ABSTRACT

Uniqueness Implies Uniqueness and Existence for Nonlocal Boundary Value Problems for Fourth Order Differential Equations

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In this dissertation, we are concerned with uniqueness and existence of solutions of certain types of boundary value problems for fourth order differential equations. In particular, we deal with uniqueness implies uniqueness and uniqueness implies existence questions for solutions of the fourth order ordinary differential equation,

$$y^{(4)} = f(x, y, y', y'', y'''),$$

satisfying nonlocal 5-point boundary conditions given by

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) - y(x_5) = y_4,$$

where $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, and $y_1,\ y_2,\ y_3,\ y_4 \in \mathbb{R}$. We also consider solutions of this fourth order differential equation satisfying nonlocal 4-point and 3-point boundary conditions given by

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y(x_3) - y(x_4) = y_4,$$

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y''(x_1) = y_3,\ y(x_2) - y(x_3) = y_4.$$

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CHAPTER ONE

Introduction

In this dissertation, we are concerned with uniqueness and existence of solutions of certain types of boundary value problems for fourth order ordinary differential equations. In particular, we deal with "uniqueness implies uniqueness" and "uniqueness implies existence" questions for solutions of the fourth order ordinary differential equation,

$$y^{(4)} = f(x, y, y', y'', y'''), \tag{1.1}$$

satisfying nonlocal 5-point boundary conditions given by

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) - y(x_5) = y_4, \tag{1.2}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2,\ y(x_4) = y_3,\ y(x_5) = y_4, \tag{1.3}$$

where $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$. We also consider solutions of (1.1) satisfying nonlocal 4-point boundary conditions given by

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y(x_3) - y(x_4) = y_4, \tag{1.4}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2,\ y(x_4) = y_3,\ y'(x_4) = y_4, \tag{1.5}$$

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_2) = y_3,\ y(x_3) - y(x_4) = y_4, \tag{1.6}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2,\ y'(x_3) = y_3,\ y(x_4) = y_4, \tag{1.7}$$

where $a < x_1 < x_2 < x_3 < x_4 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$, as well as solutions of (1.1) satisfying nonlocal 3-point boundary conditions given by

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y''(x_1) = y_3,\ y(x_2) - y(x_3) = y_4, \tag{1.8}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2, y'(x_3) = y_3,\ y''(x_3) = y_4, \tag{1.9}$$

where $a < x_1 < x_2 < x_3 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$.

Questions of uniqueness implies existence, as well as uniqueness implies uniqueness for solutions of boundary value problems for ordinary differential equations enjoy some history.

Two of the oldest works devoted to these types of questions were by Lasota and Luczynski [41] and Lasota and Opial [42] in which they dealt with uniqueness implies existence, for solutions of the second order ordinary differential equation,

$$y'' = f(x, y, y')$$

satisfying either the conjugate boundary conditions,

$$y(x_1) = y_1,\ y(x_2) = y_2,$$

or the right focal boundary conditions,

$$y(x_1) = y_1,\ y'(x_1) = y_2,$$

where $x_1 < x_2$, and $y_1, y_2 \in \mathbb{R}$.

Subsequent to those papers have been several works addressing uniqueness implies uniqueness conditions including Jackson's [34, 35] monumental works on solutions of the nth order differential equation,

$$y^{(n)} = h(x, y, y', \ldots, y^{(n-1)}) \tag{1.10}$$

satisfying k-point conjugate boundary conditions,

$$y^{(i-1)}(x_j) = y_{ij},\ 1 \le i \le m_j,\ 1 \le j \le k, \tag{1.11}$$

where $2 \le k \le n,\ m_1 + \cdots + m_k = n,\ x_1 < \cdots < x_k$, and $y_{ij} \in \mathbb{R}$, as well as the major paper by Henderson [18] for solutions of (1.10) satisfying k-point right focal boundary conditions,

$$y^{(i-1)}(x_j) = y_{ij},\ s_{j-1} + 1 \le i \le s_j,\ 1 \le j \le k, \tag{1.12}$$

where $2 \leq k \leq n,\ m_1+\cdots+m_k = n,\ s_l = s_{l-1}+m_l,\ 1 \leq l \leq k, s_0 = 0,\ x_1 < \cdots < x_k$, and $y_{ij} \in \mathbb{R}$. Other uniqueness implies uniqueness results are found in the papers by Ehme and Hankerson [7], Goecke and Henderson [10], Henderson [20, 21, 22], Henderson and McGwier [28] and Henderson and Pruet [29] for classes of boundary value problems for (1.10) that might be termed as "between" the conjugate and the right focal problems.

Uniqueness implies existence results have an equally rich history. Frequently, modified shooting methods are the main tool for establishing such results. Following the above mentioned papers by Lasota and Luczynski [41] and Lasota and Opial [42], Hartman [15] proved that, if solutions of all k-point conjugate boundary value problems, $2 \leq k \leq n$, for (1.10) are unique, when they exist, then indeed there exist solutions of all k-point conjugate boundary value problems for (1.10). Later, Hartman [17] and Klaasen [38] independently proved, if solutions of n-point conjugate boundary value problems are unique, when they exist, then there exist unique solutions of all k-point conjugate boundary value problems, $2 \leq k \leq n$. For (1.10) Henderson [19] proved an analogue of the Hartman-Klaasen result for right focal boundary value problems; in particular, if n-point right focal boundary value problems for (1.10) are unique, when they exist, then there exist solutions of all right focal boundary value problems for (1.10). Other uniqueness implies existence results have been obtained for boundary value problems for ordinary differential equations, for finite difference equations, and for dynamic equations on times scales. Many of these results appear in [1, 6, 14, 23, 26, 28, 31, 32, 36, 54].

Fourth order nonlinear boundary value problems arise naturally in the mechanics of materials, in describing the equilibrium state of an elastic beam under various boundary constraints such as either, when both ends are simply supported (so that, there are no bending moments at the ends), or perhaps when one end is simply supported and the other end is clamped by sliding clamps; see [48] or [62].

In addition, the interaction of solitary waves on shallow water has been modelled by fourth order boundary value problems as described by Marchant [51]. The literature is vast on fourth order nonlinear boundary value problems, and results for such problems have "commonly" dealt with Green's functions [2], [55], monotonicity methods for constructing sequences approximating solutions [59], uniqueness of solutions [53], *a priori* bounds on solutions leading to solutions via Leray-Schauder [50], multiple positive solutions, (including double, triple, and even infinitely many) via Leggett-Williams, Guo-Krasnosel'skii and Avery-Henderson fixed point theorems [3, 5, 43, 44, 49], optimal length intervals on which there exist unique solutions [37], and solutions via upper and lower solutions in the presence of a Nagumo condition [9].

Finally, we mention a brief history of nonlocal boundary value problems. Roughly speaking, nonlocal boundary value problems include at least one boundary condition involving data from multiple points, such as

$$y(\eta_1) - \sum_{i=2}^{m} \alpha_i y(\eta_i) = c,$$

where $\eta_i,\ 1 \le i \le m$, are points in an interval, or such as

$$\int_{\eta_1}^{\eta_2} y(x)dx = c,$$

where again, η_1 and η_2 are points in some interval. An instance in which such a boundary value problem arises, for which nonlocal discrete conditions are stated, would be found in a heat conduction problem taking place in a unit rod; the temperature along the rod, with prescribed temperatures at the ends related to the temperature at the middle point, is modelled by the boundary value problem,

$$u'' + q(t)u = 0, \quad 0 < t < 1,$$

$$u(0) - \frac{1}{2}u(\frac{1}{2}) = \lambda_1, \quad u(1) - \frac{1}{3}u(\frac{1}{2}) = \lambda_2,$$

where $q(t)$ is the source density of heat, and λ_1, λ_2 describe the relations of temperatures of the rod at the endpoints and the middle point.

An early paper on nonlocal boundary value problems was written by Il'in and Moiseev [33] for a second order Sturm-Liouville operator. Prominent in the literature in nonlocal boundary value problems are papers by Gupta *et al.* [12, 13] dealing with m-point boundary value problems for second order equations when $m > 2$. Common methods used in those papers pair inequalities with *a priori* estimates on solutions. In a similar way, R. Ma [45, 46, 47] has contributed significantly to work on multi-point boundary value problems primarily for second order equations; much of Ma's work has centered on cone theoretic applications to obtain positive and multiple positive solutions. Currently, several researchers are engaged in research on nonlocal boundary value problems that includes questions of nonlinear eigenvalue problems [11], problems on time scales [31, 32], uniqueness of solutions [57], multi-point problems at resonance for nth order equations [52], and many papers on positive solutions [39, 40, 61, 65]. And, there have been, in fact, a recent spate of papers on uniqueness implies existence for solutions of second order and third order nonlocal boundary value problems for both ordinary differential equations and for dynamic equations on time scales; see, for example [4, 24, 27, 30].

Throughout the dissertation, it is assumed that the nonlinear equation satisfies the conditions:

(A) $f : (a, b) \times \mathbb{R}^4 \to \mathbb{R}$ is continuous.

(B) Solutions of initial value problems for (1.1) are unique and exist on all of (a, b).

In addition to (A) and (B), uniqueness assumptions on solutions of (1.1):(1.j) will be made.

The majority of questions in this dissertation involve (**i**) whether uniqueness of solutions of (1.1):(1.2) implies uniqueness of (1.1):(1.4), (1.1):(1.6), (1.1):(1.8); (**ii**) whether uniqueness of solutions of (1.1):(1.4), (1.1):(1.5), (1.1):(1.6) and (1.1):(1.7) imply uniqueness of solutions (1.1):(1.2) and (1.1):(1.3); and (**iii**) whether uniqueness of solutions of (1.1):(1.2) implies existence of (1.1):(1.2), (1.1):(1.4), (1.1):(1.6) and (1.1):(1.8). Of course, a principal reason for considering questions such as (i) or (ii) would be in resolving question (iii).

Our main motivation for the results of this dissertation arises from the monumental work by Peterson [56] in which he addressed all of the questions for two-point, three-point and four-point conjugate boundary value problems for the fourth order differential equation (1.1). In particular, Peterson's [56] "uniqueness implies uniqueness" and "uniqueness implies existence" questions focused on solutions of (1.1) satisfying either some two-point conjugate conditions,

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y''(x_1) = y_3,\ y(x_2) = y_4,$$
$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y'(x_2) = y_4,$$
$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_2) = y_3,\ y''(x_2) = y_4$$

$a < x_1 < x_2 < b$, or some three-point conjugate conditions,

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y(x_3) = y_4,$$
$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_2) = y_3,\ y(x_3) = y_4,$$
$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y'(x_3) = y_4,$$

$a < x_1 < x_2 < b$, or the four-point conjugate conditions,

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) = y_4,$$

$a < x_1 < x_2 < x_3 < x_4 < b$, and in each case $y_1, y_2, y_3, y_4 \in \mathbb{R}$.

Given the hypotheses (A) and (B) and a uniqueness condition, and in a context of second order problems, Henderson [24] provided an affirmative answer to (iii). In a like manner, the paper by Clark and Henderson [4] gave an affirmative answer to questions (i), (ii) and (iii) for third order nonlocal boundary value problems.

Part I of this dissertation entails Chapter Two and Chapter Three. In Chapter Two we are primarily concerned with uniqueness of solutions of (1.1):(1.2) implying uniqueness of solutions of (1.1):(1.4) and (1.1):(1.6) and (1.1):(1.8). In Chapter Three, we are concerned with the converse questions, that is, uniqueness of solutions of (1.1):(1.4) and (1.1):(1.6) implying uniqueness of solutions of (1.1):(1.2).

In Chapter Two, first we state the Brouwer Invariance of Domain Theorem [58] and the Kamke Convergence Theorem [16]. These are fundamental to our uniqueness results as well as our existence results. Then we develop a continuous dependence theorem with respect to certain boundary value conditions, which is motivated by a similar result on continuous dependence by Clark and Henderson [4]. Later using continuous dependence, we deal with uniqueness of solutions of (1.1):(1.2) implies uniqueness of solutions of (1.1):(1.4), (1.1):(1.6) and (1.1):(1.8). In Chapter Three, we develop that uniqueness of solutions of (1.1):(1.4) and (1.1):(1.6) implies uniqueness of solutions of (1.1):(1.2).

In Part II, which entails Chapter Four, we are concerned with existence of solutions of (1.1):(1.2), (1.1):(1.4), (1.1):(1.6) and (1.1):(1.8) if we know the uniqueness of solutions of the 5-point boundary value problem (1.1):(1.2).

