ABSTRACT

Observational Constraints, Exact Plane Wave Solutions, and Exact Spherical Solutions in Einstein-Aether Theory

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There are theoretical reasons to suspect that Lorentz-invariance, a cornerstone of modern physics, may be violated at very high energy levels. To study the effects of Lorentz-invariance in the classical regime, we consider Einstein-aether theory, a modified theory of gravity in which the metric is coupled to a unit timelike vector field called the "aether." This vector field picks out a preferred frame of reference, and generates a "matter-like" stress-energy tensor $T^{æ}_{\mu\nu}$. The theory is associated with solutions for black holes and gravitational waves that differ from those of Einsteinian General Relativity. We investigate both the observational constraints on the parameters of the theory as well as the consequences of the theory for plane wave radiation, and the gravitational collapse of the aether itself. We find that the four coupling constants of the theory (c_i, i=1,2,3,4) are tightly constrained by astronomical observations, and while multiple plane wave solutions exist most of them are ruled out by observation, leaving several viable candidates, a few of which are the same as General Relativity. For vacuum spherically-symmetric solutions, for the first time we find a simple, closed-form solution for static aether which does not violate the constraints.

Observational Constraints, Exact Plane Wave Solutions, and Exact Spherical Solutions in Einstein-Aether Theory

by

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PREFACE

Every hundred years or so there comes a PhD thesis that revolutionizes every aspect of science and catapults the researcher to instant wealth and fame. Boy I'd like to write one like that!

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DEDICATION

To my Heavenly Father, my Earthly parents, my supportive friends and family, my impatient cats (RIP Cleo), and the creators and writers of the *Star Trek* franchise, for spurring my imagination.

CHAPTER ONE

Introduction

1.1 Galilean Invariance and the Principle of Relativity

The principle of relativity says that the laws of physics (i.e. the equations of a theory) do not change *form* from one reference frame to another, for some suitably-defined set of reference frames. In Newtonian mechanics, embodied by the equation

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}, \tag{1.1}$$

(where we assume constant mass) the suitable reference frames are inertial frames (frames in which Newton's first law holds) moving with respect to each other at constant velocity, and are related to each other via Galilean coordinate transformations. For an inertial frame K with coordinates $(t, \mathbf{x})$, a Galilean transformation to another frame $\bar{K}$ with coordinates $(\bar{t}, \bar{\mathbf{x}})$, moving at velocity $\mathbf{w}$ with respect to K is given by

$$\begin{aligned} \bar{t} &= t, \\ \bar{\mathbf{x}} &= \mathbf{x} - \mathbf{w}t. \end{aligned} \tag{1.2}$$

Under such a coordinate transformation, Eq.(1.1) becomes:

$$\bar{\mathbf{F}} = m\frac{d\bar{\mathbf{v}}}{d\bar{t}}, \tag{1.3}$$

where

$$\bar{\mathbf{v}} = \frac{d\bar{x}}{d\bar{t}}, \tag{1.4}$$

And $\mathbf{F}$ is an invariant (under the Galilean transformation). Since the laws of physics are the same (i.e. Eqs.(1.1) and (1.3) have the same form), Newtonian physics obey Galilean invariance. Newton assumed the existence of an absolute time and space, but he did not define what this absolute frame was or seem to consider it of great importance in itself as it could not be experimentally distinguished from other inertial frames anyway [1].

But not all of nature obeys Galilean invariance. Consider classical electrodynamics. Maxwell's equations expressed in an arbitrary inertial reference frame K with coordinate system $(t, \mathbf{x})$ have the form:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).
\end{aligned} \tag{1.5}$$

However, under a Galilean transformation of the form of Eq.(1.2), Maxwell's equations become [2]

$$\begin{aligned}
\bar{\nabla} \cdot \bar{\mathbf{E}} &= \frac{\bar{\rho}}{\epsilon_0} - \mathbf{w} \cdot (\bar{\nabla} \times \bar{\mathbf{B}}), \\
\bar{\nabla} \cdot \bar{\mathbf{B}} &= 0, \\
\bar{\nabla} \times \bar{\mathbf{E}} &= -\frac{\partial \bar{\mathbf{B}}}{\partial \bar{t}}, \\
\bar{\nabla} \times \bar{\mathbf{B}} &= \mu_0 \bar{\mathbf{J}} + \frac{1}{c^2} \frac{\partial \bar{\mathbf{E}}}{\partial \bar{t}} - \mu_0 \bar{\rho} \mathbf{w} - \frac{1}{c^2} (\mathbf{w} \cdot \bar{\nabla}) \bar{\mathbf{E}} - \frac{\mathbf{w}}{c^2} \times \left[\left(\frac{\partial}{\partial \bar{t}} - \mathbf{w} \cdot \bar{\nabla} \right) \bar{\mathbf{B}} \right].
\end{aligned} \tag{1.6}$$

Note that the magnetic Gauss's law and Faraday's law are invariant under the Galilean transformation, but that Ampere's law and the electric Gauss's law are not in general invariant under a Galilean transformation, as new terms are introduced which are proportional to $\mathbf{w}$ (or components thereof)–the velocity with which this new frame is moving with respect to the old (initially arbitrary) frame. This makes the old frame (of Eqs.(1.5)) a **preferred frame** (under Galilean transformations), in which the physics, as well as the form of physical laws, is notably simpler.

1.2 Lorentz Invariance

Lorentz and Einstein realized that Maxwell's laws obeyed a different symmetry, Lorentzian symmetry. Under a Lorentz transformation (neglecting rotations) from K to $\bar{K}$ of the form

$$\bar{t} = \gamma\left(t - \frac{\mathbf{w}\cdot\mathbf{x}}{c^2}\right), \tag{1.7}$$

$$\bar{\mathbf{x}} = \mathbf{x} + (\gamma - 1)(\mathbf{x}\cdot\hat{\mathbf{w}})\hat{\mathbf{w}} - \gamma\mathbf{w}t, \tag{1.8}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{w^2}{c^2}}}, \tag{1.9}$$

Maxwell's equations become

$$\bar{\nabla}\cdot\bar{\mathbf{E}} = \frac{\bar{\rho}}{\epsilon_0}, \tag{1.10}$$

$$\bar{\nabla}\cdot\bar{\mathbf{B}} = 0, \tag{1.11}$$

$$\bar{\nabla}\times\bar{\mathbf{E}} = -\frac{\partial\bar{\mathbf{B}}}{\partial\bar{t}}, \tag{1.12}$$

$$\bar{\nabla}\times\bar{\mathbf{B}} = \mu_0\bar{\mathbf{J}} + \frac{1}{c^2}\frac{\partial\bar{\mathbf{E}}}{\partial\bar{t}}. \tag{1.13}$$

Which is the same as Eqs.(1.5), so the form of the physical laws is invariant under a Lorentz transformation of the form of Eq.(1.7). Maxwell's equations are also rotationally-invariant, so the laws of classical electrodynamics is invariant under the full Lorentz transformations.

Like Galilean invariance, under Lorentz invariance (LI) there is no laboratory test one could perform in an inertial frame to determine whether one is at "absolute rest" or not, so there is no preferred frame. As we will see later, Einstein-aether theory explicitly defines a notion of absolute rest with respect to a preferred frame.

1.3 Why Study Lorentz Invariance Violation?

The Standard Model (SM) of particle physics is also built on LI, being a marriage of SR and quantum mechanics (QM). The SM is considered the most accurate theory in the history of science. Predictions in quantum electrodynamics (QED) are accurate up to a number of decimal places limited only by the number of perturbative terms, and deviations from LI in the matter sector (as modelled by minimum SM extensions–mSME) are very tightly constrained [17]. GR and its associated symmetries (LI, WEP, SEP) have been tested to the limits of current technology and found accurate (assuming galactic rotation curves, gravitational lensing phenomena such as the Bullet Cluster, and the acceleration of the expansion of the Universe can be adequately modelled by Dark Matter–DM–via extra mass and Dark Energy–DE–via a cosmological constant) [9]. So why study LI violation?

In the early 1970s, Thorne and Will wrote a series of papers [3, 4, 5] about a systematic way to test GR–which at the time was only just beginning to be subjected to rigorous Solar System tests–beyond the initial tests suggested by Einstein; of gravitational redshift, precession of the perihelion of Mercury, and gravitational lensing. They wrote that

> One might think that we should merely continue to measure these and other non-Newtonian, general-relativistic effects to higher and higher accuracy; and only if a discrepancy between experiment and theory is found should we begin to consider other theories.
> This would be a reasonable approach if we had enormous confidence in general relativity; but we do not–at least, some of us don't. So we would prefer to design the experiments to be as unbiased as possible; we would like to see them force us, with very few *a priori* assumptions about the nature of gravity, toward general relativity or some other theory. And, of course, this can happen only if we first open our minds to a wide variety of theoretical possibilites.

At the time that was written the lack of confidence in GR from some quarters was due to lack of rigourous experimental testing to truly distinguish GR from alternative theories of gravity (which also made similar predictions of gravitational redshift, perihelion precession, and gravitational lensing). But today physicists have great confidence in GR. However, we know that it can't be the final theory of gravity because we know that it is incomplete. Classical GR solutions like the Schwarzschild metric are geodesically incomplete and cannot predict what happens at singularities. It also allows for the possibility of closed timelike loops, violating causality. Likewise we know that the SM is incomplete (it cannot account for DM or DE). Also, the two theories do not agree with each other, and nearly every attempt to create a quantum theory of gravity requires the breaking of LI. The Lorentz transformations are continuous symmetries, whereas in quantum physics space and time are discretized. String theory, loop quantum gravity, Hořava-Lifshitz gravity, etc. all indicate a breaking of LI. While the two theories, GR and the SM are very accurate in every domain explored thus far, we know that there must be a more general, more accurate theory that encompasses them both. So in the attempt to solve the outstanding problems of modern fundamental physics–DE, DM, QG–we must consider many possibilities, as Will and Thorne wrote. This includes considering whether there really is a preferred frame in the Universe.

1.4 Metric Theories of Gravity and the PPN Formalism

The theory of general relativity (GR), which incorporates gravity and spacetime curvature, maintains local LI as well as general covariance. On a smooth spacetime, on any given point a coordinate system exists in which the metric is Minkowskian at that point, and in free-falling frames a lab-centered coordinate system can be found in which the metric is Minkowskian along its trajectory, and the laws of SR are recovered for non-gravitational physics. In local LI, the physics shouldn't depend on the relative velocity of the laboratory frame with respect to any other frame. We could thus consider GR at the linearized level to compare it to other theories, but it is more helpful to examine GR at the **Post-Newtonian** level.

Developed throughout the 20th century by different authors in various parameterizations, the modern parameterization was formulated by Will and Nordtvedt [6, 7]. The idea is to perform something like a Taylor expansion of the metric of a given matter source (such as a point mass or perfect fluid) about the Minkowski metric in orders of $\epsilon \sim v^2/c^2 \sim U \sim p/\rho \sim \Pi$ where v is the coordinate velocity of the matter, ρ is the mass density, p is the pressure, and Π is the internal energy density (neglecting gravitational self-energy), all measured in a frame co-moving with the matter. There are many variants on the formalism and many ways of parameterizing the different PPN metrics, but all metric theories of gravity can be discussed in terms of the **PPN Parameters** such as γ, β, ξ, α_n, and ζ_n.

The PPN Parameters have physical interpretations [6, 8, 9], in that γ can be thought of as a measure of how much spacetime curvature is induced by a unit mass, β tells us how much nonlinearity there is in the law of superposition for gravity, ξ is the preferred location parameter (deviations from translation invariance), α_n are the preferred frame parameters (deviations from LI), and ζ_n tell us about violations of conservation of momentum. For GR, $\gamma = \beta = 1$ and all other parameters are zero, while it varies for other theories. As we will see, in Einstein-aether theory all parameters are the same as in GR, except for the preferred frame parameters α_1 and α_2.

For example (using the parameterization of [6], in which $\xi = 0$), if we construct the PPN metric for a perfect fluid in a reference frame K with coordinates $(t, \mathbf{x})$ we have:

$$\begin{aligned} g_{00} &= 1 - 2U + 2\beta U^2 - (2\gamma + 2 + \alpha_3 + \zeta_1)\Phi_1 + \zeta_1\mathcal{A} \\ &\quad - 2[(3\gamma + 1 - 2\beta + \zeta_2)\Phi_2 + (1+\zeta_3)\Phi_3 + 3(\gamma+\zeta_4)\Phi_4], \\ g_{0j} &= \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)V_j + \frac{1}{2}(1 + \alpha_2 - \zeta_1)W_j, \\ g_{ij} &= -(1 + 2\gamma U)\delta_{ij}. \end{aligned} \tag{1.14}$$

Where U is the Newtonian potential; $\mathcal{A}$, V_j, W_j, and Φ_n (n=1,2,3,4) are other potentials generated by the matter, mass density, pressure, and internal energy.

It is not important to understand every detail of these tensor components, what is important is to see what happens after a Lorentz transformation (really a post-Galilean transformation, which is like a Taylor expansion of a Lorentz transformation) is performed on Eq.(1.14). Specifically, this will be a Lorentz boost from the frame K to a frame $\bar{K}$ moving at velocity $\mathbf{w}$ with respect to it [6]:

$$\begin{aligned}
\bar{t} &= t(1 + \frac{1}{2}w^2 + \frac{3}{8}w^4) - \mathbf{x} \cdot \mathbf{w}(1 + \frac{1}{2}w^2) + O(\epsilon^{2.5}) \times t, \\
\bar{\mathbf{x}} &= \mathbf{x} - (1 + \frac{1}{2})\mathbf{w}t + \frac{1}{2}(\mathbf{x} \cdot \mathbf{w})\mathbf{w} + O(\epsilon^2) \times \mathbf{x}.
\end{aligned} \tag{1.15}$$

After performing a transformation of the form of Eq.(1.15) on the perfect fluid PPN metric of Eq.(1.14) we find:

$$\begin{aligned}
g_{00} &= 1 - 2U + 2\beta U^2 - (2\gamma + 2 + \alpha_3 + \zeta_1)\Phi_1 + \zeta_1 \mathcal{A} \\
&\quad - 2[(3\gamma + 1 - 2\beta + \zeta_2)\Phi_2 + (1 + \zeta_3)\Phi_3 + 3(\gamma + \zeta_4)\Phi_4] \\
&\quad + (\alpha_1 - \alpha_2 - \alpha_3)w^2 U + \alpha_3 w^i w^j U_{ij} - (2\alpha_3 - \alpha_1)w^i V_i, \\
g_{0j} &= \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)V_j + \frac{1}{2}(1 + \alpha_2 - \zeta_1)W_j \\
&\quad + \frac{1}{2}(\alpha_1 - 2\alpha_2)w_j U + \alpha_2 w^i U_{ij}, \\
g_{ij} &= -(1 + 2\gamma U)\delta_{ij}.
\end{aligned} \tag{1.16}$$

Upon comparison with the original, un-transformed PPN metric we see that new terms have been introduced, all of which are proportional to $\mathbf{w}$ (or components thereof). This indicates that the (arbitrarily chosen, asymptotically Minkowskian) reference frame of Eq.(1.14) is a *preferred frame* of a general metric theory of gravity. The only way for all such terms to vanish, and thus to make a metric theory of gravity free of preferred frames and be Lorentz invariant, is for $\alpha_1 = \alpha_2 = \alpha_3 = 0$. However, if any of the α_n is non-zero, then the results of gravitational physics experiments will depend on the velocity of the reference frame with respect to the "absolute" preferred reference frame of the Universe, which breaks boost invariance.

There is a straightforward (though long and tedious) procedure for obtaining the PPN Parameters of a given theory in terms of the other parameters unique to that theory (such as coupling coefficients in the Lagrangian). The *actual* values of the PPN Parameters can be measured experimentally in a (metric) theory-independent way via Solar System observations and thus any theory that requires PPN Parameter values greater than that allowed by observation is ruled out, and this has happened to other theories in the past [6]. In practice, as will be seen in Chapter 2 of this dissertation, constraints on the PPN Parameters often serve as constraints on the parameters of the theory (other than PPN) in question rather than unambiguously invalidating it. Current constraints on the preferred frame parameters are that $|\alpha_1| < 10^{-4}$ and $|\alpha_2| < 10^{-7}$ in the weak (Solar System) field.

1.5 Preferred-Frame Effects

It can be shown [7, 8] that in a metric theory of gravity the results of Cavendish-type experiments would depend not on Newton's constant, but on an *effective* Newton's constant that would vary depending on the velocity of one's reference frame with respect to the Universe. For example, in a theory where $\gamma = \beta = 1$, α_1 and α_2 are arbitrary, and all other parameters vanish (as in Einstein-aether theory) we have [7, 8]:

$$G_{eff} = G\left[1 - \frac{1}{2c^2}\left((\alpha_1 - \alpha_2)w^2 + \alpha_2[\mathbf{w}\cdot\bar{\mathbf{n}}]^2\right)\right], \tag{1.17}$$

Where we neglect external potentials as well as the aether-induced stress-energy tensor. G is the G that appears in the Lagrangian of the theory, $\bar{\mathbf{n}}$ is a unit vector from the central mass (for example the Earth) to a test mass (for example a gravimeter), and as before $\mathbf{w}$ is the velocity of the central mass with respect to the preferred frame.

As the laboratory frame on the surface of the Earth rotates, G_{eff} varies periodically, this anisotropy in G_{eff} leads to periodic variations in the measurement of local gravitational acceleration g, analogous to tides caused by the Moon and the Sun but distinct from it. More periodic variation is caused by the Earth's motion around the sun.

It was also be shown by Nordtvedt et al [12] that a non-zero value of the preferred frame parameter α_1 induces a term in the Lagrangian of an N-body system given by:

$$\mathcal{L}_{\alpha_1} = -\frac{\alpha_1}{4} \sum_{A \neq B} \frac{GM_A M_B}{c^2 r_{AB}} (\mathbf{w}_A \cdot \mathbf{w}_B), \tag{1.18}$$

where M_A are the masses of the bodies, $\mathbf{r}_{AB}$ is the position coordinate vector in the Solar System center-of-mass frame, and $\mathbf{w}_A$ is the velocity of the A'th body with respect to the preferred frame. This leads to a perturbation of the three-body equations of motion between the Earth, Moon, and Sun, causing oscillations in the Earth-Moon distance that would be absent if there were no preferred frame. Lunar laser ranging to measure this oscillation gives the constraint:

$$|\alpha_1| \sim O(10^{-4}). \tag{1.19}$$

In a prior paper by Nordtvedt [13] it was shown that if there is a non-zero value of the preferred frame parameter α_1, then any oblate, spinning astronomical body moving at velocity $\mathbf{w}$ with respect to the preferred frame experiences a torque τ given by:

$$\tau = 2\alpha_2 T_{rot} (\hat{\omega} \cdot \mathbf{w}) \hat{\omega} \times \mathbf{w}/c^2, \tag{1.20}$$

Where T_{rot} is the rotational kinetic energy and $\bar{\omega}$ is a unit vector point along the rotational axis. This induces a precession of the body's spin axis about $\mathbf{w}$ at the rate:

$$\Omega = 2\alpha_2 \left(\frac{T_{rot}}{J_{rot}} \right) \hat{\mathbf{w}} \cdot \hat{\omega} \left(\frac{w}{c} \right)^2. \tag{1.21}$$

Where J_{rot} is the rotational angular momentum of the body. This effects the angle between the Sun's spin axis and the angular momentum axis of the Solar System as a whole, which can be measured (see [13] for the messy details) and places the constraint on α_2 given by:

$$|\alpha_2| \sim O(10^{-7}). \tag{1.22}$$

These are weak field preferred frame effects. In strong fields, for theories with a preferred frame it can be difficult to see how the predictions of those theories differ with respect to GR in general, and calculations must be done on a case-by-case basis, such as with exact solutions to the field equations, higher-order PN expansions, or numerical relativity.

1.6 Einstein-aether Theory

The Einstein field equations (EFE) of GR are given by:

$$G_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}^{matter}, \tag{1.23}$$

Where the left hand side of the equation is the "geometry side" (Einstein tensor) and the right hand side is the "matter side" (stress-energy tensor). All tensor components are at most second-order in derivatives. The Λ term is the cosmological constant term, and while it was measured to be non-zero at the end of the 1990s it is generally set to zero for non-cosmological problems because it is so small (except where otherwise noted, we will set $\Lambda = 0$ in this dissertation). The EFEs can be derived in a variety of ways, but from a theoretical physics standpoint perhaps the most germane is the Euler-Lagrange approach, whereby the EFEs are generated by varying the Einstein-Hilbert action:

$$S = S_{matter} + \int dx^4 \sqrt{-g} \mathcal{L}_{GR}, \tag{1.24}$$

$$\mathcal{L}_{GR} = \frac{c^4}{16\pi G} R, \tag{1.25}$$

where R is the Ricci scalar and sources the Einstein tensor G_{ab} while the matter action sources the stress-energy tensor T_{ab} and depends on the problem under consideration. In vacuum we neglect it. In principle there is no reason to exclude other scalars constructed from the Riemann tensor from the Lagrangian, but the Einstein-Hilbert action is the only one that generates the original EFEs. Metric theories of gravity will, in general, include other scalar terms in their Lagrangians and that is how new theories are developed. Einstein-aether theory was constructed from the ground up to be the simplest (but also general) Lorentz-violating theory with a *dynamical* preferred frame, with second-order derivatives [17, 21, 22]. [1]

In addition to the dynamical field of the metric, it introduces a dynamical timelike vector field u^a which has unit length everywhere. This vector field is dubbed "the aether" since it permeates all of spacetime (though it has nothing to do with the luminiferous aether of 19th-century physics). The aether sources a congruence of curves which "picks out" a preferred time direction at every point on the spacetime, and thus a preferred frame. An arbitrary observer (that is *not* co-moving with the aether) has velocity $\mathbf{w}$ with respect to this preferred frame, leading to the preferred frame effects of Section 1.4.

Such a vector-tensor theory can be constructed by creating scalar terms for the Lagrangian, via contractions with the Riemann and Ricci tensors. For example Will and Nordtvedt constructed a vector-tensor theory by simply including every possible scalar term second-order in derivatives in the Lagrangrian [6]:

$$\begin{aligned}\mathcal{L}_{LV} = \mathcal{L}_{matter} + k_1 R + k_2 u_a u^a R + k_3 u_a u_b R^{ab} + k_4 \nabla_b u_a \nabla^b u^a \\ + k_5 u \nabla_b u_a \nabla^a u^b + k_6 \nabla_a u^a \nabla^b u_b,\end{aligned} \tag{1.26}$$

Where the k_n would be constant coefficients that would be constrained by the observational values of the PPN Parameters. The k_3 term can be expressed as a difference of the k_5 and k_6 terms up to a total derivative (see Eq.(1.40)), so it is not an independent term and can be dropped without losing generality [22]. Using this fact, as well as a suitable redefinition of the coefficients, the Einstein-aether Lagrangian is given

[1] In most theories with a preferred frame, the rest frame of the Cosmic Microwave Background (CMB) is taken to be the preferred frame, and in fact the Earth's velocity with respect to the CMB is measured to be ~ 369 km/s [10]. While this assumption holds for Einstein-aether theory as well it is not, in principle, a necessary one.

