ABSTRACT

Gravitational Radiation and Black Hole Formation from Gravitational Collapse in Theories of Gravity with Broken Lorentz Symmetry

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Quantum gravity is expected to contain Lorentz symmetry only as an emergent low energy symmetry, with the scale at which it is broken presently inaccessible to current experiments. This dissertation is centered around understanding various physical aspects of gravitational theories that modify general relativity by explicitly breaking Lorentz symmetry in the gravitational sector (viz. Hořava-Lifshitz gravity and Einstein-æther theory) such that they are consistent with all current observations. This dissertation consists of an analytical study of black hole solutions in 2d Hořava gravity which is non-minimally coupled with a non-relativistic scalar field with a focus on understanding Hawking radiation and the properties of the universal horizons. It includes an investigation of gravitational plane wave solutions in Einstein-æther theory and their behavior, especially how they may be potentially distinguishable by present or future detectors from the standard prediction of general relativity. Lastly, it includes a numerical study of gravitational collapse of a massless scalar field in Einstein-æther theory showing the existence of outermost "dynamical Universal horizons (dUHs)". Such a dUH evolves into the causal boundary, even for excitations with arbitrarily large speeds of propagation.

Gravitational Radiation and Black Hole Formation from Gravitational Collapse in Theories of Gravity with Broken Lorentz Symmetry

by

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CHAPTER ONE

Introduction

Since the dawn of intelligent life, human beings have been intimately familiar with the ubiquitous force of gravity, often succinctly captured in the form of the following maxim "Everything that goes up, must fall down." However for much of human history this force has been misunderstood. The ancient Greek philosopher Aristotle proclaimed heavier objects fall faster compared to the lighter objects. While Galileo corrected this proclamation one and a half millenium later through repeated experiments, gravity still remained very much an earthly phenomenon. It was with the proverbial observation of a falling apple that Sir Isaac Newton famously realized the universal nature of gravity. Newton went ahead and captured the entirety of gravity through his universal law of gravitation, mathematically expressed as

$$\vec{F} = G\,\frac{M_1 M_2}{r^2}\,\hat{r}. \tag{1.1}$$

This neat and elegant formula tells us that any two objects in the universe with masses M_1 and M_2 separated by a distance r, will attract each other by a force whose magnitude is given by F acting along direction of the line joining these two objects. With this simple formula for the next two centuries physicists, mathematicians and astronomers successfully started charting the cosmos and deciphering its mysteries. Such was the power and effect of this law that when the electrostatic force was within experimental grasp, Coulomb gave a formula for electrostatic force exactly mimicking Newton's law of gravitation. Towards the end of nineteenth century, the electric and the magnetic force were no longer described by a "Newton like" law but rather a set of equations developed by Maxwell. In 1905, Einstein famously realized that Maxwell's equations are incompatible with Newtonian mechanics, which led him to a special theory of relativity [1]. Soon after Einstein realized that if the special theory of relativity is indeed the correct description of motion, then it is incompatible with all theories which relies on action-at-a-distance, including gravity as described by Newton's law. After almost a decade of deliberation and laborious research, he successfully reconciled this apparent contradiction by proposing his general theory of relativity in 1915 [2]. General relativity subsumed Newtonian gravity as an effective approximation in weak gravitational fields and low velocities much much smaller than the velocity of light. Just as Newton's law of gravitation was a shining beacon in the darkness of the cosmos for more than two centuries, Einstein's general relativity has been at the very core of our understanding of all gravitational physics, classical and quantum, since 1915.

1.1 Rudiments of General Relativity

Shortly after Einstein's paper on special relativity was published in 1905, Minkowski realized that special relativity implies space and time should no longer be considered on separate footing. Much of special relativity can be succinctly captured by demanding the separation of two events in spacetime be given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{1.2}$$

where t is the time coordinate, x, y, z are the usual Cartesian coordinates, $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ and $x^\mu = (t, x, y, z)$.

The set of transformations which leaves $\eta_{\mu\nu}$ invariant are the Lorentz transformations, which therefore are symmetries of spacetime in special relativity. Soon after this Einstein correctly concluded that the principle of equivalence (which was an accidental feature in Newtonian gravity) must be a fundamental principle in any theory of gravity. The crux of this principle states that locally an observer can never distinguish between a gravitational field and an appropriately accelerated frame. Since special relativity applies only to objects which are unaccelerated, it was obvious that to successfully incorporate gravity with principles of relativity one must modify special relativity. This modification culminated in the general theory of relativity which will be discussed now briefly.

In general relativity (GR) gravitational interaction is no longer a force or a Newtonian force, but rather an artifact of curvature of spacetime due to the presence of massive objects. The spirit of GR was elegantly captured in the famous textbook by Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler [3] as "Spacetime tells matter how to move; matter tells spacetime how to curve."

The starting point of GR is to consider that spacetime is endowed with a local dynamical metric field : $g_{\mu\nu}(x)$ and that the strong equivalence principle is obeyed. The strong equivalence principle states that (i) the weak equivalence principle (WEP) is valid (WEP states that if an uncharged body is placed at an initial event in spacetime, and is given an initial velocity there, then its subsequent world line will be independent of its internal structure and composition) and (ii) the outcome of any local test experiment either gravitational or non-gravitational is independent of the location and time of the experiment being performed anywhere in the universe and independent of the velocity of the freely falling apparatus.

These physical considerations suggest that the underlying symmetries are general covariance and local Lorentz invariance. In the spirit of classical field theory one should construct the most general action which is at most quadratic in the second derivative of the metric field in addition to the covariantized matter action. Schematically, the action looks like

$$S = \int \sqrt{-g}\, d^4x \left(\mathcal{L}_{metric} + \mathcal{L}_{matter}\right). \tag{1.3}$$

Where, $g = det\,(g_{\mu\nu})$, $\mathcal{L}_{metric}$ and $\mathcal{L}_{matter}$ are the metric and the matter Lagrangian respectively. However, the metric part of the action should be such that the equation of motion automatically satisfies the conservation of energy-momentum tensor $T^{\mu\nu}$, i.e. $\nabla_\mu T^{\mu\nu} = 0$, where ∇_μ is the covariant derivative with respect to the metric. This leads to the Einstein-Hilbert action given by

$$S = \int \sqrt{-g}\, d^4x \left[\frac{1}{8\pi G c^4} \, R + \mathcal{L}_{matter} \right] \tag{1.4}$$

where R is the Ricci scalar, G is the gravitational constant, and c is the speed of light in vacuum. The equations of motion derived by varying the metric are the famous Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \frac{8\pi G}{c^4}\, T_{\mu\nu}. \tag{1.5}$$

$R_{\mu\nu}$ is the Ricci curvature tensor. From now on in this thesis unless otherwise mentioned we will set $c = 1$.

The first exact solution of Einstein's field equations was developed by Karl Schwarzschild in 1916 [4]. It is one of the simplest, yet extremely interesting, solutions describing a spherically symmetric, static vacuum solution due to a non rotating massive object of mass m. Later in the 1960's this was identified to describe what is now known as a spherically symmetric black hole with mass m, zero angular momentum and zero electric charge. The metric is given by

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2Gm}{r}\right)}\, dr^2 + r^2 \left(d\theta^2 + sin^2\theta \, d\phi^2\right). \tag{1.6}$$

This spacetime has a curvature singularity (i.e. R diverges) at $r = 0$. The singularity at $r = 2Gm$ signals the breakdown of the (t, r, θ, ϕ) coordinate system and is not a genuine singularity. The spherical surface $r = 2Gm$ can be shown to be a null surface and acts as a one way membrane enclosing the curvature singularity. All physical information can cross this membrane from outside and fall inwards towards the singularity but nothing can escape from inside. This membrane is known as the event horizon of the Schwarzschild black hole.

Throughout the last century, numerous observational and experimental successes have firmly established GR as one of the foundational paradigms in modern physics. The story started with direct observation of bending of starlight during a solar eclipse. The next several decades saw the development of a standard model of cosmology based on GR which finally shed light into numerous questions regarding the origin and the basic nature of our universe. Quite recently, with the direct detection of gravitational waves [5] and the first direct image of a supermassive black hole [6] at the center of the galaxy M87 continue to reaffirm GR's status as pinnacle of human intellectual achievement.

1.2 Breakdown of General Relativity

Despite its many celebrated successes GR remains at its core a classical field theory. Just as it was obvious to Einstein that Newtonian gravity, while extremely successful in describing planetary motion, must necessarily be an approximation to a larger theory (i.e. GR), similarly, for the last several decades it has become increasingly clear to physicists that at extremely small length scales (or equivalently high energies) the classical theory of GR must give way to a larger quantum theory of gravity. However, all standard techniques of quantization known so far suggest that general relativity will lead to a non-renormalizable quantum theory. GR is not perturbatively renormalizable. One naive way to see this is to note that the coupling constant present in GR is Newton's gravitational constant (G_N) which in natural units (Planck constant $\hbar$ = speed of light $c = 1$) has negative mass dimension $(mass)^{-2}$ in four-dimensional spacetime, whereas it should be larger than or equal to zero for the theory to be perturbatively renormalizable [7]. This can be easily seen from the kinetic term read off from the Einstein-Hilbert action.

$$S = \frac{1}{16\pi G_N}\left\{\int_{\mathcal{M}} d^4x\sqrt{-g}\,(R-2\Lambda) + 2\oint_{\partial\mathcal{M}} d^3x\sqrt{|h|}K\right\}, \tag{1.7}$$

where Λ is the cosmological constant, h is the determinant of the induced metric on the boundary and K is the trace of the extrinsic curvature of the boundary. So, $[G_N] = 2-d$ in d dimensional spacetime. This structure distinguishes gravity from other fundamentally known interactions in nature, all of which have dimensionless coupling constants from the very beginning. So, the effective dimensionless coupling of gravity that enables perturbative expansion is given by $g_{\text{eff}} = G_N E^2$. If we consider the expansion of any given physical quantity F in terms of small gravitational constant G_N

$$F = \sum_{n=0}^{\infty} a_n \left(G_N E^2 \right)^n \tag{1.8}$$

where E is the energy of the system involved, at the energy scale where $E^2 \gg G_N^{-1}$, such expansions diverge. This seems to suggest that GR cannot be quantized in the standard way which therefore poses a serious problem in our attempt to understand quantum gravity.

There have been several independent attempts to develop a quantum theory of gravity, the most prominent one being string theory and loop quantum gravity. So far none of the attempts have yielded beyond doubt a unanimously accepted notion of quantum gravity. However, these attempts seem to require (a) that GR is indeed the correct classical description of gravity and (b) at some small enough length scale local Lorentz symmetry seems to be broken, only to emerge as an approximate symmetry at long distances. Therefore, it seems natural to ask whether one can write down classical descriptions in the spirit of GR maintaining general covariance but not local Lorentz invariance.

1.3 Violation of Local Lorentz Symmetry

Lorentz invariance (LI) is one of the fundamental symmetries of modern physics and strongly supported by observations [8]. An immediate objection to explicit breaking of Lorentz symmetry is that it seems incompatible with all of our observations so far. To date, no observational evidence shows that such a symmetry must be broken at the present experimentally achieved energy scales. However, as we shall see, while all present experiments and observations present a bound on the degree of violation of Lorentz symmetry, it is possible to write down theories that are consistent with those bounds, as well as all other experiments done so far. The constraints of such violations in the gravitational sector are much weaker than those in the matter sector because numerous observations strongly limit the violation of Lorentz symmetry among the standard model fields [9,10]. Hence, it becomes of paramount importance to closely analyze these theories and extract specific physical situations in which the predictions of these theories differ from that of GR.

There are various reasons to construct gravitational theories with broken LI. String theory, which has become one of the preeminent theories for addressing questions of quantum gravity has a fundamental length scale. At very high energy scale i.e. in the Planck regime, quantum mechanics appears to be more fundamental than symmetries of special or general relativity and continuous spacetime emerges as a classical limit of some quantum gravity having discrete substratum [11]. Therefore, many developments in this theory suggest that at a length scale which is of the order of the string length, it is not unreasonable to think that space and time are quantized. Lorentz invariance, which is a continuous symmetry may not be a fundamental symmetry of nature, but instead should be an emergent one at low energies. Another motivation comes from the modification of gravity at long distances to explain the accelerated expansion of the late-time universe. Following these lines of arguments, various Lorentz violating (LV) theories of gravity have attracted much interest in recent years. These include ghost condensation [12], Einstein-æther theory (æ-theory) [13,14] and more recently the Hořava-Lifshitz theory of gravity [15]. While the ghost condensation and Einstein-æther theory are considered as the low energy effective theories of gravity, the Hořava gravity is supposed to be ultraviolet (UV) complete [16].

This thesis consists of a set of such investigations based on two specific theories, namely Hořava-Lifshitz gravity and Einstein-æther theory. With the advent of technology relevant for experimental gravitational physics, it is extremely reasonable to expect that the ongoing experiments or the experiments planned in near future can detect these specific physical cases. Therefore these detections can firmly establish the extent to which local Lorentz symmetry should be considered a symmetry of classical gravity. The explicit breaking of local Lorentz symmetry in classical gravity will have several interesting consequences. In particular, it may provide us with the crucial insight required to find the correct quantum theory of gravity.

1.4 Plan of the Dissertation

In the rest of the thesis, a brief review of Hořava-Lifshitz theory of gravity and Einstein-æther theory of gravity will be given in chapter 2. Chapter 3 [17], chapter 4 [18], and chapter 5 [19] are based on my own research work. Chapter 3 is based on a special case of Hořava gravity, i.e. non-projectable Hořava gravity and the study of the universal horizon and Hawking radiation in that theory. Chapter 4 is based on the gravitational plane wave solutions in Einstein-æther theory of gravity. In chapter 5, we will discuss gravitational collapse and formation of black holes in Einstein-æther theory of gravity. We conclude in chapter 6 with a brief discussion of the results obtained in this dissertation and some of the future questions opened up by these investigations.

CHAPTER TWO

Review of Gravitational Physics with Broken Lorentz Symmetry

Hořava-Lifshitz gravity and Einstein-æther theory are theories of gravity that break local Lorentz symmetry but preserve general covariance. Hořava-Lifshitz gravity is a scalar-tensor theory of gravity which breaks local Lorentz symmetry by anisotropic scaling of space and time coordinates. It introduces a preferred foliation defined by a dynamical scalar field ϕ called the khronon, which labels the leaves of the foliation. Einstein-æther theory is a vector-tensor theory of gravity which introduces at every point in spacetime, a dynamical unit timelike vector field u^a to choose a preferred frame. If the vector u^a is restricted to be hypersurface orthogonal, Einstein-æther theory is recovered as an infrared (IR) limit of the extended Hořava-Lifshitz gravity [20]. Hypersurface orthogonal solutions of Einstein-æther theory are also solutions to the IR limit of Hořava-Lifshitz gravity.

2.1 Hořava-Lifshitz Gravity

The non-renormalizability of GR from a field-theoretical perspective does not rule out a quantum field theoretical description of gravity in the UV regime altogether. It has been shown that higher order derivative operators appear to stabilize perturbation theory [21]. So, several authors have suggested that the Einstein-Hilbert action should be modified by adding terms containing non-minimal functional of the metric tensor involving more than two derivatives. In 1977, K. Stelle included quadratic curvature terms to the Einstein-Hilbert action.

$$S = \int d^4x\sqrt{-g}\left(\alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2 + \gamma R\right), \tag{2.1}$$

where, α, β, and γ are the coupling constants and a common factor $\frac{1}{16\pi G}$ has been absorbed into the coefficient of each term in the action.

Stelle proved that gravity is renormalizable to all orders in perturbation theory including all fourth-order derivative operators [22]. Unfortunately, the resulting theory is not compatible with standard notions of perturbative unitarity due to the appearance of time derivative operators of order greater than two, which leads to negative kinetic energy modes known as Ostrogradsky's ghosts. In 1850, Ostrogadsky gave a very powerful theorem which states that a system is not (kinematically) stable if it is described by a non-degenerate higher time-derivative Lagrangian. [23]. This makes the theory highly unstable as the negative energy modes make the energy of the system unbounded from below. It can be shown that the Hamiltonians of these systems will include terms linear in the canonical momentum so there exists no lowest energy state rendering the system unstable. Therefore, in order to improve UV behavior at short distance one can include only higher order spatial derivative operators in the Lagrangian and time derivative operators are still kept to be second order. This approach was adopted by Hořava to propose a power counting renormalizable theory of quantum gravity [15].

Among many other ways of breaking Lorentz symmetry Hořava chose to break it by considering anisotropic scaling between space and time.,

$$t \to b^{-z}t', \quad x^i \to b^{-1}x'^i, \quad (i = 1, 2, ...d), \tag{2.2}$$

where z denotes the dynamical critical exponent. LI requires $z = 1$, and power counting renormalizability requires $z \geq d$, where d denotes the spatial dimension of the spacetime. Because of this scaling, Hořava gravity is also known as the Hořava-Lifshitz (HL) gravity as the transformation reminds us of Lifshitz scalar fields in condensed matter physics [24]. The breaking of LI becomes evident from the above scaling as we can see the dimensions of time and space now become

$$[t] = -z, \quad [x^i] = -1. \tag{2.3}$$

Whereas, in GR the theory is invariant under the diffeomorphism called general covariance

$$t \to t'(t, x^k), \quad x^i \to x'^i\left(t, x^k\right). \tag{2.4}$$

Hořava assumed a transformation that retained the spatial diffeomorphism but broke general covariance in the sense that

$$t \to t'(t), \quad x^i \to x'^i(t, x^k) \tag{2.5}$$

This modified symmetry indicates that one cannot rotate the time direction but can rescale the coordinate time in any arbitrary way. The above symmetry is called *foliation-preserving diffeomorphism.* This makes the Hořava-Lifshtiz gravity power counting renormalizable which can be understood from simple dimensional analysis to be discussed later.

2.1.1 3+1 Decomposition of General Relativity

GR faces the problem where one cannot determine uniquely the time evolution of the dynamical quantity, i.e. the gravitational field, originating from a given set of initial value of the metric field and its first time derivative. The evolved metric state at some later time can be modified by mere relabeling of the coordinate time under which the theory preserves general covariance. This is why it is necessary to separate the metric field into parts carrying the true dynamical information and the parts characterizing the coordinate system.

In the canonical formulation of GR, one needs to introduce a choice of a particular timelike direction and slice the entire manifold in a sequence of spacelike hypersurfaces. The natural way to physically foliate time and space and put them on different footings is to consider Arnowitt-Deser-Misner (ADM) decomposition. The ADM decomposition requires splitting the four-dimensional metric into a spatial scalar called the lapse function N, a spatial vector called the shift vector N^i, and a three dimensional spatial metric g_{ij}. These spacelike hypersurfaces Σ_t are given at the leaves $t = constant$, i.e.

$$\mathcal{M} = \mathbb{R} \times \Sigma_t, \tag{2.6}$$

where, $t \in \mathbb{R}$. Thus the metric now is given by

$$ds^2 = -N^2 dt^2 + g_{ij} \left(dx^i + N^i dt\right) \left(dx^j + N^j dt\right) \tag{2.7}$$

In this formalism, the covariant action of general relativity (1.7) takes the form

$$S = \frac{1}{16\pi G} \int d^3x\, dt \sqrt{{}^{(3)}g}\, N \left(K^{ij} K_{ij} - K^2 + {}^{(3)}R\right), \tag{2.8}$$

where, ${}^{(3)}g$ is the determinant of 3-metric g_{ij}, ${}^{(3)}R$ is the intrinsic curvature of the hypersurface, K is the trace of extrinsic curvature K_{ij} of the hypersurface defined by

$$K_{ij} = \frac{1}{2N} \left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right). \tag{2.9}$$

Here ∇_i represents the covariant derivative with respect to the 3-metric g_{ij} and $\dot{g}_{ij} \equiv \frac{\partial g_{ij}}{\partial t}$. Note that the kinetic terms including the extrinsic curvature terms are the only ones containing time derivatives, and ${}^{(3)}R$ contains only spatial derivatives which can be regarded as the potential energy term. Hořava-Lifshitz gravity modifies the potential term.

2.1.2 Foliation Preserving Diffeomorphism in Hořava-Lifshitz Gravity

Due to the non-relativistic nature of the theory, it is natural to use ADM-like variables in Hořava-Lifshitz gravity. Therefore in three dimensions, the dimension of N, N^i, and g_{ij} under the scaling (2.2) are

$$[N] = 0, \quad [N^i] = z - 1 = 2 \quad (z = 3), \quad [g_{ij}] = 0 \tag{2.10}$$

Considering the anisotropic scaling of space and time, the theory is no longer invariant under the full diffeomorphism allowed in general relativity, but there is a restricted diffeomorphism on the spatial hypersurfaces only defined by (2.5). The group of foliation preserving diffeomorphism of $\mathcal{M}$ is denoted by $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$. Foliations can be equipped with a Riemannian structure. The ADM variables can be considered as a decomposition of a Riemannian metric on $\mathcal{M}$ into the induced metric g_{ij} along the leaves Σ_t, the shift vector N_i, and the lapse function N. In the adapted ADM coordinate system, the infinitesimal generators of the $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ are given by

$$\delta x^i = \zeta^i(t,x), \quad \delta t = f(t) \tag{2.11}$$

The generators of $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ act on the above fields via

$$\begin{aligned}
\delta N &= \zeta^k \nabla_k N + \dot{N} f + N \dot{f}, \\
\delta N_i &= N_k \nabla_i \zeta^k + \zeta^k \nabla_k N_i + g_{ik} \dot{\zeta}^k + \dot{N}_i f + N_i \dot{f}, \\
\delta g_{ij} &= \nabla_i \zeta_j + \nabla_j \zeta_i + f \dot{g}_{ij}.
\end{aligned} \tag{2.12}$$

These transformation properties of the metric components guarantee the basic building blocks of the action invariant under the $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ to be the 3-dimensional Ricci tensor R_{ij}, the extrinsic curvature K_{ij}, the covariant derivative with respect to 3-metric ∇_i, and the 3-vector $a_i \equiv \frac{d \ln N}{dx^i}$. Therefore, the most general action possible in this scenario is

$$S = \frac{1}{16\pi G} \int d^3x\, dt \sqrt{{}^{(3)}g}\, N \left(K^{ij} K_{ij} - \lambda K^2 + F({}^{(3)}g_{ij}, N) \right), \tag{2.13}$$

where $F({}^{(3)}g_{ij}, N)$ is the potential term, and can in principle contain infinitely many terms.

