2 z_1}{\partial x^2} = 0$	$z_2''(x) = \frac{\partial^2 z_2}{\partial x^2} = 0$
$r_1(a) = 1$	$r_2(a) = 0$	$z_1(a) = -y'(a)$	$z_2(a) = 0$
$\int_a^b r_1(x)\, dx = 0$	$\int_a^b r_2(x)\, dx = 1$	$\int_a^b z_1(x)\, dx = y(a)$	$\int_a^b z_2(x)\, dx = -y(b)$

2.1 Relationships Among the Partial Derivatives

In this section, we will look at the relationships among the partial derivatives, for which we claim the following:

$$\frac{\partial y}{\partial b}(x) = -y(b)\frac{\partial y}{\partial B}(x) \tag{2.1}$$

$$\frac{\partial y}{\partial a}(x) = -y'(a)\frac{\partial y}{\partial A} + y(a)\frac{\partial y}{\partial B}(x) \tag{2.2}$$

The steps below will demonstrate the validity of (2.1).

$$\frac{\partial y}{\partial b}(x) = 2(x-a)(b-a)^{-2}[2B(b-a)^{-1} + A]$$

$$-y(b) = -2B(b-a)^{-1} + A$$

$$\frac{\partial y}{\partial B}(x) = (x-a)[2(b-a)^{-2}]$$

$$2(x-a)(b-a)^{-2}[2B(b-a)^{-1} + A] = [-2B(b-a)^{-1} + A](x-a)[2(b-a)^{-2}]$$

$$\tfrac{\partial y}{\partial b}(x) = -y(b)\tfrac{\partial y}{\partial B}(x)$$

Next, the following steps will demonstrate the validity of (2.2). From

$$\frac{\partial y}{\partial a}(x) = [4B(b-a)^{-3} - 2A(b-a)^{-2}](x-a) - [2B(b-a)^{-2} - 2A(b-a)^{-1}],$$

$$-y'(a) = \frac{-2[B - A(b-a)]}{(b-a)^2},$$

$$\frac{\partial y}{\partial A}(x) = -2(b-a)^{-1}(x-a) + 1,$$

$$y(a) = A,$$

$$\frac{\partial y}{\partial B}(x) = (x-a)[2(b-a)^{-2}],$$

we obtain

$$
\begin{aligned}
\frac{\partial y}{\partial a}(x) &= -[2B(b-a)^{-2} + 2A(b-a)^{-1}] + [4B(b-a)^{-3} - 2A(b-a)^{-2}](x-a) \\
&= -[2B(b-a)^{-2} - 2A(b-a)^{-1}] + 4B(b-a)^{-3}(x-a) - 2A(b-a)^{-2}(x-a) \\
&\quad - 4A(b-a)^{-2}(x-a) + 2A(x-a)(b-a)^{-2} \\
&= -2B(b-a)^{-2} + 2A(b-a)^{-1} + 4B(b-a)^{-3}(x-a) \\
&= [-2B(b-a)^{-2} + 2A(b-a)^{-1}][-2(b-a)^{-1}(x-a) + 1] + 2A(x-a)(b-a)^{-2} \\
&= \frac{-2[B - A(b-a)]}{(b-a)^2}[-2(b-a)^{-1}(x-a) + 1] + 2A(x-a)(b-a)^{-2} \\
&= -y'(a)\frac{\partial y}{\partial A}(x) + y(a)\frac{\partial y}{\partial B}(x).
\end{aligned}
$$

CHAPTER THREE

Extended Investigative Work

In this chapter, we extend the methods of Chapter 1 to third order problems, which involve investigating the solutions of $y'''(x) = 0$. In particular, our investigations now deal with quadratic functions and their dependency upon certain boundary conditions. The goal of this chapter, in conjunction with Chapter 1, is to determine a pattern for higher order problems. So, here we begin with the following:

$$y'''(x) = 0,$$
$$y(a) = A,$$
$$y'(a) = B,$$
$$\int_a^b y(x)\,dx = C.$$

Solutions are the quadratics which are written in the convenient the form

$$y(x, a, b, A, B) = \frac{D}{2}(x-a)^2 + E(x-a) + F.$$

We now derive a solution in terms of x, a, b, A, B, and C. Let us start with the boundary condition $y(a) = A$.

$$A = y(a) = \frac{D}{2}(a-a)^2 + E(a-a) + F = F.$$

So, we can substitute F for A in the solution to obtain

$$y(x, a, b, A, B, C) = \tfrac{D}{2}(x-a)^2 + E(x-a) + F.$$

Now, let us proceed to work with the condition $y'(a) = B$. From $y'(x) = D(x-a)^2 + E$,

$$B = y'(a) = D(a-a)^2 + E = E.$$

So, we can substitute E for B in the solution and obtain

$$y(x,a,b,A,B,C) = \tfrac{D}{2}(x-a)^2 + B(x-a) + A.$$

Finally, let us proceed to work with the condition $\int_a^b y(x)\,dx = C$.

$$\begin{aligned}
C &= \int_a^b y(x)\,dx \\
&= \int_a^b \left[\frac{D}{2}(x-a)^2 + B(x-a) + A\right] dx \\
&= \frac{D}{2}\frac{(x-a)^3}{3} + \frac{B(x-a)^2}{2} + Ax\Big|_a^b \\
&= \frac{D}{6}(b-a)^3 + \frac{B}{2}(b-a)^2 + Ab - \frac{D}{6}(a-a)^3 - \frac{B}{2}(a-a)^2 - Aa,
\end{aligned}$$

and so now, we have

$$C = \tfrac{D}{6}(b-a)^3 + \tfrac{B}{2}(b-a)^2 + A(b-a).$$

Solving for D, we have

$$D = \frac{6[C - A(b-a) - \frac{1}{2}B(b-a)^2]}{(b-a)^3}.$$

Now, with D in terms of a, b, A, B, and C, we have our solution,

$$y(x,a,b,A,B,C) = \tfrac{3[C-A(b-a)-\frac{1}{2}B(b-a)^2]}{(b-a)^3}(x-a)^2 + B(x-a) + A.$$

As in Chapter 1, we are interested in characterizing partial derivatives of $y(x,a,b,A,B,C)$ with respect to a, b, A, B, and C. First, the following are straightforward.