(in the $diag(-1,1,1,1)$ metric) as

$$\mathcal{L}_{æ} = \frac{c^4}{16\pi G}\left[-K^{ab}{}_{mn}\nabla_a u^m \nabla_b u^n + \lambda(g_{ab}u^a u^b + 1)\right] \tag{1.27}$$

Where λ is a Lagrange multiplier that enforces the timelike nature of u^a (the whole λ term can be thought of as a "potential" of the "field" $u^a u_a$ [17]), and the tensor $K^{ab}{}_{mn}$ is given by

$$K^{ab}{}_{mn} = c_1 g^{ab} g_{mn} + c_2 \delta^a_m \delta^b_n + c_3 \delta^a_n \delta^b_m - c_4 u^a u^b g_{mn}, \tag{1.28}$$

Where four coupling constants c_i's are all dimensionless, and G is related to the Newtonian constant G_N via the relation[2] [24],

$$G_N = \frac{G}{1 - \frac{1}{2}c_{14}}. \tag{1.29}$$

Combinations of the constants such as $c_1 + c_3$ show up regularly in the field equations, and so appear as $c_1 + c_3 = c_{13}$, $c_1 + c_4 = c_{14}$, etc. This makes the total action:

$$S = S_{matter} + \int dx^4 \sqrt{-g}(\mathcal{L}_{GR} + \mathcal{L}_{æ}) \tag{1.30}$$

Then, the variation of the above action with respect to g^{ab} yields

$$G_{ab} = \frac{8\pi G}{c^4} T^{matter}_{ab} + T^{æ}_{ab}. \tag{1.31}$$

So the aether vector field itself induces something like a stress-energy tensor given by

$$\begin{aligned} T^{æ}_{ab} &\equiv -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}(\mathcal{L}_{æ}))}{\delta g^{ab}} \\ &= \nabla_c\Big[J^c{}_{(a}u_{b)} + J_{(ab)}u^c - u_{(b}J_{a)}{}^c\Big] \\ &\quad + c_1\Big[(\nabla_a u_c)(\nabla_b u^c) - (\nabla_c u_a)(\nabla^c u_b)\Big] \\ &\quad + c_4 a_a a_b + \lambda u_a u_b - \frac{1}{2} g_{ab} J^d{}_c \nabla_d u^c, \end{aligned} \tag{1.32}$$

where $J^a{}_b$ and a^a are defined by

$$J^a{}_b = K^{ac}{}_{bd}\nabla_c u^d, \quad a^a = u^b \nabla_b u^a. \tag{1.33}$$

[2] Eq.(1.29) was derived at the linearized level in a frame comoving with the aether.

In addition, the variation of the action with respect to u^a yields the aether field equations,

$$\begin{aligned} Æ_a &= \frac{1}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g}\mathcal{L}_{æ}\right)}{\delta u^a} \\ &= \nabla_b J^b{}_a + c_4 a_b \nabla_a u^b + \lambda u_a = 0, \end{aligned} \tag{1.34}$$

while its variation with respect to λ gives,

$$u^a u_a = -1. \tag{1.35}$$

From Eqs.(1.34) and (1.35) we find that

$$\lambda = u_b \nabla_a J^{ab} + c_4 a^2. \tag{1.36}$$

In vacuum, the EFE in GR reduce to:

$$G_{ab} = 0. \tag{1.37}$$

However, it is important to note that the aether stress-energy tensor given by Eq.(1.32) does not vanish in vacuum, as the aether itself induces spacetime curvature much the same as matter-energy would. The non-zero right-hand side of Eq.(1.31) plus the aether dynamics of Eq.(1.34) often results in over-determined field equations, making it difficult to find exact solutions.

1.7 Implications for Quantum Gravity

If the aether vector field u^a is restricted to be hypersurface-orthogonal, then Einstein-aether theory becomes the khronometric theory of gravity. The khronometric theory differs from Einstein-aether theory in that as the aether is hypersurface-orthogonal it has zero twist, thus the c_4 term in the Lagrangian drops. This results in there being no spin-1 wave mode. Also, in the khronometric theory, wave modes can travel at infinite speeds whereas in pure Einstein-aether theory, wave modes travel at finite speeds.

Einstein-aether and the khronometric theories are the low-energy limit of a quantum gravity theory called Hořava-Lifshitz gravity [19, 45], which has extra terms in the Lagrangian that are irrelevant at low energies. Hořava-Lifshitz gravity is a power-counting renormalizable theory, and in general any solution of the khronometric theory is a solution of it.

1.8 Organization of Dissertation

In Chapter Two (which is based on [14]) we discuss the most current observational constraints on the coefficients of Einstein-aether theory, stemming from the detection of a NS-NS collision detected by LIGO, Virgo, and other astronomical observatories in both gravitational and electromagnetic spectra, which greatly constrained the (possible) difference in the speed of gravity to the speed of light to one part in 10^{15}. The work in this chapter was previously published as a first-author article by the author of this dissertation (Oost) and co-authored by Professor Shinji Mukohyama at Kyoto University's Yukawa Institute and Professor Anzhong Wang. The work was a roughly even split.

In Chapter Three (which is based on [15]) we present exact plane wave solutions in Einstein-aether theory, each solution depending on a particular choice of c_i parameters. Some of the solutions are equivalent to that of GR in which the form of the wave is arbitrary, but most of the solutions take a particular form, and some of the solutions are ruled out by observations as outlined in Chapter Two. The work in this chapter was previously published as a first-author article by the author of this dissertation, and co-authored by Madhurima Bhattacharjee (a graduate student at Baylor University), and Wang. Approximately 70% of the work was done by the author of this dissertation, with all of the equations and derivations being checked by Bhattacharjee and the initial formulation of the problem and introductory work by Wang.

In Chapter Four we present several exact spherically symmetric solutions, including a closed-form parameterization of a known static solution that had hitherto been expressed in a set of intractable inverse functions. This chapter is written by the author of this dissertation, with minor edits and the initial formulation of the problem by Wang, and with helpful discussions from Mukohyama.

In Chapter Five we discuss the future of Einstein-aether theory in light of the new constraints, and plans for more exact solutions in spherical symmetry.

1.9 Conventions

Throughout this dissertation we use the following conventions. Except where noted, we use Roman indices $(a, b, c, ...)$ or Greek indices $(\mu, \nu, ...)$ for 0 to 3, whereas Roman indices $(i, j, k, ...)$ run from 1 to 3.

The Minkowski metric η_{ab} and signature is given by:

$$\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1) \tag{1.38}$$

The symmetric metric connection Γ^a_{bc} is defined by:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\frac{\partial g_{bd}}{\partial x^c} + \frac{\partial g_{dc}}{\partial x^b} - \frac{\partial g_{bc}}{x^d} \right) \tag{1.39}$$

The Riemann curvature tensor $R^a{}_{bcd}$ is defined by:

$$\begin{aligned} \left[\nabla_c, \nabla_d\right] V^a &= \left(\partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{bd} + \Gamma^a_{de}\Gamma^e_{bc}\right) V^b \\ &= R^a{}_{bcd} V^b \end{aligned} \tag{1.40}$$

The Ricci tensor is defined by:

$$R_{ab} = R^c{}_{acd} \tag{1.41}$$

The Einstein tensor G_{ab} is given by:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \tag{1.42}$$

CHAPTER TWO

Constraints on Einstein-aether Theory After GW170817

This chapter published as [14]: J. Oost, S. Mukohyama, A. Wang, "Constraints on Einstein-aether theory after GW170817," Phys. Rev. D 97, 124023 (2018)

In this chapter, we carry out a systematic analysis of the theoretical and observational constraints on the dimensionless coupling constants c_i $(i = 1, 2, 3, 4)$ of the Einstein-aether theory, taking into account the events GW170817 and GRB 170817A. The combination of these events restricts the deviation of the speed c_T of the spin-2 graviton to the range, $-3 \times 10^{-15} < c_T - 1 < 7 \times 10^{-16}$, which for the Einstein-aether theory implies $|c_{13}| \leq 10^{-15}$ with $c_{ij} \equiv c_i + c_j$. The rest of the constraints are divided into two groups: those on the (c_1, c_{14})-plane and those on the (c_2, c_{14})-plane, except the strong-field constraints. The latter depend on the sensitivities $\sigma_{æ}$ of neutron stars, which are not known at present in the new ranges of the parameters found in this chapter.

2.1 Introduction

The invariance under the Lorentz symmetry group is a cornerstone of modern physics and strongly supported by experiments and observations [16]. Nevertheless, there are various reasons to construct gravitational theories with broken Lorentz invariance (LI) [17]. For example, if space and/or time at the Planck scale are/is discrete, as currently understood [18], Lorentz symmetry is absent at short distance/-time scales and must be an emergent low energy symmetry. A concrete example of gravitational theories with broken LI is the Hořava theory of quantum gravity [19], in which the LI is broken via the anisotropic scaling between time and space in the ultraviolet (UV), $t \rightarrow b^{-z}t$, $x^i \rightarrow b^{-1}x^i$, $(i = 1, 2, ..., d)$, where z denotes the dynamical critical exponent, and d the spatial dimensions. Power-counting renormalizability requires $z \geq d$ at short distances, while LI demands $z = 1$. For more details about Hořava gravity, see, for example, the recent review [20].

Another theory that breaks LI is the Einstein-aether theory [21], in which LI is broken by the existence of a preferred frame defined by a time-like unit vector field, the so-called aether field. The Einstein-aether theory is a low energy effective theory and passes all theoretical and observational constraints by properly choosing the coupling constants of the theory [21, 22], including the stability of the Minkowski spacetime [23], the abundance of the light elements formed in the early universe [24], gravi-Čerenkov effects [25], the Solar System observations [26], binary pulsars [28, 29], and more recently gravitational waves [30].

Among the 10 parameterized post-Newtonian (PPN) parameters [9], in the Einstein-aether theory the only two parameters that deviate from general relativity are α_1 and α_2, which measure the preferred frame effects. In terms of the four dimensionless coupling constants c_i's of the Einstein-aether theory, they are given by [26],

$$\begin{aligned}
\alpha_1 &= -\frac{8(c_3^2 + c_1 c_4)}{2c_1 - c_1^2 + c_3^2}, \\
\alpha_2 &= \frac{1}{2}\alpha_1 - \frac{(c_1 + 2c_3 - c_4)(2c_1 + 3c_2 + c_3 + c_4)}{c_{123}(2 - c_{14})},
\end{aligned} \tag{2.1}$$

where $c_{ij} \equiv c_i + c_j$ and $c_{ijk} = c_i + c_j + c_k$. In the weak-field regime, using lunar laser ranging and solar alignment with the ecliptic, Solar System observations constrain these parameters to very small values [9],

$$|\alpha_1| \leq 10^{-4}, \quad |\alpha_2| \leq 10^{-7}. \tag{2.2}$$

Considering the smallness of α_A $(A = 1, 2)$, it may be convenient to Taylor expand Eq.(2.1) with respect to α_A to obtain

$$c_2 = -\frac{c_{13}(2c_1 - c_3)}{3c_1} + \mathcal{O}(\alpha_A)\,, \quad c_4 = -\frac{c_3^2}{c_1} + \mathcal{O}(\alpha_A)\,. \tag{2.3}$$

If terms of order $\mathcal{O}(\alpha_A)$ and higher are small enough to be neglected then the four-dimensional parameter space spanned by c_i's reduces to two-dimensional one. Until recently, the strongest constraints on the Einstein-aether theory were (2.2) and thus this treatment was a good approximation. Then, using the order-of-magnitude arguments about the orbital decay of binary pulsars, Foster estimated that $|c_1 \pm c_3| \lesssim \mathcal{O}\left(10^{-2}\right)$, by further assuming that $c_i \ll 1$ [28]. More detailed analysis of binary pulsars showed that $c_{13} \lesssim \mathcal{O}\left(10^{-2}\right)$, $|c_1 - c_3| \lesssim \mathcal{O}\left(10^{-3}\right)$ (See Fig. 1 in [29]).

However, the combination of the gravitational wave event GW170817 [31], observed by the LIGO/Virgo collaboration, and the one of the gamma-ray burst GRB 170817A [32], provides much more severe constraint on c_{13}. In fact, these events imply that the speed of the spin-2 mode c_T must satisfy the bound, $-3 \times 10^{-15} < c_T - 1 < 7 \times 10^{-16}$. In the Einstein-aether theory, the speed of the spin-2 graviton is given by $c_T^2 = 1/(1 - c_{13})$ [23], so the GW170817 and GRB 170817A events imply

$$|c_{13}| < 10^{-15}. \tag{2.4}$$

This is much smaller than the limits of Eq.(2.2). As a result, if we still adopt the Taylor expansion with respect to α_A then Eq.(2.3), for example, can no longer be approximated only up to the zeroth-order of α_A. Instead, it must be expanded at least up to the fourth-order of α_1, the second-order of α_2 (plus their mixed terms), and the first-order of c_{13}, in order to obtain a consistent treatment. Otherwise, the resulting errors would become much larger than $|c_{13}|$, due to the omissions of the terms higher in α_A, and the results obtained in this way would not be trustable.

In this chapter, we shall therefore Taylor expand all constraints other than (2.4) with respect to c_{13}, keep only terms zeroth order in c_{13} by setting $c_{13} \simeq 0$ in those expressions, and let c_1, c_2 and c_{14} be restricted by those other constraints. (In particular, we shall not set $\alpha_A \simeq 0$ since this would cause large errors.) As a result, the phase space of c_i's becomes essentially three-dimensional. Moreover, it is to our surprise that the three-dimensional phase space actually becomes degenerate, in the sense that the constraints can be divided into two groups, one has constraints only on the (c_1, c_{14})-plane, and the other has constraints only on the (c_2, c_{14})-plane. Note that in [33] the case $c_{13} = \alpha_2 = 0$ was considered, so the parameter space was again reduced to two-dimensional. Then, the constraints were restricted to the (α_1, c_-)-plane, where $c_- \equiv c_1 - c_3$. It was found that in this case no bounds can be imposed on c_-.

The rest of the chapter is organized as follows: In Sec. 2.2 we first list all the relevant constraints, theoretical and observational, then consider them one by one, and finally obtain a region in the phase space, in which all theoretical and observational constraints are satisfied by the Einstein-aether theory, except for the strong-field constraints given by Eq.(2.12). These strong-field constraints depend on the sensitivities $\sigma_{æ}$ of neutron stars in the Einstein-aether theory, which depends on c_i's (and the equation of state of nuclear matter) [29] and are not known for the new ranges of the parameters found in this chapter. Thus, we shall not use these strong-field constraints to obtain further constraints on c_i's, leaving further studies to a future work. Derivations of the linearized Einstein-aether theory are relegated to Appendix A.

2.2 Constraints on Einstein-aether theory after GW170817

It is easy to show that the Minkowski spacetime is a solution of the Einstein-aether theory, in which the aether is aligned along the time direction, $\bar{u}_\mu = \delta^0_\mu$. It is then straightforward to analyze linear perturbations around the Minkowski background and investigate properties of spin-0, -1 and -2 excitations (see the last section of this chapter and/or ref. [34] for details). In particular, the coefficients of the time

---

[1] In the so-called decoupling limit $c_i \to 0$, $q_V = c_{14}$ vanishes but the limit must be taken from the positive side of $q_{S,V,T}$ and $c^2_{S,V,T}$. Similarly, if we would like to take the infinite speed limit, e.g. $c_S \to \infty$, it should also be taken from the positive side.

kinetic term of each excitation $q_{S,V,T}$ must be positive [1]:

$$q_{S,V,T} > 0\,, \tag{2.5}$$

where

$$\begin{aligned} q_S &= \frac{(1-c_{13})\,(2+c_{13}+3c_2)}{c_{123}}\,, \\ q_V &= c_{14}\,, \\ q_T &= 1-c_{13}\,. \end{aligned} \tag{2.6}$$

In addition to the ghost-free condition for each part of the linear perturbations, we must also require the theory be free of gradient instability, that is, the squared speeds must be non-negative,

$$c^2_{S,V,T} \geq 0\,, \tag{2.7}$$

where

$$\begin{aligned} c_S^2 &= \frac{c_{123}(2-c_{14})}{c_{14}(1-c_{13})(2+c_{13}+3c_2)}\,, \\ c_V^2 &= \frac{2c_1 - c_{13}(2c_1-c_{13})}{2c_{14}(1-c_{13})}\,, \\ c_T^2 &= \frac{1}{1-c_{13}}\,. \end{aligned} \tag{2.8}$$

Moreover, $c^2_{S,V,T}-1$ must be greater than -10^{-15} or so, in order to avoid the existence of the vacuum gravi-Čerenkov radiation by matter such as cosmic rays [25]. We thus impose

$$c^2_{S,V,T} \gtrsim 1\,, \tag{2.9}$$

which is stronger than (2.7).

More recently, as mentioned above, the combination of the gravitational wave event GW170817 [31], observed by the LIGO/Virgo collaboration, and the event of the gamma-ray burst GRB 170817A [32] provides a remarkably stringent constraint on the speed of the spin-2 mode, $-3\times10^{-15} < c_T - 1 < 7\times10^{-16}$, which implies the constraint (2.4).

On the other hand, applying the theory to cosmology, it was found that the gravitational constant appearing in the effective Friedman equation is given by [24],

$$G_{\text{cos}} = \frac{G_{æ}}{1 + \frac{1}{2}(c_{13} + 3c_2)}. \tag{2.10}$$

Since G_{cos} is not the same as G_N in (1.29), the expansion rate of the universe differs from what would have been expected in GR. In particular, decreasing the Hubble expansion rate during the big bang nucleosynthesis will result in weak interactions freezing-out later, and leads to a lower freeze-out temperature. This will yield a decrease in the production of the primordial ^{4}He, and subsequently a lower ^{4}He-to-hydrogen mass ratio [24]. As a result the primordial helium abundance is modified, and to be consistent with current observations [35], the ratio must satisfy the constraint,

$$\left|\frac{G_{\text{cos}}}{G_N} - 1\right| \lesssim \frac{1}{8}. \tag{2.11}$$

One could obtain other cosmological constraints on G_{cos}/G_N if we make assumptions on the dark sector of the universe [36]. While they are interesting and important, we shall not consider those additional constraints since they are model-dependent.

Moreover, for any choice of c_i's, all PPN parameters [9] of the æ-theory agree with those of GR [37, 26], except the preferred frame parameters which are given by Eq.(2.1) [26, 27, 38]. In the weak-field regime, using lunar laser ranging and solar alignment with the ecliptic, Solar System observations constrain these parameters to very small values (2.2) [9]. In the strong-field regime, using data from the isolated millisecond pulsars PSR B1937 + 21 [39] and PSR J17441134 [40], the following constraints were obtained [41],

$$|\hat{\alpha}_1| \leq 10^{-5}, \quad |\hat{\alpha}_2| \leq 10^{-9}, \tag{2.12}$$

at 95% confidence, where $(\hat{\alpha}_1, \hat{\alpha}_2)$ denotes the strong-field generalization of (α_1, α_2) [42]. In the Einstein-æther theory, they are given by [29],

$$\begin{aligned}
\hat{\alpha}_1 &= \alpha_1 + \frac{c_-(8+\alpha_1)\sigma_{æ}}{2c_1}, \\
\hat{\alpha}_2 &= \alpha_2 + \frac{\hat{\alpha}_1 - \alpha_1}{2} - \frac{(c_{14}-2)(\alpha_1 - 2\alpha_2)\sigma_{æ}}{2(c_{14} - 2c_{13})},
\end{aligned} \tag{2.13}$$

where $\sigma_{æ}$ denotes the sensitivity.

To consider the above constraints, one may first express two of the four parameter c_n's, say, c_2 and c_4, in terms of α_A's through Eqs.(2.1), and then expand c_2 and c_4 in terms of α_A, as given by Eq.(2.3). Thus, to the zeroth-order of α_A's, c_2 and c_4 are given by the first term in each of Eq.(2.3) [26, 21, 22]. In fact, this is what have been doing so far in the analysis of the observational constraints of the Einstein-aether theory [22, 34, 29, 30].

However, with the new constraint (2.4), if we still adopt the Taylor expansion with respect to α_A, then, to have a self-consistent expansion, one must expand c_2 and c_4 at least up to the fourth-order of α_1, the second-order of α_2 (plus their mixed terms, such as $\alpha_1^2\alpha_2$) [cf. Eq.(2.2)], and the first-order of c_{13}. Clearly, this will lead to very complicated analyses. In the following, instead, we simply Taylor expand constraints other than (2.4) with respect to c_{13}, keep only terms zeroth order in c_{13}, and let all the other parameters constrained by those approximated constraints. Then, keeping only the leading terms in the c_{13}-expansion is equivalent to setting

$$c_{13} = 0\,. \tag{2.14}$$

As a result, the errors are of the order of $\mathcal{O}\left(10^{-15}\right)$, as far as Eq.(2.4) is concerned. Thus, the resulting errors due to this omission is insignificant, in comparison to the bounds of the rest of the observational constraints. Hence, while the constraint $q_T > 0$ is automatically satisfied, $q_S > 0$ yields

$$\frac{2+3c_2}{c_2} > 0\,. \tag{2.15}$$

On the other hand, from Eqs.(2.8) and (2.1) we find that

$$c_V^2 = \frac{c_1}{c_{14}}, \quad \alpha_1 = -4c_{14}, \tag{2.16}$$

so the constraints (2.2), $q_V > 0$ and $c_V^2 \gtrsim 1$ lead to

$$0 < c_{14} \leq 2.5 \times 10^{-5}, \quad c_{14} \lesssim c_1 . \tag{2.17}$$

It is remarkable that these two constraints are all confined to the (c_1, c_{14})-plane, while the rest are all confined to the (c_2, c_{14})-plane, as to be shown below. As we shall see, this considerably simplifies the analysis of the whole set of the constraints listed above.

In particular, the constraint (2.11) is reduced to

$$-\frac{1}{8} \lesssim \frac{c_{14} + 3c_2}{2 + 3c_2} \lesssim \frac{1}{8}, \tag{2.18}$$

which is rewritten as

$$-\frac{2(1 + 4c_{14})}{27} \lesssim c_2 \lesssim \frac{2(1 - 4c_{14})}{21}. \tag{2.19}$$

Considering the fact that $|c_{14}|$ is as small as (2.17), we then find that

$$-\frac{2}{27} \lesssim c_2 \lesssim \frac{2}{21}, \tag{2.20}$$

which, together with the constraint (2.15), yields,

$$0 < c_2 \lesssim 0.095. \tag{2.21}$$

On the other hand, from $c_S^2 \gtrsim 1$ we also find that

$$\frac{c_2(2 - c_{14})}{c_{14}(2 + 3c_2)} \gtrsim 1 . \tag{2.22}$$

Considering the constraints (2.17) and (2.21), we find that Eq.(2.22) is equivalent to

$$0 < c_{14} \lesssim c_2 , \tag{2.23}$$

which, together with the constraint (2.21), yields

$$0 < c_{14} \lesssim c_2 \lesssim 0.095\,. \tag{2.24}$$

By setting $c_{13} = 0$ in Eq.(2.1), we also find

$$\alpha_2 \simeq \frac{c_{14}\,(c_{14} + 2c_2c_{14} - c_2)}{c_2\,(2 - c_{14})}, \tag{2.25}$$

and the second constraint in (2.2) yields

$$-10^{-7} \leq \frac{c_{14}\,(c_{14} + 2c_2c_{14} - c_2)}{c_2\,(2 - c_{14})} \leq 10^{-7}. \tag{2.26}$$

In Figs. 2.1-2.3, we show this constraint, combined with (2.24), for various scales of c_{14} in the (c_2, c_{14})-plane. The constraints in the (c_2, c_{14})-plane have simple expressions for values of c_{14} smaller than 2×10^{-7} or sufficiently larger than 2×10^{-7} (say, for c_{14} larger than 2×10^{-6}): the constraints are satisfied in either of the following two regions,

$$\begin{aligned}
\text{(i)} \qquad & 0 < c_{14} \leq 2 \times 10^{-7}\,, \\
& c_{14} \lesssim c_2 \lesssim 0.095\,, \\
\text{(ii)} \qquad & 2 \times 10^{-6} \lesssim c_{14} \lesssim 2.5 \times 10^{-5}\,, \\
& 0 \lesssim c_2 - c_{14} \lesssim 2 \times 10^{-7}\,.
\end{aligned} \tag{2.27}$$

For the constraints in the intermediate regime of c_{14} ($2 \times 10^{-7} < c_{14} \lesssim 2 \times 10^{-6}$), see the top and the middle plots in Figs. 2.1-2.3.