2.1.2.1 The Kinetic Term. The kinetic part of the Lagrangian $\mathcal{L}_K$ is the sixth-order derivative term in the momenta k,

$$\mathcal{L}_K = \frac{1}{\zeta^2} \left(K^{ij} K_{ij} - \lambda K^2 \right), \tag{2.14}$$

where ζ^2 is the gravitational coupling constant having the dimension

$$[\zeta^2] = [t].[x^i]^3 + [K]^2 = -z - 3 + 2z = z - 3. \tag{2.15}$$

For $z = 3$ the coupling constant becomes dimensionless and the theory becomes power counting renormalizable for the scaling (2.2). The parameter λ is a dimensionless coupling constant. In GR, diffeomorphism invariance guarantees $\lambda = 1$ even after radiative corrections are taken into account. But, in Hořava-Lifshitz gravity, it becomes a dynamical coupling constant due to the breaking of LI. The presence of λ reflects the fact that each of the two terms in (2.14) is separately invariant under $\text{Diff}_{\mathcal{F}}(M)$. The requirement of $\text{Diff}_{\mathcal{F}}(M)$ symmetry allows the generalized DeWitt "metric on the space of metrics"

$$G^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - g^{ij}g^{kl}, \tag{2.16}$$

to contain a free parameter λ. The generalized DeWitt metric defines the form quadratic in K_{ij} which appears in the kinetic term [25]. The kinetic terms in the action can be put into a symmetric form

$$K^{ij}K_{ij} - K^2 = K_{ij}G^{ijkl}K_{kl}. \tag{2.17}$$

The kinetic term $\mathcal{L}_K$ is independent of z and the dimension of spacetime. Theories with different z will differ from each other in the terms that are independent of time derivatives in the action.

2.1.2.2 The Potential Term. The potential term in the general action (2.13) will contain terms that are independent of time derivatives but depend on spatial derivatives. It will contain terms that are scalars under foliation preserving diffeomorphism and are functional of R_{ij}, ∇_i and a_i. The large number of potential terms that can be obtained in the gravitational sector can reduce the predictability of the theory. To eliminate the number of independent coupling constants, two additional conditions were introduced by Hořava known as the projectability and detailed balance condition [15].

2.1.2.3 Projectability Condition. The projectability condition restricts the lapse function to be a function of time only. Therefore, $N = N(t)$ remains a constant on each spacelike hypersurface. As a result, all terms involving a_i will vanish. Hence, when N depends on both space and time, it is called the non-projectable condition. Therefore, there remains only seven sixth-order non-zero terms at high energy for $d = 3$ and $z = 3$.

$$\nabla_k R_{ij} \nabla^k R^{ij}, \quad \nabla_k R_{ij} \nabla^i R^{jk}, \quad R\Delta R, \quad R^{ij}\Delta R^{ij},$$

$$R^3, \quad R^i_j R^j_k R^k_i, \quad R R_{ij} R^{ij} \tag{2.18}$$

where. $\Delta \equiv g^{ij}\nabla_i\nabla_j$. The first four terms modify the propagator, besides adding interactions, and the last three terms represent pure interaction.

2.1.2.4 Detailed Balanced Condition. To reduce the number of independent coupling constants, the detailed balanced condition was imposed by Hořava. The potential part is assumed to be obtained from a superpotential $W[g_{kl}]$ [15]

$$\mathcal{L}_V = E_{ij}\mathcal{G}^{ijkl}E_{kl},$$

$$\sqrt{g}E^{ij} = \frac{\delta W[g_{kl}]}{\delta g_{ij}}, \tag{2.19}$$

where

$$W = \frac{1}{\kappa_W^2}\int d^D x\sqrt{g}\,(R - 2\Lambda_W) \tag{2.20}$$

where $\mathcal{G}^{ijkl}$ the generalized DeWitt metric.

$$\mathcal{G}^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda g^{ij}g^{kl}. \tag{2.21}$$

This reduces the number of free parameters to five: the Newton constant G_N, cosmological constant Λ, λ in the generalized DeWitt metric and the other two constants κ_W and Λ_W from the superpotential. Even though it may appear to be a promising candidate for quantum gravity with UV completion due to the reduced free parameters, it has several problems. It lacks the Newtonian limit and sixth order derivative operators, which makes the theory power-counting non-renormalizable. It introduces one more degree of freedom called the spin-0 mode of gravitons which introduces ghosts and instability problems.

2.2 Introduction to Einstein-Æther theory of Gravity

In Einstein-æther theory, LI is broken only down to a rotation subgroup by the existence of a preferred time direction at every point of spacetime, i.e., the existence of a preferred frame of reference established by the æther vector field. This time-like unit vector field can be interpreted as a velocity four-vector of some medium substratum (æther, vacuum, or dark fluid), bringing into consideration non-uniformly-moving continuous media and their interaction with other fields. Meanwhile, this theory can be also considered as a realization of dynamic self-interaction of complex systems moving with a spacetime-dependent macroscopic velocity.

In the Einstein-aether theory, the fundamental variables of the gravitational sector are [13, 14]

$$(g_{\mu\nu}, u^{\mu}, \lambda)\,, \tag{2.22}$$

with the Greek indices $\mu, \nu = 0, 1, 2, 3$, and $g_{\mu\nu}$ is the four-dimensional metric of the space-time with the signature $(-, +, +, +)$ [26–28]. The four-vector u^{μ} represents the æther field, and λ is a Lagrangian multiplier which guarantees that the æther four-velocity is always timelike. The general action of the theory is given by,

$$S = S_{æ} + S_m, \tag{2.23}$$

where S_m denotes the action of matter, and $S_{æ}$ the gravitational action of the æ-theory, given by

$$S_{æ} = \frac{1}{16\pi G}\int\sqrt{-g}\, d^4x\Big[R(g_{\mu\nu}) + \mathcal{L}_{æ}\left(g_{\mu\nu}, u^{\alpha}, \lambda\right)\Big], \tag{2.24}$$

$$S_m = \int\sqrt{-g}\, d^4x\Big[\mathcal{L}_m\left(g_{\mu\nu}, u^{\alpha}; \psi\right)\Big]. \tag{2.25}$$

Here ψ collectively denotes the matter fields, R and g are, respectively, the Ricci scalar and determinant of $g_{\mu\nu}$, and

$$\mathcal{L}_{æ} \equiv -M^{\alpha\beta}{}_{\mu\nu}\left(D_{\alpha}u^{\mu}\right)\left(D_{\beta}u^{\nu}\right) + \lambda\left(g_{\alpha\beta}u^{\alpha}u^{\beta} + 1\right), \tag{2.26}$$

where D_{μ} denotes the covariant derivative with respect to $g_{\mu\nu}$, and $M^{\alpha\beta}{}_{\mu\nu}$ is defined as

$$M^{\alpha\beta}{}_{\mu\nu} = c_1 g^{\alpha\beta}g_{\mu\nu} + c_2\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} + c_3\delta^{\alpha}_{\nu}\delta^{\beta}_{\mu} - c_4 u^{\alpha}u^{\beta}g_{\mu\nu}. \tag{2.27}$$

Note that here we assume that matter fields couple not only to $g_{\mu\nu}$ but also to the æther field, which in general violates the weak equivalence principle [13, 14]. The four coupling constants c_i (c_1, c_2, c_3, and c_4) are all dimensionless. G is related to the Newtonian constant G_N via the relation [29],

$$G_N = \frac{G}{1 - \frac{1}{2}c_{14}}, \tag{2.28}$$

where $c_{ij} = c_i + c_j$ for i and j=1, 2, 3, 4.

The variations of the total action given by Eq. (2.23) with respect to $g_{\mu\nu}$, u^{μ} and λ yield, respectively, the field equations,

$$E^{\mu\nu} = 8\pi G T^{\mu\nu}, \tag{2.29}$$

$$Æ_{\mu} = 8\pi G T_{\mu} \tag{2.30}$$

$$g_{\alpha\beta}u^{\alpha}u^{\beta} = -1, \tag{2.31}$$

where

$$E^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - T^{\mu\nu}_{æ}, \tag{2.32}$$

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g_{\mu\nu}}, \tag{2.33}$$

$$T_{\mu} \equiv -\frac{1}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta u^{\mu}}, \tag{2.34}$$

$$\begin{aligned} T^{æ}_{\alpha\beta} \;\; \equiv \;\; & D_{\mu}\Big[J^{\mu}{}_{(\alpha}u_{\beta)} + J_{(\alpha\beta)}u^{\mu} - u_{(\beta}J_{\alpha)}{}^{\mu}\Big] \\ & + c_1\Big[\left(D_{\alpha}u_{\mu}\right)\left(D_{\beta}u^{\mu}\right) - \left(D_{\mu}u_{\alpha}\right)\left(D^{\mu}u_{\beta}\right)\Big] \\ & + c_4 a_{\alpha}a_{\beta} + \lambda u_{\alpha}u_{\beta} - \frac{1}{2} g_{\alpha\beta}J^{\delta}{}_{\sigma}D_{\delta}u^{\sigma}, \end{aligned} \tag{2.35}$$

$$Æ_{\mu} \equiv D_{\alpha}J^{\alpha}{}_{\mu} + c_4 a_{\alpha}D_{\mu}u^{\alpha} + \lambda u_{\mu}, \tag{2.36}$$

with

$$J^{\alpha}{}_{\mu} \equiv M^{\alpha\beta}{}_{\mu\nu}D_{\beta}u^{\nu}\,, \quad a^{\mu} \equiv u^{\alpha}D_{\alpha}u^{\mu}. \tag{2.37}$$

From Eqs.(2.30) and (2.31), we find that

$$\lambda = u_{\beta}D_{\alpha}J^{\alpha\beta} + c_4 a^2 - 8\pi G T_{\alpha}u^{\alpha}, \tag{2.38}$$

where $a^2 \equiv a_{\lambda}a^{\lambda}$.

2.2.1 Different Gravitational Modes in Einstein-Æther Theory

GR has two massless spin-2 modes traveling at the speed of light. Whereas, Einstein-æther theory has five massless modes for each wave vector: two spin-2 (tensor), two spin-1 (vector), and one spin-0 (scalar) mode. It can be shown that the Minkowski spacetime is a solution of the Einstein-æther theory, where the æther is aligned along the time direction. Linear perturbations around the Minkowski background yields the squared speeds (s^2) of these modes relative to the æther rest frame to be

$$\begin{aligned}
&spin-2 \qquad c_T^2 = \frac{1}{1-c_{13}} && Traceless\ metric\ mode \\
&spin-1 \qquad c_V^2 = \frac{2c_1 - c_1^2 + c_1^3}{2c_{14}(1-c_{13})} && Transverse\ æther\ mode \qquad (2.39)\\
&spin-0 \qquad c_S^2 = \frac{c_{123}(2-c_{14})}{c_{14}(1-c_{13})(2+c_{13}+3c_2)} && Trace\ mode
\end{aligned}$$

where c_T, c_V, and c_S are the speeds of the tensor, vector, and scalar modes respectively [30]. We can see from the above equations that for very small values of c_i s, the speed of the spin-2, spin-1, and spin-0 mode tends to 1, $\frac{c_1}{c_{14}}$, and $\frac{c_{123}}{c_{14}}$ respectively. If $(s^2) > 0$ and finite for all modes, the linearized field equations are hyperbolic. It implies that the theory is stable and has an initial value formulation. The presence of transverse æther and trace modes suggest that dipole and monopole radiation will also exist besides the quadruple radiation in GR.

2.2.2 Observational Constraints on Einstein-Æther Theory

Recently, the combination of the gravitational wave GW170817 [31] observed by the LIGO/Virgo collaboration, and the gamma-ray burst GRB 170817A [32] events provided a remarkably stringent constraint on the speed of the spin-2 graviton, $-3 \times 10^{-15} < c_T - 1 < 7 \times 10^{-16}$. In the Einstein-æther theory, the speed of the spin-2 graviton is given by $c_T^2 = 1/(1-c_{13})$ [30], so this implies

$$|c_{13}| < 10^{-15} \qquad (2.40)$$

Together with other observational and theoretical constraints, the parameter space of æ-theory is restricted to the intersection of the constraints [28],

$$\begin{aligned}
&|c_{13}| < 10^{-15}, \quad 0 \le c_{14} \le 2.5 \times 10^{-5}, \\
&0 \le c_2 \le 0.095, \quad c_4 \le 0\,. \qquad (2.41)
\end{aligned}$$

It should be noted that not all the points inside these ranges satisfy all the observational and theoretical constraints, and additional conditions still exist even inside these ranges. For example, for $0 \leq c_{14} \leq 2 \times 10^{-7}$ we must further require $c_{14} \lesssim c_2 \lesssim 0.095$; and for $2 \times 10^{-6} \lesssim c_{14} \lesssim 2.5 \times 10^{-5}$, we need to further require $0 \lesssim c_2 - c_{14} \lesssim 2 \times 10^{-7}$. For details, see [28].

CHAPTER THREE

Nonprojectable 2d Hořava Gravity Non-Minimally Coupled with a Non-Relativistic Scalar Field

This chapter is based on [17]: Bao-Fei Li, Madhurima Bhattacharjee, and Anzhong Wang "Universal horizons and Hawking radiation in nonprojectable 2d Hořava gravity coupled with a non-relativistic scalar field," Phys. Rev. D 96, 084006(2017).

3.1 Introduction

Quantization of gravity has been a subject of intense study for over half a century [11, 33, 34], and various candidates have been proposed, such as string/M-Theory [35–37], Loop Quantum Gravity (LQG) [38–41], Causal Dynamical Triangulation (CDT) [42], and Asymptotic Safety [43, 44], to name only a few of them. For more details, see [45]. However, our understanding of each of them is still highly limited. In particular, it is not clear how they are related (if at all), and which is *the theory* we have been looking for over these years. One of the main reasons is the absence of experimental evidence for quantum gravitational effects. In a certain sense, this is understandable, considering the fact that quantum gravitational effects are normally expected to become important only at the Planck scale, which currently is well above the range of any man-made terrestrial experiments. However, the situation has been changing recently with the arrival of precision cosmology [46–54]. Particularly, it was recently shown that one of the approaches adopted in loop quantum cosmology leads to inconsistency with current observations under certain circumstances [55, 56].

In the original incarnation of Hořava gravity [15], the theory suffered several problems, including instability in the IR, strong coupling, and inconsistency with observations [57–61]. Since then, various modifications have been proposed, and for a recently updated review we refer readers to [16]. Among several important issues, quantization of Hořava gravity has been considered only in some particular cases, despite the vast literature on the theory. In particular, in (3+1)-dimensional spacetimes with the projectability and detailed balance conditions, the renormalizability of Hořava gravity was shown to reduce to one of the corresponding (2+1)-dimensional topologically massive gravity [62]. The latter is expected to be renormalizable [63], although a rigorous proof is still absent. Lately, it was shown that the theory is renormalizable even without the detailed balance condition, by properly choosing a gauge that ensures the correct anisotropic scaling of the propagators and their uniform falloff at large frequencies and momenta [64].

The quantization of Hořava gravity has been studied with and without the projectability condition in (1+1)-dimensional (2d) spacetimes [65, 66]. Due to the foliation-preserving diffeomorphism, the theory is non-trivial even in 2d spacetimes, in contrast to the relativistic case [67–69], although the total degree of freedom of the theory is still zero [65, 66]. In particular, in the projectable case, when only gravity is present, the system can be quantized by following the canonical Dirac quantization [70], and the corresponding wavefunction is normalizable [65]. It is remarkable to note that in this case the corresponding Hamiltonian can be written in terms of a simple harmonic oscillator, whereby the quantization can be carried out quantum mechanically in the standard way. When minimally coupled to a scalar field, the momentum constraint can be solved explicitly in the case where the fundamental variables are functions of time only. In this case, the coupled system can also be quantized by following the Dirac process, and the corresponding wavefunction is also normalizable.

In the non-projectable case, the analysis of the 2d Hamiltonian structure shows that there are two first-class and two second-class constraints [66]. Then, following Dirac one can quantize the theory by first requiring that the two second-class constraints be strongly equal to zero, which can be carried out by replacing the Poisson bracket by the Dirac bracket [70]. The two first-class constraints give rise to the

Wheeler-DeWitt equations. A remarkable feature is that orderings of the operators from a classical Hamiltonian to a quantum mechanical one play a fundamental role in order for the Wheeler-DeWitt equation to have nontrivial solutions. In addition, the space-time is well quantized, even when it is classically singular. It was also shown that the 2d projectable Hořava gravity is exactly equal to the 2d CDT [71]. Such studies were further generalized to the case coupled with a scalar field [72]. In addition, the quantization of a 2d Friedman-Robertson-Walker universe was studied in [73, 74].

In this chapter, we investigate the 2d Hořava gravity with the non-projectable condition, but focus on two related issues: the existence of universal horizons and their Hawking radiations. The existence of black holes in gravitational theories with LI is closely related to the existence of light cones [75]. It was expected that black holes should not exist in theories of gravity where LI is broken as particles in such theories can have speeds larger than that of light. Such particles are always able to cross event horizons and escape to infinity, even if they are inside the event horizon initially. Therefore, it was very surprising to discover that black holes exist even in such theories, but now with universal horizons as the boundaries of black holes [76,77], instead of Killing horizons [75].

Since then, universal horizons and their thermodynamics have been studied intensively (see, for example, [16] and references therein). In particular, it was shown that universal horizons exist in the three well-known black hole solutions: the Schwarzschild, Schwarzschild anti-de Sitter, and Reissner-Nordström [78], which are also solutions of Hořava gravity [79]. At the universal horizon, the first law of black hole mechanics exists for the neutral Einstein-aether black holes [80], provided that the surface gravity is defined by [81],

$$\kappa_{UH} \equiv \frac{1}{2} u^{\alpha} D_{\alpha} \left(u_{\lambda} \zeta^{\lambda} \right), \tag{3.1}$$

which was obtained by considering the peering behavior of ray trajectories of constant khronon field ϕ. However, for the charged Einstein-æther black holes, such a first law is still absent [82]. The universal horizon radiates as a black-body at a fixed temperature [83]. However, different species of particles, in general, experience different temperatures [84],

$$T_{UH}^{z\geq 2} = \frac{2(z-1)}{z}\left(\frac{\kappa_{UH}}{2\pi}\right), \tag{3.2}$$

where κ_{UH} is the surface gravity calculated from Eq. (3.1) and z is the exponent of the dominant term in the UV regime. When $z = 2$, we have the standard result, $T_{UH}^{z=2} = \frac{\kappa_{UH}}{2\pi}$, which was first obtained in [81, 83]. More careful studies of ray trajectories showed that the surface gravity for particles with a non-relativistic dispersion relation is indeed given by [85],

$$\kappa_{UH}^{z\geq 2} = \frac{2(z-1)}{z}\kappa_{UH}. \tag{3.3}$$

The same results were also obtained in [86]. It is remarkable to note that in terms of $\kappa_{UH}^{z\geq 2}$ and $T_{UH}^{z\geq 2}$, the standard relationship between the temperature and surface gravity of a black hole still holds here.

The existence of universal horizons is closely related to the existence of a globally defined time-like khronon field φ [16]. Then, all the particles are assumed to move in the increasing direction of φ. At the beginning, universal horizons were studied in the framework of the Einstein-æther theory with spherical symmetry, in which the time-like æther naturally plays the role of the khronon field [76, 77]. To generalize such concepts to other theories, including Hořava gravity, in which the æther field is not a part of the theory, one can consider the khronon field as a test field [87], a role similar to a Killing vector field ξ_μ, which satisfies the Killing equations, $\nabla_{(\nu}\xi_{\mu)} = 0$, on a given spacetime background $g_{\mu\nu}$. In this chapter, we shall adopt this generalization, and assume that the test khronon field satisfies the same equations as the æther field, the most general second-order partial differential equations in terms of the æther four-velocity [88]. For more detail, we refer readers to [16] and references therein. We shall study universal horizons and their thermodynamics in 2d non-projectable Hořava gravity, coupled with a non-relativistic scalar field. The chapter is organized as follows: in section 3.2, we present the general action of the coupled system and derive the corresponding Hamiltonian structure and field equations. In section 3.3, we find various diagonal and non-diagonal stationary solutions of the coupled system in [89]. In section 3.4 we first study the existence of universal horizons in a representative spacetime found in section 3.3, and then study its Hawking radiation by using the Hamilton-Jacobi method. To compare it with the relativistic case, Hawking radiation at Killing horizons is also studied in this section. The main conclusions are presented in section 3.5.