If $r_1(x) := \frac{\partial y}{\partial A}$, then $r_1''' = 0$, $r_1(a) = 1$, $r_1'(a) = 0$, and $\int_a^b r_1(x)\,dx = 0$.

If $r_2(x) := \frac{\partial y}{\partial B}$, then $r_2''' = 0$, $r_2(a) = 0$, $r_2'(a) = 1$, and $\int_a^b r_2(x)\,dx = 0$.

If $r_3(x) := \frac{\partial y}{\partial C}$, then $r_3''' = 0$, $r_3(a) = 0$, $r_3'(a) = 0$, and $\int_a^b r_3(x)\,dx = 1$.

The computations for the partial derivatives are provided below. So let us observe the smooth dependence of the partial derivative of A on its boundary conditions.

$$\begin{aligned}
\frac{\partial y}{\partial A}(x) &= \frac{-3(x-a)^2}{(b-a)^2} + 1, \\
\frac{\partial y}{\partial A}(a) &= \frac{-3(a-a)^2}{(b-a)^2} + 1 = 1, \\
\left(\frac{\partial y}{\partial A}\right)'(x) &= \frac{-3(2)(x-a)}{(b-a)^2} = \frac{-6(x-a)}{(b-a)^2}, \\
\left(\frac{\partial y}{\partial A}\right)'(a) &= \frac{-6(a-a)}{(b-a)^2} = 0,
\end{aligned}$$

and

$$\int_a^b \frac{-3(x-a)^2}{(b-a)^2} + 1\, dx = \left. \frac{-(x-a)^3}{(b-a)^2} \right|_a^b = 0.$$

Now, let us proceed to observe the smooth dependence of the partial derivative of B on its boundary conditions.

$$\begin{aligned}
\frac{\partial y}{\partial B}(x) &= \frac{-3(x-a)^2}{2(b-a)} + (x-a), \\
\frac{\partial y}{\partial B}(a) &= \frac{-3(a-a)^2}{2(b-a)} + (a-a) = 0, \\
\left(\frac{\partial y}{\partial B}\right)'(x) &= \frac{-3(2)(x-a)}{2(b-a)} + 1 = \frac{-3(x-a)}{(b-a)} + 1, \\
\left(\frac{\partial y}{\partial B}\right)'(a) &= \frac{-3(a-a)}{(b-a)} + 1 = 1,
\end{aligned}$$

and

$$\int_a^b \frac{-3(x-a)^2}{2(b-a)} + (x-a)\, dx = \left. \frac{-(x-a)^3}{2(b-a)} + \frac{(x-a)^2}{2} \right|_a^b = 0.$$

Finally, let us observe the smooth dependence of the partial derivative of C on its boundary conditions.

$$\frac{\partial y}{\partial C}(x) = \frac{3(x-a)^2}{(b-a)^3},$$

$$\frac{\partial y}{\partial C}(a) = \frac{3(a-a)^2}{(b-a)^3} = 0,$$

$$\left(\frac{\partial y}{\partial C}\right)'(x) = \frac{6(x-a)}{(b-a)^3},$$

$$\left(\frac{\partial y}{\partial C}\right)'(a) = \frac{6(a-a)}{(b-a)^3} = 0,$$

and

$$\int_a^b \frac{3(x-a)^2}{(b-a)^3}\,dx = \left.\frac{(x-a)^3}{(b-a)^3}\right|_a^b = 1.$$

Like we noticed in the Basic Investigative Work, we also notice that the boundary conditions for the partial derivatives with respect to A, B, and C can also be represented in a matrix,

$$\begin{bmatrix} r_1(a) & r_2(a) & r_3(a) \\ r_1'(a) & r_2'(a) & r_3'(a) \\ \int_a^b r_1(x)\,dx & \int_a^b r_2(x)\,dx & \int_a^b r_3(x)\,dx \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, for the solution $y(x,a,b,A,B,C) = \frac{3[C-A(b-a)-\frac{1}{2}B(b-a)^2]}{(b-a)^3}(x-a)^2 + B(x-a) + A$, we claim:

If $z_1(x) := \frac{\partial y}{\partial a}$, then $z_1''' = 0$, $z_1(a) = -y'(0) = -B$, $z_1'(a) = -y''(a)$, $\int_a^b z_1(x)\,dx = y(a)$,

and if $z_2(x) := \frac{\partial y}{\partial b}$, then $z_2''' = 0$, $z_2(a) = 0$, $z_2'(a) = 0$, $\int_a^b z_2(x)\,dx = -y(b)$.

We begin with $z_1(x) = \frac{\partial y}{\partial a}$.

$$\begin{aligned} z_1(x) &= \frac{3C(2)(x-a)(-1)(b-a)^3 - 3C(x-a)^2(3)(b-a)^2(-1)}{(b-a)^6} \\ &\quad - \frac{3A(2)(x-a)(-1)(b-a)^2 - 2(b-a)(-1)3A(x-a)^2}{(b-a)^4} \\ &\quad - \frac{3B(2)(x-a)(-1)(2)(b-a) - 2(-1)(3B)(x-a)^2}{4(b-a)^2} \\ &\quad + B(-1). \end{aligned}$$

Simplifying, we get that

$$\begin{aligned} z_1(x) &= \frac{-6C(x-a)}{(b-a)^3} + \frac{9C(x-a)^2}{(b-a)^4} + \frac{6A(x-a)}{(b-a)^2} \\ &\quad - \frac{6A(x-a)^2}{(b-a)^3} + \frac{3B(x-a)}{(b-a)} - \frac{3B(x-a)^2}{2(b-a)^2} - B \end{aligned}$$

To confirm the statement $z_1(a) = -y'(a)$, we will substitute $x = a$, and we have

$$\begin{aligned} z_1(a) &= \frac{-6C(a-a)}{(b-a)^3} + \frac{9C(a-a)^2}{(b-a)^4} + \frac{6A(a-a)}{(b-a)^2} \\ &\quad - \frac{6A(a-a)^2}{(b-a)^3} + \frac{3B(a-a)}{(b-a)} - \frac{3B(a-a)^2}{2(b-a)^2} - B \\ &= -B, \end{aligned}$$

and

$$-y'(x) = \frac{3\{C - \left[\frac{B}{2}(b-a)^2 + A(b-a)\right]\}}{(b-a)^3}(2)(x-a) + B,$$

so that

$$\begin{aligned} -y'(a) &= \frac{-6\{C - [\frac{B}{2}(b-a)^2 + A(b-a)]\}}{(b-a)^3}(a-a) - B \\ &= -B. \end{aligned}$$

Therefore, $z_1(a) = -y'(a)$.

