ABSTRACT

Holography and Black Holes in Gravitational Theories without Lorentz Symmetry

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Recently, relativistic gauge/gravity duality, the best understood example of which is AdS/CFT correspondence, has been extended to its nonrelativistic version. In this dissertation, we study the holographic duality between nonrelativistic quantum field theories and gravitational theories which break Lorentz symmetry. In particular, we find that high-order operators dramatically modify a probe scalar field in the UV limit. Then, according to the gauge/gravity duality, this in turn affects the two-point correlation functions on the boundary. Black holes also exist in these theories with causal boundaries termed universal horizons. We present two new classes of charged black hole solutions in the framework of the Einstein-Maxwell-æther theory. Furthermore, we construct the Smarr formulas and study the temperatures at both Killing and universal horizons.

Holography and Black Holes in Gravitational Theories without Lorentz Symmetry

by

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To my parents

CHAPTER ONE

Introduction

Gravity, electromagnetism, strong and weak forces are four fundamental forces in nature. Interestingly, gravity was the first one to be recognized by humans, but our understanding about it is still quite limited. Though gravity is the weakest force, its universality and the massive nature of the earth made it recognizable from early times.

1.1 Historical Review of Gravity

In the 17th century, English physicist and mathematician Sir Isaac Newton proposed his theory of gravity: Newton's law of universal gravitation. His theory enjoyed its success in the framework of classical physics, for example, by predicting the existence of Neptune. However this theory has its own limited domain of applicability. As we already know, Newton's law of universal gravitation fails to work when one considers strong gravitational effects or if the speed of a body is comparable to the speed of light. Before 1915, people already knew Euclidean geometry and Riemannian geometry were reasonable, but always thought that only Euclidean geometry was real: real space should be flat. In 1915, Einstein generalized Newton's gravity with a different point of view and created a new theory of gravity, the general theory of relativity. He accurately pointed out that when there is no gravitational field, spacetime is flat and Euclidean geometry is able to describe it; however when the gravitational field exists, Riemannian geometry is real and spacetime is curved.

General relativity is still a classical field theory whose action is described by the Einstein-Hilbert action. The dynamical variable is the metric, and the Ricci scalar contains no higher than second order derivatives. The equation of motion can be derived from the action. One side of the equation of motion is the Einstein tensor which describes the curvature of spacetime, and the other side is the energy momentum tensor, which describes the distribution of matter. Einstein's field equation can be interpreted as matter distribution determines curvature of spacetime.

1.2 The Success and Problem of General Relativity

The predictions of general relativity have so far been tested by most observations and experiments. Although general relativity is not the only theory describing gravitation today, it is the simplest one that is consistent with the observational data. Einstein's theory of general relativity in astrophysics has several crucial applications. It directly deduces that some large stars will end up as black holes: a region of spacetime so distorted that even light cannot escape. The deflection of light in a gravitational field creates a gravitational lens phenomenon. General relativity also predicts the existence of gravitational waves, which have been confirmed by observation [1]. In addition, general relativity is also the theoretical basis of the expanding universe in modern cosmology.

Despite its great success, general relativity has its problems. The gravitational coupling constant G has a negative dimension of mass/energy squared: it is a perturbatively nonrenormalizable theory. In 1977, Stelle showed that adding two quadratic curvature invariants makes the theory renormalizable [2]. However, in the new Lagrangian, due to the existence of fourth time derivatives of metric, the propagator has a negative sign – the ghost, which makes the theory not unitary. In fact, in 1850, Ostrogradsky already showed that if a system contains time derivatives higher than second order, it is not stable unless it is degenerate [3].

1.3 UV Completion

From a quantum field theory point of view, general relativity should be the low energy limit of its UV completion. For UV completion, let us begin with several well known examples in quantum field theory before we enter into details of general relativity and its UV completion. The first example considered here is Schrödinger's equation with some potential given by (in natural units)

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi + V(r)\psi. \tag{1.1}$$

This is a non-relativistic equation. The general Hamiltonian not only contain $\frac{\vec{p}^2}{2m}$, but should be given as

$$H = \frac{\vec{p}^2}{2m}[1 - \frac{\vec{p}^2}{4m^2} + \frac{\vec{p}^4}{8m^4} + \ldots] + V(r). \tag{1.2}$$

The rest of the terms are higher order corrections. But even if we neglect all the higher order terms, Schrödinger's equation still makes many quantum predictions. The reason is that when $|\vec{p}| \ll m$, in the non-relativistic limit, all the higher order terms have very small contributions compared to $\frac{\vec{p}^2}{2m}$. At the same time, this example tells us that when momentum is greater than or close to the mass, perturbation theory breaks down: new physics beyond that energy scale should be taken into account. If we want to make perturbation theory still work beyond a certain energy scale (at high energy), it is called the UV completion of that theory. After Schrödinger's pioneering work on quantum mechanics, Dirac proposed his relativistic equation, the Dirac equation, given by (in natural units)

$$(i\partial\!\!\!/ - m)\psi = 0. \tag{1.3}$$

It is predictive at high energies and is the UV completion of the Schrödinger equation.

Our second example is the Four-Fermi theory of weak interaction. The energy scale of this theory is around 300 GeV. Although the Four-Fermi theory is really predictive below that energy, it preturbatively breaks down as energy approaches the 300 GeV level. The UV completion of the Four-Fermi theory is the famous Glashow-Weinberg-Salam model of electroweak theory which unifies the electromagnetic and weak interactions [4–6]. At high energy, new physics appears since massive vector bosons are produced: electroweak physics dominates at high energy.

Now let us come back to gravity. General relativity is an effective theory, and for energy less than Planck energy, quantum effects in gravity are very small and higher order terms are suppressed. When energies are higher than the Planck energy $\sim 1.2 \times 10^{19}$ GeV, or lengths are smaller than the Planck length $\sim 1.6 \times 10^{-33}$ cm, quantum effects become important and should be taken into account. A possible UV completion of general relativity is string theory, which originates from quantum field theory, but replaces the concept of point particles with strings, objects with one dimensional extended.

1.4 Hořava-Lifshitz gravity

In 2009, P. Hořava proposed a possible candidate for a UV completion of gravity [7]. In order to avoid the ghost problem by holding the time derivative operator up to second order, one can only add higher spatial derivatives into the Lagrangian to make this gravity theory power-counting renormalizable. But the price paid here is to break Lorentz invariance, which is one of the most important symmetries in modern physics and passes all the experimental tests. Nevertheless, it is entirely possible to break Lorentz invariance at ultra-high energy and restore this symmetry at low energy. Hořava borrowed the idea of anisotropic scaling of space and time from condensed matter systems,

$$\mathbf{x} \to b\mathbf{x}, \quad t \to b^z t, \tag{1.4}$$

where z is the dynamical critical exponent measuring the degree of anisotropy between space and time. At high energies, power-counting renormalizable condition requires z to be greater or equal to the spatial dimension; however, at low energies z approaches to 1 to restore the relativistic scaling of space and time. The anisotropic scaling breaks diffeomorphism invariance, which is not the fundamental symmetry in Hořava's theory. The new symmetry is described by the group Diff(M, F) of foliation-preserving diffeomorphisms in which time plays a special role,

$$\delta t = f(t), \quad \delta x^i = \zeta^i(t, \mathbf{x}), \tag{1.5}$$

under this Diff(M, F), the lapse function N, shift vector N^i and the 3-metric g_{ij} transform as

$$\begin{aligned}
\delta N &= \zeta^j \nabla_j N + \dot{N} f + N \dot{f}, \\
\delta N_i &= N_k \nabla_i \zeta^k + \zeta^k \nabla_k N_i + g_{ik} \dot{\zeta}^k + \dot{N}_i f + N_i \dot{f}, \\
\delta g_{ij} &= \nabla_i \zeta_j + \nabla_j \zeta_i + f \dot{g}_{ij},
\end{aligned} \tag{1.6}$$

where $\dot{f} = df/dt$, $N_i = g_{ik} N^k$ and ∇_i denotes the covariant derivative with respect to the 3-metric g_{ij}. E. M. Lifshitz was the first one to construct a scalar field theory by using anisotropic scaling, hence the gravity theory is called Hořava-Lifshitz (HL) gravity.