3.2 2d Hořava Gravity Coupled with a Scalar Field

The general gravitational action of Hořava gravity is given by

$$S_{HL} = \zeta^2 \int dt\, dx\, N\sqrt{g}\,(\mathcal{L}_K - \mathcal{L}_V), \tag{3.4}$$

where ζ^2 denotes the coupling constant of Hořava gravity, N the lapse function in the Arnowitt-Deser-Misner (ADM) decomposition [90], and $g \equiv \det(g_{ij})$, here g_{ij} is the spatial metric defined on the leaves $t =$ constant. $\mathcal{L}_K$ is the kinetic part of the action, given by

$$\mathcal{L}_K = K_{ij}K^{ij} - \lambda K^2, \tag{3.5}$$

where λ is a dimensionless constant, and K_{ij} denotes the extrinsic curvature tensor of the leaves $t =$ constant, given by

$$K_{ij} = \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right), \tag{3.6}$$

and $K \equiv g^{ij}K_{ij}$. Here $\dot{g}_{ij} \equiv \partial g_{ij}/\partial t$, ∇_i denotes the covariant derivative with respect to the metric g_{ij}, and N^i the shift vector, with $N_i \equiv g_{ij}N^j$.

$\mathcal{L}_V$ denotes the potential part of the action, and in 2d spacetimes, it takes the form [66],

$$\mathcal{L}_V = 2\Lambda - \beta a_i a^i, \tag{3.7}$$

where Λ denotes the cosmological constant, and β is another dimensionless coupling constant.

On the other hand, the action for a non-relativistic scalar field takes the form

$$S_\phi = \int dt\, dx\, N\sqrt{g}\left\{\frac{1}{2}\left(\partial_\perp\phi\right)^2 - \alpha_0\left(\nabla_i\phi\right)^2 - V(\phi) - f(\phi)R\right\}, \tag{3.8}$$

where $\partial_\perp \equiv N^{-1}(\partial_t - N^i\nabla_i)$, α_0 is a dimensionless coupling constant. In the relativistic case, it is equal to 1/2. The function $f(\phi)$ is arbitrary and depends on ϕ only, and R denotes the Ricci scalar of the 2d spacetimes. The scalar field couples non-minimally to gravity through the term $f(\phi)R$. So, the total action is

$$S = S_{HL} + S_\phi = \zeta^2 \int dt\, dx\, N\sqrt{g}\mathcal{L}. \tag{3.9}$$

3.2.1 Hamiltonian Structure

The 2d spacetimes are described by the general metric,

$$ds^2 = -N^2 dt^2 + \gamma^2\left(dx + N^1 dt\right)^2, \tag{3.10}$$

subjected to the gauge freedom (2.5), where N, N^1 and γ are in general functions of t and x, and $\gamma \equiv \sqrt{g_{11}}$. To be as general as possible, we shall not impose any gauge conditions in this section. Then, the action (3.4) takes the form

$$S_{HL} = \int dt dx N\gamma\left[(1-\lambda)K^2 - 2\Lambda + \beta a_1 a^1\right]. \tag{3.11}$$

where $a_1 = (\ln N)'$, and

$$K = -\frac{1}{N}\left(\frac{\dot{\gamma}}{\gamma} - \frac{N_1'}{\gamma^2} + \frac{N_1\gamma'}{\gamma^3}\right), \tag{3.12}$$

with $\gamma' \equiv \partial\gamma/\partial x$, and $\dot{\gamma} \equiv \partial\gamma/\partial t$. In terms of N, N_1 and γ, the matter action takes the form [1]

$$S_\phi = \int dtdxN\gamma \left\{ \frac{1}{2N^2}\left(\dot{\phi} - \frac{N_1\phi'}{\gamma^2}\right)^2 - \frac{\alpha_0}{\gamma^2}\phi'^2 - V(\phi) - f(\phi)R \right\}, \tag{3.13}$$

where

$$R = \frac{2}{N\gamma}\left[\partial_\mu(N\gamma n^\mu K) - \left(\frac{N'}{\gamma}\right)'\right]. \tag{3.14}$$

Here $n^\mu \equiv N^{-1}(1, -N^1)$ denotes the normal vector to the hypersurfaces $t =$ Constant. Then, we find the conjugate momenta of N, N_1, γ, and ϕ are given by

$$\pi_N \equiv \frac{\partial\mathcal{L}}{\partial\dot{N}} = 0, \quad \pi_{N_1} \equiv \frac{\partial\mathcal{L}}{\partial\dot{N}_1} = 0,$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\gamma}} = 2K(\lambda - 1) - 2f'\frac{\dot{\phi}}{N} + 2f'\frac{\phi' N_1}{N\gamma^2},$$

$$\pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\gamma}{N}(\dot{\phi} - N_1\frac{\phi'}{\gamma^2}) + 2f'\gamma K. \tag{3.15}$$

After a Legendre transformation, it can be shown that the Hamiltonian can be cast in the form

$$\mathcal{H}_0 = N\mathcal{H} + N_1\mathcal{H}^1 - 2\beta\left(\frac{N'}{\gamma}\right)', \tag{3.16}$$

where

$$\mathcal{H}^1 = -\frac{\pi'}{\gamma} + \frac{\pi_\phi\phi'}{\gamma^2}, \tag{3.17}$$

$$\begin{aligned}\mathcal{H} \;\; = \;\; & -\frac{\pi_\phi\pi}{2f'} + \frac{(\lambda-1)\pi_\phi}{f'}K + (1-\lambda)K^2\gamma + 2\Lambda\gamma + \alpha_0\frac{\phi'^2}{\gamma} + \gamma V(\phi) \\ & -\frac{\gamma}{2}\left(\frac{\pi_\phi}{\gamma} - 2f'K\right)^2 - 2\left(\frac{f'\phi'}{\gamma}\right)' + \beta\frac{N'^2}{N\gamma} + 2\beta\left(\frac{N'}{N\gamma}\right)'. \end{aligned} \tag{3.18}$$

[1]It is more convenient to use N_i as the fundamental variables, instead of N^i. Therefore, in this chapter the fundamental components of the metric will be chosen as (N, N_1, γ).

Here K can be expressed in terms of the canonical fields and their conjugate momenta,

$$K = \frac{\pi\gamma + 2f'\pi_\phi}{4\gamma f'^2 - 2\gamma(1-\lambda)}. \tag{3.19}$$

A straightforward evaluation of Poisson brackets between momentum constraints yields

$$\left\{\mathcal{H}^1(x), \mathcal{H}^1(x')\right\} = \left(\frac{\mathcal{H}^1(x')}{\gamma^2(x')} + \frac{\mathcal{H}^1(x)}{\gamma^2(x)}\right)\partial_{x'}\delta(x-x'), \tag{3.20}$$

which is the same as in the pure gravity case [66]. The Poisson bracket between $\mathcal{H}$ and $\mathcal{H}^1$ will not vanish on the constraint surface because of the appearance of terms related to the lapse function N in the Hamiltonian constraint $\mathcal{H}$. Therefore, we need to redefine the momentum constraint by adding a term proportional to the primary constraint π_N, which generates the diffeomorphisms of N,

$$\tilde{\mathcal{H}}^1 = \mathcal{H}^1 + \frac{N'}{\gamma^2}\pi_N. \tag{3.21}$$

In principle, one can also add a term generating diffeomorphisms of N_1. However, in the present case, since the Hamiltonian constraint doesn't depend on N_1, this term is not mandatory. In terms of $\tilde{\mathcal{H}}^1$, the structure of Eq. (3.20) will not change, while one can show that $\tilde{\mathcal{H}}^1$ now commutes with $\mathcal{H}$ on the constraint surface,

$$\left\{\tilde{\mathcal{H}}^1(x), \mathcal{H}(x')\right\} = -\left(4\mathrm{c}\pi + \frac{2\mathrm{b}\pi_\phi}{\gamma}\right)\tilde{\mathcal{H}}^1(x)\delta(x-x') + \frac{\mathcal{H}(x)}{\gamma^2(x)}\partial_x\delta(x-x'). \tag{3.22}$$

Here $\mathrm{c} \equiv -\alpha/2 - 2\xi^2\alpha^2$ and $\mathrm{b} \equiv \alpha\xi(2\beta-1) - \frac{1}{2\xi}[1+2\alpha(1-\lambda)]$, where $\alpha^{-1} \equiv 4\xi^2 + 2(\lambda-1)$. Note that in writing the above expression, we have set $f(\phi) = \xi\phi$ for the sake of simplicity. Thus, the total Hamiltonian of the coupled system can be written as

$$\mathcal{H}_t = N\mathcal{H} + N_1\tilde{\mathcal{H}}^1 + \sigma\pi_N + \sigma_1\pi_{N_1}. \tag{3.23}$$

For this coupled system, there are two first-class constraints $\tilde{\mathcal{H}}^1$ and π_{N_1}, and two second-class constraints $\mathcal{H}$ and π_N.

Note that no other constraints will be generated by the equations of motion (E.O.M.) of the said four constraints because the secondary constraint $\tilde{\mathcal{H}}^1$ will not give rise to any tertiary constraints due to Eq. (3.20) and Eq. (3.22), while on the other hand the preservation of $\mathcal{H}$ will only produce two differential equations for lapse function N and Lagrange multiplier σ since $\mathcal{H}$ is a second-class constraint. Thus, the Dirac procedure of finding all the constraints in the Hamiltonian formulation terminates at the level of secondary constraints, and the physical degrees of freedom in the configuration space is one which is due to the introduction of the scalar field into the whole system, while in the pure gravity case it is zero [66].

3.2.2 Field Equations

The variations of the total action S with respect to N, N_1, γ and ϕ, yield, respectively,

$$
\begin{aligned}
&(1-\lambda)\gamma K^2 + 2\beta\left(\frac{N'}{N\gamma}\right)' + \frac{\beta N'^2}{N^2\gamma} + \gamma\left(2\Lambda + V\right) + \frac{\gamma}{2N^2}\left(\dot{\phi} - \frac{N_1\phi'}{\gamma^2}\right)^2 \\
&+ \frac{\alpha_0\phi'^2}{\gamma} + \frac{2K}{N}\left(f'\dot{\phi}\gamma - \frac{f'\phi' N_1}{\gamma}\right) - \left(\frac{2f'\phi'}{\gamma}\right)' = 0, \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2(1-\lambda)K'}{\gamma} + \frac{\phi'}{N\gamma}\left(\dot{\phi} - \frac{N_1\phi'}{\gamma^2}\right) + \frac{2f'\phi' K}{\gamma} + \left(\frac{2f'\dot{\phi}}{N\gamma} - \frac{2f'\phi' N_1}{N\gamma^3}\right)' \\
&+ \frac{2\gamma'}{N\gamma^3}\left(f'\dot{\phi}\gamma - \frac{f'\phi' N_1}{\gamma}\right) = 0, \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
&2(1-\lambda)\left(\dot{K} + \frac{N_1 K'}{\gamma^2} - \frac{NK^2}{2}\right) - \frac{\beta N'^2}{N\gamma^2} + \frac{1}{2N}\left(\dot{\phi} - \frac{N_1\phi'}{\gamma^2}\right)^2 \\
&+ \frac{2N_1\phi'}{N\gamma^2}\left(\dot{\phi} - \frac{N_1\phi'}{\gamma^2}\right) - N(2\Lambda + V) + 2f'\dot{\phi}K + 2f'\phi'\frac{N_1 K}{\gamma^2} \\
&+ 2f'\phi'\frac{N'}{\gamma^2} + \alpha_0\phi'^2\frac{N}{\gamma^2} - 2K\left(f'\dot{\phi} - \frac{f'\phi' N_1}{\gamma^2}\right) + \left(\frac{2f'\dot{\phi}\gamma}{N\gamma} - \frac{2f'\phi' N_1}{N\gamma^2}\right)_{,t} \\
&- \frac{2N_1'}{\gamma^2}\left(f'\dot{\phi}\gamma - f'\phi'\frac{N_1}{\gamma}\right) + \frac{4N_1\gamma'}{N\gamma^4}\left(f'\dot{\phi}\gamma - f'\phi'\frac{N_1}{\gamma}\right) \\
&+ \left(\frac{2N_1 f'\dot{\phi}}{N\gamma^2} - \frac{4f'\phi' N_1^2}{N\gamma^4}\right)' = 0, \qquad (3.26)
\end{aligned}
$$

$$\left(\frac{\gamma\dot{\phi}}{N}-\frac{N_1\phi'}{N\gamma}\right)_{,t}-\left(\frac{N_1\dot{\phi}}{N\gamma}-\frac{N_1^2\phi'}{N\gamma^3}\right)'-2\alpha_0\left(\frac{N\phi'}{\gamma}\right)'+N\gamma V'$$
$$-2f''\dot{\phi}\gamma K+2(f'\gamma K)^{\cdot}+2f''\phi'\frac{N_1K}{\gamma}-2\left(\frac{f'N_1K}{\gamma}\right)'$$
$$+2f''\phi'\frac{N'}{\gamma}-2\left(\frac{f'N'}{\gamma}\right)'=0. \tag{3.27}$$

Here $f'(\phi)\equiv df(\phi)/d\phi$, etc. Note that Eqs. (3.24)-(3.27) hold for any function $f(\phi)$.

3.3 Stationary Spacetimes

In this section, we will study stationary spacetimes of the 2d Hořava gravity coupled with a non-relativistic scalar field, presented in the last section. Setting all the time derivative terms to zero in Eqs. (3.24)-(3.27), and

$$f(\phi)=\xi\phi, \tag{3.28}$$

where ξ is a constant, we find that

$$(1-\lambda)\gamma K^2+2\beta\left(\frac{N'}{N\gamma}\right)'+\frac{\beta N'^2}{N^2\gamma}+\frac{N_1^2\phi'^2}{2N^2\gamma^3}+\frac{\alpha_0\phi'^2}{\gamma}-\frac{2K\xi\phi'N_1}{N\gamma}$$
$$+\gamma(2\Lambda+V)-\left(\frac{2\xi\phi'}{\gamma}\right)'=0, \tag{3.29}$$

$$\frac{2(1-\lambda)K'}{\gamma}-\frac{N_1\phi'^2}{N\gamma^3}+\frac{2\xi\phi'K}{\gamma}-\left(\frac{2\xi\phi'N_1}{N\gamma^3}\right)'-\frac{2\xi\phi'\gamma'N_1}{N\gamma^4}=0, \tag{3.30}$$

$$2(1-\lambda)\left(\frac{N_1K'}{\gamma^2}-\frac{NK^2}{2}\right)-\frac{\beta N'^2}{N\gamma^2}-\frac{3N_1^2\phi'^2}{2N\gamma^4}+\alpha_0\phi'^2\frac{N}{\gamma^2}+$$
$$4\xi\phi'\frac{N_1K}{\gamma^2}+2\xi\phi'\frac{N'}{\gamma^2}-N(2\Lambda+V)+\frac{2\xi N_1'\phi'N_1}{\gamma^3}-\frac{4\xi\gamma'\phi'N_1^2}{N\gamma^5}$$
$$-\left(\frac{4\xi\phi'N_1^2}{N\gamma^4}\right)'=0, \tag{3.31}$$

$$\left(\frac{N_1^2\phi'}{N\gamma^3}\right)'-2\alpha_0\left(\frac{N\phi'}{\gamma}\right)'+N\gamma V'-2\xi\left(\frac{N_1K}{\gamma}\right)'-2\xi\left(\frac{N'}{\gamma}\right)'=0. \tag{3.32}$$

3.3.1 Diagonal Solutions

When the metric is diagonal, we have

$$N_1 = 0, \tag{3.33}$$

so the extrinsic curvature K vanishes and Eq. (3.30) holds identically, while Eqs. (3.29), (3.31) and (3.32) reduce, respectively, to

$$2\beta\left(\nu'' - \nu'\mu'\right) + \beta\nu'^2 - 2\xi\left(\phi'' - \phi'\mu'\right) + \alpha_0\phi'^2 = -(V + 2\Lambda)e^{2\mu}, \tag{3.34}$$

$$\beta\nu'^2 - 2\xi\phi'\nu' - \alpha_0\phi'^2 = -(V + 2\Lambda)e^{2\mu}, \tag{3.35}$$

$$2\xi\left(\nu'' + \nu'^2 - \nu'\mu'\right) + 2\alpha_0\left(\phi'' - \phi'\mu' + \nu'\phi'\right) = e^{2\mu}V', \tag{3.36}$$

where $\nu \equiv \ln N$ and $\mu \equiv \ln\gamma$.

It should be noted that static diagonal solutions were studied recently in [89] with $\Lambda = 0 = \xi$. However, comparing Eq. (3.34) with Eq. (12) given in [89], it can be seen that the second-order derivative term ν'' (or N'') is missing there. This is because, when taking the variation of the total action with respect to N, the authors of [89] incorrectly assumed that a_1 is independent of N. Unfortunately, as a result, all the solutions resulted from Eq. (12) given in [89] in general are not solutions of the field equations of the 2d Hořava gravity coupled with a non-relativistic scalar field.

Using the gauge freedom given by Eq. (2.5), without loss of the generality, we can always set $\mu = -\nu$, i.e,

$$N = \frac{1}{\gamma} = e^{\nu}. \tag{3.37}$$

To solve Eqs. (3.34)-(3.36), let us further consider the case where $V = -2\Lambda$, so that Eqs. (3.34)-(3.36) reduce to

$$2\beta\left(\nu'' + \nu'^2\right) + \beta\nu'^2 - 2\xi\left(\phi'' + \phi'\nu'\right) + \alpha_0\phi'^2 = 0, \tag{3.38}$$

$$\beta\nu'^2 - 2\xi\phi'\nu' - \alpha_0\phi'^2 = 0, \tag{3.39}$$

$$\nu'' + 2\nu'^2 + \frac{\alpha_0}{\xi}(\phi'' + 2\nu'\phi') = 0. \tag{3.40}$$

Then, from Eqs. (3.38) and (3.39) we find that

$$\nu'' + 2\nu'^2 - \frac{\xi}{\beta}(\phi'' + 2\nu'\phi') = 0. \tag{3.41}$$

Thus, Eqs. (3.40) and (3.41) show that there are two possibilities,

$$(i)\ \alpha_0\beta + \xi^2 \neq 0; \quad (ii)\ \alpha_0\beta + \xi^2 = 0. \tag{3.42}$$

3.3.1.1 $\alpha_0\beta + \xi^2 \neq 0$. In this case we must have

$$\nu'' + 2\nu'^2 = 0, \tag{3.43}$$

$$\phi'' + 2\nu'\phi' = 0, \tag{3.44}$$

which have the solutions,

$$\begin{aligned} N &= \sqrt{C_0 x + C_1}, \\ \phi &= \phi_0 \ln(C_0 x + C_1) + \phi_1, \end{aligned} \tag{3.45}$$

where C_i and ϕ_i are the integration constants. Without loss of generality, we can always set $C_0 = 1$, so the metric and scalar field finally take the form

$$\begin{aligned} ds^2 &= -(x - x_0)\,dt^2 + \frac{dx^2}{x - x_0}, \\ \phi &= \phi_0 \ln(x - x_0) + \phi_1, \end{aligned} \tag{3.46}$$

where $x_0 \equiv -C_1$. Clearly, the scalar field is singular at $x = x_0$, and so is the corresponding spacetime.

3.3.1.2 $\alpha_0\beta + \xi^2 = 0$. In this case, there are only two independent equations which are Eqs. (3.39) and (3.40). Now if we substitute the relation $\alpha_0 = -\xi^2/\beta$ into these equations and define a new constant $\kappa = \xi/\beta$, one can easily arrive at

$$\nu'^2 - 2\kappa\phi'\nu' + \kappa^2\phi'^2 = 0, \tag{3.47}$$

$$\nu'' + \nu'^2 - \kappa\phi'' - \kappa^2\phi'^2 = 0. \tag{3.48}$$

The first equation tells us that ν' and ϕ' are linearly dependent, i.e.,

$$\nu = \frac{\xi}{\beta}\left(\phi - \phi_0\right), \tag{3.49}$$

which also makes the second equation hold identically, where ϕ_0 is a constant. Therefore, in the current case *for any chosen ϕ, the solution (3.49) will satisfy the field equations (3.38)-(3.40)*. The corresponding metric takes the form,

$$ds^2 \;=\; -e^{\frac{2\xi(\phi-\phi_0)}{\beta}}dt^2 + e^{-\frac{2\xi(\phi-\phi_0)}{\beta}}dx^2, \tag{3.50}$$

for $\alpha_0 = -\xi^2/\beta$.

