ABSTRACT

Hubbard Trees and Their Properties

Cordell Hammon, Ph.D.

Mentor: Jonathan Meddaugh, Ph.D.

A Hubbard tree is a set of points at the core of dendritic Julia sets. These trees encapsulate all the information about the larger Julia set, but in a much smaller, easier to understand structure. We discuss the structure of Hubbard trees, in particular, we provide a useful definition of branch point and endpoint. Afterwards, we demonstrate the existence of some trees that cannot be Hubbard trees in any meaningful context. Later, we turn our attention to the structure of inverse limits of Hubbard trees, making use of the definition of branch point and endpoint tenured earlier in the work and demonstrate that one Hubbard tree, with minor alterations, can generate infinitely many mutually non-homeomorphic inverse limits.

Hubbard Trees and Their Properties

by

Cordell Hammon, B.S.

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Approved by the Dissertation Committee

Jonathan Meddaugh, Ph.D., Chairperson

Brian Raines, D. Phil.

David Ryden, Ph.D.

Scott Varda, Ph.D.

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J. Larry Lyon, Ph.D., Dean

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To Ayra

CHAPTER ONE

Introduction and Preliminaries

1.1 Overview

There is an old joke that goes:

Q: What does the "B." in Benoit B. Mandelbrot stand for?

A: Benoit B. Mandelbrot.

The black shape seen in Figure 1.1 is called the Mandelbrot set and is named after Benoit B. Mandelbrot who popularized it in 1982 [Man82], although it was likely first described by Brooks and Matelski in 1981 [BM81]. We say "likely first described" because the question of "who discovered the Mandelbrot set?" is surprisingly difficult to answer. The 1990 article by Horgan [Hor90] does a fantastic job of digging into the history of the beloved shape.

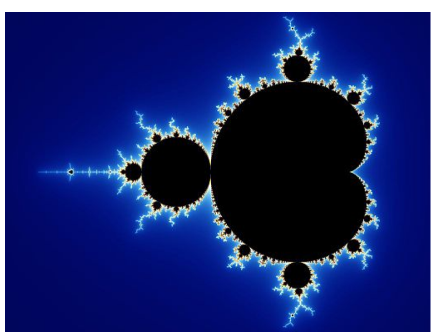


Figure 1.1. The Mandelbrot Set

The beginning joke is funny (or, at the very least, funny to some mathematicians) because the Mandelbrot set is "quasi-self-similar". That is, when one zooms in on certain portions of the Mandelbrot set, one can find another almost exact copy of the Mandelbrot set. In fact any neighborhood of a point in the boundary of the Mandelbrot set, contains infinitely many embedded copies of the Mandelbrot set [Mil89]. Such a zoom in can be found in Figure 1.2. This figure is magnified roughly two hundred thousand times compared to Figure 1.1. Both figures were created by Wolfgang Beyer with the program Ultra Fractal 3., CC BY-SA 3.0 http://creativecommons.org/licenses/by-sa/3.0/, via Wikimedia Commons.

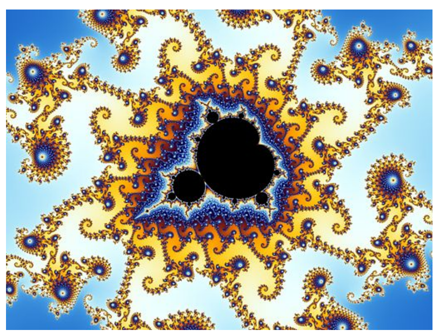


Figure 1.2. A zoomed in Mandelbrot set centered at $-.743643135 + .131825963i$

To see a gif zooming in on the Mandelbrot set scan the QR Code in Figure 1.3. The gif was created by Simpsons contributor at English Wikipedia, Public domain, via Wikimedia Commons. For those people reading a digital version of this dissertation, you can view the gif by clicking here.

The Mandelbrot set, denoted $\mathcal{M}$, is of primary importance in the study of iteration of complex polynomials and, not surprisingly, is constructed via iteration of complex valued

Figure 1.3. A QR Code for a gif zooming into the Mandelbrot Set

polynomials. For a given complex number, c we define $f_c(z) : \mathbb{C} \to \mathbb{C}$ by $f_c(z) = z^2 + c$. The Mandelbrot set is defined as follows:

Definition 1.1.1: $\mathcal{M} = \{c \in \mathbb{C} : \lim_{n\to\infty} f_c^n(0) \neq \infty\}$.

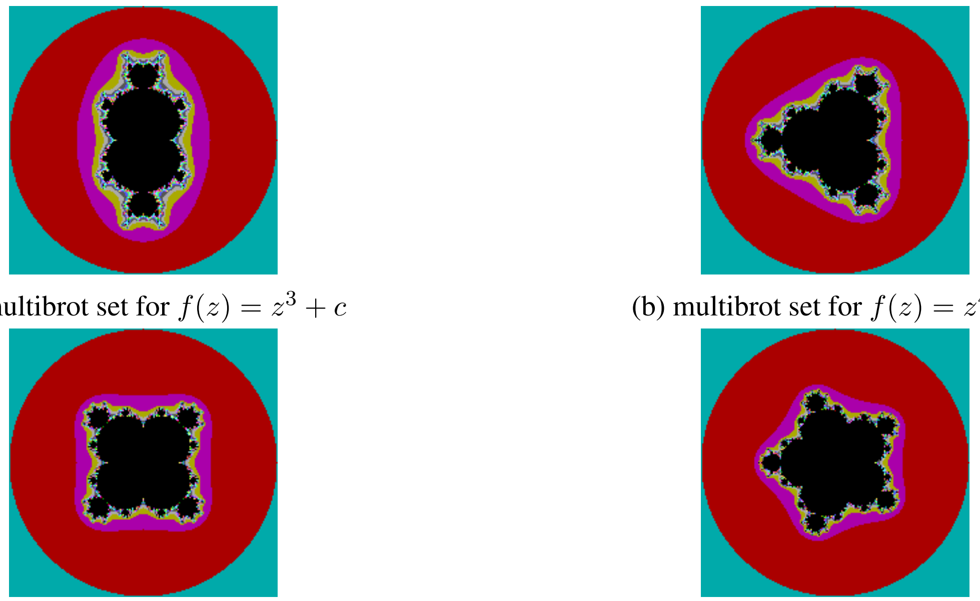
So, for example, consider $c = i$. Then $f_i(z) = z^2 + i$. Now consider the points to which 0 maps under iteration of f_i.

- $f_i(0) = 0^2 + i = i$,
- $f_i(i) = i^2 + i = -1 + i$,
- $f_i(-1 + i) = (-1 + i)^2 + i = -i$,
- $f_i(-i) = (-i)^2 + i = -1 + i$

From here, if we continue to iterate f_i we get repeating outputs $-i, -1 + i, -i, -1 + i, \ldots$ and so it is clear that $\lim_{n\to\infty} f_i^n(0) \neq \infty$. Thus i is in the Mandelbrot set. In this way, the Mandelbrot set is made by varying c and iterating $f_c(z) = z^2 + c$ starting with $z = 0$.

While there is only one set known as "The Mandelbrot Set" (the set associated with the function $f(z) = z^2 + c$), there are similar sets corresponding to different functions

$f(z) = z^d + c$. These sets are often called "multibrot sets" to distinguish them from "The" Mandelbrot set. Figure 1.4 shows the multibrot sets for the powers 3, 4, 5, and 6. All multibrot set images were created by Cuddlyable3, Public domain, via Wikimedia Commons.



(a) multibrot set for $f(z) = z^3 + c$

(b) multibrot set for $f(z) = z^4 + c$

(c) multibrot set for $f(z) = z^5 + c$

(d) multibrot set for $f(z) = z^6 + c$

Figure 1.4. Multibrot sets for $f(z) = z^d + c$ for $d \in \{3, 4, 5, 6\}$

To see a video of various multibrot sets for $f(z) = z^d + c$ as d varies from zero to eightscan the QR Code in Figure 1.5. For those people reading a digital version of this dissertation, you can view the video by clicking here. The video was created by George, Public domain, via Wikimedia Commons.

There is another structure ; similar to the Mandelbrot set and closely related but discovered nearly a half century earlier. This structure is the Julia set of a polynomial. Whereas

Figure 1.5. A QR Code for a video of various multibrot sets for $f(z) = z^d + c$ as d varies from zero to eight

$\mathcal{M}$ is made by fixing $z = 0$ and varying c in $f_c(z) = z^2 + c$, the Julia set of $f_c(z)$ is made by fixing c and varying the inputs z.

Definition 1.1.2: For a complex polynomial, f, the filled Julia set of f, denoted $K(f)$, is the set $\{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) \neq \infty\}$. The Julia set of f, denoted $J(f)$, is the boundary of $K(f)$, i.e. $J(f) = \partial K(f)$.

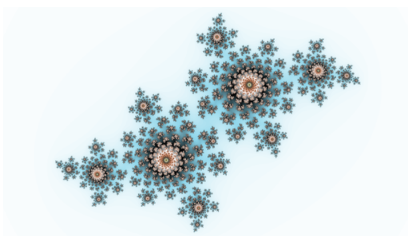
If the polynomial is of the form $f_c(z) = z^2 + c$, we write K_c and J_c for the filled Julia set of $f_c(z)$ and the Julia set of $f_c(z)$.

Julia sets are named after Gaston Julia, who first described them in [Jul18]. Examples of Julia sets can be found in Figure 1.6. Figure 1.6b is cropped from the original by Morn, CC BY-SA 4.0 https://creativecommons.org/licenses/by-sa/4.0, via Wikimedia Commons. Figure 1.6c was made by Adam majewski, CC BY 3.0 https://creativecommons.org/licenses/by/3.0, via Wikimedia Commons.

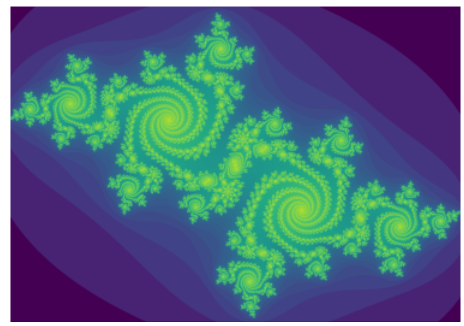
So, although there is only one Mandelbrot set, each complex parameter c has its own Julia set. As is evident from the definitions, the Mandelbrot sets and Julia sets are closely linked. In fact, a result from [Jul18] which states that K_c is connected if and only if K_c

contains 0, gives an alternate characterization of the Mandelbrot set, equivalent to that found in Definition 1.1.1.

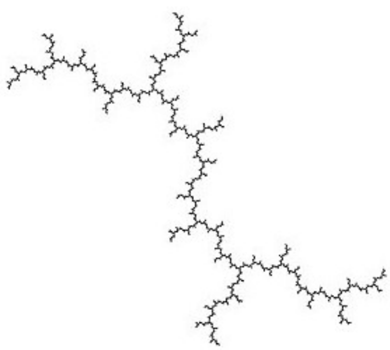
Theorem 1.1.3: $\mathcal{M} = \{c \in \mathbb{C} : 0 \in K_c\} = \{c \in \mathbb{C} : K_c \text{ is connected}\}$.



(a) Julia set of $c = -0.4 + 0.6i$



(b) Julia set of $c \approx -0.5125 + 0.5213i$



(c) The Julia set of $c = i$

Figure 1.6. Examples of Julia sets

But there is a deeper connection still. In [Tan90] it was shown that, if $c \in \mathcal{M}$, then sufficiently small neighborhoods of c in $\mathcal{M}$ and J_c are asymptotically similar [that is, they behave similarly under iteration of f_c and more visually inspiring, they neighborhoods around c in $\mathcal{M}$ and J_c look indistinguishable]. For some truly inspiring visualizations of the similarities between J_c and $\mathcal{M}$ around c we point the interested reader to [Tan90], in particular Figures 9 - 12. Along these lines of connection, it has been shown that boundary of the Mandelbrot set has Hausdorff dimension 2 and, there exists a dense set of points in c in $\partial\mathcal{M}$ for which J_c has Hausdorff dimension 2 [Shi98].

The Julia set found in Figure 1.6c is K_i [i.e. the filled Julia set for $f_i(z) = z^2 + i$]. A special property of this filled Julia set is that it has no interior [i.e. $J_i = K_i$]. Such Julia sets are called *dendritic Julia sets* and always arise when 0 is strictly pre-periodic under f_c but can arise when 0 is periodic under f_c. If 0 is strictly pre-periodic under f_c, then c is a *Misurewicz point* so named after Michał Misurewicz who studied pre-periodic points in [Mis81]. The name "dendritic" Julia set follows from the fact that these Julia sets are, in fact, dendrites. We fully define the term dendrite in Definition 1.2.2.

In 1984 and 1985 Adrien Douady and John Hubbard released the monumental *Étude dynamique des polynômes complexes. Partie I.* and *Étude dynamique des polynômes complexes. Partie II.* ([DH84; DH85]). These books are collectively referred to as the "Orsay Notes" since they are based on notes written by Douady for a course he taught on holomorphic dynamical systems at Paris-Sud 11 University, Orsay in 1983-84.

In the Orsay Notes, Douady and Hubbard focus on holomorphic dynamical systems and, in particular, the iteration of those polynomials $x^2 + c$ whose Julia set is dendritic. They showed that to each dendritic Julia set, J_c, there belongs a core, called a Hubbard tree. The Hubbard tree is the the convex hull — in J_c — of the orbit of 0 under f_c. When studying dendritic Julia sets, it is often sufficient to study the Hubbard tree, a much simpler structure. In fact, the Hubbard tree captures all the dynamics of the dendritic Julia set in the sense that, if T_c is the Hubbard tree of $f_c(z) = z^2 + c$, then $J_c = \overline{\bigcup_{n\in\omega} f_c^{-n}(T)}$. For more information about Hubbard trees and their relation to Julia sets, we point the interested reader to [Poi10].

Like many things in mathematics, the concept of "Hubbard tree" has been generalized. These generalized Hubbard trees — which are officially called "generalized Hubbard trees" — are trees, T, along with associated functions, f, such that the pair (T, f) shares many of the same properties that a standard Hubbard tree and its quadratic polynomial possess and are the main focus of this work. The full definition of generalized Hubbard tree is given in Definition 1.2.17.

However, Hubbard trees and Julia sets are not the only tools used to understand dynamical systems. One such useful tool is that of the inverse limit. Inverse limits originated in the field of category theory but have been quickly adopted by those studying dynamical systems. We give the dynamical systems definition here.

Definition 1.1.4: Given a sequence of spaces $\langle X_i \rangle_{i\in\mathbb{N}}$ and functions $f_i : X_{i+1} \to X_i$, we say the *inverse limit* of the system $\{X_i, f_i\}$, denoted $\varprojlim\{X_i, f_i\}$, is the subset of the product space $\prod X_i$ to which the point x belongs if and only if $f_i(x_{i+1}) = x_i$ for all $i \in \mathbb{N}$.

This is especially useful if each X_i, f_i is the same (e.g. the polynomial $f(z) = z^2 + c$ iterating on $\mathbb{C}$). In this case, instead of writing $\varprojlim\{X_i, f_i\}$ we simply write $\varprojlim\{X, f\}$. In this case, an inverse limit is a single object that is related to a dynamical system and that captures all the dynamics of the associated system. Each point in the inverse limit space is a thread that contains the entire history of a single point in the underlying space X. Similarly, for any point $p \in X$, one can consider the set of all the points in $\varprojlim\{X, f\}$ which have p in their first coordinate. This set of points is the set of all possible paths one can take to arrive at p.

For the purposes of this paper, each space X_i is the same space and each function f_i is the same

Inverse limits of Julia sets and other closely related structures have been studied by many (e.g. [Bal07], [RŠ07], [Ing95], [Ing00]). In this paper we will study, in part, the various functions f that can be associated with a single tree T such that (T, f) is a generalized Hubbard tree, as well as the various inverse limits which arise from these different functions.

1.2 Preliminary Definitions, Notation, and Concepts

We will use $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ are the sets of positive integers, integers, real numbers, and complex numbers, respectively, and we let $\omega = \mathbb{N} \cup \{0\}$ denote the set of nonnegative integers.

For a function $f : X \to X$, f^n denotes the composition of f with itself n times (with f^0 being the identity function). For $x \in X$, the *orbit of x under f* is the set $Orb_f(x) = \{f^n(x)\}_{n\in\omega}$. If f is clear from context we write $Orb(x)$ for $Orb_f(x)$. If $f^n(x) = x$ for some $n \in \mathbb{N}$, then x is said to be a *periodic point of period n*. If n is minimal in this regard, we say that x has *prime period n*. If there exists a minimal $n \in \mathbb{N}$ such that $f^n(x)$ is periodic under f but $f^{n-1}(x)$ is not periodic under f, we say that x is *pre-periodic* with respect to f.

We adopt the following notation concerning sequences. A sequence in a space X is a function whose domain is some (possibly unbounded) interval in $\mathbb{Z}$. A sequence is *right-infinite* if its domain is of the form $[n, \infty)$ for some $n \in \mathbb{Z}$. A right-infinite sequence with unspecified domain will be assumed to have domain equal to ω. Similarly, a *left-infinite sequence* is one with domain of the form $(-\infty, n]$ for some $n \in \mathbb{Z}$ and a left-infinite sequence with unspecified domain will be assumed to have domain $(-\infty, 0]$. A *bi-infinite sequence* is one with domain equal to $\mathbb{Z}$. A *finite sequence* is one with domain of the form $[n, m]$ for some $n < m \in \mathbb{Z}$ and is said to have length $m - n + 1$. A finite sequence of length l and unspecified domain will be assumed to have domain equal to $[0, l-1]$.

It will be useful in the development of our results to discuss the restriction of a (finite or infinite) sequence to a smaller domain. If $[n, m]$ is a subset of the domain of a sequence z, we use $z_{[n,m]}$ to denote the restriction of z to the domain $[n, m]$ and z_n to denote the image of n. If z is left or right infinite, we use $z_{(-\infty,n]}$ or $z_{[n,\infty)}$ to denote the appropriate restricted sequence.

For clarity and convenience, it will be useful to tell whether a sequence is bi-inifite, left-infinite, or right-infinite at a quick glance. To make this distinction, we will use a bar, left-arrow, or right-arrow over a symbol to emphasize that it is bi-infinite ($\overline{z}$), left infinite ($\overleftarrow{z}$), or right infinite ($\overrightarrow{z}$), respectively. If $\overline{z}$ is a bi-infinite sequence and $n \in \mathbb{Z}$, we will use $\overleftarrow{z}_n$ to denote the left-infinite sequence $\overline{z}_{(-\infty,n]}$ and $\overrightarrow{z}_n$ is the right infinite string $\overline{z}_{[n,\infty)}$. We will also use this system of notation for domain-restriction on left/right-infinite sequences, e.g. if $\overrightarrow{z}$ is a right-infinite sequence, then $\overrightarrow{z}_n = \overrightarrow{z}_{[n,\infty)}$ and $\overleftarrow{z}_n = \overleftarrow{z}_{(-\infty,n]}$ (and similarly for left-infinite sequences).

Finally, we define the *shift map,* σ on the space of right- or bi-infinite sequences as follows. If α is a right-infinite (or bi-infinite sequence), then the shift of α, denoted $\sigma(\alpha)$, is the right-infinite (or bi-infinite) sequence β such that $\beta_n = \alpha_{n+1}$ for all n in the domain of α. So, if $\alpha = \alpha_0\alpha_1\alpha_2\ldots$ then $\sigma(\alpha) = \alpha_1\alpha_2\alpha_3\ldots$ Note that σ is invertible on the space of bi-infinite sequences.

We end our barrage of definitions associated with sequence with an operation that helps us work with sequences.

Definition 1.2.1: We use $\frown$ to notate string concatenation. For example, if $\alpha = 110$ and $\beta = ABC$ then $\alpha{}^\frown\beta = 110ABC$. Furthermore, we use an overline to denote infinite repeating strings. So $\overline{1} = 111111\ldots$

Our primary method for investigation of Julia sets will be through the use of *abstract dendritic Julia sets*, which we formally define in Definition 1.2.10 after the following preliminary notions which are adapted from [Bal07].