Another theory which is closely related to HL gravity is the Einstein-æther theory. It breaks local Lorentz invariance by a globally well-defined unit timelike vector u^μ- the æther field [8,9]. Jacobson showed, in the infrared, HL gravity is identical to the hypersurface-orthogonal Einstein-æther theory [10, 11].

For the recent developments of HL gravity, see the review article of [12].

1.5 Black Holes and their Thermodynamics

Black hole physics plays a central role in the understanding of quantum gravity because it ties together quantum theory, thermodynamics and gravity. The classical understanding of a black hole is that nothing can escape from it once it enters inside the event horizon, which is the causal boundary separating the interior of a black hole from the outside. Initially it was believed that a black hole was not a thermodynamic system since there was no thermal radiation from it. The temperature of a classical black hole seemed to be absolute zero and there was no entropy associated with it. Nevertheless, one can dump a cup of tea (with some finite entropy) into a black hole to decrease entropy which violates the second law of thermodynamics. So a black hole should be a thermodynamic system with some entropy. It turns out that when we consider quantum effects (vacuum fluctuations) near the horizon, a black hole does emit particles.

According to the black hole no-hair theorem, a stationary black hole is fully characterized by only three parameters: mass, charge, and angular momentum. Black hole entropy should also depend on these three observable parameters. Hawking's area theorem says: the area of the future event horizon of a black hole, assuming the cosmic censorship and weak energy condition, cannot decrease. One can make an analogy with the second law of classical thermodynamics, and entropy should be a monotonic function of the area of the event horizon to satisfy the generalized second law (GSL) [13, 14], which is characterized by the relation

$$\triangle\left(S_{BH}+S_{matter}\right)\geqslant 0. \tag{1.7}$$

In the early 1970s, Bekenstein and Hawking showed the connection between black holes and thermodynamics [13–16],

$$k_B T=\frac{\hbar\kappa}{2\pi c}, \tag{1.8}$$

$$S=\frac{A_H c^3 k_B}{4\hbar G}, \tag{1.9}$$

where T and S are temperature and entropy of a black hole; κ and A_H are its surface area and horizon area; c is the speed of light; k_B and G are Boltzmann's constant and Newton's constant, respectively.

Consider Eq. (1.9), for the simplest case – the Schwarzschild black hole, the horizon area on the right hand only depends on one parameter of black hole, its mass. For the most general stationary black hole, the Kerr-Newman black hole, the horizon area depends on all three parameters mentioned above.

We summarize the four laws of classical thermodynamics and the four laws of black hole mechanics in Table 1.1 to end this section.

Table 1.1: Four laws of classical thermodynamics are in parallel with four laws of black hole mechanics.

Law	Classical Thermodynamics	Black Hole Mechanics
0th	The temperature is constant through all objects in thermal equilibrium.	The surface gravity of a black hole is constant over the event horizon.
1st	$dE = TdS$ + work done terms.	$dM = \frac{\kappa}{8\pi} dA$ + work done terms.
2nd	The entropy of an isolated system is either increased or stays the same.	The area of the event horizon of an isolated black hole is either increased or stays the same.
3rd	It is impossible to reach absolute zero temperature in a finite number of processes.	It is impossible to reduce the surface gravity of a black hole to zero in a finite number of processes.

1.6 Universal Horizon

Due to the higher spatial derivatives in HL gravity, the dispersion relation in HL gravity is really different from that in general relativity, and can be written in the form as:

$$E^2 = m^2 + {c_n}^2 p^2 + \frac{a_n p^4}{{M_*}^4} + \frac{b_n p^6}{{M_*}^6}, \tag{1.10}$$

where E and p are the energy and spatial momentum of the particle, and c_n, a_n, b_n are coefficients depending on the species of the particle considered. From Eq. (1.10) we can see that both phase and group velocities of particles can be infinitely high. This feature leads to the fact that the causal structure of HL gravity is totally different from that in general relativity, see Figure 1.1. Events inside light cones are causally connected in general relativity. However, in HL gravity, there is an absolute time. Particles can travel with arbitrary large velocities but have to move forward in time. The causal structure is similar to that in the Newtonian case. One might think that usual black holes defined in general relativity cannot exist in HL gravity because the event horizon cannot trap particles with infinitely large velocities.

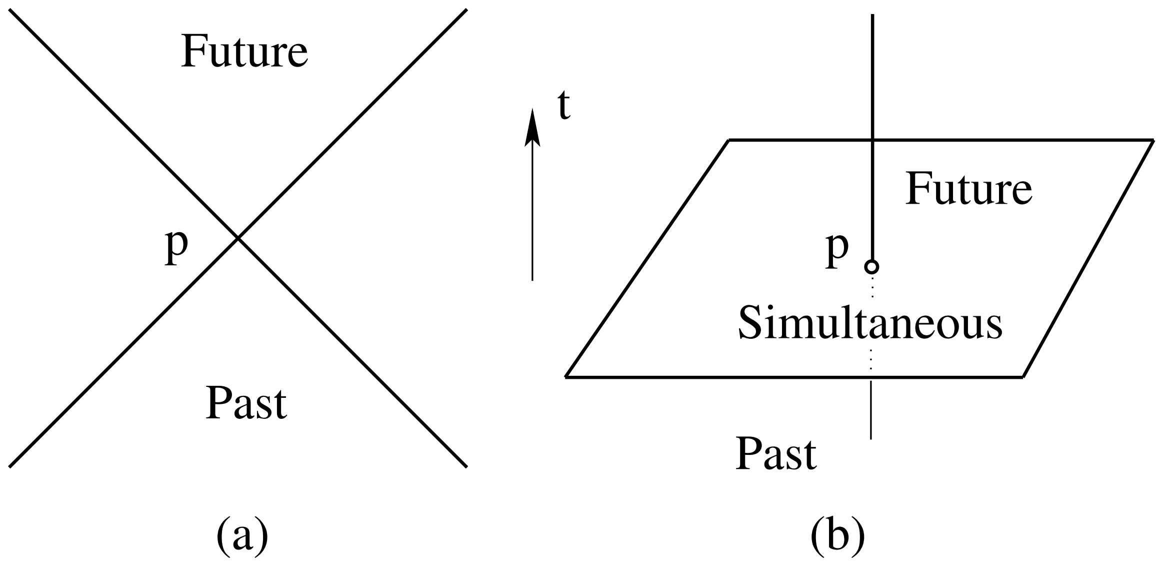

Figure 1.1: (a) The causal structure of relativistic spacetime. Particles travel at no greater than the speed of light. (b) The causal structure of Newtonian spacetime. Velocities of particles can become enormously high, but particles are constrained to move forward in time. This figure is adopted from [20].

However, causal boundaries still exist for gravitational theories which break Lorentz invariance. Blas and Sibiryakov showed that the universal horizon is the causal boundary disconnected regions from the asymptotic infinity [17]. It is like a one-way membrane: once particles are inside the universal horizon, they are destined to hit the black hole singularity [18].