In Part III, we talk about local existence of boundary value problems. First we state an equivalence between the solutions of boundary value problems and solutions of integral equations. Also, we state the Contraction Mapping Theorem. Then in terms of upper bounds of the integral of the Green's function and its derivatives for the fourth order boundary value problem, we apply the Contraction Mapping Theorem to obtain the local existence and uniqueness for solutions of 5-point nonlocal boundary value problems for fourth order equations, when function f is continuous and satisfies a Lipschitz condition. Then as a consequence of the earlier uniqueness implies existence results, we also have local existence and uniqueness of solutions of 3-point and 4-point nonlocal boundary value problems.

CHAPTER TWO

Uniqueness

In this chapter we will consider solutions of the fourth order differential equation,

$$y^{(4)} = f(x, y, y', y'', y'''), \tag{2.1}$$

satisfying nonlocal 5-point boundary conditions given by

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) - y(x_5) = y_4, \tag{2.2}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2,\ y(x_4) = y_3,\ y(x_5) = y_4, \tag{2.3}$$

where $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$. We also consider solutions of (2.1) satisfying nonlocal 4-point boundary conditions given by

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y(x_3) - y(x_4) = y_4, \tag{2.4}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2,\ y(x_4) = y_3,\ y'(x_4) = y_4, \tag{2.5}$$

or

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_2) = y_3,\ y(x_3) - y(x_4) = y_4, \tag{2.6}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2,\ y'(x_3) = y_3,\ y(x_4) = y_4, \tag{2.7}$$

where $a < x_1 < x_2 < x_3 < x_4 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$, as well as solutions of (2.1) satisfying nonlocal 3-point boundary conditions given by

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y''(x_1) = y_3,\ y(x_2) - y(x_3) = y_4, \tag{2.8}$$

$$y(x_1) - y(x_2) = y_1,\ y(x_3) = y_2,\ y'(x_3) = y_3,\ y''(x_3) = y_4, \tag{2.9}$$

where $a < x_1 < x_2 < x_3 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$.

In particular, this chapter is devoted to uniqueness implies uniqueness relationships among solutions of (2.1) satisfying any of the boundary conditions, (2.2)-(2.9).

We will concentrate on uniqueness of solutions of nonlocal 5-point boundary value problems implying uniqueness of solutions of nonlocal 4-point and nonlocal 3-point boundary value problems. Questions of this type are not without motivation. As discussed in greater detail in the Introduction, numerous papers have been devoted to such uniqueness questions, such as the papers by Jackson [34], [35], for conjugate boundary value problems, the papers by Henderson [10], [20] and [24], for right focal boundary value problems, and the recent papers [4], [27], for nonlocal boundary value problems. Behind these uniqueness results is the role of continuous dependence of solutions on boundary conditions. This continuous dependence arises somewhat from applications of the Brouwer Invariance of Domain Theorem [58] in conjunction with continuous dependence of solutions on initial conditions. The continuous dependence on initial conditions is a consequence of the Kamke Convergence Theorem [16]. We will include in this chapter statements of both the Brouwer Theorem and the Kamke Theorem.

The work by Peterson [56] dealing with 2-point, 3-point and 4-point conjugate boundary value problems for (2.1) is also a primary motivation for the results of this chapter. In particular, we model our results along the lines of Peterson's work concerning solutions of (2.1) satisfying the 4-point conjugate boundary conditions,

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) = y_4, \tag{2.10}$$

where $a < x_1 < x_2 < x_3 < x_4 < b$, as well as the 3-point (respectively 2-1-1, 1-2-1, and 1-1-2) conjugate boundary conditions,

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y(x_3) = y_4, \tag{2.11}$$

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_2) = y_3,\ y(x_3) = y_4, \tag{2.12}$$

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y'(x_3) = y_4, \tag{2.13}$$

where $a < x_1 < x_2 < x_3 < b$, and finally the 2-point (respectively, 3-1, 2-2, and 1-3) conjugate boundary conditions,

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y''(x_1) = y_3,\ y(x_2) = y_4, \tag{2.14}$$

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y'(x_2) = y_4, \tag{2.15}$$

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_2) = y_3,\ y''(x_2) = y_4, \tag{2.16}$$

where $a < x_1 < x_2 < b$, and in each case $y_1, y_2, y_3, y_4 \in \mathbb{R}$.

In this chapter concerning (2.1), we will assume the following conditions.

(A) $f : (a, b) \times \mathbb{R}^4 \to \mathbb{R}$ is continuous.

(B) Solutions of initial value problems for (2.1) are unique and exist on all of (a, b).

(C) Given $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, if $y(x)$ and $z(x)$ are two solutions of (2.1) satisfying

$$y(x_1) = z(x_1),\ y(x_2) = z(x_2),\ y(x_3) = z(x_3),\ y(x_4) - y(x_5) = z(x_4) - z(x_5),$$

then $y(x) = z(x),\ a < x < b$.

(D) Given $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, if $y(x)$ and $z(x)$ are two solutions of (2.1) satisfying

$$y(x_1) - y(x_2) = z(x_1) - z(x_2),\ y(x_3) = z(x_3),\ y(x_4) = z(x_4),\ y(x_5) = z(x_5),$$

then $y(x) = z(x),\ a < x < b$.

We now state The Brouwer Theorem on Invariance of Domain [58], and the Kamke Convergence Theorem [16].

Theorem 2.1. *(Brouwer Invariance of Domain Theorem) If $\varphi : G \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuous and one to one, and if G is an open set, then $\varphi(G)$ is an open set and φ is homeomorphism.*

Theorem 2.2. *(Kamke Convergence Theorem) Assume that in the equation*

$$y^{(n)} = g_k(x, y, y', ..., y^{(n-1)}), \ k = 0, 1, 2, ..., \qquad (n)_k$$

the functions $g_k(x, u_1, u_2, ..., u_n)$ *are continuous on* $I \times \mathbb{R}^n$*, where* I *is an interval of the reals, assume that solutions of initial value problems are unique, and assume that*

$$\lim_{k\to\infty} g_k(x, u_1, u_2, ..., u_n) = g_0(x, u_1, u_2, ..., u_n)$$

uniformly on each compact subset of $I \times \mathbb{R}^n$*. Assume that* $\{(x_k, y_{1k}, ..., y_{nk})\}_{k=0}^{\infty}$ *is a sequence in* $I \times \mathbb{R}^n$ *with*

$$\lim_{k\to\infty} (x_k, y_{1k}, y_{2k}, ..., y_{nk}) = (x_0, y_{10}, y_{20}, ..., y_{n0}).$$

For each $k \leq 1$*, let* $y_k(x)$ *be a solution of* $(n)_k$ *satisfying* $y_k^{(i-1)}(x_k) = y_{ik}, \ 1 \leq i \leq n$*, defined on its maximal interval* $I_k \subset I$ *with* $x_k \in I_k$*. Let* $y_0(x)$ *be the solution of* $(n)_0$ *satisfying* $y_0^{(i-1)}(x_0) = y_{i0}, \ 1 \leq i \leq n$*, on its maximal interval* $I_0 \subset I$*. Then for any compact interval* $[c, d] \subset I_0$*, it follows that* $[c, d] \subset I_k$*, for all sufficiently large* k *and* $\lim_{k\to\infty} y_k^{(i-1)}(x) = y_0^{(i-1)}(x)$ *uniformly on* $[c, d]$*, for each* $1 \leq i \leq n$*.*

We now present our first continuous dependence result.

Theorem 2.3. *Assume (A), (B), (C), and let* $z(x)$ *be an arbitrary solution of (2.1). Then, for any* $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ *and* $a < c < x_1$*, and* $x_5 < d < b$*, and given any* $\epsilon > 0$*, there exists* $\delta(\epsilon, [c, d]) > 0$*, so that* $|x_i - t_i| < \delta, \ 1 \leq i \leq 5, \ |z(x_i) - y_i| < \delta, \ i = 1, 2, 3,$ *and* $|z(x_4) - z(x_5) - y_4| < \delta$ *imply that (2.1) has a solution* $y(x)$ *with*

$$y(t_i) = y_i, \quad i = 1, 2, 3,$$

$$y(t_4) - y(t_5) = y_4,$$

and $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$ *on* $[c, d]$*,* $i = 1, 2, 3, 4.$

Proof. Fix a point $t_0 \in (a,b)$. Define the open subset $G \subset \mathbb{R}^9$ by,

$G = \{(t_1,t_2,t_3,t_4,t_5,c_1,c_2,c_3,c_4) | a < t_1 < t_2 < t_3 < t_4 < t_5 < b$ and $c_1,c_2,c_3,c_4 \in \mathbb{R}\}$.

For

$$m = (t_1,t_2,t_3,t_4,t_5,c_1,c_2,c_3,c_4) \in G,$$

then define a mapping $\varphi : G \to \mathbb{R}^9$ by

$$\varphi(m) = (t_1,t_2,t_3,t_4,t_5,y(t_1),y(t_2),y(t_3),y(t_4)-y(t_5))$$

where $y(x)$ is the solution of the (2.1) with

$$y^{(i-1)}(t_0) = c_i,\ \ i = 1,2,3,4.$$

It follows from Theorem 2.2 that solutions of initial value problems for (2.1) depend continuously on initial conditions. Consequently φ is a continuous function.

We claim that φ is one to one. In that direction, assume that

$$\varphi(s_1,s_2,s_3,s_4,s_5,d_1,d_2,d_3,d_4) = \varphi(t_1,t_2,t_3,t_4,t_5,c_1,c_2,c_3,c_4).$$

Then

$$s_i = t_i,\ \ i = 1,2,3,4,5,$$

$$y(s_i) = y(t_i) = w(t_i),\ \ i = 1,2,3,$$

$$y(t_4) - y(t_5) = w(t_4) - w(t_5),$$

where $y(x)$ and $w(x)$ are solutions of (2.1) with

$$y^{(i-1)}(t_0) = \ d_i, i = 1,2,3,4,$$

$$w^{(i-1)}(t_0) = \ c_i, i = 1,2,3,4.$$

By assumptions (B) and (C), we have $y(x) = w(x)$ on (a,b), which implies $d_i = c_i,\ \ i = 1,2,3,4$. Therefore φ is one to one. So by the Brouwer Invariance of Domain Theorem, we have $\varphi(G)$ is open and φ^{-1} is continuous on $\varphi(G)$.

We shall show that the theorem is true through the use of continuity of φ^{-1}. So let $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ be chosen, and let $a < c < x_1,\ x_5 < d < b$, and $\epsilon > 0$ be given. By continuity respect to initial conditions, there exists an $\eta > 0$ such that if $|z^{(i-1)}(t_0) - c_i| < \eta,\ i = 1, 2, 3, 4$, then

$$|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon \text{ on } [c, d],\ i = 1, 2, 3, 4,$$

where $y(x)$ is the solution of (2.1) with $y^{(i-1)}(t_0) = c_i,\ i = 1, 2, 3, 4$.

Now since $(x_1, x_2, x_3, x_4, x_5, z(x_1), z(x_2), z(x_3), z(x_4) - z(x_5)) \in \varphi(G)$ and $\varphi(G)$ is open, there exists a $\delta > 0$ such that, if

$$|t_i - x_i| < \delta, \quad i = 1, 2, 3, 4, 5,$$

$$|y_i - z(x_i)| < \delta, \quad i = 1, 2, 3, 4,$$

$$|z(x_4) - z(x_5) - y_4| < \delta,$$

then we have $(t_1, t_2, t_3, t_4, t_5, y_1, y_2, y_3, y_4) \in \varphi(G)$, and by the continuity of φ^{-1}, we have $\varphi^{-1}(t_1, t_2, t_3, t_4, t_5, y_1, y_2, y_3, y_4)$ belongs to the open cube of half-edge η centered at

$$\varphi^{-1}(x_1, x_2, x_3, x_4, x_5, z(x_1), z(x_2), z(x_3), z(x_4) - z(x_5))$$
$$= (x_1, x_2, x_3, x_4, x_5, z(t_0), z'(t_0), z''(t_0), z'''(t_0)).$$

Say that $\varphi^{-1}(t_1, t_2, t_3, t_4, t_5, y_1, y_2, y_3, y_4) = (t_1, t_2, t_3, t_4, t_5, d_1, d_2, d_3, d_4)$. So, from above there is a solution $y(x)$ of (2.1) satisfying

$$y^{(i-1)}(t_0) = d_i, \quad i = 1, 2, 3, 4, \text{ and}$$
$$|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon \text{ on } [c, d], \quad i = 1, 2, 3, 4.$$

Moreover

$$\begin{aligned}(t_1, t_2, t_3, t_4, t_5, y(t_1), y(t_2), y(t_3), y(t_4) - y(t_5)) &= \varphi(t_1, t_2, t_3, t_4, t_5, d_1, d_2, d_3, d_4) \\ &= (t_1, t_2, t_3, t_4, t_5, y_1, y_2, y_3, y_4).\end{aligned}$$

In particular,

$$y(t_i) = y_i, \ i = 1, 2, 3,$$

$$y(t_4) - y(t_5) = y_4,$$

and $|y^{(i-1)}(x) - z^{(i-1)}z(x)| < \epsilon$ on $[c, d], \ i = 1, 2, 3, 4.$

The proof is complete. □

Motivation for the main results of this chapter is the following result by Peterson [56].