A *continuum* is a compact, connected, metric space. A continuum X is *uniquely arcwise connected* provided that, for any two distinct points $p, q \in X$, there is exactly one arc in X with p and q as endpoints. We will use $[p, q]$ to denote this arc, and define $(p, q) = [p, q] \setminus \{p, q\}$.

Definition 1.2.2: A *dendrite* is a uniquely arcwise connected, locally connected continuum.

Note that finite acyclic graphs consisting of exactly one non-trivial component are dendrites. However, dendrites can also have significantly more complicated structure.

Definition 1.2.3: We say that a function $f : D \to D$ is *locally one-to-one* at a point $x \in D$ if there is a neighborhood U of x such that $f|_U$ is one-to-one. A point at which f is not locally one-to-one will be called a *turning point* of f. A map from a dendrite into itself will be called *unimodal* if it has at most one turning point. A map $f : D \to D$ is *locally eventually onto* (l.e.o) if for any open set $U \subset D$ there exists $n \in \omega$ with $f^n(U) = D$.

Lemma 1.2.4: Let D be a dendrite and $f : D \to D$ be continuous. Then f is injective on any connected subset which does not contain a turning point.

Proof. Suppose x and y are distinct points of D with $f(x) = f(y)$. Let $p = f(x)$ and let $I = f([x, y])$. If I is a singleton, then every point in $[x, y]$ is a turning point. Otherwise, since I is a dendrite, I has at least two endpoints (one of which may be p). Let q be an endpoint of I distinct form p and let $r \in [x, y]$ be a preimage of q. Then f cannot be locally one-to-one at r so r is a turning point. □

Definition 1.2.5: Let $f : D \to D$ be a unimodal map on a dendrite with turning point t. We define a *leg* of D (with respect to t) to be a component of $D - \{t\}$ and a *pseudoleg* of D to be a union of legs of D on which f is one-to-one.

Definition 1.2.6: Let $f : D \to D$ be a unimodal map on a dendrite with turning point t and let $\mathcal{S}$ be a partition of $D \setminus \{t\}$ into (pseudo)legs.

The *itinerary of a point* $x \in D$ with respect to f and $\mathcal{S}$ is the sequence $\iota(x, f, \mathcal{S}) = \iota_0\iota_1\iota_2\ldots$ of labels defined by $\iota_n = k$ if and only if $f^n(x) \in S_k$.

The *kneading sequence* of the function f with respect to $\mathcal{S}$, written $\tau(f, \mathcal{S})$, is defined as the sequence $\iota(t, f, \mathcal{S})$. If f, $\mathcal{S}$, and t are obvious from context, we write $\iota(x)$ for $\iota(x, f, \mathcal{S})$ and τ for $\tau(f, \mathcal{S})$.

Definition 1.2.7: If $\alpha = a_0a_1\ldots a_n$ is periodic with $\alpha_0 = *$, and $\beta = b_0b_1\ldots$ are two strings, we define the product $\alpha \circ \beta = b_0a_1a_2\ldots a_nb_1a_1a_2\ldots a_nb_3\ldots$.

We say a kneading sequence is a *composite sequence* if it is the product of two strings, and *prime* otherwise.

It is common in the literature to use a $*$ in place of $\circ$ in the definition above. We use $\circ$ instead since for us $*$ signifies the third symbol of our alphabet.

Definition 1.2.8: Let $f : D \to D$ be a unimodal map on a dendrite with turning point t and let $\mathcal{S}$ be a partition of $D \setminus \{t\}$ into (pseudo)legs. The map f has the *unique itinerary property* with respect to $\mathcal{S}$ provided that if $\iota(x) = \iota(y)$, then $x = y$.

Definition 1.2.9: A unimodal map $f : D \to D$ with turning point t is said to be *self-similar* if there is a partition $\mathcal{S}$ of $D \setminus \{t\}$ into (pseudo)legs so that for each (pseudo)leg M, $f(M \cup \{t\}) = D$.

Definition 1.2.10: An *abstract dendritic Julia set* is a pair (D, f) where D is a dendrite and $f : D \to D$ is a unimodal map such that there is a partition $\mathcal{S}$ of $D \setminus \{t\}$ into (pseudo)legs with respect to which f is self-similar and has the unique itinerary property.

An abstract dendritic Julia set is said to be *quadratic* provided that no point has more than two pre-images under f. In this case, we say that the dendrite is a triple, (D, H, f) such that $f : D \to D$ is a unimodal map such that there is a partition $\{S_1, S_0\}$ of $D \setminus \{t\}$ into (pseudo)legs such that $f(S_0 \cup \{t\}) = f(S_1 \cup \{t\}) = D$. H is the Hubbard tree of f and is the convex hull of $Orb_f(t)$ in D.

From here on, anytime we discuss an abstract dendritic Julia set, we mean a quadratic abstract dendritic Julia set.

Note that the class of quadratic abstract dendritic Julia sets contains the class of dendritic Julia sets arising from the iteration of quadratic complex polynomials [Bal10, Theorem 2.5]. In this case, for itinerary purposes, we denote the set containing the turning point with "$*$". So our partition of D is $\mathcal{S} = \{S_0, S_1, *\}$.

Furthermore, in this case, the kneading sequence τ of such a Julia set satisfies the criterion set forth in the following definition.

Definition 1.2.11: A sequence τ is said to be *acceptable* if it begins with $*$, is not constant, $\tau_n = *$ implies $\sigma^n(\tau) = \tau$, and for every $n \in \mathbb{N}$ with $\sigma^n(\tau) \neq \tau$, there exists $k \in \omega$ with $* \neq \tau_k \neq \sigma^n(\tau)_k \neq *$.

It will often be useful to talk about the *space* of itineraries. Let $P = \{0, 1, *\}$ with topology given by $\{\varnothing, \{0\}, \{1\}, \{0, 1, *\}\}$. Then P^ω (with the product topology) is called itinerary space. When considering τ as an element of the itinerary space, acceptability simply indicates that the iterates of τ are either equal to τ or can be separated by open sets in P^ω.

We can identify a quadratic abstract Julia set (D, H, f) with a subset of itinerary space defined solely in terms of its kneading sequence as follows.

Definition 1.2.12: Let τ be the kneading sequence of the quadratic abstract dendritic Julia set (D, f). An element $\alpha \in P^\omega$ is called τ-*admissible* provided that

1. $\sigma^n(\alpha) = \tau$ if $\alpha_n = *$ and,

2. for all $i \in \omega$ with $\sigma^i(\alpha) \neq \tau$, there exists $n \in \omega$ with $* \neq \alpha_{i+n} \neq \tau_n \neq *$.

Note that τ-admissibility is also a condition concerned with the ability to separate a point from the kneading sequence.

Definition 1.2.13: Let τ be an acceptable sequence in P^ω. Define $D_\tau = \{\alpha \in P^\omega : \alpha \text{ is } \tau\text{-admissible}\}$.

With this terminology in place, the following result is a rephrasing of results of Baldwin [Bal10, Theorems 2.2 and 2.4].

Theorem 1.2.14: (D, H, f) is a quadratic abstract dendritic Julia set if and only it is conjugate to (D_τ, σ) for some acceptable sequence τ. In particular, this conjugacy holds if and only if τ is the kneading sequence of (D, f).

It is worth noting that the family of quadratic abstract dendritic Julia sets contains the family of dendritic Julia sets that arise in complex dynamics, as is seen in the following theorem, again slightly paraphrased from [Bal10].

Theorem 1.2.15: Let $f_c(z) = z^2 + c$. If J_c is a dendrite, then there is an acceptable τ such that (J_c, f_c) is conjugate to (D_τ, σ).

Since we will be identifying dendrites with sets of right-infinite sequences in the itinerary space, the usual notation for inverse limits becomes cumbersome. Since the bonding map that we will be using is the shift map σ, the following conventions are useful.

Remark 1.2.16: Let A be a set of right-infinite sequences in P^ω and $\sigma : A \to A$ the shift map. Then the inverse limit of the inverse system $\{A, \sigma\}$ can be identified with the set of bi-infinite sequences

$$\{\overline{a} \in P^{\mathbb{Z}} : \overrightarrow{a}_{-n} \in A \text{ for all } n \in \omega\}$$

by identifying the point $\langle \overrightarrow{a_i} \rangle_{i \in \omega} \in \varprojlim\{A, \sigma\}$ with the point $\overline{a} \in P^{\mathbb{Z}}$ whose j-th coordinate is the 0-th coordinate of $\overrightarrow{a_{-j}}$ if j is non-positive and is equal to the j-th coordinate of $\overrightarrow{a_0}$ if j is non-negative.

As we have seen, for each dendritic Julia set there is an associated Hubbard tree. As we are concerned with *abstract* dendritic Julia sets, we will make use of the generalized notion of Hubbard trees.

Definition 1.2.17: A *generalized Hubbard tree* (hereafter referred to as "Hubbard tree") is a pair (T, f) where T is a tree and $f : T \to T$ is a function with a distinguished point $t \in T$, satisfying the following conditions:

1. f is continuous (and surjective);
2. every point in T has at most two preimages under f;
3. f is locally one-to-one at all points other than t;

4. all endpoints of T are in the orbit of t;

5. t is periodic or preperiodic, but not fixed;

6. if $x \neq y$ are branch points or in the orbit of t, then there is an integer $n \geq 0$ such that $t \in f^n([x, y])$.

Note that throughout the paper we will sometimes use the term “Hubbard Tree” to refer to the dynamical system (T, f) and sometimes to refer to the underlying space, T. Context will make evident what we mean.

Just like in the case of absract dendritic Julia sets, we partition generalized Hubbard trees into three sets $\mathcal{S} = \{S_1, S_0, *\}$ where $*$ is the set containing the turning point t, S_1 is the component of $T \setminus *$ which contains $f(t)$ and $S_0 = T \setminus (S_1 \cup *)$.

CHAPTER TWO

Branch Points and The Structure of Hubbard Trees

2.1 Branch Points

Branch points and endpoints are essential in the study of Hubbard trees. A common definition of endpoints comes from [Bin51] and is as follows:

Definition 2.1.1: We say a point x in a continuum X is an *endpoint* if for any two subcontinuua, A, B of X, each of which contain x, either $A \subseteq B$ or $B \subseteq A$.

The problem, though, is that under this definition, the point x in Figure 2.1 is not an endpoint since the arc from x to y and the arc from x to z both contain x, but neither contains the other.

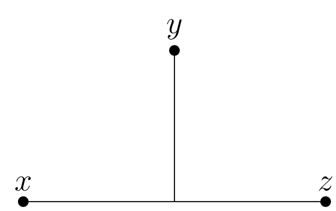


Figure 2.1. Comparison Between Different Definitions of Endpoint

Definition 2.1.1 is useful is the context of arclike continua, but we need a definition for endpoints better suited for our purposes. It warrants mentioning, that endpoint only makes sense in the context of 1-dimensional continua. But our definition also must ensure that not all 1-dimensional continua have endpoints. Circles should not have endpoints. So, in order to properly define *endpoint* we need to have a definition of dimension. We defer to classical notion of dimension coming from [Leb21].

Definition 2.1.2: We call a space X is n*-dimensional* if, given any open cover $\mathcal{U}$ of X, there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ such that no point in X is contained in more than $n+1$ sets in $\mathcal{V}$ and n is maximal with respect to this property.

Definition 2.1.3: We say a point x in a 1-dimensional continuum X is of *degree at least* n if there exist n subcontinua of X, call them $A^1, A^2, \ldots, A^n$, such that $\{x\} \subsetneq A^i$ for each $i \in \{1, \ldots, n\}$ and $A^i \cap A^j = \{x\}$ for $1 \leq i < j \leq n$.

We further say that x is of *degree exactly* n, if x is degree at least n and not degree at least $n+1$, i.e. there is no collection of $n+1$ subcontinua satisfying the aforementioned criteria. Typically, instead of saying “of degree exactly n”, we say “of degree n”. If x is of degree $n > 2$, then we say x is a *branch point*, and if x is a point of degree exactly 1, then we call it an *endpoint*. If there exist infinitely many subcontinua $A^1, A^2 \ldots$ such that $\{x\} \subsetneq A^i$ for each $i \in \mathbb{N}$ and $A^i \cap A^j = \{x\}$ for $i, j \in \mathbb{N}$, then we call x a branch point of degree infinity. This can be countable or uncountable infinity. An example of a branch point with uncountable degree is the vertex of the cone of the Cantor set. However, for our purposes here, every degree infinity point from hereafter will have countable degree.

We denote the degree of x in X via $\deg_X(x)$. If the ambient space is made clear from context, we will write $\deg(x)$.

This definition of branch point is consistent with the notion of saying x is the branch point of an n-od, but not of an $(n+1)$-od. The definition of endpoint is similar to the standard notion of endpoint found in [BM95; Bin51]. The difference is that under Definition 2.1.3 we would say that the point x in Figure 2.1 is an endpoint.

Armed with an understanding of branch points and endpoints, we can now discuss the structure of Hubbard trees.

2.2 The Structure of Hubbard Trees

Recall that when we say "Hubbard Tree" we usually mean "generalized Hubbard tree" which is a pair (T, f) where T is a tree and $f : T \to T$ is a function with a unique "turning point" t, satisfying the following conditions:

1. f is continuous and surjective;
2. every point in T has at most two preimages under f;
3. f is locally one-to-one at all points other than t;
4. all endpoints of T are in the orbit of t;
5. t is periodic or preperiodic, but not fixed;
6. if $x \neq y$ are branch points or in the orbit of t, then there is an integer $n \geq 0$ such that $t \in f^n([x, y])$.

Some examples of Hubbard trees can be seen in Figure 2.2.

The following notion from [BKS09] is helpful. Let (T, f) and (T_0, f_0) be two Hubbard trees with turning points t and t_0 respectively. Let $P = \{p : p \text{ is a branch point of } T \text{ or in } Orb_f(t)\}$ and $Q = \{q : q \text{ is a branch point of } T_0 \text{ or in } Orb_{f_0}(t_0)\}$. We say (T, f) and (T_0, f_0) are equivalent if there is a bijection, $g : P \to Q$ which is respected by the

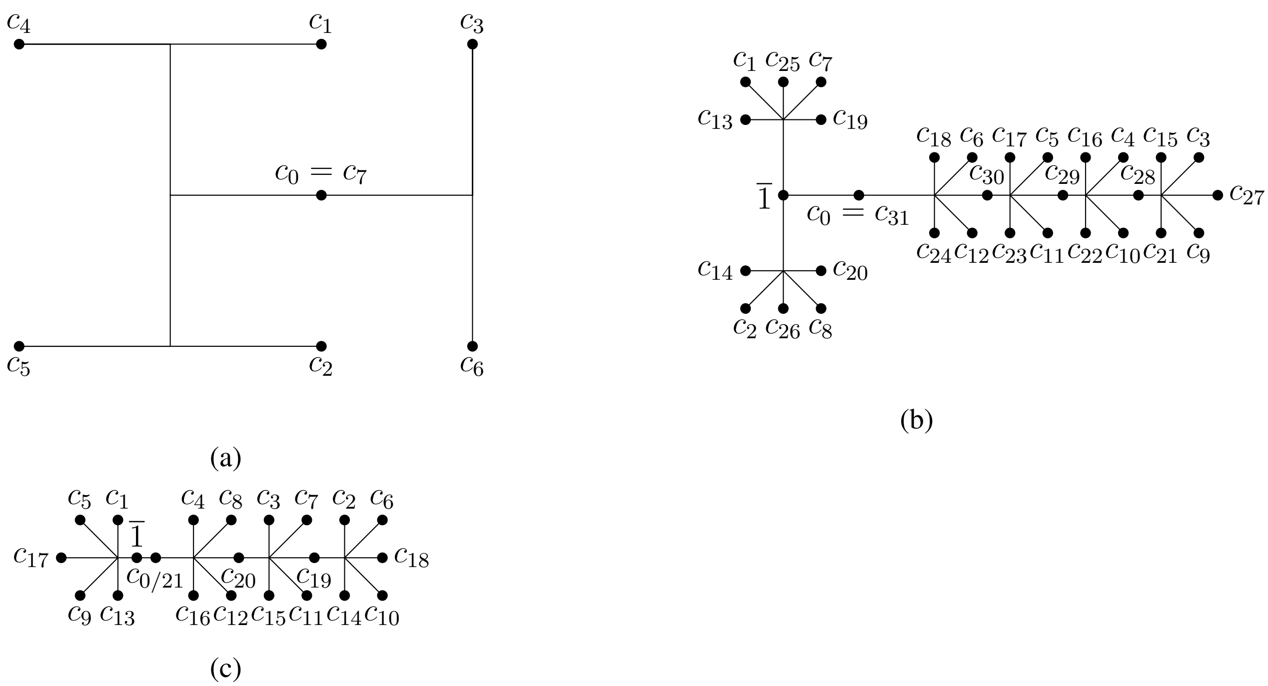


Figure 2.2. Examples of Hubbard trees

dynamics, and if $p_1, p_2 \in P$ such that $(p_1, p_2) \cap P = \varnothing$ then $(g(p_1), g(p_2)) \cap Q = \varnothing$. This is weaker than a topological conjugation.

Recall that two systems (X, f), (Y, g) are topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $f = h^{-1} \circ g \circ h$. However, for (T, f) and (T_0, f_0) to be considered equivalent, all we need is for $f|_P$ and $f_0|_Q$ behave the same; $f|_{T \setminus P}$ and $f_0|_{T_0 \setminus Q}$ can behave very differently.

The following Lemma, rephrased from [BKS09, Lemma 2.3], details some of the basic properties of Hubbard trees.

Lemma 2.2.1: The turning point t divides the tree into two parts and $f(t)$ is an endpoint. Each branch point is periodic or pre-periodic and never maps onto the critical point. If b is a periodic branch point and p is in the orbit of b, then $\deg(p) = \deg(b)$. Lastly, any arc which does not contain the turning point in its interior maps homeomorphically onto its image.

Also recall that we can break the T into three sets, $*, S_1$, and S_0 where $* = \{t\}$, S_1 is the component of $T \setminus *$ which contains $f(t)$ and S_0 is the other component of T. With this in mind, we can prove the following useful fact.

Lemma 2.2.2: $f|_{\overline{S_i}}, i \in \{0,1\}$ is an embedding.

Proof. $f|_{\overline{S_i}}$, $i \in \{0,1\}$ is injective, surjective onto its range, and continuous. Moreover, $\overline{S_i}$ is a continuum. The continuous image of a continuum is a continuum, so $f(\overline{S_i})$ is a continuum. Thus $f|_{\overline{S_i}} : \overline{S_i} \to f(\overline{S_i})$ is a homeomorphism by [Mun00, Theorem 26.6]. $\square$

Moreover, the fracturing of T into $*, S_1$, and S_0 allows us to, given a point $p \in T$, define the itinerary of p as the infinite string $\iota(p) = p_0p_1p_2\dots$ where

$$p_i = \begin{cases} * \text{ if } f^i(p) \in * \\ 1 \text{ if } f^i(p) \in S_1 \\ 0 \text{ if } f^i(p) \in S_0 \end{cases}$$

The itinerary of the turning point t is called the *kneading sequence* of (T, f) and is typically denoted τ. Lastly, we often refer to the turning point as the *critical point* — owing to the fact that in Hubbard trees of quadratic polynomials $f(z) = z^2 + c$, the turning point of T is the critical point of f — and denote the critical point c_0. Further, we often write c_i to denote the $f^i(c_0)$. Generally speaking, a Hubbard tree can have multiple points which each share the same itinerary, however, if there are two points p, p' with the same itinerary, than every point in $[p, p']$ has the same itinerary [BKS09, Lemma 2.8]. If a Hubbard tree is such that no two points have the same itinerary, we say that the Hubbard tree has the *unique itinerary property.*

Remark 2.2.3: The language of itineraries allows a convenient rephrasing of the sixth property of Hubbard trees: Let (T, f) be a Hubbard tree and let $I = \{p \in T : p$ is a branch point or in $Orb(t)\}$. Then no two points in I have the same itinerary.

Definition 2.2.4: We say a kneading sequence, τ is *acceptable* if $\tau_0 = *, \tau \neq \overline{*}$, if $\tau_n = *$ then $\tau_{n+j} = \tau_j$ for all $j \in \omega$, and for all $n \in \omega$ there exists some $j \in \omega$ such that $* \neq \sigma^n(\tau)_j \neq \tau_j \neq *$.