The main idea is as follows: A timelike scalar field ϕ – the khronon field, is introduced in a given spacetime [19]. All particles are constrained to move along the increasing direction of this field, so the khronon field earns the physical meaning of universal time. A timelike unit vector u_μ, normal to the surface ϕ = constant, is defined as

$$u_\mu = \frac{\phi_{,\mu}}{\sqrt{X}}, \tag{1.11}$$

where $\phi_{,\mu} = \partial\phi/\partial x^\mu$, $X = -g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$. In Figure 1.2, the location of the universal horizon is where u_μ and the Killing vector are orthogonal to each other

$$u \cdot \zeta = u_\mu \zeta^\mu = 0, \tag{1.12}$$

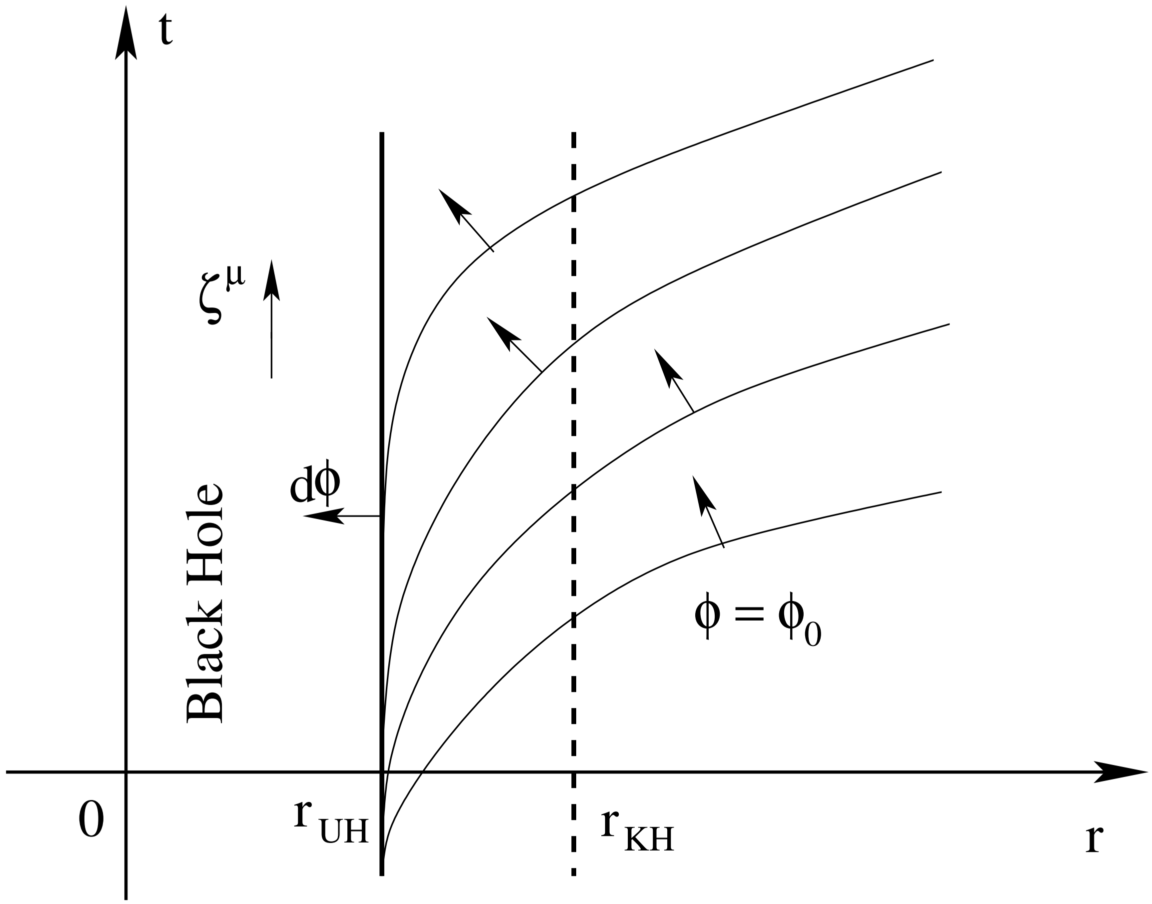

Figure 1.2: The bending of the universal time $\phi =$ constant hypersurfaces, and the location of the universal horizon (Vertical solid line at $r = r_{UH}$) is always inside the Killing horizon (Vertical dashed line at $r = r_{KH}$). Particles are constrained to move forward in time: the increasing direction of ϕ. The Killing vector field always points upward throughout the whole spacetime. This figure is adopted from [20].

where ζ is the asymptotically timelike Killing vector. Since u_μ is a globally well-defined timelike unit vector, if we want Eq. (1.12) to hold, ζ^μ needs to be spacelike vector and it should be always inside the Killing horizon.

As stated in Section 1.5, entropies should be associated with causal boundaries in a given spacetime. In gravitational theories without Lorentz symmetry, one should ascribe the thermodynamic properties not to the Killing horizon, but to the universal horizon.

1.7 Organization of Dissertation

The following chapters will attack topics mentioned in previous sections not only in the framework of HL gravity, but also in Einstein-æther gravity. In particular, Chapter Two is based on [21], in which we study the effects of high-order operators on the nonrelativistic Lifshitz holography in HL gravity. According to the holographic correspondence, the author and collaborators show that these operators affect the two-point correlation functions. This chapter is a published paper co-authored by the author of this dissertation. Dr. A. Wang and Dr. G. Cleaver are Baylor physics professors. Dr. A. Wang pointed out the direction of this project and Dr. G. Cleaver gave some advise. The rest four people are the research performers. Dr. J. Yang and Dr. M. Tian were two visiting scholars at Baylor University from China. X. Wang and Y. Deng were Baylor physics Ph.D. students. Graduate students need work with others to grow and become more experienced. All four visiting scholars and graduate students checked each others' calculations to make sure the results were correct. They are approximately equal contributors to this paper. X. Wang is the author of this dissertation.

Chapter Three is based on [22], in which the author and collaborators present two new classes of exact charged black hole solutions in the framework of the Einstein-Maxwell-æther theory. Smarr formulas are constructed and the temperatures of the horizons are calculated. We find the temperature we obtained is not proportional to its surface gravity at either universal horizon or Killing horizon. This chapter is a published paper co-authored by the author of this dissertation. Dr. A. Wang is a Baylor physics professor. Dr. A. Wang pointed out the direction of this project. The rest two people are the research performers. Dr. C. Ding was a visiting scholar to Baylor University from China. X. Wang was a Baylor physics Ph.D. student. All two visiting scholar and graduate student checked each others' calculations to make sure the results were correct. They are approximately equal contributors to this paper. X. Wang is the author of this dissertation.

Chapter Four is based on [23], in which we study the quantum tunneling effect of relativistic and nonrelativistic particles at both Killing and universal horizons in the framework of the Einstein-Maxwell-æther theory. This chapter is a published paper co-authored by the author of this dissertation. Dr. A. Wang is a Baylor physics professor. Dr. A. Wang pointed out the direction of this project. The rest three people are the research performers. Dr. C. Ding and Dr. T. Zhu were two visiting scholars to Baylor University from China. X. Wang was a Baylor physics Ph.D. student. All three visiting scholars and graduate student checked each others' calculations to make sure the results were correct. They are approximately equal contributors to this paper. X. Wang is the author of this dissertation.

Conclusions and outlook will be presented in Chapter Five.