Theorem 2.4. *Assume (A) and (B) are satisfied, and in addition that solutions of 4-point conjugate boundary value problems (2.1):(2.10) are unique, when they exist. Then solutions of 3-point and 2-point conjugate boundary value problems for (2.1) are unique, when they exist.*

It follows that, if in addition to (A) and (B), condition (C) is also assumed then solutions of (2.1):(2.10) are unique. In fact, we prove this now.

Theorem 2.5. *Assume conditions (A), (B) and (C) are satisfied. Then solutions of conjugate boundary value problems for (2.1) are unique, when they exist.*

Proof. In view of Theorem 2.4, it suffices to prove that solutions of 4-point conjugate boundary value problems (2.1):(2.10) are unique.

Assume for the sake of contradiction that there exist distinct solutions $y(x)$ and $z(x)$ of (2.1) and successive points $a < x_1 < x_2 < x_3 < x_4 < b$ so that

$$y(x_i) = z(x_i), \quad i = 1, 2, 3, 4,$$

and $y(x) \neq z(x)$, for all $x \in (x_1, x_4) \setminus \{x_2, x_3\}$.

Assume without loss of generality that $y(x) > z(x)$ on (x_3, x_4). Then, the function $w(x) = y(x) - z(x)$ has a local positive maximum at some point $c \in (x_3, x_4)$, and $w(x_3) = w(x_4) = 0$. By continuity, there exist points $x_3 < \tau_1 < c < \tau_2 < x_4$ so that $w(\tau_1) = w(\tau_2)$. In particular, we have

$$y(x_1) = z(x_1),$$

$$y(x_2) = z(x_2),$$

$$y(x_3) = z(x_3),$$

and

$$y(\tau_1) - y(\tau_2) = z(\tau_1) - z(\tau_2).$$

By (C), $y(x) = z(x)$ on (a, b), which is a contradiction.

The theorem is proved. □

We now proceed to show that the assumptions of Theorem 2.3 also yield uniqueness of the 4-point and 3-point nonlocal boundary value problems for (2.1).

Theorem 2.6. *Assume (A), (B) and (C) are satisfied. Then solutions of (2.1):(2.4) are unique when they exist.*

Proof. Suppose (2.1):(2.4) has two solutions $y(x)$ and $z(x)$, and let us say,

$$z(x_1) = y(x_1),$$

$$z'(x_1) = y'(x_1),$$

$$z(x_2) = y(x_2),$$

$$z(x_3) - z(x_4) = y(x_3) - y(x_4),$$

for some $a < x_1 < x_2 < x_3 < x_4 < b$. By uniqueness of conjugate boundary value problems (2.1):(2.14) and (2.1):(2.15), respectively, $z''(x_1) \neq y''(x_1)$ and $z'(x_2) \neq y'(x_2)$.

Without loss of generality, we assume $y(x) > z(x)$ on $(a, x_2)\backslash\{x_1\}$. Then $y(x) < z(x)$ on (x_2, b). Fix $a < \tau < x_1$. By Theorem 2.3, for $\epsilon > 0$ sufficiently small, there exist a $\delta > 0$ and a solution $z_\delta(x)$ of (2.1) satisfying

$$\begin{aligned} z_\delta(\tau) &= z(\tau), \\ z_\delta(x_1) &= z(x_1) + \delta, \\ z_\delta(x_2) &= z(x_2) = y(x_2), \\ z_\delta(x_3) - z_\delta(x_4) &= z(x_3) - z(x_4) \\ &= y(x_3) - y(x_4), \end{aligned}$$

and $|z_\delta^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon, i = 1, 2, 3, 4,$ on $[\tau, x_4]$. For ϵ small, there exists $\tau < \sigma_1 < x_1 < \sigma_2 < x_2$ so that

$$\begin{aligned} z_\delta(\sigma_1) &= y(\sigma_1), \\ z_\delta(\sigma_2) &= y(\sigma_2), \\ z_\delta(x_2) &= = y(x_2), \\ z_\delta(x_3) - z_\delta(x_4) &= y(x_3) - y(x_4). \end{aligned}$$

By assumption (C), $z_\delta(x) = y(x)$ on (a, b). However, $z_\delta(x_1) = z(x_1) + \delta = y(x_1) + \delta > y(x_1)$, which is a contradiction.

So solutions of (2.1):(2.4) are unique. $\square$

Having established uniqueness of (2.1):(2.4), we now exhibit that such solutions depend continuously on boundary conditions.

Theorem 2.7. *Assume (A), (B), (C), and let $z(x)$ be an arbitrary solution of (2.1). Then, for any $a < x_1 < x_2 < x_3 < x_4 < b$ and $a < c < x_1$, and $x_4 < d < b$, and given any $\epsilon > 0$, there exists $\delta(\epsilon, [c, d]) > 0$, so that $|x_i - t_i| < \delta,\ 1 \le i \le 4,\ |z^{(i-1)}(x_1) - y_i| < \delta,\ i = 1, 2,\ |z(x_2) - y_3| < \delta$ and $|z(x_3) - z(x_4) - y_4| < \delta$ imply that (2.1) has a solution*

$y(x)$ with

$$y^{(i-1)}(t_1) = y_i, \quad i = 1, 2,$$

$$y(t_2) = y_3,$$

$$y(t_3) - y(t_4) = y_4,$$

and $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$ *on* $[c, d]$, $i = 1, 2, 3, 4.$

Proof. Fix a point $t_0 \in (a, b)$. Define the open subset $G \subset \mathbb{R}^8$ by,

$$G = \{(t_1, t_2, t_3, t_4, c_1, c_2, c_3, c_4) | a < t_1 < t_2 < t_3 < t_4 < b \text{ and } c_1, c_2, c_3, c_4 \in \mathbb{R}\}.$$

For $m = (t_1, t_2, t_3, t_4, c_1, c_2, c_3, c_4) \in G$, then define a mapping $\varphi : G \to \mathbb{R}^8$ by

$$\varphi(m) = (t_1, t_2, t_3, t_4, y(t_1), y'(t_1), y(t_2), y(t_3) - y(t_4)),$$

where $y(x)$ is the solution of the (2.1) with

$$y^{(i-1)}(t_0) = c_i, i = 1, 2, 3, 4.$$

It follows from Theorem 2.2 that solutions of initial value problems for (2.1) depend continuously on initial conditions. Consequently φ is a continuous function.

We claim that φ is one to one. Assume that

$$\varphi(s_1, s_2, s_3, s_4, d_1, d_2, d_3, d_4) = \varphi(t_1, t_2, t_3, t_4, c_1, c_2, c_3, c_4).$$

Then

$$s_i = t_i, \; i = 1, 2, 3, 4,$$

$$y(s_i) = y(t_i) = w(t_i), \; i = 1, 2,$$

$$y'(s_1) = y'(t_1) = w'(t_1),$$

$$y(t_3) - y(t_4) = w(t_3) - w(t_4),$$

where $y(x)$ and $w(x)$ are solutions of (2.1) with

$$y^{(i-1)}(t_0) = d_i, i = 1, 2, 3, 4,$$
$$w^{(i-1)}(t_0) = c_i, i = 1, 2, 3, 4.$$

By Theorem 2.6 we have $y(x) = w(x)$ on (a, b), which implies $d_i = c_i,\ i = 1, 2, 3, 4$. Therefore φ is one to one. So by the Brouwer Invariance of Domain Theorem, we have $\varphi(G)$ is open and φ^{-1} is continuous on $\varphi(G)$.

We shall show that the theorem is true through the use of continuity of φ^{-1}. So let $a < x_1 < x_2 < x_3 < x_4 < b$ be chosen, and let $a < c < x_1,\ x_4 < d < b$, and $\epsilon > 0$ be given. By continuity respect to initial conditions, then there exists an $\eta > 0$ such that if $|z^{(i-1)}(t_0) - c_i| < \eta,\ i = 1, 2, 3, 4$, then

$$|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon \text{ on } [c, d],\ i = 1, 2, 3, 4,$$

where $y(x)$ is the solution of (2.1) with $y^{(i-1)}(t_0) = c_i,\ i = 1, 2, 3, 4$.

Now since $(x_1, x_2, x_3, x_4, z(x_1), z'(x_1), z(x_2), z(x_3) - z(x_4)) \in \varphi(G)$ and $\varphi(G)$ is open, then there exists a $\delta > 0$ such that, if

$$|t_i - x_i| < \delta, \quad i = 1, 2, 3, 4,$$
$$|z^{(i-1)}(x_i) - y_i| < \delta, \quad i = 1, 2,$$
$$|z(x_2) - y_3| < \delta,$$
$$|z(x_3) - z(x_4) - y_4| < \delta,$$

then we have $(t_1, t_2, t_3, t_4, y_1, y_2, y_3, y_4) \in \varphi(G)$, and by the continuity of φ^{-1}, we have $\varphi^{-1}(t_1, t_2, t_3, t_4, y_1, y_2, y_3, y_4)$ belongs to the open cube of half-edge η centered at

$$\varphi^{-1}(x_1, x_2, x_3, x_4, z(x_1), z'(x_1), z(x_2), z(x_3) - z(x_4))$$
$$= (x_1, x_2, x_3, x_4, z(t_0), z'(t_0), z''(t_0), z'''(t_0)).$$

Say that $\varphi^{-1}(t_1, t_2, t_3, t_4, y_1, y_2, y_3, y_4) = (t_1, t_2, t_3, t_4, d_1, d_2, d_3, d_4)$. So, from above there is a solution $y(x)$ of (2.1) satisfying

$$y^{(i-1)}(t_0) = d_i, \quad i = 1, 2, 3, 4, \text{ and}$$
$$|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon \text{ on } [c, d], \quad i = 1, 2, 3, 4.$$

Moreover

$$\begin{aligned}(t_1, t_2, t_3, t_4, y(t_1), y'(t_1), y(t_2), y(t_3) - y(t_4)) &= \varphi(t_1, t_2, t_3, t_4, d_1, d_2, d_3, d_4) \\ &= (t_1, t_2, t_3, t_4, y_1, y_2, y_3, y_4).\end{aligned}$$

In particular,

$$y(t_1) = y_1,$$
$$y'(t_1) = y_2,$$
$$y(t_2) = y_3,$$
$$y(t_3) - y(t_4) = y_4,$$

and $|y^{(i-1)}(x) - z^{(i-1)}z(x)| < \epsilon$ on $[c, d]$, $i = 1, 2, 3, 4$.

The proof is complete. $\square$

Theorem 2.8. *Assume (A), (B) and (C) are satisfied. Then solutions of (2.1):(2.6) are unique when they exist.*

Proof. Suppose (2.1):(2.6) has two solutions $y(x)$ and $z(x)$, and let us say,

$$z(x_1) = y(x_1),$$
$$z(x_2) = y(x_2),$$
$$z'(x_2) = y'(x_2),$$
$$z(x_3) - z(x_4) = y(x_3) - y(x_4),$$

for some $a < x_1 < x_2 < x_3 < x_4 < b$. By uniqueness of conjugate boundary value problems (2.1):(2.15) and (2.1):(2.16), respectively, $z'(x_1) \neq y'(x_1)$ and $z''(x_2) \neq y''(x_2)$.

Without loss of generality, we assume $y(x) > z(x)$ on $(x_1, b)\backslash\{x_2\}$. Then $y(x) < z(x)$ on (a, x_1). Fix $x_1 < \tau < x_2$. By Theorem 2.3, for $\epsilon > 0$ sufficiently small, there exists a $\delta > 0$ and a solution $z_\delta(x)$ of (2.1) satisfying

$$\begin{aligned} z_\delta(x_1) &= z(x_1) = y(x_1), \\ z_\delta(\tau) &= z(\tau), \\ z_\delta(x_2) &= z(x_2) + \delta, \\ z_\delta(x_3) - z_\delta(x_4) &= z(x_3) - z(x_4) \\ &= y(x_3) - y(x_4), \end{aligned}$$

and $|z_\delta^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon, i = 1, 2, 3, 4,$ on $[\tau, x_4]$. For ϵ small, there exists $x_1 < \sigma_1 < x_2 < \sigma_2 < x_4$ so that

$$\begin{aligned} z_\delta(x_1) &= y(x_1), \\ z_\delta(\sigma_1) &= y(\sigma_1), \\ z_\delta(\sigma_2) &= y(\sigma_2), \\ z_\delta(x_3) - z_\delta(x_4) &= y(x_3) - y(x_4). \end{aligned}$$

By assumption (C), $z_\delta(x) = y(x)$ on (a, b). However, $z_\delta(x_2) = z(x_2) + \delta = y(x_2) + \delta > y(x_2)$, which is a contradiction.