This definition indicates that any shift of τ can be sepeartaed from τ in P^ω. While this may seem random and arbitrary, it leads to the following beautiful result which is a combination of Theorem 2.29 and Proposition 3.17 of [Bal07].

Lemma 2.2.5: If τ is acceptable, then there is a Hubbard tree with kneading sequence τ which has the unique itinerary property.

With this in mind, for the rest of the chapter, we treat all Hubbard trees as having the unique itinerary property. If a tree has the unique itinerary property, then we refer to a point and its itinerary interchangeably. So, if a point p has itinerary $\overline{110}$ then we refer to it as both p *and* $\overline{110}$ and make no distinction.

Kneading sequences fully classify Hubbard trees, as the following Theorem from [BKS09, Proposition 3.5] says:

Theorem 2.2.6: Any two Hubbard trees with the same kneading sequence are equivalent.

Not only can kneading sequences be used to tell if two Hubbard trees are equivalent, a kneading sequence can be used to construct a Hubbard tree. In fact:

Theorem 2.2.7: Every periodic or pre-periodic kneading sequence is realized by a unique (up to equivalence) abstract Hubbard tree.

But how can we use the kneading sequence alone to build the Hubbard tree? For any three points, s, t, u in a tree T, the intersection $[s, t] \cup [t, u] \cup [u, v]$ is a single point. If the three points are colinear, then this intersection gives the point in the middle. If they are not colinear, then this intersection is the unique branch point which separates each point in $\{s, t, u\}$ from the other two.

We use what is called the voting sequence which, as shown in shown in [Bal07], gives the itinerary of the point at this intersection. The voting sequence is defined as follows:

Definition 2.2.8: Given three points s, t, u of a Hubbard tree and a critical point, τ, the *voting sequence* of s, t, u is the infinite string $V(s, t, u)$ where

$$V(s,t,u) = \begin{cases} s_0 \frown V(\sigma(s), \sigma(t), \sigma(u)) \text{ if} & s_0 = t_0 = u_0 \\ \tau \text{ if} & \{s_0, t_0, u_0\} = \{0, 1, *\} \\ s_0 \frown V(\sigma(s), \sigma(t), \sigma(\tau)) \text{ if} & s_0 = t_0 \neq u_0 \\ s_0 \frown V(\sigma(s), \sigma(\tau), \sigma(u)) \text{ if} & s_0 = u_0 \neq t_0 \\ t_0 \frown V(\sigma(\tau), \sigma(t), \sigma(u)) \text{ if} & t_0 = u_0 \neq s_0 \end{cases}$$

Notice, then, that order of the triple does not matter. So $V(s,t,u) = V(s,u,t) = V(t,s,u)$ etc.

Example. Suppose $\tau = \overline{*110}, s = \overline{10*1}, t = \overline{110*}$, and $u = \overline{0*11}$. Then since $s_0 = t_0 \neq u_0$ we have $V(s,t,u) = 1^\frown V(\sigma(s), \sigma(t), \sigma(\tau))$. Now, since $\sigma(s) = u, \sigma(t) = s$, and $\sigma(\tau) = t$, we have $V(s,t,u) = 1^\frown V(u,s,t) = 1^\frown V(s,t,u)$. Thus $V(s,t,u) = \overline{1}$.

If $|Orb(\tau)| = n$ then τ and its shifts give us the itinerary of n different points in the Hubbard tree. Importantly, among these n itineraries are the itineraries of every endpoint. For each branch point, b, $T \setminus \{b\}$ has at least three components, so there exists at least three end points e_1, e_2, e_3 with $b = [e_1, e_2] \cap [e_2, e_3] \cap [e_1, e_3]$. Thus, running the voting sequence on every possible set of three distinct shifts of τ gives us the itinerary every branch point in the Hubbard tree. Running the voting sequence on every possible triple comprised of branch points or endpoints gives us enough information to determine where each branch point must lay in relation to each other. This then defines the tree. With this in mind, we can use the voting sequence to build a tree from a kneading sequence.

So, in our previous example, we know there is a branch point with itinerary $\bar{1}$ that lies between s, t and u. If we run the voting sequence on all possible sets of 3 shifts of τ we find that there is only one branch point and its itinerary is $\bar{1}$.

The tree made from $\tau = \overline{*110}$ can be found in Figure 2.3.

A fixed branch point with itinerary $\bar{1}$ is not unique to this kneading sequence. The following Lemma is a rephrasing of [Bal07, Theorem 1.22].

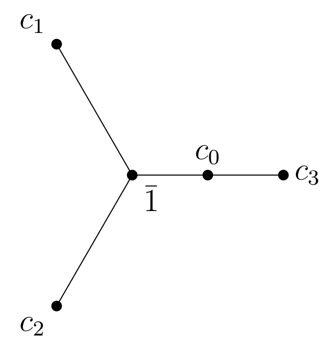


Figure 2.3. Hubbard tree with kneading sequence $\overline{*110}$

Lemma 2.2.9: In every Hubbard tree, there exists a fixed point with itinerary $\overline{1}$. Moreover, if the tree's kneading sequence τ begins $*1^n0\ldots$ then the fixed branch point will be degree $n+1$.

We always call this point *the fixed branch point* even though its degree may be only two.

Knowing that every Hubbard tree has a fixed point in S_1 begs the question, do they all have a fixed point in S_0?

Theorem 2.2.10: If (H, f) is a Hubbard tree with a fixed point in S_0, then that point must be an endpoint.

Proof. Suppose (H, f) is a Hubbard tree with kneading sequence τ and a point p with $p = \overline{0}$. By way of contradiction, suppose that no endpoint has itinerary $\overline{0}$. Then, since p is not an endpoint of H, there exists an endpoint e such that $p \in [c_0, e]$. Then $V(p, \tau, e)$ gives the itinerary of $[c_0, p] \cap [p, e] \cap [c_0, e] = p$. Since $e \neq \overline{0}$, there is some minimal n such that $f^n(e) \notin S_0$. Then $V(\overline{0}, \tau, e) = 0^{n-1\frown} V(\overline{0}, \sigma(\tau), f^n(e))$. Since $\sigma(\tau)$ and $f^n(e)$ both do not start with a 1, we get

$$V(\overline{0},\tau,e) = 0^{n-1}1^\frown V(\sigma(\tau),\sigma^2(\tau),f^{n+1}(e))$$

or $0^{n-1}\tau$. Neither of these sequences are the itinerary of p, a contradiction. Thus, p is an endpoint. □

Corollary 2.2.11: If a Hubbard tree has a fixed point in S_0, then the kneading sequence terminates in $\overline{0}$.

Proof. As previously seen, if there is a point with itinerary $\overline{0}$ then it is an endpoint. The kneading sequence must contain the itineraries of each endpoint and thus must contain $\overline{0}$. □

So if we know an endpoint has itinerary $\overline{0}$, then we know something about the kneading sequence. This works in reverse too. If we know something about the longest block of 0s in the kneading sequence, then we know something about the endpoints.

Theorem 2.2.12: Let (H, f) be a Hubbard tree with kneading sequence τ. Let $n \in \mathbb{N}, k \in \mathbb{N} \cup \{\infty\}$ be such that $\tau_{[n,k)}$ is a string of all 0s and $k-n$ is maximal with respect to this property (note that since the turning point of a Hubbard tree must be (pre)periodic, this is well defined). If there are multiple blocks of the same length, let n be minimal. Then c_n is an endpoint of H. If $\tau_k = *$ then c_{n+1} is not an endpoint.

Proof. Let n and k be as stated. If $k = \infty$, then from Theorem 2.2.10 we know that c_n is an endpoint of H. So, suppose $k < \infty$. If c_n were not an endpoint, then there would be be some endpoint e with $c_n \in (c_0, e)$ which means $[c_0, e] \cap [c_n, e] \cap [c_0, c_n] = c_n$. With this in mind, we can demonstrate that c_n is an endpoint by verifying $V(c_0, c_n, e) \neq c_n$ for all endpoints e.

Furthermore, instead of running the voting sequence $V(c_0, c_n, c_j)$ for all $j \in \omega$, we are able to restrict to $V(c_0, c_n, c_j)$ for all $j < n$. This is because f is locally injective at all points except c_0. So the only non-endpoint that can map to an endpoint is c_0. So, if there is some $t > 0$ such that c_{n+t} is an endpoint, then there must be some $m < t$ such that $c_{n+m} = c_0$. So if there is some $j > n$ such that c_j is an endpoint satisfying $c_n \in (c_0, c_j)$, then there is some $j' < n$ with $c_{j'} = c_j$.

With all this in mind, fix $0 < j < n$. If $c_j \in S_1$ then $V(\tau, c_j, c_n) = \tau \neq c_n$. If, on the other hand, $c_j \in S_0$, let m be minimal so that $f^m(c_j) \notin S_0$. By assumption, $m < k - n$ so $f^m(c_n) \in S_0$. Thus $V(\tau, c_j, c_n) = 0^{m-1\frown} V(\sigma(\tau), f^m(c_j), f^m(c_n)) = 0^{m-1}1^\frown V(\sigma^2(\tau), f^{m+1}(c_j), \sigma(\tau)) \neq c_n$. Thus there is no endpoint e such that $c_n \in (c_0, e)$. Therefore, c_n is an endpoint.

Now, we show c_{n+1} is not an endpoint if $\tau_k = *$ that is, if the block of 0's is the last block before τ repeats. Notice that $f^{k-n-1}(c_n) \in S_0$ and $f^{k-n-1}(c_{n+1}) = c_0$. So $V(\tau, c_n, c_{n+1}) = 0^{k-n-1\frown} V(\sigma(\tau), f^{k-n-1}(c_n), \tau) = 0^{k-n-1}\tau$. $\square$

We end our discussion on the structure of Hubbard trees with a demonstration that these trees can have a variety of forms.

Theorem 2.2.13: Hubbard trees can have arbitrarily many branch points of arbitrarily large degree.

Proof. To construct a Hubbard Tree with n colinear branch points of degree m in S_0, a fixed branch point of degree x in S_1, x many branch points of degree m in S_1, and no other branch points, one can use the following kneading sequence:

$$\tau = \overline{*(1^x 0^n)^{m-1}}.$$

(See A.0.1).

An example of such a Hubbard tree with 4 branch points of degree 6 can be found in Subfigure 2.2b which has kneading sequence

$$\tau = \overline{*(110000)^5}.$$

Moreover, to construct a Hubbard Tree with exactly n branch points, each of degree m one can let $x = 1$. In this case we get the following kneading sequence:

$$\tau = \overline{*(10^{n-1})^{m-1}}.$$

(See A.0.2).

An example of a Hubbard tree with exactly 4 branch points each of degree 6 can be found in Subfigure 2.2c. This tree has kneading sequence $\tau = \overline{*(1000)^5}$. $\square$

CHAPTER THREE

Trees That Cannot Be Hubbard Trees

Results like those in Section 2.2 may lead one to believe that any tree can be a Hubbard tree if it is equipped with the right function. But this is not true. In order to see why, we first introduce a few notions.

Definition 3.0.1: We say two points p, q from a tree, T, are in a *free arc* if (p, q) contains no branch points of T.

Definition 3.0.2: We say a tree T is *Hubbardizable* if there exists a function $f : T \to T$ such that (T, f) is a generalized Hubbard tree.

Definition 3.0.3: Given a point p in a tree T, the *trees emanating from* p are the closures of the components of $T \setminus \{p\}$.

Definition 3.0.4: For a tree, T we use end(T) to denote the set of endpoints of T.

Lemma 3.0.5: If T, W are trees with $W \subset T$, then $|\text{end}(W)| \leq |\text{end}(T)|$.

Proof. Let b be a branch point in W. Since $W \subset T$, b is also a branch point of T. For each endpoint, $e \in \text{end}(W)$ there is at least one endpoint $e' \in \text{end}(T)$ such that $e \in [b, e']$.

So, for each endpoint e in W choose such a corresponding endpoint e' in T. This gives an injection from end(W) to end(T) proving the claim. $\square$

Definition 3.0.6: Given a Hubbard Tree, (H, f), let b be the fixed point in S_1. The existence of such a point is guaranteed by Lemma 2.2.1. We define $T_0, T_1, ...T_n$ be the trees emanating from b enumerated in the order which the critical point travels to them, i.e. $S_0 \subset T_0$. For simplicity of notation, we define $T_{n+1} = T_0$ and we use T_0^* to denote $T_0 \cap S_1$.

Lemma 3.0.7: For all $0 < i \leq n$, $f(T_i) \subseteq T_{i+1}$ and $f(T_0^*) \subseteq T_1$.

Proof. Fix $0 \leq i \leq n$. If $0 = i$ let $T_i = T_0^*$. Then $T_i \subset S_1$ and T_i has $\overline{1}$ as an endpoint. By definition of T_{i+1} we have $f(T_i) \cap T_{i+1} \neq \varnothing$. Suppose there exists a point $p \in T_i$ with $f(p) \notin T_{i+1}$. Let $q \in T_i \setminus \{\overline{1}\}$ be such that $f(q) \in T_{i+1}$. Then, $f([p, q])$ meets two different trees T_j and T_{i+1} where $j \neq i + 1$ and thus $\overline{1} \in (f(p), f(q))$. So there exists a point $c \in (p, q)$ with $f(c) = b$. But $f(b) = b$ and $f|_{T_i}$ is injective, a contradiction. $\square$

Lemma 3.0.8: Let T, W be two trees. If $|\text{end}(T)| > |\text{end}(W)|$ then T cannot map injectively into W.

Proof. Suppose $f : T \to W$ is injective. Let $W' = f(T)$. Then f is a bijection between the continuua T and W' and is thus a homeomorphism between the two. So T and W' must have the same number of endpoints. But, by Lemma 3.0.5, $|\text{end}(W')| \leq |\text{end}(W)| < |\text{end}(T)|$, a contradiction. $\square$

This immediately leads to a nice corollary:

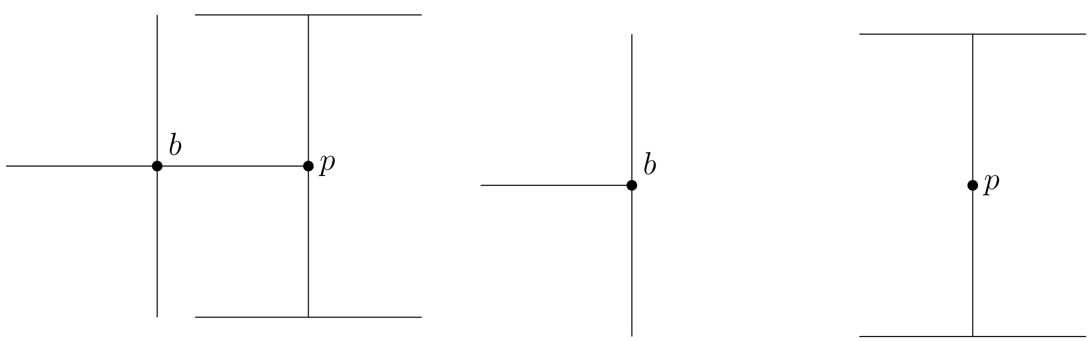
Corollary 3.0.9: For $0 < i \leq n$, $|\text{end}(T_i)| \leq |\text{end}(T_{i+1})|$ and $|\text{end}(T_0^*)| \leq |\text{end}(T_1)|$.

We are now ready to see some trees that are not Hubbardizable.

3.1 Two Sufficient Conditions for non-Hubbardizablity

Lemma 3.1.1: Let T be a tree such that T has at least two branch points and there exists a branch point b with $\deg(b)$ strictly larger than that of all other branch points. Let p be a branch point in a free arc with b. Then $T \setminus (p, b)$ has two components. Let A_1 be the component which contains b and let A_2 be the other component. Lastly, suppose that there are two subtrees of A_2 emanating from p which have more end points than every subtree of A_1 emanating from b. Then T is not Hubbardizable.

An example of such a tree can be found in Figure 3.1.



(a) The Tree (b) A_1 (c) A_2

Figure 3.1. A tree, T, for which there does not exist a function $f : T \to T$ such that (T, f) is a Hubbard tree

Proof. Assume hypotheses as stated. Let $k = \deg(b)$ and let V, W be two trees in A_2 emanating from p that have more endpoints than any tree in A_1 emanating from p. By way of contradiction, suppose there exists an associated function f such that (T, f) is a Hubbard tree. As shown in Lemma 2.2.9, every Hubbard tree has a fixed point with itinerary $\overline{1}$. Here, the fixed point must be b since no neighborhood of b can map injectively to any neighborhood of a point of lower degree.

Recall that T_0 is the tree emanating from $\overline{1}$ which contains c_0 and by Corollary 3.0.9, $|\text{end}(T_i)| \leq |\text{end}(T_{i+1 \mod k})|$ and $|\text{end}(T_0^*)| \leq |\text{end}(T_1)|$. With this in mind, c_0 cannot lie in A_1 since otherwise the tree emanating from b which contains p would be some T_i with more end points than T_0. Moreover, c_0 cannot lie in A_2 since in this case T_0^* must contain one of the trees emanating from p that has more end points than any subtree of A_1 emanating from b. But then, $|\text{end}(T_0^*)| > |\text{end}(T_1)|$, a contradiction. Thus $c_0 \in (b, p)$. Since $f(c_0) \in S_1$, we also have $f(c_0) \in A_1$.

Now that we have determined the locations of c_0 and $f(c_0)$, we aim to find the location of $f(p)$. Suppose $f(p) = b$. Then, since $f|_{S_0}$ is an embedding by Lemma 2.2.2, at most one tree emanating from p has a point which maps to c_0, and thus there is at most one tree in A_2 emanating from p whose image under f meets S_0. But, that means that at least one of W, V maps injectively into S_1, and more specifically, maps injectively into a subtree of A_1 emanating from $b = f(p)$. But, by Lemma 3.0.8, no such injection can exist. So $f(p)$ cannot be b.

Suppose $f(p)$ is a branch point in $A_1 \setminus \{b\}$. Then $f(p) \in T_i$ for some $0 < i \leq k-1$. By similar reasoning to the above, all of the subtrees in A_2 emanating from p (with possibly one exception) must then map injectively into T_i. But, this means at least one of W, V must map injectively into T_i contradicting Lemma 3.0.8.

Lastly, suppose $f(p)$ is some other branch point in A_2. Call this point q. Let T_p^q be the tree emanating from p which contains q. Then, since $T_p^q \subset S_0$ and $f|_{S_0}$ is a homeomorphism, we have $f|_{T_p^q}$ is a homeomorphism. So no point in T_p^q except p can map to q.

By way of contradiction, suppose $f(T_p^q) \not\subseteq T_p^q$. Then there is a point in T_p^q,that maps outside of T_p^q which means there must be a point $w \in T_p^q$ such that $f(w) = p$. Then, since $p \in (c_0, w)$ we have $q = f(p) \in (c_1, f(w)) = (c_1, p)$. But then $q \notin T_p^q$, a contradiction. So $f(T_p^q) \subsetneq T_p^q$ and, in particular, $f(T_p^q)$ contains fewer branch points of S_0 than T_p^q does meaning f cannot be injective, a contradiction.

Thus the only point to which p can map is itself, and so p is a branch point with itinerary $\overline{0}$, contradicting Theorem 2.2.10.

Thus, there can be no function associated with T that satisfies all the condition to be a Hubbard Tree, so T is not Hubbardizable. $\square$

Although the hypotheses in Lemma 3.1.1 are sufficient to tell that a tree is not Hubbardizable, they are not necessary as we will see in the following.

Definition 3.1.2: For $k \geq 3$, let F_0^k be a tree with a single branch point of degree k. Continuing recursively, for $n \geq 1$ let F_n^k be obtained by taking F_{n-1}^k and adding $k-1$ arcs to each endpoint, thus turning every endpoint in F_{n-1}^k into a branch point of degree k in F_n^k. Figure 3.2 shows F_1^4, F_2^4, and F_3^4.

Lemma 3.1.3: F_i^k is Hubbardizable if and only if $i < 2$.