1.8 Conventions

- Einstein summation convention is adopted:

$$T_\mu{}^\mu \equiv \sum_{\mu=0}^{3} T_\mu{}^\mu. \tag{1.13}$$

- The signature of metric is $(- \; + \; + +)$:

$$\eta_{\mu\nu} = diag(-1, +1, +1, +1). \tag{1.14}$$

- The Christoffel symbol constructed from the metric is given by

$$\Gamma^\lambda_{\mu\nu} = \frac{g^{\lambda\sigma}}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \tag{1.15}$$

- The Riemann tensor obtained from the Christoffel symbol is given by

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\lambda\mu}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\sigma\mu}. \tag{1.16}$$

- Take a contraction of the Riemann tensor to get the Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}. \tag{1.17}$$

- Finally, the Einstein tensor $G_{\mu\nu}$ is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{1.18}$$

CHAPTER TWO

Nonrelativistic Lifshitz Holography in HL Gravity

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In this chapter, we study the effects of high-order operators on the non-relativistic Lifshitz holography in the framework of the Hořava-Lifshitz (HL) theory of gravity, which naturally contains high-order operators in order for the theory to be power-counting renormalizble, and provides an ideal place for such studies. In particular, we show that the Lifshitz space-time is still a solution of the full theory of the HL gravity. The effects of the high-order operators on the spacetime itself is simply to shift the Lifshitz dynamical exponent. However, while in the infrared the asymptotic behavior of a (probe) scalar field near the boundary is similar to that studied in the literature, it gets dramatically modified in the UV limit, because of the presence of the high-order operators in this regime. Then, according to the gauge/gravity duality, this in turn affects the two-point correlation functions.

2.1 Introduction

Non-relativistic gauge/gravity duality has attracted lot of attention recently, as it may provide valuable tools to study strongly coupling systems encountered in condensed matter physics [24], which otherwise are not tractable with our current understanding. If such a duality indeed exists, instead of directly studying those strongly coupling systems, one can study the corresponding weakly coupling systems of gravity, which are much easier to handle, and often well within our abilities.

The non-relativistic quantum field theories (NQFT) are usually assumed to possess either the Schrödinger [25] or the Lifshitz [26] symmetry. In the latter, the symmetry algebra consists of the rotations M_{ij}, spatial translations P_i, time translations H, and dilatations D. These generators satisfy the standard commutation relations for M_{ij}, P_k and H [28], while with D the relations read,

$$[D, M_{ij}] = 0, \quad [D, P_i] = iP_i, \quad [D, H] = izH, \tag{2.1}$$

where z denotes the Lifshitz dynamical exponent, and determines the relative scaling between the time and spatial coordinates [27],

$$x^i \to \ell x^i, \quad t \to \ell^z t. \tag{2.2}$$

This algebra is often called the Lifshitz algebra, as it generalizes the symmetry of Lifshitz fixed points [24].

The gauge/gravity duality requires that the space-time in the gravitational side must possess the same symmetry. However, the symmetry of a spacetime is usually defined by the existence of Killing vectors ζ_μ [29], satisfying the Killing equations,

$$\zeta_{\mu;\nu} + \zeta_{\nu;\mu} = 0, \tag{2.3}$$

where a semicolon ";" denotes the covariant derivative with respect to the spacetime metric $g_{\mu\nu}$. It was found that this can be realized in the Lifshitz spacetime [26],

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2, \tag{2.4}$$

where $d\vec{x}^2 \equiv \sum_{i=1}^{d} dx^i dx^i$. Then, the Killing vectors $\zeta^\mu \partial_\mu \equiv (M, P, H, D)$ of the above spacetime, given by,

$$\begin{aligned} &M_{ij} = -i\left(x_i \partial_j - x_j \partial_i\right), \quad P_i = -i\partial_i, \\ &H = -i\partial_t, \quad D = -i\left(zt\partial_t + x^i \partial_i - r\partial_r\right), \end{aligned} \tag{2.5}$$

produce precisely the required Lifshitz algebra, where $x_i \equiv \delta_{ij} x^j$. The corresponding NQFT lives on the boundary $r = \infty$.

Note that the metric is invariant under the rescaling (2.2), provided that r is scaling as $r \to \ell^{-1} r$. Clearly, this is non-relativistic for $z \neq 1$, and to produce such a spacetime in Einstein's theory of general relativity (GR), matter fields must be present, in order to create such a preferred direction. In [26], this was realized by two p-form gauge fields with $p = 1, 2$, and was soon generalized to other cases [30].

On the other hand, to construct a viable theory of quantum gravity, Hořava [7] recently proposed a theory based on the anisotropic scaling (2.2), the so-called Hořava-Lifshitz (HL) theory of quantum gravity, and has attracted a great deal of attention, due to its several remarkable features [31]. The HL theory is based on the perspective that Lorentz symmetry should appear as an emergent symmetry at long distances, but can be fundamentally absent at short ones [32]. In the UV regime, the system exhibits a strong anisotropic scaling between space and time, given by Eq. (2.2). To have the theory be power-counting renormalizable, the Lifshitz dynamical exponent z must be no less than D in the $(D+1)$-dimensional spacetime [7,33]. At long distances, high-order curvature corrections become negligible, and the lowest order terms take over, whereby the Lorentz invariance is expected to be "accidentally restored."

Since in the HL gravity the anisotropic scaling (2.2) is built in [1] , it is natural to expect that the HL gravity provides a minimal holographic dual for non-relativistic Lifshitz-type field theories. Indeed, recently it was showed that the Lifshitz spacetime (2.4) is a vacuum solution of the HL gravity in (2+1) dimensions, and that the full structure of the $z = 2$ anisotropic Weyl anomaly can be reproduced in dual field theories [34], while its minimal relativistic gravity counterpart yields only one of two independent central charges in the anomaly. This speculation has been further confirmed by the existence of other types of the Lifshitz spacetimes, including Lifshitz solitons [35, 36].

[1] It should be noted that in the HL gravity, all the spatial coordinates $\left(r, x^i\right)$ are scaling as $x^n \to \ell x^n$, where $n = r, i, (i = 1, 2, 3, ..., d)$. This is different from that of the metric (2.4), in which r must be scaling as $r \to \ell^{-1} r$, in order to keep the metric invariant. Therefore, in principle the Lifshitz dynamical exponent z appearing in (2.4) is different from that considered in the HL theory: $x^n \to \ell x^n, \quad t \to \ell^z t$.

In this chapter, we study another important issue: the effects of high-order operators in non-relativistic Lifshitz holography. Since high-order operators necessarily appear in the HL gravity in order to be power-counting renormalizable, it provides an ideal place to study such effects. In the framework of GR, this was studied in [37], and found that these effects only shift the values of z. In this chapter, we shall first show that this is true also in the HL gravity. Then, we study the effects on a scalar field and the corresponding two-point correlation functions. We find that, while in the infrared the asymptotic behavior of a (probe) scalar field near the boundary is similar to that studied in [26], it is dramatically modified in the UV limit, because of the presence of the high-order operators in this regime. Then, according to the gauge/gravity duality, this in turn affects the two-point correlation functions. This is expected, as in the UV the high-order operators will dominate, and the asymptotic behavior of the scalar field will be determined by these high-order operators.

Specifically, the chapter is organized as follows: In Section 2.2, we shall give a brief introduction to the non-projectable HL gravity in (2+1)-dimensional spacetimes, and find out the stability and ghost-free conditions in terms of the independently coupling constants of the theory. In Section 2.3, we show that the Lifshitz spacetime (2.4) is not only a solution of the HL gravity in the IR limit, but also a solution of the full theory. The only difference is that the Lifshitz dynamical exponent z is shifted. In Section 2.4, we study a scalar field propagating on the Lifshitz background (2.4). To compare our results with the ones obtained in [26], in this section (and also the next) we set $z = 2$. In Section 2.5, we calculate the two-point correlation functions, and find their main properties in the IR as well as in the UV limit.