So solutions of (2.1):(2.6) are unique. □

Having established uniqueness of solutions of (2.1):(2.6), we now exhibit that such solutions depend continuously on boundary conditions.

Theorem 2.9. *Assume (A), (B), (C), and let $z(x)$ be an arbitrary solution of (2.1). Then, for any $a < x_1 < x_2 < x_3 < x_4 < b$ and $a < c < x_1$, and $x_4 < d < b$, and given* *any* $\epsilon > 0$, *there exists* $\delta(\epsilon, [c,d]) > 0$, *so that* $|x_i - t_i| < \delta,\ 1 \le i \le 4,\ |z(x_i) - y_i| < \delta,\ i = 1,2,\ |z'(x_2) - y_3| < \delta$ *and* $|z(x_3) - z(x_4) - y_4| < \delta$ *imply that (2.1) has a solution* $y(x)$ *with*

$$y(t_i) = y_i, \quad i = 1,2,$$

$$y'(t_2) = y_3,$$

$$y(t_3) - y(t_4) = y_4,$$

and $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$ *on* $[c,d],\ i = 1,2,3,4.$

Proof. Fix a point $t_0 \in (a,b)$. Define the open subset $G \subset \mathbb{R}^8$ by,

$$G = \{(t_1,t_2,t_3,t_4,c_1,c_2,c_3,c_4) | a < t_1 < t_2 < t_3 < t_4 < b \text{ and } c_1,c_2,c_3,c_4 \in \mathbb{R}\}.$$

For $m = (t_1,t_2,t_3,t_4,c_1,c_2,c_3,c_4) \in G$, then define a mapping $\varphi : G \to \mathbb{R}^8$ by

$$\varphi(m) = (t_1,t_2,t_3,t_4,y(t_1),y(t_2),y'(t_2),y(t_3) - y(t_4)),$$

where $y(x)$ is the solution of the (2.1) with

$$y^{(i-1)}(t_0) = c_i, i = 1,2,3,4.$$

It follows from Theorem 2.2 that solutions of initial value problems for (2.1) depend continuously on initial conditions. Consequently φ is a continuous function.

We claim that φ is one to one. Assume that

$$\varphi(s_1,s_2,s_3,s_4,d_1,d_2,d_3,d_4) = \varphi(t_1,t_2,t_3,t_4,c_1,c_2,c_3,c_4).$$

Then

$$s_i = t_i,\ i = 1,2,3,4,$$

$$y(s_i) = y(t_i) = w(t_i),\ i = 1,2,$$

$$y'(s_2) = y'(t_2) = w'(t_2),$$

$$y(t_3) - y(t_4) = w(t_3) - w(t_4),$$

where $y(x)$ and $w(x)$ are solutions of (2.1) with

$$y^{(i-1)}(t_0) = \; d_i, i = 1, 2, 3, 4,$$
$$w^{(i-1)}(t_0) = \; c_i, i = 1, 2, 3, 4.$$

By Theorem 2.8 we have $y(x) = w(x)$ on (a, b), which implies $d_i = c_i, \; i = 1, 2, 3, 4$. Therefore φ is one to one. So by the Brouwer Invariance of Domain Theorem, we have $\varphi(G)$ is open and φ^{-1} is continuous on $\varphi(G)$.

We shall show that the theorem is true through the use of continuity of φ^{-1}. So let $a < x_1 < x_2 < x_3 < x_4 < b$ be chosen, and let $a < c < x_1, \; x_4 < d < b$, and $\epsilon > 0$ be given. By continuity respect to initial conditions, then there exists an $\eta > 0$ such that if $|z^{(i-1)}(t_0) - c_i| < \eta, \; i = 1, 2, 3, 4$, then

$$|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon \text{ on } [c, d], \; i = 1, 2, 3, 4,$$

where $y(x)$ is the solution of (2.1) with $y^{(i-1)}(t_0) = c_i, \; i = 1, 2, 3, 4$.

Now since $(x_1, x_2, x_3, x_4, z(x_1), z(x_2), z'(x_2), z(x_3) - z(x_4)) \in \varphi(G)$ and $\varphi(G)$ is open, then there exists a $\delta > 0$ such that, if

$$|t_i - x_i| < \delta, \quad i = 1, 2, 3, 4,$$
$$|y_i - z(x_i)| < \delta, \quad i = 1, 2,$$
$$|z'(x_2) - y_3| < \delta,$$
$$|z(x_3) - z(x_4) - y_4| < \delta,$$

then we have $(t_1, t_2, t_3, t_4, y_1, y_2, y_3, y_4) \in \varphi(G)$, and by the continuity of φ^{-1}, we have $\varphi^{-1}(t_1, t_2, t_3, t_4, y_1, y_2, y_3, y_4)$ belongs to the open cube of half-edge η centered at

$$\varphi^{-1}(x_1, x_2, x_3, x_4, z(x_1), z(x_2), z'(x_2), z(x_3) - z(x_4))$$
$$= (x_1, x_2, x_3, x_4, z(t_0), z'(t_0), z''(t_0), z'''(t_0)).$$

Say that $\varphi^{-1}(t_1, t_2, t_3, t_4, y_1, y_2, y_3, y_4) = (t_1, t_2, t_3, t_4, d_1, d_2, d_3, d_4)$. So, from above there is a solution $y(x)$ of (2.1) satisfying

$$y^{(i-1)}(t_0) = d_i, \quad i = 1, 2, 3, 4, \text{ and}$$
$$|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon \text{ on } [c, d], \quad i = 1, 2, 3, 4.$$

Moreover

$$\begin{aligned}(t_1, t_2, t_3, t_4, y(t_1), y(t_2), y'(t_2), y(t_3) - y(t_4)) &= \varphi(t_1, t_2, t_3, t_4, d_1, d_2, d_3, d_4)\\ &= (t_1, t_2, t_3, t_4, y_1, y_2, y_3, y_4).\end{aligned}$$

In particular,

$$y(t_i) = y_i, \ i = 1, 2,$$
$$y'(t_2) = y_3,$$
$$y(t_3) - y(t_4) = y_4,$$

and $|y^{(i-1)}(x) - z^{(i-1)}z(x)| < \epsilon$ on $[c, d]$, $i = 1, 2, 3, 4$.

The proof is complete. □

We now establish that under the uniqueness condition of (C), we also have uniqueness of solutions of 3-point nonlocal boundary value problems.

Theorem 2.10. *Assume (A), (B) and (C) are satisfied. Then solutions of (2.1):(2.8) are unique when they exist.*

Proof. Suppose (2.1):(2.8) has two solutions $y(x)$ and $z(x)$ satisfying

$$y(x_1) = z(x_1), \ y'(x_1) = z'(x_1), \ y''(x_1) = z''(x_1), \ y(x_2) - y(x_3) = z(x_2) - z(x_3),$$

for some $a < x_1 < x_2 < x_3 < b$. Now $y'''(x_1) \neq z'''(x_1)$, and we may assume $y'''(x_1) > z'''(x_1)$.

By Theorem 2.7, solutions of (2.1):(2.4) depend continuously on their boundary conditions. Fix $x_1 < \rho < x_2$. For $\epsilon > 0$ small, there is a $\delta > 0$ and a solution $z_\delta(x)$

satisfying

$$\begin{aligned} z_\delta(x_1) &= z(x_1) = y(x_1), \\ z'_\delta(x_1) &= z'(x_1) + \delta, \\ z_\delta(\rho) &= z(\rho), \\ z_\delta(x_2) - z_\delta(x_3) &= z(x_2) - z(x_3) \\ &= y(x_2) - y(x_3). \end{aligned}$$

and $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon,\ i = 1, 2, 3, 4,$ on $[x_1, x_3]$. For ϵ sufficiently small, there exist points $a < \tau_1 < x_1 < \tau_2 < \rho$, which are in a neighborhood of x_1, such that $y(x)$ and $z_\delta(x)$ both satisfy,

$$\begin{gathered} z_\delta(\tau_1) = y(\tau_1), \\ z_\delta(x_1) = y(x_1), \\ z_\delta(\tau_2) = y(\tau_2), \\ z_\delta(x_2) - z_\delta(x_3) = y(x_2) - y(x_3). \end{gathered}$$

So we have $z_\delta(x) = y(x)$ on (a, b) by hypothesis (C). But

$$z'_\delta(x_1) = z'(x_1) + \delta = y'(x_1) + \delta > y'(x_1).$$

This is a contradiction. So (2.1):(2.8) has at most one solution. □

As in the previous cases, once uniqueness of solutions of (2.1):(2.8) has been established, we have a result for continuous dependence of solutions on boundary conditions. We omit the proof.

Theorem 2.11. *Assume (A), (B), (C), and let* $z(x)$ *be an arbitrary solution of (2.1). Then, for any* $a < x_1 < x_2 < x_3 < b$ *and* $a < c < x_1$, *and* $x_3 < d < b$, *and given any* $\epsilon > 0$, *there exists* $\delta(\epsilon, [c, d]) > 0$, *so that* $|x_i - t_i| < \delta,\ 1 \le i \le 3,\ |z^{(i-1)}(x_1) - y_i| <$

$\delta,\ i = 1, 2, 3,$ *and* $|z(x_2) - z(x_3) - y_4| < \delta$ *imply that (2.1) has a solution* $y(x)$ *with*

$$y^{(i-1)}(t_1) = y_i, \quad i = 1, 2, 3,$$

$$y(t_2) - y(t_3) = y_4,$$

and $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$ *on* $[c, d],\ i = 1, 2, 3, 4.$

In terms of the uniqueness condition (D), there are dual uniqueness results which we now state.

Theorem 2.12. *Assume (A), (B) and (D) are satisfied. Then solutions of (2.1):(2.5) are unique when they exist.*

Theorem 2.13. *Assume (A), (B) and (D) are satisfied. Then solutions of (2.1):(2.7) are unique when they exist.*

Theorem 2.14. *Assume (A), (B) and (D) are satisfied. Then solutions of (2.1):(2.9) are unique when they exist.*

We conclude this chapter by noting that in the presence of assumption (C) and (D), solutions of (2.1) satisfying any of (2.2)–(2.9) depend continuously on boundary conditions. Verification follows along the lines of Theorem 2.3, Theorem 2.7 and Theorem 2.9. This is due to the uniqueness of these solutions.

CHAPTER THREE

Uniqueness 2

This chapter is devoted to, in some sense, a question converse to Theorems 2.6, 2.8 and 2.10. In particular, in those theorems, we proved that, under hypothesis (C) and (D), solutions of 4-point and 3-point nonlocal boundary value problems for (2.1) are unique, when they exist. Put more clearly, if solutions of (2.1):(2.2) and (2.1):(2.3) are unique, then solutions of (2.1) satisfying any of (2.4)-(2.9) are unique.

In this chapter, our assumptions will be on uniqueness of solutions of 4-point and 3-point nonlocal boundary value problems to establish uniqueness of solutions of 5-point nonlocal boundary value problems for the equation (2.1). Relative to equation (2.1), we again assume the conditions:

(A) $f : (a, b) \times \mathbb{R}^4 \to \mathbb{R}$ is continuous.

(B) Solutions of initial value problems for (2.1) are unique and exist on all of (a, b).

Fundamental to our arguments is a Kamke type of convergence result for boundary value problems due to Vidossich [60], as well as a precompactness condition on bounded sequences of solutions of (2.1) due to to Jackson and Schrader; see Agarwal [1].