Next, let's confirm $z_1'(a) = -y''(a)$.

$$z_1'(x) = \frac{-6C}{(b-a)^3} + \frac{18C(x-a)}{(b-a)^4} + \frac{6A}{(b-a)^2} - \frac{12A(x-a)}{(b-a)^3} + \frac{3B}{(b-a)} - \frac{6B(x-a)}{2(b-a)^2}.$$

Then,

$$
\begin{aligned}
z'(a) &= -\frac{6C}{(b-a)^3} + \frac{6A}{(b-a)^2} + \frac{3B}{(b-a)} \\
&= \frac{-6C + 6A(b-a) + 3B(b-a)^2}{(b-a)^3}.
\end{aligned}
$$

Next,

$$
y''(x) = \frac{6\{C - \left[\frac{B}{2}(b-a)^2 + A(b-a)\right]\}}{(b-a)^3}
$$

and so,

$$
y''(a) = \frac{6\{C - \left[\frac{B}{2}(b-a)^2 + A(b-a)\right]\}}{(b-a)^3}.
$$

We have verified that the statement $z_1'(a) = -y''(a)$ is valid.

Finally, we verify that $\int_a^b z_1(x)dx = y(a)$.

$$
\begin{aligned}
\int_a^b z_1(x), dx = [&\frac{-6C(x-a)^2}{2(b-a)^3} + \frac{9C(x-a)^3}{3(b-a)^{-1}} + \frac{6A(x-a)^2}{2(b-a)^2} \\
&- \frac{6A(x-a)^3}{3(b-a)^3} + \frac{3B(x-a)^2}{2(b-a)} - \frac{3B(x-a)^3}{2(3)(b-a)^2} - B(x)]\Big|_a^b \\
&= \frac{-6C}{2(b-a)} + \frac{9C}{3(b-a)} + \frac{6A}{2} - \frac{6A}{3} + \frac{3B(b-a)}{2} - \frac{B}{2}(b-a) - Bb + Ba \\
&= A \\
&= y(a).
\end{aligned}
$$

Therefore, our claims concerning $z_1(x)$ have been validated.

We, now, turn our attention to the claims for $z_2(x) = \frac{\partial y}{\partial b}$.

$$
\begin{aligned}
z_2(x) &= \frac{\partial y}{\partial b}(x) \\
&= 3C(x-a)^2(-3)(b-a)^{-4} - 3A(x-a)^2(-2)(b-a)^{-3} \\
&\quad - \frac{3B}{2}(x-a)^2(-1)(b-a)^{-2} \\
&= \frac{-9C(x-a)^2}{(b-a)^4} + \frac{6A(x-a)^2}{(b-a)^3} + \frac{3B(x-a)^2}{2(b-a)^2}.
\end{aligned}
$$

From that, let us confirm the validity of $z_2(a) = 0$:

$$\begin{aligned} z_2(a) &= \frac{-9C(a-a)^2}{(b-a)^4} + \frac{6A(a-a)^2}{(b-a)^3} + \frac{3B(a-a)^2}{2(b-a)^2} \\ &= 0. \end{aligned}$$

Next, let us confirm the validity of $z_2'(a) = 0$:

$$\begin{aligned} z_2'(x) &= -9C(x-a)^2(-4)(b-a)^{-5} + 6A(x-a)^2(-3)(b-a)^{-4} \\ &\quad + \frac{3B(x-a)^2(-2)(b-a)^{-3}}{2} \\ &= \frac{36C(x-a)^2}{(b-a)^5} - \frac{18A(x-a)^2}{(b-a)^4} - \frac{3B(x-a)^2}{(b-a)^3}, \end{aligned}$$

from which we obtain

$$\begin{aligned} z_2'(a) &= \frac{36C(a-a)^2}{(b-a)^5} - \frac{18A(a-a)^2}{(b-a)^4} - \frac{3B(a-a)^2}{(b-a)^3} \\ &= 0. \end{aligned}$$

Finally, let us confirm the validity of the statement $\int_a^b z_2(x)\,dx = -y(b)$.

$$\begin{aligned} \int_a^b z_2(x)\,dx &= \frac{-9C(x-a)^3}{(b-a)^4} + \frac{6A(x-a)^3}{3(b-a)^3} + \frac{3B(x-a)^3}{3(2)(b-a)^2}\Bigg|_a^b \\ &= \frac{-9C(b-a)^3}{3(b-a)^4} + \frac{6A(b-a)^3}{3(b-a)^3} + \frac{B(b-a)^3}{2(b-a)^2} \\ &= \frac{-3C}{(b-a)} + 2A + \frac{B(b-a)}{2}, \end{aligned}$$

and

$$\begin{aligned} -y(b) &= -[\frac{3C(b-a)^2}{(b-a)^3} - \frac{3A(b-a)^2}{(b-a)^2} - \frac{3B(b-a)^2}{2(b-a)} + B(b-a) + A] \\ &= \frac{-3C}{(b-a)} + 3A + \frac{3B(b-a)}{2} - B(b-a) - A \\ &= \frac{-3C}{(b-a)} + 2A + \frac{B(b-a)}{2} \end{aligned}$$

Therefore,

$$\int_a^b z_2(x)\,dx = -y(b).$$

Comparing our computational results of this chapter with those of Chapter 2, we do indeed recognize some patterns among the results. These patterns will be dealt with in generalization in the next chapter.

CHAPTER FOUR

Generalizations

In this chapter, we deal with generalization of the results in Chapters 2 and 3. In particular, we consider certain $(n-1)^{st}$ degree polynomials as solutions of boundary value problems, and we deal with characterizations of derivatives of the solutions with respect to the boundary conditions.

Hartman [8] states in his book a theorem which he attributes to Guiseppe Peano concerning differentiation with respect to initial conditions of solutions of initial value problems. We will make much use of the Peano theorem in the context of $(n-1)^{st}$ degree polynomials. We state the Peano theorem in that context as a lemma.