Proof. We first prove F_i^k is not Hubbardizable for $i \geq 2$. In order to do so, fix $k > 2, i \geq 2$ and suppose that there is a function, f such that (F_i^k, f) is a Hubbard tree. Figure 3.3 is

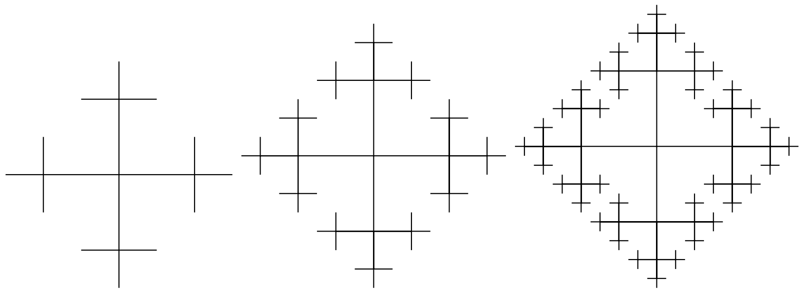


Figure 3.2. The trees F_1^4, F_2^4, and F_3^4

a labelled diagram accompanying the proof. While the tree pictured is F_2^4, the proof is sufficiently general.

Let B_0 denote the central branch point of F_0^k and let B_j denote the set of branch points in F_j^k that are not branch points of F_{j-1}^k. In this way, the points in Figure 3.3 labeled $\alpha, \beta, \gamma, \delta$ comprise B_1.

Claim: B_0 must be fixed.

Proof of claim: By way of contradiction, suppose $f(B_0) \neq B_0$. Notice that every tree emanating from B_0 has the same number of endpoints. Call this number E. As such, the injective images of these trees must have at least E endpoints. Let W denote the tree emanating from $f(B_0)$ which contains B_0.

W is only one tree emanating from $f(B_0)$ with at least n endpoints. So W is the only tree with enough endpoints to contain the injective images of the trees emanating from B_0 and as such, by Lemma 3.0.5, all trees emanating from B_0 (except possibly the tree containing c_0) must map into W. Let $\alpha, \beta, \gamma \in B_1$. Then these three branch points are in a free arc with B_0.

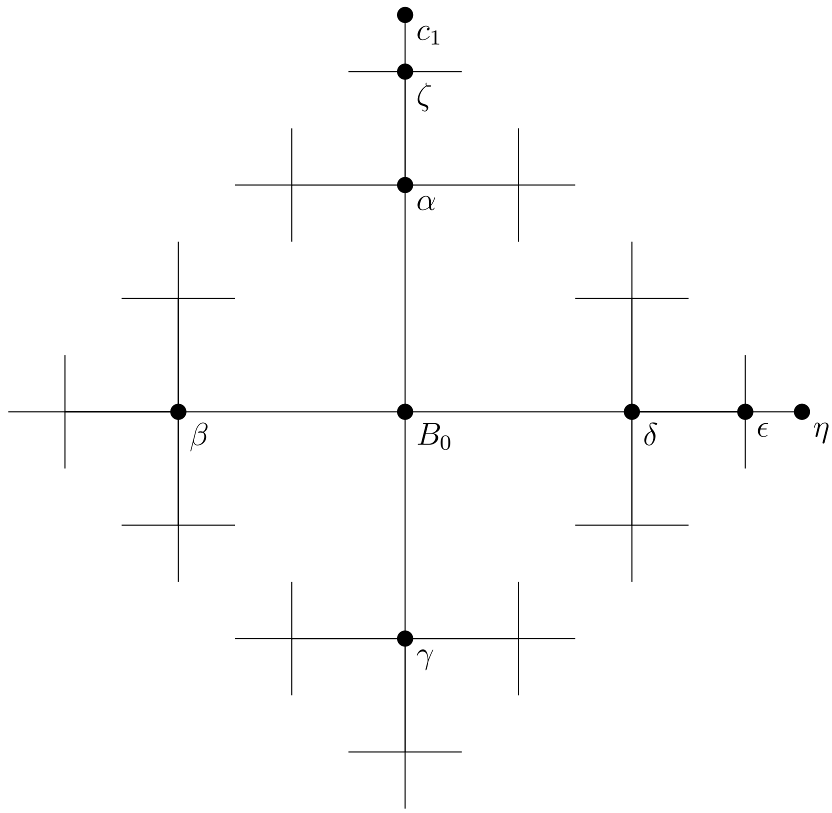


Figure 3.3. F_2^4 with branch points labeled

Let z be the branch point in W closest to $f(B_0)$. Note that z may be one of α, β, γ. So, we must have that at least two of $f((B_0, \alpha))$, $f((B_0, \beta))$, and $f((B_0, \gamma))$ meet $(f(B_0), z)$. But then f is not locally injective at B_0, so B_0 is both a branch point and the critical point, contradicting Lemma 2.2.1. This proves the claim.

Claim: $f(B_1) = B_1$.

Proof of claim: As we have already seen, $f(B_0) = B_0$. Observe that, for $j > 0$, a point $z \in B_j$ if and only if there are $j - 1$ many branch points between z and B_0.

Let z be a point in B_1. If $z \in S_1$ then $z \in T_n$ for some $n > 0$. As such, by Lemma 3.0.7, T_n embeds into T_{n+1}. So the number of branch points in $[z, B_0]$ must equal the number of branch points in $[f(z), B_0]$. Thus both z and $f(z)$ belong to B_1.

If $z \in S_0$, then we have two possibilities, either c_0 is in (B_0, z) or c_0 is not in (B_0, z). If $c_0 \notin (B_0, z)$ then $f|_{[B_0,z]}$ is an embedding and by similar reasoning to the above, both z and $f(z)$ are in B_0.

If, on the other hand, $c_0 \in (B_0, z)$ then $f|_{[B_0,z]}$ is not injective. $f([B_0, c_0]) = [B_0, c_1]$ and $f([c_0, z]) = [c_1, f(z)]$.

By definition of F_i^k every branch point has degree k so the number of trees emanating from z is k. Exactly $k - 1$ of the trees have the same number of endpoints (the only exception is the tree containing c_0 and B_0 since this tree contains all of S_1). The number of endpoints is $(k - 1)^{i-1}$, but for the sake of conciseness, let $r = (k - 1)^{i-1}$.

Let κ denote the union of these $k - 1$ trees. Then $f|_\kappa$ is an embedding, so $f(z)$ must be a branch point from which exactly k trees emanate. Among these, $k - 1$ must have at least r endpoints, so $f(z)$ is either in B_0 or B_1.

If $f(z) \neq T_1 \cap B_1$ then no points in κ can map into T_1. Then the only endpoint in T_1 that has a preimage is c_1, a contradiction. So, if $z \in B_1 \cap S_0$, then $f(z) \in B_1 \cap T_1$.

Thus, $f(B_1) = B_1$.

Then, since no two branch points can share the same itinerary, there must be a branch point $b_1 \in B_1$ such that $c_0 \in (B_0, b_1)$. Otherwise, every branch point in B_1 would have the same itinerary, $\overline{1}$.

We have that T_1 is the tree emanating from B_0 which contains c_1. Let $p \in T_1 \cap B_1$. Note that p is unique and $f(b_1) = p$. Let q be a branch point in B_2 such that $q \in (p, c_1)$. Such a

q exists by definition of F_i^k since $i > 2$. Since $q \in T_1$ all of q's preimages must be in T_0 by Lemma 3.0.7. Moreover, since $q \in (p, c_1)$, the preimage of q under $f|_{S_0}$ must be in (c_0, b_1). Similarly, the preimage of q under $f|_{S_1}$ must be in (c_0, B_0). However, neither of these arcs contain branch points and $f([c_0, B_0]) = [c_1, B_0]$ and $f([c_0, b_1]) = [c_1, p])$. So none of the branch points (or endpoints) in a free arc with q other than those in $[B_0, c_1]$ have preimages, so f is not surjective. Thus, F_i^k is not a Hubbard tree for $i > 2$.

What remains to be show is that F_i^k *is* Hubbardizable for $i < 2$. We demonstrate this by defining the function f such that (F_i^k, f) is a Hubbard tree. F_0^k is a k-od. To find a function f that makes (F_0^k, f) a Hubbard tree, let B_0 be the branch point and arbitrarily label the arcs emanating from B_0 as $a_0, a_1, \dots, a_{k-1}$. Pick a point in the interior of a_0 to be c_0 and let q be the endpoint of $a_0 \setminus \{B_0\}$. For $0 < j \leq k-1$ let f map a_j linearly over $a_{j+1 \mod k}$ fixing B_0. Let f map $[B_0, c_0]$ linearly over a_1 and map $[c_0, q]$ linearly over $[c_1, c_0]$. Then (F_0^k, f) is a Hubbard tree with kneading sequence $\tau = \overline{*1^{k-1}0}$. Such a Hubbard tree can be seen in Figure 2.3.

To define a function f for which (F_1^k, f) is a Hubbard tree, we follow a process similar to the one used in the case of F_0^k. Let B_0 be the central branch point, and arbitrarily label the trees emanating from B_0 as $T_0, T_1, \dots, T_{k-1}$. For $j > 0$, let f map T_j homeomorphically over $T_{j+1 \mod k}$. Let z be the branch point in T_0. Pick some point in (B_0, z) to be the critical point, c_0. Pick some endpoint in T_1 to be $c_1 = f(c_0)$ and pick some endpoint, e, in T_{k-1} to map to c_0 Lastly, let f map $[B_0, c_0]$ linearly over $[B_0, c_1]$ and let f map the tree emanating from c_0 which contains z homeorphically of the tree $T_1 \cup [B_0, c_0]$. Then (F_1^k, f) is a Hubbard tree with kneading sequence $\tau = \overline{*(1^{k-1}0)^{k-1}}$. An example of F_1^3 can be found in Figure 2.2a.

For an explanation of how to build F_0^k and F_1^k from the respective kneading sequences, $\overline{*1^{k-1}0}$ and $\overline{*(1^{k-1}0)^{k-1}}$, see A.0.3. $\square$

We have now seen two examples of trees that are not Hubbardizable, and while the author longs for the discovery of a necessary and sufficient condition for Hubbardizability, he contents himself with what follows: a full classification of all Hubbardizable trees with fewer than four branch points.

3.2 A Full Classification of all Hubbardizable Trees with Fewer than Four Branch Points

Before we fully classify all such trees, we make note of a convenient theorem.

Theorem 3.2.1: Let T be a tree whose branch points are all of the same degree and such that there is some arc $[a, b]$ in T which contains every branch point. Then T is Hubbardizable

Proof. As seen in the proof of Theorem 2.2.13, to construct a Hubbard tree with exactly n branch points, each of degree d, one can use the kneading sequence

$$\tau = \overline{*(10^{n-1})^{d-1}}.$$

Each of these branch points has the itinerary $\overline{0^j10^{n-1-j}}$ for some $0 \leq j \leq n-1$. To see that these are co-linear, first observe that if $1 \leq w < j$, then $V(\tau, \overline{0^j10^{n-1-j}}, \overline{0^w10^{n-1-w}}) = \overline{0^w10^{n-1-w}}$. So, if p is the branch point with itinerary $\overline{0^j10^{n-1-j}}$ and q is the branch point with itinerary $\overline{0^w10^{n-1-w}}$ then $q \in (c_0, p)$. Thus, if r is the branch point with itinerary $\overline{0^{n-1}1}$ then all branch points in S_0 are contained in $(c_0, r]$.

Finally, since there is only one branch point in S_1, call it b, all branch points are contained in $[b, r]$. $\square$

Zero Branch Points

Theorem 3.2.2: Let T be a non-degenerate tree with zero branch points. Then T is Hubbardizable.

Proof. The only non-degenerate tree with 0 branch points is the arc. It can be made, as a Hubbard tree, with the kneading sequence $\tau = *1\overline{0}$. This is the Hubbard tree of the polynomial $f(z) = z^2 - 2$ as shown in [Dev03, Example 5.11].

Moreover, it can be made a Hubbard tree with a different kneading sequence: $\tau = \overline{*10}$. To do this, label the end points of the arc a, b and define some point in (a, b) to be the critical point c_0. Define a function f such that $f(c_0) = a, f(a) = b$, and $f(b) = c_0$. Moreover, define it so f maps $[a, c_0]$ linearly over $[b, a]$ and f maps $[c_0, b]$ linearly over $[a, b]$.

In this way, the arc is a single tree, which can be paired with multiple functions each of which yields a dynamically distinct generalized Hubbard tree. $\square$

One Branch Point

Theorem 3.2.3: Let T be a tree with only one branch point. Then T is Hubbardizable

Proof. The only Hubbard trees with one branch points are n-ods. As we have previously seen in Lemma 3.1.3, these are Hubbardizable. $\square$

Two Branch Points

Theorem 3.2.4: Let T be a tree with exactly two branch points a, b. Let $D = \{\deg(a), \deg(b)\}$. Then T is Hubbardizable if and only if $|D| = 1$ or $|D| = 2$ and $\min(D) = 3$.

Proof. Let T be a tree with exactly two branch points a, b. Begin by supposing that (T, f) is a Hubbard tree. We show that if $|D| = 2$ then $\min(D) = 3$. This then proves the claim that if (T, f) is a Hubbard tree with two branch points, then they are the same degree or the smaller degree is 3. So suppose $|D| = 2$. Without loss of generality, let $\deg(a) > \deg(b)$. Since f is locally injective at all but one point, no point (except the critical point) can map to a point of lower degree. Since a has higher degree than b, we must have $f(a) = a$. By Lemma 2.2.9 there is only one fixed branch point in any Hubbard tree, so we have $f(b) \neq b$ and thus $f(b) = a$.

Since a is the fixed branch point in S_1, the kneading sequence must begin $*1^n0$ where $n = \deg(a) - 1$. Thus every endpoint in $\overline{S_1}$ is visited by the orbit of c_0 before it ever gets to S_0. So if an endpoint in a free arc with b maps into S_1, then it must be the last endpoint visited by c_0.

So if an endpoint in a free arc with b is not be the last endpoint visited by c_0 then it must map into S_0. By way of contradiction, suppose there are two endpoints, p, q from S_0 that mapp into S_0. Then $[b, p]$ and $[b, q]$ are two arcs in S_0 that cover $[a, c_0]$, a contradiction.

So, there are at most two endpoints in a free arc with b, one that maps into S_1 and one that maps into S_0 and thus $\deg(b) = 3$.

To complete the proof we show that if $|D| = 1$ or $|D| = 2$ and $\min(D) = 3$ that T is Hubbardizable. As shown in Theorem 3.2.1, if $|D| = 1$ then there is a kneading sequence which gives rise to T. If $|D| = 2$ with $\min(D) = 3, \max(D) = k$ then T can be realized by the kneading sequence $\tau = \overline{*1^{k-1}001}$. See A.0.4 $\square$

Three Branch Points

Lemma 3.2.5: Suppose that (T, f) is a Hubbard tree with exactly three branch points, a, b, c such that $b \in (a, c)$ and c_0 in (a, b). If $b, c \in S_0$ then $f(b) = a, f(c) = b$. If $b, c \in S_1$, then $c \neq \overline{1}$ and if $b = \overline{1}$ then $f(c) = a$.

Proof. Assume hypotheses as stated. Since f is locally injective at the branch points, $f(\{a, b, c\}) \subseteq \{a, b, c\}$. If $b, c \in S_0$, then b cannot map to c else $f([c_0, b]) = [c_1, c] \supset [c_0, b]$ and thus there is a fixed point in (c_0, b). But, by Theorem 2.2.10, if there is a fixed point in S_0, then it must be an endpoint. Thus, $f(b) = a$.

Since branch points in S_0 cannot be fixed, $f(c) \neq c$. Furthermore, we cannot have $f(c) = a$ otherwise b and c share the same itinerary, a contradiction. Thus $f(c) = b$.

If $b, c \in S_1$ then we cannot have $c = \overline{1}$ because otherwise, c_1 would be in a free arc with c. But then $f([c, c_0]) = [c, c_1]$ would contain $f(b)$, a contradiction. If $b = \overline{1}$ then $f(c) = a$ otherwise b and c would share an itinerary. $\square$

Lemma 3.2.6: Suppose that (T, f) is a Hubbard tree with exactly three branch points, a, b, c such that $b \in (a, c)$ and c_0 in (a, b). If $b \neq \overline{1}$ and $f(b) = \overline{1}$, then $\deg(b) = 3$.

Proof. If $f(b) = \overline{1}$ then $\overline{1}$ must be a branch point and b must not be in S_1. Thus, there is only one branch point in S_1. Without loss of generality, assume that this branch point is a. Since $b \in (a, c)$ we must have $c \in S_0$ as well and because c's itinerary must be different than that of b, $f(c) \neq a$. So $f(c) \in S_0$.

Then the kneading sequence must begin $*1^{k-1}0$ where $k = \deg(a)$. In this way, every endpoint in S_1 is visited by c_0 before it ever maps into S_0. So if an endpoint in S_0 maps into S_1, it either maps to an endpoint already visited by c_0 or a non-endpoint. Thus, if an endpoint in S_0 maps into S_1, it must be the last endpoint visited by c_0.

By way of contradiction, suppose there are at least two endpoints p and q in a free arc with b. Because $f|_{S_0}$ is injective and c_0, p, q, c are all in different trees emanating from b, we must have that $f(c_0), f(p), f(q), f(c)$ are all in different trees emanating from $f(b)$. Since $f(c) \in S_0$, we must have $f(c_0), f(p), f(q)$ all in S_1. But then p and q are both the last endpoint visited by c_0, a contradiction. Thus, there is at most one endpoint in a free arc with b, so $\deg(b) = 3$. $\square$

Lemma 3.2.7: Suppose that (T, f) is a Hubbard tree with exactly three branch points, a, b, c such that $b \in (a, c), c_0 \in (b, c)$, and $b = \overline{1}$. Then $f(a) = c, \deg(a) = 3$ and $\deg(c) \in \{3, 4\}$.

Proof. Observe that $a \in S_1$ so $f(a) \neq a$ and $f(a) \neq b$ otherwise a and b would have the same itinerary. So $f(a) = c$.

Since $f(a) \in S_0$, a must be in T_k where $k = \deg(b)$. Recall that T_k is the only tree in S_1 emanating from b whose image can meet S_0. By Lemma 3.0.7, T_{k-1} is the only tree in S_1 emanating form b that can map into T_k. By assumption, T_{k-1} is an arc. Every endpoint in T_k is in a free arc with a, so there is at most one endpoint in a free arc with a whose preimage is in S_1.

Since f is injective on $\overline{S_0}$, then endpoints in a free arc with c must map into different trees emanating from $f(c) = b$. So at most one such endpoint can map to an endpoint in a free arc with a. Thus, there is at most one endpoint in a free arc with a whose preimage is in S_0. So there are at most two endpoints in a free arc with a, so $\deg(a) = 3$.

These are the only endpoints in S_1 that can map into S_0. So there are at most two endpoints in a free arc with c whose preimage is in S_1. Again, since f is locally injective at c, each endpoint in a free arc with c must map into different trees emanating from b, so there can be only one such endpoint which maps into T_0, there is at most one such endpoint that maps to an endpoint in S_0. Thus, there are at most three endpoints in a free arc with c so $\deg(c) \leq 4$. $\square$

Theorem 3.2.8: Suppose that T is a tree with exactly three branch points, a, b, c such that c_0 in (a, b). Let $D = \{\deg(a), \deg(b), \deg(c)\}$. Then T is Hubbardizable if and only if one of the following conditions holds:

1. $|D| = 1$,
2. $|D| = 2$ and $\min(D) = 3$ or,
3. $D = \{3, 4, k\}$ for some $k > 4$ and $\deg(b) = k$.

Proof. Let T be a tree with three branch points a, b, c with $b \in (a, c)$. Suppose that there is a function f such that (T, f) is a Hubbard tree. We have multiple cases in which not all the degrees are the same:

- $\deg(a) = \deg(c) > \deg(b)$,
- $\deg(a) = \deg(b) > \deg(c)$ [or by symmetry $\deg(c) = \deg(b) > \deg(a)$],
- $\deg(a) > \max\{\deg(b), \deg(c)\}$ [or by symmetry $\deg(c) > \max\{\deg(a), \deg(b)\}$] and,

- $\deg(b) > \max\{\deg(a), \deg(c)\}$,

We will go through these one-by-one and find out what restrictions the Hubbardizability of T puts on the degrees of the branch points in each case.