3.3.2 Non-diagonal Solutions

In this case, using the gauge transformations (2.5), without loss of generality, we can always set

$$\gamma = 1, \tag{3.51}$$

so the metric takes the form

$$ds^2 = -N^2(x)dt^2 + \big(dx + h(x)dt\big)^2. \tag{3.52}$$

Then, Eqs. (3.29)-(3.32) reduce to

$$\begin{aligned}&(1-\lambda)K^2 + 2\beta\left(\frac{N'}{N}\right)' + \frac{\beta N'^2}{N^2} + 2\Lambda + V(\phi) + \frac{h^2\phi'^2}{2N^2} + \alpha_0\,\phi'^2\\ &-\frac{2\,K\,\xi\,\phi'\,h}{N} - 2\,\xi\,\phi'' = 0,\end{aligned} \tag{3.53}$$

$$2(1-\lambda)K' - \frac{h\phi'^2}{N} + 2\,\xi\,\phi' K - \left(\frac{2\,\xi\,\phi' h}{N}\right)' = 0, \tag{3.54}$$

$$\begin{aligned}&2(1-\lambda)\left(hK' - \frac{NK^2}{2}\right) - \frac{\beta N'^2}{N} - \frac{3h^2\phi'^2}{2N} - N(2\Lambda + V)\\ &+\alpha_0\,\phi'^2 N + 4\,\xi\,\phi'\,h\,K + 2\,\xi\,\phi'\,N' + 2\,\xi\,h'\,\phi'\,h - \left(\frac{4\,\xi\,\phi'\,h^2}{N}\right)' = 0,\end{aligned} \tag{3.55}$$

$$\left(\frac{h^2\,\phi'}{N}\right)' - 2\,\alpha_0\,(N\phi')' + NV' - 2\,\xi(hK)' - 2\,\xi N'' = 0, \tag{3.56}$$

where

$$K = \frac{h'}{N}. \tag{3.57}$$

To solve the above equations, we shall consider some particular cases.

3.3.2.1 $N(x) = 1$. In this case, let us first consider the solution with $\phi = \phi_0$, where ϕ_0 is a constant. Then, from Eq. (3.53) we find that

$$h'^2 = \frac{2\hat{\Lambda}}{\lambda - 1}, \tag{3.58}$$

where $\hat{\Lambda} \equiv \Lambda + V(\phi_0)/2$. The above equation has the solution,

$$h(x) = \pm\sqrt{\frac{2\hat{\Lambda}}{\lambda - 1}}\,x = \pm\eta x. \tag{3.59}$$

It can be shown that in this case a Killing horizon exists, located at $x_{KH} = \pm\eta^{-1}$.

3.3.2.2 $\xi = 0$. When $\xi = 0$, Eqs. (3.53)-(3.56) reduce to

$$(1-\lambda)\left(\frac{h'}{N}\right)^2 + 2\beta\left(\frac{N'}{N}\right)' + \frac{\beta N'^2}{N^2} + \hat{V} + \alpha_0\phi'^2 + \frac{h^2\phi'^2}{2N^2} = 0, \tag{3.60}$$

$$2(1-\lambda)\left(\frac{h''}{N} - \frac{h'N'}{N^2}\right) - \frac{h\phi'^2}{N} = 0, \tag{3.61}$$

$$2(1-\lambda)\left(\frac{h''}{N} - \frac{h'N'}{N^2} - \frac{h'^2}{2hN}\right) - \frac{\beta N'^2}{hN} - \frac{3h\phi'^2}{2N} + \frac{N}{h}\left(\alpha_0\phi'^2 - \hat{V}\right) = 0, \tag{3.62}$$

$$\left(\frac{h^2\phi'}{N}\right)' - 2\alpha_0(N\phi')' + N\hat{V}' = 0, \tag{3.63}$$

where $\hat{V} \equiv V + 2\Lambda$. To solve the above equations, let us consider the case,

$$N = h, \quad \hat{V} = 0, \tag{3.64}$$

for which the above equations reduce to

$$2\beta\nu'' + (1-\lambda+\beta)\nu'^2 = -\frac{1+2\alpha_0}{2}\phi'^2, \tag{3.65}$$

$$2(1-\lambda)\nu'' = \phi'^2, \tag{3.66}$$

$$2(1-\lambda)\nu'' - (1-\lambda+\beta)\nu'^2 = \frac{3-2\alpha_0}{2}\phi'^2, \tag{3.67}$$

$$(1-2\alpha_0)\left(e^{\nu}\phi'\right)' = 0, \tag{3.68}$$

where $\nu = \ln N$. To solve the above equations, let us consider the cases $\alpha_0 = 1/2$ and $\alpha_0 \neq 1/2$, separately.

Case B.2.1) $\alpha_0 = 1/2$**:** This is the relativistic case, and Eq. (3.68) is satisfied identically, while from Eqs. (3.65) and (3.67), we find

$$(1-\lambda+\beta)\nu'' = 0. \tag{3.69}$$

If $\lambda \neq \beta + 1$, it can be shown that the above equations have only the trivial solution in which ν and ϕ are all constants. On the other hand, when $\lambda = \beta + 1$, Eqs. (3.65)-(3.67) reduce to a single equation,

$$2\beta\nu'' = -\phi'^2, \; (\beta = \lambda - 1), \tag{3.70}$$

for the two arbitrary functions ν and ϕ. Again, similar to Case A.2 considered in the last subsection, the solutions are not uniquely determined. In fact, for any given ϕ, the solution

$$\nu(x) = -\frac{1}{2\beta}\int^{x} dx' \int^{x'} \phi'^2(x'')dx'' + C_1 x + C_0 \tag{3.71}$$

will satisfy the field equations (3.65) and (3.67), where C_1 and C_0 are two integration constants.

Case B.2.2) $\alpha_0 \neq 1/2$**:** In this case, from Eq. (3.68) we find

$$\phi' = C_0 e^{-\nu}, \tag{3.72}$$

where C_0 is another constant. Substituting it into Eq. (3.66), we obtain

$$NN'' - N'^2 + \mathcal{D} = 0, \tag{3.73}$$

where $\mathcal{D} \equiv C_0^2/(2(\lambda - 1))$. The above equation has two particular solutions,

$$N_A(x) = \frac{1}{2C_1^2} e^{C_1(x+C_2)} - \frac{\mathcal{D}}{2} e^{-C_1(x+C_2)}, \tag{3.74}$$

$$N_B(x) = \frac{1}{2C_1^2} e^{-C_1(x+C_2)} - \frac{\mathcal{D}}{2} e^{C_1(x+C_2)}, \tag{3.75}$$

where C_1 and C_2 are two integration constants. Correspondingly, the scalar field ϕ is given, respectively, by

$$\phi_A(x) = -\frac{2}{\sqrt{\mathcal{D}}} \tanh^{-1} \left(\frac{e^{C_1(C_2+x)}}{\sqrt{\mathcal{D}} C_1} \right), \tag{3.76}$$

$$\phi_B(x) = \frac{2}{\sqrt{\mathcal{D}}} \tanh^{-1} \left(C_1 \sqrt{\mathcal{D}} e^{C_1(C_2+x)} \right). \tag{3.77}$$

3.4 Universal Horizons and Hawking Radiation

In this section, we shall consider two issues, universal horizons and the corresponding Hawking radiations. As a representative case, we shall focus on the solution given by Eqs.(3.52) and (3.59) with $N = 1$. Without loss of generality, we consider only the case with "-" sign, that is,

$$ds^2 = -dt^2 + (dx - \eta x dt)^2 = -\left(1 - \eta^2 x^2\right) dt^2 - 2\eta x dt dx + dx^2, \tag{3.78}$$

where $-\infty < t, x < \infty$. The corresponding inverse metric is given by

$$g^{tt} = -1, \quad g^{tx} = -\eta x, \quad g^{xx} = 1 - \eta^2 x^2, \tag{3.79}$$

which is non-singular, except at the infinities $x = \pm\infty$. The latter are coordinate singularities, similar to the 4d de Sitter space. In fact, the extrinsic curvature and 2d Ricci scalar are all finite, and given by $-\eta$ and $2\eta^2$, respectively. However, there exist two cosmological Killing horizons, located, respectively, at $x_{KH} = \pm\eta^{-1}$. Similar to the 4d de Sitter space, the time-translation Killing vector, $\xi^\mu = \delta^\mu_t$, is time-like only in the region $x^2 < x_{KH}^2$. In the regions $x^2 > x_{KH}^2$, the Killing vector becomes spacelike, and only in these regions can the universal horizon exist, as the latter is defined by [16]

$$(\xi \cdot u) = 0. \tag{3.80}$$

Since the four-velocity u of the khronon field is always time-like, Eq. (3.80) has solutions only when ξ becomes spacelike, which are the regions in which $x^2 > x_{KH}^2$ holds.

To see the difference between the physics at Killing horizons and that at universal horizons, let us first consider Hawking radiation at the Killing horizon.

3.4.1 Hawking Radiation at the Killing Horizon

As shown in [84], at a Killing horizon only relativistic particles are radiated quantum mechanically. So, in this subsection we consider only the relativistic limit in which the dispersion relation of radiated massless scalar particles satisfies $k^2 \equiv k_\lambda k^\lambda = 0$. Considering only the positive outgoing particles, $k_t = -\omega < 0$, we find

$$k_x^{\pm} = \frac{\omega(h \pm 1)}{1 - h^2}\ , \tag{3.81}$$

which is singular for k_x^+ at the Killing horizon at which we have $h\left(x_{KH}\right) = 1$. Then, from the following formula [84],

$$2\mathrm{Im}\mathcal{S} = \mathrm{Im} \oint \mathrm{k_x^+ dx} = \frac{\omega}{\mathrm{T_{KH}}}, \tag{3.82}$$

we find that

$$T_{KH} = -\frac{h'(x_{KH})}{2\pi} = \frac{\eta}{2\pi}, \tag{3.83}$$

where $x_{KH} = -\eta^{-1}$. On the other hand, the surface gravity at the Killing horizon is given by [75]

$$\kappa_{KH} \equiv \sqrt{-\frac{1}{2}\left(D_\mu \chi_\nu\right)\left(D^\mu \chi^\nu\right)} = \eta, \tag{3.84}$$

where D_μ denotes the covariant derivative with respect to the 2d metric $g_{\mu\nu}$, and $\chi^\mu = \delta^\mu_t$ is the timelike Killing vector. Therefore, the standard form holds

$$T_{KH} = \frac{\kappa_{KH}}{2\pi}. \tag{3.85}$$

3.4.2 Hawking Radiation at the Universal Horizon

The existence of a universal horizon is closely related to the existence of a globally defined timelike scalar field φ [16,87],

$$u_\mu = \frac{\partial_\mu \varphi}{\sqrt{-g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi}}, \quad u_\lambda u^\lambda = -1, \tag{3.86}$$

where the equation of φ is given by the action [91],

$$S_u = \int dt dx N\gamma\Big[\frac{\kappa_1}{2}F^{\alpha\beta}F_{\alpha\beta} + \kappa_2(D_\alpha u^\alpha)^2 + \sigma(u^\alpha u_\alpha + 1)\Big], \tag{3.87}$$

where $F_{\alpha\beta} \equiv D_\alpha u_\beta - D_\beta u_\alpha$, σ is a Lagrange multiplier, and $\kappa_{1,2}$ are two coupling constants. It should be noted that the action (3.87) remains unchanged under the transformations

$$\varphi = \mathcal{F}(\tilde{\varphi}), \tag{3.88}$$

where $\mathcal{F}(\tilde{\varphi})$ is a monotonically increasing or decreasing function of $\tilde{\varphi}$ only. In the following, we shall use this property to choose $\mathcal{F}(\tilde{\varphi})$ so that $d\varphi$ is along the same direction as dt in the regions we are interested in.

Under the background (3.78), we find that the equations of motion are given by

$$\kappa_1(1-\eta^2x^2)u_0'' - \sigma u_0 = 0, \tag{3.89}$$

$$\kappa_1\eta x u_0'' + \kappa_2(u^1)'' - \sigma u_1 = 0, \tag{3.90}$$

$$u_0^2 + 2\eta x u_0 u_1 - (1-\eta^2x^2)u_1^2 - 1 = 0. \tag{3.91}$$

Generally, these coupled non-linear equations are difficult to solve. One simple solution can be obtained when $\kappa_1 = 0$, in which we find $\sigma u_0 = 0$. Since $u_0 \neq 0$ we must have $\sigma = 0$, and Eqs. (3.89)-(3.91) have the solution [2],

$$\begin{aligned} u^0 &= \frac{\eta x u^1 - \sqrt{G(x)}}{\eta^2 x^2 - 1}, \quad u^1 = cx + d, \\ G(x) &\equiv \left(c^2 - \eta^2\right) x^2 + 2cdx + \left(d^2 + 1\right), \end{aligned} \tag{3.92}$$

or inversely

$$\begin{aligned} u_0 &= -\sqrt{G(x)}, \\ u_1 &= \frac{-(cx+d) + \eta x \sqrt{G(x)}}{\eta^2 x^2 - 1}, \end{aligned} \tag{3.93}$$

where c and d are two integration constants. In asymptotically flat spacetimes, these two constants can be determined by requiring that [76,87]: (a) the solution be aligned asymptotically with the time translation Killing vector; and (b) the khronon have a regular future sound horizon. However, the spacetime we are studying is asymptotically de Sitter, and these conditions cannot be applied to the present case. Instead, we shall leave this possibility open, as long as it allows a globally defined khronon field φ since only the latter is essential for the existence of the universal horizon, as explained previously in the introduction. Then, one may ask what is their physical meanings. To see these, let us first calculate the quantity [92]

$$\nabla_\alpha u_\beta = c s_\alpha s_\beta + \hat{c} u_\alpha s_\beta, \tag{3.94}$$

where

$$\hat{c} \equiv \frac{x\eta^2 - c(cx+d)}{\sqrt{1 + (cx+d)^2 - x^2\eta^2}}. \tag{3.95}$$

---

[2] Eq. (3.91) is a quadratic equation for u_0, so in general it has two solutions. In the following we shall consider only the one with the minus sign, as the one with the plus sign will give the same results.

Thus, c is directly related to the expansion of the æther. In fact, we have $\theta \equiv g^{\alpha\beta}\nabla_\alpha u_\beta = c$. On the other hand, assuming that the æther is moving along the trajectory $x^\mu = x^\mu(\tau)$, where τ is the proper time measured by æther, from Eq. (3.92) we find

$$u^1 \equiv \left.\frac{dx(\tau)}{d\tau}\right|_{c=0} = d, \tag{3.96}$$

that is, the parameter d is directly related to the constant part of the velocity of the æther.

In order to have the solution (3.92) well-defined for all the values of $x \in (-\infty, \infty)$, we must assume that $G(x) \geq 0$, which yields

$$c^2 \geq \left(1 + d^2\right)\eta^2. \tag{3.97}$$

On the other hand, the universal horizon is located at [16], $(u \cdot \xi) = -\sqrt{G(x)} = 0$. Since $G(x) \geq 0$ for $x \in (-\infty, \infty)$, we must have [78]

$$G\left(x_{UH}\right) = 0, \qquad \left.\frac{dG(x)}{dx}\right|_{x=x_{UH}} = 0, \tag{3.98}$$

at the universal horizon $x = x_{UH}$. Inserting Eq. (3.92) into the above equations, we find that

$$c = \epsilon_c \eta\sqrt{1+d^2}, \quad x_{UH} = -\epsilon_c \frac{\sqrt{1+d^2}}{\eta d}, \tag{3.99}$$

where $\epsilon_c = \text{Sign}(c)$. It is interesting to note that the above solution for c saturates the bound of Eq. (3.97). We also note that

$$x_{UH}^2 - x_{KH}^2 = \frac{1}{(\eta d)^2} > 0, \tag{3.100}$$

as expected.

On the other hand, from Eqs. (3.86) and (3.88), we find that the khronon field takes the form

$$\varphi = t + f(x), \tag{3.101}$$

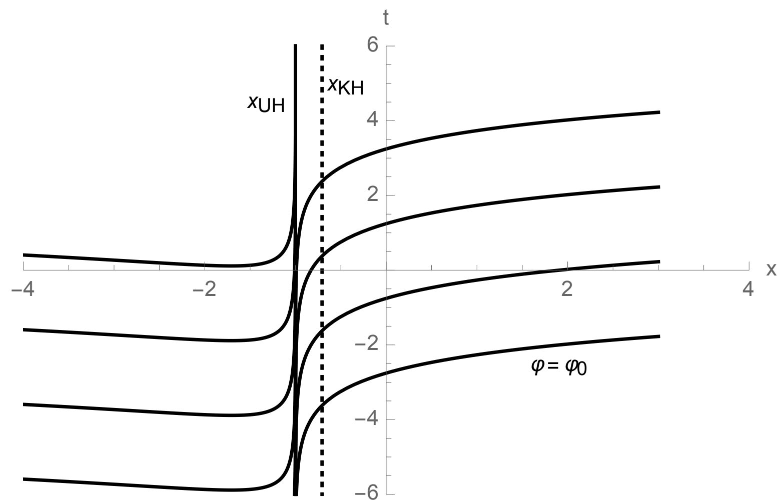

Figure 3.1: The curves of $\varphi =$ Constant. In this figure, we choose $\epsilon_c = 1,\ d = 1,\ \eta = \sqrt{2}$. The universal horizon (dot dashed vertical line) is located at $x_{UH} = -1$, and the dashed line denotes the location of the cosmological Killing horizon located at $x_{KH} = -\frac{1}{\sqrt{2}}$.

where we had chosen $\mathcal{F} = -\tilde{\varphi}$, and dropped the tilde from $\tilde{\varphi}$ for the sake of simplicity, without causing any confusion. The function f satisfies the differential equation

$$f'(x) = \frac{u^1 - \eta x\sqrt{G(x)}}{(\eta^2 x^2 - 1)\sqrt{G(x)}}. \tag{3.102}$$

In Fig. 3.1, we show the curves of constant φ, from which it can be seen clearly the peeling behavior of the curves of constant φ at the universal horizon, while these curves are well-behaved across the Killing horizon.

From Eq. (3.92), we can construct a spacelike unit vector $s_\mu = s_0\delta^t_\mu + s_1\delta^x_\mu$, which is orthogonal to u^μ. It can be shown that s_μ has the non-vanishing components

$$\begin{aligned} s_0 &= -(cx+d), \\ s_1 &= \frac{\eta x u^1 - \sqrt{G(x)}}{\eta^2 x^2 - 1}. \end{aligned} \tag{3.103}$$

Then, we can project k^μ onto u^α and s^α, and obtain

$$\begin{aligned} k_u &\equiv (k\cdot u) = -\omega u^0 + k_x u^1, \\ k_s &\equiv (k\cdot s) = -\omega u_1 - k_x u_0. \end{aligned} \tag{3.104}$$

To proceed further, we need to consider the æther four-velocity u_μ in the regions $x > x_{UH}$ and $x < x_{UH}$, separately. In particular, we have

$$u_0 = -|d\eta x + \sqrt{d^2+1}|,$$

$$u_1 = d,$$

$$u^0 = \sqrt{d^2+1},$$

$$u^1 = \eta\sqrt{d^2+1}x + d,$$

$$f' = -\frac{d}{x\eta d + \sqrt{d^2+1}},$$

$$f = -\frac{1}{\eta}\ln\left(\eta x d + \sqrt{d^2+1}\right), \tag{3.105}$$

for $x > x_{UH}$. When $x < x_{UH}$, we find that u_0 and u^1 remain the same while u^0, u_1, f' and f are changed to

$$u^0 = \frac{\eta^2 x^2\sqrt{d^2+1} + 2d\eta x + \sqrt{d^2+1}}{\eta^2 x^2 - 1},$$

$$u_1 = -\frac{2\eta x\sqrt{d^2+1} + d\eta^2 x^2 + d}{\eta^2 x^2 - 1},$$

$$f' = \frac{d}{x\eta d + \sqrt{d^2+1}} + \frac{2\eta x}{1 - x^2\eta^2},$$

$$f = \frac{1}{\eta}\ln\left(\frac{dx\eta + \sqrt{d^2+1}}{1 - x^2\eta^2}\right),\ (x < x_{UH}). \tag{3.106}$$

At the universal horizon, similar to the (3+1)-dimensional case [84], relativistic particles cannot be emitted in the form of Hawking radiation. Thus, in the following we consider only the particles with the following non-relativistic dispersion relation [84],

$$k_u^2 = k_s^2 + a_2\frac{k_s^4}{k_0^2}, \tag{3.107}$$

where a_2 is a dimensionless constant of order one, and k_0 is the cutoff energy scale. For $k \ll k_0$, the particles become relativistic. Then, from Eq. (3.104) we find

$$\begin{aligned} k_u &= -\frac{1}{u_0}(k_s u^1 - \omega), \\ k_x &= -\frac{1}{u_0}(\omega u_1 + k_s). \end{aligned} \tag{3.108}$$

Combined with the dispersion relation (3.107), we find that k_s has a simple pole at the universal horizon $x = x_{UH}$ with $u_0(x_{UH}) = 0$. Thus, we assume that near the universal horizon we have

$$k_s = -\frac{b(x)}{u_0}, \tag{3.109}$$

where $b(x = x_{UH}) \neq 0$. To calculate the temperature given by Eq. (3.82) but now at the universal horizon, in principle we only need the Laurent expansion of k_x in the neighborhood of the universal horizon. Setting $\epsilon = x - x_{UH}$, for the special case given by Eq. (3.105), we find

$$\begin{aligned} u_0 &= -d\eta\epsilon, \\ u^1 &= -\frac{1}{d} + \epsilon\eta\sqrt{d^2+1}, \\ b(x) &= b_0 + b_1\epsilon + \mathcal{O}\left(\epsilon^2\right), \\ k_x &= \frac{b_0}{\eta^2 d^2 \epsilon^2} + \frac{1}{\epsilon}\left(\frac{\omega}{\eta} + \frac{b_1}{\eta^2 d^2}\right) + \mathcal{O}\left(1\right), \end{aligned} \tag{3.110}$$

for $x > x_{UH}$, where

$$\begin{aligned} b_0 &= \pm\frac{k_0}{\sqrt{a_2}d}, \\ b_1 &= \eta d^2\omega - \eta \mathrm{d} \mathrm{b}_0\sqrt{\mathrm{d}^2+1}. \end{aligned} \tag{3.111}$$

When $x < x_{UH}$, the Taylor expansions of u^1 and $b(x)$ remain the same as in Eq. (3.110) while u_0 and k_x are changed to

$$\begin{aligned} u_0 &= d\eta\epsilon, \\ k_x &= \frac{b_0}{\eta^2 d^2 \epsilon^2} + \frac{1}{\epsilon}\left(-\frac{\omega}{\eta} + \frac{b_1}{\eta^2 d^2}\right) + \mathcal{O}\left(1\right), \end{aligned} \tag{3.112}$$

Correspondingly, with the help of dispersion relation Eq. (3.107), one can show

$$
\begin{aligned}
b_0 &= \pm\frac{k_0}{\sqrt{a_2}d}, \\
b_1 &= -\eta d^2\omega - \eta \mathrm{d}\mathrm{b}_0\sqrt{\mathrm{d}^2+1}.
\end{aligned}
\tag{3.113}
$$

In order to figure out the temperature at the universal horizon, one needs to analytically continue the radial momentum k_x to the complex plane, combining Eqs. (3.110) and (3.112), it's easy to conclude that, by setting $x = x_{UH} + \epsilon e^{i\theta}$, for $\theta \in (0, 2\pi)$

$$
k_x = \frac{b_0}{\eta^2 d^2 \epsilon^2 e^{2i\theta}} + \frac{2\omega}{\eta\epsilon e^{i\theta}} - \frac{b_0\sqrt{d^2+1}}{\eta d\epsilon}. \tag{3.114}
$$

Then, using Eq. (3.82),

$$
\frac{\omega}{T_{KH}} = \mathrm{Im}\oint \mathrm{k}_\mathrm{x}^+ \mathrm{dx} = \frac{4\pi\omega}{\eta}, \tag{3.115}
$$

we find that

$$
T_{UH} = \frac{\eta}{4\pi}. \tag{3.116}
$$

The surface gravity at the universal horizon is given by [16] [3],

$$
\kappa_{UH} = \frac{1}{2}D_u(u\cdot\zeta) = \frac{\eta}{2}, \tag{3.117}
$$

from which we find that the standard relation

$$
T_{UH} = \frac{\kappa_{UH}}{2\pi}, \tag{3.118}
$$

is satisfied at the universal horizon. This is similar to the (3+1)-dimensional case [81, 83, 84]. For the more general case with the dispersion relation,

$$
k_u^2 = k_s^2 \sum_{n=0}^{2z} a_n \left(\frac{k_s}{k_0}\right)^n \tag{3.119}
$$

it can be shown that the (3+1)-dimensional results [84] can be also obtained

$$
T_{UH}^{z\geq 2} = \frac{\kappa_{UH}^{z\geq 2}}{2\pi} = \left(\frac{2(z-1)}{z}\right)\left(\frac{\kappa_{UH}}{2\pi}\right). \tag{3.120}
$$

[3] It should be noted that κ_{UH} given by Eq. (3.117) can also be obtained by considering the peeling behavior of the khronon field φ given by Eq. (3.101), as it was done in [81].