Because of the anisotropic scaling (2.2) (see also Footnote 1), the gauge symmetry of the theory is broken down to the foliation-preserving diffeomorphism, Diff(M, $\mathcal{F}$),

$$\delta t = -f(t), \quad \delta x^i = -\zeta^i(t, \mathbf{x}), \tag{2.6}$$

for which the lapse function N, shift vector N^i, and 3-spatial metric g_{ij}, first introduced in the Arnowitt-Deser-Misner (ADM) decompositions [38], transform as

$$\begin{aligned}
\delta N &= \zeta^k \nabla_k N + \dot{N} f + N \dot{f}, \\
\delta N_i &= N_k \nabla_i \zeta^k + \zeta^k \nabla_k N_i + g_{ik} \dot{\zeta}^k + \dot{N}_i f + N_i \dot{f}, \\
\delta g_{ij} &= \nabla_i \zeta_j + \nabla_j \zeta_i + f \dot{g}_{ij},
\end{aligned} \tag{2.7}$$

where $\dot{f} \equiv df/dt$, ∇_i denotes the covariant derivative with respect to g_{ij}, $N_i = g_{ik} N^k$, and $\delta g_{ij} \equiv \tilde{g}_{ij}\left(t, x^k\right) - g_{ij}\left(t, x^k\right)$, etc.

Due to the Diff(M, $\mathcal{F}$) diffeomorphisms (2.6), one more degree of freedom appears in the gravitational sector - a spin-0 graviton. Using the gauge freedom (2.6), without loss of the generality, one can always set

$$N^i = 0, \tag{2.8}$$

for which the remaining gauge freedom is

$$t = \hat{f}(t'), \quad x^i = \hat{\zeta}^i(x'). \tag{2.9}$$

In the rest of this section, we shall leave the gauge choice open, and in particular not restrict ourselves to the gauge (2.8).

The Riemann and Ricci tensors R_{ijkl} and R_{ij} of the 2D leaves t = constant are uniquely determined by the 2D Ricci scalar R via the relations [39],

$$\begin{aligned}
R_{ijkl} &= \frac{1}{2}\left(g_{ik} g_{jl} - g_{il} g_{jk}\right) R, \\
R_{ij} &= \frac{1}{2} g_{ij} R, \ (i, j = 1, 2).
\end{aligned} \tag{2.10}$$

The general action of the HL theory without the projectability condition in (2+1)-dimensional spacetimes is given by [35]

$$S = \zeta^2 \int dt d^2x N\sqrt{g}\Big(\mathcal{L}_K - \mathcal{L}_V + \zeta^{-2}\mathcal{L}_M\Big), \tag{2.11}$$

where $g = \det(g_{ij})$, $\zeta^2 = 1/(16\pi G)$, and

$$\begin{aligned}
\mathcal{L}_K &= K_{ij}K^{ij} - \lambda K^2, \\
\mathcal{L}_V &= \gamma_0\zeta^2 + \beta a_i a^i + \gamma_1 R \\
&\quad + \frac{1}{\zeta^2}\Big[\gamma_2 R^2 + \beta_1 \left(a_i a^i\right)^2 + \beta_2 \left(a^i{}_i\right)^2 \\
&\quad + \beta_3 a_i a^i a^j{}_j + \beta_4 a^{ij} a_{ij} + \beta_5 a^i a_i R + \beta_6 a^i{}_i R\Big],
\end{aligned} \tag{2.12}$$

with $\Delta \equiv g^{ij}\nabla_i\nabla_j$, and

$$\begin{aligned}
K_{ij} &= \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right), \\
a_i &= \frac{N_{,i}}{N}, \quad a_{ij} = \nabla_i a_j.
\end{aligned} \tag{2.13}$$

$\mathcal{L}_M$ is the Lagrangian of matter fields. Then, the corresponding field equations and conservation laws are given explicitly in [35].

2.2.1 Stability and Ghost-free Conditions

It is easy to show that the Minkowski spacetime

$$\left(\bar{N}, \bar{N}^i, \bar{g}_{ij}\right) = \left(1, 0, \delta_{ij}\right), \tag{2.14}$$

is a solution of the above HL gravity with $\gamma_0 = 0$. Then its linear perturbations are given by

$$\begin{aligned}
&\delta N = n, \quad \delta N_i = \partial_i B - S_i, \\
&\delta g_{ij} = -2\psi\delta_{ij} + \left(\partial_i\partial_j - \delta_{ij}\partial^2\right) E + 2F_{(i,j)},
\end{aligned} \tag{2.15}$$

where $F_{(i,j)} \equiv (F_{i,j} + F_{j,i})/2$, and

$$\partial^i S_i = \partial^i F_i = 0. \tag{2.16}$$

It is interesting to note that in the decompositions (2.15) no tensor mode appears in δg_{ij}. This is closely related to the fact that in (2+1)-dimensional spacetimes, spin-2 massless gravitons do not exist.

Then the infinitesimal gauge transformations (2.7) can be written as

$$f = \epsilon(t), \quad \zeta^i = \partial^i \zeta + \eta^i, \ \ (\partial_i \eta^i = 0), \tag{2.17}$$

under which the quantities defined in Eq. (2.15) transfer as,

$$\begin{aligned}
&\tilde{n} = n + \dot{\epsilon}, \qquad \tilde{B} = B + \dot{\zeta}, \\
&\tilde{E} = E + \zeta, \qquad \tilde{\psi} = \psi - \frac{1}{2}\partial^2 \zeta, \\
&\tilde{S}_i = S_i + \dot{\eta}_i, \quad \tilde{F}_i = F_i + \eta_i.
\end{aligned} \tag{2.18}$$

Thus, from the above we can construct three scalar and one vector gauge-invariants,

$$\begin{aligned}
&\Psi \equiv \psi + \frac{1}{2}\partial^2 E, \quad \Phi \equiv B - \dot{E}, \\
&\Upsilon \equiv \partial^2 n, \qquad\qquad \Phi_i \equiv S_i - \dot{F}_i.
\end{aligned} \tag{2.19}$$

Using the above gauge freedom, without loss of the generality, we can set

$$E = 0, \quad F_i = 0, \tag{2.20}$$

which will uniquely fix the gauge freedom represented by ζ and η_i, while leave $\epsilon(t)$ unspecified. To further study the above linear perturbations, let us consider the scalar and vector perturbations separately.