Lemma 1. [Peano] *Suppose that* $p(x)$ *is an* $(n-1)^{st}$ *degree polynomial (i.e.* $p^{(n)}(x) = 0$*) satisfying, for some* $x_0 \in \mathbb{R}$,

$$\begin{aligned} &p(x_0) = c_1, \\ &p'(x_0) = c_2, \\ &\vdots \\ &p^{(n-1)}(x_0) = c_n. \end{aligned}$$

Then,

(i) For $i = 1, \ldots, n$, $\frac{\partial}{\partial c_i} p(x)$ *exists and* $\alpha_i(x) := \frac{\partial}{\partial c_i} p(x)$ *is an* $(n-1)^{st}$ *degree polynomial in* x *and satisfies,*

$$\alpha_i^{(j-1)}(x_0) = \left\{ \begin{matrix} 0, & j \neq i \\ 1, & j = i \end{matrix} \right\} := \delta_{ij}, j = 1, \ldots, n,$$

and

(ii) $\frac{\partial}{\partial x_0}p(x)$ *exists and* $\beta(x) := \frac{\partial}{\partial x_0}p(x)$ *is an* $(n-1)^{st}$ *degree polynomial in* x *and satisfies* $\beta^{(j-1)}(x_0) = -p^{(j)}(x_0), j = 1, \ldots, n.$

For the results of this section, we will rely at crucial places upon the following lemma.

Lemma 2. *If* $p(x)$ *is an* $(n-1)^{st}$ *degree polynomial (i.e.* $p^{(n)}(x) = 0$*), such that for some real numbers* $a < b$,

$$p^{(i-1)}(a) = 0, i = 1, \ldots, n-1, \int_a^b p(x)\, dx = 0,$$

then $p(x) = 0$*, for all* $x \in \mathbb{R}$.

Proof. Let $p(x)$ be such a polynomial. Then, for some real constants $a_0, a_1, \ldots, a_{n-1}$,

$$p(x) = a_0 + a_1(x-a) + \frac{a_2}{2!}(x-a)^2 + \frac{a_3}{3!}(x-a)^3 + \ldots + \frac{a_{n-1}}{(n-1)!}(x-a)^{n-1}.$$

Then,

$$\begin{aligned}
p'(x) &= a_1 + a_2(x-a) + \frac{a_3}{2!}(x-a)^2 + \cdots + \frac{a_{n-1}}{(n-2)!}(x-a)^{n-2},\\
p''(x) &= a_2 + a_3(x-a) + \cdots + \frac{a_{n-1}}{(n-3)!}(x-a)^{n-3},\\
&\vdots\\
p^{(n-2)}(x) &= a_{n-2} + a_{n-1}(x-a).
\end{aligned}$$

From the boundary conditions,

$$\begin{aligned}
0 =& p(a) = a_0,\\
0 =& p'(a) = a_1,\\
0 =& p''(a) = a_2,\\
&\vdots\\
0 =& p^{(n-2)}(a) = a_{n-2},
\end{aligned}$$

and so $p(x) = \frac{a_{n-1}}{(n-1)!}(x-a)^{n-1}$.

Also, from the integral boundary condition,

$$\begin{aligned} 0 &= \int_a^b p(x)\,dx \\ &= \int_a^b \frac{a_{n-1}}{(n-1)!}(x-a)^{n-1}\,dx \\ &= \frac{a_{n-1}}{n!}(x-a)^n\Big|_a^b \\ &= \frac{a_{n-1}}{n!}(b-a)^n. \end{aligned}$$

Since, $b-a>0$, we conclude $a_{n-1}=0$. In particular, we have $a_0 = a_1 = \cdots = a_{n-1} = 0$, so that that polynomial $p(x)=0$, for all $x \in \mathbb{R}$. □

Note: If $p(x) \not\equiv 0$, then at least one of $p^{(i-1)}(a) \neq 0$, $i = 1, \ldots, n-1$, or $\int_a^b p(x)dx \neq 0$.

Hereafter, we assume $n \geq 2$ is a fixed natural number.

Theorem 3. *Let $y(x)$ be an $(n-1)^{st}$ degree polynomial (i.e. $y^{(n)}(x)=0$) on $\mathbb{R}$. Let $a<b$ be given, so that $y(x) = y(x, a, b, A_1, \ldots, A_{n-1}, A_n)$, where $y^{(i-1)}(a) = A_i, i = 1, \ldots, n-1, and \int_a^b y(x)\,dx = A_n$. Then,*

(I) For $i = 1, \ldots, n-1$, $\frac{\partial y}{\partial A_i}$ exists on $\mathbb{R}$, and $r_i := \frac{\partial y}{\partial A_i}$ is an $(n-1)^{st}$ degree polynomial in x and satisfies the boundary conditions,

$$r_i^{(j-1)}(a) = \delta_{ij}, j = 1, \ldots, n-1, and \int_a^b r_i(x)\,dx = 0,$$

and $r_n := \frac{\partial y}{\partial A_n}$ exists on $\mathbb{R}$, as an $(n-1)^{st}$ degree polynomial in x, and satisfies the boundary conditions,

$$r_n^{(j-1)}(a) = 0, j = 1, \ldots, n-1, and \int_a^b r_n(x)\,dx = 1.$$

(II) $z_1 := \frac{\partial y}{\partial a}$ and $z_2 := \frac{\partial y}{\partial b}$ both exist on $\mathbb{R}$, as $(n-1)^{st}$ degree polynomials, and satisfy the respective boundary conditions,

$$z_1^{(j-1)}(a) = -y^{(j)}(a), j = 1, \ldots, n-1, and \int_a^b z_1(x)\,dx = y(a),$$

$$z_2^{(j-1)}(a) = 0, j = 1, \ldots, n-1, \textit{and} \int_a^b z_2(x)\, dx = -y(b).$$

(III) The partial derivatives satisfy,

$$\frac{\partial y}{\partial a} = -\sum_{j=1}^{n-1} y^{(j)}(a) \frac{\partial y}{\partial A_j} + y(a) \frac{\partial y}{\partial A_n},$$

$$\frac{\partial y}{\partial b} = -y(b) \frac{\partial y}{\partial A_n}.$$

4.1 Proofs of $\frac{\partial y}{\partial A_1}$ and $\frac{\partial y}{\partial A_n}$

Proof. In part (I), we will provide the details for $\frac{\partial y}{\partial A_1}$ and $\frac{\partial y}{\partial A_n}$. First, we make the argument for $\frac{\partial y}{\partial A_1} = r_1$. Let $h \neq 0$ be given, and define

$$r_{1h}(x) = \frac{1}{h}[y(x, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n) - y(x, a, b, A_1, A_2, \ldots, A_{n-1}, A_n)].$$

We need to establish

$$\lim_{h \to 0} r_{1h}(x)$$

exists.