3.5 Summary

In this chapter, we studied the non-projectable Hořava gravity coupled with a non-relativistic scalar field, in which the coupling is in general non-minimal through the interaction term $f(\phi)R$. The Hamiltonian structure of this coupled system is very similar to that of the pure gravity case. There exist two first-class constraints and two second-class constraints. (The combinations of two second-class constraints will generate two global first-class constraints which account for global time reparametrization symmetry of Hořava gravity as first pointed out in [93]). Therefore, the number of local degree of freedom is one due to the presence of the scalar field.

We also found diagonal static solutions for the couplings $f(\phi) = \xi\phi$, and showed that Killing horizons exist in such solutions, but the scalar field turns out to be singular at these Killing horizons. For the non-diagonal stationary solutions, when the lapse function and the spatial metric component g_{11} are set to one, we found that the solutions represent black holes, in which both Killing and universal horizons exist. At the Killing horizon, the temperature of Hawking radiation is proportional to its surface gravity defined as in the relativistic case [cf. Eq.(3.84)] [75].

To study locations of the universal horizons, we first considered a test timelike scalar field in such a fixed background [87], and found solutions of the test field, whereby the universal horizons located at $\chi{\cdot}u = 0$ were found. By using the Hamilton-Jacobi method [84], we calculated the temperature at the universal horizon, and found that it is proportional to the modified surface gravity defined by Eq. (3.117). For $z = 2$ in the dispersion relation (3.119), the modified surface gravity given by Eq. (3.117) satisfies the standard relation with its temperature, $T_{UH} = \kappa_{UH}/(2\pi)$, similar to the (3+1)-dimensional case [81, 83]. But, in more general cases, both of them will depend on z, as shown by Eq. (3.120), although the standard relation, $T_{UH}^{z\geq 2} = \kappa_{UH}^{z\geq 2}/(2\pi)$, is still expected to hold [85, 86].

CHAPTER FOUR

Gravitational Plane Wave Solutions in Einstien-Æther Theory

This chapter is based on [18]: Jacob Oost, Madhurima Bhattacharjee, and Anzhong Wang "Gravitational plane waves in Einstein–æther theory," General Relativity and Gravitation (2018) 50:124

4.1 Introduction

The recent observations of gravitational waves (GWs) emitted from distant binary systems of black holes [94–96] and neutron stars [31] have opened a new era of gravitational wave astronomy. These observations have once again verified the general theory of relativity and its predictions, thus making GR to appear to be the true theory of classical gravity. Hence, any alternative competing theory of gravity other than GR must now predict gravitational wave solutions, otherwise the theory must be ruled out.

Therefore it would be interesting to study whether or not spacetimes of gravitational plane waves are compatible with the presence of the timelike æther field in Einstein-æther theory. However, due to the existence of a global timelike æther field in Einstein-æther theory, this issue is not trivial. Gravitational plane waves, by definition, propagate along congruences defined by a null vector. Plane gravitational waves can be defined by analogy with plane electromagnetic waves; the analogy depends on the symmetry properties of such waves. It is not expected to find plane gravitational waves in nature, but we presume that gravitational waves from a finite source at very large distance from the observer must appear to be approximately planar. The plane-wave solutions, nevertheless, provide useful and interesting models for studying the properties of gravitational waves.

4.1.1 Polarizations and Interaction of Gravitational Plane Waves

The spacetimes for gravitational plane waves can be cast in various forms, depending on the choice of the coordinates and gauge-fixing [97–99]. In this chapter, we shall adopt the form originally due to Baldwin, Jeffery, Rosen (BJR) [100, 101], which can be cast as [97, 102]

$$\begin{aligned} ds^2 &= -2e^{-M}dudv + e^{-U}\Big[e^V \cosh W dy^2 - 2\sinh W dydz \\ &\quad + e^{-V}\cosh W dz^2\Big], \end{aligned} \tag{4.1}$$

where M, U, V and W are functions of u only, which in general represents a gravitational plane wave propagating along the null hypersurfaces u = constant. The corresponding spacetimes belong to Petrov Type N [97–99][1]. Choosing a null tetrad defined as

$$\begin{aligned} &l^\mu \equiv B\delta^\mu_v, \quad n^\mu \equiv A\delta^\mu_u, \quad m^\mu = \zeta^2\delta^\mu_2 + \zeta^3\delta^\mu_3, \\ &\bar{m}^\mu = \overline{\zeta^2}\delta^\mu_2 + \overline{\zeta^3}\delta^\mu_3, \end{aligned} \tag{4.2}$$

where A and B must be chosen so that $M \equiv \ln(AB)$, and

$$\begin{aligned} \zeta^2 &\equiv \frac{e^{(U-V)/2}}{\sqrt{2}}\left(\cosh\frac{W}{2} + i\sinh\frac{W}{2}\right), \\ \zeta^3 &\equiv \frac{e^{(U+V)/2}}{\sqrt{2}}\left(\sinh\frac{W}{2} + i\cosh\frac{W}{2}\right), \end{aligned} \tag{4.3}$$

we find that the Weyl tensor has only one independent component, represented by Ψ_4, and is given by [97]

$$C^{\mu\nu\alpha\beta} = 4\Big[\Psi_4 l^{[\mu}m^{\nu]}l^{[\alpha}m^{\beta]} + \bar{\Psi}_4 l^{[\mu}\bar{m}^{\nu]}l^{[\alpha}\bar{m}^{\beta]}\Big],$$

[1] By rescaling the null coordinate $u \to u' = \int e^{-M(u)}du$, without loss of the generality, one can always set $M = 0$.

$$\Psi_4 = -\frac{1}{2}A^2\Bigg\{\cosh W V_{uu} + \cosh W\big(M_u - U_u\big)V_u + 2\sinh W V_u W_u$$
$$+i\Big[W_{uu} + \big(M_u - U_u\big)W_u - \sinh W\cosh W V_u^2\Big]\Bigg\}, \qquad (4.4)$$

where $[A, B] \equiv (AB - BA)/2$, and $V_u \equiv \partial V/\partial u$, etc. To see the physical meaning of Ψ_4, following [97, 102], let us first introduce the orthogonal spacelike unit vectors, $E^{\mu}_{(a)}$ $(a = 2, 3)$, in the (y, z)-plane via the relations,

$$E^{\mu}_{(2)} \equiv \frac{m^{\mu} + \bar{m}^{\mu}}{\sqrt{2}},$$
$$E^{\mu}_{(3)} \equiv \frac{m^{\mu} - \bar{m}^{\mu}}{i\sqrt{2}}. \qquad (4.5)$$

We find that the Weyl tensor can be written in the form

$$C^{\mu\nu\alpha\beta} = \frac{1}{2}\Big[e_{+}^{\mu\nu\alpha\beta}\left(\Psi_4 + \bar{\Psi}_4\right) + ie_{\times}^{\mu\nu\alpha\beta}\left(\Psi_4 - \bar{\Psi}_4\right)\Big],$$

where

$$e_{+}^{\mu\nu\alpha\beta} \equiv 4\Big(l^{[\mu}E^{\nu]}_{(2)}l^{[\alpha}E^{\beta]}_{(2)} - l^{[\mu}E^{\nu]}_{(3)}l^{[\alpha}E^{\beta]}_{(3)}\Big),$$
$$e_{\times}^{\mu\nu\alpha\beta} \equiv 4\Big(l^{[\mu}E^{\nu]}_{(2)}l^{[\alpha}E^{\beta]}_{(3)} + l^{[\mu}E^{\nu]}_{(3)}l^{[\alpha}E^{\beta]}_{(2)}\Big). \qquad (4.6)$$

Making a rotation in the $\big(E_{(2)}, E_{(3)}\big)$-plane,

$$E_2 = E'_{(2)}\cos\varphi + E'_{(3)}\sin\varphi,$$
$$E_3 = -E'_{(2)}\sin\varphi + E'_{(3)}\cos\varphi, \qquad (4.7)$$

we find that

$$e_{+} = e'_{+}\cos 2\varphi + e'_{\times}\sin 2\varphi,$$
$$e_{\times} = -e'_{+}\sin 2\varphi + e'_{\times}\cos 2\varphi. \qquad (4.8)$$

In particular, if we choose φ such that

$$\varphi = \frac{1}{2}\tan^{-1}\left(\frac{\mathrm{Im}\left(\Psi_4\right)}{\mathrm{Re}\left(\Psi_4\right)}\right), \tag{4.9}$$

we obtain

$$C^{\mu\nu\alpha\beta} = \frac{1}{2}\left|\Psi_4\right| e'_{+}{}^{\mu\nu\alpha\beta}. \tag{4.10}$$

Thus, the amplitude of the Weyl tensor is proportional to the absolute value of Ψ_4, and the angle defined by Eq.(4.9) is the polarization angle of the gravitational plane wave in the plane spanned by $\left(E_{(2)}, E_{(3)}\right)$, which is orthogonal to the propagation direction l^μ of the gravitational plane wave. It is interesting to note that the unit vectors $E^\mu_{(2)}$ and $E^\mu_{(3)}$ are parallelly transported along l^ν,

$$l^\nu D_\nu E^\mu_{(2)} = 0 = l^\nu D_\nu E^\mu_{(3)}. \tag{4.11}$$

Therefore, the angle defined by Eq. (4.9) is invariant with respect to the parallelly transported basis $\left(E_{(2)}, E_{(3)}\right)$ along the propagation direction l^μ of the gravitational plane wave[2]. This is an important property belonging only to single gravitational plane waves.

When $W = 0$, from Eq. (4.5) we find that

$$\mathrm{Im}\left(\Psi_4\right) = 0,\ (W = 0), \tag{4.12}$$

and $\varphi = 0$. Then, the polarization is along the $E^\mu_{(2)}$-direction, which is usually referred to as the "+" polarization, characterized by the non-vanishing of the function V. The other polarization of the gravitational plane wave, often referred to as the "×" polarization, is represented by the non-vanishing of the function W, for which generically we have $\mathrm{Im}\left(\Psi_4\right) \neq 0$ $(W \neq 0)$ (cf. Fig. 1 given in [102]).

[2]Polarizations of GWs in weak-field approximations were also studied in [103] in the framework of Einstein-æther theory.

When M, U, V and W are functions of v only, the gravitational plane wave is now propagating along the null hypersurfaces $v =$ constant. In this case, by rescaling the null coordinate $v \to v' = \int e^{-M(v)} dv$, one can always set $M(v) = 0$.

When gravitational plane waves moving in both of the two null directions are present, the metric coefficients M, U, V and W are in general functions of u and v. An interesting case is the collision of two gravitational plane waves moving along opposite directions, which generically produces spacetime singularities due to their mutual foci [104]. Another remarkable feature is that one of the gravitational plane waves can serve as a medium for the other, due to their non-linear interaction, so the polarizations of the gravitational plane wave can be changed. The change of polarizations due to the nonlinear interaction is exactly a gravitational analogue of the Faraday rotation, but with the other gravitational plane wave as the magnetic field and medium [97, 102].

4.2 Linearly Polarized Gravitational Plane Waves

In this section, we shall consider gravitational plane waves moving along the hypersurfaces $u =$constant only with one direction of polarization, which are usually called linearly polarized gravitational plane waves. Without loss of the generality, we shall consider only gravitational plane waves with the "+" polarization. Then, by rescaling the u coordinate, without loss of the generality, we can always set $M = 0$, so the metric takes the form,

$$ds^2 = -2dudv + e^{-U(u)}\Big(e^{V(u)}dy^2 + e^{-V(u)}dz^2\Big). \tag{4.13}$$

We also assume that the æther moves only in the (u, v)-plane, so its four-velocity u_μ takes the general form

$$u^\mu = \frac{1}{\sqrt{2}}(e^{-h}, e^h, 0, 0). \tag{4.14}$$

Since the spacetime is only of u dependence, it is easy to see that $h = h(u)$. Then, the non-vanishing components of the Einstein and æther tensors $G_{\mu\nu}$ and $T^{æ}_{\mu\nu}$ and the æther vector $Æ_{\mu}$ are given, respectively, by Eqs. (A.1) and (A.2). In the vacuum case, we have $T^{m}_{\mu\nu} = 0$, $T_{\mu} = 0$, and the Einstein-æther equations (2.29) reduce to

$$G_{\mu\nu} = T^{æ}_{\mu\nu}, \tag{4.15}$$

which yield five equations[3], given by Eqs. (A.4)-(A.8). The æther equations $Æ_{\mu} = 0$ yield the same equation as given by Eq. (A.6).

It is remarkable to note that there are five independent field equations for the three unknowns, U, V, and h. Therefore, in contrast to the situation of GR, in which there is only one independent field equation, given by Eq. (4.16), for two unknown functions U and V, here in the framework of the Einstein-æther theory, we are facing an over-determined problem, instead of under-determined, and clearly only for particular cases the above equations allow solutions for U, V and h.

From the constraint Eqs. (2.40) we can see that the current observations of GW170817 and GRB170817A practically requires $c_{13} \simeq 0$. In addition, for the spin-2 gravitons to move precisely with the speed of light, we also need to set $c_{13} = 0$. However, in order for our results to be as generally applicable as possible, in the rest of this section we shall not impose this condition, and consider all the possible solutions with both $c_{13} = 0$ and $c_{13} \neq 0$, separately.

4.2.1 Solutions with $c_{13} = 0$

When $c_{13} = 0$, Eqs. (A.4)-(A.8) reduce to

$$2U_{uu} - \left(U_u^2 + V_u^2\right) + 2c_{14}\left(h_{uu} - h_u U_u - h_u^2\right) = 0, \tag{4.17}$$

[3] It is interesting to note that in Einstein's theory the field equations $G_{\mu\nu} = 0$ yields only a single equation [97, 102],

$$2U_{uu} - U_u^2 = V_u^2, \tag{4.16}$$

for the two unknown functions $U(u)$ and $V(u)$. In this sense, the problem is underdetermined in Einstein's theory. Thus, for any given gravitational wave $V(u)$, we can always integrate the above equation to find $U(u)$.

$$c_2\Big(U_{uu} - 2h_uU_u - U_u^2\Big) + \big(c_2 - c_{14}\big)\Big(h_{uu} - h_uU_u - 2h_u^2\Big) = 0, \tag{4.18}$$

$$c_2U_{uu} + \big(c_2 - c_{14}\big)\Big(h_{uu} - h_uU_u - h_u^2\Big) = 0, \tag{4.19}$$

$$c_2\Big(2U_{uu} - U_u^2 - 4h_uU_u\Big) + 2c_2h_{uu} - \big(3c_2 + c_{14}\big)h_u^2 = 0. \tag{4.20}$$

Then, from Eqs. (4.18) and (4.19) we find

$$c_2\Big(U_u^2 + 2U_uh_u\Big) + \big(c_2 - c_{14}\big)h_u^2 = 0, \tag{4.21}$$

$$c_2\Big(U_{uu} + U_u^2\Big) + \big(c_2 + c_{14}\big)U_uh_u + \big(c_2 - c_{14}\big)h_{uu} = 0. \tag{4.22}$$

To study the above equations further, we need to distinguish the cases $c_2 \neq c_{14}$ and $c_2 = c_{14}$ separately.

4.2.1.1 $c_2 \neq c_{14}$. In this case, from Eqs. (4.21) and (4.22) we find that

$$h_u^2 = \frac{c_2}{c_{14} - c_2}\Big(U_u^2 + 2U_uh_u\Big), \tag{4.23}$$

$$h_{uu} = \frac{1}{c_{14} - c_2}\Bigg\{c_2\Big(U_{uu} + U_u^2\Big) + \big(c_2 + c_{14}\big)U_uh_u\Bigg\}. \tag{4.24}$$

Inserting the above expressions into Eq. (4.20), we find

$$c_2c_{14}\Big(U_{uu} - U_u^2 - 2U_uh_u\Big) = 0, \tag{4.25}$$

from which we can see that there are three different cases that need to be considered separately

$$i)\ c_2c_{14} \neq 0, \quad ii)\ c_2 = 0,\ c_{14} \neq 0, \quad iii)\ c_2 \neq 0,\ c_{14} = 0. \tag{4.26}$$

Case i) $c_2c_{14} \neq 0$**:** In this case we have

$$U_{uu} = U_u^2 + 2U_uh_u, \tag{4.27}$$

which has the solution

$$U_u = \alpha_0 e^{U+2h}, \tag{4.28}$$

where α_0 is an integration constant. Then Eq. (4.18) reduces to

$$h_{uu} - 2h_u^2 - h_u U_u = 0, \tag{4.29}$$

which has the solution

$$h_u = \alpha_1 e^{2h+U}, \tag{4.30}$$

where α_1 is an integration constant. Notice that $h_u \propto U_u$. In fact we may write

$$h = \alpha U + h_0, \tag{4.31}$$

where α and h_0 are constants. By substituting Eqs. (4.27) and (4.31) into Eq. (4.19) or (4.20) we find that

$$\alpha = -\frac{\sqrt{c_2}}{\sqrt{c_2} \pm \sqrt{c_{14}}}. \tag{4.32}$$

By substituting Eqs. (4.27) and (4.31) into Eq. (4.17) we find

$$V = \beta U + V_0, \tag{4.33}$$

where V_0 is another integration constant, and

$$\beta \equiv \pm\sqrt{1 + 4\alpha + 2c_{14}\alpha^2}. \tag{4.34}$$

Now combining Eqs. (4.31) and (4.28) we find

$$U_u = \hat{\alpha}_0 e^{(2\alpha+1)U}, \tag{4.35}$$

where $\hat{\alpha}_0 \equiv \alpha_0 e^{2h_0}$. Thus, we obtain

$$U(u) = -\frac{1}{2\alpha+1} \ln\left[-\alpha_0(2\alpha+1)(u-u_0)\right], \tag{4.36}$$

where u_0 is a constant of integration. Once $U(u)$ is given the functions $h(u)$ and $V(u)$ can be read off from Eqs. (4.31) and (4.33), respectively, i.e.,

$$V(u) = \frac{\beta}{2\alpha+1} \ln\left[-\alpha_0(2\alpha+1)(u-u_0)\right] + V_0,$$

$$h(u) = -\frac{\alpha}{2\alpha+1} \ln\left[-\alpha_0(2\alpha+1)(u-u_0)\right] + h_0, \tag{4.37}$$

where β is given by Eq.(4.34) in terms of α and c_{14}.