Scalar perturbations. Under the gauge (2.20), the remaining scalars are n, B and ψ, with which it can be shown that the gravitational sector of the action to the second-order takes the form,

$$\begin{aligned}
S_g^{(2)} &= \zeta^2 \int dt d^2x \Bigg\{ 2(1-2\lambda)\dot{\psi}^2 + 2(1+2\lambda)\dot{\psi}\partial^2 B \\
&+ (1-\lambda)(\partial^2 B)^2 + \beta n \partial^2 n - 2\gamma_1 n \partial^2 \psi \\
&- \frac{1}{\zeta^2}\left[4\gamma_2(\partial^2\psi)^2 + (\beta_2+\beta_4)(\partial^2 n)^2 + 2\beta_6(\partial^2 n)(\partial^2\psi)\right] \Bigg\}.
\end{aligned} \tag{2.21}$$

Its variations with respect to ψ, B and n yield, respectively,

$$\ddot{\psi}+\frac{1}{2}\partial^2\dot{B}+\frac{\gamma_1}{2(1-2\lambda)}\partial^2 n+\frac{4\gamma_2\partial^4\psi+\beta_6\partial^4 n}{2\zeta^2(1-2\lambda)}=0, \tag{2.22}$$

$$(1-2\lambda)\dot{\psi}+(1-\lambda)\partial^2 B=0, \tag{2.23}$$

$$\beta n-\gamma_1\psi-\frac{\beta_2+\beta_4}{\zeta^2}\partial^2 n-\frac{\beta_6}{\zeta^2}\partial^2\psi=0. \tag{2.24}$$

From Eq. (2.23) we can find B in terms of ψ, and then substituting it into (2.21) we obtain,

$$\begin{aligned} S_g^{(2)} &= \zeta^2\int dtd^2x\Bigg\{\frac{1-2\lambda}{1-\lambda}\dot{\psi}^2+\beta n\partial^2 n-2\gamma_1 n\partial^2\psi \\ &\quad -\frac{1}{\zeta^2}\Big[4\gamma_2(\partial^2\psi)^2+(\beta_2+\beta_4)(\partial^2 n)^2+2\beta_6(\partial^2 n)(\partial^2\psi)\Big]\Bigg\}. \end{aligned} \tag{2.25}$$

Then, the ghost-free condition require

$$\frac{1-2\lambda}{1-\lambda}\geq 0, \tag{2.26}$$

that is,

$$(i)\ \lambda>1 \quad \text{or} \quad (ii)\ \lambda\leq\frac{1}{2}. \tag{2.27}$$

From Eqs. (2.22)-(2.24), on the other hand, we can get a master equation for ψ, which in momentum space can be written in the form

$$\ddot{\psi}_k+\omega_k^2\psi_k=0, \tag{2.28}$$

where

$$\begin{aligned} \omega_k^2 &= \frac{1-\lambda}{1-2\lambda}\left(\frac{4\gamma_2 k^4}{\zeta^2}+\Big(\frac{\beta_6 k^4}{\zeta^2}-\gamma_1 k^2\Big)\frac{\gamma_1-\frac{\beta_6 k^2}{\zeta^2}}{\beta+\frac{(\beta_2+\beta_4)k^2}{\zeta^2}}\right) \\ &= \begin{cases} -\frac{1-\lambda}{1-2\lambda}\frac{\gamma_1^2 k^2}{\beta}, & \mathrm{k}^2/\zeta\ll 1, \\ \frac{1-\lambda}{1-2\lambda}\left(4\gamma_2-\frac{\beta_6^2}{\beta_2+\beta_4}\right)\frac{k^4}{\zeta^2}, & \mathrm{k}^2/\zeta\gg 1. \end{cases} \end{aligned} \tag{2.29}$$

Thus, to have the mode be stable in the infrared (IR), we must require

$$\beta<0, \tag{2.30}$$

while its stability condition in the ultraviolet (UV) requires

$$\gamma_2 \geq \frac{\beta_6^2}{4(\beta_2 + \beta_4)}. \tag{2.31}$$

In the intermediate range, by properly choosing other free parameters the mode can always be made stable, and such a requirement does not impose any severe constraints. So, in the following we do not consider it further, and simply assume that it is always satisfied.

It should be noted that the conditions (2.27), (2.30) and (2.31) are valid only for the cases $\lambda \neq 1$, for which Eq. (2.30) tells that β must be strictly negative, and in particular cannot be zero.

When $\lambda = 1$, from Eq. (2.23) we find that

$$\dot{\psi} = 0, \tag{2.32}$$

that is, ψ does not represent a propagative mode, and we can always set it to zero by properly choosing the boundary conditions. Then, Eqs. (2.22) and (2.24) reduce to,

$$\dot{B} - \gamma_1 n - \frac{\beta_6}{\zeta^2}\partial^2 n = 0, \tag{2.33}$$

$$\frac{\beta_2 + \beta_4}{\zeta^2}\partial^2 n - \beta n = 0. \tag{2.34}$$

From the last equation, we can see that n does not represent a propagative mode either, and can be set to zero by properly choosing the boundary conditions. Then, Eq. (2.33) yields $\dot{B} = 0$, that is, B is also not a propagative mode.

Therefore, in the case $\lambda = 1$ there is no gravitational propagative mode, similar to the relativistic case [39]. As a result, *all the parameters in this case are free, as long as the stability and ghost-free conditions are concerned.*

As a corollary, we find that the HL theory with $\beta = 0$ is viable only when $\lambda = 1$. Otherwise, the corresponding scalar mode will become unstable, as one can see clearly from Eq. (2.29).

Vector perturbations. Under the gauge (2.20), the remaining vector is S_i, with which it can be shown that the gravitational sector of the action to the second-order takes the form,

$$S_g^{(2)} = -\frac{\zeta^2}{2}\int dtd^2xN\sqrt{g}\big(S^i\partial^2 S_i\big), \tag{2.35}$$

from which we find that,

$$\partial^2 S^i = 0. \tag{2.36}$$

That is, there is no propagative vector mode in the HL gravity; even the Lorentz symmetry is violated.

In summary, the above analysis shows: *(i) In the case $\lambda \neq 1$, only spin-0 gravitons exist in the (2+1)-dimensional non-projectable HL gravity.* Their stability and ghost-free conditions require the independent coupling constants must satisfy the conditions of Eqs. (2.27), (2.30) and (2.31). *(ii) In the case $\lambda = 1$, the gravitational sector of the HL gravity has no free propagation mode, similar to its relativistic counterpart.* Then, all the free parameters in this case are free, as long as the stability and ghost-free conditions are concerned.

2.2.2 Detailed Balance Condition

To reduce the number of the coupling constants, Hořava imposed the detailed balance condition [7]. The main idea is to introduce a superpotential W on the leaves $t =$ constant,

$$W = \int d^2x\sqrt{g}\mathcal{L}_W\left(R_{ij}, a_k, \nabla_l\right), \tag{2.37}$$

so that the potential part of the action is given by

$$\hat{\mathcal{L}}_V^{(DB)} = E_{ij}G^{ijkl}E_{kl}, \quad E_{ij} \equiv \frac{1}{\sqrt{g}}\frac{\delta W}{\delta g^{ij}}, \tag{2.38}$$

where G^{ijkl} denotes the generalized de Witt metric on the space of metrics, and is given by

$$G^{ijkl} \equiv \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda g^{ij}g^{kl}. \tag{2.39}$$

Power-counting renormalizibility requires that the dimension of $\mathcal{L}_W$ must be greater or equal to $2d$, that is, $[\mathcal{L}_W] \geq 2d$. Taking the lowest dimension, one can see that in (2+1)-dimensional space-times, $\mathcal{L}_W$ in general can be cast in the form,

$$\mathcal{L}_W = w\left(R + \mu a_i a^i - 2\Lambda_W\right), \tag{2.40}$$

where w, μ and Λ_W are three coupling constants. Plugging the above into Eq. (2.38) and taking Eq. (2.6) into account, we find that

$$\begin{aligned} E_{ij} &= w\left[\mu\left(a_i a_j - \frac{1}{2}g_{ij}a_k a^k\right) + \Lambda_W g_{ij}\right], \\ \hat{\mathcal{L}}_V^{(DB)} &= \frac{w^2}{2}\left[\mu^2\left(a_i a^i\right)^2 + 4\left(1-2\lambda\right)\Lambda_W^2\right]. \end{aligned} \tag{2.41}$$

To have a healthy IR limit, the detailed balance condition is frequently allowed to be broken softly [7,40,41] by adding all the low dimensional relevant terms, R, $a_i a^i$, Λ, into $\hat{\mathcal{L}}_V^{(DB)}$, so that the potential is finally given by

$$\mathcal{L}_V^{(DB)} = 2\Lambda + \beta a_i a^i + \gamma_1 R + \frac{\beta_1}{\zeta^2}\left(a_i a^i\right)^2, \tag{2.42}$$

where $\beta_1 \equiv w^2\mu^2/2$ and $\Lambda \equiv \gamma_0\zeta^2/2$. Comparing it with $\mathcal{L}_V$ given by Eq. (2.12), one can see that this is equivalent to set $\gamma_2 = 0 = \beta_n$ $(2 \leq n \leq 6)$.