1. If $\deg(a) = \deg(c) > \deg(b)$, then we have two possibilities. Either one of a or c is fixed and both points map to the fixed point — without loss of generality $f(a) = a = f(c)$— or a and c map to each other — $f(a) = c$ and $f(c) = a$.

   If a is fixed, then since $f(a) = f(c)$ we have $c_0 \in (a, c)$ and so $f([a, c]) = [a, c_1]$ and so b maps injectively into $[a, c_1]$, but b is a branch point, and there are no branch points in $[a, c_1]$, a contradiction.

   Thus, $f(a) = c, f(c) = a$. Then c_0 is either in (a, b) or (b, c), but by symmetry, we can assumes $c_0 \in (b, c)$.

   Then $f(b) \neq c$ or else a and b would have the same itinerary, contradicting the unique itinerary property of Hubbard trees. So, either $f(b) = b$ or $f(b) = a$. Suppose b is fixed. Every endpoint in a free arc with a must map into different trees emanating from c and, since $[a, b]$ maps over $[c, b]$, these endpoints cannot map into the tree emanating from c containing b. Similarly, since $f([c_0, c]) = [c_1, a] \ni b$ every endpoint in a free arc with c must map into different trees emanating from a which do not contain b. But then all of the end points in a free arc with a would have the same itinerary, again violating the unique itinerary property of Hubbard trees.

   So $f(b)$ must equal a. Then, since $f([a, b]) = [a, c] \supset [a, b]$, the fixed point in S_1 must be in (a, b). Since $\overline{1} \in (c_0, c_1)$, we have c_1 is an endpoint in a free arc with a.

Again, since $f([c_0, a]) = [c_1, c]$, we have that every endpoint in a free arc with a must map into different trees emanating from c which do not contain b.

Let e be and endpoint in a free arc with b. Since f is locally injective at b, and the points $\overline{1}, c_0, e$ are all on opposite sides b, we have $f(\overline{1}), f(c_0)$, and $f(e)$ must all be on opposite sides of $f(b)$. Since b and $\overline{1}$ are in the same tree emanating from $f(b)$, we have that no endpoint in a free arc with b can map to an endpoint in a free arc with b. Thus, any endpoint in a free arc with b must have a preimage in S_0.

But, since $f([c_0, c]) = [c_1, a]$ we have every endpoint on opposite sides of c must map to the opposite sides of a (and not the side containing c_1). So there can be only one endpoint in a free arc with c that can map to an endpoint in a free arc with b. So, of all the endpoints in a free arc with b, at most one can have a preimage in S_0.

Thus, there can be at most one endpoint in a free arc with b.Thus $\deg(b) = 3$. So if T is Hubbardizable and $\deg(a) = \deg(c) > b$, then $\deg(b) = 3$, so $|D| = 2$ with $\min(D) = 3$.

2. If $\deg(b) = \deg(a) > \deg(c)$ then we must have $f(\{a, b\}) \subseteq \{a, b\}$. Thus $c_0 \in (a, b)$ else a and b would share the same itinerary. We now work through a few possibilities.

   (a) $f(a) = a$. By Lemma 3.2.5, we have $f(b) = a$ and $f(c) = b$. By Lemma 3.2.6 we have $\deg(b) = 3$ and thus $\deg(a) = 3$ but by assumption, this means $\deg(c) < 3$, a contradiction.

   (b) $f(b) = b$. We must have $b \in S_1$ and so $a \in S_0, c \in S_1$. By Lemma 3.2.5, we must have $f(c) = a$. This is exactly the same as the hypotheses in Lemma

3.2.7 but with the labels of a and c reversed. So we find $\deg(c) = 3$ and $\deg(a) \in \{3, 4\}$. However, to be consistent with our assumption $\deg(b) = \deg(a) > \deg(c)$ we have $\deg(a) = 4, \deg(b) = 4, \deg(c) = 3$.

(c) $f(a) = b, f(b) = a$. In this case the fixed branch point is actually degree two and lies in (a, b). Since c cannot map to a without having the same itinerary as b we must have $f(c) = b$. In order to get a picture of this Hubbard tree, we now try to find out which of a, b belongs to S_1. There exists a unique point, p whose itinerary begins $1*$. Since there is only one point in S_1 that maps to c_0, every point in $[c_0, p)$ must map to a point in S_1. If there were a branch point in $[c_0, p]$ then one of a, b must be in $[c_0, p]$, but since they map to each other, this branch point would map into S_0, so $[c_0, p]$ is a free arc. Every point in $S_1 \setminus [p, c_0]$ must map into S_0. So, if $b \in S_1$ (and by extension $c \in S_1$, then $f(c) \in S_0$. But $f(c) = b \in S_1$, a contradiction. Thus $b \in S_0$ and $a \in S_1$. So c_1 is in a free arc with a.

Since each endpoint in a free arc with a must map into different trees emanating from b, there can be at most one such endpoint that maps to an endpoint in a free arc with c. Every endpoint in a free arch with b must map into different trees emanating from a, but since $[b, c]$ maps onto $[a, b]$, there are no endpoints in a free arc with b that map to endpoints in a free arc with c/ Lastly, since each endpoint in a free arc with c must themselves map into different trees emanating from b. So there is at most one endpoint in a free arc with c whose preimage is in S_0. Thus $\deg(c) = 3$.

So if $\deg(a) = \deg(b) > \deg(c)$ then $|D| = 2$ and $\min(D) = 3$.

3. If $\deg(a) > \max\{\deg(b), \deg(c)\}$, then $a = \overline{1}$ and so c_1 is an endpoint in a free arc with a. Since the branch points must have different itineraries and $\{f(a), f(b), f(c)\} \subseteq \{a, b, c\}$, the branch points cannot all be on the same side of c_0. So $c_0 \in (a, b)$ or $c_0 \in (b, c)$. By way of contradiction, suppose $c_0 \in (b, c)$. Then $b \in (a, c_0)$ so $f(b) \in (f(a), f(c_0)) = (a, c_1)$. But $[a, c_1]$ is a free arc, a contradiction. So $c_0 \in (a, b)$.

   Now we figure out where b and c map under f. Since both branch points are in S_0, they cannot be fixed. So $f(c) = b$ or $f(c) = a$. Suppose $f(c) = a$. Then, since $b \in (c_0, c)$, $f(b) \in (f(c_0), f(c)) = (c_1, a)$. But, again, $[a, c_1]$ is a free arc, a contradiction. So $f(c) = b$. Thus $f(b)$ must be a since, if $f(b) = c$ then both branch points would have itinerary $\overline{0}$.

   Suppose $\deg(b) \geq 4$. Then there are at least two end points p, q which are in a free arc with b. Since $f(b) = a$ we must have that $f(p)$ and $f(q)$ map into different T_i's.

   One of p or q has to get visited by c_0 first. Without loss of generality, c_0 visits p first. $f(p)$ must be in some $T_i, i > 0$. That is to say, $f(p) \in [e, a]$ for some endpoint e in a free arc with a. If $f(p)$ is some non-endpoint, then since the only non-endpoint that can map to an endpoint is c_0, we have that $q \notin Orb(p) \subseteq Orb(c_0)$, a contradiction.

   If, on the other hand, $f(p) = e$, then since e has already been visited by c_0 we have that c_0 is pre-periodic and only visits endpoints. But then $Orb(c_0)$ cannot contain q since $f(p) = e$ and so $q \notin Orb(e)$, a contradiction. Thus $\deg(b) = 3$.

Thus, since $f(c) = b$ and f is locally injective at c, $deg(c) \leq \deg(b)$ and so $\deg(c) = 3$. Thus, if T is Hubbardizable and $\deg(a) > \max\{\deg(b), \deg(c)\}$ then $\deg(b) = \deg(c) = 3$, so $|D| = 2$ and $\min(D) = 3$.

4. If $\deg(b) > \max\{\deg(a), \deg(c)\}$, then we have $b = \overline{1}$. c_0 must either be in (a, b) or in (b, c) otherwise all the branch points would be in S_1 and would each have itinerary $\overline{1}$.

   Without loss of generality $c_0 \in [b, c]$. Then by Lemma 3.2.7 we have $\deg(a) = 3, \deg(c) \leq 4$. So either $|D| = 2$ and $\min(D) = 3$ or $|D| = \{3, 4, k\}$ for some $k > 4$ and $\deg(b) = k$.

So, after all this case-checking we have arrived at the following: if T is a Hubbardizable tree with three branch points, a, b, c and $D = \{\deg(a), \deg(b), \deg(c)\}$ then either

1. $|D| = 1$,

2. $|D| = 2$ and $\min(D) = 3$ or,

3. $D = \{3, 4, k\}$ for some $k > 4$.

Now, let T be a tree with three branch points, a, b, c with $b \in (a, c)$ such that either

1. $|D| = 1$,

2. $|D| = 2$ and $\min(D) = 3$ or,

3. $D = \{3, 4, k\}$ for some $k > 4$ and $\deg(b) = k$.

We show T is Hubbardizable. We do so by providing a kneading sequence which gives rise to T and as before, we will go case-by-case and subcase-by-subcase.

1. If $|D| = 1$ then as seen in Theorem 3.2.1, T is Hubbardizable.

2. If $|D| = 2$ and $\min(D) = 3$ we have a few cases:

   (a) If $\deg(a) = \deg(c) > \deg(b)$ then T can be made with the kneading sequence $\tau = \overline{*(10)^n 1101}$ (see A.0.5).

   (b) If $\deg(a) = \deg(b) > \deg(c)$ [or by symmetry $\deg(c) = \deg(b) > \deg(a)$], then T can be made with kneading sequence $\tau = \overline{*(10)^{k-2}100010}$ (see A.0.6).

   (c) $\deg(a) > \max\{\deg(b), \deg(c)\}$ [or by symmetry $\deg(c) > \max\{\deg(a), \deg(b)\}$], then T can be made with the kneading sequence $\tau = \overline{*1^{k-1}0001}$ (see A.0.7).

   (d) $\deg(b) > \max\{\deg(a), \deg(c)\}$, then T can be made with the kneading sequence $\tau = \overline{*1^{k-1}01011}$ (see A.0.8).

3. If $|D| = \{3, 4, k\}$ and $\deg(b) = k$, then T can be made with the kneading sequence $\tau = \overline{*1^{k-1}001011}$ (see A.0.8).

Thus, if $|D| = 1, |D| = 2$ and $\min(D) = 3$, or $|D| = \{3, 4, k\}$ and $\deg(b) = k$, then T is Hubbardizable. $\square$

CHAPTER FOUR

Branch Points and Endpoints of Hubbard Tree Inverse Limits

Our goal is to eventually construct an infinite family of homeomorphic abstract dendritic Julia sets which have, at their cores, homeomorphic generalized Hubbard trees, but such that the inverse limit of each dendritic Julia set is *not* homeomorphic to any other. To show that these aforementioned inverse limits are not homeomorphic requires careful examination of branch points and endpoints. With that in mind, we now turn our attention to branch points and endpoints of inverse limits of Hubbard trees. For this chapter, let (H, f) be a Hubbard tree.

Definition 4.0.1: Define $I_0 = \{p \in H : p \text{ is a branch point or in the critical orbit}\}$, and for $n > 0$, define $I_n = f^{-1}(I_{n-1})$.

Definition 4.0.2: For $\bar{x} \in \hat{H}$, define

$$\varphi(\bar{x}) = \min\{D | \deg(\overrightarrow{x}_{-n}) = D \text{ for infinitely many } n \in \omega\}.$$

Observe that since the set $\{D|$ there exists $x \in H$ with $\deg(x) = D\}$ is a subset of the set of degrees of branch points in H and is thus finite, $\varphi(\bar{x})$ is well defined.

Lemma 4.0.3: If $\bar{x}$ is a point in $\hat{H}$ such that $\overrightarrow{x}_{-n}$ is a branch point for all $n \in \omega$, then $\deg(\overrightarrow{x}_{-n}) = \varphi(\bar{x})$ for all $n \in \omega$. Furthermore, $\overrightarrow{x}_{-n}$ is periodic for all $n \in \omega$.

Proof. Since each branch point in H is (pre)periodic, and there are finitely many such points, then there is some $w \in \omega$ such that, for all branch points, $b \in H, \sigma^w(b)$ is periodic.

With this in mind, if $\overrightarrow{x}_{-n}$ is a pre-periodic branch point, then $\overrightarrow{x}_{-(n+w)}$ must not be a branch point. So if $\overrightarrow{x}_{-n}$ is a branch point for all $n \in \omega$, then $\overrightarrow{x}_{-n}$ must be periodic for all $n \in \omega$.

Moreover, since σ is locally injective at all branch points (since it is locally injective at all non-critical points), the degree of a periodic branch point must be the same as the degree of all points in its orbit. Thus, if $\overrightarrow{x}_{-n}$ is a branch point for all $n \in \omega$ then $\deg(\overrightarrow{x}_{-n}) = \deg(\overrightarrow{x}_{-m})$ for all $n, m \in \omega$. This degree, by definition, is $\varphi(\bar{x})$. $\square$

Lemma 4.0.4: Let $[a, b] \subset H$. Then $f^{-1}([a, b])$ has at most two components.

Proof. Suppose that $f^{-1}([a, b])$ has three or more components. Then there is an $i \in \{0, 1\}$ such that $S_i \cap f^{-1}([a, b])$ has at least two components. Call these two components W, Y. Then, since $f|_{S_i}$ is an embedding by Lemma 2.2.1, $f(W \cup Y)$ is a disconnected subset of $[a, b]$. So there is some $z \in (a, b)$ such that z separates $f(W)$ and $f(Y)$ and $f^{-1}(z) \cap S_i = \varnothing$.

Let w be a point in $f(W)$ and let $w' \in W$ such that $f(w') = w$. Define y and y' similarly. Then by the intermediate value theorem, there is some $z' \in (w', y')$ such that $f(z') = z$. But since $w', y' \in S_i$ we must have $z' \in S_i$, contradicting the fact that $f^{-1}(z) \cap S_i = \varnothing$. Thus $f^{-1}([a, b])$ has at most two components, and at most one component in each of S_1, S_0. $\square$

Lemma 4.0.5: Let $[a, b] \subset H$ be an arc such that no point of (a, b) is in I_0. If $a \neq c_1$ and $b \neq c_1$, then $f^{-1}([a, b])$ is either homeomorphic to $[a, b]$ under f or is the union of two disjoint sets each of which is homeomorphic to $[a, b]$ under f, or made of two disjoint sets — one of which is homeomorphic to $[a, b]$ under f while the other is a singleton.

If $c_1 \in \{a, b\}$, then $f^{-1}([a, b]) \setminus \{c_0\}$ is made of two disjoint sets, U, V such that $U \cup \{c_0\}$ and $V \cup \{c_0\}$ are both homeomorphic to $[a, b]$ under f.

Proof. To start, suppose $c_1 \notin [a, b]$. If $f^{-1}([a, b])$ has only one component then we are done, so suppose there are two nonempty components U, V.

Suppose V is not a singleton and $f(V) \neq [a, b]$. The set V must be closed, since V is the preimage of a closed set under the continuous function $f|_V$. So $f(V) \subset [a, b]$ is closed, and is thus an arc. Then, since $f(V) \neq [a, b]$, at least one of the endpoints must not have a preimage in V. We have two cases. Either:

1. neither end point has a preimage in V or,

2. without loss of generality, b does not have a preimage in V, but a does.

As we will see, both of these lead to contradictions. In case 1, let $z \in V$. Then $f(z) \in [a, b]$. Then, since $[c_0, z] \subseteq S_i$ for some $i \in \{0, 1\}$, we have $f|_{[c_0,z]}$ is an embedding, so $f([c_0, z])$ is an arc. Moreover, $f([c_0, z]) \cap [a, b] = [c_1, f(z)] \cap [a, b] \neq \varnothing$. Then either $a \in [c_1, z] \cap [a, b]$, $b \in [c_1, z] \cap [a, b]$ or $[c_1, z] \cap [a, b] = \{z\}$. In the first two cases we arrive at a contradiction because one of a or b has a preimage in V, and in the latter case we find $f(z)$ is a branch point, but (a, b) misses I_0.

If we are in case 2, then $a \in f(V)$ and $b \notin f(V)$. Since V is an arc, $f(V)$ must also be an arc. So, let p be such that $f(V) = [a, p]$. Let $p' \in V$ be the point that maps to p.

Since V misses I_0 we have p' is neither a branch point, nor an end point of H. But, since $f|_V$ is an embedding, and p is an endpoint of $f(V)$, we must have p' is an endpoint of V. So we can find an arc $[d, p']$ such that $[d, p'] \cap V = \{p'\}$. By definition of V we must have $f([d, p']) \cap [a, b] = \{p\}$, but this implies p is a branch point. Thus, either p is in (a, b) and is a branch point or $p = a$. But both cases lead to a contradiction. Thus, if V is not a singleton, then $f(V) = [a, b]$.

At least one of U, V must not be a singleton, so either both U, V are homeomorphic to $[a, b]$ under f or one of them is homeomorphic to $[a, b]$ under f and the other is a singleton.

Now, suppose $a = c_1$. Then $f^{-1}([a, b])$ is an arc containing c_0. In S_1 we have $f([\overline{1}, c_0]) = [\overline{1}, c_1] \supseteq [a, b]$. So b has a preimage in S_1. Moreover, b must have a preimage in S_0 since (a, b) misses I_0. If b had no preimage in S_0, then $f(S_0) \subset (a, b)$. But S_0 contains points on the critical orbit, so then (a, b) would also contains points in the critical orbit. Then $f^{-1}([a, b]) \setminus \{c_0\}$ is the union of two sets U, V such that $U \cup \{c_0\}$ and $V \cup \{c_0\}$ are both homeomorphic to $[a, b]$ under f. □

Theorem 4.0.6: For each $\bar{x} \in \hat{H}$, $\deg(\bar{x}) = \varphi(\bar{x})$.

Proof. Fix $\bar{x} \in \hat{H}$ and let $d = \varphi(\bar{x})$. We first show that $\deg(\bar{x}) \geq d$ by building subcontinuua $A^1, \ldots, A^d \subset \hat{H}$ with $\{\bar{x}\} \subsetneq A^i$ and $A^i \cap A^j = \{\bar{x}\}$ for all $1 \leq i < j \leq d$.

Fix $n \in \omega$ such that $\overrightarrow{x}_{-n}$ is point of degree d. So there exists d many subcontinuua of H, call them $K^1_{-n}, \ldots, K^d_{-n}$, such that each contains $\{\overrightarrow{x}_{-n}\}$ as a proper subset and, $K^i_{-n} \cap K^j_{-n} = \{\overrightarrow{x}_{-n}\}$ for all $1 \leq i < j \leq d$. What's more, without loss of generality, each K^i_{-n} can be chosen so that each K^i_{-n} is an arc and $K^i_{-n} \setminus \{\overrightarrow{x}_{-n}\}$ misses I_0.

Now, for $m < n$ define $K^i_{-m} := f^{n-m}(K^i_{-n})$. For $m > n$ define K^i_{-m} to be the component of $f^{-(m-n)}(K^i_{-n})$ which contains $\overrightarrow{x}_{-m}$. This component is homeomorphic to K^i_{-n} by Lemma 4.0.5. Define $A^i = \varprojlim\{K^i_{-n}, f\}$. By construction, $\langle A^i \rangle_{i \leq d}$ is a set of d nondegenerate subcontinuua of $\hat{H}$ whose pairwise intersection is $\bar{x}$. Thus $\deg(\bar{x}) \geq d$.

All that is left to show is that $\bar{x}$ is point of degree exactly d. With this in mind, suppose $\bar{x}$ is degree at least $d+1$ and let $A^i, i \leq d+1$ be a set of proper subcontinuua of $\hat{H}$ each of which properly contain $\bar{x}$ and $A^i \cap A^j = \{\bar{x}\}$ for all $1 \leq i < j \leq d+1$.