Case ii) $c_2 = 0,\ c_{14} \neq 0$**:** In this case from Eqs. (4.18) and (4.19) we find that $h_u = 0$, that is

$$h(u) = h_0, \tag{4.38}$$

where h_0 is a constant. Then, Eqs. (4.18)-(4.19) are satisfied identically, while Eq. (4.17) reduces to

$$2U_{uu} - U_u^2 = V_u^2, \tag{4.39}$$

which is the same as in GR, that is, in the present case the functions U and V are not uniquely determined. For any given $U(u)$, one can integrate the above equation to obtain $V(u)$.

Case iii) $c_2 \neq 0,\ c_{14} = 0$**:** In this case from Eqs. (4.18) and (4.19) we find that $U_u + h_u = 0$, which has the solution

$$U = -h + U_0, \tag{4.40}$$

where U_0 is a constant. Inserting the above expression into Eq. (4.18) we find that $h_u = 0$, that is,

$$h = h_0. \tag{4.41}$$

Then, from Eq. (4.17) we obtain

$$V = V_0, \tag{4.42}$$

where V_0 is a constant. By rescaling the y and z coordinates, without loss of the generality, we can always set $V_0 = U_0 = 0$, so the solution represents the Minkowski spacetime. That is, in the current case only the trivial Minkowski solution is allowed.

4.2.1.2 $c_2 = c_{14}$. In this case, from Eq.(4.19) we find that

$$c_2 U_{uu} = 0. \tag{4.43}$$

Therefore, depending on the values of c_2, we have two different cases.

Case i) $c_2 = c_{14} \neq 0$**:** In this case, we must have $U_{uu} = 0$, which has the general solution

$$U(u) = \alpha_0 u + U_0, \tag{4.44}$$

where α_0 and U_0 are two integration constants. On the other hand, from Eq. (4.18) we find that

$$h(u) = -\frac{\alpha_0}{2} u + h_0, \tag{4.45}$$

while Eq. (4.20) is satisfied identically. Then, from Eq.(4.17) we find that

$$V(u) = \pm\sqrt{\frac{(c_2 - 2)\alpha_0^2}{2}}\, u + V_0, \tag{4.46}$$

where V_0 is another integration constant.

Case ii) $c_2 = c_{14} = 0$**:** In this case, Eqs. (4.18)-(4.20) are satisfied identically for any given $h(u)$, while Eq. (4.17) reduces to

$$2U_{uu} - U_u^2 = V_u^2, \tag{4.47}$$

which is the same as in GR, that is, in the present case the functions U, V and $h(u)$ are not uniquely determined. For any given $U(u)$ and $h(u)$, one can integrate Eq. (4.47) to obtain $V(u)$.

4.2.2 Solutions with $c_{13} \neq 0$

When $c_{13} \neq 0$, from Eqs. (A.7) and (A.8) we find that

$$V_{uu} - U_u V_u - 2h_u V_u = 0, \tag{4.48}$$

which has the solution

$$V_u = \alpha_0 e^{U+2h}, \tag{4.49}$$

where α_0 is an integration constant. Inserting the above expression into Eqs. (A.4)-(A.8), we obtain the following four independent equations for U and h:

$$2U_{uu} - U_u^2 + 2c_{14}\Big(h_{uu} - h_u U_u - h_u^2\Big) = V_u^2, \tag{4.50}$$

$$c_2\Big(U_{uu} - 2h_u U_u - U_u^2\Big) + \big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_u U_u - 2h_u^2\Big) = 0, \tag{4.51}$$

$$2\big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_u U_u - h_u^2\Big) + 2c_2 U_{uu} + c_{13} U_u^2 + = -c_{13} V_u^2, \tag{4.52}$$

$$\big(c_{13} + 2c_2\big)\Big(2U_{uu} - U_u^2 - 4h_u U_u\Big) + 4c_2 h_{uu} - 2\big(3c_2 - c_{13} + c_{14}\big)h_u^2 = -c_{13} V_u^2. \tag{4.53}$$

Combining Eqs. (4.50) and (4.52) we find

$$c_{123} U_{uu} = (c_{13}c_{14} + c_2 + c_{13} - c_{14})(h_u^2 + h_u U_u - h_{uu}), \tag{4.54}$$

and by using Eqs. (4.50) and (4.53) we obtain

$$c_{123} U_u^2 = (c_{13}c_{14} + 2c_{13} - 2c_{14})(h_u^2 + h_u U_u - h_{uu}) + (c_{13} - c_{14} - c_2)h_u^2 - 2c_{123} h_u U_u. \tag{4.55}$$

To study the above equations further, we need to consider separately the cases $c_{123} = 0$ and $c_{123} \neq 0$.

4.2.2.1 $c_{123} = 0$.

In this case, from Eqs.(4.50) and (4.52) we find

$$c_{14}(c_{13} - 1)(h_{uu} - h_u^2 - h_u U_u) = 0. \tag{4.56}$$

The possibility of $c_{13} = 1$ is ruled out by observation [28], as mentioned above, leaving the possibilities

$$c_{14} = 0, \tag{4.57}$$

or

$$h_{uu} - h_u U_u - h_u^2 = 0. \tag{4.58}$$

Case A.1 $c_{14} = 0$: In the case of Eq. (4.57) we find that Eqs. (4.51) and (4.53) reduce to

$$U_{uu} = 2h_u U_u + U_u^2, \tag{4.59}$$

and

$$h_{uu} = 2h_u^2 + h_u U_u, \tag{4.60}$$

respectively, where we have used the fact that Eq. (4.50) reduces to $2U_{uu} = U_u^2 + V_u^2$. Then, both h_u and U_u are proportional to e^{2h+U}, and hence by Eq. (4.49) we find

$$h = \alpha V + h_0 \quad U = \beta V + U_0, \tag{4.61}$$

where h_0 and U_0 are two integration constants, and the constants α and β can be determined by substituting Eq. (4.61) and Eq. (4.59) into Eq. (4.50) or Eq. (4.52), which yields

$$\alpha = \frac{1-\beta^2}{4\beta}. \tag{4.62}$$

Inserting the above expressions into Eq. (4.49), we find that

$$V = -\frac{2\beta}{1+\beta^2} \ln\left[\hat{\alpha}_0 \left(u_0 - u\right)\right], \tag{4.63}$$

where $\hat{\alpha}_0 \equiv \alpha_0 \left(2\alpha + \beta\right) e^{U_0 + 2h_0}$ and u_0 is an integration constant. Therefore, in this case the solutions are given by Eqs. (4.61)-(4.63).

Case A. 2 $c_{14} \neq 0$: In this case we find that

$$h_u = \alpha_1 e^{h+U}, \tag{4.64}$$

and by Eq. (4.53) that

$$h_u^2 \left(\frac{c_{14}}{c_{13}} - 2 \right) = 0. \tag{4.65}$$

If $h_u = 0$ ($\alpha_1 = 0$) then by Eq. (4.51) we have

$$U_u = \alpha_2 e^U, \tag{4.66}$$

and using this result with Eq. (4.50) we have

$$U = \pm V + U_0. \tag{4.67}$$

Inserting the above expressions into Eq. (4.49), we find that

$$V = \mp \ln \left[\mp \hat{\alpha}_0 (u - u_0) \right], \tag{4.68}$$

where $\hat{\alpha}_0 \equiv \alpha_0 e^{2h_0 + U_0}$ and where the choice of upper or lower sign must hold for both Eqs. (4.67) and Eq. (4.68). Thus, in this case, the general solutions are given by

$$(U, V, h) = (\pm V + U_0, V, h_0), \tag{4.69}$$

where V is given by Eq. (4.68), and U_0 and h_0 are two integration constants. However, if $h_u \neq 0$ then Eq. (4.51) reduces to

$$U_{uu} - 2h_u^2 - U_u^2 - 2h_u U_u = 0, \tag{4.70}$$

and we add the LHS of Eq. (4.58) (which is zero) twice to the LHS of Eq. (4.70) to get

$$U_{uu} + 2h_{uu} - 4h_u^2 - 4h_u U_u - U_u^2 = 0, \tag{4.71}$$

which simplifies to

$$2h_{uu} + U_{uu} = (2h_u + U_u)^2. \tag{4.72}$$

If we define a function $f(u)$ such that $f(u) = 2h(u) + U(u)$, then Eq. (4.72) can be written as

$$f_{uu} = f_u^2, \tag{4.73}$$

which has the solution

$$f = -\ln\left(-\alpha_3(u - u_0)\right), \tag{4.74}$$

where α_3 and u_0 are integration constants. If we multiply both sides of Eq. (4.64) by e^h we have

$$h_u e^h = \alpha_1 e^{2h+U}, \tag{4.75}$$

and making use of Eq. (4.74) we find

$$h_u e^h = -\frac{\alpha_1}{\alpha_3}\frac{1}{u - u_0}, \tag{4.76}$$

whereupon we find by integration

$$h = \ln\left(-\frac{\alpha_1}{\alpha_3}\ln\left(u - u_0\right) + h_0\right). \tag{4.77}$$

So, for the functions U and V we have

$$U = -\ln(-\alpha_3(u - u_0)) - 2h, \tag{4.78}$$

$$V = -\frac{\alpha_0}{\alpha_3}\ln(u - u_0) + V_0. \tag{4.79}$$

By substituting these results into Eq. (4.50) we find that $\alpha_3 = \pm\alpha_0$.

4.2.2.2 $c_{123} \neq 0$. In this case we can substitute Eqs. (4.54) and (4.55) into Eq. (4.51), and by defining

$$Q \equiv c_{123} - c_{14} + \frac{c_2}{c_{123}}(c_{13} - c_{14} - c_2), \tag{4.80}$$

we have

$$Q(h_{uu} - 2h_u^2 - h_u U_u) = 0. \tag{4.81}$$

Therefore we must consider the cases where $Q \neq 0$ and $Q = 0$.

Case B.1 $Q \neq 0$: In this case, we have

$$h_u = \alpha_1 e^{2h+U} \propto V_u. \tag{4.82}$$

Using this result with Eqs. (4.54) and (4.55) we find also that

$$U_u = \alpha_2 e^{2h+U} \propto V_u, \tag{4.83}$$

and thus we can set

$$h = \alpha V + h_0, \quad U = \beta V + U_0, \tag{4.84}$$

for some constants α, β, h_0 and U_0. Substituting Eqs. (4.84) and (4.74) into Eqs. (4.50) and (4.52), we find that α and β must satisfy the relations,

$$\beta^2 + 4\alpha\beta + 2c_{14}\alpha^2 - 1 = 0, \tag{4.85}$$

$$2(c_{14} - c_{13} - c_2)\alpha^2 - 4c_2\alpha\beta - (c_{13} + 2c_2)\beta^2 - c_{13} = 0, \tag{4.86}$$

which uniquely determine α and β, but the expressions for them are too long to be presented here. Inserting the above expressions into Eq. (4.49), we find that

$$V = -\frac{1}{2\alpha + \beta} \ln\left[\hat{\beta}_0(u_0 - u)\right], \tag{4.87}$$

where $\hat{\beta}_0 \equiv \alpha_0(2\alpha + \beta)e^{U_0 + 2h_0}$. Therefore, in the present case, once α and β are determined by Eqs. (4.85) and (4.86), the functions $V(u)$, $U(u)$ and the aether field $h(u)$ are given, respectively, by Eqs. (4.84) and (4.87).

Case B.2 $Q = 0$: We introduce a new parameter δ such that $\delta = 2c_2 + c_{13}$ and solve for c_{14} in terms of other c_is. From Eq. (4.80) we find that

$$c_{14}\delta = c_{13}(c_2 + \delta). \tag{4.88}$$

If we consider $\delta = 0$, then we have $c_2 = 0$ since $c_{13} \neq 0$. But by $\delta = 2c_2 + c_{13}$, we must have $c_{13} = 0$, which violates our assumption, and so we *must* have

$$\delta \neq 0, \tag{4.89}$$

and thus

$$c_{14} = c_{13}\left(1 + \frac{c_2}{\delta}\right), \tag{4.90}$$

is a general solution for the $Q = 0$ case. However, we can still have $c_2 = 0$ in general. If that is the case then we have $c_{13} = c_{14}$ and from Eq. (4.52) we find that

$$V_u^2 = -U_u^2. \tag{4.91}$$

So to have real functions we must have U and V constant in u. Then by considering Eqs. (4.54) and (4.55) with a vanishing U_u we have

$$h_{uu} - h_u^2 = 0, \tag{4.92}$$

which has the solution

$$h = -\ln(\alpha(u - u_0)) + h_0, \ (c_2 = 0), \tag{4.93}$$

where α and h_0 are the integration constants. So, in the case of $c_2 = 0$ we have a static Minkowskian spacetime with a dynamical æther.

If $c_2 \neq 0$, then we find from Eqs. (4.54) and (4.55) that

$$U_{uu} - U_u^2 = \frac{2c_2}{\delta}(h_u^2 + h_u U_u - h_{uu}) + \frac{2c_2}{\delta}h_u^2 + 2h_u U_u, \tag{4.94}$$

and

$$2U_{uu} - U_u^2 = +\frac{2c_2}{\delta}h_u^2 + 2h_u U_u + \mathcal{D}(h_u^2 + h_u U_u - h_{uu}), \tag{4.95}$$

where

$$\mathcal{D} \equiv \frac{2c_2 c_{13}^2}{c_{123}\delta} + \frac{1}{\delta}(c_{13}^2 + 2c_2). \tag{4.96}$$

These expressions can be substituted into Eqs. (4.50) and (4.53) to find

$$V_u^2 = \left(c_{13}\frac{(c_2+\delta)}{c_{123}} - \frac{2c_2}{\delta}\right) + \frac{2c_2}{\delta}h_u^2 + 2h_u U_u, \tag{4.97}$$

and

$$V_u^2 = \left(c_{13}\frac{(c_2+\delta)}{c_{123}} - \frac{2c_2}{c_{13}}\right) + \frac{2c_2}{\delta}h_u^2 + 2h_u U_u. \tag{4.98}$$

Equating these two gives us

$$c_2(h_{uu} - h_u^2 - h_u U_u) = 0. \tag{4.99}$$

Since now we have $c_2 \neq 0$, then we must have

$$h_u = \alpha e^{h+U}. \tag{4.100}$$

In this case, Eq. (4.52) reduces to $V_u^2 = -\frac{2c_2}{c_{13}}U_{uu} - U_u^2$, and by Eq. (4.50) we also have

$$V_u^2 = 2U_{uu} - U_u^2, \tag{4.101}$$

Therefore we we must have

$$U_{uu} = 0, \tag{4.102}$$

since $c_{123} \neq 0$ in this case. As U_u must be a constant, then by Eq. (4.101) we find that V_u must be also a constant, and to keep the constants real we must have U_u and V_u vanish, as before. Considering this result, Eq. (4.51) reduces to

$$h_u^2 = 0. \tag{4.103}$$

Therefore, when $c_2 \neq 0$, the spacetime must be Minkowski and the aether field is simply given by $h(u) = h_0$, this is, the solution in the present case is

$$(U, V, h) = (U_0, V_0, h_0)\,,\ (c_2 \neq 0), \tag{4.104}$$

where $U_0,\ V_0$ and h_0 are all constants.

4.3 Summary

In this chapter, we have found all vacuum solutions of the linearly polarized gravitational plane waves in Einstein-æther theory. In general, such waves need to satisfy five independent Einstein-æther field equations, given by Eqs. (A.4)-(A.8), for three unknown functions $U(u)$, $V(u)$, and $h(u)$. Therefore, the problem in the Einstein-æther theory is over-determined, and it is expected that gravitational plane waves exist only for some particular choices of the coupling constants c_i. This is in sharp contrast to Einstein's general relativity, in which the problem is actually under-determined, i.e. the vacuum Einstein field equations $G_{\mu\nu}$ only yield one independent equation,

$$2U_{uu} - U_u^2 = V_u^2, \tag{4.105}$$

for the two unknown functions U and V. Thus, for any given $V(u)$, one can integrate Eq. (4.105) to find the metric coefficient $U(u)$. This implies that Einstein's theory allows the existence of any form of gravitational plane waves. This is no longer true in Einstein-æther theory, due to the presence of the time-like æther field.

In particular, in order to have arbitrary forms of gravitational plane waves exist in Einstein-æther theory, the coupling constants c_i must be chosen so that one of the following two conditions must be satisfied

$$\begin{aligned} &(i)\ \ c_{13} = c_2 = 0, \ \ c_{14} \neq 0, \ \ h(u) = h_0, \quad \text{or} \\ &(ii)\ c_{13} = c_2 = c_{14} = 0, \qquad \forall\, h(u). \end{aligned} \tag{4.106}$$

In the former case it can be seen that the æther must be a constant, while in the latter the æther has no contributions to the spacetime, and $T^{æ}_{\mu\nu} = 0$ identically, as can be seen from Eq. (A.1). In addition to the above two cases, in which any form of gravitational plane waves are allowed to exist in Einstein-æther theory, there exist also several particular cases in which the spacetime and the æther field take particular forms. Non-trivial solutions exist in the other six particular cases,

$$
\begin{aligned}
&(iii) \qquad c_{13}=0,\ c_2\neq c_{14},\ c_2c_{14}\neq 0,\\
&(iv) \qquad c_{13}=0,\ c_2=c_{14}\neq 0,\\
&(v) \qquad c_{13}\neq 0,\ c_{123}=c_{14}=0,\\
&(vi) \qquad c_{13}\neq 0,\ c_{123}=0,\ c_{14}\neq 0, h_u=0,\\
&(vii) \qquad c_{13}\neq 0,\ c_{123}=0,\ c_{14}\neq 0, h_u\neq 0,\\
&(viii) \qquad c_{13}\neq 0,\ c_{123}\neq 0,\ Q\neq 0,
\end{aligned} \tag{4.107}
$$

in which the particular solutions of the vacuum Einstein-æther field equations are given, respectively, by Eqs. (4.36)-(4.37); Eqs. (4.44)-(4.46); Eqs. (4.61)-(4.63); Eqs. (4.68)-(4.69); Eqs. (4.77)-(4.79), and Eqs. (4.84)-(4.87), where Q is defined by Eq. (4.80).

In the rest of the cases, the solutions are either not allowed or simply represent the Minkowski spacetime with either a constant or dynamical æther field. Some of these cases are problematic, as outlined in Jacobson's review article [13,14]. Any case in which $c_{123}=0$ results in α_2 diverging (suggesting that the current PPN analysis is not valid here).[4] In terms of the coupling constants of the Einstein-æther theory, α_2 is given by [105]

$$
\alpha_2=\frac{\alpha_1}{2}-\frac{(c_1+2c_3-c_4)(2c_1+3c_2+c_3+c_4)}{c_{123}(2-c_{14})}, \tag{4.108}
$$

where $\alpha_1=\frac{-8(c_3^3+c_1c_4)}{2c_1-c_1^2+c_3^2}$. While any case in which $c_{14}=0$ results in the speeds of the scalar and vector modes (Eqs. (2.39)) diverging (suggesting that wave equations for these modes do not exist).

In Case (iv), the squared speed of the spin-0 mode is given by $c_S^2=(2-c_2)/(2+3c_2)$. Thus, to have $c_S\geq 1$, we must require $c_2=c_{14}<0$, which is in conflict

[4]Einstein-æther theory has only two post-Newtonian (PPN) parameters that deviate from GR. They are α_1 and α_2, which measure the preferred frame effects.

with the observational constraints of Eq. (2.41). Therefore, this case is ruled out by observations.

If we require that the speeds of the scalar, vector and tensor modes are all precisely equal to one, then we find that

$$c_{13} = c_4 = 0, \quad c_2 = \frac{c_1}{1 - 2c_1}, \;\; (c_T = c_V = c_S = 1), \tag{4.109}$$

which is satisfied only by Case (iii), and the corresponding solutions are still quite different from those of GR, even if all of these gravitational modes now move at the same speed as the spin-2 graviton in GR.

CHAPTER FIVE

Dynamical Universal Horizons in Einstein-Æther Theory

This chapter is based on [19]: Madhurima Bhattacharjee, Shinji Mukohyama, Mew-Bing Wan, and Anzhong Wang "Gravitational collapse and formation of universal horizons in Einstein–æther theory," Phys. Rev. D 98, 064010(2018)

5.1 Introduction

Faster-than-light propagation does not violate causality [16]. In particular, gravitational theories with broken LI still allow the existence of black holes [76–78, 80, 82, 84, 85, 87, 106–126]. However, instead of Killing horizons, now the boundaries of black holes are hypersurfaces termed *universal horizons*, which can trap excitations traveling at arbitrarily high velocities. (For more details, see, for example, [16] for a recent review.) This universal horizon may radiate thermally at a fixed temperature and strengthen a possible thermodynamic interpretation though there is no universal light cone [83].