Theorem 3.1. *(Vidossich) For each* $n > 0$, *let* $g_n : [c, d] \times \mathbb{R}^N \to \mathbb{R}$ *be continuous, let* $L_n : C([c, d] \times \mathbb{R}^N, \mathbb{R}) \to \mathbb{R}^N$ *be continuous, and let* $r_n \in \mathbb{R}^N$. *Assume that*
(a) $\lim_n r_n = r_0$;
(b) $\lim_n g_n = g_0$ *and* $\lim_n L_n = L_0$ *uniformly on compact subsets of* $[c, d] \times \mathbb{R}^N$, *respectively;*

(c) Each initial value problem

$$x' = g_n(t,x), \quad x(a) = u,$$

has at most one local solution for $u \in \mathbb{R}^N$*;*

(d) The functional boundary value problem

$$x' = g_0(t,x), \quad L_0(x) = r,$$

has at most one solution for each $r \in \mathbb{R}^N$*.*

Let x_0 *be the solution to* $x' = g_0(t,x)$, $L_0(x) = r_0$*. Then for each* $\epsilon > 0$*, there exists* n_ϵ *such that the functional boundary value problem,*

$$x' = g_n(t,x), \quad L_n(x) = r_n,$$

has a solution x_n *, for* $n > n_\epsilon$*, satisfying the condition*

$$\|x_0 - x_n\|_\infty < \epsilon.$$

Theorem 3.2. *(Jackson-Schrader) Assume that with respect to (2.1), conditions (A) and (B) hold. In addition, assume that solutions of 4-point conjugate boundary value problems (2.1):(2.10) are unique. If* $\{y_k(x)\}$ *is sequence of solutions of (2.1) for which there exists an interval* $[c,d] \subset (a,b)$ *and there exists an* $M > 0$ *such that* $|y_k(x)| < M$*, for all* $x \in [c,d]$ *and for all* $k \in \mathbb{N}$*, then there exists a subsequence* $\{y_{k_j}(x)\}$ *such that, for* $i = 0,1,2,3$*,* $\{y_{k_j}^{(i)}(x)\}$ *converges uniformly on each compact subinterval of* (a,b)*.*

In the context of conjugate boundary value problems for (2.1), Peterson [56] established a converse to Theorem 2.4.

Theorem 3.3. *Assume with respect to (2.1), conditions (A) and (B) are satisfied. Assume also that solutions of (2.1) satisfying 2-point conjugate boundary conditions (i.e. (2.14), (2.15), (2.16)), are unique, when they exist. Then solutions of 4-point and 3-point conjugate boundary value problem for (2.1) (i.e. (2.10), (2.11), (2.12), (2.13)) are unique when they exist.*

We remark that Peterson also proved a uniqueness implies existence result.

Theorem 3.4. *Assume with respect to (2.1), conditions (A) and (B) are satisfied. Assume also that solutions of (2.1):(2.10) are unique, when they exist. Then there exist solutions of 4-point, 3-point and 2-point conjugate boundary value problems for (2.1). (i.e. for any of (2.11), ..., (2.16))*

We also mention here a uniqueness result due to Henderson and Jackson [25].

Theorem 3.5. *Assume (A) and (B) and uniqueness of solutions of 3-point conjugate boundary value problems for (2.1). Then solutions of 2-point conjugate boundary value problems for (2.1) are unique.*

As a consequence, we have a summary statement.

Theorem 3.6. *Assume (A)and (B) are satisfied and that given* $k \in \{2, 3, 4\}$ *solutions of k-point conjugate boundary value problems for (2.1) are unique. Then, each 2-point, 3-point and 4-point conjugate boundary value problems for (2.1) has a unique solution.*

We finally remark that, if solutions of (2.1):(2.4), (2.1):(2.5) and (2.1):(2.6), (2.1):(2.7) are unique, then solutions, by the Mean Value Theorem, of 3-point conjugate boundary value problems are unique.

We now provide a type of converse to Theorems 2.6, 2.8 and 2.10.

Theorem 3.7. *Assume (A) and (B)are satisfied. Assume solutions of (2.1) satisfying any of (2.4)-(2.7)are unique, when they exist. Then the solutions of (2.1):(2.2) and (2.1):(2.3) are unique.*

Proof. We establish the result for only (2.1):(2.2). Suppose (2.1):(2.2) has two distinct solutions $y(x)$ and $z(x)$, for some $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ and some $y_1, y_2, y_3, y_4 \in \mathbb{R}$. That is,

$$y(x_i) = z(x_i),\ i = 1, 2, 3,\ \text{and}\ y(x_4) - y(x_5) = z(x_4) - z(x_5).$$

By assumptions (A) and (B) and uniqueness of solutions of 4-point nonlocal boundary value problems, (2.1):(2.4), (2.1):(2.5), (2.1):(2.6) and (2.1)(2.7), we know solutions of all conjugate boundary problems for (2.1) are unique, and hence all conjugate boundary problems have unique solutions.

For each $n \geq 1$, let $y_n(x)$ be the solution of the conjugate boundary value problems for (2.1) satisfying conditions of (2.11),

$$\begin{aligned} y_n(x_3) &= y(x_3) = z(x_3), \\ y_n'(x_3) &= y'(x_3) - n, \\ y_n(x_4) &= y(x_4), \\ y_n(x_5) &= y(x_5). \end{aligned}$$

It follows from uniqueness of solutions of 4-point conjugate problems that, for $n \geq 1$,

$$y(x) < y_n(x) < y_{n+1}(x)$$

on (a, x_3).

For each $n \geq 1$, let

$$E_n = \{x : x_1 \leq x \leq x_2 | \text{ where } y_n(x) \leq z(x)\}.$$

We claim that $E_n \neq \emptyset$, for each $n \geq 1$. In that direction, suppose there exists n_0 so that $E_{n_0} = \emptyset$. Then $y_{n_0}(x) > z(x)$ on $[x_1, x_2]$.

Next, for all $\epsilon \geq 0$, let y_ϵ be the solution of (2.1):(2.11) satisfying the 2-1-1 conjugate boundary conditions,

$$\begin{aligned} y_\epsilon(x_3) &= y(x_3) = z(x_3), \\ y_\epsilon'(x_3) &= y'(x_3) - \epsilon, \\ y_\epsilon(x_4) &= y(x_4), \\ y_\epsilon(x_5) &= y(x_5). \end{aligned}$$

Note when $\epsilon = 0,\ y_\epsilon(x) = y(x)$.

Define

$$S = \{\epsilon \geq 0 |\text{ for some } x_1 \leq x \leq x_2,\ y_\epsilon(x) \leq z(x)\},$$

$S \neq \emptyset$ since $0 \in S$. Now since $E_{n_0} = \emptyset$, S is bounded above.

Let $\epsilon_0 = \sup S$, and consider the solution, $y_{\epsilon_0}(x)$ of equation (2.1). We claim there exists $\tau \in (x_1, x_2)$ so that $y_{\epsilon_0}(\tau) \leq z(\tau)$. If not, then $y_{\epsilon_0}(x) > z(x)$, for all $x_1 \leq x \leq x_2$. By continuous dependence of solutions of (2.1):(2.11) on boundary conditions, there exists $0 < \epsilon_1 < \epsilon_0$, so that $y_{\epsilon_1}(x) > z(x)$ for all $x_1 \leq x \leq x_2$. Therefore ϵ_1 is an upper bound of S. But by assumption $\epsilon_0 = \sup S$, whereas $0 < \epsilon_1 < \epsilon_0$. This is a contradiction. Therefore there exists $\tau \in (x_1, x_2)$ so that $y_{\epsilon_0}(\tau) \leq z(\tau)$.

Next, if $y_{\epsilon_0}(\tau) < z(\tau)$, then by continuity, there exists an interval $[\tau - \rho, \tau + \rho]$ so that $y_{\epsilon_0}(x) < z(x)$ on $[\tau - \rho, \tau + \rho]$. So there exists $\epsilon_0 < \epsilon_2$ so that $y_{\epsilon_2}(x) \leq z(x)$, on some interval $[\tau - \eta, \tau + \eta] \subset [\tau - \rho, \tau + \rho] \subset [x_1, x_2]$. So $\epsilon_2 \in S$. But $\epsilon_2 > \epsilon_0$, and so we contradict that ϵ_0 is the least upper bound of S.

Now for this $\tau \in (x_1, x_2)$, $y_{\epsilon_0}(\tau) = z(\tau)$, and $y_{\epsilon_0}(x) \geq z(x)$ for all $x \in [x_1, x_2] \setminus \{\tau\}$.

In particular,

$$y_{\epsilon_0}(\tau) = z(\tau),$$

$$y'_{\epsilon_0}(\tau) = z'(\tau),$$

$$y_{\epsilon_0}(x_3) = z(x_3),$$

$$y_{\epsilon_0}(x_4) - y_{\epsilon_0}(x_5) = z(x_4) - z(x_5).$$

By the uniqueness of solutions of 4-point nonlocal conjugate boundary problems, we reach a contradiction. So $E_n \neq \emptyset$, for all $n \geq 1$.

Thus, $E_{n+1} \subset E_n \subset (x_1, x_2)$, for each $n \geq 1$, and each E_n is also compact. Hence,

$$\bigcap_{n=1}^{\infty} E_n := E \neq \emptyset.$$

Next, we observe that the set E consists of a single point $\{x_0\}$ with $x_1 < x_0 < x_2$. To see this, suppose there are points $t_1, t_2 \in E$ with $x_1 < t_1 < t_2 < x_2$.

We claim that the interval $[t_1, t_2] \subseteq E$. Suppose to the contrary that there exists $\tau \in (t_1, t_2)$ such that $\tau \notin E$. Then, there exists an $N \in \mathbb{N}$ such that, for each $n \geq N,\ y_n(\tau) > z(\tau)$. By continuity, there exists a $\lambda > 0$ such that, for each $n \geq N$,

$$z(x) < y_n(x) < y_{n+1}(x), \quad x \in [\tau - \lambda, \tau + \lambda].$$

With the solution $y_\epsilon(x)$ of (2.1) as defined above, we define a new set

$$S' = \{\epsilon \geq 0 |\ \text{for some } \tau - \lambda \leq x \leq \tau - \lambda,\ y_\epsilon(x) \leq z(x)\}.$$

Again $0 \in S'$, and so $S' \neq \emptyset$. In this case N is an upper bound of S'. We reach the same contradiction as above in showing the foregoing sets E_n are nonnull. We conclude that the interval $[t_1, t_2] \subseteq E$, and the claim is verified.

However, $[t_1, t_2] \subseteq E$ implies that the sequence $\{y_n(x)\}$ is uniformly bounded on $[t_1, t_2]$. It follows from Theorem 3.2 that there is a subsequence $\{y_{n_j}(x)\}$ such that for each $i = 0, 1, 2, 3,\ \{y_{n_j}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b). However,

$$\lim_{j\to\infty} y'_{n_j}(x_3) = \lim_{j\to\infty} y'(x_3) - n_j = -\infty;$$

this is a contraction.

Thus we conclude,

$$E = \{x_0\},$$

with $x_1 < x_0 < x_2$, and we also have

$$\lim_{n\to\infty} y_n(x_0) \leq z(x_0).$$

Now, let $y_0(x)$ be the solution of the 4-point conjugate boundary value problem for (2.1) satisfying

$$\lim_{n\to\infty} y_n(x_0) = y_0(x_0),\ y_0(x_3) = y(x_3) = z(x_3),\ y_0(x_4) = y(x_4),\ y_0(x_5) = y(x_5).$$

By the Vidossich Theorem, Theorem 3.1, $\{y_n^{(i)}(x)\}$ converges to $y_0^{(i)}(x),\ i = 0, 1, 2, 3,$ on each compact subinterval of (a, b).

So $y_0(x_0) \leq z(x_0)$, which we claim leads to contradictions. There are two cases to resolve. First, assume $y_0(x_0) = z(x_0)$. Then we have two solutions $y_0(x)$ and $z(x)$ of equation (2.1) satisfying

$$y_0(x_0) = z(x_0),\ y_0'(x_0) = z'(x_0),\ y_0(x_3) = z(x_3),$$
$$y_0(x_4) - y_0(x_5) = y(x_4) - y(x_5) = z(x_4) - z(x_5),$$

and so by uniqueness of 4-point nonlocal boundary value problems (2.1):(2.4), $y_0(x) \equiv z(x)$ on (a, b). This is a contradiction. So $\lim_{n\to\infty} y_n(x_0) \neq z(x_0)$.

The remaining case is that $y_0(x_0) < z(x_0)$. In this case, by the continuity of $y_0(x)$, there exists $\delta > 0$ with $[x_0 - \delta, x_0 + \delta] \subset (x_1, x_2)$ on which $y_0(x) < z(x)$. Since $\lim_n y(x) = y_0(x)$ uniformly on each compact subinterval of (a, b), it follows that $[x_0 - \delta, x_0 + \delta] \subset E$. This is a contradiction.

From this final contradiction, we conclude that $y_0(x_0) \leq z(x_0)$ is impossible.This resolves all situations, and we conclude solutions of (2.1):(2.2) are unique. Of course, completely symmetric arguments yield that solutions of (2.1):(2.3) are unique. $\square$

CHAPTER FOUR

Existence

Having established in Chapters Three and Four, under the assumptions (A) and (B), the equivalence of the uniqueness of solutions for (2.1):(2.2) and (2.1):(2.3) with that of the uniqueness of solutions for (2.1):(2.4) - (2.1):(2.9), we now deal with uniqueness of solutions implying their existence for these problems.