Let $(n_k)_{k \in \omega}$ be a sequence of indices such that $\deg(\overrightarrow{x}_{-n_k}) = d$ for all $k \in \omega$ and $\pi_{-n_k}(A^i) \neq H$ for all $1 \leq i \leq d+1, k \in \omega$. Note that since each A^i is a proper subcontinuum of $\hat{H}$, for each $i \leq d+1$ there must be some $i_j \in \omega$ such that for all $m > i_j, \pi_m(A^i) \neq H$, else A^i would be $\hat{H}$. Moreover, since A^i is not a singleton, $\pi_{-m}(A^i)$ is not a singleton for all $m \in \omega$. To see why, notice that if $\pi_{-m}(A^i)$ were a singleton and $\pi_{-(m+1)}(A^i)$ were not, then since each point in H has at most two preimages, $\pi_{-(m+1)}(A^i)$ would have at most two points. But $\pi_{-(m+1)}(A^i)$ is a continuum and thus, must be a singleton. So, if $\pi_{-m}(A^i)$ is a singleton for any m, then it is a singleton for every m. But A^i is not a singleton, so $\pi_{-m}(A^i)$ is not a singleton for any m.

Then, for all k, since $\deg(\overrightarrow{x}_{-n_k}) = d$, there must be a pair A^{i_k}, A^{j_k} from our set of continuua in $\hat{H}$ such that $(\pi_{-n_k}(A^{i_k}) \cap \pi_{-n_k}(A^{j_k})) \setminus \{\overrightarrow{x}_{-n_k}\} \neq \varnothing$. But, since there are infinitely many indices and only finitely many pairs of continuua, there must be a pair of continuua A^i, A^j such that $(\pi_{-n_k}(A^i) \cap \pi_{-n_k}(A^j)) \setminus \{\overrightarrow{x}_{-n_k}\} \neq \varnothing$ for infinitely many k. But, if $(\pi_{-n_k}(A^i) \cap \pi_{-n_k}(A^j)) \setminus \{\overrightarrow{x}_{-n_k}\} \neq \varnothing$ for some k, then $(\pi_{-n_r}(A^i) \cap \pi_{-n_r}(A^j)) \setminus \{\overrightarrow{x}_{-n_r}\} \neq \varnothing$ for all $r < k$.

Thus, for all $r \in \omega$, $(\pi_{-n_r}(A^i) \cap \pi_{-n_r}(A^j)) \setminus \{\overrightarrow{x}_{-n_r}\} \neq \varnothing$. But this would mean that $A^i \cap A^j \setminus \{\bar{x}\} \neq \varnothing$, a contradiction. Thus, $\deg(\bar{x}) = \varphi(\bar{x})$. $\square$

Corollary 4.0.7: $\overline{x}$ is an endpoint of $\hat{H}$ if and only if $\overrightarrow{x}_{-n}$ is an endpoint of H for infinitely many $n \in \omega$.

Corollary 4.0.8: The number of branch points in $\hat{H}$ is exactly the number of periodic branch points in H, and the degree of a branch point in $\hat{H}$ is exactly the degree of all of its projections.

Proof. Let $\bar{b} \in \hat{H}$ be a branch point. Then $\deg(\bar{b}) \geq 3$. As such, by Lemma 4.0.3 and Theorem 4.0.6 this must mean that $\overrightarrow{b}_{-n}$ is a periodic branch point in H with $\deg_H(\overrightarrow{b}_{-n}) = \deg_{\hat{H}}(\bar{b})$ for all $n \in \omega$.

Each branch point in $\hat{H}$ can be uniquely identified by its zeroth projection. To see why, suppose $\bar{b}, \bar{x}$ are two branch points in $\hat{H}$ with $\overrightarrow{b}_0 = \overrightarrow{x}_0$. Then for all $n \in \omega$, $\overrightarrow{b}_{-n}$ and $\overrightarrow{x}_{-n}$ are both periodic points in the orbit of $\overrightarrow{b}_0$ which map to $\overrightarrow{b}_0$ in exactly n iterates, and thus must be the same point. So, for every branch point $\bar{b} \in \hat{H}$, there exists a unique periodic branch point $\beta \in H$ such that $\bar{b}_0 = \beta$.

Moreover, by the construction of inverse limits, for every periodic branch point β in H, there must be some $\bar{b} \in \hat{H}$ whose projections are all the periodic preimages of β. In particular, for each $\beta \in H$, there is some $\bar{b} \in \hat{H}$ such that $\bar{b}_0 = \beta$.

Thus, the number of branch points in $\hat{H}$ is exactly the number of periodic branch points in H and the degree of a branch point in $\hat{H}$ is is the same as the degree of all of its projections. $\square$

CHAPTER FIVE

Constructing Infinite Families with Pairwise Non-Homeomorphic Inverse Limits

Recall that as abstract dendritic Julia set is a triple, (D, H, f) where f is the associated function and H is the associated Hubbard tree. In this chapter our goal is to, given a dendritic Julia set, (D, H, f), construct two infinite families. Each element in the first family will have the form (D_i, H_i, f_i) and the family will have the property $D_i \cong D$ for all i, but $H_i \cong H_j$ if and only if $i = j$ and $\varprojlim\{D_i, f_i\} \cong \varprojlim\{D_j, f_j\}$ if and only if $i = j$. As a bonus, each member (D_i, H_i, f_i) of this constructed family is an actual dendritic Julia set and Hubbard tree of some complex quadratic polynomial.

Each element in the second family will, again, have the form (D_i, H_i, f_i), but, perhaps more surprisingly, both $D_i \cong D$ and $H_i \cong H$ for all i, but $\varprojlim\{D_i, f_i\} \cong \varprojlim\{D_j, f_j\}$ if and only if $i = j$.

To this end we need four things. The first is a rephrasing of [Bal12, Theorem 4.2]:

Theorem 5.0.1: If (D_1, H_1, f_1) and (D_2, H_2, f_2) are two dendritic Julia sets such that $\varprojlim\{D_1, f_1\} \cong \varprojlim\{D_2, f_2\}$, then $\varprojlim\{H_1, f_1\} \cong \varprojlim\{H_2, f_2\}$.

The second and third things we need are lemmas that helps prove the fourth.

Lemma 5.0.2: In an dendritic Julia set D, f is locally two-to-one at c_0.

Proof. Since D is self-similar under f we have $f(S_1 \cup \{c_0\}) = f(S_2 \cup \{c_0\}) = D$. So every point in $D \setminus \{c_1\}$ has exactly two preimages, one in S_0 and one in S_1. So any point in a neighborhood of c_1 has two preimages in a neighborhood of c_0, so f is locally two-to-one at c_0. $\square$

Lemma 5.0.3: If (H, f) has prime kneading sequence, then for any point $p \in H$ the set of of points that eventually map to p is arc-dense in D.

Proof. As shown in in [BKS11], $D = \overline{\bigcup_{n\in\omega} f^{-n}(H)}$. So the set of points in D that eventually map into H is dense in D. Moreover, $f^{-1}(H)$ is the union of the preimage of H in $\overline{S_0}$ together with the preimage of H in $\overline{S_1}$. Both of these are connected and contain c_0 so $f^{-1}(H)$ is connected.

So, since $\bigcup_{n\in\omega} f^{-n}(H)$ is a connected dense subset of D, we have that $\bigcup_{n\in\omega} f^{-n}(H)$ contains every point in D except possibly some endpoint. So any arc in D that does not contain an endpoint of D must map into H in finitely many iterations. Since the kneading sequence if prime, the system (H, f) is locally eventually onto so any arc in H eventually covers H. The result follows. $\square$

The fourth thing we need allows us to determine the structure of the dendrite from the Hubbard tree:

Theorem 5.0.4: Let D be an abstract dendritic Julia set with associated function f, Hubbard Tree, H, and critical point c_0. Let $\Delta = \{d :$ there is a cyclic branch point $b \in H$ with $\deg_H(b) = d\}$. Then D will have an arc dense set of branch points of degree d for each $d \in \Delta$. Moreover, if c_0 is periodic then D will have an arc dense set of branch points of degree infinity. If c_0 is pre-periodic, let $j = \max\{\deg_H(f^i(c_0))\}_{i\in\mathbb{N}}$. Then D will have an arc dense set of branch points of degree $2j$. D will contain no branch points other than those that are previously mentioned.

Proof. Let us first consider $Orb(c_0)$. Let c_i be the endpoint of H last visited by c_0. If c_0 is periodic, then let n be minimal such that $f^n(c_i) = c_0$. Then f^n is a homeomorphism on a sufficiently small neighborhood of c_i in D. Thus, since D is self-similar under $f, \deg_D(c_i) = \deg_D(c_0)$. However, by Lemma 5.0.2, f is locally two-to-one at c_0. So we have $\deg_D(c_0) = 2\deg_D(c_1) = 2\deg_D(c_i) = 2\deg_D(c_0)$ and so $\deg_D(c_0) = \infty$.

If c_0 is pre-periodic, let $j = \max\{\deg_H(f^i(c_0))\}_{i\in\mathbb{N}}$ and let m be maximal so that $c_m \neq c_i$ for all $i < m$ but $c_{m+1} = c_i$ for some $i < m+1$, i.e. let c_m be the point in $Orb(c_0)$ last visited by c_0. Because f is locally injective for all $x \in H \setminus \{c_0\}, \deg_H(x) \leq \deg_H(f(x))$ for all $x \in H \setminus \{c_0\}$. Then the degree of c_m in H must be maximal among $Orb(c_0)$ so c_m is a point degree j. Since f is a local homeomorphism at all points except c_0, we have $\deg_D(c_i) = j$ for all $i > 0$. But since f is locally 2-1 at c_0 and f is locally 1-1 on $D - \{c_0\}$ we must have that c_0 is a branch point in D of degree $2j$.

Since the set of precritical points is arc dense in D [Bal07, Proposition 1.17] we have that there is an arc dense set of degree infinity points (if c_0 is periodic) or degree $2j$ points (if c_0 is pre-periodic).

Let us now consider the branch points of H. Fix a branch point $b \in H$ let $d = \max\{\deg_H(f^j(b)) : j \in \omega\}$. (Note: every branch point is either periodic or pre-periodic, so this maximum is well defined). Let p be a periodic branch point in $Orb(b)$. Then $\deg_H(p) = d$ Since f is self similar on D, we have $\deg_D(b) = \deg_D(p)$. We want to show that $\deg_D(b) = \deg_H(p)$. Since $H \subseteq D$ we must have $\deg_H(p) \leq \deg_D(p)$. Suppose

$\deg_D(p) > \deg_H(p)$. Let $a \in D \setminus H$ be such that $[p, a]$ is an arc emanating from p. Then $[p, a] \cap H = p$.

Since the set of precritical points is dense in D, there is a minimal n such that $f^n([a, p])$ contains c_0. As such, there is a minimal $m \in \mathbb{N}$ such that $f^m([a, p)) \cap H \neq \varnothing$. But, on a sufficiently small neighborhood of p, f^m is a injective. So, if $f^{m-1}((p, a])$ maps into H under f, then there must be an arc emanating from $f^{m-1}(p)$ in H that maps out of H under f. But H is closed under f, a contradiction. So, $\deg_D(p) = \deg_H(p)$. Thus $\deg_D(b) = \max\{\deg_H(f^j(b)) : j \in \omega\}$.

By Lemma 5.0.3 we must have that for all $x \in D$ that eventually map to b (except for those that are pre-critical, mapping to c_0 before eventually mapping to b), $\deg_D(x) = d$. The self-similarity of D ensures that this set is arc-dense. So D contains an arc dense set of branch points of degree d for each $d \in \Delta$.

Suppose D had branch points of some other degree not in $\Delta \cup \{\deg_D(c_0)\}$. Then since D is self similar, we would know that these branch points are arc dense in D and so are also arc dense in H, though their degree in H would have to be two, because the degrees of all the branch points in H have already been taken into consideration during construction of Δ. Similarly, these branch points in D cannot not be precritical, since we have already covered the degrees of the precritical points. The arcs in $D \setminus H$ emanating from these branch points in H could never map into H (since f is 1-1 everywhere except c_0 and the points are not precritical), and so the itinerary of every point on these arcs would be the same as the itinerary of the branch point, contradicting the arc-density of precritical points in D. $\square$

Corollary 5.0.5: In any dendritic Julia set D, the set $\{d : \exists b \in D \text{ with } \deg_D(b) = d\}$ is finite.

Proof. Suppose the set is infinite. Then by Theorem 5.0.4, the set $\Delta = \{d :$ there is a cyclic branch point $b \in H$ with $\deg_H(b) = d\}$ must be infinite. But there are only finitely many branch points in H. $\square$

So now, given a Hubbard tree (H, f) we can determine the structure of the dendrite. We now seek to construct Hubbard trees whose dendrites will be homeomorphic to some given dendrite. This task breaks into two cases: if the given dendrite has degree infinity points, then the turning point in our Hubbard tree must be periodic, and otherwise the turning point must be pre-periodic. This gives us two cases, periodic kneading sequences, and pre-periodic kneading sequences. We are now ready to construct the first family.

5.1 Different Hubbard Tree, Same Dendrite, Different Inverse Limit

In order to achieve our goal of constructing an infinite family of pairwise homeomorphic abstract dendritic Julia sets with pairwise non-homeomorphic Hubbard trees, we rely heavily on internal addresses introduced in [BS08] which is defined as follows:

Definition 5.1.1: For a kneading sequence τ define

$$\rho_\tau : \mathbb{N} \to \mathbb{N} \cup \{\infty\}, \quad \rho_\tau(n) = \inf\{k > n : \tau_k \neq \tau_{k-n}\}.$$

If the kneading sequence τ is made clear from context, we typically write ρ instead of ρ_τ. For $k \geq 1$ we call $Orb_\rho(k) = k \to \rho(k) \to \rho^2(k) \to \dots$ the ρ-orbit of k. The ***internal address*** of a kneading sequence τ is the ρ-orbit of 1.

If there exists j for which $\rho^j(1) = \infty$ then we say that the internal address is finite $1 \to \rho(1) \to \cdots \to \rho^{j-1}(1)$. As a result Orb_ρ is a finite or infinite sequence that never contains ∞.

Notice that if τ is a periodic kneading sequence with prime period k then $Orb_\rho(\tau)$ is finite.

Along with the definition of an internal address comes a notion of an internal address and its associated kneading sequence being admissible.

Definition 5.1.2: An internal address ϱ is admissible if, for each $m \in \mathbb{N}$ at least one of the following is false:

1. ϱ contains m,

2. there exists some $k < m$ such that k divides m and $\rho(k) > m$,

3. $\rho(m) = \infty$ or if $r \in \{1, \dots, m\}$ is congruent to $\rho(m) \mod m$, then $m \notin Orb_\rho(r)$.

If an internal address is admissable, we also say that the associated kneading sequence is admissable.

The reason we bring this up is that we aim to, given a set, Δ, of degrees of branch points, construct an admissible internal address whose associated abstract dendritic Julia set has periodic branch points of degree d for each $d \in \Delta$.

Algorithm 5.1.3: Let $(d_1, d_2, ...d_n)$ be a list of (not necessarily unique) natural numbers such that $d_i \geq 3$ for all $i \in \{1, ..., n\}$. We define a function

$$IA : \{\text{finite lists of finite numbers greater than two}\} \rightarrow \{\text{finite internal addresses}\}$$

by $IA(d_1, d_2, \cdots d_n) = k_0 \rightarrow k_1 \rightarrow k_2 \rightarrow ... \rightarrow k_n$ where k_j is defined by the sequence $k_0 = 1$, $k_j = k_{j-1}(d_j - 1) + 1$.

Example 5.1.4: Suppose our list of numbers is $(5, 4, 3)$. Then $IA(5, 4, 3) = 1 \rightarrow 5 \rightarrow 16 \rightarrow 33$. We can then convert this, if we so desired, to a kneading sequence as in Algorithm 2.3 of [BKS11]. To do so we start with the string "1", and we repeat this string until we reach character number $k_1 = 5$. Here we flip the character to its opposite (i.e. if regular repetition would make the 5^{th} character a 1, we make the 5^{th} character a 0 and vice-versa). So our new string is "11110". We now copy and paste this string until we reach the $k_2 = 16^{th}$ character, at which point we flip the 16^{th} character to its opposite. So our string is now "1111011110111100". We now repeat this string until we reach the $k_3 = 33^{rd}$) character, but since 33 is the largest number in the internal address, and must correspond to the period of the critical point we write a $*$ for the 33^{rd} character. We make the whole sequence repeat, and we have our kneading sequence, $\tau = \overline{*11110111101111001111011110111100}$.

Lemma 5.1.5: In a kneading sequence made from any internal address in the range of IA every block of 1's is of length $k_1 - 1$.

Proof. Let τ be the kneading sequence associated with an internal address $k_1 \rightarrow \cdots \rightarrow k_n$ in the range of IA. By definition, $k_j \cong 1 \mod k_{j-1}$ so we are guaranteed to iterate through the sequence $\tau_1\tau_2, \cdots \tau_{k_{j-1}}$ a whole number of times before flipping character k_j. This means that each block of 1's is an exact copy of the very first block of 1's which is of length $k_1 - 1$. □

Lemma 5.1.6: Any tree whose internal address is in the range of IA will have $k_n - n + 1$ endpoints where n is the index of the last element in the internal address.

Proof. Let τ be the kneading sequence associated with the given internal address. By construction of τ, the last block of 0s before repetition has length n and is longer than any other block of 0s. Let w be the index of the first 0 in this block. Thus, by Theorem 2.2.12, $\sigma^w(\tau)$ is the itinerary of an endpoint of H and $\sigma^{w-1}(\tau)$ is not the itinerary of an endpoint of H.

The kneading sequence has prime period k_n, and each shift of the critical point is an endpoint except for those shifts whose itineraries begin $0^{n-m}*$ for some $1 \leq m < n$, i.e. $f^j(c_0)$ is an endpoint if and only if $0 < j \leq k_n - n + 1$, so there are exactly $k_n - n + 1$ many endpoints. □

Lemma 5.1.7: In any tree whose internal address is in the range of IA, for any endpoint, p', in S_0 with itinerary p that maps into S_1 we have $V(\sigma(\tau), \tau, \sigma(p)) \neq \overline{1}$. That is to say, $f(p') \in T_1$ (the tree emanating from $\overline{1}$ containing c_1).

Proof. Any endpoint in S_0 that maps into S_1 must map to an endpoint by the same reasoning as in the proof of Lemma 5.1.6. But every block of 0's is followed by a $*$ or $k_1 - 1$ many 1s. So if we map to an endpoint in S_1, the itinerary of that endpoint must begin with exactly k_1 many 1s.

So, $\sigma(p)$ begins $1^{k_1-1}0\dots$. Thus,

$$V(\sigma(\tau), \tau, \sigma(p)) = 1^{k_1-1}0^\frown V(\sigma^{k_1+1}(\tau), \sigma^{k_1}(\tau), \sigma^{k_1+1}(p)) \neq \overline{1}.$$

Thus, $[c_1, f(p')] \cap [c_1, c_0] \cap [c_0, f(p')]$ is not the fixed branch point, so $f(p')$ is either in T_1 or T_0^*? But, since $\sigma(p)$ begins $1^{k_1-1}0\dots$, we have that $f(p'), f^2(p'), \dots, f^{k_1-1}(p')$ are all in S_1, but $f^{k_1}(p')$ is in S_0. By Lemma 3.0.7 T_0^* maps injectively into T_1 which maps injectively into T_2 and, continuing in this manner, T_{k_1} maps injectively into T_0. So, any point in T_0^* requires at least k_1 many iterations of f to map into S_0. Since p' maps into S_0 in fewer iterates, p' must be in T_1. $\square$

Lemma 5.1.8: Any internal address in the range of IA is admissible.

Proof. Let τ be the kneading sequence associated with an internal address $k_1 \to \dots \to k_n$ in the range of IA. It has been shown (6.5, 6.6, [Sch17]) that the internal address associated with τ is admissible if no shift $\sigma^k(\tau)$ exceeds τ (without $*$) with respect to lexicographic ordering.

To see that $\sigma^0(\tau)$ is indeed maximal, lexicographical speaking, we begin by recalling that every string of 1s in τ is exactly $k_1 - 1$ characters long by Lemma 5.1.5. So the largest lexicographic shift of τ we must start with $k_1 - 1$ many 1s. So the first 0 is at position k_1 and there is a 0 at every whole number multiple of k_1.