Once Lorentz Invariance is broken, different species of particles can travel with different (sometimes arbitrarily large) speeds. This suggests that black holes may exist only at low energies. At high energies, signals with sufficiently large speed emanated from inside an event horizon (EH) can escape to infinity. However, in contrast to this physical intuition, it was found that there still exist absolute causal boundaries, the so-called *universal horizons* (UHs), and even particles with infinitely large velocities would propagate along these boundaries and not escape to infinity [76, 77, 80]. This is closely related to the causality in LV theories of gravity. Since the speeds of particles can be arbitrarily large, similar to Newton's theory, to preserve causality, it is necessary to introduce a scalar field with a globally timelike gradient, the so-called *khronon*, which defines an absolute time, and all particles are assumed to move along its direction of increase, such that the causality in the sense of the past and the future

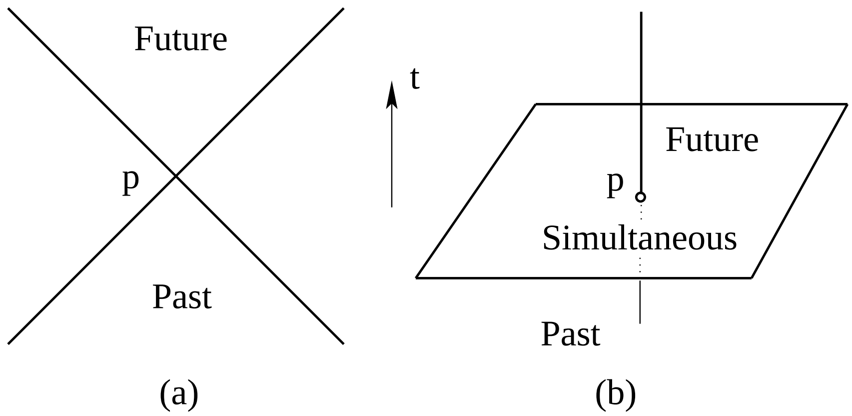

Figure 5.1: (a) The light cone of the event p in special relativity. (b) The causal structure of the point p in Newtonian theory.

is assured (Fig. 5.1). Then, in asymptotically flat stationary spacetimes, there might exist a surface on which the timelike translation Killing vector becomes orthogonal to the gradient of the khronon (Fig. 5.2).

Hence, a particle must cross this surface and move inevitably inward (towards the increasing direction of the khronon), no matter how large its speed. This is a one-way membrane, and particles even with infinitely large speed cannot escape from it once they are trapped inside. The membrane acts as an absolute horizon to all particles. UHs have been extensively studied (see e.g. [16] and references therein), including their thermodynamics [81, 83, 84].

In GR, it is well known that EHs can be formed from gravitational collapse of realistic matter, which implies that black holes with EHs as their boundaries exist in our universe. However, since particles with speeds larger than that of light exist in LV theories, such particles can cross the EHs and escape to infinity, even though initially located inside EHs. So, EHs in such theories are no longer one-way membranes. Instead, black hole boundaries are replaced by UHs, as argued above. Therefore, from the same astrophysical considerations as in GR, a key issue is whether UHs can also form from gravitational collapse in our universe [108,116,127]. In this chapter, we shall address this important issue in the framework of æther theory, which propagates three

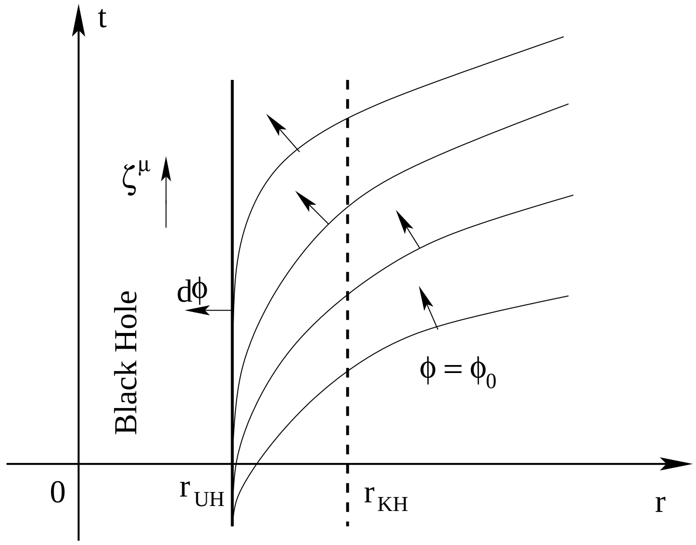

Figure 5.2: ϕ denotes the khronon field, and t and r are the Painlev-Gullstrand coordinates. The location of the Killing horizon, $r = r_{KH}$ is denoted by the vertical dashed line and the location of the universal horizon, $r = r_{UH}$ is denoted by the vertical solid line. r_{UH} is always less than $r = r_{KH}$. The Killing vector $\zeta = \delta^\mu{}_t$ always points upward at each point of the plane.

kinds of modes: the usual spin-2 graviton, the spin-1 and the spin-0 gravitons [13]. We numerically show the formation of dynamical UHs (dUHs), the generalization of UHs to dynamical spacetimes with spherical symmetry. We also find that the proper distance of the outermost dUH from the apparent (or spin-0) horizon keeps increasing on æther-orthogonal time slices. To our knowledge, this is the first time to show explicitly that dUHs can be formed from gravitational collapse.

5.2 Æther Theory and Spherical Collapse

Gravitational collapse of a spherical massless scalar field in æther theory has already been studied in some detail [27,128]. In particular, it was shown that for two different sets of c_is given by Eqs. (16) in [27], the first of which we will call GEJ1

$$c_1 = \frac{1}{3}, \quad c_2 = -\frac{1}{4}, \quad c_3 = \frac{1}{6}, \quad c_4 = -\frac{1}{12}, \tag{6.1}$$

and Eqs. (34) in [27] which we will call GEJ2, with $c_1 = 0.7$, $c_3 = c_4 = 0$, and

$$c_2 = \frac{-{c_1}^3}{2 - 4c_1 + 3{c_1}^2}. \tag{6.2}$$

Such that for GEJ2, the spin-0 horizon coincides with the usual metric horizon. i.e. $c_s = 1 = c_T$.

Both apparent horizons (AHs) and spin-0 horizons (S0Hs) are formed during the collapse [27], and the configurations finally settle down to the regular static black holes found numerically in [129]. For another set of c_i's, the collapse instead results in the temporary formation of a white hole horizon [128], although the corresponding static black hole exists [77]. It should be noted that neither GEJ1 nor GEJ2 satisfies the constraints defined by Eq. 2.41.

Therefore, in this chapter our goals are two-fold: First, we show that even within the range of the new constraints, AHs and S0Hs can be still formed from gravitational collapse. Second, dynamical UHs can be also formed. To achieve these goals, we choose to study the same setup as that studied in [27,128], closely following their notation and conventions. This will in particular allow us to check our numerical codes.

We choose the surfaces of constant time orthogonal to u^μ and the gauge that leads to the form of metric

$$ds^2 = \gamma_{ab} dx^a dx^b + \Phi^2 \left(d\theta^2 + \sin\theta^2 d\varphi^2\right), \tag{6.3}$$

where,

$$\gamma_{ab}\, dx^a dx^b = -\alpha^2 dt^2 + \left(dr + \beta^r dt\right)^2, \quad a, b = 0, 1; \tag{6.4}$$

where α (lapse function), β^r (shift vector) and Φ (area radius) are functions of $x^a = (t, r)$ only; and $u_\mu dx^\mu = u_a dx^a = -\alpha dt$, for which the time evolution vector is given by $t^\mu = \alpha u^\mu + \beta^\mu$ with $\beta^\mu \partial_\mu = \beta^r \partial_r$.

For the massless scalar field χ we have the matter Lagrangian

$$\mathcal{L}_m = -D_\nu \psi D^\nu \psi / (16\pi G_{æ}), \tag{6.5}$$

where $\psi \equiv \sqrt{8\pi G_{æ}}\,\chi$. The evolved quantities are then (ψ, P, K, a_r, Φ), where $P \equiv \mathcal{L}_u \psi$, K is the trace of the extrinsic curvature of constant-t surfaces, and a_r is the acceleration of the æther field. The dynamical equations and constraints are given, respectively, by [27]

$$\begin{aligned}
\dot{\psi} &= \alpha P + \beta^r \psi', && (6.6)\\
\dot{P} &= \beta^r P' + \alpha \left(PK + a^r \psi' + \psi'' + \frac{2\Phi'}{\Phi}\psi' \right), && (6.7)\\
\dot{K} &= \beta^r K' + \frac{\alpha}{3} K^2 + \frac{\alpha}{\Delta}\Bigg[2P^2 + 3\left(1 - c_{13}\right) Q^2 \\
&\quad + \left(c_{14} - 2\right)\left(a_r' + 2 a_r \frac{\Phi'}{\Phi} + a_r^2 \right) \Bigg], && (6.8)\\
\dot{a}_r &= \beta^r a_r' + \alpha \Bigg[\left(\frac{2K}{3} - Q \right) a_r + \frac{c_{13}}{c_{14}\left(1 - c_{13}\right)} P \psi' \\
&\quad - \frac{c_{123}}{c_{14}\left(1 - c_{13}\right)} K' \Bigg], && (6.9)\\
\dot{\Phi} &= \beta^r \Phi' + \alpha \Phi \left(\frac{Q}{2} - \frac{K}{3} \right), && (6.10)
\end{aligned}$$

and

$$\begin{aligned}
Q' &= -3Q\frac{\Phi'}{\Phi} + \frac{1}{1 - c_{13}} \left(\frac{\Delta}{3} K' - P\psi' \right), && (6.11)\\
\frac{\alpha'}{\alpha} &= a_r, && (6.12)\\
\beta^{r\prime} &= \alpha \left(Q + \frac{K}{3} \right), && (6.13)\\
\mathcal{C} &= \Phi'' + \frac{\Phi'^2 - 1}{2\Phi} + c_{14} a_r \Phi' + \frac{\Phi}{4}\Bigg[c_{14}\left(2a_r' + a_r^2\right) \\
&\quad + P^2 + \psi'^2 + \frac{3}{2}\left(1 - c_{13}\right) Q^2 - \frac{\Delta}{3} K^2 \Bigg] = 0, && (6.14)
\end{aligned}$$

with $Q \equiv K^r_r - K/3$, $\Delta \equiv 2 + c_{13} + 3c_2$, $\dot{\psi} \equiv \partial_t \psi$, $\psi' \equiv \partial_r \psi$, and so on. We consider the collapse of a massless scalar field of the form [27]

$$\psi = a_0 \, exp \left[-\frac{(r^2 - r_0^2)}{s^4} \right]. \tag{6.15}$$

It is initially at rest and it is a spherical shell of radius $r_0 = 10$, thickness $s = 4$ and amplitude $a_0 = 0.15$. P, Q, and K are chosen to be zero at a moment of time symmetry. a_r and ψ can be freely specified, so a_r is chosen to vanish and ψ is specified by Eq. (6.15).

The locations of the S0Hs is defined by

$$\tilde{\gamma}^{ab} n_a n_b = 0, \tag{6.16}$$

and AHs is defined by

$$\gamma^{ab} n_a n_b = 0 \tag{6.17}$$

where $n_a \equiv \partial_a \Phi$, $\tilde{\gamma}^{ab} = (\tilde{\gamma}^{-1})^{ab}$ and $\tilde{\gamma}_{ab} \equiv \gamma_{ab} + (1 - c_S^2) \, u_a u_b$. Hereafter, by a S0H/AH we shall denote an outer S0H/AH.

In stationary spacetimes, UHs are defined by

$$u_a \zeta^a = 0, \tag{6.18}$$

where $\zeta^a \partial_a$ is the time translation Killing vector [16, 76]. However, when spacetimes are dynamical, such a vector does not exist any longer. Following [16, 116] in defining a dUH, we first introduce the Kodama vector [130] (See also Refs. [131, 132]),

$$k^a \equiv \epsilon_{\perp}^{ab} n_b = \frac{(-\Phi_{,r}, \Phi_{,t})}{\alpha} \tag{6.19}$$

where $\epsilon_{\perp}^{ab}$ is the Levi-Civita tensor with $\epsilon_{\perp}^{01} = -1/\sqrt{-\gamma}$. It is clear that $k^a n_a = 0$. For asymptotically flat spacetimes there always exists a region with sufficiently large Φ in which n_a (k^a) is spacelike (time-like). An AH may form, say, at $r = r_{\mathrm{AH}}$, where n_a becomes null. Then, in the trapped region with $\gamma^{ab} n_a n_b < 0$, n_a (k^a) becomes timelike (spacelike). Hence, we define the location of a dUH as the surface at which

$$u_a k^a = 0, \tag{6.20}$$

where in the current case

$$u_a k^a = \Phi_{,r}. \tag{6.21}$$

Since u_a is globally timelike, Eq. (6.20) is possible only when k^a is spacelike. Clearly, this can be true only inside AH, that is, we must have $r_{\mathrm{dUH}} < r_{\mathrm{AH}}$. Eq. (6.20) may have multiple roots, and what is relevant is the outermost dUH, i.e. the one with the largest r (but not necessarily with the largest Φ). For the outermost dUH, we have $\Phi_{\mathrm{dUH}} < \Phi_{\mathrm{AH}}$ since $\Phi_{,r} > 0$ for $r > r_{\mathrm{dUH}}$. In the stationary spacetimes, the Kodama vector coincides with the time translation vector, and the above definition reduces to static spacetimes, and later generalized to various stationary spacetimes (see [16] and references therein).

5.3 Numerical Setup and Results

Our simulations are performed with a finite-differencing code. The initial data, numerical schemes and boundary conditions used in our code also closely follow [27]. The set of PDEs are solved on a uniformly spaced r-domain, where r is the proper radial coordinate spanning $[0, r_{\max}]$, $\quad r_{\max} = 80$ (or $= 320$) with a spacing of $\Delta r = 0.003125, 0.00625, 0.0125$, the high, medium and low resolutions, respectively. The timestep size is set to $0.2\times\Delta r$. In our code, the dynamical variables are integrated in time using an iterated Crank-Nicholson scheme with two iterations. We apply the 4th-order Kreiss-Oliger dissipation with an amplitude of 0.9 to the time-integration equations so as to damp out spurious high-frequency unstable modes of the solution. The non-dynamical variables Q, α and β^r are integrated through the r-domain at every time step using the trapezoidal method. The integration for α is done from $r = r_{\max}$, whereas that for Q and β^r are done from $r = 0$. Specifically, for smoothness

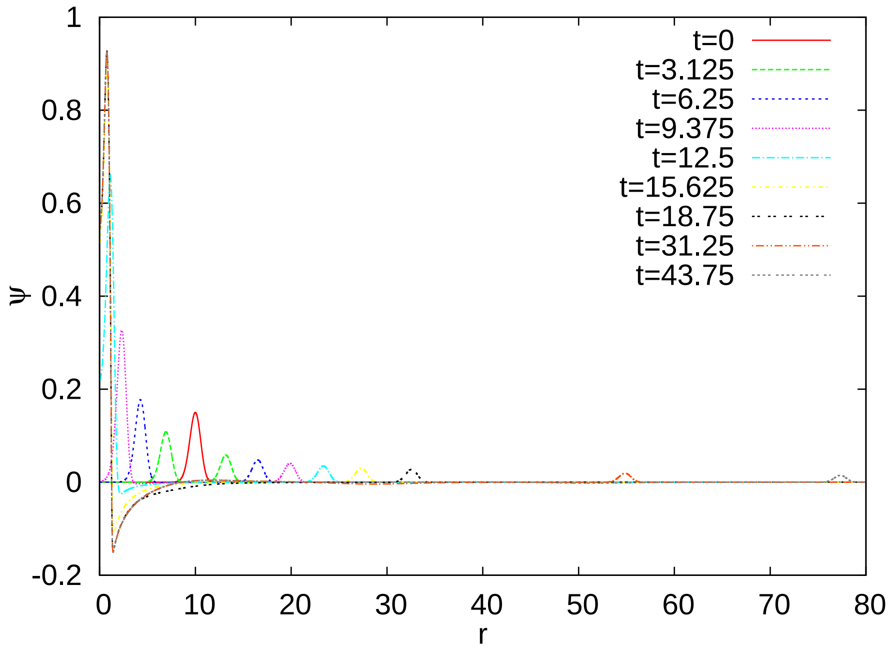

Figure 5.3: Evolution of the scalar field profile, Ψ for the case GEJ1, using a medium-resolution simulation.

we assume Q to be an even function of r and vanish at $r = 0$, and β^r an odd function of r. The boundary conditions for both dynamical and non-dynamical variables are imposed at every time step. We shall choose three sets of c_i's, GEJ1, GEJ2, and NC, where NC denotes the choice, $c_{13} = 0,\ c_2 = 2c_{14} = 2.0 \times 10^{-7}$, which satisfies the constraints of Eq. (2.41). For all three sets, the æther field is stable throughout and beyond the collapse of the scalar field to the central region. During the collapsing process, our code converges in a 2nd-order manner in line with the designed order of convergence of the numerical schemes. We further validate our code by reproducing the results of [27] for the parameter sets of GEJ1 and GEJ2. Different boundary conditions for Q and β^r at $r = 0$ or $r = \Delta r$ are tested, and we find that different boundary treatments do not affect the behavior of the PDE system in the bulk of the r-domain.

In our simulations for all the cases, the scalar field splits into two pieces, with one collapsing under its self-gravity toward $r = 0$ and the other traveling to $r \to \infty$

(Fig. 5.3). As the collapsing piece reaches the central region, we see the formations of the apparent, spin-0, and dynamical universal horizons at finite areal radii.

Fig. 5.4 shows the profiles of $\gamma^{ab}n_a n_b$, $\tilde{\gamma}^{ab}n_a n_b$, and $u_a k^a (= \partial_r \Phi)$ of GEJ1 shortly after the respective horizons are formed. The finite areal radii of these horizons are robust with respect to the resolutions used in this study, indicating that the system has almost completely converged at $\Delta r = 0.0125$, i.e., the low resolution (Fig. 5.4). From tests carried out using $r_{\rm max} = 80, 320$ at the medium resolution, we also see that the results are robust with respect to the size of the r-domain. At $t = 16.25$, a dUH forms at $r \approx 1.40$ ($\Phi \approx 0.95$).

For GEJ2, we track the collapsing process using our high-resolution simulation and similarly find the formation of all three horizons (Fig. 5.5). As noted in [27], the AH and S0H in this case coincide since $c_S^2 = 1$ and thus $\tilde{\gamma}^{ab} = \gamma^{ab}$. Hereafter, all results are obtained using high-resolution simulations, except for those with $r_{max} = 320$.

For NC, the AH forms at $t \approx 14$ and becomes quasi-stationary beginning at $t \approx 25$ with $\Phi \approx 0.8818$ (Fig. 5.6a). The S0H forms at $t \approx 14.625$ and achieves quasi-stationarity from $t \approx 31.25$ with $\Phi \approx 0.8210$ (Fig. 5.6b). At $t \approx 18.5$, a dUH forms as a double root of $\partial_r \Phi$ at $\Phi \approx 0.660$ ($r \approx 2.0$) (Fig. 5.6c). After that, the double root splits into two single roots, i.e. the inner (smaller r, larger Φ) and outer (larger r, smaller Φ) dUHs, and then the areal radius of the outer dUH decreases until it becomes almost constant at $t \approx 31.25$ with $\Phi \approx 0.6232$. The areal radius of the inner dUH becomes almost constant already at $t \approx 21.25$ with $\Phi \approx 0.6538$. At $t \approx 28.69$, an additional pair of dUHs forms outside the already existing pair and thus one of the new pair of dUHs becomes the outermost dUH. The areal radii of the new pair are between those of the old pair. At $t \approx 40.25$, one more pair of dUHs forms outside the two pairs and thus one of the newest pair becomes the outermost dUH. The areal radii of the newest pair are between those of the second pair (Fig. 5.6c). As time increases, the number of such pairs of dUHs keeps increasing, and one of the newest

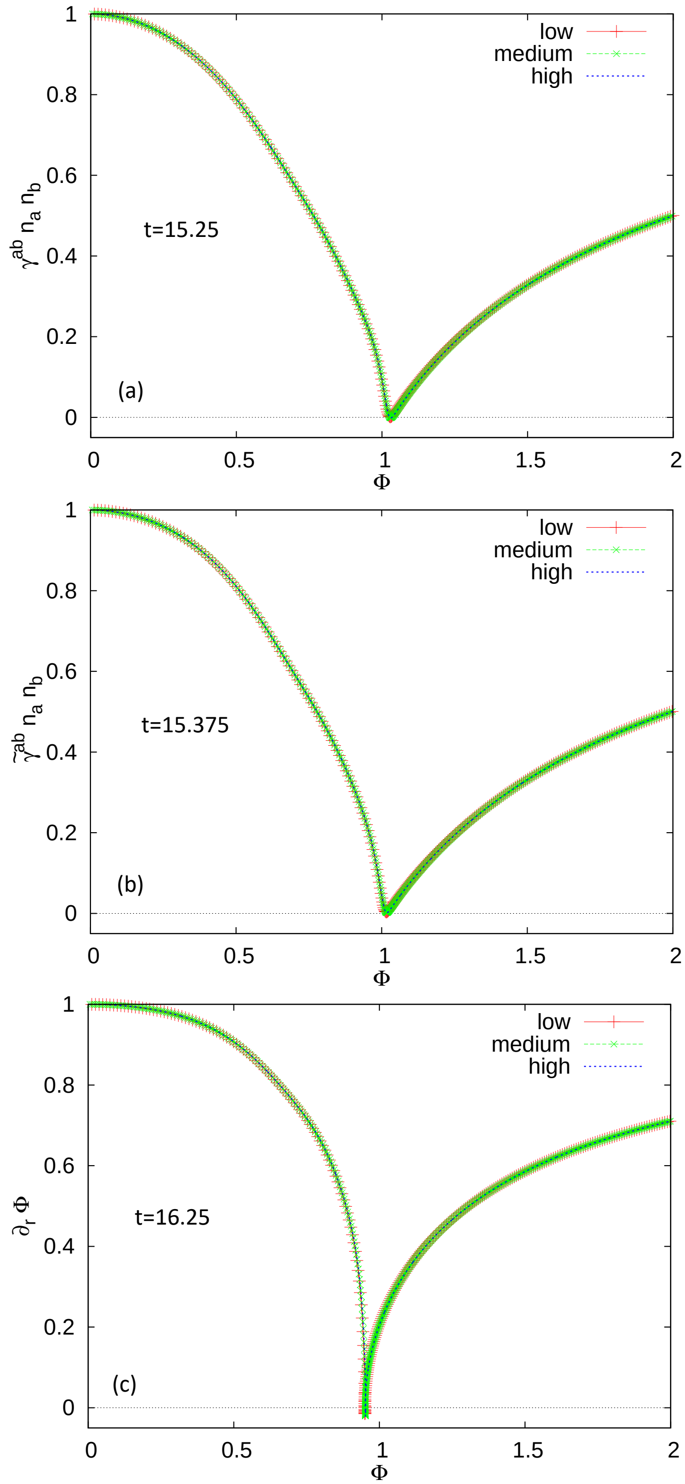

Figure 5.4: Formation of (a) AH, (b) S0H, and (c) dUH for GEJ1 at the respective times indicated in each panel. The almost complete overlap of the curves obtained from simulations with low, medium and high resolutions show that the system has almost completely converged at the low resolution of this study.