As was discussed in great detail in Chapter One, much study has been devoted to uniqueness implies existence questions for boundary value problems, with the first work by Lasota and Opial [42], and then followed by landmark papers by Hartman [17] and Klaasen [37] for conjugate problems. Later, Henderson [19] obtained close analogues for right focal boundary value problems. Since then, similar questions have been resolved in the context of finite difference equations as well as dynamic equations on time scales; see [6], [26] and [31, 32].

For the results of this chapter, continuous dependence, as in Theorems 2.3 and others, plays a role, as does the precompactness condition in Theorem 3.2. We state here for convenience our assumptions for this chapter.

(A) $f : (a, b) \times \mathbb{R}^4 \to \mathbb{R}$ is continuous.

(B) Solutions of initial value problems for (2.1) are unique and exist on all of (a, b).

(C) Given $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, if $y(x)$ and $z(x)$ are two solutions of (2.1) satisfying

$$y(x_1) = z(x_1),\ y(x_2) = z(x_2),\ y(x_3) = z(x_3),\ y(x_4) - y(x_5) = z(x_4) - z(x_5),$$

then $y(x) = z(x),\ a < x < b$.

(D) Given $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, if $y(x)$ and $z(x)$ are two solutions of (2.1) satisfying

$$y(x_1) - y(x_2) = z(x_1) - z(x_2),\ y(x_3) = z(x_3),\ y(x_4) = z(x_4),\ y(x_5) = z(x_5),$$

then $y(x) = z(x),\ a < x < b$.

Of course, from the results of Chapters Two and Three, solutions of 5-point, 4-point and 3-point nonlocal boundary value problems are unique, when they exist. Our first existence result deals with solutions of (2.1):(2.2).

Theorem 4.1. *Assume hypotheses (A), (B) and (C) are satisfied with respect to equation (2.1). Then, given* $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of (2.1):(2.2) on* (a, b).

Proof. Let $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ and $y_1, y_2, y_3, y_4 \in \mathbb{R}$ be selected. We note, as in Chapter Three that 4-point, 3-point, as well as 2-point, conjugate boundary value problems have unique solutions; that is, solutions of (2.1) satisfying any of (2.10) to (2.16) have unique solutions.

Let $z(x)$ be the solution of (2.1) satisfying the 4-point conjugate boundary conditions at $x_1,\ x_2,\ x_3$ and x_4,

$$z(x_2) = y_2,\ z(x_3) = y_3,\ z(x_4) = y_4,\ z(x_5) = 0.$$

Observe that $z(x_4) - z(x_5) = y_4$. Next, define the set

$$S = \{u(x_1) |\ u(x) \text{ is a solution of equation (2.1) satisfying}$$

$$u(x_2) = z(x_2),\ u(x_3) = z(x_3),\ u(x_4) - u(x_5) = z(x_4) - z(x_5)\}.$$

We observe first that S is nonempty, since $z(x_1) \in S$.

Next, choose $s_0 \in S$. Then, there is a solution $u_0(x)$ of (2.1) satisfying

$$u_0(x_1) = s_0,$$

$$u_0(x_2) = z(x_2),$$

$$u_0(x_3) = z(x_3),$$

$$u(x_4) - u(x_5) = z(x_4) - z(x_5).$$

By the continuous dependence theorem, Theorem 2.3, there exists a $\delta > 0$ such that, for each $0 \leq |s - s_0| < \delta$, there is a solution $u_s(x)$ of (2.1) satisfying

$$u_s(x_1) = s,\ u_s(x_2) = u_0(x_2) = z(x_2),\ u_s(x_3) = u_0(x_3) = z(x_3),$$

and

$$u_s(x_4) - u_s(x_5) = u_0(x_4) - u_0(x_5) = z(x_4) - z(x_5),$$

or in other words, $s \in S$; in particular, $(s_0 - \delta, s_0 + \delta) \subset S$, and so S is an open subset of $\mathbb{R}$.

The remainder of the argument is devoted to showing that S is also a closed subset of $\mathbb{R}$. To that end, we assume for the purpose of contradiction that S is not closed. Then there exists an $r_0 \in \overline{S} \setminus S$ and a strictly monotone sequence $\{r_k\} \subset S$ such that $\lim_{n\to\infty} r_k = r_0$.

We may assume, without loss of generality, that $r_k \uparrow r_0$. By the definition of S, we denote, for each $k \in \mathbb{N}$, by $u_k(x)$ the solution of equation (2.1) satisfying

$$u_k(x_1) = r_k,\ u_k(x_2) = z(x_2),$$

$$u_k(x_3) = z(x_3),\ u_k(x_4) - u_k(x_5) = z(x_4) - z(x_5).$$

By uniqueness of solutions of (2.1):(2.2), we have for each $k \in \mathbb{N}$,

$$u_k(x) < u_{k+1}(x) \text{ on } (a, x_2).$$

We now claim that $\{u_k(x)\}$ is not uniformly bounded above on each compact subinterval of (a, x_1) and (x_1, x_2). Suppose there exists a subinterval $[c, d] \subset (a, x_1)$ so that

$\{u_k(x)\}$ is uniformly bounded above on $[c, d]$. That is there exists $H > 0$ so that $u_k(x) \leq H$, for all $c \leq x \leq d$ and $k \geq 1$. In particular,

$$u_1(x) \leq u_k(x) \leq H \text{ for all } c \leq x \leq d \text{ and } k \geq 1.$$

But by the precompactness condition of Theorem 3.2, we know there exists a subsequence $\{u_{k_j}(x)\}$ such that $\{u_{k_j}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), $i = 0, 1, 2, 3$.

Suppose $u_{k_j}^{(i)}(x) \to v^{(i)}(x)$ uniformly on the compact interval $[x_1, x_5]$, for $i = 0, 1, 2, 3$. Then by Theorem 3.1, $v(x)$ is a solution of equation (2.1) satisfying

$$v(x_1) = \lim_{j\to\infty} u_{k_j}(x_1) = \lim_{j\to\infty} r_{k_j} = r_0,$$

$$v(x_2) = z(x_2),\ v(x_3) = z(x_3),$$

and

$$v(x_4) - v(x_5) = \lim_{j\to\infty} (u_{k_j}(x_4) - u_{k_j}(x_5)) = z(x_4) - z(x_5).$$

Therefore $r_0 \in S$. But this is contradictory to the assumption $r_0 \notin S$. So $\{u_k(x)\}$ is not uniformly bounded above on each compact subinterval of (a, x_1). The argument relative to (x_1, x_2) is exactly analogous.

Next let $w(x)$ be the solution of the 3-point conjugate boundary problem for equation (2.1) satisfying,

$$w(x_1) = r_0,\ w'(x_1) = 0,\ w(x_2) = y_2,\ w(x_3) = y_3.$$

It follows that, for some K large, there exists points $a < \tau_1 < x_1 < \tau_2 < x_2$ so that

$$u_K(\tau_1) = w(\tau_1),\ u_K(\tau_2) = w(\tau_2).$$

Also,

$$u_K(x_2) = z(x_2) = w(x_2),\ u_K(x_3) = z(x_3) = w(x_3).$$

By uniqueness of solutions of 4-point conjugate boundary value problems for (2.1), we have $u_K \equiv w$. However,

$$w(x_1) = r_0 > r_K = u_K(x_1),$$

which gives a contradiction. Thus S is also a closed subset of $\mathbb{R}$.

In summary, S is a nonempty subset of $\mathbb{R}$ that is both open and closed. We have $S = \mathbb{R}$. By choosing $y_1 \in S$, there is a corresponding solution $y(x)$ of equation (2.1) such that

$$y(x_1) = y_1,\ y(x_2) = z(x_2) = y_2,\ y(x_3) = z(x_3) = y_3,$$
$$y(x_4) - y(x_5) = z(x_4) - z(x_5) = y_4.$$

This completes the proof. □

We now turn to existence of solutions for 4-point and 3-point nonlocal boundary value problems for (2.1). We first address the 4-point problems.

Theorem 4.2. *Assume (A), (B) and (C) are satisfied with respect to (2.1). Given points* $a < x_1 < x_2 < x_3 < x_4 < b$, *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of (2.1):(2.4) on* (a, b).

Proof. Let $a < x_1 < x_2 < x_3 < x_4 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$ be selected. Also, fix $a < \tau < x_1$. We repeat again that 4-point, 3-point, as well as 2-point, conjugate boundary value problems for (2.1) have unique solutions.

Let $z(x)$ be the solution of the nonlocal 5-point boundary value problem (2.1):(2.2) obtained in Theorem 4.1 and satisfying,

$$z(\tau) = 0,\ z(x_1) = y_1,\ z(x_2) = y_3,\ z(x_3) - z(x_4) = y_4.$$

This time, define the set

$$S = \{u'(x_1) |\ u(x) \text{is a solution of (2.1) satisfying}$$

$$u(x_1) = z(x_1), u(x_2) = z(x_2), u(x_3) - u(x_4) = z(x_3) - z(x_4)\}.$$

Again S is nonempty since $z'(x_1) \in S$.

Next, choose $s_0 \in S$. Then, there is a solution $u_0(x)$ of (2.1) satisfying

$$u_0(x_1) = z(x_1),$$

$$u_0'(x_1) = s_0,$$

$$u_0(x_2) = z(x_2),$$

$$u_0(x_3) - u_0(x_4) = z(x_3) - z(x_4).$$

By the continuous dependence theorem, Theorem 2.7, there exists a $\delta > 0$ such that, for each $0 \leq |s - s_0| < \delta$, there is a solution $u_s(x)$ of (2.1) satisfying

$$u_s(x_1) = u_0(x_1) = z(x_1),\ u_s'(x_1) = s,\ u_s(x_2) = u_0(x_2) = z(x_2),$$

and

$$u_s(x_3) - u_s(x_4) = u_0(x_3) - u_0(x_4) = z(x_3) - z(x_4),$$

or in other words, $s \in S$; in particular, $(s_0 - \delta, s_0 + \delta) \subset S$, and so S is an open subset of $\mathbb{R}$.

As in the proof of Theorem 4.1, the remainder of the argument is devoted to showing S is also a closed subset of $\mathbb{R}$. We assume, for contradiction, that S is not closed. Then, there is an $r_0 \in \overline{S} \backslash S$ and a strictly monotone sequence $\{r_k\} \subset S$ such that $\lim_{k\to\infty} r_k = r_0$. Again, we may assume $r_k \uparrow r_0$.

By the definition of S, we denote, for each $k \in \mathbb{N}$, by $u_k(x)$ the solution of (2.1) satisfying

$$u_k(x_1) = z(x_1),\ u_k'(x_1) = r_k,\ u_k(x_2) = z(x_2),$$

$$u_k(x_3) - u_k(x_4) = z(x_3) - z(x_4).$$

By uniqueness of solutions of (2.1):(2.2), we have

$$u_k(x) > u_{k+1}(x) \quad \text{on} \quad (a, x_1),\ \text{and}\ u_k(x) < u_{k+1}(x) \quad \text{on} \quad (x_1, x_2).$$

We claim that $\{u_k(x)\}$ is not uniformly bounded below on each compact subinterval of (a, x_1) and is not uniformly bounded above on each compact subinterval of (x_1, x_2). Suppose there exists a subinterval $[c, d] \subset (a, x_1)$ so that $\{u_k(x)\}$ is uniformly bounded below on $[c, d]$. That is there exists $H > 0$ so that $u_k(x) \geq H$ for all $c \leq x \leq d$ and $k \geq 1$. In particular,

$$u_1(x) \geq u_k(x) \geq H \text{ for all } c \leq x \leq d \text{ and } k \geq 1.$$

So by the precompactness condition of Theorem 3.2, we know there exists a subsequence $\{u_{k_j}(x)\}$ such that $\{u_{k_j}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), $i = 0, 1, 2, 3$.

Suppose $u_{k_j}^{(i)}(x) \to v^{(i)}(x)$ uniformly on the compact interval $[x_1, x_4]$, where $i = 0, 1, 2, 3$. Then by Theorem 3.1, $v(x)$ is a solution of equation (2.1) satisfying

$$v(x_1) = z(x_1),\ v'(x_1) = \lim_{j\to\infty} u'_{k_j}(x_1) = r_0,\ v(x_2) = z(x_2),$$

and

$$v(x_3) - v(x_4) = \lim_{j\to\infty} (u_{k_j}(x_3) - u_{k_j}(x_4)) = z(x_3) - z(x_4).$$

Therefore $r_0 \in S$, which is a contradiction to the assumption $r_0 \notin S$. So $\{u_k(x)\}$ is not uniformly bounded below on each compact subinterval of (a, x_1). The argument that $\{u_k(x)\}$ is not uniformly bounded above on each compact subinterval on (x_1, x_2) is completely analogous.