The constraints (2.12) with (2.13) in principle constrain the parameters c_i's. However, the sensitivities $\sigma_{æ}$ of a neutron star, which depend on c_i's and the equation of state of nuclear matter [29], are not known so far within the new ranges of the parameters given above. Therefore, instead of using (2.12) to constrain the parameters c_i's, we simply rewrite them in term of c_i's and the sensitivities $\sigma_{æ}$ for future

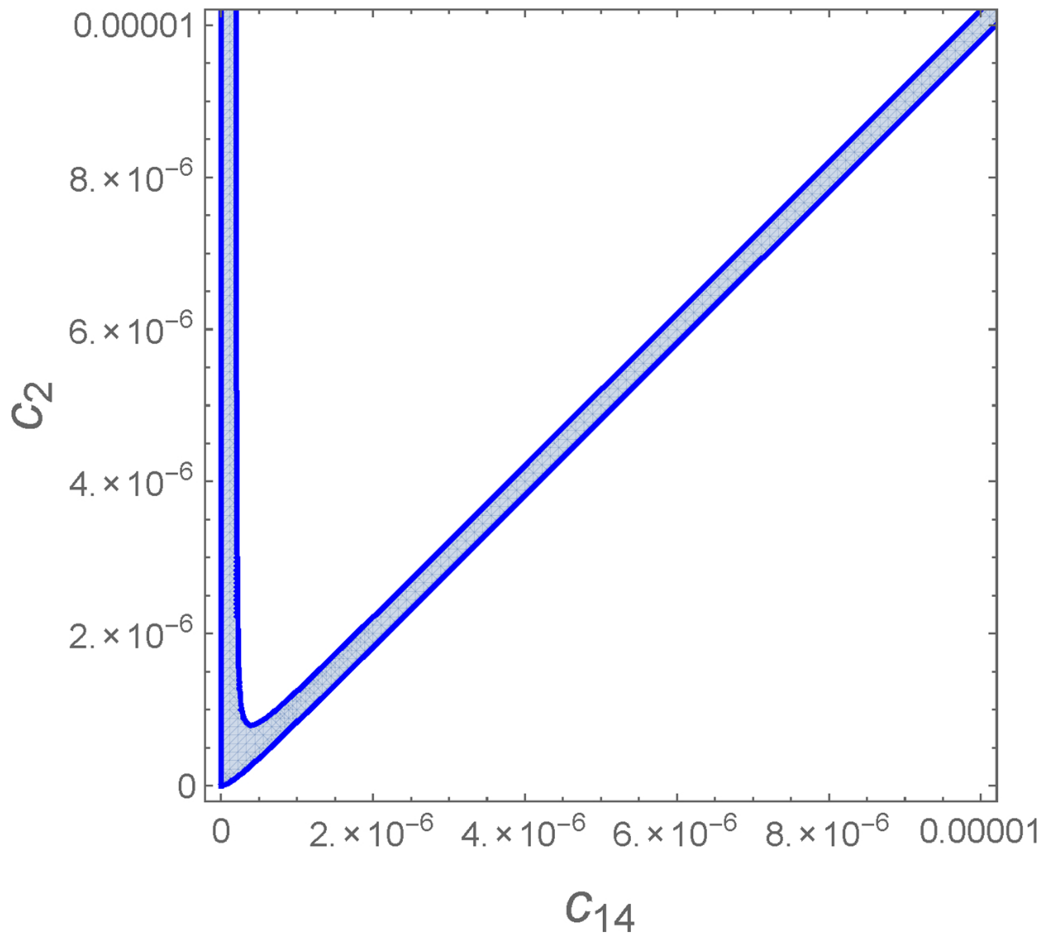

Figure 2.1: In this figure, we plot the constraint $|\alpha_2| \leq 10^{-7}$ given by Eq.(2.26), together with Eq.(2.24), in the (c_2, c_{14})-plane.

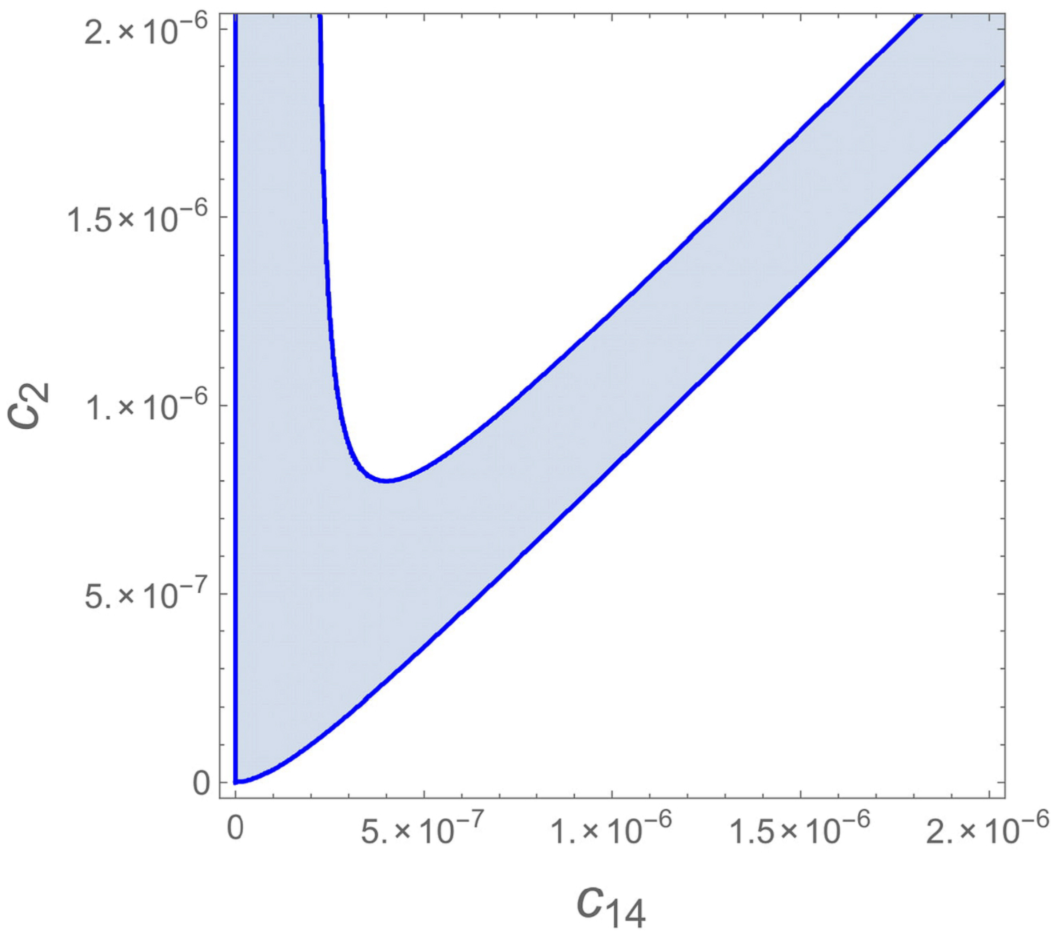

Figure 2.2. A version of Fig.2.1 with a different scale

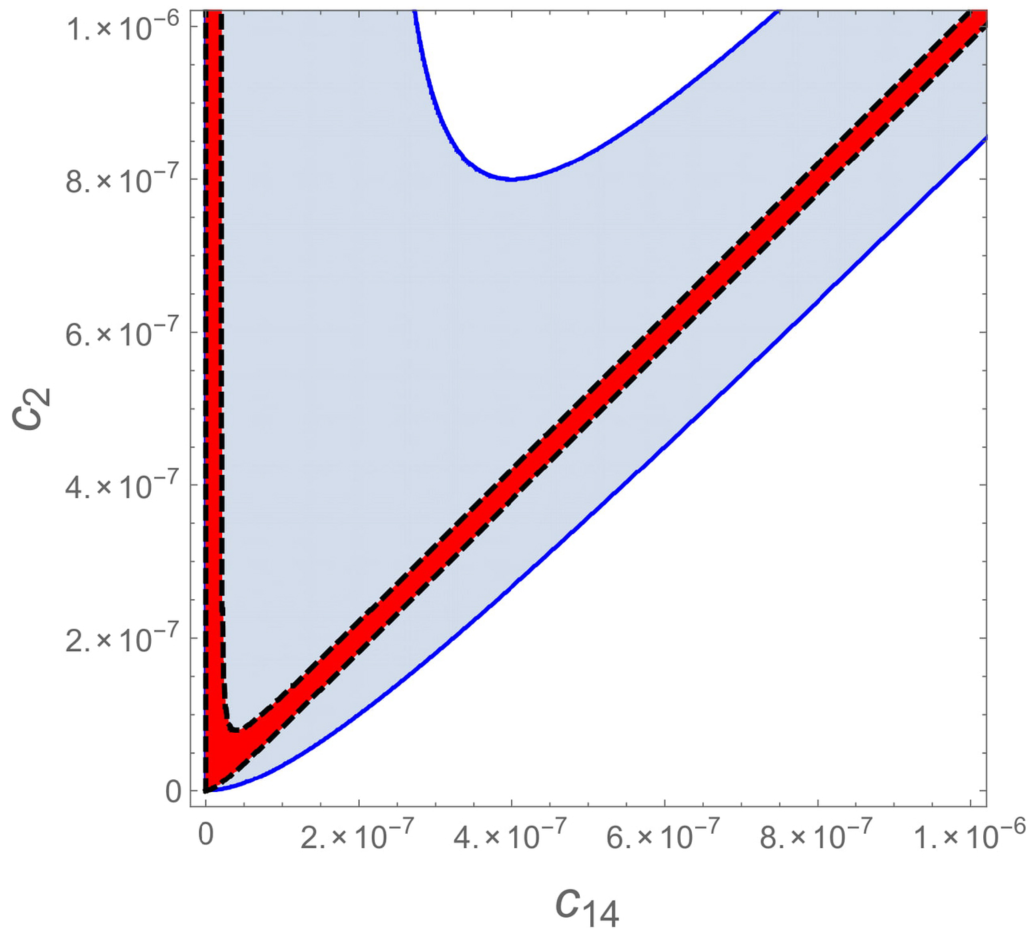

Figure 2.3: A version of Fig.2.1 with a different scale. In this plot, the region $|\alpha_2| \leq 10^{-8}$ marked with red color and dashed line boundary is also shown.

references. Setting $c_{13} = 0$ in Eq.(2.13), we find that

$$\begin{aligned} \hat{\alpha}_1 &= \alpha_1 \left[1 + \sigma_{æ} \left(1 + \frac{8}{\alpha_1} \right) \right] , \\ \hat{\alpha}_2 &= \alpha_2 \left[1 + \sigma_{æ} \left(1 + \frac{8}{\alpha_1} \right) \right] . \end{aligned} \tag{2.28}$$

Since $|\alpha_1| \leq 10^{-4}$, the constraints (2.12) are reduced to

$$|\alpha_1 + 8\sigma_{æ}| \leq 10^{-5} , \quad \left| \frac{\alpha_2}{\alpha_1} \right| \times |\alpha_1 + 8\sigma_{æ}| \leq 10^{-9} . \tag{2.29}$$

As already mentioned above, we leave the analysis of these two constraints that involves the computation of the sensitivities $\sigma_{æ}$ to a future work.

CHAPTER THREE

Exact Plane Wave Solutions in Einstein-aether Theory

This chapter published as [15]: J. Oost, M. Bhattacharjee, A. Wang, "Plane-fronted gravitational waves with parallel rays in Einstein-aether theory", Gen. Relativ. Grav. 50 (2018) 124

In this chapter, we systematically study spacetimes of gravitational plane waves in Einstein-aether theory. Due to the presence of the timelike aether vector field, now the problem in general becomes overdetermined. In particular, for the linearly polarized plane waves, there are five independent vacuum Einstein-aether field equations for three unknown functions. Therefore, solutions exist only for particular choices of the four free parameters c_i's of the theory. We find that there exist eight cases, in two of which any form of gravitational plane waves can exist, similar to that in general relativity, while in the other six cases, gravitational plane waves exist only in particular forms. Beyond these eight cases, solutions either do not exist or are trivial (simply representing a Minkowski spacetime with a constant or dynamical aether field.).

3.1 Introduction

The introduction of the aether vector field allows for some novel effects, e.g., matter fields can travel faster than the speed of light [77], and new gravitational wave polarizations can spread at different speeds [23]. It should be noted that the faster-than-light propagation does not violate causality [20]. In particular, gravitational theories with broken LI still allow the existence of black holes [78]. However, instead of Killing horizons, now the boundaries of black holes are hypersurfaces termed *universal horizons*, which can trap excitations traveling at arbitrarily high velocities (For more details, see, for example, [20] for a recent review.). This universal horizon may radiate thermally at a fixed temperature and strengthen a possible thermodynamic interpretation though there is no universal light cone [63].

Another interesting issue is whether or not spacetimes of gravitational plane waves are compatible with the presence of the timelike aether field. This becomes more interesting after the recent observations of several gravitational waves (GWs) emitted from remote binary systems of either black holes [79, 80, 81, 82] or neutron stars [31]. The sources of these GWs are far from us, and when they arrive to us, they can be well approximated by gravitational plane waves. However, this issue is not trivial, specially for the Einstein-aether theory, in which a globally time-like aether field exists, while such plane waves, by definition, move along congruences defined by a null vector.

In this chapter, we shall focus ourselves on this issue. In particular, we shall show that the system of the differential equations for gravitational plane waves in the Einstein-aether theory is in general overdetermined, that is, we have more independent differential equations than the number of independent functions that describe the spacetime and aether, sharply in contrast to that encountered in Einstein's General Relativity (GR), in which the problem is usually underdetermined, that is, we have less independent differential equations than the number of independent functions that describe the spacetime [76, 83, 84]. In particular, for the linearly polarized gravitational plane waves, there are five independent vacuum field equations for three unknown functions in the Einstein-aether theory, while there is only one independent vacuum field equation for two unknown functions in GR.

The rest of the chapter is organized as follows: In Sec. 3.2, we give a summary on the gravitational plane waves with two independent polarization directions, and define the polarization angle with respect to a parallelly transported basis along the path of the propagating gravitational plane wave. Such a description is valid for any metric theory, including GR and Einstein-aether theory. In Sec. 3.3, we systematically study the linearly polarized gravitational plane waves in Einstein-aether theory, and find that such gravitational plane wave solutions exist only for particular choices of the free parameter c_i's of the theory. We identify all these particular cases, and find that there are in total eight cases. Cases beyond these either do not allow such solutions to exist or are trivial, in the sense that their spacetime is Minkowski (though sometimes with a dynamical aether field). The full details of all tensor components and field equations not stated in Chapter Three are relegated to Appendix B.

3.2 Polarizations and Interaction of Gravitational Plane Waves

The spacetimes for gravitational plane waves can be cast in various forms, depending on the choice of the coordinates and gauge-fixing [76, 83, 84]. In this chapter, we shall adopt the form originally due to Baldwin, Jeffery, Rosen (BJR) [87, 88], which can be cast as [85, 76]

$$\begin{aligned} ds^2 &= -2e^{-M}dudv + e^{-U}\Big[e^{V}\cosh W dy^2 - 2\sinh W dydz \\ &\quad + e^{-V}\cosh W dz^2\Big], \end{aligned} \tag{3.1}$$

where M, U, V and W are functions of u only, which in general represents a gravitational plane wave propagating along the null hypersurfaces u = constant. The corresponding spacetimes belong to Petrov Type N [76, 83, 84] [1]. Choosing a null tetrad defined as,

$$\begin{aligned} & l^\mu \equiv B\delta^\mu_v, \quad n^\mu \equiv A\delta^\mu_u, \quad m^\mu = \zeta^2\delta^\mu_2 + \zeta^3\delta^\mu_3, \\ & \bar{m}^\mu = \overline{\zeta^2}\delta^\mu_2 + \overline{\zeta^3}\delta^\mu_3, \end{aligned} \tag{3.2}$$

where A and B must be chosen so that $M \equiv \ln(AB)$, and

$$\begin{aligned} \zeta^2 &\equiv \frac{e^{(U-V)/2}}{\sqrt{2}}\left(\cosh\frac{W}{2} + i\sinh\frac{W}{2}\right), \\ \zeta^3 &\equiv \frac{e^{(U+V)/2}}{\sqrt{2}}\left(\sinh\frac{W}{2} + i\cosh\frac{W}{2}\right), \end{aligned} \tag{3.3}$$

[1] By rescaling the null coordinate $u \to u' = \int e^{-M(u)}du$, without loss of the generality, one can always set $M = 0$.

we find that the Weyl tensor has only one independent component, represented by Ψ_4, and is given by [76],

$$\begin{aligned}
C^{\mu\nu\alpha\beta} &= 4\Big[\Psi_4 l^{[\mu}m^{\nu]}l^{[\alpha}m^{\beta]} + \bar{\Psi}_4 l^{[\mu}\bar{m}^{\nu]}l^{[\alpha}\bar{m}^{\beta]}\Big],\\
\Psi_4 &= -\frac{1}{2}A^2\Bigg\{\cosh W V_{uu} + \cosh W\big(M_u - U_u\big)V_u\\
&\quad +2\sinh W V_u W_u + i\Big[W_{uu} + \big(M_u - U_u\big)W_u\\
&\quad -\sinh W\cosh W V_u^2\Big]\Bigg\},
\end{aligned} \tag{3.4}$$

where $[A,B] \equiv (AB - BA)/2$, and $V_u \equiv \partial V/\partial u$, etc. To see the physical meaning of Ψ_4, following [85, 76], let us first introduce the orthogonal spacelike unit vectors, $E^{\mu}_{(a)}$ $(a = 2,3)$, in the (y,z)-plane via the relations,

$$E^{\mu}_{(2)} \equiv \frac{m^{\mu} + \bar{m}^{\mu}}{\sqrt{2}}, \quad E^{\mu}_{(3)} \equiv \frac{m^{\mu} - \bar{m}^{\mu}}{i\sqrt{2}}, \tag{3.5}$$

we find that the Weyl tensor can be written in the form,

$$C^{\mu\nu\alpha\beta} = \frac{1}{2}\Big[e^{\mu\nu\alpha\beta}_{+}\big(\Psi_4 + \bar{\Psi}_4\big) + ie^{\mu\nu\alpha\beta}_{\times}\big(\Psi_4 - \bar{\Psi}_4\big)\Big], \tag{3.6}$$

where

$$\begin{aligned}
e^{\mu\nu\alpha\beta}_{+} &\equiv 4\Big(l^{[\mu}E^{\nu]}_{(2)}l^{[\alpha}E^{\beta]}_{(2)} - l^{[\mu}E^{\nu]}_{(3)}l^{[\alpha}E^{\beta]}_{(3)}\Big),\\
e^{\mu\nu\alpha\beta}_{\times} &\equiv 4\Big(l^{[\mu}E^{\nu]}_{(2)}l^{[\alpha}E^{\beta]}_{(3)} + l^{[\mu}E^{\nu]}_{(3)}l^{[\alpha}E^{\beta]}_{(2)}\Big).
\end{aligned} \tag{3.7}$$

Making a rotation in the $\big(E_{(2)}, E_{(3)}\big)$-plane,

$$\begin{aligned}
E_2 &= E'_{(2)}\cos\varphi + E'_{(3)}\sin\varphi,\\
E_3 &= -E'_{(2)}\sin\varphi + E'_{(3)}\cos\varphi,
\end{aligned} \tag{3.8}$$

we find that

$$\begin{aligned}
e_{+} &= e'_{+}\cos 2\varphi + e'_{\times}\sin 2\varphi,\\
e_{\times} &= -e'_{+}\sin 2\varphi + e'_{\times}\cos 2\varphi.
\end{aligned} \tag{3.9}$$

In particular, if we choose φ such that

$$\varphi = \frac{1}{2}\tan^{-1}\left(\frac{\mathrm{Im}\,(\Psi_4)}{\mathrm{Re}\,(\Psi_4)}\right), \tag{3.10}$$

we obtain

$$C^{\mu\nu\alpha\beta} = \frac{1}{2}\left|\Psi_4\right| e'^{\ \mu\nu\alpha\beta}_{+}. \tag{3.11}$$

Thus, the amplitude of the Weyl tensor is proportional to the absolute value of Ψ_4, and the angle defined by Eq.(3.10) is the polarization angle of the gravitational plane wave in the plane spanned by $\left(E_{(2)}, E_{(3)}\right)$, which is orthogonal to the propagation direction l^μ of the gravitational plane wave. It is interesting to note that the unit vectors $E^\mu_{(2)}$ and $E^\mu_{(3)}$ are parallelly transported along l^ν,

$$l^\nu D_\nu E^\mu_{(2)} = 0 = l^\nu D_\nu E^\mu_{(3)}. \tag{3.12}$$

Therefore, the angle defined by Eq.(3.10) is invariant with respect to the parallelly transported basis $\left(E_{(2)}, E_{(3)}\right)$ along the propagation direction l^μ of the gravitational plane wave [2]. This is an important property belonging only to single gravitational plane waves.

When $W = 0$, from Eq.(3.5) we find that

$$\mathrm{Im}\,(\Psi_4) = 0,\ (W = 0), \tag{3.13}$$

and $\varphi = 0$. Then, the polarization is along the $E^\mu_{(2)}$-direction, which is usually referred to as the "+" polarization, characterized by the non-vanishing of the function V. The other polarization of the gravitational plane wave, often referred to as the "×" polarization, is represented by the non-vanishing of the function W, for which generically we have $\mathrm{Im}\,(\Psi_4) \neq 0$ ($W \neq 0$) (cf. Fig. 1 given in [85]).

[2] Polarizations of GWs in weak-field approximations were also studied in [30] in the framework of Einstein-aether theory.

When M, U, V and W are functions of v only, the gravitational plane wave is now propagating along the null hypersurfaces $v =$ constant. In this case, by rescaling the null coordinate $v \to v' = \int e^{-M(v)} dv$, one can always set $M(v) = 0$.

When gravitational plane waves moving in both of the two null directions are present, the metric coefficients M, U, V and W are in general functions of u and v. An interesting case is the collision of two gravitational plane waves moving along the opposition directions, which generically produces spacetime singularities due to their mutual focuses [89]. Another remarkable feature is that one of the gravitational plane waves can serve as a medium for the other, due to their non-linear interaction, so the polarizations of the gravitational plane wave can be changed. The change of polarizations due to the nonlinear interaction is exactly a gravitational analogue of the Faraday rotation, but with the other gravitational plane wave as the magnetic field and medium [85, 76, 76].

3.3 Linearly Polarized gravitational plane waves

In this section, we shall consider gravitational plane waves moving along the hypersurfaces u =constant only with one direction of polarizations, which are usually called linearly polarized gravitational plane waves. Without loss of the generality, we shall consider only gravitational plane waves with the "+" polarization. Then, by rescaling the u coordinate, without loss of the generality, we can always set $M = 0$, so the metric takes the form,

$$ds^2 = -2dudv + e^{-U(u)}\left(e^{V(u)}dy^2 + e^{-V(u)}dz^2\right). \tag{3.14}$$

We also assume that the aether moves only in the (u, v)-plane, so its four-velocity u_μ takes the general form,

$$u^\mu = \frac{1}{\sqrt{2}}(e^{-h}, e^h, 0, 0). \tag{3.15}$$

Since the spacetime is only of u dependence, it is easy to see that $h = h(u)$. Then, the non-vanishing components of the Einstein and aether tensors $G_{\mu\nu}$ and $T^{æ}_{\mu\nu}$ and the aether vector $Æ_\mu$ are given, respectively, by Eqs.(B.1) and (B.2). In the vacuum case, we have $T^m_{\mu\nu} = 0,\ T_\mu = 0$, and the Einstein-aether equations (1.31) reduce to

$$G_{\mu\nu} = T^{æ}_{\mu\nu}, \tag{3.16}$$

which yields five equations, given by Eqs.(B.4)-(B.7). The aether equations $Æ_\mu = 0$ yield the same equation as given by Eq.(B.5). It is interesting to note that in Einstein's theory the field equations $G_{\mu\nu} = 0$ yields only a single equation [85, 76],

$$2U_{uu} - U_u^2 = V_u^2, \tag{3.17}$$

for the two unknown functions $U(u)$ and $V(u)$. In this sense, the problem is underdetermined in Einstein's theory. Thus, for any given gravitational wave $V(u)$, we can always integrate the above equation to find $U(u)$.