Similarly, in τ each block of two 0s are such that the last 0 in the block is a whole number multiple of k_2 apart from one another. That is, there is a 0 at every whole number multiple of k_2. So, in order for a shift of τ to be the largest lexicographic shift possible it must have the first occurrence of a second 0 to be at position k_2. Similarly there are 0s at every multiple of k_3, and these 0s come at the end of a block of length 3, so the maximal shift will be such that the first occurrence of a third 0 is at position k_3. Continuing in this way, there is a 0 at every multiple of k_j and these 0s come at the end of a block of length j, so the furthest such a 0 can be from the start of the string is at position k_j. There is only one shift that can accomplish all of these, and that is the zeroth shift. Thus $\sigma^0(\tau) = \max\{\sigma^i(\tau)\}$ with respect to lexicographical ordering, so τ and its internal address are admissible. □

Now that we know that the kneading sequence is admissible, we introduce a nifty fact, proven in the the same paper that originally defined internal addresses:

Theorem 5.1.9: A Hubbard tree can be realized by a quadratic polynomial if and only if the associated kneading sequence does not fail the admissibility condition for any $m \in \mathbb{N}$.

Theorem 5.1.10: Let D be a dendritic Julia set with associated Hubbard tree (H, f) such that D has a branch point of infinite degree (i.e. c_0 is periodic under f). Let $\Delta = \{d_1, d_2, ...d_n\}$ be the set of finite degrees of branch points in D. Then $IA(d_1, d_2, ...d_n)$ is the internal address of a Hubbard tree (H', f') whose associated dendrite D' is homeomorphic to D.

Proof. Lemma 5.1.8 shows that the internal address is admissible.

In [BS08, Proposition 4.19], it is shown that if for any $m \geq 1$, we let $r \in \{1, 2, ..., m\}$ be congruent to $\rho(m)$ modulo m, and define

$$q(m) = \begin{cases} \frac{\rho(m)-r}{m} + 1 & \text{if } m \in Orb_\rho(r) \\ \\ \frac{\rho(m)-r}{m} + 2 & \text{if } m \notin Orb_\rho(r) \end{cases}$$

then, for any entry m in the internal address with $q(m) \geq 3$, there is a branch point of exact period m whose degree is $q(m)$ (unless m is the period of the critical point). Moreover, if z is a branch point of exact period m, then m is in the internal address.

In this way, the internal address contains a substantial portion of the dynamical information about the periodic branch points of the tree.

The internal address constructed by this algorithm creates a Hubbard tree (H', f') that contains periodic branch points of the appropriate degrees since, by construction $q(k_{j-1}) = d_j$.

Hence, H' is such that, for each $d_j \in \Delta$, there is a periodic branch point, $p \in H$ whose degree is d_j.

Let D' be the dendritic Julia set associated with (H', f'). By Theorem 5.0.4, D' will have an arc-dense set of branch points of degree d_j for each $d_j \in \Delta$ and, since the critical point of H' is periodic under f', an arc dense set of degree infinity branch points.

In [CD94, Theorem 6.2], it is shown that if two abstract dendritic Julia sets, J, L are such that $\{n \in \mathbb{N}$: there exists an arc dense set of branch points in J with degree $n\} = \{n \in \mathbb{N}$: there exists an arc dense set of branch points in L with degree $n\}$, then J and

L are homeomorphic. Thus, since they share the same set of degrees of arc dense branch points, $D' \cong D$. $\square$

Corollary 5.1.11: Every abstract dendritic Julia set with an arc dense set of degree infinity branch points is homeomorphic to the Julia set of some quadratic polynomial.

Proof. Let D be an abstract dendritic Julia set with an arc dense set of degree infinity branch points. Note that dendritic Julia sets have a finite set of degrees its branch points can take, so we can apply Theorem 5.1.10 to the finite list of finite degree branch points in D to make an admissible internal address of a Hubbard tree whose associated dendrite is homeomorphic to D. Since the internal address is admissible, the Hubbard tree can be realize by a quadratic polynomial by Theorem 5.1.9. $\square$

Theorem 5.1.12: Let (D, H, f) be an abstract dendritic Julia set with Hubbard tree H and function f with a periodic critical point. Then there is an infinite family $\langle (D_i, H_i, f_i) \rangle_{i \in \omega}$ all of which have a critical point that is periodic under f_i, and $H_i \cong H_j$ if, and only if, $i = j$ and $D_i \cong D$ for all i.

Moreover, $\varprojlim \{D_i, f_i\} \cong \varprojlim \{D_j, f_j\}$ if and only if $i = j$.

Proof. If the set of finite degrees of arc-dense branch points in D is $\{d_1, d_2, \ldots, d_n\}$, then we can get homeomorphic dendrites by using Algorithm 5.1.3 on $(d_1, d_2, \cdots, d_n)$ or $(d_1, d_1, d_2, \cdots, d_n)$, or $(d_1, d_1, d_1, d_2, \cdots, d_n)$, etc. But, by construction, each of these Hubbard trees will have sets of periodic branch points of degree d_1 with distinct cardinalities and since these are the only branch points of degree d_1 in the Hubbard tree, the underlying trees will be non-homeomorphic.

Furthermore, for the family of Hubbard trees (H_i, f_i) let B_i be the set of periodic branch points in (H_i, f_i). Then we have $|B_i| = |B_j|$ if and only if $i = j$. But by Theorem 4.0.6 this must mean $\varprojlim\{H_i, f_i\} \cong \varprojlim\{H_j, f_j\}$ if and only if $i = j$, and so, by Theorem 5.0.1 we have $\varprojlim\{D_i, f_i\} \cong \varprojlim\{D_j, f_j\}$ if and only if $i = j$. □

5.2 Same Hubbard tree, Same Dendritic Julia set, Different Inverse Limits

In this section we demonstrate that for each dendritic Julia set D, there is a countable collection of distinct kneading sequences whose associated dendrites are homeomorphic to D but which have mutually non-homeomorphic inverse limits. In order to do so, we first prove that each Hubbard tree is compatible with a countable family of dynamically distinct unimodal maps. In particular, the postcritical periods of this family are not bounded, which will be crucial to our main result.

We first we recall Lemma 2.2.5 which says that if τ is acceptable, then there is a Hubbard tree with kneading sequence τ which has the unique itinerary property.

Lemma 5.2.1: Let (H, f) be a Hubbard Tree with a prime kneading sequence. Then there exist infinitely many unimodal maps $g : H \to H$ sharing the same critical point as f such that (H, g) is a Hubbard tree with the unique itinerary property.

Proof. Let (H, f) be a Hubbard Tree with prime kneading sequence, τ. Since prime kneading sequences are acceptable [Bal07, Theorem 3.10] (H, f) is equivalent to a Hubbard tree with the unique itinerary property, so we may assume that it has the unique itinerary property. Let c_0 be the critical point of (H, f) and let c_i be the last endpoint visited by its orbit. Let b be the point in I_0 nearest c_i. Since $f(b)$ and $f(c_i)$ have different itineraries, we can choose a point $q \in (f(b), f(c_i))$ which is not on the critical orbit but which is precritical. Moreover, since Hubbard trees with prime kneading sequences are locally eventually onto, we can choose a precritical q that maps near the fixed branch point before it maps to c_0. In particular, we can choose a q whose precritical itinerary contains arbitrarily many 1's. Note that for any $n \in \mathbb{N}$ there is a neighborhood near $\overline{1}$ such that all points in the neighborhood have an itinerary that starts with at least n many 1's.

If the orbit of q meets $[b, c_i)$, choose $j \in \mathbb{N}$ minimal such that $f^j(q)$ is the point in $Orb(q) \cap [b, c_i)$ closest to c_i and let $w = f^j(q)$. Otherwise let $w = b$. Since $f((b, c_i))$ contains q, there is a point, $s \in (b, c_i)$ that maps to q. We have two cases. If $s \in (w, c_i)$ then let $r = w$. Otherwise, if $w \in (s, c_i)$, then we find a point $t \in (w, c_i)$ such that t eventually maps to q. Such a t must exist by LEO. Let $r = t$. Then $[f(r), q]$ is a non-degenerate arc in H.

Define $f_q : H \to H$ as follows. First, fix a linear map $l : [r, c_i] \to [f(r), q]$ with $l(r) = f(r)$ and $l(c_i) = q$. Then we define

$$f_q(x) = \begin{cases} f(x) : & x \in H \setminus (r, c_i] \\ l(x) : & x \in [r, c_i] \end{cases}$$

We now show (H, f_q) is a Hubbard Tree by verifying conditions (1)-(6) in Definition 1.2.17.

1. (continuous and surjective) Notice that for $0 \leq j \leq i$, $f_q(c_j) = f(c_j)$, and thus $\{c_0, c_1, \ldots c_i\} \in f_q(H)$. Since this is the collection of endpoints of H, it follows that

$f_q(H) = H$, i.e. f_q is surjective. What's more, the function f_q is continuous and well-defined as it is piecewise continuous and $f(r) = l(r)$.

2. (every point has at most two preimages) Fix $p \in H$. Observe that $f_q^{-1}(p) = (f^{-1}(p) \setminus [r, c_i]) \cup l^{-1}(p)$. Notice that $f^{-1}(p) \setminus [r, c_i]$ has at most two elements and $l^{-1}(p)$ at most one. Indeed, if $f^{-1}(p) \setminus [r, c_i]$ has two elements, then $p \notin [f(r), f(c_i)]$ (else p has three preimages under f), and thus $l^{-1}(p)$ is empty. Thus $f_q^{-1}(p)$ consists of at most two elements.

3. (locally 1-1 at all points except c_0) Observe that $c_0 \notin (r, c_i]$ and thus $f_q(c_0) = f(c_0)$ is an endpoint of H. Since c_0 is *not* an endpoint, it follows that f_q is not locally one-to-one at c_0.

   Now, consider $p \in H \setminus \{c_0\}$.

   If $p \notin [r, c_i]$ then, since $[r, c_i]$ is closed in H and does not contain p, it follows that there is a neighborhood of p on which $f_q = f$. Thus f_q is locally one-to-one at p if and only if f is. Since $p \neq c_0$, f_q is indeed locally one-to-one at p.

   If $p \in (r, c_i]$ then, since $(r, c_i]$ contains no branch points of H by construction, it follows that there is a neighborhood of p on which f_q is equal to l, and thus f_q is locally one-to-one at p.

   Finally, if $p = r$ then, we can choose a neighborhood U of r that does not include any other branch points, endpoints, or c_0. Suppose $x, y \in U$ with $x \neq y$, and $f_q(x) = f_q(y)$. Then at least one of x, y is in $[r, c_i]$ because otherwise, $f(x) = f_q(x) = f_q(y) = f(y)$ but f is one-to-one on U. They cannot both be in $[r, c_i]$ since $f_q|_{[r,c_i]} = l$

is linear, and thus one-to-one. Without loss of generality, $x \in [r, c_i]$ and $y \notin [r, c_i]$. But then $f_q(y) = f(y)$ and since $x \in [r, c_i]$, $f_q(x) \in [f(r), q]$ and thus $f(y) \in [f(r), q] \subseteq f([r, c_i])$. Thus f is not one-to-one on $[y, c_i]$, and thus $c_0 \in (y, c_i)$. Since $c_0 \notin (r, c_i]$, we see that $c_0 \in (y, r) \subseteq (y, x) \subseteq U$, a contradiction.

4. (all endpoints are in $Orb(c_0)$) $f^k(c_0) = f_q^k(c_0)$ for all $0 \leq k \leq i$ and after i iterates, c_0 has visited every endpoint.

5. (c_0 is periodic or pre-periodic, but not fixed) Notice that $f_q(c_0) = f(c_0)$, and thus c_0 is not fixed. Since, for $k > i$, $f_q^k(c_0) = f_q^{k-i-1}(q)$ and f_q is equal to f on the orbit of q, it follows that c_0 is either periodic (if q is precritical) or pre-periodic (if q is (pre-)periodic).

6. (weak unique itinerary property) Let x, y be distinct points, each of which is a branch point or a point on the critical orbit under f_q. If they are both branch points, then $f^k(x) = f_q^k(x)$ and $f^k(y) = f_q^k(y)$ for all $k \in \mathbb{N}$ and so there exists an $n \in \mathbb{N}$ with $c_0 \in f_q^n([x, y])$.

   If at least one of x, y is not a branch point, (say x), then x is in the critical orbit, and so there is some $n \in \omega$ so that $f^n([x, y])$ contains c_0.

Thus, (H, f_q) is a Hubbard Tree and since q's itinerary under f (and thus under f_q) is different than the itinerary of $f(c_i)$ under f, (H, f_q) is a Hubbard tree, homeomorphic to (H, f) with the same critical point but with a different kneading sequence.

Since Hubbard trees with prime kneading sequences are locally eventually onto [Bal07, Theorems 4.10, 4.13], we can choose our point q so that the pre-critical itinerary of q contains an arbitrarily long block of 1's. In particular, q can be chosen so that the longest block of 1's in pre-critical itinerary of q is longer than the longest block of 1's and $*$ appearing in τ. So, for example, if $\tau = \overline{*110111111}$, then the longest block of 1's and $*$ is 9 characters long. Then, since the new kneading sequence under f_q is periodic, in any shift of the new kneading sequences, this longer block of 1's must overlap a zero. So the new kneading sequence is acceptable.

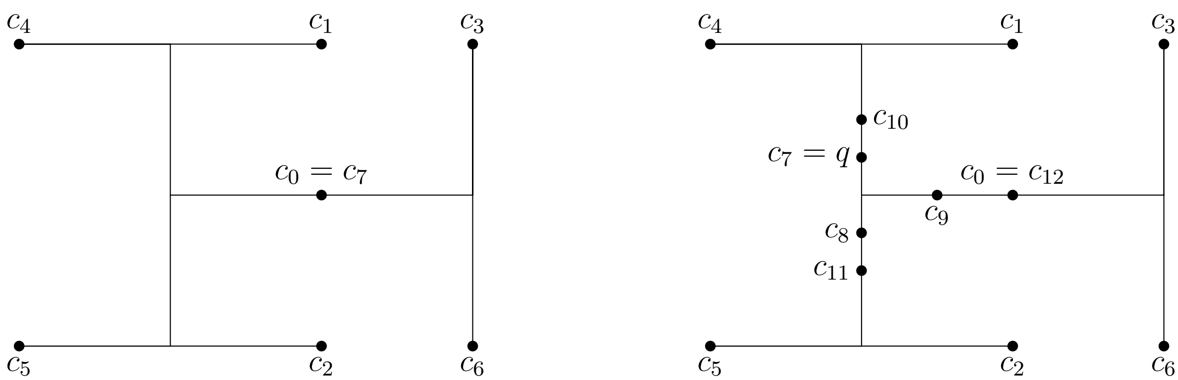
Since any tree with an acceptable kneading sequence is equivalent to a tree with the same kneading sequence and the unique itinerary property. Thus, there exists a Hubbard Tree (H, g), homeomorphic to (H, f), with the same kneading sequence and same critical point as (H, f_q) that has the unique itinerary property.

The choice of q is arbitrary so there exist infinitely many such kneading sequences giving rise to infinitely many such trees. $\square$

Figure 5.1 provides a concrete example of Lemma 5.2.1 in action. It starts with a Hubbard tree whose kneading sequence is $\overline{*110110}$ and alters the function f by mapping c_6 to a precritical point q. This new Hubbard tree (H, f_q) has kneading sequence $\overline{*11011011111}$.

Knowing that we can get an infinite family of kneading sequences that each give rise to a homeomorphic tree with the same critical point leads to a natural question: what can we say about the dendrites?

Corollary 5.2.2: Let (H, f) be a Hubbard tree with kneading sequence ν. Let τ be a different kneading sequence for H as constructed in Lemma 5.2.1, giving rise to (H, g). Then the associated dendrites D_ν and D_τ are homeomorphic.



(a) A Hubbard Tree with kneading sequence $\overline{*110110}$

(b) A Hubbard Tree with kneading sequence $\overline{*11011011111}$

Figure 5.1. Using 5.2.1 to construct a Hubbard tree with different kneading sequence

Proof. Under both τ and ν we have that c_0 is periodic or pre-periodic with the same eventual orbit period. Moreover, the orbit of each branch point is unaffected by the change from ν to τ since the process of altering the function in Lemma 5.2.1 only alters the orbits of points whose orbits intersect the arc $(r, c_i]$ which does not contain any branch points. So by Theorem 5.0.4 we have that the dendrites D_τ and D_ν will both contain the same sets of arc dense branch point degrees and will thus be homeomorphic by [CD94, Theorem 6.2]. $\square$

Theorem 5.2.3: Let D be a dendritic Julia set with acceptable, periodic kneading sequence, τ. Then there are countably many other acceptable kneading sequences $\langle \tau_i \rangle_{i \in \omega}$ with associated dendrites, $\langle D_i \rangle_{i \in \omega}$, such that each D_i is homeomorphic to D and the associated Hubbard trees, $\langle H_i \rangle_{i \in \omega}$, are also homeomorphic with the same critical point, but the inverse limit spaces, $\varprojlim \{D_i, \sigma_i\}$ are mutually non-homeomorphic.

Proof. Assume hypotheses as stated. Then D contains a periodic Hubbard tree, H_τ. Theorem 5.2.1 shows that there are countably many periodic kneading sequences $\langle \tau_i \rangle_{i\in\omega}$ that give rise to the same tree, and thus, by Corollary 5.2.2, to the same dendrite.

Moreover, we can make it so that the period of τ_i is less than the period of τ_{i+1} for all i. Recall that Corollary 4.0.7 says that a point $\overline{x} \in \hat{H}_\tau$ is an endpoint if and only if it projects to an endpoint infinitely often. So then $|\text{end}(\hat{H}_{\tau_i})| < |\text{end}(\hat{H}_{\tau_{i+1}})|$ for all i. and thus $\varprojlim\{H, f_i\} \ncong \varprojlim\{H, f_j\}$ for all $i \neq j$. Since the Hubbard tree inverse limit is a topological invariant of the dendrite inverse limit, the same dendrite can have countably many non-homeomorphic inverse limits. $\square$

CHAPTER SIX

Further Work

6.1 Future Directions For Chapter Three

Much of Chapter Three is concerned with fully describing all Hubbardizable trees with fewer than four branch points. While methods similar to those found in Chapter Three can be used to fully describe all Hubbardizable trees with four, five, six branch points, etc. to do so would require a staggering amount of case-checking.

It would be remarkable to see a relatively simple necessary and sufficient condition for Hubbardizablity, although the author can provide no useful direction.

6.2 Future Directions For Chapter Five

In Chapter Five we learned a process to take a given Hubbard tree (H, f), slightly alter the function to make a new function g such that (H, g) is also a Hubbard tree. Along these lines, we present a conjecture that seems true based on empirical evidence, but we have been unable to prove.

Much like the "altering of f" done in Chapter Five, this conjecture (if true) provides a step by step method to start with a Hubbard tree (H, f) and create infinitely many Hubbard trees (H, f_i) such that $\varprojlim\{H, f_i\} \cong \varprojlim\{H, f_j\}$ if and only if $i = j$. The only stipulation, is that under the original function f, the critical point must be periodic. The full conjecture is as follows:

Conjecture 6.2.1: Let (H, f) be a Hubbard tree such that the critical point is pre-periodic under f. Based on Definition 5.1.1, the internal address of (H, f) is an infinite list of increasing integers

$$\langle I_k \rangle_{k \in \omega} = 1, \rho(1), \rho^2(1), \rho^3(1), \ldots$$

There exists an $n \in \omega$ such that for all $m \geq n$ there exists a Hubbard tree (H_m, f_m) such that $H_m \cong H$ and the internal address of (H_m, f_m) is $\langle I_k \rangle_{k=0}^{m}$ (that is, we truncate the internal address of (H, f) after the first m entries). Moreover, $\varprojlim\{H_m, f_m\} \cong \varprojlim\{H_r, f_r\}$ if and only if $r = m$ and $\varprojlim\{H_m, f_m\} \ncong \varprojlim\{H, f\}$ for all m.

Along these lines, we would love to prove a pre-periodic counter part to Theorem 5.2.3. It might look something like the following:

Conjecture 6.2.2: Let (D, H, τ) be a dendritic Julia set with pre-periodic kneading sequence. Then there are countably many other dendritic Julia sets $\langle (D_i, H_i, \tau_i) \rangle_{i \in \omega}$ such that for all $i \in \omega$ we have τ_i is pre-periodic, $H_i \cong H$, $D_i \cong D$ and the inverse limit spaces, $\varprojlim\{D_i, \sigma_i\}$ are mutually non-homeomorphic.