We note that $r_{1h}(x)$ is an $(n-1)^{st}$ degree polynomial in x, because both summands in the definition of $r_{1h}(x)$ are $(n-1)^{st}$ degree polynomials. Next, from the notation for y, we have

$$\begin{aligned} r_{1h}(a) &= \frac{1}{h}[y(a, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n) - y(a, a, b, A_1, A_2, \ldots, A_{n-1}, A_n)] \\ &= \frac{1}{h}[A_1 + h - A_1] \\ &= 1. \end{aligned}$$

Similarly,

$$r_{1h}^{(j-1)}(a) = \frac{1}{h}[A_j - A_j] = 0, \;\; j = 2, \ldots, n-1.$$

And

$$\begin{aligned} \int_a^b r_{1h}\, dx &= \frac{1}{h}\int_a^b [y(x, a, b, A_1 + h, A_2, \ldots, A_n) - y(x, a, b, A_1, A_2, \ldots, A_n)]\, dx \\ &= \frac{1}{h}[A_n - A_n] \\ &= 0. \end{aligned}$$

Our next steps involve conversion from the boundary value notation used for $y(x)$ to the initial value notation of Lemma 1. First, $y(x, a, b, A_1, A_2, \ldots, A_{n-1}, A_n)$ means it is an $(n-1)^{st}$ degree polynomial in x, i.e.,

$$y^{(n)}(x, a, b, A_1, A_2, \ldots, A_{n-1}, A_n) = 0.$$

Also,

$$
\begin{aligned}
y(a, a, b, A_1, A_2, \ldots, A_{n-1}, A_n) &= A_1, \\
y'(x, a, b, A_1, \ldots, A_{n-1}, A_n)|_{x=a} & \\
= y'(a, a, b, A_1, \ldots, A_{n-1}, A_n) &= A_2, \\
&\vdots \\
y^{(n-2)}(a, a, b, A_1, \ldots, A_{n-1}, A_n) &= A_{n-1},
\end{aligned}
$$

and

$$\int_a^b y(x, a, b, A_1, \ldots, A_{n-1}, A_n)\, dx = A_n$$

Yet, we do not know the value of $y^{(n-1)}(a, a, b, A_1, \ldots, A_{n-1}, A_n)$. So, let us set

$$y^{(n-1)}(a, a, b, A_1, \ldots, A_{n-1}, A_n) = \gamma_n.$$

Next similarly,

$$y(x, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n)$$

is an $(n-1)^{st}$ degree polynomial in x, and also

$$
\begin{aligned}
y(a, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n) &= A_1 + h, \\
y'(a, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n) &= A_2, \\
&\vdots \\
y^{(n-2)}(a, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n) &= A_{n-1},
\end{aligned}
$$

and

$$\int_a^b y(x, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n)\, dx = A_n.$$

Again, we do not know the value of $y^{(n-1)}(a, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n)$. Let us call the difference

$$y^{(n-1)}(a, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n) - \gamma_n = \epsilon_n(h) = \epsilon_n.$$

Then,

$$y^{(n-1)}(a, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n) = \epsilon_n + \gamma_n.$$

Then by continuity, $\epsilon_n = \epsilon_n(h) \to 0$ as $h \to 0$. Now using the notation of Lemma 1 and viewing the solutions y as solutions of intial value problems; that is, denoting $y(x, a, b, A_1, A_2, \ldots, A_{n-1}, A_n)$ by $p(x, a, b, A_1, A_2, \ldots, A_{n-1}, \gamma_n)$, we have

$$\begin{aligned} r_{1h}(x) &= \frac{1}{h}[y(x, a, b, A_1 + h, A_2, \ldots, A_{n-1}, A_n) - y(x, a, b, A_1, A_2, \ldots, A_{n-1}, A_n] \\ &= \frac{1}{h}[p(x, a, A_1 + h, A_2, \ldots, A_{n-1}, \gamma_n + \epsilon_n) - p(x, a, A_1, A_2, \ldots, A_{n-1}, \gamma_n)]. \end{aligned}$$

We use a telescoping sum to write

$$\begin{aligned} r_{1h}(x) &= \frac{1}{h}[\{p(x, a, A_1 + h, A_2, \ldots, A_{n-1}, \gamma_n + \epsilon_n) \\ &\quad -p(x, a, A_1, A_2, \ldots, A_{n-1}, \gamma_n + \epsilon_n)\} \\ &\quad +\{p(x, a, A_1 + h, A_2, \ldots, A_{n-1}, \gamma_n + \epsilon_n) \\ &\quad -p(x, a, A_1 + h, A_2, \ldots, A_{n-1}, \gamma_n)\}]. \end{aligned}$$

By Lemma 1 and the Mean Value Theorem,

$$\begin{aligned} r_{1h}(x) &= \frac{1}{h}\alpha_1(x, p(x, a, A_1 + \bar{h}, A_2, \ldots, A_{n-1}, \gamma_n + \epsilon_n))(A_1 + h - A_1) \\ &\quad +\frac{1}{h}\alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n + \bar{\epsilon}_n))(\gamma_n + \epsilon_n - \gamma_n) \end{aligned}$$

where α_1 is defined to be $\frac{\partial p}{\partial c_1}$ and α_n is defined to be $\frac{\partial p}{\partial c_n}$, and $A_1 + \bar{h}$ is between A_1 and $A_1 + h$, and $\gamma_n + \bar{\epsilon}_n$ is between γ_n and $\gamma_n + \epsilon_n$ In particular,

$$\begin{aligned} r_{1h}(x) &= \alpha_1(x, p(x, a, A_1 + \bar{h}, A_2, \ldots, A_{n-1}, \gamma_n + \epsilon_n)) \\ &\quad +\frac{\epsilon_n}{h}\alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n + \bar{\epsilon}_n)) \end{aligned}$$

where $\alpha_1(x, p(x, a, A_1 + \bar{h}, A_2, \ldots, A_{n-1}, \gamma_n + \epsilon_n))$ is an $(n-1)^{st}$ degree polynomial in x and satisfies,