It is remarkable to note that there are five independent field equations for the three unknowns, U, V and h. Therefore, in contrast to the situation of GR, in which there is only one independent field equation, given by Eq.(3.17), for two unknown functions U and V, here in the framework of the Einstein-aether theory, we are facing an overdetermined problem, instead of underdetermined, and clearly only for particular cases the above equations allow solutions for U, V and h.

From the constraint of Eq.(2.4) we can see that the current observations of GW170817 and GRB 170817A practically requires $c_{13} \simeq 0$. In addition, for the spin-2 gravitons to move precisely with the speed of light, we also need to set $c_{13} = 0$. However, in order for our results to be as much applicable as possible, in the rest of this section we shall not impose this condition, and consider all the possible solutions with both $c_{13} = 0$ and $c_{13} \neq 0$, separately.

3.3.1 Solutions with $c_{13}=0$

When $c_{13}=0$, Eqs.(B.4)-(B.7) reduce to,

$$2U_{uu}-\left(U_u^2+V_u^2\right)+2c_{14}\left(h_{uu}-h_uU_u-h_u^2\right)=0, \tag{3.18}$$

$$c_2\left(U_{uu}-2h_uU_u-U_u^2\right)+\left(c_2-c_{14}\right)\left(h_{uu}-h_uU_u-2h_u^2\right)=0, \tag{3.19}$$

$$c_2U_{uu}+\left(c_2-c_{14}\right)\left(h_{uu}-h_uU_u-h_u^2\right)=0, \tag{3.20}$$

$$c_2\left(2U_{uu}-U_u^2-4h_uU_u\right)2c_2h_{uu}-\left(3c_2+c_{14}\right)h_u^2=0. \tag{3.21}$$

Then, from Eqs.(3.19) and (3.20) we find

$$c_2\left(U_u^2+2U_uh_u\right)+\left(c_2-c_{14}\right)h_u^2=0, \tag{3.22}$$

$$c_2\left(U_{uu}+U_u^2\right)+\left(c_2+c_{14}\right)U_uh_u+\left(c_2-c_{14}\right)h_{uu}=0. \tag{3.23}$$

To study the above equations further, we need to distinguish the cases $c_2\neq c_{14}$ and $c_2=c_{14}$, separately.

3.3.1.1 When $c_2\neq c_{14}$. In this case, from Eqs.(3.22) and (3.23) we find that

$$h_u^2=\frac{c_2}{c_{14}-c_2}\left(U_u^2+2U_uh_u\right), \tag{3.24}$$

$$h_{uu}=\frac{1}{c_{14}-c_2}\left\{c_2\left(U_{uu}+U_u^2\right)+\left(c_2+c_{14}\right)U_uh_u\right\}. \tag{3.25}$$

Inserting the above expressions into Eq.(3.21), we find

$$c_2c_{14}\left(U_{uu}-U_u^2-2U_uh_u\right)=0, \tag{3.26}$$

from which we can see that there are three different cases that need to be considered separately,

$$i)\ c_2c_{14}\neq 0,\quad ii)\ c_2=0,\ c_{14}\neq 0,\quad iii)\ c_2\neq 0,\ c_{14}=0. \tag{3.27}$$

Case i) $c_2c_{14}\neq 0$: In this case we have

$$U_{uu}=U_u^2+2U_uh_u, \tag{3.28}$$

which has the solution

$$U_u = \alpha_0 e^{U+2h}, \tag{3.29}$$

where α_0 is an integration constant. Then Eq.(3.19) reduces to

$$h_{uu} - 2h_u^2 - h_u U_u = 0, \tag{3.30}$$

which has the solution

$$h_u = \alpha_1 e^{2h+U}, \tag{3.31}$$

where α_1 is an integration constant. Notice that $h_u \propto U_u$. In fact we may write

$$h = \alpha U + h_0, \tag{3.32}$$

where α and h_0 are constants. By substituting Eqs.(3.28) and (3.32) into Eq.(3.20) or (3.21) we find that

$$\alpha = -\frac{\sqrt{c_2}}{\sqrt{c_2} \pm \sqrt{c_{14}}}. \tag{3.33}$$

By substituting Eqs (3.28) and (3.32) into Eq.(3.18) we find

$$V = \beta U + V_0, \tag{3.34}$$

where V_0 is another integration constant, and

$$\beta \equiv \pm\sqrt{1 + 4\alpha + 2c_{14}\alpha^2}. \tag{3.35}$$

Now combining Eqs.(3.32) and (3.29) we find

$$U_u = \hat{\alpha}_0 e^{(2\alpha+1)U}, \tag{3.36}$$

where $\hat{\alpha}_0 \equiv \alpha_0 e^{2h_0}$. Thus, we obtain

$$U(u) = -\frac{1}{2\alpha+1}\ln\left[-\alpha_0(2\alpha+1)(u-u_0)\right], \tag{3.37}$$

where u_0 is a constant of integration. Once $U(u)$ is given the functions $h(u)$ and $V(u)$ can be read off from Eqs.(3.32) and (3.34), respectively, that is,

$$\begin{aligned} V(u) &= -\frac{\beta}{2\alpha+1}\ln\left[-\alpha_0(2\alpha+1)(u-u_0)\right] + V_0, \\ h(u) &= -\frac{\alpha}{2\alpha+1}\ln\left[-\alpha_0(2\alpha+1)(u-u_0)\right] + h_0, \end{aligned} \tag{3.38}$$

where β is given by Eq.(3.35) in terms of α and c_{14}.

Case ii) $c_2 = 0,\ c_{14} \neq 0$**:** In this case from Eqs.(3.19) and (3.20) we find that $h_u = 0$, that is

$$h(u) = h_0, \tag{3.39}$$

where h_0 is a constant. Then, Eqs.(3.19) - (3.20) are satisfied identically, while Eq.(3.18) reduce to

$$2U_{uu} - U_u^2 = V_u^2, \tag{3.40}$$

which is the same as in GR, that is, in the present case the functions U and V are not uniquely determined. For any given $U(u)$, one can integrate the above equation to obtain $V(u)$.

Case iii) $c_2 \neq 0,\ c_{14} = 0$**:** In this case from Eqs.(3.19) and (3.20) we find that $U_u + h_u = 0$, which has the solution,

$$U = -h + U_0, \tag{3.41}$$

where U_0 is a constant. Inserting the above expression into Eq.(3.19) we find that $h_u = 0$, that is,

$$h = h_0. \tag{3.42}$$

Then, from Eq.(3.18) we obtain

$$V = V_0, \tag{3.43}$$

where V_0 is a constant. By rescaling y and z coordinates, without loss of the generality, we can always set $V_0 = U_0 = 0$, so the solution represents the Minkowski spacetime. That is, in the current case only the trivial Minkowski solution is allowed.

3.3.1.2 When $c_2 = c_{14}$. In this case, from Eq.(3.20) we find that

$$c_2 U_{uu} = 0. \tag{3.44}$$

Therefore, depending on the values of c_2, we have two different cases.

Case i) $c_2 = c_{14} \neq 0$**:** In this case, we must have $U_{uu} = 0$, which has the general solution,

$$U(u) = \alpha_0 u + U_0, \tag{3.45}$$

where α_0 and U_0 are two integration constants. On the other hand, from Eq.(3.19) we find that

$$h(u) = -\frac{\alpha_0}{2} u + h_0, \tag{3.46}$$

while Eq.(3.21) is satisfied identically. Then, from Eq.(3.18) we find that

$$V(u) = \pm\sqrt{\frac{(c_2 - 2)\alpha_0^2}{2}}\, u + V_0, \tag{3.47}$$

where V_0 is another integration constant.

Case ii) $c_2 = c_{14} = 0$**:** In this case, Eqs.(3.19) - (3.21) are satisfied identically for any given $h(u)$, while Eq.(3.18) reduces to

$$2U_{uu} - U_u^2 = V_u^2, \tag{3.48}$$

which is the same as in GR, that is, in the present case the functions U, V and $h(u)$ are not uniquely determined. For any given $U(u)$ and $h(u)$, one can integrate Eq.(3.48) to obtain $V(u)$.

3.3.2 Solutions with $c_{13} \neq 0$

When $c_{13} \neq 0$, from Eqs.(B.6) and (B.7) we find that

$$V_{uu} - U_u V_u - 2h_u V_u = 0, \tag{3.49}$$

which has the solution,

$$V_u = \alpha_0 e^{U+2h}, \tag{3.50}$$

where α_0 is an integration constant. Inserting the above expression into Eqs.(B.4)-(B.7), we obtain the following four independent equations for U and h,

$$V_u^2 = 2U_{uu} - U_u^2 + 2c_{14}\Big(h_{uu} - h_u U_u - h_u^2\Big) = V_u^2, \tag{3.51}$$

$$0 = c_2\Big(U_{uu} - 2h_u U_u - U_u^2\Big) + \big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_u U_u - 2h_u^2\Big), \tag{3.52}$$

$$0 = 2\big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_u U_u - h_u^2\Big) + 2c_2 U_{uu} + c_{13}U_u^2 + c_{13}V_u^2, \tag{3.53}$$

$$\begin{aligned} 0 = {} & \big(c_{13} + 2c_2\big)\Big(2U_{uu} - U_u^2 - 4h_u U_u\Big) + 4c_2 h_{uu} \\ & - 2\big(3c_2 - c_{13} + c_{14}\big)h_u^2 + c_{13}V_u^2, \end{aligned} \tag{3.54}$$

Combining Eqs.(3.51) and (3.53) we find

$$c_{123}U_{uu} = (c_{13}c_{14} + c_2 + c_{13} - c_{14})(h_u^2 + h_u U_u - h_{uu}), \tag{3.55}$$

and by using Eqs.(3.51) and (3.54) we obtain

$$\begin{aligned} c_{123}U_u^2 = {} & (c_{13}c_{14} + 2c_{13} - 2c_{14})(h_u^2 + h_u U_u - h_{uu}) \\ & + (c_{13} - c_{14} - c_2)h_u^2 - 2c_{123}h_u U_u. \end{aligned} \tag{3.56}$$

To study the above equations further, we need to consider separately the cases $c_{123} = 0$ and $c_{123} \neq 0$.

3.3.2.1 When $c_{123} = 0$. In this case, from Eqs.(3.51) and (3.53) we find

$$c_{14}(c_{13} - 1)(h_{uu} - h_u^2 - h_u U_u) = 0. \tag{3.57}$$

The possibility of $c_{13} = 1$ is ruled out by observation [14], as mentioned above, leaving the possibilities

$$c_{14} = 0, \tag{3.58}$$

or

$$h_{uu} - h_u U_u - h_u^2 = 0. \tag{3.59}$$

Case A.1 $c_{14} = 0$: In the case of Eq.(3.58) we find that Eqs.(3.52) and (3.54) reduce to

$$U_{uu} = 2h_u U_u + U_u^2, \tag{3.60}$$

and

$$h_{uu} = 2h_u^2 + h_u U_u, \tag{3.61}$$

respectively, where we have used the fact that Eq.(3.51) reduces to $2U_{uu} = U_u^2 + V_u^2$. Then, both h_u and U_u are proportional to e^{2h+U}, and hence by Eq.(3.50) we find

$$h = \alpha V + h_0 \quad U = \beta V + U_0, \tag{3.62}$$

where h_0 and U_0 are two integration constants, and the constants α and β can be determined by substituting Eq.(3.62) and Eq.(3.60) into Eq.(3.51) or Eq.(3.53), which yields

$$\alpha = \frac{1-\beta^2}{4\beta}. \tag{3.63}$$

Inserting the above expressions into Eq.(3.50), we find that

$$V = -\frac{2\beta}{1+\beta^2} \ln\left[\hat{\alpha}_0 \left(u_0 - u\right)\right], \tag{3.64}$$

where $\hat{\alpha}_0 \equiv \alpha_0 \left(2\alpha + \beta\right) e^{U_0 + 2h_0}$ and u_0 is an integration constant. Therefore, in this case the solutions are given by Eqs.(3.62)-(3.64).

Case A. 2 $c_{14} \neq 0$: In this case we find that

$$h_u = \alpha_1 e^{h+U}, \tag{3.65}$$

and by Eq.(3.54) that

$$h_u^2 \left(\frac{c_{14}}{c_{13}} - 2\right) = 0. \tag{3.66}$$

If $h_u = 0$ $(\alpha_1 = 0)$ then by Eq.(3.52) we have

$$U_u = \alpha_2 e^{U}, \tag{3.67}$$

and using this result with Eq.(3.51) we have

$$U = \pm V + U_0. \tag{3.68}$$

Inserting the above expressions into Eq.(3.50), we find that

$$V = \mp \ln \left[\mp \hat{\alpha}_0 (u - u_0) \right], \tag{3.69}$$

where $\hat{\alpha}_0 \equiv \alpha_0 e^{2h_0 + U_0}$ and where the choice of upper or lower sign must hold for both Eqs(3.68) and Eq.(3.69). Thus, in this case, the general solutions are given by

$$(U, V, h) = \left(\pm V + U_0, V, h_0 \right), \tag{3.70}$$

where V is given by Eq.(3.69), and U_0 and h_0 are two integration constants.

However, if $h_u \neq 0$ then Eq.(3.52) reduces to

$$U_{uu} - 2h_u^2 - U_u^2 - 2h_u U_u = 0, \tag{3.71}$$

and we add the LHS of Eq.(3.59) (which is zero) twice to the LHS of Eq.(3.71) to get

$$U_{uu} + 2h_{uu} - 4h_u^2 - 4h_u U_u - U_u^2 = 0, \tag{3.72}$$

which simplifies to

$$2h_{uu} + U_{uu} = (2h_u + U_u)^2. \tag{3.73}$$

If we define a function f(u) such that

$$f(u) = 2h(u) + U(u), \tag{3.74}$$

then Eq.(3.73) can be written as

$$f_{uu} = f_u^2, \tag{3.75}$$

which has the solution

$$f = -\ln \left(-\alpha_3 (u - u_0) \right), \tag{3.76}$$

where α_3 and u_0 are integration constants. If we multiply both sides of Eq.(3.65) by e^h we have

$$h_u e^h = \alpha_1 e^{2h+U}, \tag{3.77}$$

and making use of Eq.(3.76) we find

$$h_u e^h = -\frac{\alpha_1}{\alpha_3}\frac{1}{u-u_0}, \tag{3.78}$$

whereupon integration we find

$$h = \ln\left(-\frac{\alpha_1}{\alpha_3}\ln(u-u_0) + h_0\right). \tag{3.79}$$

So, for the functions U and V we have

$$U = -\ln(-\alpha_3(u-u_0)) - 2h, \tag{3.80}$$

$$V = -\frac{\alpha_0}{\alpha_3}\ln(u-u_0) + V_0. \tag{3.81}$$

By substituting these results into Eq.(3.51) we find that $\alpha_3 = \pm\alpha_0$.

3.3.2.2 When $c_{123} \neq 0$. In this case we can substitute Eqs.(3.55) and (3.56) into Eq.(3.52) and by defining

$$Q \equiv c_{123} - c_{14} + \frac{c_2}{c_{123}}(c_{13} - c_{14} - c_2), \tag{3.82}$$

we have

$$Q(h_{uu} - 2h_u^2 - h_u U_u) = 0. \tag{3.83}$$

And so we must consider the cases where $Q \neq 0$ and $Q = 0$.

Case B.1 $Q \neq 0$: Then, we have

$$h_u = \alpha_1 e^{2h+U} \propto V_u. \tag{3.84}$$

Using this result with Eqs.(3.55) and (3.56) we find also that

$$U_u = \alpha_2 e^{2h+U} \propto V_u, \tag{3.85}$$

and thus we can set

$$h = \alpha V + h_0, \quad U = \beta V + U_0, \tag{3.86}$$

for some constants α, β, h_0 and U_0. Substituting Eqs.(3.86) and (3.76) into Eqs.(3.51) and (3.53), we find that α and β must satisfy the relations,

$$\beta^2 + 4\alpha\beta + 2c_{14}\alpha^2 - 1 = 0, \tag{3.87}$$

$$2(c_{14} - c_{13} - c_2)\alpha^2 - 4c_2\alpha\beta$$
$$- (c_{13} + 2c_2)\beta^2 - c_{13} = 0, \tag{3.88}$$

which uniquely determine α and β, but the expressions for them are too long to be presented here. Inserting the above expressions into Eq.(3.50), we find that

$$V = -\frac{1}{2\alpha + \beta} \ln\left[\hat{\beta}_0(u_0 - u)\right], \tag{3.89}$$

where $\hat{\beta}_0 \equiv \alpha_0(2\alpha + \beta)e^{U_0 + 2h_0}$. Therefore, in the present case, once α and β are determined by Eqs.(3.87) and (3.88), the functions $V(u)$, $U(u)$ and the aether field $h(u)$ are given, respectively, by Eqs.(3.86) and (3.89).

Case B.2 $Q = 0$: It will be helpful to try to solve for c_{14} as a function of the other c_i's, and to introduce a new parameter δ such that:

$$\delta = 2c_2 + c_{13}. \tag{3.90}$$

Then we find from Eq.(3.82) that

$$c_{14}\delta = c_{13}(c_2 + \delta), \tag{3.91}$$

If we consider $\delta = 0$, then we have $c_2 = 0$ since $c_{13} \neq 0$. But by Eq.(3.90) this means we must have $c_{13} = 0$, which violates our assumption, and so we *must* have

$$\delta \neq 0, \tag{3.92}$$

and thus

$$c_{14} = c_{13}\left(1 + \frac{c_2}{\delta}\right), \tag{3.93}$$

is a general solution for the $Q = 0$ case. However, we can still have $c_2 = 0$ in general. If that is the case then we have $c_{13} = c_{14}$ and we find from Eq.(3.53) that

$$V_u^2 = -U_u^2, \tag{3.94}$$

and so to have real functions we must have U and V constant in u. Then by considering Eqs.(3.55) and (3.56) with a vanishing U_u we have

$$h_{uu} - h_u^2 = 0, \tag{3.95}$$

which has the solution

$$h = -\ln(\alpha(u - u_0)) + h_0, \ (c_2 = 0), \tag{3.96}$$

where α and h_0 are the integration constants. So, in the case of $c_2 = 0$ we have a static Minkowskian spacetime with a dynamical aether.

If $c_2 \neq 0$, then we find from Eqs.(3.55) and (3.56) that

$$U_{uu} - U_u^2 = \frac{2c_2}{\delta}(h_u^2 + h_u U_u - h_{uu}) + \frac{2c_2}{\delta}h_u^2 + 2h_u U_u, \tag{3.97}$$

and

$$2U_{uu} - U_u^2 = +\frac{2c_2}{\delta}h_u^2 + 2h_u U_u + \mathcal{D}(h_u^2 + h_u U_u - h_{uu}), \tag{3.98}$$

where

$$\mathcal{D} \equiv \frac{2c_2 c_{13}^2}{c_{123}\delta} + \frac{1}{\delta}(c_{13}^2 + 2c_2). \tag{3.99}$$

These expressions can be substituted into Eqs.(3.51) and (3.54) to find

$$V_u^2 = \left(c_{13}\frac{(c_2+\delta)}{c_{123}} - \frac{2c_2}{\delta}\right) + \frac{2c_2}{\delta}h_u^2 + 2h_u U_u, \tag{3.100}$$

and

$$V_u^2 = \left(c_{13}\frac{(c_2+\delta)}{c_{123}} - \frac{2c_2}{c_{13}}\right) + \frac{2c_2}{\delta}h_u^2 + 2h_u U_u. \tag{3.101}$$

Equating these two gives us

$$c_2(h_{uu} - h_u^2 - h_u U_u) = 0. \tag{3.102}$$

Since now we have $c_2 \neq 0$, then we must have

$$h_u = \alpha e^{h+U}. \tag{3.103}$$

In this case, Eq.(3.53) reduces to

$$V_u^2 = -\frac{2c_2}{c_{13}} U_{uu} - U_u^2, \tag{3.104}$$

and by Eq.(3.51) we also have

$$V_u^2 = 2U_{uu} - U_u^2, \tag{3.105}$$

by the result of which we must have

$$U_{uu} = 0, \tag{3.106}$$

since $c_{123} \neq 0$ in this case. As U_u must be a constant, then by Eq.(3.105) we find that V_u must be also a constant, and to keep the constants real we must have U_u and V_u vanish, as before. Considering this result, Eq.(3.52) reduces to

$$h_u^2 = 0. \tag{3.107}$$

Therefore, when $c_2 \neq 0$, the spacetime must be Minkowski and the aether field is simply given by $h(u) = h_0$, this is, the solution in the present case is

$$(U, V, h) = (U_0, V_0, h_0)\,,\ (c_2 \neq 0), \tag{3.108}$$

where $U_0,\ V_0$ and h_0 are all constants.

CHAPTER FOUR

Exact Solutions with Static Aether in Spherical Symmetry

In this chapter, we consider three different forms of metrics for spherically symmetric spacetimes in Einstein-aether theory, the conformally flat, Schwarzschild and Painlevè-Gullstrand coordinates, and present both time-dependent and time-independent exact solutions. In particular, in the conformally flat coordinates we find a static solution in closed form, which satisfies all the observational constraints of the theory and reduces to the Schwarzschild vacuum solution in the decoupling limits.

4.1 Spherical Symmetry

Static spherically symmetric spacetimes in æ-theory have been studied by various authors. In particular, static spherically-symmetric solutions were found in [55], while numerical black hole solutions were found in [56, 57], which were shown that they are indeed the end states of the gravitational collapse of a massless scalar field [58]. Recently, it was found that white holes can also be formed from gravitational collapse [59]. It has also been shown by various authors that while the speeds of the spin-0, 1 and 2 modes can be arbitrarily high, the theory would still possess black holes but now with boundaries of particles with arbitrarily large velocities, dubbed universal horizons [78], and numeric simulations showed that such black holes can also be the end states of a collapsing scalar field [61]. With the establishment of the existence of universal horizons, a natural question is, do universal horizons have a thermal interpretation, similar to Killing horizons in GR? Similar to the Killing hori-

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[1] In such studies, two exact solutions were found in [62], one with $c_{14} = 0$ and the other with $c_{123} = 0$, where $c_{i...j} \equiv c_i + ... + c_j$, and c_i's are the four dimensionless coupling constants of æ-theory. These solutions were further generalized to couple with an electromagnetic field, respectively, in four-, three- and high dimensions [64, 65, 66]. Unfortunately, these solutions do not satisfy the current constraints of the theory given in Chapter Two or [14].

zons, it was argued that universal horizons should possess such properties, and showed explicitly that the first law of black hole mechanics indeed exists for static and neutral universal horizons [62, 63] [1], but it is still an open question how to generalize such a first law to charged and/or rotating universal horizons [67, 68, 69, 70, 71, 72, 73]. Lately, exact plane wave solutions to the full (non-linearized) theory were found in [15], while time-dependent exact spherically-symmetric cosmological solutions were found in [60].

In this chapter we study spherically symmetric vacuum solutions of Einstein-aether theory, both time-dependent and time-independent, and in three different sets of coordinates, conformally flat, Schwarzschild and Painlevè-Gullstrand. In all of these coordinates, we assume that the aether is comoving. Specifically, the chapter is organized as follows.

In Sec. 4.2. we derive several spherical solutions in conformally-flat coordinates including a static solution in closed form and two variants of a time-dependent FLRW vacuum cosmological solution with an accelerating expansion, one with constant positive curvature and the other with constant negative curvature. In Sec. 4.3. we present a static solution in Painlevè-Gullstrand coordinates, and in Sec. 4.4 we re-derive the solution found in [55] as well as some time-dependent solutions in Schwarzschild-type coordinates. While the field equations in each coordinate system are presented in full, the individual Einstein and stress-energy components are relegated to Appendix B.

In [55] it was argued that there is a unique static solution. There, a static aether solution was found parametrically in terms of an inverse function of the form,

$$A = A(Y) \tag{4.1}$$

$$r = r(Y) \tag{4.2}$$

Where the metric components were in terms of $A(r)$, but the lack of a closed form expression made analysis complicated. In addition to other solutions we present a static aether solution in closed form in isotropic coordinates. It is an open question whether a coordinate transformation can be found that makes our solution equivalent to that of [55].