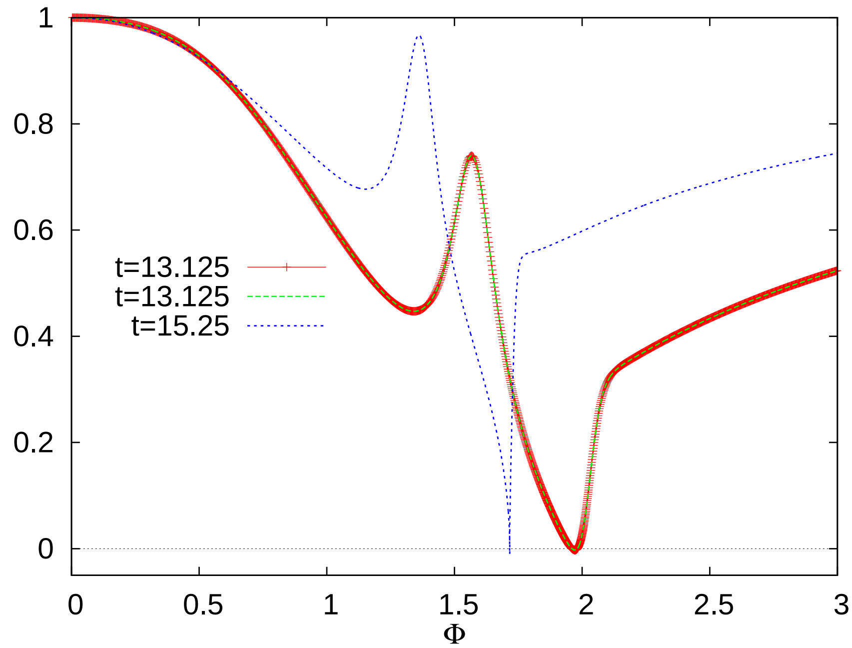

Figure 5.5: Formation of AH, S0H and dUH for GEJ2 at the respective times indicated in the legend. The red line with crosses represents the profile for $\gamma^{ab}n_a n_b$, the dashed green line for $\tilde{\gamma}^{ab}n_a n_b$, and the dashed blue line for $\partial_r\Phi$.

pair becomes the outermost dUH. This demonstrates that even after the first pair of dUHs (denoted by the two black squares in Fig. 5.6c has become stationary, the region outside i.e. with larger r (but with Φ's between the first pair of dUHs) is still highly dynamical. It is interesting to note that static black holes (in the decoupling limit) also have infinite layers of UHs [76]. In Fig. 5.7, we show some physical quantities nearby the locations of the dUHs. While their magnitudes are much higher than those in the surrounding regions, they do not exhibit any blow-up in time, indicating that the spacetime is regular at the locations of these horizons. We note that since we have imposed the smoothness condition at $r = 0$, our simulations do not show any blow-up of the curvature at $r = 0$.

Using the result of the medium-resolution simulation with $r_{\max} = 320$, we plot the change in the proper distance of the *outermost* dUH from both AH and S0H in Fig. 5.8. The fact that these distances become longer and longer as time

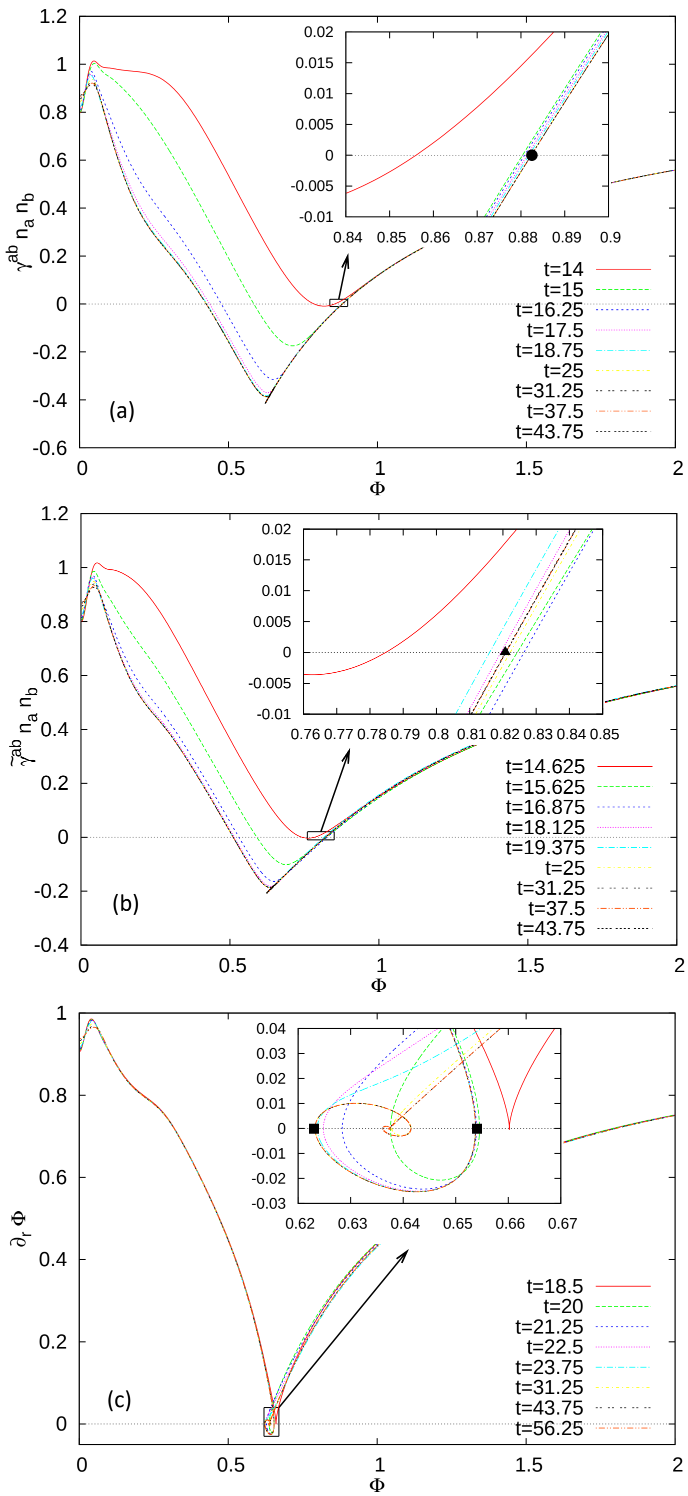

Figure 5.6: Locations of (a) AH (black dot in inset), (b) S0H (black triangle in inset), and (c) dUHs (black squares in inset) for NC. The red line in each plot indicates the profiles shortly after the respective horizons form.

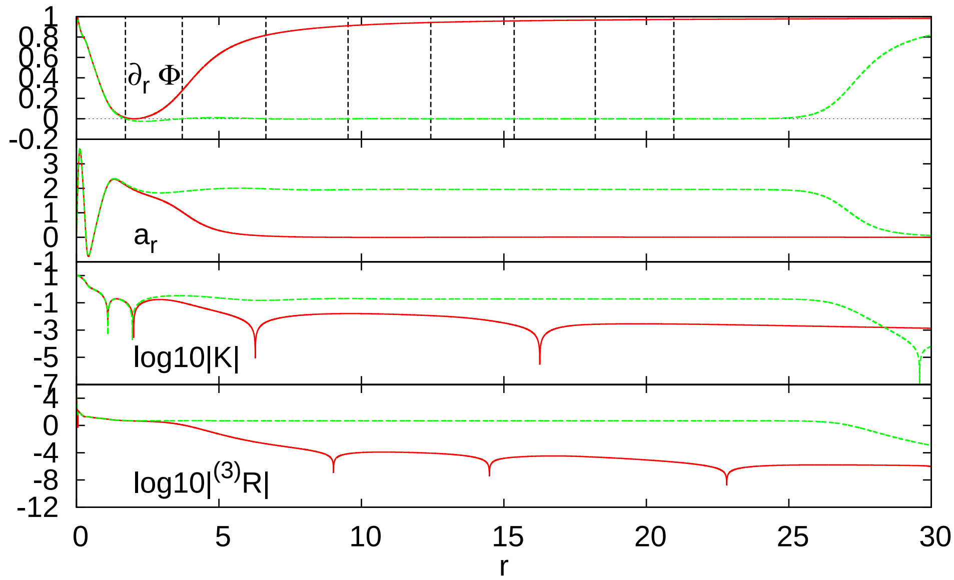

Figure 5.7: Some physical quantities vs r at $t = 18.5$ (solid red line) and $t = 56.25$ (dashed green line) for NC. The dashed black vertical lines in the top-most panel indicate the locations of various dUHs at $t = 56.25$.

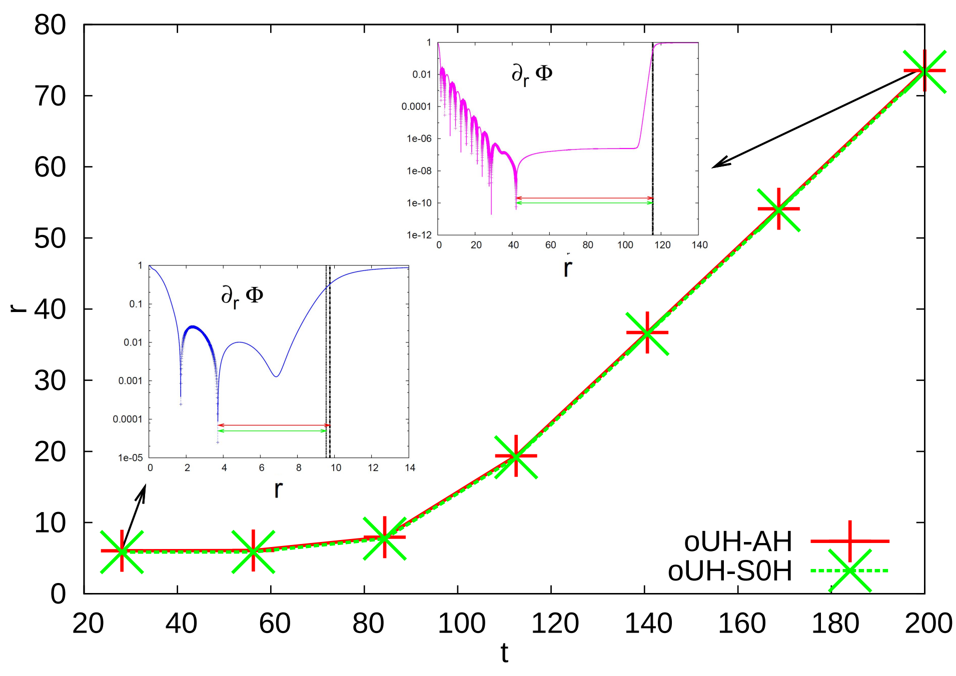

Figure 5.8: Proper distance r of the *outermost* dUH from AH labeled by oUH-AH and that from S0H labeled by oUH-S0H for NC.

progresses indicates that the *outermost* dUH is evolving into the causal boundary, even for excitations with large speeds of propagation.

5.4 Summary

In GR, EHs can be formed from gravitational collapse of realistic matter, so it strongly suggests that black holes with EHs as their boundaries exist in our universe. However, particles with speeds larger than that of light exist in gravitational theories with broken Lorentz symmetry, so those EHs are no longer the one-way membranes to such particles as they can cross those boundaries and escape to infinity, even if they are initially located inside them. Instead, now the black hole boundaries are defined by UHs. Therefore, astrophysically it is important to show UHs can also be formed from the gravitational collapse of realistic matter, so that even with respect to these particles black holes exist in our universe [108, 116, 127].

CHAPTER SIX

Conclusions and Future Directions

In the last four years, the direct detection of gravitational waves and a black hole opened a new window to the universe and have given gravitational physics a new dimension. The observation of a gravitational wave from a primordial black hole and/or the quantum fluctuations in the early universe will be a breakthrough and take us closer to the mystery of the origin of the universe. Thus it is very exciting to study theories of gravity that can be a true description of our universe. Most importantly it is essential to verify if general relativity is actually the true theory of gravity. Although all predictions made by Einstein's GR have passed all experimental tests so far, making it the most successful theory of gravity at low energy, there has been no success in the several attempts to quantize GR. Also, GR is unable to describe the physics at the singularity inside a black hole where it breaks down. Any observation which deviates from predictions of GR will provide a cause to consider alternate theories of gravity. Even in the absence of such groundbreaking observations, a systematical study of alternate theories to GR can lead to a more detailed theoretical understanding, vindicating GR's position as the correct classical description of gravity. Therefore, it is essential to keep investigating alternative theories of gravity that can lead to a sensible quantum theory of gravity and which also satisfy all experimental tests such that GR can emerge as an approximation to an alternate theory of gravity.

Black holes are the most intriguing and enigmatic objects of the universe. Therefore it is of utmost importance to study the existence of these objects in an alternative theory of gravity. In this thesis, we have mainly studied the existence of black holes and gravitational radiation in two alternative theories of gravity, namely

Hořava-Lifshitz theory and Einstein-æther theory. We studied the universal horizon for black holes and the existence of gravitational plane wave solutions in theories of gravity with broken Lorentz invariance.

We have found that in Hořava-Lifshitz gravity the existence of universal horizons and their thermodynamics are independent of the dimensions of spacetimes concerned. Therefore, the 2d Hořava gravity provides an ideal place to address the important issues of formation of black holes and their thermodynamics, which often technically become very complicated in higher dimensional spacetimes. Unlike in GR, in 2d, now we have a local degree of freedom due to the presence of the scalar field.

We have found that gravitational plane wave solutions exist in Einstein æther theory, but only for certain choices of the coupling constants c_i s. This is good news, as this prevents the æther theory from being ruled out completely as a potential alternative theory of gravity. We know that the æther field is always unit normed and timelike, while the gravitational plane waves propagate only along a null direction. So, spacetimes, in general, depend on both u and v coordinates due to the scattering between plane gravitational waves over the æther field. Thus, it would be very interesting to study the interactions of a plane gravitational wave with the æther and other matter fields, as well as with a gravitational plane wave propagating in the opposite direction. In this scenario, we can pay particular attention to Faraday rotations and the difference from those found in GR [97, 102], due to the presence of the timelike æther field, which violates LI.

Lastly, we have numerically studied the gravitational collapse of a massless scalar field with spherical symmetry in æther theory, and shown explicitly that all three kinds of horizons, *apparent, spin-0* and *dynamical universal*, can be formed from gravitational collapse, by considering three representative sets, GEJ1, GEJ2, and NC, of the free parameters c_i's. In the cases of GEJ1 and GEJ2, the collapse finally settles down to the regular static black holes found numerically in [129], although neither of these two cases satisfies the constraints of Eq. (2.41). In the case of NC, which satisfies Eq. (2.41), all three kinds of horizons are formed, and the spacetime in the neighborhoods of these horizons is well-behaved and regular, while the spacetime outside the apparent and spin-0 horizons soon settles down to a static configuration.

To conclude, we know, all experimental tests of GR so far are in the range of μm-AU, where gravity is very weak. If we think in terms of curvatures, the range in which we have tested GR appears to be very restricted. Neutron stars and stellar and intermediate-mass black holes can exhibit curvatures which are many orders of magnitudes larger than the usual weak-field experiments. Therefore, it will be interesting to understand the phenomena that occur in their vicinity in alternative theories of gravity. They are most likely the new frontier in gravitational physics. Also, an important prediction of GR is the existence of only two gravitational wave polarization tensor modes. Gravitational wave polarizations describe the distortion of the pattern of these waves as they propagate through spacetime. However, any alternate generic metric theory of gravity will allow only four additional polarization modes viz. two vector and two scalar modes [133]. Therefore, in the future, it would be interesting to study polarization in alternative theories of gravity as the observation of vector or scalar modes would be in direct conflict with GR.

APPENDIX

APPENDIX A

Field Equations for the Linearly Polarized Gravitational Plane Wave Background in Einstein-Æther Theory

For the spacetime of Eq. (4.13), the non-vanishing components of the Einstein tensor $G_{\mu\nu}$ and $T^{æ}_{\mu\nu}$ are given by

$$
\begin{aligned}
G_{00} &= \frac{1}{2}\Big(2U_{uu} - U_u^2 - V_u^2\Big),\\
T^{æ}_{00} &= -\frac{1}{8}\Big[2c_2U_{uu} + c_{13}\Big(V_u^2 + U_u^2\Big)\\
&\quad +2\big(c_{13}+c_2+3c_{14}\big)\Big(h_{uu} - h_uU_u - h_u^2\Big)\Big],\\
T^{æ}_{01} &= \frac{e^{-2h}}{4}\Big[c_2\Big(U_{uu} - 2h_uU_u - U_u^2\Big)\\
&\quad +\big(c_2+c_{13}-c_{14}\big)\Big(h_{uu} - h_uU_u - 2h_u^2\Big)\Big],\\
T^{æ}_{11} &= -\frac{e^{-4h}}{8}\Big[2c_2U_{uu} + c_{13}\Big(U_u^2 + V_u^2\Big)\\
&\quad +2\big(c_2+c_{13}-c_{14}\big)\Big(h_{uu} - h_uU_u - h_u^2\Big)\Big],\\
T^{æ}_{22} &= \frac{e^{V-U-2h}}{8}\Big[c_{13}\Big(2V_{uu} - V_u^2 - 2U_uV_u - 4h_uV_u\Big)\\
&\quad -\big(c_{13}+2c_2\big)\Big(2U_{uu} - U_u^2 - 4h_uU_u\Big)\\
&\quad -4c_2h_{uu} + 2\big(3c_2 - c_{13} + c_{14}\big)h_u^2\Big],\\
T^{æ}_{33} &= -\frac{e^{-(V+U+2h)}}{8}\Big[c_{13}\Big(2V_{uu} + V_u^2 - 2U_uV_u - 4h_uV_u\Big)\\
&\quad +\big(c_{13}+2c_2\big)\Big(2U_{uu} - U_u^2 - 4h_uU_u\Big)\\
&\quad +4c_2h_{uu} - 2\big(3c_2 - c_{13} + c_{14}\big)h_u^2\Big],
\end{aligned}
\tag{A.1}
$$

and $E_\mu = \big(E_0, E_1, 0, 0\big)$, where

$$
\begin{aligned}
E_0 &= -Æ_1e^{2h} = -\frac{e^{-h}}{4\sqrt{2}}\Big[2c_2U_{uu} + c_{13}\Big(U_u^2 + V_u^2\Big)\\
&\quad +2\big(c_2+c_{13}-c_{14}\big)\Big(h_{uu} - h_uU_u - h_u^2\Big)\Big].
\end{aligned}
\tag{A.2}
$$

In the vacuum case, we have $T^m_{\mu\nu} = 0$, and the Einstein-æther equations Eq. (2.29) reduce to

$$G_{\mu\nu} = T^{æ}_{\mu\nu}, \tag{A.3}$$

which yield five independent equations

$$2U_{uu} - \Big(V_u^2 + U_u^2\Big) + 2c_{14}\Big(h_{uu} - h_u U_u - h_u^2\Big) = 0, \tag{A.4}$$

$$c_2\Big(U_{uu} - 2h_u U_u - U_u^2\Big) + \big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_u U_u - 2h_u^2\Big) = 0, \tag{A.5}$$

$$2c_2 U_{uu} + c_{13}\Big(U_u^2 + V_u^2\Big) + 2\big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_u U_u - h_u^2\Big) = 0, \tag{A.6}$$

$$c_{13}\Big(2V_{uu} - V_u^2 - 2U_u V_u - 4h_u V_u\Big) - \big(c_{13} + 2c_2\big)\Big(2U_{uu} - U_u^2 - 4h_u U_u\Big) - 4c_2 h_{uu} + 2\big(3c_2 - c_{13} + c_{14}\big)h_u^2 = 0, \tag{A.7}$$

$$c_{13}\Big(2V_{uu} + V_u^2 - 2U_u V_u - 4h_u V_u\Big) + \big(c_{13} + 2c_2\big)\Big(2U_{uu} - U_u^2 - 4h_u U_u\Big) + 4c_2 h_{uu} - 2\big(3c_2 - c_{13} + c_{14}\big)h_u^2 = 0, \tag{A.8}$$

where in Eq. (A.4) we have used the fact that $T^{æ}_{00}$ can be expressed in terms of $T^{æ}_{11}$ which is equal to zero.

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