$$\alpha_1^{(j-1)}(a) = \delta_{ij}, j = 1, \ldots, n,$$

and $\alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n + \bar{\epsilon}_n))$ is the $(n-1)^{st}$ degree polynomial in x and satisfies,

$$\alpha_n^{(j-1)}(a) = \delta_{nj}, j = 1, \ldots, n.$$

To establish the existence of $\lim_{h \to 0} r_{1h}(x)$, we must show $\lim_{h \to 0} \frac{\epsilon_n}{h}$ exists. Now, $\alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n + \bar{\epsilon}_n))$ is a nontrivial $(n-1)^{st}$ degree polynomial in x, because $\alpha_n^{(n-1)}(a, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n + \bar{\epsilon}_n)) = 1$. Since $\alpha_n^{(j-1)}(a, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n + \bar{\epsilon}_n) = 0$, for $j = 1, \ldots, n-1$, it follows from Lemma 2 that

$$\int_a^b \alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n + \bar{\epsilon}_n))\, dx \neq 0.$$

Now, above $\int_a^b r_{1h}(x)\, dx = 0$, and so we obtain

$$\frac{\epsilon_n}{h} = \frac{-\int_a^b \alpha_1(x, p(x, a, A_1 + \bar{h}, \ldots, A_{n-1}, \gamma_n + \epsilon_n))\, dx}{\int_a^b \alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n + \bar{\epsilon}_n))\, dx}.$$

It follows from continuity that we can let $h \to 0$ to obtain

$$\begin{aligned} \lim_{h \to 0} \frac{\epsilon_n}{h} &= \frac{-\int_a^b \alpha_1(x, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n))\, dx}{\int_a^b \alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n))\, dx} \\ &= \frac{-\int_a^b \alpha_1(x, y(x, a, b, A_1, \ldots, A_{n-1}, A_n))\, dx}{\int_a^b \alpha_n(x, y(x, a, b, A_1, \ldots, A_{n-1}, A_n))\, dx} \\ &:= V. \end{aligned}$$

Hence, if we define $r_1(x) = \lim_{h \to 0} r_{1h}(x)$, then $r_1(x) = \frac{\partial y}{\partial A_1}(x, a, b, A_1, \ldots, A_n)$, and also

$$\begin{aligned} r_1(x) &= \lim_{h \to 0} r_{1h}(x) \\ &= \alpha_1(x, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n)) + V\alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \gamma_n) \\ &= \alpha_1(x, y(x, a, b, A_1, \ldots, A_n)) + V_{\alpha_n}(x, y(x, a, b, A_1, \ldots, A_n)) \end{aligned}$$

which is an $(n-1)^{st}$ degree polynomial in x, and due to the boundary conditions satisfied by $r_{1h}(x)$, we also have

$$\begin{aligned} r_1(a) &= \lim_{h \to 0} r_{1h}(a) = 1, \\ r_1^{(j-1)}(a) &= \lim_{h \to 0} r_{1h}^{(j-1)}(a) = 0, j = 2, \ldots, n-1, \end{aligned}$$

and

$$\int_a^b r_1(x)\,dx = \lim_{h\to 0}\int_a^b r_{1h}(x)\,dx = 0.$$

This completes the proof for $\frac{\partial y}{\partial A_1}$.

The arguments of the proofs for $\frac{\partial y}{\partial A_i}$, $i = 2, \ldots, n-1$, are analagous to the above, but the arguments for $\frac{\partial y}{\partial A_n}$ have some differences, which merit inclusion of the details.

Again, let $h \neq 0$ be given, and this time define

$$r_{nh}(x) = \frac{1}{h}[y(x, a, b, A_n, \ldots, A_{n-1}, A_n + h) - y(x, a, b, A_1, \ldots, A_{n-1}, A_n)]$$

Our goal is to establish $\lim_{h\to 0} r_{nh}(x)$ exists and to determine its boundary values. As before, $r_{nh}(x)$ is an $(n-1)^{st}$ degree polynomial in x.

This time, from the notation for y, we see

$$r_{nh}^{(j-1)}(a) = \frac{1}{h}[A_j - A_j] = 0, j = 1, \ldots, n-1,$$

and

$$\begin{aligned}\int_a^b r_{nh}(x)\,dx &= \frac{1}{h}\int_a^b [y(x, a, b, A_n, \ldots, A_{n-1}, A_n + h) - y(x, a, b, A_1, \ldots, A_{n-1}, A_n)]\,dx \\ &= \frac{1}{h}[A_n + h - A_n] \\ &= 1.\end{aligned}$$

Again, we let

$$\gamma_n = y^{(n-1)}(a, a, b, A_1, \ldots, A_{n-1}, A_n),$$

but for this case, we set

$$\delta_n = \delta_n(h) = y^{(n-1)}(a, a, b, A_1, \ldots, A_{n-1}, A_n + h) - \gamma_n.$$

As before by continuity, $\delta_n = \delta_n(h) \to 0$, as $h \to 0$. We convert from boundary value notation y to initial value notation p, followed by Lemma 1 and the Mean

Value Theorem to obtain

$$\begin{aligned} r_{nh}(x) &= \frac{1}{h}[(p(x,a,A_1,\ldots,A_{n-1},\gamma_n+\delta_n)-p(x,a,A_1,\ldots,A_{n-1},\gamma_n)] \\ &= \frac{1}{h}\alpha_n(x,p(x,a,A_1,\ldots,A_{n-1},\gamma_n+\bar{\delta}_n))(\gamma_n+\delta_n-\gamma_n) \\ &= \frac{\delta_n}{h}\alpha_n(x,p(x,a,A_1,\ldots,A_{n-1},\gamma_n+\bar{\delta}_n)), \end{aligned}$$

where $\gamma_n+\bar{\delta}_n$ is between γ_n and $\gamma_n+\delta_n$, and $\alpha_n(x,p(x,a,A_1,\ldots,A_{n-1},\gamma_n+\bar{\delta}_n))$ is the nontrivial $(n-1)^{st}$ degree polynomial as in the previous case, and as in that case

$$\int_a^b \alpha_n(x,p(x,a,A_1,\ldots,A_{n-1},\gamma_n+\bar{\delta}_n))\,dx \neq 0.$$