The most general form for a spherically-symmetric metric can be written as,

$$ds^2 = g_{AB}(t,r)dx^A dx^B + R(t,r)^2 d\Omega^2, \tag{4.3}$$

where $A, B = t, r$, and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. This metric clearly is invariant under the coordinate transformations,

$$t = f(\bar{t}, \bar{r}), \quad r = g(\bar{t}, \bar{r}), \tag{4.4}$$

or inversely,

$$\bar{t} = F(t,r), \quad \bar{r} = G(t,r), \tag{4.5}$$

where $f,\ g,\ F$ and G are arbitrary functions of their indicated arguments. By properly choosing these functions, we are able to fix two of the four arbitrary functions $g_{tt},\ g_{tr},\ g_{rr}$ and $R(t,r)$.

4.2 Conformally-flat Metric

In this case, we shall use the gauge freedom (4.4) to set,

$$g_{rr} = R(t,r), \quad g_{tr} = 0, \tag{4.6}$$

so that the metric (4.3) takes the form,

$$ds^2 = -e^{2\mu(r,t)}dt^2 + e^{2\nu(t,r)}d\sigma^2, \tag{4.7}$$

Where $d\sigma^2$ is the spatial part of the metric, defined as,

$$d\sigma^2 \equiv dr^2 + r^2\left(d\theta^2 + \sin\theta^2 d\phi^2\right). \tag{4.8}$$

We assume that the aether is comoving in this system of coordinates, that is,

$$u_a = e^{\mu}\delta^t_a. \tag{4.9}$$

4.2.1 Field Equations

To write down the field equations, we find convenient first to introduce the constant α and the function Σ as,

$$
\begin{aligned}
\alpha^2 &\equiv 3\left(1+\frac{3c_2+c_{13}}{2}\right), &(4.10)\\
\Sigma(t) &\equiv 3\dot{\nu}^2+2\ddot{\nu}-2\dot{\mu}\dot{\nu}. &(4.11)
\end{aligned}
$$

In this chapter we use the convention that a prime as in μ' denotes a partial derivative with respect to the radial coordinate r, and that a dot as in $\dot{\mu}$ denotes a partial derivative with respect to the time coordinate t.

Then, the non-vanishing equation for the aether dynamics is,

$$
0=(3c_2+c_{13}+c_{14})\mu'\dot{\nu}+c_{14}\dot{\mu}'-\beta\dot{\nu}', \tag{4.12}
$$

Where $\beta \equiv 3c_2+c_{13}$. The non-vanishing Einstein-aether vacuum equations $G_{ab}=T^{æ}_{ab}$ are the tt, tr, rr, $\theta\theta$ components, given, respectively, by,

$$
\begin{aligned}
e^{2\nu}\left[\alpha^2\dot{\nu}^2\right] &= e^{2\mu}\left[c_{14}\left(\frac{\mu'^2}{2}+\mu'\nu'+\mu''+2\frac{\mu'}{r}\right)\right.\\
&\quad\left.+\nu'^2+2\nu''+4\frac{\nu'}{r}\right], &(4.13)\\
c_{14}(\dot{\mu}'+\mu'\dot{\nu}) &= 2(\mu'\dot{\nu}-\dot{\nu}'), &(4.14)\\
\frac{\alpha^2}{3}e^{2\nu}\Sigma(t) &= e^{2\mu}\left[\nu'^2+2\mu'\nu'+\frac{2}{r}(\mu'+\nu')\right.\\
&\quad\left.+c_{14}\frac{\mu'^2}{2}\right], &(4.15)\\
\frac{\alpha^2}{3}e^{2\nu}\Sigma(t) &= e^{2\mu}\left[\mu'^2+\mu''+\nu''+\frac{\mu'+\nu'}{r}\right.\\
&\quad\left.-c_{14}\frac{\mu'^2}{2}\right]. &(4.16)
\end{aligned}
$$

4.2.2 Time-Independent Solutions

With no time-dependence, the five equations are reduced to three, (tt, rr, and $\theta\theta/\phi\phi$), given by the right-hand sides of Eqs.(4.13)-(4.16). We add the tt and $\theta\theta$ equations to find,

$$0 = f'' + f'^2 + \frac{3}{r} f', \tag{4.17}$$

where

$$f' = \mu' + \nu'. \tag{4.18}$$

We would begin solving Eq.(4.17 by dividing by f', but first we must consider the case in which $f' = 0$, or $\mu' = -\nu'$.

4.2.2.1 When $\mu' = -\nu'$. In this case, the tt and $rr/\theta\theta/\phi\phi$ field equations simplify to:

$$0 = (c_{14} - 2)\left(2\mu'' + 4\frac{\mu'}{r} - \mu'^2\right) \tag{4.19}$$

$$0 = (c_{14} - 2)\mu'^2 \tag{4.20}$$

In the case where $c_{14} = 2$ then all of the field equations are satisfied identically for any f and hence any μ, and the metric may be written as:

$$ds^2 = -h(r)dt^2 + \frac{1}{h(r)} d\sigma^2 \tag{4.21}$$

For any function $h(r)$.

However, if $c_{14} \neq 2$ then we see from Eq.(4.20) that $\mu' = 0$, and thus both μ and ν are constants, which makes this spacetime equivalent to the Minkowski metric by a re-scaling of t and r.

4.2.2.2 When $\mu' \neq -\nu'$. To solve Eq.(4.17) we divide both side of the equation by f' and integrate, leading to,

$$\ln\left(L_0 f'\right) = -f - 3\ln\left(\frac{r}{r_0}\right), \tag{4.22}$$

where L_0 and r_0 are the integration constants with dimensions of length. The above equation can be rewritten as,

$$e^f df = \frac{1}{L_0}\left(\frac{r_0}{r}\right)^3 dr, \tag{4.23}$$

which can be solved for f by integration, and is given by,

$$f = \mu + \nu = \ln\left(f_0\left(K_0 \pm \frac{r_0^2}{r^2}\right)\right), \tag{4.24}$$

where K_0 and $f_0 \equiv r_0/2L_0$ are dimensionless constants. This is our general solution for $f(r)$ and we will use it to find μ and ν. We subtract the rr equation from the tt equation, and solve for the expression $2\nu'' + 2\nu'/r$, leading to,

$$2\nu'' + 2\frac{\nu'}{r} = (2 - c_{14})\mu'\nu' + (2 - 2c_{14})\frac{\mu'}{r} - c_{14}\mu''. \tag{4.25}$$

Now, solving the $\theta\theta$ equation, we find,

$$2\nu'' + 2\frac{\nu'}{r} = -2\mu'^2 - 2\mu'' - 2\frac{\mu'}{r} + c_{14}\mu'^2. \tag{4.26}$$

The combination of Eqs.(4.25) and (4.26) yields,

$$0 = (c_{14} - 2)\left[\mu'^2 + \mu'' + 2\frac{\mu'}{r} + \mu'\nu'\right]. \tag{4.27}$$

We now consider the two cases, $c_{14} = 2$ and t $c_{14} \neq 2$, separately.

In the $c_{14} = 2$ case, the (tt, rr, $\theta\theta/\phi\phi$) equations reduce to,

$$0 = f'^2 + 2f'' + 4\frac{f'}{r}, \tag{4.28}$$

$$0 = f' + \frac{2}{r}, \tag{4.29}$$

$$0 = f'' + \frac{f'}{r} \tag{4.30}$$

which has the solution given by Eq.(4.24) where we must have $K_0 = 0$ in order to satisfy the field equations. However, these equations are only in terms of $f = \mu + \nu$, so there is still one degree of freedom in our choice of μ or ν. If we choose the parameterization:

$$\mu = \mu(r), \tag{4.31}$$

$$\nu = \log\left(\frac{r_0}{r}\right)^2 - \mu(r), \tag{4.32}$$

then we can write the metric as

$$ds^2 = -e^{2\mu}dt^2 + \left(\frac{r_0}{r}\right)^4 \frac{1}{e^{2\mu}}d\sigma^2. \tag{4.33}$$

However, this is the same as the solution given by Eq.(4.21) if we perform the change of coordinates $1/r \to R$:

$$ds^2 = -e^{2\mu(R)}dt^2 + \frac{1}{e^{2\mu(R)}}d\sigma^2. \tag{4.34}$$

In the $c_{14} \neq 2$ case, from Eq.(4.27) we find that

$$0 = \mu'^2 + \mu'' + 2\frac{\mu'}{r} + \mu'\nu', \tag{4.35}$$

which has the solution,

$$f \equiv \mu + \nu = \ln\left(\frac{f_0 r_0 q}{\mu' r^2}\right), \tag{4.36}$$

where f_0 and r_0 are the arbitrary constants from Eq.(4.24) and q is an arbitrary dimensionless constant. Note that we already have a solution for f in Eq.(4.24), where f is a function of r only. In Eq.(4.36) we have an expression for f as a function of both r and μ'. We set these two expressions equal to each other and solve for μ',

$$\mu' = \frac{r_0 q}{K_0 r^2 - r_0^2}. \tag{4.37}$$

In the special case where $K_0 = 0$ we find that μ is linear in r, and we can find ν by using Eq.(4.24), giving the metric

$$ds^2 = -e^r dt^2 + \frac{e^{-r}}{r^4}d\sigma^2, \tag{4.38}$$

where we have re-scaled t and r to suppress arbitrary constants. However, this solution does not solve the field equations unless $c_{14} = 2$, so this is just a special case of the metric given by Eq.(4.33).

For general K_0, we integrate Eq.(4.37) to find

$$\mu = \frac{q}{2} \ln \left| \frac{r - r_0}{r + r_0} \right|, \tag{4.39}$$

where we have absorbed a factor of $\sqrt{K_0}$ into the arbitrary constants q and r_0. We can now solve for ν by using Eqs.(4.24) and (4.39), and find,

$$\nu = \ln \left[f_0 \left| 1 - \frac{r_0}{r} \right|^{1-q/2} \left| 1 + \frac{r_0}{r} \right|^{1+q/2} \right] \tag{4.40}$$

These solutions for μ and ν solve the field equations exactly provided that q is given by,

$$q^2 = \frac{8}{2 - c_{14}}. \tag{4.41}$$

The choice of positive or negative square root is arbitrary as both choices lead to the same spacetime upon taking the $c_{14} \to 0$ limit to fix r_0. So the spacetime is given by,

$$ds^2 = -\left| \frac{1 - \frac{r_0}{r}}{1 + \frac{r_0}{r}} \right|^q dt^2 + \left| 1 - \frac{r_0}{r} \right|^{2-q} \left| 1 + \frac{r_0}{r} \right|^{2+q} d\sigma^2, \tag{4.42}$$

Where we have suppressed the arbitrary constant f_0 by re-scaling t and r. We can identify the values of q and r_0 by considering the $c_{14} \to 0$ limit. In isotropic coordinates, the Schwarzschild metric is given by,

$$ds^2 = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right)^2 dt^2 + \left(1 + \frac{m}{2r} \right)^4 d\sigma^2. \tag{4.43}$$

In the $c_{14} \to 0$ limit, $q \to 2$, so the spacetime given by Eq.(4.42) does indeed reduce to the isotropic Schwarzschild solution given by Eq.(4.43) provided $r_0 = \frac{m}{2}$ and we only consider $r > \frac{m}{2}$, which is the only region in which the isotropic coordinates are valid for the GR Schwarzschild solution. Then, the metric (4.42) takes the form,

$$ds^2 = -\left| \frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right|^q dt^2 + \left| 1 - \frac{m}{2r} \right|^{2-q} \left| 1 + \frac{m}{2r} \right|^{2+q} d\sigma^2, \tag{4.44}$$

where q is given by Eq.(4.41).

4.2.3 Metric Singularities and Schwarzschild Form

The spacetime given by Eq.(4.44) has singularities at $r = \frac{m}{2}$ and at $r = 0$. Both are curvature singularities as can by seen by considering the Ricci scalar, however this is easier to see in a a coordinate system similar to the Schwarzschild form. Consider the coordinate transformation:

$$\bar{r} = r\left(1 + \frac{m}{2r}\right)^2, \tag{4.45}$$

upon which the metric becomes

$$ds^2 = -\left(1 - \frac{2m}{\bar{r}}\right)^{q/2} dt^2 + \left(1 - \frac{2m}{\bar{r}}\right)^{-q/2} d\bar{r}^2 + \left(1 - \frac{2m}{\bar{r}}\right)^{1-q/2} \bar{r}^2 d\Omega^2. \tag{4.46}$$

Under this coordinate transformation the absolute value brackets are nullified and the coordinate t is spacelike for $\bar{r} < 2m$. Then the Ricci scalar is given by:

$$R = \frac{m^2(4 - q^2)}{2\bar{r}^4}\left(1 - \frac{2m}{\bar{r}}\right)^{\frac{q}{2}-2}, \tag{4.47}$$

and the Kretschmann scalar is given by

$$K = \frac{m^2}{4\bar{r}^4}\left(1 - \frac{2m}{\bar{r}}\right)^{q-4}(A\bar{r}^2 + B\bar{r} + C), \tag{4.48}$$

where

$$A = 48q^2, \tag{4.49}$$

$$B = -32mq(q^2 + 3q + 2), \tag{4.50}$$

$$C = m^2(2 + q)^2(7q^2 + 4q + 12). \tag{4.51}$$

Obviously both the Ricci and Kretschmann scalars have curvature singularities at the origin, and upon carefully taking the limit when $\bar{r}$ approaches $2m$ we see that there are curvature singularities at $\bar{r} = 2m$ as well. When c_{14} is set to zero then we have:

$$R = 0, \tag{4.52}$$

$$K = \frac{48m^2}{\bar{r}^6}, \tag{4.53}$$

which are the correct values for the Schwarzschild solution's Ricci and Kretschmann scalars (expressed in Schwarzschild coordinates). As can be seen from Eq.(4.46) the area of a sphere centered on the origin is given by:

$$\mathcal{A} = 4\pi\bar{r}^2\left(1-\frac{2m}{\bar{r}}\right)^{1-q/2}. \tag{4.54}$$

When $c_{14}=0$ then $\bar{r}$ becomes the areal radial coordinate and a sphere with coordinate radius $\bar{r}=2m$ has the area $4\pi\bar{r}^2$ as expected. However the area of a sphere at $\bar{r}=2m$ is infinite for any non-zero value of c_{14}. This shows that while the spacetime of Eqs.(4.44, 4.46) do approach the Schwarzschild solution as c_{14} approaches zero, the approach is not completely continuous. This areal radius reaches a minimum at:

$$\bar{r}_{min} = 2m\left(\frac{2+q}{4}\right) \tag{4.55}$$

Which is outside the curvature singularity located at $\bar{r}=2m$.

It is important that this $\bar{r}_{min}$ is outside the curvature singularity, because there is a horizon at $\bar{r}_{min}$, and thus the singularity is hidden behind a horizon. To see this, we look at the expansion of null geodesics (we follow [74]). Let the metric of Eq.(4.46) be written as

$$ds^2 = -u_a u_b + s_a s_b + \Gamma_{ab} \tag{4.56}$$

Where u_a is given by Eq.(4.9), $s_a = e^{\nu}\delta^r_a$ and Γ_{ab} is the 2-sphere metric. Then let the outgoing null geodesics have the tangent vector k_a given by:

$$k_a = \frac{1}{\sqrt{2}}(u_a+s_a). \tag{4.57}$$

Then the expansion of null geodesics is given by

$$\Theta = \Gamma^{ab}\nabla_a k_b = \frac{(m(2+q)-2\bar{r})\left|1-\frac{2m}{\bar{r}}\right|^{q/4}}{\bar{r}\sqrt{2}|2m-\bar{r}|} \tag{4.58}$$

Which equals zero at $\bar{r}_{min}$.

The structure of the spacetime is wormhole-like outside $\bar{r}=2m$, and with a throat at $\bar{r}_{min}$ (see Fig.(4.1). This means the throat is on a horizon, contrary to

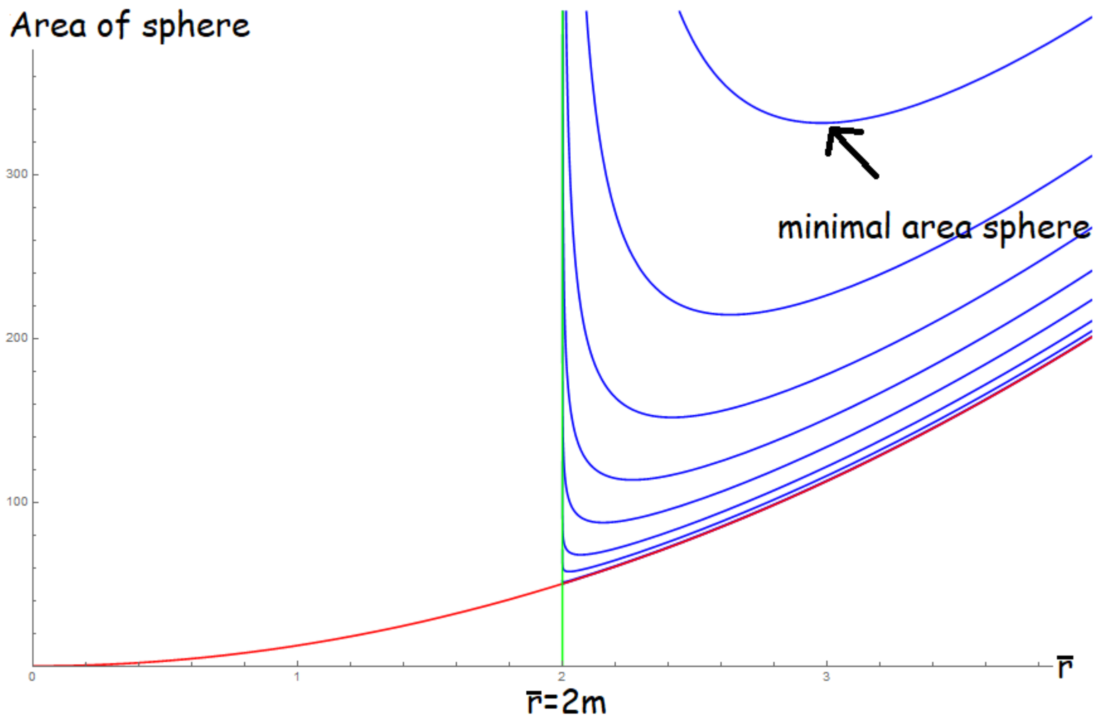

Figure 4.1: Plot of the area of sphere centered on the origin vs. areal radius. The wormhole-like geometry is evident outside $\bar{r} = 2m$, with a throat a the minimum radius $\bar{r}_{min}$. The solutions in blue are for various values of $0 < c_{14} < 2$ and they approach the Schwarzschild solution (in red) as c_{14} approaches zero.

what Eling and Jacobson wrote in the paper in which they derived this solution in a different coordinate system [55]. In that paper they considered the Killing horizon at $\bar{r} = 2m$ and while they implied that the expansion of null geodesics would be zero at $\bar{r}_{min}$, they explicitly said that $\bar{r}_{min}$ was not on a horizon, so here we have clarified the horizon structure of the spacetime. Unlike in the Schwarzschild solution of GR, the Killing horizon and the event horizon do not coincide.

4.2.4 Time-Dependent Solutions

If we consider solutions such that e^{μ} and e^{ν} are separable in t and r, then we seek solutions of the form,

$$\mu(r,t) = \mu_0(r) + \mu_1(t), \tag{4.59}$$

$$\nu(r,t) = \nu_0(r) + \nu_1(t), \tag{4.60}$$

so that all mixed-partial derivatives of μ and ν are zero. However, redefining the time coordinate t by t',

$$t' \equiv \int e^{2\mu_1(t)} dt, \tag{4.61}$$

we can see that, without loss of generality, we can set $\mu_1 = 0$, and look for solutions of the form,

$$\mu(r,t) = \mu_0(r), \tag{4.62}$$

$$\nu(r,t) = \nu_0(r) + \nu_1(t). \tag{4.63}$$

If $\dot{\nu} = 0$ then the equations of motion reduce to the static case, so we assume that $\dot{\nu} \neq 0$. In this case, when seeking solutions of the form of Eqs.(4.62)-(4.63) the tr and aether equations reduce to,

$$c_{14}\mu'\dot{\nu} = 2\mu'\dot{\nu}, \tag{4.64}$$

$$c_{14}\mu'\dot{\nu} = -\beta\mu'\dot{\nu}. \tag{4.65}$$

Thus, there are three possibilities,

$$(i) \quad c_{14} = 2 = -\beta; \quad (ii) \quad \mu' = 0; \quad (iii) \quad \dot{\nu} = 0, \tag{4.66}$$

where (iii) is just the static case. So, in the following we shall not consider it.

4.2.4.1 When $c_{14} = 2$, $\beta = -2$. This case just reduces to the static case considered in Section III.B, as by the definition of α^2 given in Eq.(4.10), $\alpha = 0$ now. Then, the left-hand sides of the field equations, containing all of the time-dependence, vanish identically, and the solution for all $c_{14} = 2$ cases for the isotropic metric is given by Eq.(4.7).

4.2.4.2 When $\mu' = 0$. In this case the three relevant equations are the ones of tt, rr, and $\theta\theta$ components,

$$\alpha^2 \dot{\nu_1}^2 e^{2\nu_1} = e^{2\mu_0 - 2\nu_0}\left(\nu_0'^2 + 2\nu_0'' + 4\frac{\nu_0'}{r}\right), \tag{4.67}$$

$$\alpha^2 e^{2\nu_1}\left(\dot{\nu_1}^2 + \frac{2}{3}\ddot{\nu_1}\right) = e^{2\mu_0 - 2\nu_0}\left(\nu_0'^2 + 2\frac{\nu_0'}{r}\right), \tag{4.68}$$

$$\alpha^2 e^{2\nu_1}\left(\dot{\nu_1}^2 + \frac{2}{3}\ddot{\nu_1}\right) = e^{2\mu_0 - 2\nu_0}\left(\nu_0'' + \frac{\nu_0'}{r}\right). \tag{4.69}$$

Note that for each equation, the left-hand side is t-dependent and the right-hand side is r-dependent, thus both sides must be equal to the same constant. Setting

$$K_0^2 = e^{2\nu_1}\left[\alpha^2 \dot{\nu_1}^2\right], \tag{4.70}$$

$$K_1^2 = e^{2\nu_1}\left[\alpha^2\left(\dot{\nu_1}^2 + \frac{2}{3}\ddot{\nu_1}\right)\right], \tag{4.71}$$

where K_0 is associated with the tt equation and K_1 is associated with the rr and $\theta\theta$ equation, from Eqs.(4.68) and (4.69) we have

$$\nu_0'' = \nu_0'^2 + \frac{\nu_0'}{r}, \tag{4.72}$$

and thus it can be shown that $K_0^2 = K_1^2/3$. Eq.(4.72) has the general solution,

$$\nu_0(r) = \ln\left(\frac{r_1}{r^2 - r_0^2}\right), \tag{4.73}$$

where r_1 and r_0 are integration constants. Next we solve for $\nu_1(t)$ using Eq.(4.70),

$$e^{2\nu_1}\alpha^2 \dot{\nu_1}^2 = K_0^2, \tag{4.74}$$

which as the solution,

$$\nu_1(t) = \ln\left[\frac{K_0}{\alpha}(t - t_0)\right], \tag{4.75}$$

where t_0 is an integration constant, and t $K_0 \neq 0$. It is straightforward to show that the solutions given by Eqs.(4.73)-(4.75) solve the field equations (4.67)-(4.69) provided that,

$$\frac{K_0^2 r_1^2}{\alpha} = 12 r_0^2 e^{2\mu_0}. \tag{4.76}$$

Thus we do not have the freedom to set either K_0 or r_0 to zero. Then, the complete solution for $\mu(t,r)$ and $\nu(t,r)$ can be expressed as,

$$\mu(t,r) = \mu_0, \tag{4.77}$$

$$\nu(t,r) = \ln\left[\sqrt{\frac{12}{\alpha}}\frac{r_0(t-t_0)}{r^2-r_0^2}\right] + \mu_0. \tag{4.78}$$

If we reset the zero point of the time coordinate, and then rescale the time-coordinate by a factor of $e^{-\mu_0}$, we can express the line element as

$$ds^2 = -dt^2 + \frac{12r_0^2t^2}{\alpha(r^2-r_0^2)^2}\left(dr^2 + r^2d^2\Omega\right). \tag{4.79}$$

It can be shown that the spacetime described by the above metric is conformally flat, that is, the Weyl tensor vanishes identically, and the spacetime is singular at $t=0$, as can be seen from the Ricci and Kretschmann scalars, now given by,

$$R = -\frac{3\beta}{t^2}, \quad K = \frac{1}{t^4}\left(\frac{4}{3}\alpha^2 - 8\alpha + 12\right), \tag{4.80}$$

where $\beta = 3c_2 + c_{13}$, as defined previously.