2.3 Lifshitz Spacetimes in (2+1) Dimensions

In this section we are going to study static vacuum spacetimes with the ADM variables given by

$$\begin{aligned} N &= r^z f(r), \quad N^i = 0, \\ g_{ij} &= \text{diag}\left(\frac{g^2(r)}{r^2}, r^2\right), \end{aligned} \tag{2.43}$$

in the coordinates (t, r, x), where z is the dynamical Lifshitz exponent. Then, we find that

$$\begin{aligned} R_{ij} &= \frac{rg' - g}{r^2 g}\delta^r_i\delta^r_j + \frac{r^2\left(rg' - g\right)}{g^3}\delta^\theta_i\delta^\theta_j, \\ a_i &= \frac{(zf + rf')}{rf}\delta^r_i, \quad K_{ij} = 0. \end{aligned} \tag{2.44}$$

Inserting the above into the general action (2.11), for the vacuum case $\mathcal{L}_M = 0$, we obtain

$$S_g = -V_x \zeta^2 \int dt dr r^z f g \mathcal{L}_V \Big(f^{(n)}, g^{(m)}, r \Big), \tag{2.45}$$

where $V_x \equiv \int dx$, $I^{(n)} \equiv d^n I(r)/dr^n$. Then, it can be shown that in the present case there are only two independent equations, which can be cast in the forms,

$$\sum_{n=0}^{3} (-1)^n \frac{d^n}{dr^n} \left(\frac{\delta \mathcal{L}_g}{\delta f^{(n)}} \right) = 0, \tag{2.46}$$

$$\sum_{n=0}^{3} (-1)^n \frac{d^n}{dr^n} \left(\frac{\delta \mathcal{L}_g}{\delta g^{(n)}} \right) = 0, \tag{2.47}$$

where $\mathcal{L}_g \equiv r^z f g \mathcal{L}_V$.

The Lifshitz spacetime corresponds to

$$f = f_0, \quad g = g_0, \tag{2.48}$$

where f_0 and g_0 are two constant. Then, the corresponding metric can be cast in the form,

$$ds^2 = L^2 \left\{ -\left(\frac{r}{\ell}\right)^{2z} dt^2 + \left(\frac{\ell}{r}\right)^2 dr^2 + \left(\frac{r}{\ell}\right)^2 dx^2 \right\}, \tag{2.49}$$

where $L \equiv (f_0 g_0^z)^{1/(z+1)}$, $\ell \equiv (g_0/f_0)^{1/(1+z)}$. Inserting Eq. (2.48) into Eqs. (2.46) and (2.47), we obtain

$$\begin{aligned} &2\zeta^2 \Lambda g_0^4 - \zeta^2 g_0^2 \left[z(2+z)\beta + 2\gamma_1 \right] - z^3(4+3z)\beta_1 + 4\gamma_2 \\ &+ z\Big[z(3+2z)\beta_2 + z\left(z^2-2\right)\beta_3 - (2+z)\left(\beta_4 - 2\beta_5 + 2\beta_6\right) \Big] = 0, \end{aligned} \tag{2.50}$$

$$\begin{aligned} &2\zeta^2 \Lambda g_0^4 - z\zeta^2 g_0^2 \left(z\beta + 2\gamma_1\right) - 4\gamma_2 + 2z\left(4\gamma_2 + \beta_6\right) \\ &-z^2 \Bigg\{ \beta_2 + 3\beta_4 - 4\beta_5 + 4\beta_6 + z\Big[3z\beta_1 - 2\beta_2 - (z-2)\beta_3 + 2\beta_5 \Big] \Bigg\} = 0. \end{aligned} \tag{2.51}$$

In the IR limit, all the fourth-order terms become negligible, and the above equations reduce to

$$2\Lambda g_0^2 - \left[z(2+z)\beta + 2\gamma_1 \right] = 0, \tag{2.52}$$

$$2\Lambda g_0^2 - z\left(z\beta + 2\gamma_1\right) = 0, \tag{2.53}$$

which have the solutions,

$$z = \frac{\gamma_1}{\gamma_1 - \beta}, \quad \Lambda = \frac{\gamma_1^2(2\gamma_1 - \beta)}{2g_0^2(\gamma_1 - \beta)^2}. \tag{2.54}$$

These are exactly what were obtained in [34].

When the higher-order operators are not negligible, the sum of Eqs. (2.50) and (2.51) yields,

$$\begin{aligned}
&\Lambda = \frac{\zeta^2\left[z\beta + (1-z)\,\gamma_1\right]}{\Delta}\Big\{z^4\left[z\beta - (1+3z)\,\gamma_1\right]\beta_1 \\
&+z^2\left[z\beta + \left(2z^2+z+1\right)\gamma_1\right]\beta_2 + z^4\left[\beta + (z-1)\,\gamma_1\right]\beta_3 \\
&+z^2\left[z\,(z+2)\,\beta + (1-z)\,\gamma_1\right]\beta_4 + z^3\left[(z+2)\,(z-1)\,\beta + 4\gamma_1\right]\beta_5 \\
&+z\left[z\,(z+2)\,(z+1)\,\beta - 2\gamma_1(z^2+1)\right]\beta_6 \\
&-4\left[z\left(z^2+z-1\right)\beta + (z-1)\,\gamma_1\right]\gamma_2\Big\},
\end{aligned} \tag{2.55}$$

where

$$\begin{aligned}
\Delta &= 2\Big\{2z^3\beta_1 - 2z^2\beta_2 - z\,(z-3)\,\beta_6 \\
&\quad + (1-z)\left[z^2\beta_3 + z\beta_4 - 4\gamma_2\right] - z\left[2 + z\,(z-1)\right]\beta_5\Big\}^2.
\end{aligned} \tag{2.56}$$

The difference of Eqs. (2.50) and (2.51), on the other hand, yields,

$$az^3 + bz^2 + cz + d = 0, \tag{2.57}$$

where

$$\begin{aligned}
a &= -2\beta_1 + \beta_3 + \beta_5, \\
b &= 2\beta_2 - \beta_3 + \beta_4 - \beta_5 + \beta_6, \\
c &= -\alpha^2(\beta - \gamma_1) - 4\gamma_2 - \beta_4 + 2\beta_5 - 3\beta_6, \\
d &= 4\gamma_2 - \alpha^2\gamma_1, \quad \alpha \equiv \zeta g_0,
\end{aligned} \tag{2.58}$$

which can be used to determine the dynamical exponent z in terms of the coupling constants. In general, it has three different solutions for any given set of the coupling constants. On the other hand, Eq. (2.57) can be also used to determine the integration constant g_0 for any given z and a set of the coupling constants. In this case, we have

$$g_0^2 = \frac{az^3 + bz^2 + \hat{c}z + 4\gamma_2}{\zeta^2[\gamma_1 - (\gamma_1 - \beta)z]}, \tag{2.59}$$

where $\hat{c} \equiv -4\gamma_2 - \beta_4 + 2\beta_5 - 3\beta_6$. Clearly, for the metric to have a proper signature, z has to be chosen so that $g_0^2 > 0$ for any given set of coupling constants (β_i, γ_j).