Next let $w(x)$ be the solution of the 3-point conjugate boundary problem for equation (2.1) satisfying,

$$w(x_1) = z(x_1),\ w'(x_1) = r_0,\ w(x_2) = z(x_2),\ w(x_3) = z(x_3).$$

It follows that, for some K large, there exists points $a < \tau_1 < x_1 < \tau_2 < x_2$ so that

$$u_K(\tau_1) = w(\tau_1),\ u_K(\tau_2) = w(\tau_2).$$

Also,

$$u_K(x_1) = z(x_1) = w(x_1),\ u_K(x_2) = z(x_2) = w(x_2).$$

By uniqueness of solutions of 4-point conjugate boundary value problems for (2.1), we have $u_K \equiv w$. However,

$$w'(x_1) = r_0 > r_K = u'_K(x_1),$$

which is a contradiction. Thus S is also a closed subset of $\mathbb{R}$.

In summary, S is a nonempty subset of $\mathbb{R}$ that is both open and closed. We have $S = \mathbb{R}$. By choosing $y_2 \in S$, there is a corresponding solution $y(x)$ of equation (2.1) such that

$$y(x_1) = z(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = z(x_2) = y_3,$$

$$y(x_3) - y(x_4) = z(x_3) - z(x_4) = y_4.$$

This completes the proof. $\square$

Theorem 4.3. *Assume (A), (B) and (C) are satisfied with respect to (2.1). Given points* $a < x_1 < x_2 < x_3 < x_4 < b$, *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of (2.1):(2.6) on* (a, b).

Proof. Let $a < x_1 < x_2 < x_3 < x_4 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$ be selected. Also, fix $x_1 < \tau < x_2$. We repeat again that 4-point, 3-point, as well as 2-point, conjugate boundary value problems for (2.1) have unique solutions.

Let $z(x)$ be the solution of the nonlocal 5-point boundary value problem (2.1):(2.2) obtained in Theorem 4.1 and satisfying,

$$z(x_1) = y_1,\ z(\tau) = 0,\ z(x_2) = y_2,\ z(x_3) - z(x_4) = y_4.$$

This time, define the set

$$S = \{u'(x_2)|\ u(x) \text{is a solution of (2.1) satisfying}$$

$$u(x_1) = z(x_1), u(x_2) = z(x_2), u(x_3) - u(x_4) = z(x_3) - z(x_4)\}.$$

Again S is nonempty since $z'(x_2) \in S$. It follows from Theorem 2.9 that solutions of (2.1):(2.6) depend continuous on boundary conditions. The standard argument then yields that S is an open subset of $\mathbb{R}$.

As in the proof of Theorem 4.1, the remainder of the argument is devoted to showing S is also a closed subset of $\mathbb{R}$. We assume, for contradiction, that S is not closed. Then, there is an $r_0 \in \overline{S} \backslash S$ and a strictly monotone sequence $\{r_k\} \subset S$ such that $\lim_{k\to\infty} r_k = r_0$. Again, we may assume $r_k \uparrow r_0$.

By the definition of S, we denote, for each $k \in \mathbb{N}$, by $u_k(x)$ the solution of (2.1) satisfying

$$u_k(x_1) = z(x_1),\ u_k(x_2) = z(x_2),\ u_k'(x_2) = r_k,$$

$$u_k(x_3) - u_k(x_4) = z(x_3) - z(x_4).$$

By uniqueness of solutions of (2.1):(2.2), we have

$$u_k(x) > u_{k+1}(x) \quad \text{on} \quad (x_1, x_2),\ \text{and}\ u_k(x) < u_{k+1}(x) \quad \text{on} \quad (x_2, x_3).$$

We claim that $\{u_k(x)\}$ is not uniformly bounded below on each compact subinterval of (x_1, x_2) and is not uniformly bounded above on each compact subinterval of (x_2, x_3). Suppose there exists a subinterval $[c, d] \subset (x_1, x_2)$ so that $\{u_k(x)\}$ is uniformly bounded above on $[c, d]$. That is there exists $H > 0$ so that $u_k(x) \geq H$ for all $c \leq x \leq d$ and $k \geq 1$. In particular,

$$u_1(x) \geq u_k(x) \geq H \text{ for all } c \leq x \leq d \text{ and } k \geq 1.$$

So by the precompactness condition of Theorem 3.2, we know there exists a subsequence $\{u_{k_j}(x)\}$ such that $\{u_{k_j}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), $i = 0, 1, 2, 3$.

Suppose $u_{k_j}^{(i)}(x) \to v^{(i)}(x)$ uniformly on the compact interval $[x_1, x_4]$, where $i = 0, 1, 2, 3$. Then by Theorem 3.1, $v(x)$ is a solution of equation (2.1) satisfying

$$v(x_1) = z(x_1),\ v(x_2) = z(x_2),\ v'(x_2) = \lim_{j\to\infty} u_{k_j}'(x_2) = r_0,$$

and

$$v(x_3) - v(x_4) = \lim_{j\to\infty} (u_{k_j}(x_3) - u_{k_j}(x_4)) = z(x_3) - z(x_4).$$

Therefore $r_0 \in S$, which is a contradiction to the assumption $r_0 \notin S$. So $\{u_k(x)\}$ is not uniformly bounded below on each compact subinterval of (x_1, x_2). The argument that $\{u_k(x)\}$ is not uniformly bounded above on each compact subinterval on (x_2, x_3) is completely analogous.

Next let $w(x)$ be the solution of the 3-point conjugate boundary problem for equation (2.1) satisfying,

$$w(x_1) = z(x_1),\ w(x_2) = z(x_2),\ w'(x_2) = r_0,\ w(x_3) = z(x_3).$$

It follows that, for some K large, there exists points $x_1 < \tau_1 < x_2 < \tau_2 < x_3$ so that

$$u_K(\tau_1) = w(\tau_1),\ u_K(\tau_2) = w(\tau_2).$$

Also,

$$u_K(x_1) = z(x_1) = w(x_1),\ u_K(x_2) = z(x_2) = w(x_2).$$

By uniqueness of solutions of 4-point conjugate boundary value problems for (2.1), we have $u_K \equiv w$. However,

$$w'(x_2) = r_0 > r_K = u_K'(x_2),$$

which is a contradiction. Thus S is also a closed subset of $\mathbb{R}$.

In summary, S is a nonempty subset of $\mathbb{R}$ that is both open and closed. We have $S = \mathbb{R}$. By choosing $y_3 \in S$, there is a corresponding solution $y(x)$ of equation (2.1) such that

$$y(x_1) = z(x_1) = y_1,\ y(x_2) = z(x_2) = y_2,\ y'(x_2) = y_3,$$

$$y(x_3) - y(x_4) = z(x_3) - z(x_4) = y_4.$$

This completes the proof. $\square$

In a like manner, we now show that there are solutions for the 3-point nonlocal boundary value problems.

Theorem 4.4. *Assume (A), (B) and (C) are satisfied with respect to (2.1). Given points* $a < x_1 < x_2 < x_3 < b$, *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of (2.1):(2.8) on* (a, b).

Proof. Let $a < x_1 < x_2 < x_3 < b$, and $y_1, y_2, y_3, y_4 \in \mathbb{R}$ be selected. Also, fix $a < \tau < x_1$. As before, we repeat that 4-point, 3-point, as well as 2-point, conjugate boundary value problems for (2.1) have unique solutions.

Let $z(x)$ be the solution of the 4-point nonlocal boundary value problem (2.1):(2.6) obtained in Theorem 4.3 and satisfying,

$$z(\tau) = 0,\ z(x_1) = y_1,\ z'(x_1) = y_2,\ z(x_2) - z(x_3) = y_4.$$

Now, define the set

$$S = \{u''(x_1) |\ u(x) \text{ is a solution of (2.1) satisfying}$$
$$u(x_1) = z(x_1),\ u'(x_1) = z'(x_1),\ u(x_2) - u(x_3) = z(x_2) - z(x_3)\}.$$

This time $z''(x_1) \in S$, and so S is nonempty. It follows from Theorem 2.11 that solutions of (2.1):(2.8) depend continuous on boundary conditions. The standard argument then yields that S is an open subset of $\mathbb{R}$.

As in the previous theorems, the remainder of the argument is devoted to showing S is also a closed subset of $\mathbb{R}$. We assume, for contradiction, that S is not closed. Then there is an $r_0 \in \overline{S} \backslash S$ and a strictly monotone sequence $\{r_k\} \subset S$ such that $\lim_{k\to\infty} r_k = r_0$. Again, we may assume $r_k \uparrow r_0$.

By the definition of S, we denote, for each $k \in \mathbb{N}$, by $u_k(x)$ the solution of (2.1) satisfying

$$u_k(x_1) = z(x_1),\ u'_k(x_1) = r_k,\ u''_k(x_1) = r_k,$$
$$u_k(x_2) - u_k(x_3) = z(x_2) - z(x_3).$$

Since $r_k < r_{k+1}$, by uniqueness of solutions of (2.1):(2.4) and (2.1):(2.6), we have

$$u_k(x) < u_{k+1}(x) \quad \text{on} \quad (a, x_2)\backslash\{x_1\}.$$

We now claim that $\{u_k(x)\}$ is not uniformly bounded above on each compact subinterval of (a, x_1) and (x_1, x_2). Suppose there exists a subinterval $[c, d] \subset (a, x_1)$ so that $\{u_k(x)\}$ is uniformly bounded above on $[c, d]$. That is there exists $H > 0$ so that $u_k(x) \leq H$ for all $c \leq x \leq d$ and $k \geq 1$. Then

$$u_1(x) \leq u_k(x) \leq H \text{ for all } c \leq x \leq d \text{ and } k \geq 1.$$

So by the precompactness condition of Theorem 3.2, we know there exists a subsequence $\{u_{k_j}(x)\}$ such that $\{u_{k_j}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), $i = 0, 1, 2, 3$.

Suppose $u_{k_j}^{(i)}(x) \to v^{(i)}(x)$ uniformly on the compact interval $[x_1, x_3]$, for $i = 0, 1, 2, 3$. Then by Theorem 3.1, $v(x)$ is a solution of equation (2.1) satisfying

$$v(x_1) = z(x_1),\ v'(x_1) = z'(x_1),\ v''(x_1) = \lim_{j\to\infty} u''_{k_j}(x_1) = r_0,$$

and

$$v(x_2) - v(x_3) = \lim_{j\to\infty}(u_{k_j}(x_2) - u_{k_j}(x_3)) = z(x_2) - z(x_3).$$

Therefore $r_0 \in S$. But this contradicts the assumption $r_0 \notin S$, so $\{u_k(x)\}$ is not uniformly bounded above on each compact subinterval of (a, x_1) and (x_1, x_2).

Next let $w(x)$ be the solution of the 2-point conjugate boundary problem for equation (2.1) satisfying,

$$w(x_1) = z(x_1),\ w'(x_1) = z'(x_1),\ w''(x_1) = r_0,\ w(x_3) = z(x_3).$$

It follows that, for some K large, there exists points $a < \tau_1 < x_1 < \tau_2 < x_2$ so that

$$u_K(\tau_1) = w(\tau_1),\ u_K(\tau_2) = w(\tau_2),$$

Also,

$$u_K(x_1) = z(x_1) = w(x_1),\ u'_K(x_1) = z'(x_1) = w'(x_1).$$

By uniqueness of solutions of 3-point conjugate boundary value problems for (2.1), we have $u_K \equiv w$. However,

$$w''(x) = r_0 > r_K = u_K''(x_1),$$

which is a contradiction. Thus S is also a closed subset of $\mathbb{R}$.

In summary, S is a nonempty subset of $\mathbb{R}$ that is both open and closed. We have $S = \mathbb{R}$. By choosing $y_3 \in S$, there is a corresponding solution $y(x)$ of equation (2.1) such that

$$y(x_1) = z(x_1) = y_1,\ y'(x_1) = z'(x_1) = y_2,\ y''(x_1) = y_3,$$
$$y(x_2) - y(x_3) = z(x_2) - z(x_3) = y_4.$$

This completes the proof. □

There is a list of dual uniqueness implies existence results for (2.1) with respect to solutions satisfying conditions (2.3), (2.5), (2.7) and (2.9). We list these results without proof. For these results, rather than hypothesis (C), we will assume the dual condition (D).

Theorem 4.5. *Assume hypotheses (A), (B) and (D) are satisfied with respect to equation (2.1). Then, given* $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$, *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of (2.1):(2.3) on* (a, b).