To study this solution further, let us consider the energy conditions. We define a timelike vector field t^a in the (t,r)-plane,

$$t^a = A\delta^a_t + B\delta^a_r, \quad A^2 = v^2 + B^2e^{2\nu}, \tag{4.81}$$

from which we find that $t^at_a = -v^2$, where v is an arbitrary non-vanishing real function of x^a. A stress-energy tensor that obeys the weak energy condition ensures that all observers following timelike trajectories will see only positive energy density, that is,

$$T^{æ}_{ab}t^at^b \geq 0. \tag{4.82}$$

However, for the spacetime of Eq.(4.79) we have,

$$T^{æ}_{ab}t^at^b = -3\beta\left[\frac{6B^2r_0^2}{\alpha(r^2-r_0^2)^2} + \frac{v^2}{2t^2}\right], \tag{4.83}$$

which is always non-positive. Thus, the aether field in the current case always violates the weak energy condition.

A stress-energy tensor that obeys the strong energy condition ensures that gravity will always be attractive, not repulsive. It is given by the inequality,

$$T^{\text{æ}}_{ab}t^a t^b - \frac{1}{2}T t^a t_a \geq 0. \tag{4.84}$$

Again, in the current cae, the above condition is violated, as now we have,

$$T^{\text{æ}}_{ab}t^a t^b - \frac{1}{2}T t^a t_a = -\frac{18B^2\beta r_0^2}{\alpha(r^2 - r_0^2)^2} < 0. \tag{4.85}$$

In addition, the above spacetime actually belongs to the Friedmann universe. To show this, we change to a new radial coordinate R (unrelated to the Ricci scalar) defined by,

$$dR^2 = \frac{dr^2}{(r^2 - r_0^2)^2}. \tag{4.86}$$

This can be integrated with partial fraction decomposition, yielding,

$$R = \frac{1}{2r_0}\ln\left(\frac{|1 - r/r_0|}{|1 + r/r_0|}\right). \tag{4.87}$$

This evaluates to different inverse hyperbolic functions depending on whether r is larger or smaller than r_0. While this function cannot be evaluated at $r = r_0$ we will see that the final expression for the factor $r^2/(r^2 - r_0^2)^2$ in front of the angular part of the metric is the same for both cases. For R we have,

$$R = \begin{cases} -\frac{1}{r_0}\tanh^{-1}(r/r_0), & r < r_0, \\ -\frac{1}{r_0}\coth^{-1}(r/r_0), & r > r_0. \end{cases} \tag{4.88}$$

While for r we have,

$$r = \begin{cases} -r_0\tanh(r_0 R), & r < r_0, \\ -r_0\coth(r_0 R), & r > r_0. \end{cases} \tag{4.89}$$

And it is straightforward to show that,

$$\frac{r^2}{(r^2 - r_0^2)^2} = \begin{cases} \sinh^2{(2r_0 R)}/4r_0^2, & r < r_0, \\ \sinh^2{(2r_0 R)}/4r_0^2, & r > r_0. \end{cases} \tag{4.90}$$

Hence the metric may now be written as,

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + dR^2 + \frac{\sinh^2{(2r_0 R)}}{4r_0^2} d\Omega^2 \right), \tag{4.91}$$

where

$$a(\eta) = \gamma r_0 \exp{(\gamma r_0(\eta - \eta_0))}, \tag{4.92}$$

and

$$d\eta^2 = \frac{\alpha}{12r_0^2} \frac{dt^2}{t^2}. \tag{4.93}$$

To relate R to the areal radius Ψ, we consider h_{ab}, the induced metric on the 2-sphere hypersurface of constant η and R, such that,

$$h_{ab} = \frac{\sinh^2{(2r_0 R)}}{4r_0^2} (\delta_a^\theta \delta_b^\theta + \sin{\theta}^2 \delta_a^\phi \delta_b^\phi). \tag{4.94}$$

Then the surface area of a sphere at Ψ is given by,

$$\begin{aligned} 4\pi\Psi^2 \quad &= \int_0^{2\pi} \int_0^{\pi} \sqrt{h} d\theta d\phi \\ &= \frac{\sinh^2{(2r_0 R)}}{4r_0^2} \int_0^{2\pi} \int_0^{\pi} \sin{\theta} d\theta d\phi \\ &= 4\pi \frac{\sinh^2{(2r_0 R)}}{4r_0^2}. \end{aligned} \tag{4.95}$$

Thus the areal radius is given by,

$$\Psi = \frac{\sinh{(2r_0 R)}}{2r_0}, \tag{4.96}$$

Leading to the transformation,

$$dR^2 = \frac{d\Psi^2}{1 + 4r_0^2 \Psi^2}. \tag{4.97}$$

If we now rewrite Ψ as r (unrelated to the starting coordinate r) for convenience, then we can write the metric as,

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + \frac{dr^2}{1 + 4r_0^2 r^2} + r^2 d\Omega^2 \right). \tag{4.98}$$

Remember that r_0 was an integration constant, and from Eqs.(4.75-4.76) we see that we cannot set r_0 to zero. If we set $r_0^2 = 1/4$ then the metric of Eq.(4.98) would be the traditional form for an FLRW metric of constant negative curvature ($k = -1$). A spacetime very similar to this one was found in [60]. If we set $r_0^2 = -1/4$ then we have an FLRW metric of constant positive curvature, but that would induce a complex conformal factor $a(\eta)$. Remember that $a(\eta)$ is given by Eq.(4.92), and the full expression for γ is given by,

$$\gamma = 2 \left(\frac{6}{2 + 3c_2 + c_{13}} \right)^{\frac{1}{4}}. \tag{4.99}$$

4.3 Painlevè-Gullstrand Coordinates

In this section, using the gauge freedom (4.4), we choose the gauge

$$g_{rr} = 1, \quad R(t, r) = r, \tag{4.100}$$

so the metric takes the Painlevè-Gullstrand (PG) form,

$$ds^2 = -e^{2\mu(r)} dt^2 + 2e^{\nu(r)} dr dt + d\sigma^2. \tag{4.101}$$

Recall $d\sigma^2 = dr^2 + r^2 d\Omega^2$. For this metric we only consider time-independent solutions, and assume that the aether is comoving,

$$u^a = e^{-\mu} \delta^a_t. \tag{4.102}$$

So, the aether is aligned with the timelike Killing vector of the metric, which is itself hypersurface-orthorgonal.

4.3.1 Field Equations

To simplify the field equations, we first define the quantity Δ,

$$\Delta \equiv e^{2\mu} + e^{2\nu}. \tag{4.103}$$

Then, it can be shown that the aether dynamical equations are identically zero for any μ and ν, and the remaining field equations are the ones given by the components, $(tt,\ rr,\ \theta\theta)$,

$$\begin{aligned}
0 = &\, e^{4\mu-2\nu}\left[c_{14}(2r^2\mu'' + 4r\mu' + r^2\mu'^2)\right] \\
&+ e^{2\mu}\left[c_{14}(2r^2\mu'' + 4r\mu' + r^2\mu'^2 + 2r^2\mu'(\mu'-\nu'))\right. \\
&\left.+4r(\mu'-\nu')\right] - 2\Delta,
\end{aligned} \tag{4.104}$$

$$\begin{aligned}
0 = &\, e^{2\mu}\left[c_{14}(4r\mu' + 2r^2\mu'' - r^2\mu'^2) - 8r\mu'\right] \\
&\, e^{2\nu}\left[c_{14}(4r\mu' + 2r^2\mu'' + 2r^2\mu'(\mu'-\nu')) - 4r\nu'\right] \\
&- e^{4\mu-2\nu}\left[4r\mu' + c_{14}r^2\mu'^2\right] + 2\Delta,
\end{aligned} \tag{4.105}$$

$$\begin{aligned}
0 = &\, \Delta\left[(c_{14}-2)r^2\mu'^2 - 2r^2\mu'' - 2r\mu'\right] \\
&+ e^{2\nu}\left[2r\nu'(1+r\mu') - 2r\mu' - 2r^2\mu'^2\right].
\end{aligned} \tag{4.106}$$

As evident from Eqs.(B.33)-(B.42), the Einstein-aether equations require that

$$\Delta \neq 0, \tag{4.107}$$

although this is not evident from the field equations after simplification. So, as we proceed we must reject outright any solution that violates Eq.(4.107).

4.3.2 Solutions

Our strategy is to first solve the tt equation for ν'. The result is

$$\begin{aligned}
\nu' = &\, \frac{-2e^{4\nu} + c_{14}(4r\mu' + r^2\mu'^2 + 2r^2\mu'')e^{4\mu}}{2re^{2\mu+2\nu}(2 + c_{14}r\mu')} \\
&+ \frac{e^{2\mu+2\nu}(c_{14}(r\mu'(4+3r\mu') + 2r^2\mu''))}{2re^{2\mu+2\nu}(2 + c_{14}r\mu')} \\
&+ \frac{e^{2\mu+2\nu}(4r\mu' - 2)}{2re^{2\mu+2\nu}(2 + c_{14}r\mu')}.
\end{aligned} \tag{4.108}$$

Note that in deriving the above expression, we assume that

$$r\mu' \neq -\frac{2}{c_{14}}. \tag{4.109}$$

When $2+c_{14}r\mu' = 0$, the solutions are different. So, let us pause here for a while, and first consider the case $2+c_{14}r\mu' = 0$.

4.3.2.1 When $2+c_{14}r\mu' = 0$. In this case we can easily integrate to find,

$$\mu = \frac{2}{c_{14}} \ln\left(\frac{r_0}{r}\right). \tag{4.110}$$

By substituting this into the tt/tr equations and solving for ν we find,

$$\nu = \frac{1}{2}\ln\left[\frac{2(1-c_{14})}{c_{14}}\left(\frac{r_0}{r}\right)^{4/c_{14}}\right] \tag{4.111}$$

Unfortunately this does not solve the $\theta\theta/\phi\phi$ equation for any choice of c_{14} or r_0. If instead we insert the μ from Eq.(4.110) into the $\theta\theta/\phi\phi$ equation and solve for ν we find:

$$\nu = \frac{1}{2}\ln\left[\frac{2}{c_{14}}\frac{R_0 - \frac{1}{2}c_{14}(r_0 r)^{4/c_{14}}}{r^{8/c_{14}}}\right] \tag{4.112}$$

where R_0 is an integration constant. However this fails to solve the tt, tr, and rr equations. So we cannot have $2+c_{14}r\mu' = 0$.

4.3.2.2 When $2+c_{14}r\mu' \neq 0$. This is the case in which Eq.(4.108) holds. We substitute the value for ν' from this equation into the rr equation and solve for $e^{2\nu}$. The result is,

$$e^{2\nu} = e^{2\mu}\left(2r\mu' + \frac{c_{14}}{2}r^2\mu'^2\right). \tag{4.113}$$

We can substitute this value for $e^{2\nu}$ into Eq.(4.108), and then we have expressions for both ν' and $e^{2\nu}$ in terms of μ and its derivatives. The full expression for ν' is given by,

$$\begin{aligned}\nu' &= \frac{c_{14}\mu''(2+r\mu'(4+c_{14}r\mu'))}{\mu'(2+c_{14}r\mu')(4+c_{14}r\mu')} \\ &\quad + \frac{4(c_{14}-1)+c_{14}r\mu'(2+r\mu')(4+c_{14}r\mu')}{r(2+c_{14}r\mu')(4+c_{14}r\mu')}.\end{aligned} \tag{4.114}$$

If we put these expressions for ν' and e^{ν} into either the tt or rr equations, we find,

$$
\begin{aligned}
0 &= (c_{14}-2)\left(4\mu' + 4r\mu'^2 + c_{14}r^2\mu'^3 + 2r\mu''\right) \\
&\times \left[8 + 2(8+c_{14})r\mu' + 8c_{14}r^2\mu'^2 + c_{14}^2 r^3\mu'^3\right]. \qquad (4.115)
\end{aligned}
$$

So we have three possibilities,

$$0 = c_{14} - 2, \qquad (4.116)$$

$$0 = 8 + 2(8+c_{14})r\mu' + 8c_{14}r^2\mu'^2 + c_{14}^2 r^3\mu'^3, \qquad (4.117)$$

$$0 = 4\mu' + 4r\mu'^2 + c_{14}r^2\mu'^3 + 2r\mu''. \qquad (4.118)$$

We have already studied a case like Eq.(4.116). Not only was it unphysical but also it violated the constraint of Eq.(4.108). Equation (4.117) is cubic in $r\mu'$ so it is straightforward to find expressions for $r\mu'$, and then separate variables and integrate. Equation (4.118) leads to the static aether solution found in [55]. Let us first consider Eq.(4.117).

First we define $f(r)$ such that,

$$f = r\mu'. \qquad (4.119)$$

Then Eq.(4.117) may be written as,

$$0 = (f-\beta_0)(f-\beta_1)(f-\beta_2), \qquad (4.120)$$

Where

$$
\begin{aligned}
\beta_0 &= -\frac{4}{c_{14}}, \\
\beta_1 &= -\frac{2+\sqrt{4-2c_{14}}}{c_{14}}, \\
\beta_2 &= \frac{-2+\sqrt{4-2c_{14}}}{c_{14}}. \qquad (4.121)
\end{aligned}
$$

Generically, the solution to each case is of the form,

$$\mu = \ln\left(\frac{r}{r_0}\right)^{\beta_i}, \quad e^{2\mu} = \left(\frac{r}{r_0}\right)^{2\beta_i}, \qquad (4.122)$$

where β_i is any of the ones given in Eq.(4.121). When we insert Eq.(4.122) into Eq.(4.113) we find that,

$$e^{2\nu} = \frac{1}{2}\beta_i(4 + c_{14}\beta_i)\left(\frac{r}{r_0}\right)^{2\beta_i}. \tag{4.123}$$

Since any solution in which $e^{2\nu} = 0$ is equivalent to the Minkowski metric, we ignore the case of β_0, as this would make $e^{2\nu} = 0$, as can be seen from Eq.(4.123). If we insert the others β_i into Eq.(4.123), then we have,

$$e^{2\nu} = -\left(\frac{r}{r_0}\right)^{2\beta_i}. \tag{4.124}$$

Unfortunately this violates the constraint of Eq.(4.109), so we must reject this, and assume that Eq.(4.117) does not hold.

Coincidentally, if $c_{14} = 2$, then the metric components given by Eqs.(4.122, 4.123) satisfy the field equations identically for any choice of β, save for those that violate the constraint of Eq.(4.108). So the metric given by:

$$ds^2 = -\left(\frac{r}{r_0}\right)^{2\beta} dt^2 + 2\sqrt{\frac{\beta}{2}(4 + c_{14}\beta)}\left(\frac{r}{r_0}\right)^{\beta} drdt + d\sigma^2, \tag{4.125}$$

is a valid, though un-physical, solution.

This brings us to Eq.(4.118), which we rewrite it as,

$$0 = 4r\mu' + 4r^2\mu'^2 + c_{14}r^3\mu'^3 + 2r^2\mu''. \tag{4.126}$$

By Eq.(4.119) we can say that,

$$r^2\mu'' = rf' - f, \tag{4.127}$$

and thus we can rewrite Eq.(4.126) as

$$f' = -\frac{f}{r}\left(1 + 2f + \frac{c_{14}}{2}f^2\right). \tag{4.128}$$

But this is precisely the same equation as for the static case in the Schwarzschild coordinates, given by Eq.(26) of [55]. Then, we can find the corresponding solutions by proceeding exactly in the same way as done in [55]. In particular, the solution for μ is given by,

$$\mu(f) = \ln\left[\left(f_0 \frac{1 - f/f_-}{1 - f/f_+}\right)^{\frac{f_+ f_-}{f_+ - f_-}}\right], \tag{4.129}$$

$$f_\pm = \frac{-1 \pm \sqrt{1-\alpha}}{\alpha}, \tag{4.130}$$

$$\frac{r_0}{r} = \left(\frac{f}{f - f_-}\right)\left(\frac{f - f_-}{f - f_+}\right)^{\frac{1}{2(1+f_+)}}. \tag{4.131}$$

This is not surprising given that the Schwarzschild metric in the Painlevè-Gullstrand coordinates has the same tt component as the Schwarzschild metric in the Schwarzschild coordinates. However, as can by seen by comparing Eq.(4.114) and Eq.(4.147) the g_{tr} component in the Painlevè-Gullstrand coordinates is different from the g_{rr} component in the Schwarzschild coordinates.

4.4 The Schwarzschild Coordinates

The Schwarzschild coordinates correspond to the choice,

$$g_{tr} = 0, \quad R(t,r) = r, \tag{4.132}$$

for which the metric takes the form,

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\nu(t,r)}dr^2 + r^2 d^2\Omega. \tag{4.133}$$

Let the aether vector field take the form,

$$u_a = e^{\mu}\delta^t_a. \tag{4.134}$$

In the static case, the spacetime were already studied in [55]. So, in the following, we shall pay particular attention for the non-static case.

4.4.1 Field Equations

For simplicity, we define the quantities,

$$Q \equiv \frac{\mu'^2}{2} - \mu'\nu' + \mu'', \tag{4.135}$$

$$H \equiv \frac{\dot{\nu}^2}{2} - \dot{\mu}\dot{\nu} + \ddot{\nu}. \tag{4.136}$$

The non-vanishing equation for the aether dynamics is,

$$0 = (2c_{13} - (c_2 + c_{13} - c_{14})r\mu')\dot{\nu} + r(c_{123}\dot{\nu}' - c_{14}\dot{\mu}'). \tag{4.137}$$

The non-vanishing Einstein-aether vacuum equations $G_{ab} = T^{æ}_{ab}$ are the tt, tr, rr, $\theta\theta$ components, given, respectively by,

$$\begin{aligned} 0 &= e^{2\mu}\left[c_{14}Q + 2c_{14}\frac{\mu'}{r} - \frac{2\nu'}{r} + \frac{1}{r^2}\right] \\ &\quad - e^{2\nu}\frac{c_{123}}{2}\dot{\nu}^2 - \frac{e^{2(\mu+\nu)}}{r^2}, \end{aligned} \tag{4.138}$$

$$0 = c_{14}(\dot{\mu}' - \mu'\dot{\nu}) - \frac{2\dot{\nu}}{r}, \tag{4.139}$$

$$\begin{aligned} 0 &= e^{2\nu}\left[c_{123}H\right] + \frac{e^{2\nu+2\mu}}{r^2} \\ &\quad - e^{2\mu}\left[\frac{2\mu'}{r} + \frac{1}{r^2} + \frac{c_{14}}{2}\mu'^2\right], \end{aligned} \tag{4.140}$$

$$\begin{aligned} 0 &= e^{2\mu}\left[\frac{\mu'^2}{2}(c_{14} - 1) + \frac{\nu' - \mu'}{r} - Q\right] \\ &\quad + e^{2\nu}\left[\frac{\dot{\nu}^2}{2}(1 - c_{13}) + (c_2 + 1)H\right]. \end{aligned} \tag{4.141}$$

4.4.2 Time-Independent Case

The static solution was already found in [55] but with a different (though equivalent) parameterization of the metric. In the static case, all time-derivatives go to zero and the (tt, rr, $\theta\theta$) equations become,

$$\frac{e^{2\nu}}{r^2} = c_{14}Q + 2c_{14}\frac{\mu'}{r} - 2\frac{\nu'}{r} + \frac{1}{r^2}, \tag{4.142}$$

$$\frac{e^{2\nu}}{r^2} = c_{14}\frac{\mu'^2}{2} + 2\frac{\mu'}{r} + \frac{1}{r^2}, \tag{4.143}$$

$$0 = \frac{\mu'^2}{2}(c_{14} - 1) + c_{14}\mu'' - c_{14}\mu'\nu'. \tag{4.144}$$

We subtract the rr equation from the tt equation to find,

$$2\frac{\nu'}{r} = c_{14}\mu'' - c_{14}\mu'\nu' + 2\frac{\mu'}{r}(c_{14} - 1), \tag{4.145}$$

and rearrange the $\theta\theta$ equation to have,

$$2\frac{\nu'}{r} = 2\mu'' + \mu'^2(2 - c_{14}) + 2\frac{\mu'}{r} - 2\mu'\nu'. \tag{4.146}$$

Setting them equal to each other we find,

$$\nu' = \frac{\mu''}{\mu'} + \mu' + \frac{2}{r}, \tag{4.147}$$

which is not explicitly dependent on the c_i. We can rewrite the $\theta\theta$ equation as,

$$\nu'\left(\frac{1}{r} + \mu'\right) = \frac{\beta}{2}\mu'^2 + \frac{\mu'}{r} + \mu''. \tag{4.148}$$

Inserting our expression for ν' into this equation and after simplification, we find

$$r^2\mu'' + 2r\mu' + 2r^2\mu'^2 + \frac{c_{14}}{2}r^3\mu'^3 = 0, \tag{4.149}$$

which is equivalent to equation (26) of [55], provided we make the substitutions,

$$c_{14} \to c_1, \quad \mu \to \frac{A}{2}. \tag{4.150}$$

We can solve this using an equivalent process as the authors of [55] did. We define $f = r\mu'$, then find that Eq.(4.149) becomes

$$\frac{df}{dr} = -\frac{f}{r}\left(1 + 2f + \alpha f^2\right), \tag{4.151}$$

but now with $\alpha \equiv c_{14}/2$. From the chain role, $\frac{d\mu}{dr} = \frac{d\mu}{df}\frac{df}{dr}$, and the definition of f we find

$$\frac{d\mu}{df} = -\frac{1}{1 + 2f + \alpha f^2}. \tag{4.152}$$

We use partial fraction decomposition to solve the above equation, and find

$$\mu(f) = \ln\left[\left(f_0\frac{1 - f/f_-}{1 - f/f_+}\right)^{\frac{f_+ f_-}{f_+ - f_-}}\right], \tag{4.153}$$

where f_0 is an integration constant whose square is unity. The equivalent equation in [55] is (34), and this solution matches it exactly, bearing in mind that,

$$f_{\pm} = \frac{-1 \pm \sqrt{1-\alpha}}{\alpha}. \tag{4.154}$$

Then we can solve Eq.(4.151) and find,

$$\frac{r_0}{r} = \left(\frac{f}{f - f_-}\right)\left(\frac{f - f_-}{f - f_+}\right)^{\frac{1}{2(1+f_+)}}, \tag{4.155}$$

which is equivalent to Eq.(35) iof [55].