Now, for this case $\int_a^b r_{nh}(x)\,dx = 1$, and so we have

$$\frac{\delta_n}{h} = \frac{1}{\int_a^b \alpha_n(x,p(x,a,A_1,\ldots,A_{n-1},\gamma_n+\bar{\delta}_n))\,dx},$$

from which by continuity, we may let $h\to 0$ to obtain,

$$\begin{aligned} \lim_{h\to 0}\frac{\delta_n}{h} &= \frac{1}{\int_a^b \alpha_n(x,y(x,a,A_1,\ldots,A_n))\,dx} \\ &:= K. \end{aligned}$$

As a consequence $\lim_{h\to 0} r_{nh}(x)$ exists, and in particular,

$$\begin{aligned} r_n(x) &:= \lim_{h\to 0} r_{nh}(x) \\ &= \frac{\partial y}{\partial A_n}(x,a,b,A_1,\ldots,A_n). \end{aligned}$$

Ultimately,

$$\begin{aligned} r_n(x) &= \lim_{h\to 0} r_{nh}(x) \\ &= K\alpha_n(x,y(x,a,b,A_1,\ldots,A_n)), \end{aligned}$$

which is an $(n-1)^{st}$ degree polynomial in x, and from the boundary conditions satisfied by $r_{nh}(x)$, we have

$$r_n^{(j-1)}(a) = 0, j = 1,\ldots,n-1,$$

and

$$\int_a^b r_n(x)\,dx = 1.$$

This completes the proof for $\frac{\partial y}{\partial A_n}$, which also completes the proof for part (I).

4.2 Proofs of $\frac{\partial y}{\partial a}$ and $\frac{\partial y}{\partial b}$

For part (II) of the theorem, we include the arguments for $\frac{\partial y}{\partial a}$ with the details for $\frac{\partial y}{\partial b}$ being along the same lines.

Let $h \neq 0$ be given, and define

$$z_{1h}(x) = \frac{1}{h}[y(x, a+h, b, A_1, \ldots, A_n) - y(x, a, b, A_1, \ldots, A_n)]$$

As usual, our goal is to establish $\lim_{h\to 0} z_{1h}(x)$ exists and to characterize it by its boundary values. Again, $z_{1h}(x)$ is an $(n-1)^{st}$ degree polynomial in x.

Our first concern is with the boundary values satisfied by $z_{1h}(x)$. By the First Mean Value Theorem for Integrals [2], and for $h \neq 0$, there is a c_h inclusively between a and $a+h$ such that,

$$\begin{aligned}
\int_a^b z_{1h}(x)\,dx &= \frac{1}{h}\int_a^b [y(x, a+h, b, A_1, \ldots, A_n) - y(x, a, b, A_1, \ldots, A_n)]\,dx \\
&= \frac{1}{h}[\int_a^{a+h} y(x, a+h, b, A_1, \ldots, A_n)\,dx \\
&\quad + \int_{a+h}^b y(x, a+h, b, A_1, \ldots, A_n)\,dx \\
&\quad - \int_a^b y(x, a, b, A_1, \ldots, A_n)\,dx] \\
&= \frac{1}{h}\left[\int_a^{a+h} y(x, a+h, b, A_1, \ldots, A_n)\,dx + A_n - A_n\right] \\
&= \frac{1}{h} y(c_h, a+h, b, A_1, \ldots, A_n)(a+h-a) \\
&= y(c_h, a+h, b, A_1, \ldots, A_n).
\end{aligned}$$

By continuity, we can compute $\lim_{h\to 0}\int_a^b z_{1h}(x)\,dx = y(a, a, b, A_1, \ldots, A_n) = y(a)$.

Next, fix $j = 1, \ldots, n-1$. By the Mean Value Theorem, we have

$$\begin{aligned}
z_{1h}^{(j-1)} &= \frac{1}{h}\left[y^{(j-1)}(a, a+h, b, A_1, \ldots, A_n) - y^{(j-1)}(a, a, b, A_1, \ldots, A_n)\right] \\
&= \frac{1}{h}[y^{(j-1)}(a, a+h, A_1, \ldots, A_n) - y^{(j-1)}(a+h, a+h, b, A_1, \ldots, A_n)] \\
&= -\frac{1}{h} y^{(j)}(d_h, a+h, b, A_1, \ldots, A_n) \cdot h \\
&= -y^{(j)}(d_h, a+h, b, A_1, \ldots, A_n),
\end{aligned}$$

where d_h is between $a + h$ and a.

Passing to the limit, we have

$$\lim_{h\to 0} z_{1h}^{(j-1)}(a) = -y^{(j)}(a, a, b, A_1, \ldots, A_n) = -y^{(j)}(a).$$

Finally, we turn to $\lim_{h\to 0} z_{1h}(x)$. We let

$$\beta_n = y^{(n-1)}(a, a, b, A_1, \ldots, A_{n-1}, A_n),$$

and

$$\sigma_n = \sigma_n(h) = y^{(n-1)}(a + h, a + h, b, A_1, \ldots, A_{n-1}, A_n) - \beta_n.$$

By continuity, $\sigma_n \to 0$ as $h \to 0$. As in above part (I), we convert to notation of solutions of initial value problems, followed by Lemma 1, and we have

$$\begin{aligned}
z_{1h}(x) &= \frac{1}{h}[y(x, a + h, b, A_1, \ldots, A_{n-1}, A_n) - y(x, a, b, A_1, \ldots, A_{n-1}, A_n)] \\
&= \frac{1}{h}[p(x, a + h, A_1, \ldots, A_{n-1}, \beta_n + \sigma_n) - p(x, a, A_1, \ldots, A_{n-1}, \beta_n)] \\
&= \frac{1}{h}[p(x, a + h, A_1, \ldots, A_{n-1}, \beta_n + \sigma_n) - p(x, a, A_1, \ldots, A_{n-1}, \beta_n + \sigma_n) \\
&\quad + p(x, a, A_1, \ldots, A_{n-1}, \beta_n + \sigma_n) - p(x, a, A_1, \ldots, A_{n-1}, \beta_n)] \\
&= \frac{1}{h}[\beta(x, p(x, a + \bar{h}, A_1, \ldots, A_{n-1}, \beta_n + \sigma_n))h \\
&\quad + \alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \beta_n + \bar{\sigma}_n))\sigma_n] \\
&= \beta(x, p(x, a + \bar{h}, A_1, \ldots, A_{n-1}, \beta_n + \sigma_n)) \\
&\quad + \frac{\sigma_n}{h}\alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \beta_n + \bar{\sigma}_n)),
\end{aligned}$$

where $\beta(x, p(x, a + \bar{h}, A_1, \ldots, A_{n-1}, \beta_n + \sigma_n))$ is an $(n-1)^{st}$ degree polynomial in x and satisfies

$$\beta^{(j-1)}(a, p(x, a + \bar{h}, A_1, \ldots, A_{n-1}, \beta_n + \sigma_n)) = -p^{(j)}(a) = -y^{(j)}(a), j = 1, \ldots, n,$$