Theorem 4.6. *Assume (A), (B) and (D) are satisfied with respect to (2.1). Given points* $a < x_1 < x_2 < x_3 < x_4 < b$, *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of (2.1):(2.5) on* (a, b).

Theorem 4.7. *Assume (A), (B) and (D) are satisfied with respect to (2.1). Given points* $a < x_1 < x_2 < x_3 < x_4 < b$, *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of (2.1):(2.7) on* (a, b).

Theorem 4.8. *Assume (A), (B) and (D) are satisfied with respect to (2.1). Given points* $a < x_1 < x_2 < x_3 < b$, *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$, *there exists a unique solution of (2.1):(2.9) on* (a, b).

CHAPTER FIVE

Local Existence Theorems

In this chapter, we discuss the local existence of solutions of nonlocal boundary value problems associated with the nonlinear equation

$$y^{(4)} = f(x, y, y', y'', y'''). \tag{5.1}$$

Theorem 5.1. *Assume that $f(x, u_1, u_2, u_3, u_4) : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous. Then a function $y(x) \in C^{(4)}[a, b]$ is a solution of the boundary value problem for (5.1) satisfying*

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) - y(x_5) = y_4, \tag{5.2}$$

where $a = x_1 < x_2 < x_3 < x_4 < x_5 = b$ and $y_1, y_2, y_3, y_4 \in \mathbb{R}$, if and only if $y(x) \in C^{(3)}[a, b]$ is a solution of the integral equation

$$y(x) = w(x) + \int_a^b G(x, s) f(s, y(s), y'(s), y''(s), y'''(s))ds, \tag{5.3}$$

on $[a, b]$, where $G(x, s)$ is the Green's function for

$$y^{(4)} = 0,\ y(x_1) = 0,\ y(x_2) = 0,\ y(x_3) = 0,\ y(x_4) - y(x_5) = 0, \tag{5.4}$$

and $w(x)$ is the solution of

$$y^{(4)} = 0,\ y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) - y(x_5) = y_4. \tag{5.5}$$

Proof. First assume that $y(x) \in C^{(4)}[a, b]$ is a solution of the stated boundary problem. Then $y(x) \in C^{(3)}[a, b]$, and

$$y^{(4)}(x) = f(x, y(x), y'(x), y''(x), y'''(x)) := h(x) \in C[a, b],$$

and satisfies

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) - y(x_5) = y_4.$$

Thus, we are considering a solution of

$$y^{(4)} = h(x),\ y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) - y(x_5) = y_4.$$

It follows that

$$y(x) = w(x) + \int_a^b G(x,s)h(s)ds \ = \ w(x) + \int_a^b G(x,s)f(s, y(s), y'(s), y''(s), y'''(s))ds.$$

Conversely, let's suppose that $y(x) \in C^{(3)}[a,b]$ and satisfies the integral equation (5.3) on $[a,b]$. Now because $w(x)$ is a solution of (5.5) and because of properties of $G(x,s)$, it follows that

$$y^{(4)} = f(x, y(x), y'(x), y''(x), y'''(x))$$

on $[a,b]$, so that $y(x) \in C^{(4)}[a,b]$. Moreover, from the properties of $w(x)$ and

$$\int_a^b G(x,s)f(s, y(s), y'(s), y''(s), y'''(s))ds,$$

we also have $y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,\ y(x_4) - y(x_5) = y_4$.

Therefore, $y(x)$ is a solution of the boundary value problem (5.1):(5.2). $\square$

The following fixed point theorem, known as the Contraction Mapping Theorem, will be fundamental in obtaining our local existence results.

Theorem 5.2. *Let $< M, d >$ be a complete metric space and let $T : M \to M$ be such that there exists $0 \le \alpha < 1$, with $d(T(x), T(y)) \le \alpha d(x,y)$, for all $x, y \in M$. Then T has a unique fixed point in M.*

Now, by the properties of the Green's function for (5.4), there exist constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, independent of $x_1 < x_2 < x_3 < x_4 < x_5$, such that

$$\int_a^b |G(x,s)|ds \le \gamma_1(b-a)^4,$$
$$\int_a^b |\frac{\partial G(x,s)}{\partial x}|ds \le \gamma_2(b-a)^3,$$
$$\int_a^b |\frac{\partial^2 G(x,s)}{\partial x^2}|ds \le \gamma_3(b-a)^2,$$
$$\int_a^b |\frac{\partial^3 G(x,s)}{\partial x^3}|ds \le \gamma_4(b-a).$$

Theorem 5.3. *Let* $f(x,u_1,u_2,u_3,u_4) : [a,b]\times\mathbb{R}^4 \to \mathbb{R}$ *be continuous and satisfy a Lipschitz condition,*

$$|f(x,y_1,y_2,y_3,y_4)-f(x,z_1,z_2,z_3,z_4)| \le K|y_1-z_1|+L|y_2-z_2|+M|y_3-z_3|+N|y_4-z_4|$$

on $[a,b]\times\mathbb{R}^4$. *Then if*

$$K\gamma_1(b-a)^4 + L\gamma_2(b-a)^3 + M\gamma_3(b-a)^2 + N\gamma_4(b-a) < 1,$$

the boundary value problem

$$y^{(4)} = f(x,y,y',y'',y'''),$$
$$y(x_1)=y_1,\ y(x_2)=y_2,\ y(x_3)=y_3,\ y(x_4)-y(x_5)=y_4,$$

has a unique solution for all $a=x_1<x_2<x_3<x_4<x_5=b,$ *and* $y_1,y_2,y_3,y_4\in\mathbb{R}$.

Proof. First define a mapping: $T : C^{(3)}[a,b]\to C^{(3)}[a,b]$ via

$$(Th)(x) = w(x) + \int_a^b G(x,s)f(s,h(s),h'(s),h''(s),h'''(s))ds,$$

for $a\le x\le b,\ h\in C^{(3)}[a,b],$ where $G(x,s)$ is the Green's function for (5.4) and $w(x)$ is determined by (5.5).

We shall show that T is a contraction mapping with respect to the metric on $C^{(3)}[a,b]$,

$$d(h,g) = \|h-g\| = K|h-g|_\infty + L|h'-g'|_\infty + M|h''-g''|_\infty + N|h'''-g'''|_\infty,$$

where $|h|_\infty = \max_{a\le x\le b}|h(x)|$.

So, let $h, g \in C^{(3)}[a,b]$. Then, for $a \le x \le b$,

$$\begin{aligned}
|(Th)(x) - (Tg)(x)| &= \int_a^b |G(x,s)||f(s,h(s),h'(s),h''(s),h'''(s)) \\
&- f(s,g(s),g'(s),g''(s),g'''(s))|ds \\
&\le \int_a^b |G(x,s)|[K|h(s)-g(s)| + L|h'(s)-g'(s)| \\
&\quad +M|h''(s)-g''(s)| + N|h'''(s)-g'''(s)|\ ]ds \\
&\le \int_a^b |G(x,s)|[K|h-g|_\infty + L|h'-g'|_\infty \\
&\quad +M|h''-g''|_\infty + N|h'''-g'''|_\infty\ ]ds \\
&= \int_a^b |G(x,s)|\|h-g\|ds \\
&\le \gamma_1(b-a)^4\|h-g\|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|(Th)'(x) - (Tg)'(x)| &\le \|h-g\| \int_a^b |\frac{\partial G(x,s)}{\partial x}|ds \\
&\le \gamma_2(b-a)^3\|h-g\|,
\end{aligned}$$

$$\begin{aligned}
|(Th)''(x) - (Tg)''(x)| &\le \|h-g\| \int_a^b |\frac{\partial^2 G(x,s)}{\partial x^2}|ds \\
&\le \gamma_3(b-a)^2\|h-g\|,
\end{aligned}$$

$$\begin{aligned}
|(Th)'''(x) - (Tg)'''(x)| &\le \|h-g\| \int_a^b |\frac{\partial^3 G(x,s)}{\partial x^3}|ds \\
&\le \gamma_4(b-a)\|h-g\|.
\end{aligned}$$

Each of these bounds are independent of x. Hence

$$|Tg - Th|_\infty \le \gamma_1(b-a)^4\|h-g\|,$$

$$|(Tg)' - (Th)'|_\infty \le \gamma_2(b-a)^3\|h-g\|,$$

$$|(Tg)'' - (Th)''|_\infty \le \gamma_3(b-a)^2\|h-g\|,$$

$$|(Tg)''' - (Th)'''|_\infty \le \gamma_4(b-a)\|h-g\|.$$

Consequently,

$$\begin{aligned} d(Th, Tg) &= \|Th - Tg\| \\ &= K|Th - Tg|_\infty + L|(Th)' - (Tg)'|_\infty \\ &\quad +M|(Th)'' - (Tg)''|_\infty + N|(Th)''' - (Tg)'''|_\infty \\ &\leq (K\gamma_1(b-a)^4 + L\gamma_2(b-a)^3 + M\gamma_3(b-a)^2 + N\gamma_4(b-a))\|h - g\| \\ &= (K\gamma_1(b-a)^4 + L\gamma_2(b-a)^3 + M\gamma_3(b-a)^2 + N\gamma_4(b-a))d(h, g). \end{aligned}$$

Hence, if

$$K\gamma_1(b-a)^4 + L\gamma_2(b-a)^3 + M\gamma_3(b-a)^2 + N\gamma_4(b-a) < 1,$$

then T is a contraction mapping, and thus there exists a unique fixed point $y(x) \in C^{(3)}[a, b]$. In particular, there is a unique $y(x)$ satisfying

$$y(x) = (Ty)(x) = w(x) + \int_a^b G(x, s)f(s, y(s), y'(s), y''(s), y'''(s))ds,$$

and by Theorem 5.1, $y(x)$ is the unique solution of (5.1):(5.2). $\square$

In view of the above result establishing local existence and uniqueness of solutions of (5.1):(5.2), and from the uniqueness implies existence results of Chapter Four, we can state as corollaries, some local existence and uniqueness results for 4-point and 3-point nonlocal boundary value problems for (5.1).

Corollary 5.1. *Let $f(x, u_1, u_2, u_3, u_4) : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous and satisfy a Lipschitz condition,*

$$|f(x, y_1, y_2, y_3, y_4) - f(x, z_1, z_2, z_3, z_4)| \leq K|y_1 - z_1| + L|y_2 - z_2| + M|y_3 - z_3| + N|y_4 - z_4|$$

on $[a, b] \times \mathbb{R}^4$. Then if

$$K\gamma_1(b-a)^4 + L\gamma_2(b-a)^3 + M\gamma_3(b-a)^2 + N\gamma_4(b-a) < 1,$$

the boundary value problem for (5.1) satisfying

$$y(x_1) = y_1, y'(x_1) = y_2, y(x_2) = y_3, y(x_3) - y(x_4) = y_4,$$

has a unique solution for all $a < x_1 < x_2 < x_3 < x_4 < b$ *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$.

Corollary 5.2. *Let* $f(x, u_1, u_2, u_3, u_4) : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ *be continuous and satisfy a Lipschitz condition,*

$$|f(x, y_1, y_2, y_3, y_4) - f(x, z_1, z_2, z_3, z_4)| \leq K|y_1 - z_1| + L|y_2 - z_2| + M|y_3 - z_3| + N|y_4 - z_4|$$

on $[a, b] \times \mathbb{R}^4$. *Then if*

$$K\gamma_1(b-a)^4 + L\gamma_2(b-a)^3 + M\gamma_3(b-a)^2 + N\gamma_4(b-a) < 1,$$

the boundary value problem for (5.1) satisfying

$$y(x_1) = y_1, y(x_2) = y_2, y'(x_2) = y_3, y(x_3) - y(x_4) = y_4,$$

has a unique solution for all $a < x_1 < x_2 < x_3 < x_4 < b$ *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$.

Corollary 5.3. *Let* $f(x, u_1, u_2, u_3, u_4) : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ *be continuous and satisfy a Lipschitz condition,*

$$|f(x, y_1, y_2, y_3, y_4) - f(x, z_1, z_2, z_3, z_4)| \leq K|y_1 - z_1| + L|y_2 - z_2| + M|y_3 - z_3| + N|y_4 - z_4|$$

on $[a, b] \times \mathbb{R}^4$. *Then if*

$$K\gamma_1(b-a)^4 + L\gamma_2(b-a)^3 + M\gamma_3(b-a)^2 + N\gamma_4(b-a) < 1,$$

the boundary value problem for (5.1) satisfying

$$y(x_1) = y_1, y'(x_1) = y_2, y''(x_1) = y_3, y(x_2) - y(x_3) = y_4,$$

has a unique solution for all $a < x_1 < x_2 < x_3 < b$ *and* $y_1, y_2, y_3, y_4 \in \mathbb{R}$.

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