Note that, when $c_{14} = 2$, instead of Eq.(4.153) now we have

$$\mu(f) = \ln\left[\left(f_0 \frac{f + f_+}{f + f_-}\right)^{\frac{1}{f_- - f_+}}\right], \tag{4.156}$$

where now $f_{\pm}$ are defined by

$$f_{\pm} = \frac{3}{4} \pm \frac{\sqrt{41}}{4}, \tag{4.157}$$

and instead of Eq.(4.155) we have

$$\frac{r_0}{r} = f^{\frac{2}{f_+ f_-}}\left(\frac{(f - f_+)^{1/f_+}}{(f - f_-)^{1/f_-}}\right)^{\frac{2}{f_+ - f_-}}. \tag{4.158}$$

4.4.3 Time-Dependent Cases

If we consider solutions such that e^{μ} and e^{ν} are separable in t and r, then we seek solutions of the form,

$$\mu(r,t) = \mu_0(r) + \mu_1(t), \tag{4.159}$$

$$\nu(r,t) = \nu_0(r) + \nu_1(t). \tag{4.160}$$

However, as shown previously, by redefining the time coordinate, we can always set $\mu_1 = 0$, without loss of generality. So, in the following we only consider the case,

$$\mu(r,t) = \mu_0(r), \tag{4.161}$$

$$\nu(r,t) = \nu_0(r) + \nu_1(t). \tag{4.162}$$

Now we have $\dot{\mu} = 0$ and if $\dot{\nu} = 0$ then the equations of motion reduce to the static case, so we assume that $\dot{\nu} \neq 0$. In proceeding we will not defer to the constraints on the c_i as outlined in [14] in order to have as complete a set of solutions as possible. In this case, when seeking solutions of the form of Eqs.(4.161)-(4.162) the tr and aether equations reduce to,

$$\frac{2}{r} = -c_{14}\mu', \tag{4.163}$$

$$\frac{2c_{13}}{r} = \mu'(c_{14} - c_{123}). \tag{4.164}$$

We now consider separately the cases $c_{13} = 0$ and $c_{13} \neq 0$.

4.4.3.1 When $c_{13} = 0$. By Eq.(4.163) we must have

$$c_{14} \neq 0, \quad \mu' \neq 0. \tag{4.165}$$

Then, from Eq.(4.164) we have,

$$c_2 = c_{14}, \tag{4.166}$$

for which Eq.(4.163) yields,

$$\mu = \ln\left(\frac{U_0}{r^\alpha}\right), \quad \alpha \equiv \frac{2}{c_2}, \tag{4.167}$$

where U_0 is an arbitrary constant. Then, the tt, rr, $\theta\theta$ equations (4.138), (4.140) and (4.141) become

$$\frac{U_0^2}{r^{2\alpha+2}}(\alpha - 1) = e^{2\nu}\left[\frac{\dot{\nu}^2}{\alpha} + \frac{U_0^2}{r^{2\alpha+2}}\right], \tag{4.168}$$

$$\frac{U_0^2}{r^{2\alpha+2}}(\alpha - 1) = e^{2\nu}\left[-\frac{2}{\alpha}\ddot{\nu} - \frac{\dot{\nu}^2}{\alpha} - \frac{U_0^2}{r^{2\alpha+2}}\right], \tag{4.169}$$

$$\begin{aligned}\frac{U_0^2}{r^{2\alpha+2}}(\alpha - 1)(r\nu' + \alpha) = e^{2\nu}\Bigg[&\frac{2}{\alpha}\ddot{\nu} + \frac{\dot{\nu}^2}{\alpha} \\ &+ \ddot{\nu} + \dot{\nu}^2\Bigg].\end{aligned} \tag{4.170}$$

By combining the tt and rr equations we find

$$\frac{\dot{\nu}_1(t)^2 + \ddot{\nu}_1(t)}{\alpha} = -\frac{U_0^2}{r^{2\alpha+2}}, \tag{4.171}$$

where we have explicitly written the expressions for $\dot{\nu}$ in terms of $\nu_1(t)$ to emphasize the t-dependence. Since the left-hand side (LHS) is purely t-dependent, and the right-hand side (RHS) is purely r-dependent, then both sides must be equal to some constant. Since $U_0 \neq 0$, the only way to ensure that the RHS of Eq.(4.171) is constant is to set,

$$\alpha = -1, \tag{4.172}$$

for which Eq.(4.171) reduces to

$$\dot{\nu}^2 + \ddot{\nu} = U_0^2. \tag{4.173}$$

By using Eq.(4.173) with either the tt or rr equation, we arrive at

$$2U_0^2 = e^{2\nu_1}\left(\dot{\nu}_1^2 - U_0^2\right), \tag{4.174}$$

which yields,

$$\nu_1(t) = \ln\left\{\sqrt{2}\sinh\left[U_0\left(t_0 \pm t\right)\right]\right\}, \tag{4.175}$$

where t_0 is an arbitrary constant. On the other hand, from Eq.(4.170) we find

$$e^{2\nu}\left(\dot{\nu}^2 - U_0^2\right) = 2U_0^2(1 - r\nu'). \tag{4.176}$$

Comparing this to Eq.(4.174), we find $\nu_0(r) = \text{const.}$, so that

$$\nu(t,r) = \ln\left\{\sqrt{2}\sinh\left[U_0\left(t_0 \pm t\right)\right] + V_0\right\}, \tag{4.177}$$

where V_0 is a constant. Eqs.(4.167) and (4.177) satisfy all of the field equations, provided that $V_0 = 0$, with no other constraints on the remaining arbitrary constants. Thus for the case $c_{13} \neq 0$, the solution is

$$\begin{aligned} \mu(r) &= \ln\left(U_0 r\right), \\ \nu(t,r) &= \ln\left[\sqrt{2}\sinh\left(U_0(t_0 \pm t)\right)\right]. \end{aligned} \tag{4.178}$$

However, using the gauge freedom for the choice of t, we can always set $U_0 = 1$ and $t_0 = 0$, so the metric finally takes the form,

$$ds^2 = -r^2 dt^2 + 2\sinh^2(t)\, dr^2 + r^2 d^2\Omega. \tag{4.179}$$

In this case the Ricci scalar is given by

$$R = \frac{4\sinh^2(t) - 3}{\sinh^2(t) r^2} \tag{4.180}$$

and thus it has curvature singularities at an initial time $t = 0$ and at the origin.

4.4.3.2 When $c_{13} \neq 0$. In this case we combine the tt and rr equations, and find

$$c_{123}(\dot{\nu_1}^2 + \ddot{\nu_1}) = -\frac{2U_0^2}{r^{2\alpha+2}}. \tag{4.181}$$

Substituting it into the tt equation, and then subtracting it from the $\theta\theta$ equation, we find obtain,

$$2U_0^2 r\nu_0'(r)e^{-2\nu_0(r)} = e^{2\nu_1(t)}\left[\frac{2c_{13}}{1+c_{13}}U_0^2\right]. \tag{4.182}$$

The right-hand is always different form zero, so the above equation holds only when $\nu_1 = \text{const}$. Then, the solution becomes static, and we have

$$r\nu_0'(r)e^{-2\nu_0(r)} = \frac{e^{2\nu_1}}{2U_0^2}\left[\frac{2c_{13}}{1+c_{13}}U_0^2\right] \equiv A_0, \tag{4.183}$$

where A_0 is a non-zero constant. While this equation does yield a different static solution for ν than was found previously, it does not solve all of the field equations.

CHAPTER FIVE

Conclusions

5.1 Constraints on Æ-theory

In chapter 2, we have considered various constraints on the Einstein-aether theory, as listed in Eqs.(2.4), (2.5)-(2.12), which represent the major constraints from the self-consistency of the theory to various observations. The severest one is from the recent gravitational wave event, GW170817 [31], observed by the LIGO/Virgo collaboration, and the gamma-ray burst observation of GRB 170817A [32], given by Eq.(2.4) due to the constraint on the deviation of the speed of the spin-2 graviton from that of light.

In the previous studies, all analyses were done by expanding the two parameters c_2 and c_4 in terms of α_1 and α_2 through the relations given by Eq.(2.3), and then keeping only the leading terms, so finally one obtains [26, 22],

$$c_2 = -\frac{c_{13}(2c_1 - c_3)}{3c_1}, \quad c_4 = -\frac{c_3^2}{c_1}, \ (\alpha_1 = \alpha_2 = 0). \tag{4.1}$$

Clearly, in this approach the errors due to the omission of the higher-order terms are of the order of $\mathcal{O}(\alpha_1) \simeq 10^{-4}$, which is too large in comparing with the new constraint (2.4) from the observations of gravitational waves [31, 32].

In that chapter, instead, for any given constraint, say, $F(c_i) = 0$, we have expanded it only in terms of $\epsilon \equiv c_{13}$,

$$\begin{aligned} F(c_1, c_2, c_{14}, \epsilon) = F(c_1, c_2, c_{14}, 0) \\ + F_{,\epsilon}(c_1, c_2, c_{14}, 0)\epsilon + ... = 0, \end{aligned} \tag{4.2}$$

and leave all the other parameters free. Then, keeping only the leading term, we can see that the resulting errors due to this omission is of the order of $\mathcal{O}\left(10^{-15}\right)$, which is insignificant in comparing with the rest of constraints. In doing so, the reduced phase space is in general three-dimensional. However, it is remarkable that the constraints are then divided into two groups, one is confined on the (c_1, c_{14})-plane, and the other on the (c_2, c_{14})-plane. In the former, the constraints are given by Eq.(2.17). We can also transfer this constraint to the (c_4, c_{14})-plane, which is simply equal to,

$$c_4 \lesssim 0, \quad 0 < c_{14} \leq 0.25 \times 10^{-4}. \tag{4.3}$$

(See footnote 1 for a comment on the $c_{14} \to 0$ limit.)

On the other hand, the cosmological constraint from the measurements of the primordial helium-4 abundance restricts c_2 to the range given by Eq.(2.21), while the constraint $c_S^2 \gtrsim 1$ further requires (see footnote 1 again),

$$0.095 \gtrsim c_2 \gtrsim c_{14} > 0. \tag{4.4}$$

However, the severest constraint on c_2 comes from Eq.(2.26), from which we find the constraints (2.27) for $c_{14} \in [0,\ 2 \times 10^{-7}]$ and $c_{14} \in [2 \times 10^{-6},\ 2.5 \times 10^{-5}]$, respectively. In the intermediate regime, $c_{14} \in (2 \times 10^{-7},\ 2 \times 10^{-6})$, the constraints are illustrated in Figs. 2.2-2.3.

It should be noted that the constraints given above do not include the strong-field regime constraints (2.12), because they depend on the sensitivities of neutron stars in the theory, which are not known so far for the parameters given in the above new ranges [29]. Therefore, instead using them to put further constraints on the parameter c_i's, we have used them to find the upper bounds on the sensitivity parameter $\sigma_{æ}$, given by Eq.(2.29), i.e.,

$$|\alpha_1 + 8\sigma_{æ}| \leq 10^{-5}\,, \quad \left|\frac{\alpha_2}{\alpha_1}\right| \times |\alpha_1 + 8\sigma_{æ}| \leq 10^{-9}\,, \tag{4.5}$$

although they are not free parameters, and normally depend on c_i's, as shown explicitly in [29]. Eq.(4.5) represents very severe constraints, and imposes tight bounds on the radiation of neutron stars in the Einstein-aether theory, through the emissions of the different species of the spin-0, spin-1 and spin-2 gravitons. Therefore, it would be very interesting to calculate $\sigma_{æ}$ in the new ranges of the free parameters c_i's, and then comparing such obtained values of $\sigma_{æ}$ with the constraints (4.5).

Finally, we note that recently constraints of the khronometric theory [90] was studied numerically in [43]. When the aether is hypersurface-orthogonal,

$$u_{[\alpha}\nabla_{\beta}u_{\lambda]} = 0, \tag{4.6}$$

it can be shown that u_μ can be always written in terms of a timelike scalar field ϕ, *the khronon*, in the form [44],

$$u_\mu = \frac{\phi_{,\mu}}{\sqrt{-\phi_{,\alpha}\phi^{,\alpha}}}, \quad \phi_{,\alpha}\phi^{,\alpha} < 0. \tag{4.7}$$

Then, we find that,

$$\omega^2 \equiv a^\mu a_\mu + \left(\nabla_\alpha u_\beta\right)\left(\nabla^\alpha u^\beta\right) - \left(\nabla_\alpha u_\beta\right)\left(\nabla^\beta u^\alpha\right), \tag{4.8}$$

vanishes identically. As a result, one can add the following term to the general action (1.27) [45, 46],

$$\Delta S_{æ} \equiv c_0 \int dx^4 \sqrt{-g}\, \omega^2, \tag{4.9}$$

where c_0 is an arbitrary dimensionless constant. Hence, among the four coupling constants c_i $(i = 1, 3, 4)$ of the Einstein-aether theory, only the three combinations $(c_{14},\ c_{13},\ c_2)$ have physical meaning in the khronometric theory [90]. This theory was also referred to as the "T-theory" in [45] [1].

In view of the above considerations, it is clear that the spin-1 graviton appearing in the Einstein-aether theory is absent in the khronometric theory (in addition, an instantaneous mode appears in the khronometric theory [47, 20, 48], while this mode is absent in the Einstein-aether theory [22, 23]). As a result, all the constraints from the spin-1 mode should be dropped, in order to obtain the constraints on the khronometric theory. In other words, the constraints obtained in that chapter projected onto the

[1] It is interesting to note that the khronometric theory can be considered as the low energy limit of the non-projectable version of the Hořava gravity [45, 90, 46, 20].

three dimensional subspace (c_{14}, c_{13}, c_2) are more stringent than the constraints found in [43].

5.2 Plane Wave Solutions

In chapter 3, we have studied gravitational plane waves in Einstein-aether theory, and found all vacuum solutions of the linearly polarized gravitational plane waves. In general, such waves need to satisfy five independent Einstein-aether field equations, given by Eqs.(B.4) -(B.7), for three unknown functions $(U(u), V(u), h(u))$. Therefore, the problem in the Einstein-aether theory is overdetermined, and it is expected that gravitational plane waves exist only for some particular choices of the coupling constants c_i. This is sharply in contrast to Einstein's general relativity, in which the problem is actually underdetermined, i.e. the vacuum Einstein field equations $G_{\mu\nu}$ only yield one independent equation,

$$2U_{uu} - U_u^2 = V_u^2, \tag{4.10}$$

for the two unknown functions U and V. Thus, for any given $V(u)$, one can integrate Eq.(4.10) to find the metric coefficient $U(u)$. This implies that Einstein's theory allows the existence of any form of gravitational plane waves. This is no longer true in Einstein-aether theory, due to the presence of the time-like aether field. In particular, in Einstein-aether theory in order to have arbitrary forms of gravitational plane waves exist, the coupling constants c_i must be chosen so that one of the following two conditions must be satisfied,

$$\begin{aligned} &(i)\ \ c_{13} = c_2 = 0, \ \ c_{14} \neq 0, \ \ h(u) = h_0, \quad \text{or} \\ &(ii)\ c_{13} = c_2 = c_{14} = 0, \qquad \forall\, h(u). \end{aligned} \tag{4.11}$$

In the former case it can be seen that the aether must be a constant, while in the latter the aether has no contributions to the spacetime, and $T^{æ}_{\mu\nu} = 0$ identically, as can be seen from Eq.(B.1).

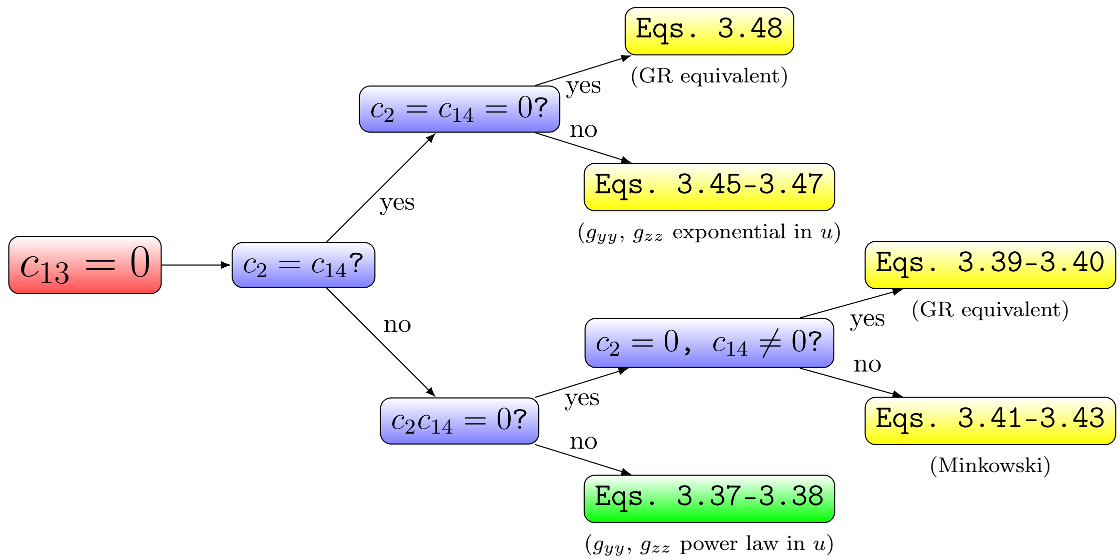

Figure 5.1: A decision tree of all solutions for which $c_{13} = 0$, as virtually required by the constraints of Chapter Two. The solution which obeys all constraints exactly is highlighted in green.

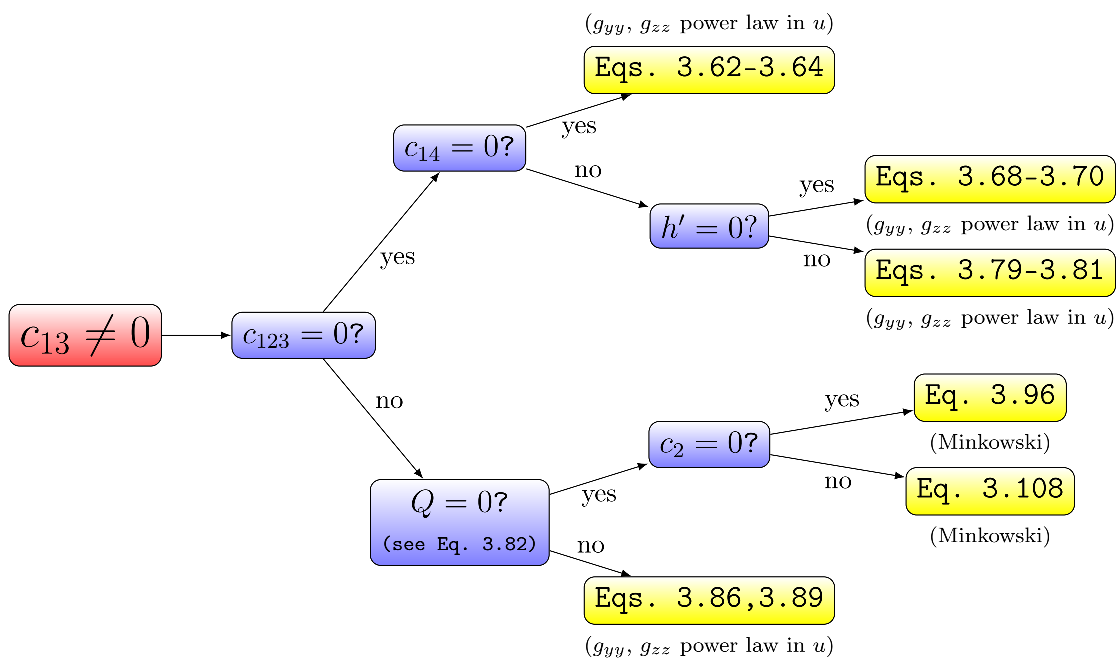

Figure 5.2. A decision tree of all solutions for which $c_{13} \neq 0$, for completeness

In general, the family of solutions for all cases in which $c_{13} = 0$ are given by Fig.(5.1), and the family of solutions for all cases in which $c_{13} \neq 0$ is given by Fig.(5.2). Any solution which is disallowed by the strict observance of the constraints of Chapter Two is highlighted in yellow, and the solution which strictly obeys all constraints is highlighted in green.

Some of these cases are problematic even without considering the observational constraints, as outlined in Jacobson's review article [22]. Any case in which $c_{123} = 0$ results in α_2 diverging (suggesting that the current PPN analysis is not valid here), while any case in which $c_{14} = 0$ results in the speeds of the scalar and vector modes diverging (suggesting that wave equations for these modes do not exist).

In the case of the solution given by Eqs.(3.45-3.47), the squared speed of the spin-0 mode is given by $c_S^2 = (2-c_2)/(2+3c_2)$. Thus, to have $c_S \geq 1$, we must require $c_2 = c_{14} < 0$, which is in conflict with the observational constraints of Chapter Two. Therefore, this case is ruled out by observations.

The solution given by Eqs.(3.37-3.38) with α and β given by Eqs.(3.33,3.35) stands as the only solution found that strictly obeys all constraints and has the metric given by:

$$\begin{aligned} ds^2 &= -2dudv + (u)^{\frac{1-\beta}{1+2\alpha}}dy^2 + (u)^{\frac{1+\beta}{1+2\alpha}}dz^2, \\ \alpha &= -\frac{\sqrt{c_2}}{\sqrt{c_2} \pm \sqrt{c_{14}}}, \\ \beta &\equiv \pm\sqrt{1 + 4\alpha + 2c_{14}\alpha^2}, \end{aligned} \tag{4.12}$$

Where arbitrary constants have been absorbed by a translation of u by the appropriate constant factor and a rescaling of y and z by the appropriate constant factors. Choice of $\pm$ sign doesn't matter for solving the equations, so this metric represents four different, though related, solutions. If we require that the speeds of the scalar, vector and tensor modes (at the linearized level) are all precisely equal to one, then we find that

$$c_{13} = c_4 = 0, \quad c_2 = \frac{c_1}{1 - 2c_1}, \quad (c_T = c_V = c_S = 1), \tag{4.13}$$

which is also satisfied only by Eq.(4.12), and the corresponding solutions are still quite different from those of GR, even all of these gravitational modes now move at the same speed as that of the spin-2 graviton in GR.

It should be noted that the results obtained in that chapter are quite understandable, since the aether field is always unity and timelike, while the gravitational plane waves move only along a null direction. Then, due to their mutual scattering, it is expected that oppositely moving gravitational plane waves exist generically, and the spacetimes must depend on both u and v. Therefore, if only a single gravitational wave moving along a fix null direction is allowed to exist, it is clear that only for particular choices of the coupling constants c_i's, can compatible solutions exist.

Thus, it would be very interesting to study the interactions of a plane gravitational wave with the aether and other matter fields, as well as with a gravitational plane wave moving in the opposite direction, by paying particular attention on Faraday rotations and the difference from those found in GR [85, 76, 76], due to the presence of the timelike aether field, which violates LI.

5.3 Spherically-Symmetric Solutions

We have derived some exact, spherically-symmetric solutions with a static aether (aligned with the timelike Killing vector of the metric) in several coordinate systems, as depicted in Figs. 5.3-5.5.

Some of the solutions found were un-physical in that they required c_2, c_{14}, or c_{13} to have values not allowed by the experimental constraints [14]. But given that the physically-valid solutions sometimes had corresponding un-physical solutions with similar structure, perhaps the structure of these un-physical solutions might induce a search for physical solutions of a similar kind.

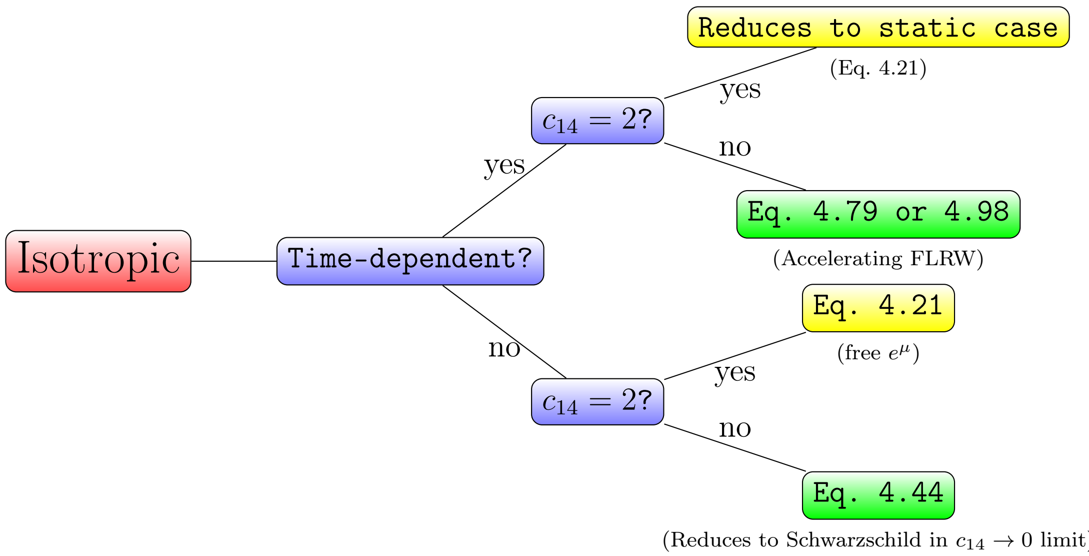

Figure 5.3. Solutions found in the isotropic coordinates.