The difficult part of the proof is to show that the inverse limits of the Hubbard trees (and thereby the inverse limits of the dendrites) are non-homeomorphic. In Theorem 5.2.3 we could distinguish inverse limits by the number of endpoints, this technique will not work to distinguish inverse limits of Hubbard trees with pre-periodic kneading sequences since many of these inverse limits will have no endpoints at all. We believe that any method of distinguishing these inverse limits will rely on folding points; an idea that was first introduced by Raines in [Rai04] and is defined as follows:

Definition 6.2.3: We say $x \in X$ is a *folding point* if for all open covers $\mathcal{U}$ of X, there exists an open cover $\mathcal{C} \prec \mathcal{U}$ and a pair of sets $C_0, C_1 \in \mathcal{C}$ with $x \in C_1$ and there exists an open cover $\mathcal{V} \prec \mathcal{C}$ such that for all open covers $\mathcal{D} \prec \mathcal{V}$ there exists a chain $D_1, \ldots, D_m \in \mathcal{D}$ and a number $1 < j < m$ with

- $D_p \cap D_q \neq \varnothing$ if and only if $|p - q| \leq 1$,
- $\bigcup_{i=1}^{m} D_i \subseteq C_0 \cup C_1$,
- $D_1 \cup D_m \subseteq C_0 \setminus C_1$ and $D_j \subseteq C_1 \setminus C_0$.

We believe that folding points will be key because we also believe the following.

Conjecture 6.2.4: A point $\bar{x} \in \hat{H}$ is a folding point if and only if $\overrightarrow{x}_{-n} \in Orb(c_0)$ for infinitely many $n \in \omega$.

We have made many attempts to prove this conjecture, and while this *should* be true (morally speaking) it has proven difficult to fully demonstrate outright. But any young, enterprising topologist should be able to make some progress on the claim. This claim, together with methods similar to those found in the proof of Lemma 5.2.1 should be sufficient to prove Conjecture 6.2.2.

APPENDIX

APPENDIX A

Kneading Sequence Arguments

This appendix contains many arguments about creating a tree from a given kneading sequence. If you would like to play with kneading sequences yourself and discover which sequences give rise to which trees, click this link.

This links to a web page in which you can enter a periodic kneading sequence and generate the tree. The way this is done is by implementing the voting sequence on all possible triples of distinct shifts of the kneading sequence. This generates a list of all the itineraries of branch points of T. It then runs the voting sequence on all possible triples of distinct points from the critical orbit and branch points to find out which points are in a free arc with each other. If two points p, q are in a free arc then $V(p, q, x) \in \{p, q\}$ for all x that are either branch points or in the critical orbit. Once it is known which points are in free arcs with one another, it builds the appropriate tree. The source code can be found by clicking here. But we walk through some psuedo code below. While the itineraries of every point is infinitely long, every branch point or point in the critical orbit is (pre-)periodic so we can compute the itineraries in finite time.

Psuedo-code

1. Fix $n = |Orb(\tau)|$

2. Make a list, "orbTau" of the n shifts of τ

3. Make a list, "BranchPoints", which is a copy of orbTau

4. for i in range[0, n-2]:

   for j in range[i+1, n-1]:

   for k in range[j+1, n]:

   compute V(orbTau[i]. orbTau[j], orbTau[k])

   If V(orbTau[i]. orbTau[j], orbTau[k]) not in BranchPoints, add it to BranchPoints

5. Make a list called "freeArcs"

6. for i in range[0, len(BranchPoints)-2]:

   for j in range[i+1, len(BranchPoints)-1]:

   compute V(BranchPoints[i], BranchPoints[j], BranchPoints[k]) for all k

   If V(BranchPoints[i], BranchPoints[j], BranchPoints[k]) = BranchPoints[i] or BranchPoints[j] for all k, then BranchPoints[i], BranchPoints[j] are in a free arc so add the tuple (BranchPoints[i], BranchPoints[j]) to freeArcs

7. Use the list of edges in freeArcs to construct the tree

What follows is proofs of various claims about certain trees being generated from various kneading sequences throughout the work.

Claim A.0.1: (From Theorem 2.2.13) To construct a Hubbard Tree with n colinear branch points of degree m in S_0, a fixed branch point of degree x in S_1, x many branch points of degree m in S_1, and no other branch points, one can use the following kneading sequence:

$$\tau = \overline{*(1^x 0^n)^{m-1}}.$$

Proof. For every branch point $b \in S_0$, there exists at least two endpoints e_1, e_2 in S_0 such that $\{b\} = [c_0, e_1] \cap [c_0, e_2] \cap [e_1, e_2]$. As such, to verify claims about branch points in S_0 we need only concern ourselves with voting sequences of the form $V(\tau, e_1, e_2)$ where e_1, e_2 are endpoint in S_0.

Let $\tau = \overline{*(1^x0^n)^{m-1}}$ for some n, x, m. By Theorem 2.2.12, we have that the sequence $\overline{0^n\tau}$ is the itinerary of an endpoint and the sequence $\overline{0^{n-1}\tau}$ is the itinerary of some non-endpoint in the critical orbit. So, we have $(n)(m-2)+1$ many endpoints in S_0.

Let $p(k,i)$ denote the itinerary of the point in the critical orbit whose itinerary starts with the i^{th} 0 in the k^{th} block of 0s. So, for example, if $\tau = \overline{*1100011000}$ then $p(1,1) = \overline{00011000*11}$ and $p(2,3) = \overline{0*110001100}$.

Fix $1 \leq j \leq i \leq n$ and let $p(k,j), p(w,i)$ be the itineraries of two distinct endpoints in S_0. By inspection, $V(\tau, p(k,j), p(w,i)) = \overline{0^{n-i+1}1^x0^{i-1}}$. In the case where $j = i = 1, V(\tau, p(k,j), p(w,i)) = \overline{0^n1^x}$.

By the unique itinerary property of branch points, this gives the itinerary of n distinct branch points in S_0. To show these all must have degree m, it suffices to show that $\deg(\overline{0^n1^x}) = m$ since the branch points are periodic. There are $m-1$ many endpoints in S_0 whose itineraries also begin with 0^n. Let e be such an endpoint (then $\iota(e) = p(w,1)$ for some w, and let q be another end point in T. We have two cases:

1. $q \in S_1$. By inspection, $V(\iota(e), \iota(q), \overline{0^n1^x}) = \overline{0^n1^x}$ and so the branch point with itinerary $\overline{0^n1^x}$ is in $[e,q]$.

2. $q \in S_0$. Then $\iota(q) = p(k,i)$ for some i and $1 \leq k \leq n$. By inspection, $V(p(w,1), p(k,j), \overline{0^n1^x}) = \overline{0^n1^x}$ so the branch point with itinerary $\overline{0^n1^x}$ is in $[q,e]$

Thus, for any endpoint whose itinerary starts with n copies of 0, and any end point $q \in orb(\tau)$ we have $\overline{0^n 1^x} \in [e, q]$. So there are no branch points between $\overline{0^n 1^x}$ and e. Thus there are m arcs emanating from $\overline{0^n 1^x}$ (to the $m-1$ end points and to τ). So the degree of the branch point with itinerary $\overline{0^n 1^x}$ is m, and thus the degree all branch points of the form $\overline{0^{n-i+1} 1^x 0^{i-1}} = m$. But these are all the branch points in S_0. □

Claim A.0.2: (From Theorem 2.2.13) Moreover, to construct a Hubbard Tree with exactly n branch points, each of degree m one can let $x = 1$. In this case we get the following kneading sequence:

$$\tau = \overline{*(10^{n-1})^{m-1}}.$$

Proof. Let T be a Hubbard tree with kneading sequence τ. Since the kneading sequence begins $*10\ldots$, the fixed point in S_1 must have degree 2 by Lemma 2.2.9. Moreover, by Theorem 2.2.12, the last block of 0s in τ corresponds to an endpoint of T. I.e. the point in the orbit of τ with itinerary $0^{n-1}\tau$ is an endpoint. Thus, every preceding shift of τ is an endpoint.

Let p', q' be distinct endpoints in S_1 with itineraries p, q respectively. Then p, q both begin $10^{n-1}\ldots$. By inspection, $V(p, q, \tau) = \overline{10^{n-1}}$. Since this is the only itinerary that one can get as a result of applying the voting sequence to two endpoints in S_1 and c_0, we must have that S_1 has only one branch point. Since there are $m-1$ many endpoints in S_1, this branch point must have degree m (since it is in a free arc with the $m-1$ endpoints and c_0).

Now, let us consider branch points in S_0. Let p', q' be distinct endpoints in S_0 with itineraries p, q respectively. Then p begins $0^{j_p}1\dots$ or $0^{j_p}*\dots$ and q begins $0^{j_q}1\dots$ or $0^{j_q}*\dots$. We cannot have both p and q begin $0^{j_p}*\dots$ and $0^{j_q}*\dots$ since we require p' and q' to be endpoints. (Note that any point in the orbit of τ whose itinerary begins 0^k* for $k < n-1$ is not an endpoint).

Let $j = \min\{j_p, j_q\}$. Then, by inspection $V(p,q,\tau) = \overline{0^j 1 0^{n-1-j}}$. As such, every branch point in S_0 has itinerary $\overline{0^j 1 0^{n-1-j}}$ for some $1 \leq j \leq n-1$. Each of these branch points must be in the orbit of the branch point from S_1 (since the itineraries are shifts of the itinerary of the branch point in S_1). Since the branch point in S_1 is periodic of degree m, we have that every branch point in S_0 is periodic of degree m.

Thus, T has exactly n branch points, each of degree m. □

Claim A.0.3: (From Lemma 3.1.3) F_0^k can be made by $\tau = \overline{*1^{k-1}0}$ and F_1^k can be made by $\tau = \overline{*(1^{k-1}0)^{k-1}}$.

Proof. We begin by letting T be a Hubbard tree with kneading sequence $\tau = \overline{*1^{k-1}0}$. We show that this kneading sequence yields a $k-$od. By Theorem 2.2.12, the point in the critical orbit with itinerary 0τ is an endpoint, and as such, there is only one endpoint in S_0, and thus no endpoints in S_0. Moreover, since the preimage of an endpoint is either an endpoint or the critical point, this means that every point in the critical orbit (except the critical point itself) is an endpoint. So, the Hubbard tree with kneading sequence τ must have exactly k endpoint.

By Lemma 2.2.9 we know that there is a branch point in S_1 whose degree is k. But since there are only k endpoints, every endpoint must be in a free arc with this point. So there is only one branch point and it is of degree k. Thus, any Hubbard tree with kneading sequence τ must be a k-od.

Now, let T be a Hubbard tree with kneading sequence $\tau = \overline{*(1^{k-1}0)^{k-1}}$. Again, by Theorem 2.2.12 we have that the point in the critical orbit with itinerary 0τ is an endpoint. So we find that every point in the critical orbit (except c_0 itself) is an endpoint. So there are exactly $k(k-1)$ endpoints in T. Note that this is the same number of endpoints in F_1^k.

By Theorem 2.2.9, there is a fixed branch point in S_1 whose degree is k. By inspection, if p, q are itineraries of distinct endpoints in S_0, then $V(p, q, \tau) = \overline{01^{k-1}}$. As such, there is exactly one branch point in S_0 and it's itinerary is $\overline{01^{k-1}}$. This branch point is degree k (since there are $k-1$ endpoints in S_0 and this branch point is in a free arc with each of them, and in a free arc with c_0).

Now, let p, q be itineraries of distinct endpoints in S_1. Notice, then, that p and q must both start with a block of 1s. These blocks can either be the same length, or different lengths. If they are the same length, say length j, then $V(p, q, \tau) = \overline{1^j 0 1^{k-1-j}}$. Since each of these branch points is in the periodic orbit of the branch point with itinerary $\overline{01^{k-1}}$, we have that each of these branch points is degree k. If the starting blocks of 1s are differing lengths, then $V(p, q, \tau) = \overline{1}$.

Lastly, to truly verify that $T \cong F_1^k$ we show that the branch point $\overline{1}$ is the "central branch point". With this in mind, let p, q be itineraries of distinct branch points in the orbit of $\overline{01^{k-1}}$. Then $V(p, q, \overline{1}) = \overline{1}$. Thus, if p', q' are the branch points with itineraries p, q and b is the branch point with itinerary $\overline{1}$, then $b \in [p', q']$. In this way, b is the central branch point of T and thus $T \cong F_1^k$. $\square$

Claim A.0.4: (From Theorem 3.2.4) If $|D| = 2$ and $\min(D) = 3, max(D) = k$, then T can be realized by the kneading sequence $\tau = \overline{*1^{k-1}001}$.

Proof. Since τ being $*1^{k-1}0$ the fixed branch point has degree k. By Theorem 2.2.12 the point 001τ is an endpoint. We can determine if the point 01τ is an endpoint by running $V(\tau, 001\tau, 01\tau) = 0\overline{1}$. Since $01\tau \notin [\tau, 001\tau]$ we have that 01τ must be and endpoint. Thus there are only two endpoints in S_0 and so the branch point between them, $0\overline{1}$ mus have degree 3.

To verify that there are no other branch points notice that if p, q are two points in the critical orbit whose itineraries start with 1 then $V(\tau, p, q) = \overline{1}$. Thus there is only one branch point in S_1. So the tree has only two branch points, one of degree k and one of degree 3. □

Claim A.0.5: (From Theorem 3.2.8, Case 2a) This tree can be made with $\tau = \overline{*(10)^n 1101}$.

Proof. To prove the claim we need to show that there are only three branch points. two in S_1 and one in S_0, one of the branch points in S_1 has degree three, and the other two branch points have degree $n + 1$.

The fixed point in S_1 has degree two. Let p be a shift of τ proceeding 1101τ. Then $V(\tau, p, 1101\tau) = 1\overline{10}$. So 1101τ is the itinerary of an endpoint in a free arc with the branch point whose itinerary is $1\overline{10}$. Thus any shift of τ proceeding 1101τ is the itinerary of an endpoint.

But, $V(101\tau, 101101\tau, \tau) = 101\tau$ so 101τ is *not* the itinerary of an endpoint. Thus, there are exactly $2n + 1$ many endpoints in the tree, n in S_0 and $n + 1$ in S_1. If p, q are the itineraries of two distinct endpoints in S_0, then $V(p, q, \tau) = \overline{01}$. There are n endpoints in

S_0 and so the branch point in S_2 must have degree $n+1$. Since this branch point is periodic, any branch point in its orbit must also have degree $n+1$, so if a is the branch point with itinerary $\overline{10}$, then $\deg(a) = n+1$.

Lastly, to see that b, the branch point with itinerary $1\overline{10}$, has degree three, one can run the voting sequence and verify that b is in a free arc with c_0, a, the endpoint whose itinerary is 1101τ and no other points. Thus $\deg(b) = 3$. $\square$

Claim A.0.6: (From Theorem 3.2.8, Case 2b) If $\deg(a) = \deg(b) > \deg(c)$ [or by symmetry $\deg(c) = \deg(b) > \deg(a)$], then T can be made with kneading sequence $\tau = \overline{*(10)^{k-1}0010}$.

Proof. The fixed branch point has degree two. The longest block of 0s in τ occurs at 00010τ. By Theorem 2.2.12 we have that this point must be an endpoint. Thus every proceeding shift of τ (except τ itself) is also an endpoint. $V(100010\tau, 10\tau, \tau) = 10\tau$. Thus $10\tau \in (100010\tau, \tau)$ and so 10τ is not an endpoint. Thus, there are $k-1$ many endpoints in S_1. Let p, q be two such endpoints, then $V(p, q, \tau) = \overline{10}$. Therefore, $\overline{10}$ is the only branch point in S_1 and it has degree k. Since it is periodic, there must be a branch point $\overline{01}$ in S_0 with degree k.

We now turn our attention to finding all the branch points in S_0. By computing $V(0010\tau, s, \tau)$ for any s, a proceeding shift of τ which starts with 0, we find $0010\tau \notin (\tau, s)$, so 0010τ is an endpoint. But, $V(010\tau, 0100010\tau, \tau) = 010\tau$ so 010τ is not an endpoint. Thus, the number of endpoints in S_0 is k.

$V(00010\tau, 0010\tau, \tau) = 0\overline{01}$ so $0\overline{01}$ is a branch point in S_0. $\overline{10}, \overline{01}$, and $0\overline{01}$ are the only branch points in T. Moreover, if p is some shift of τ other than 00010τ or 0010τ, then

$V(00010\tau, 0\overline{01}, p) = 0\overline{01}$and $V(0010\tau, , \tau, p) = 0\overline{01}$. Thus, $0\overline{01}$ is in a free arc with the endpoints 00010τ and 0010τ. Similarly, $0\overline{01}$ is in a free arc with $\overline{01}$. These are the only endpoints or branch points with which $0\overline{01}$ is ain a free arc, so $\deg(0\overline{01}) = 3$. □

Claim A.0.7: From Theorem 3.2.8, Case 2c(From Theorem 3.2.8, Case 2c) Such a tree can be made by $\overline{*1^{k-1}0001}$.

Proof. In any tree with this kneading sequence. there is a fixed branch point in S_1 with degree k by Lemma 2.2.9. Let p, q be two distinct shifts of τ which start with 1. Then $V(p, q, \tau) = \overline{1}$ and so there is only one branch point in S_1.

Moreover, running the voting sequence on τ along with any two shifts thereof which begin with 0 yield the following:

$$V(0001\tau, 001\tau, \tau) = 00\overline{1}, \quad V(001\tau, 01\tau, \tau) = 0\overline{1}, \quad V(0001\tau, 01\tau, \tau) = 0\overline{1}.$$

From this we gather that no shift of τ starting with 0 corresponds to an endpoint and that the branch point with itinerary $0\overline{1}$ separates the endpoint with itinerary 01τ from the other two endpoints in S_2. Similarly, the branch point with itinerary $00\overline{1}$ separates the end points with itineraries 000τ and 00τ. Lastly, $V(00\overline{1}, 0\overline{1}, \tau) = 0\overline{1}$. From here it follows that both branch points in S_0 have degree three. □

Claim A.0.8: (From Theorem 3.2.8, Item 2d and Item 3) A kneading sequence of the form $\overline{*1^{k-1}001011}$ yields $\deg(a) = 3, \deg(b) = k, \deg(c) = 4$. And $\overline{*1^{k-1}01011}$ yields $\deg(a) = 3, \deg(b) = k, \deg(c) = 3$

Proof. We start by investigating $\tau = \overline{*1^{k-1}001011}$. By Lemma 2.2.9 there is a fixed branch point in S_1 of degree k . All that remains is to verify that there is a branch point in S_1 of degree 3 and a branch point in S_0 of degree 4. We can run the voting sequence on τ along with any two shifts thereof which begin with 0. Doing so yields the following:

$$V(001011\tau, 01011\tau, \tau) = 0\overline{1}, \quad V(01011\tau, 011\tau, \tau) = 0\overline{1}, \quad V(001011\tau, 011\tau, \tau) = 0\overline{1}.$$

From this we gather every shift of τ starting with 0 corresponds to an endpoint and there is only one branch point in S_2. This branch point must be degree 4 as desired.

Since 011τ is the itinerary of an endpoint, any proceeding shift of τ must also correspond to an endpoint. Notice, however, that $V(11001011\tau, 11\tau, \tau) = 11\tau$ so the shift of τ which begins 11τ does not correspond to an endpoint. Thus there are k many endpoints in S_1. Finally, $V(1011\tau, 1001011\tau, \tau) = 10\overline{1}$. For every other possible choice of two shifts of τ, p, q, $V(p, q, \tau) \neq 10\overline{1}$. Thus the branch point with itinerary $101\overline{1}$ is degree three, thus proving the claim.

We now consider $\tau = \overline{*1^{k-1}01011}$. The exact same analysis as in the previous yields a fixed branch point in S_1 of degree k, a branch point in S_1 with itinerary $10\overline{1}$ of degree three, and a branch point in S_0 with itinerary $0\overline{1}$ of degree three. $\square$

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