$\alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \beta_n + \bar{\sigma}_n))$ is an $(n-1)^{st}$ degree polynomial in x and satisfies

$$\alpha_n^{(j-1)}(a, p(x, a, A_1, \ldots, A_{n-1}, \beta_n + \bar{\sigma}_n)) = \delta_{nj}, j = 1 \ldots, n,$$

and $\beta_n + \bar{\sigma}_n$ lies between β_n and $\beta_n + \sigma_n$, and $a + \bar{h}$ lies between a and $a + h$. For $\lim_{h\to 0} z_{1h}(x)$ to exist, it suffices for $\lim_{h\to 0} \frac{\sigma_n}{h}$ to exist.

As we know from the previous cases, Lemma 2 implies

$$\int_a^b \alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \beta_n + \bar{\sigma}_n))\, dx \neq 0$$

and so

$$\frac{\sigma_n}{h} = \frac{\int_a^b z_{1h}(x)\, dx - \int_a^b \beta(x, p(x, a + \bar{h}, A_1, \ldots, A_{n-1}, \beta_n + \sigma_n))\, dx}{\int_a^b \alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \beta_n + \bar{\sigma}_n))\, dx}.$$

Passing to the limit, we have from above calculations,

$$\begin{aligned}\lim_{h\to 0} \frac{\sigma_n}{h} &= \frac{y(a, a, b, A_1, \ldots, A_n) - \int_a^b \beta(x, y(x, a, b, A_1, \ldots, A_n))\, dx}{\int_a^b \alpha_n(x, y(x, a, b, A_1, \ldots, A_n))\, dx} \\ &:= N.\end{aligned}$$

From the above expression,

$$\begin{aligned} z_{1h}(x) &= \beta(x, p(x, a + \bar{h}, A_1, \ldots, A_{n-1}, \beta_n + \sigma_n)) \\ &\quad + \frac{\epsilon_n}{h} \alpha_n(x, p(x, a, A_1, \ldots, A_{n-1}, \beta_n + \bar{\sigma}_n)),\end{aligned}$$

so that

$$\begin{aligned} z_1(x) &:= \lim_{h\to 0} z_{1h}(x) \\ &= \frac{\partial y}{\partial a}(x, a, b, A_1, \ldots, A_n) \\ &= \beta(x, y(x)) + N\alpha_n(x, y(x)),\end{aligned}$$

which is an $(n-1)^{st}$ degree polynomial in x, and $z_1(x)$ satisfies the boundary conditions,

$$z_1^{(j-1)}(a) = \lim_{h\to 0} z_{1h}^{(j-1)}(a) = -y^{(j)}(a), j = 1, \ldots, n-1,$$
$$\int_a^b z_1(x)\, dx = \lim_{h\to 0} \int_a^b z_{1h}(x)\, dx = y(a).$$

The proof of part (II) is complete.

For part (III) of the theorem, concerning the first equality, both sides are $(n-1)^{st}$ degree polynomials in x. Moreover, from parts (I) and (II),

$$\frac{\partial y}{\partial a}^{(i-1)}(a) = -y^{(i)}(a)$$
$$= -\sum_{j=1}^{n-1} y^{(j)}(a)\frac{\partial y}{\partial A_j}^{(i-1)}(a) + y(a)\frac{\partial y}{\partial A_n}^{(i-1)}(a),\ \ i = 1, \ldots, n-1,$$

and

$$\int_a^b \frac{\partial y}{\partial a}(x)\,dx = y(a)$$
$$= -\sum_{j=1}^{n-1} y^{(j)}(a)\int_a^b \frac{\partial y}{\partial A_j}(x)\,dx + y(a)\int_a^b \frac{\partial y}{\partial A_n}(x)\,dx.$$

And so the first equality of part (III) holds by Lemma 2. Simliarly, the second equality of part (III) is true. □

CHAPTER FIVE

Extensions and Applications

The results obtained in this work have potential impact in their extensions to nonlinear differential equations satisfying integral boundary conditions, as well as, in modeling hybrid phenonmena now being modeled by dynamic equations on time scales [3].

Using the results and methods of proof from this work will lead to research involving differentiability of solutions of nonlinear ordinary differential equations of the form

$$y'' = f(x, y, y'), a \leq x \leq b, \tag{5.1}$$

satisfying Dirichlet and nonlocal integral conditions of the respective forms,

$$y(a) = A \quad and \quad \int_a^b y(x)\, dx = B, \tag{5.2}$$

where $A, B \in \mathbb{R}$, and

(i) $f(x, u_1, u_2) : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and

(ii) $\frac{\partial}{\partial u_i} f(x, u_1, u_2) : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, i=1,2.

In particular, for solutions $y(x) = y(x, a, b, A, B)$ of (5.1),(5.2), the results of this work will motivate research directed toward characterizing the partial derivatives $\frac{\partial y}{\partial a}$, $\frac{\partial y}{\partial b}$, $\frac{\partial y}{\partial A}$, and $\frac{\partial y}{\partial B}$. Such extensions will most likely include techniques from variational analysis.

Dynamic equations [1] on time scales deal with unification of continuous and discrete analysis via a hybrid analysis of differential equations and finite difference equations. These dynamic equations make use of a so-called "delta derivatives." One scenario in which dynamic equations on a time scale have been employed involves modeling insect population that are sometimes considered as continuous while in season (or may follow a difference scheme with variable step-size), then die out in the winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. Just as in this work, certain polynomials are characterized with respect to their smooth dependence on boundary conditions, so also can polynomials be characterized by differences with respect to discrete boundary conditions. A dynamic equation unification of these characterization might also provide insight into how the hybrid insect population dynamics change with respect to both continuous and discrete boundary conditions.

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