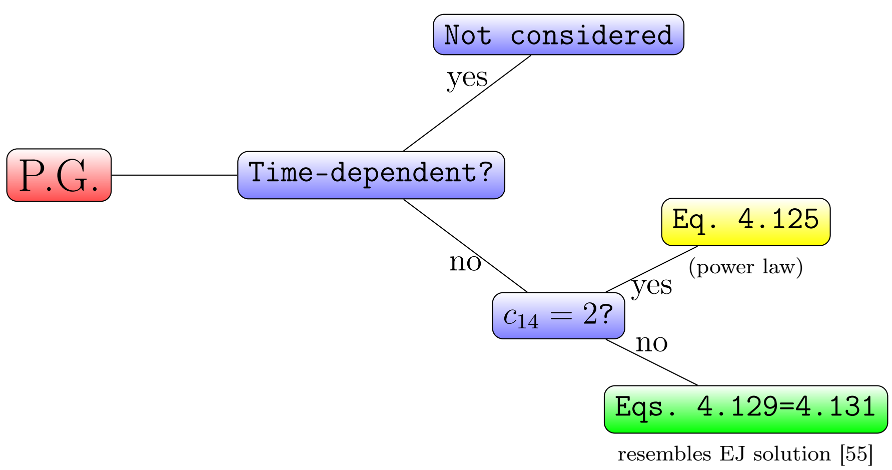

Figure 5.4. Solutions found in the Painlevè-Gullstrand coordinates.

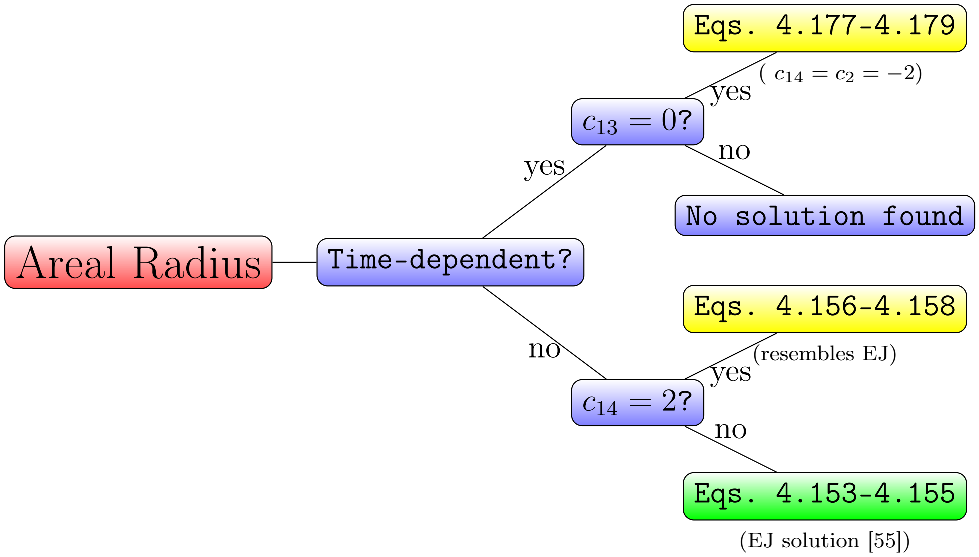

Figure 5.5. Solutions found in the areal radius coordinates.

In isotropic coordinates we found exact time-dependent (Eq.(4.79)) and time-independent (Eq.(4.44)) solutions that do not violate the constraints on the c_i. A coordinate transformation was found that brings the time-dependent solution to an FLRW metric with negative constant curvature (Eq.(4.98)) similar to that found in [60], or an FLRW metric with constant positive curvature (and a generally complex scale factor). These are vacuum solutions without a cosmological constant and yet with an accelerating expansion of the universe, valid for any physical values of c_2 and c_{13}.

The static solution reduces to the Schwarzschild solution of GR as $c_{14} \to 0$, and is a solution with a static metric and an aether vector aligned with the metric's timelike Killing vector. According to [55] there is a unique solution of this kind, yet we could not find a coordinate transformation that brings our solution to the one presented in [55] or Appendix A.

This solution lends itself to more tractable analysis and comparison with the Schwarzschild metric of GR.

5.4 Future Work

Future Solar System tests of the preferred frame parameters $\alpha_{1,2}$ will allow tighter constraints on the c_i, and updated constraints on the spread of primordial He-4 abundances will also help to further constrain the theory. A more complete PPN analysis of Einstein-aether theory in which special cases such as $c_{123} = 0$ are considered would help to generalize the observational constraint analysis. LIGO and other gravitational wave observatories continue to search for signals with polarizations other than the h_x and h_+ allowed by pure GR. Better knowledge of the experimental constraints can guide the search for more viable spherically-symmetric and plane wave solutions to the field equations. Further analysis of the existing plane wave solutions, as well as the search for new solutions in alternative coordinate systems such as the Brinkmann metric, and especially the collision of two plane waves in Einstein-aether theory, will allow greater comparison with the plane waves of pure GR.

The closed-form solution for general c_i values found in isotropic coordinates could make a vacuum region solution for future star solutions in Einstein-aether theory, for those star solutions where the aether is aligned with the timelike Killing vector. Allowing a radial tilt to the aether will lead to greater freedom in finding new spherically-symmetric and black hole solutions, as will considering a wider range of coordinate systems. With more observationally-viable, closed-form solutions one can perform perturbations to see the stability of black holes in Einstein-aether theory. Further analytical solutions incorporating a cosmological constant or scalar field are further opportunities to explore the theoretical and observational implications of the aether field.

APPENDICES

APPENDIX A

Mathematica Scripts

A.1 Linear Perturbations around the Minkowski Background

It is easy to show that the Minkowski spacetime is a solution of the Einstein-aether theory, in which the aether is aligned along the time direction, $\bar{u}_\mu = \delta^0_\mu$. Let us consider the linear perturbations,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad u_\mu = \bar{u}_\mu + w_\mu, \tag{A.1}$$

where

$$\begin{aligned} h_{0i} &= \partial_i B + B_i \,, \quad w_i = \partial_i v + v_i \,, \\ h_{ij} &= 2\psi\delta_{ij} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right) E \\ &\quad + \frac{1}{2}(\partial_i E_j + \partial_j E_i) + \gamma_{ij} \,, \end{aligned} \tag{A.2}$$

with $\Delta \equiv \delta^{ij}\partial_i\partial_j$ and the constraints

$$\begin{aligned} \partial^i v_i &= \partial^i B_i = \partial^i E_i = 0 \,, \\ \partial^i \gamma_{ij} &= 0 \,, \quad \gamma^i{}_i = 0 \,, \end{aligned} \tag{A.3}$$

where all the spatial indices are raised or lowered by δ^{ij} or δ_{ij}, for example $\partial^i v_i \equiv \delta^{ij}\partial_j v_i$, and so on. Therefore, we have six scalars, h_{00}, w^0, B, v, ψ and E; three transverse vectors, B_i, v_i and E_i; and one transverse-traceless tensor, γ_{ij}. Under the following coordinate transformations,

$$t' = t + \xi^0 \,, \quad x'^i = x^i + \xi^i + \partial^i \xi \,, \tag{A.4}$$

where $\partial_i \xi^i = 0$, these quantities change as

$$
\begin{aligned}
h'_{00} &= h_{00} - 2\dot{\xi}^0 \,, \qquad w'^0 = w^0 + \dot{\xi}^0 \,, \\
E' &= E + 2\xi \,, \quad \psi' = \psi + \xi^0 + \frac{1}{3}\Delta\xi \,, \quad v' = v + \dot{\xi} \,, \\
B' &= B - \xi^0 + \dot{\xi} \,, \qquad (A.5) \\
B'_i &= B_i + \dot{\xi}_i \,, \quad E'_i = E_i + 2\xi_i \,, \\
v'_i &= v_i + \dot{\xi}_i \,, \qquad (A.6) \\
\gamma'_{ij} &= \gamma_{ij} \,. \qquad (A.7)
\end{aligned}
$$

For the scalar part, let us choose the gauge

$$
E = B = 0 \,, \tag{A.8}
$$

which are equivalently to choose the arbitrary functions ξ^0 and ξ as $\xi = -E/2$ and $\xi^0 = B + \dot{\xi}$, so that the gauge freedom is completely fixed [1]. Then, integrating out the variables h_{00}, w^0 and v, we find that the quadratic action of the scalar part takes the form,

$$
\begin{aligned}
S^{(2,S)}_{æ} &= \frac{1}{8\pi G_{æ}} \int d^4x \left[\frac{(1-c_{13})(2+c_{13}+3c_2)}{c_{123}} \dot{\psi}^2 \right. \\
&\qquad \left. + \frac{2-c_{14}}{c_{14}} \psi \Delta \psi \right] . \qquad (A.9)
\end{aligned}
$$

Thus, the ghost-free condition requires

$$
q_S \equiv \frac{(1-c_{13})(2+c_{13}+3c_2)}{c_{123}} > 0 \,. \tag{A.10}
$$

Then, the variation of $S^{(2,S)}_{æ}$ with respect to ψ yields the field equation, $\ddot{\psi} - c_S^2 \Delta\psi = 0$, where

$$
c_S^2 \equiv \frac{c_{123}(2-c_{14})}{c_{14}(1-c_{13})(2+c_{13}+3c_2)} \,. \tag{A.11}
$$

[1] In [34], the gauge $v = B = 0$ was adopted. However, as it can be seen from Eq.(A.5), in this case ξ is fixed up to an arbitrary function $\hat{\xi}\left(x^k\right)$, that is, $\xi = \hat{\xi}\left(x^k\right) - \int v dt$, while ξ^0 is completely fixed by $\xi^0 = B + \dot{\xi}$.

For the vector part, we choose the gauge $\xi_i = -E_i/2$, so that $E'_i = 0$. Then, after integrating out B_i, we find that the quadratic action of the vector part takes the form,

$$S_{æ}^{(2,V)} = \frac{1}{16\pi G_{æ}} \int d^4x \Bigg[c_{14} \dot{v}^i \dot{v}_i + \frac{2c_1 - c_{13}c_-}{2(1-c_{13})} v^i \Delta v_i \Bigg]. \tag{A.12}$$

Clearly, the ghost-free condition of the vector part now requires

$$q_V \equiv c_{14} > 0\,. \tag{A.13}$$

Then, the variation of $S_{æ}^{(2,V)}$ with respect to v_i yields the field equation, $\ddot{v}_i - c_V^2 \Delta v_i = 0$, where

$$c_V^2 \equiv \frac{2c_1 - c_{13}c_-}{2c_{14}(1-c_{13})}\,. \tag{A.14}$$

Similarly, the quadratic action of the tensor part takes the form,

$$S_{æ}^{(2,T)} = \frac{1}{64\pi G_{æ}} \int d^4x \Big[(1-c_{13})\, \dot{\gamma}^{ij} \dot{\gamma}_{ij} + \phi^{ij} \Delta \gamma_{ij} \Big]. \tag{A.15}$$

Thus, the ghost-free condition of the tensor part requires

$$q_T \equiv 1 - c_{13} > 0\,. \tag{A.16}$$

Then, the variation of $S_{æ}^{(2,T)}$ with respect to γ_{ij} yields the field equation, $\ddot{\gamma}_{ij} - c_T^2 \Delta \gamma_{ij} = 0$, where

$$c_T^2 = \frac{1}{1-c_{13}}\,. \tag{A.17}$$

APPENDIX B

Einstein and Stress-Energy Tensor Components

B.1 Plane Wave Spacetime

For the plane wave spacetime of Eq.(3.14), the non-vanishing components of the Einstein tensor $G_{\mu\nu}$ and the aether stress-energy tensor $T^{æ}_{\mu\nu}$ are given by,

$$
\begin{aligned}
G_{00} &= \frac{1}{2}\Big(2U_{uu} - U_u^2 - V_u^2\Big), \\
T^{æ}_{00} &= -\frac{1}{8}\Big[2c_2U_{uu} + c_{13}\Big(V_u^2 + U_u^2\Big) \\
&\quad +2\big(c_{13} + c_2 + 3c_{14}\big)\Big(h_{uu} - h_uU_u - h_u^2\Big)\Big], \\
T^{æ}_{01} &= \frac{e^{-2h}}{4}\Big[c_2\Big(U_{uu} - 2h_uU_u - U_u^2\Big) \\
&\quad +\big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_uU_u - 2h_u^2\Big)\Big], \\
T^{æ}_{11} &= -\frac{e^{-4h}}{8}\Big[2c_2U_{uu} + c_{13}\Big(U_u^2 + V_u^2\Big) \\
&\quad +2\big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_uU_u - h_u^2\Big)\Big], \\
T^{æ}_{22} &= \frac{e^{V-U-2h}}{8}\Big[c_{13}\Big(2V_{uu} - V_u^2 - 2U_uV_u - 4h_uV_u\Big) \\
&\quad -\big(c_{13} + 2c_2\big)\Big(2U_{uu} - U_u^2 - 4h_uU_u\Big) \\
&\quad -4c_2h_{uu} + 2\big(3c_2 - c_{13} + c_{14}\big)h_u^2\Big], \\
T^{æ}_{33} &= -\frac{e^{-(V+U+2h)}}{8}\Big[c_{13}\Big(2V_{uu} + V_u^2 - 2U_uV_u - 4h_uV_u\Big) \\
&\quad +\big(c_{13} + 2c_2\big)\Big(2U_{uu} - U_u^2 - 4h_uU_u\Big) \\
&\quad +4c_2h_{uu} - 2\big(3c_2 - c_{13} + c_{14}\big)h_u^2\Big],
\end{aligned}
\tag{B.1}
$$

and the aether dynamics tensor $Æ_\mu = \big(Æ_0, Æ_1, 0, 0\big)$, where

$$
\begin{aligned}
Æ_0 &= -Æ_1e^{2h} = -\frac{e^{-h}}{4\sqrt{2}}\Big[2c_2U_{uu} + c_{13}\Big(U_u^2 + V_u^2\Big) \\
&\quad +2\big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_uU_u - h_u^2\Big)\Big].
\end{aligned}
\tag{B.2}
$$

In the vacuum case, we have $T^{m}_{\mu\nu} = 0$, and the Einstein-aether equations (1.31) reduce to

$$G_{\mu\nu} = T^{æ}_{\mu\nu}, \tag{B.3}$$

which yield five independent equations,

$$\begin{aligned}
&2U_{uu} - \Big(V_u^2 + U_u^2\Big) + 2c_{14}\Big(h_{uu} - h_u U_u - h_u^2\Big) = 0,\\
&c_2\Big(U_{uu} - 2h_u U_u - U_u^2\Big)\\
&\qquad + \big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_u U_u - 2h_u^2\Big) = 0, &\text{(B.4)}\\
&2c_2 U_{uu} + c_{13}\Big(U_u^2 + V_u^2\Big)\\
&\qquad + 2\big(c_2 + c_{13} - c_{14}\big)\Big(h_{uu} - h_u U_u - h_u^2\Big) = 0, &\text{(B.5)}\\
&c_{13}\Big(2V_{uu} - V_u^2 - 2U_u V_u - 4h_u V_u\Big)\\
&\qquad - \big(c_{13} + 2c_2\big)\Big(2U_{uu} - U_u^2 - 4h_u U_u\Big)\\
&\qquad - 4c_2 h_{uu} + 2\big(3c_2 - c_{13} + c_{14}\big)h_u^2 = 0, &\text{(B.6)}\\
&c_{13}\Big(2V_{uu} + V_u^2 - 2U_u V_u - 4h_u V_u\Big)\\
&\qquad + \big(c_{13} + 2c_2\big)\Big(2U_{uu} - U_u^2 - 4h_u U_u\Big)\\
&\qquad + 4c_2 h_{uu} - 2\big(3c_2 - c_{13} + c_{14}\big)h_u^2 = 0 &\text{(B.7)}
\end{aligned}$$

where in Eq.(B.4) we have used the fact that $T^{æ}_{00}$ can be expressed in terms of $T^{æ}_{11}$ which is equal to zero.

B.2 Conformally-flat Spherically-Symmetric Spacetime

To list the Einstein tensor components for the spacetime of Eqs.(4.7,4.8), we first define Σ such that,

$$\Sigma = 3\dot{\nu}^2 + 2\ddot{\nu} - 2\dot{\mu}\dot{\nu} \tag{B.8}$$

Then the non-zero components of the Einstein tensor are,

$$G_{00} = 3\dot{\nu}^2 - e^{2\mu-2\nu}\left[\nu'^2 + 2\nu'' + 4\frac{\nu'}{r}\right], \tag{B.9}$$

$$G_{01} = 2\mu'\dot{\nu} - 2\dot{\nu}', \tag{B.10}$$

$$G_{11} = \nu'^2 + 2\mu'\nu' + \frac{2}{r}(\mu' + \nu') - \Sigma e^{-2\mu+2\nu} \tag{B.11}$$

$$G_{22} = r^2\Big[\mu'^2 + \mu'' + \nu'' + \frac{\mu' + \nu'}{r} - \Sigma e^{-2\mu+2\nu}\Big], \tag{B.12}$$

$$G_{33} = sin^2\theta\,(G_{22}) \tag{B.13}$$

For the aether stress-energy tensor components, we first define β such that

$$\beta = 3c_2 + c_{13}. \tag{B.14}$$

Then the non-zero components of the aether stress-energy tensor are,

$$\begin{aligned} T^{æ}_{00} &= e^{2\mu-2\nu}\Big[c_{14}\Big(\frac{\mu'^2}{2} + \mu'\nu' + \mu'' + 2\frac{\mu'}{r}\Big)\Big] \\ &\quad - \frac{3}{2}\beta\dot{\nu}^2, \end{aligned} \tag{B.15}$$

$$T^{æ}_{01} = c_{14}\left(\dot{\mu}' + \mu'\dot{\nu}\right), \tag{B.16}$$

$$T^{æ}_{11} = \frac{\beta}{2}\Sigma e^{-2\mu+2\nu} - \frac{c_{14}}{2}\mu'^2, \tag{B.17}$$

$$T^{æ}_{22} = r^2\Big[T^{æ}_{11} + c_{14}\mu'^2\Big] \tag{B.18}$$

$$T^{æ}_{33} = sin^2\theta\,\big(T^{æ}_{22}\big) \tag{B.19}$$

B.3 Schwarzschild-type Spacetime

Given how often they repeat, I define the quantities,

$$Q = \frac{\mu'^2}{2} - \mu'\nu' + \mu'' \tag{B.20}$$

$$H = \frac{\dot{\nu}^2}{2} - \dot{\mu}\dot{\nu} + \ddot{\nu} \tag{B.21}$$

The non-zero components of the Einstein tensor are,

$$G_{00} = \frac{1}{r^2} e^{2(\mu-\nu)} \left[e^{2\nu} + 2r\nu' - 1\right], \tag{B.22}$$

$$G_{01} = \frac{2\dot{\nu}}{r}, \tag{B.23}$$

$$G_{11} = \frac{1}{r^2} \left[1 - e^{2\nu} + 2r\mu'\right], \tag{B.24}$$

$$\begin{aligned} G_{22} = r^2 \Big[& e^{-2\nu} \Big(Q + \Big(\frac{\mu'^2}{2} - \frac{\nu' - \mu'}{r} \Big) \Big) \\ & - e^{-2\mu} \Big(H + \frac{\dot{\nu}^2}{2} \Big) \Big], \end{aligned} \tag{B.25}$$

$$G_{33} = sin^2\theta \, (G_{22}) \tag{B.26}$$

The non-zero components of the aether stress-energy tensor are,

$$T^{\text{æ}}_{00} = e^{2\mu-2\nu} c_{14} \Big[Q + \frac{2\mu'}{r} \Big] - \frac{c_{123}}{2} \dot{\nu}^2, \tag{B.27}$$

$$T^{\text{æ}}_{01} = c_{14} \left(\dot{\mu}' - \mu' \dot{\nu} \right), \tag{B.28}$$

$$T^{\text{æ}}_{11} = e^{-2\mu+2\nu} c_{123} H - \frac{c_{14}}{2} \mu'^2, \tag{B.29}$$

$$\begin{aligned} T^{\text{æ}}_{22} = r^2 \Big[& e^{-2\mu} \Big(c_2 H - \frac{c_{13}}{2} \dot{\nu}^2 \Big) \\ & - e^{-2\nu} \frac{c_{14}}{2} \mu'^2 \Big], \end{aligned} \tag{B.30}$$

$$T^{\text{æ}}_{33} = sin^2\theta \left(T^{\text{æ}}_{22} \right) \tag{B.31}$$

B.4 Painlevè-Gullstrand Coordinates

Recall from Eq.(4.103) that Δ is defined by,

$$\Delta = e^{2\mu} + e^{2\nu} \tag{B.32}$$

The non-zero components of the Einstein tensor are,

$$
\begin{aligned}
G_{00} &= \frac{1}{\Delta^2 r^2}[e^{4\mu+2\nu}(1-2r(\mu'+\nu') \\
&\quad + e^{2\mu+4\nu}], && \text{(B.33)} \\
G_{01} &= \frac{1}{\Delta^2 r^2}[e^{2\mu+3\nu}(-1+2r(\mu'-\nu')) \\
&\quad - e^{5\nu}], && \text{(B.34)} \\
G_{11} &= \frac{1}{\Delta^2 r^2}[e^{2\mu+2\nu}(4r\mu'-1)+2e^{4\mu}r\mu' \\
&\quad + e^{4\nu}(2r\nu'-1)], && \text{(B.35)} \\
G_{22} &= \frac{1}{\Delta^2 r^2}[e^{4\mu}(r\mu'+r^2\mu'^2+r^2\mu'') \\
&\quad + e^{2\mu+2\nu}(r(1+r\mu')(2\mu'-\nu') \\
&\quad + r^2\mu'')], && \text{(B.36)} \\
G_{33} &= sin^2\theta(G_{22}). && \text{(B.37)}
\end{aligned}
$$

The non-zero components of the aether stress-energy tensor are,

$$
\begin{aligned}
T_{00}^{æ} &= \frac{c_{14}}{2\Delta^2 r^2}[e^{6\mu}(4\mu'+r\mu'^2+2r\mu'') \\
&\quad + (e^{4\mu+2\nu})(4\mu'+3r\mu'^2-2r\mu'\nu' \\
&\quad + 2r\mu'')], && \text{(B.38)} \\
T_{01}^{æ} &= \frac{c_{14}}{2\Delta^2 r^2}[e^{2\mu+3\nu}(4\mu'+3r\mu'^2+2r\mu''-2r\mu'\nu') \\
&\quad - (e^{4\mu+\nu})(4\mu'+r\mu'^2+2r\mu'\nu')], && \text{(B.39)} \\
T_{11}^{æ} &= \frac{c_{14}}{2\Delta^2 r^2}[e^{4\nu}(2\mu'+r\mu'^2+r\mu'') \\
&\quad + (e^{2\mu+2\nu})(4\mu'-r\mu'^2+2r\mu'\nu') \\
&\quad - 4e^{4\mu}r\mu'^2)], && \text{(B.40)} \\
T_{22}^{æ} &= \frac{c_{14}e^{2\mu}r^2\mu'^2}{2\Delta}, && \text{(B.41)} \\
T_{33}^{æ} &= sin^2\theta(Tæ). && \text{(B.42)}
\end{aligned}
$$

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