When the fourth-order corrections are small, we can expand z near its IR fixed point, z_0, given by Eq. (2.54). Writing the fourth-order coupling constants in the form $s = s_0 + \epsilon\hat{s}$, where $\epsilon \ll 1$, we find that

$$\begin{aligned}
z &= z_0 + \epsilon\delta z,\\
a &= \epsilon(-2\hat{\beta}_1 + \hat{\beta}_3 + \hat{\beta}_5),\\
b &= \epsilon(2\hat{\beta}_2 - \hat{\beta}_3 + \hat{\beta}_4 - \hat{\beta}_5 + \hat{\beta}_6),\\
c &= c_0 + \epsilon(-4\hat{\gamma}_2 - \hat{\beta}_4 + 2\hat{\beta}_5 - 3\hat{\beta}_6),\\
d &= d_0 + 4\epsilon\hat{\gamma}_2,
\end{aligned} \tag{2.60}$$

where

$$z_0 = \frac{\gamma_1}{\gamma_1 - \beta}, \quad c_0 = -\alpha^2(\beta - \gamma_1), \quad d_0 = -\alpha^2\gamma_1.$$

Thus, to the first-order of ϵ Eq. (2.57) yields,

$$\begin{aligned}
&(-2\hat{\beta}_1 + \hat{\beta}_3 + \hat{\beta}_5)z_0^3 + (2\hat{\beta}_2 - \hat{\beta}_3 + \hat{\beta}_4 - \hat{\beta}_5 + \hat{\beta}_6)z_0^2\\
&+(-4\hat{\gamma}_2 - \hat{\beta}_4 + 2\hat{\beta}_5 - 3\hat{\beta}_6)z_0 + 4\hat{\gamma}_2 + c_0\delta z = 0,
\end{aligned} \tag{2.61}$$

from which we find that,

$$\begin{aligned}
\delta z = \frac{1}{\alpha^2(\beta - \gamma_1)^4}\Big\{&\gamma_1[\beta^2\left(\beta_4 - 2\beta_5 + 3\beta_6\right) - \beta\gamma_1\left(-2\beta_2 + \beta_3 + \beta_4 - 3\beta_5 + 5\beta_6\right)\\
&+ 2\gamma_1^2\left(\beta_1 - \beta_2 - \beta_5 + \beta_6\right)] + 4\beta\gamma_2\left(\beta - \gamma_1\right)^2\Big\}.
\end{aligned} \tag{2.62}$$

Note that in writing the above expression, without causing any confusion, we had dropped hats from all fourth-order parameters. To study the behavior of z in the UV, let us consider some particular cases.

2.3.1 Solutions with Softly-breaking Detailed Balance Condition

When the softly-breaking detailed balance condition is imposed, we have $\gamma_2 = \beta_i = 0,\ (i \geq 2)$. Then, Eqs. (2.57) and (2.55) reduce, respectively, to

$$z^3 + \frac{\alpha^2}{2\beta_1}\left(\beta - \gamma_1\right) z + \frac{\alpha^2}{2\beta_1}\gamma_1 = 0, \tag{2.63}$$

$$\Lambda = \frac{\zeta^2}{4z^2\beta_1}\left[z\beta + (1-z)\gamma_1\right]\left[z\beta - (1+3z)\gamma_1\right]. \tag{2.64}$$

Eq. (2.63) in general has three roots, and depending on the signature of $\mathcal{D}$, the nature of these roots are different, where

$$\mathcal{D} \equiv \frac{\alpha^4}{16\beta_1^2}\left[\gamma_1^2 - \frac{2\alpha^2\left(\gamma_1 - \beta\right)^3}{27\beta_1}\right]. \tag{2.65}$$

Let us consider the cases $\mathcal{D} = 0,\ \mathcal{D} > 0$ and $\mathcal{D} < 0$, separately.

When $\mathcal{D} = 0$, we find that

$$\beta_1 = \frac{2\alpha^2\left(\gamma_1 - \beta\right)^3}{27\gamma_1^2}, \tag{2.66}$$

and Eq. (2.63) has three real roots, two of which are equal and given by

$$z_1 = \frac{3\gamma_1}{\beta - \gamma_1}, \quad z_2 = z_3 = -\frac{3\gamma_1}{2\left(\beta - \gamma_1\right)}. \tag{2.67}$$

Clearly, by properly choosing β and γ_1, they can take any real values, $z_i \in (-\infty, \infty)$.

When $\mathcal{D} > 0$, Eq. (2.63) has only one real root, which can be written as

$$z = \sqrt[3]{\mathcal{D}^{1/2} - \frac{q}{2}} - \sqrt[3]{\mathcal{D}^{1/2} + \frac{q}{2}}, \tag{2.68}$$

where $q \equiv \alpha^2\gamma_1/(2\beta_1)$. In this case it is clear that z can also take any real values for different choices of $(\beta, \gamma_1, \beta_1)$. In particular, it has an extreme at $\beta = \gamma_1$, given by $z_m = -q^{1/3}$.

When $\mathcal{D} < 0$, Eq. (2.63) has three real and different roots, given by

$$z_n = \sqrt{\frac{2\alpha^2\left(\gamma_1 - \beta\right)}{3\beta_1}}\cos\left(\theta + \frac{2n\pi}{3}\right), (n = 0, 1, 2), \tag{2.69}$$

where θ is defined as

$$\theta = \frac{1}{3}\arccos\left[\frac{\alpha^2\gamma_1}{4\beta_1}\left(\frac{6\beta_1}{\alpha^2(\gamma_1-\beta)}\right)^{3/2}\right]. \tag{2.70}$$

Again, similar to the last two subcases, by choosing different values of the coupling constants, we can have different values of z_n. For example, taking $\alpha^2 = 4,\ \beta = -1,\ \beta_1 = 0.00001,\ \gamma_1 = 1$, we obtain $z_1 \simeq 632.205$.

2.3.2 Solutions with $\mathcal{L}_V = \mathcal{F}(R)$

Another interesting case is the $\mathcal{F}(R)$ models [42], for which we have

$$\mathcal{L}_V = \mathcal{F}(R), \tag{2.71}$$

where $\mathcal{F}(R)$ can be any function of R (possibly subjected to some stability and ghost-free conditions). In particular, one can take the form,

$$\mathcal{F}(R) = 2\Lambda + \gamma_1 R + \beta\mathcal{A}^2 + \frac{\gamma_2}{\zeta^2}R^2, \tag{2.72}$$

which corresponds to the potential given by Eq. (2.12) with $\beta_i = 0,\ (i = 1, ..., 6)$, where $\mathcal{A}^2 \equiv a_i a^i$. Note that in writing the above expression, we had kept the $a_i a^i$ term, in order to have a healthy IR limit for any given coupling constant λ [34,35].

In this case, Eqs. (2.50) and (2.51) have the solutions,

$$\begin{aligned} z &= 1 - \frac{\alpha^2\beta}{4\gamma_2 - \alpha^2(\gamma_1-\beta)}, \\ \Lambda &= \frac{\zeta^2}{2\alpha^4}\left\{\alpha^2\left[z(2+z)\beta + 2\gamma_1\right] - 4\gamma_2\right\}. \end{aligned} \tag{2.73}$$

2.3.3 Solutions with $\mathcal{L}_V = \mathcal{G}(\mathcal{A})$

Similar to the last case, the function $\mathcal{G}(\mathcal{A})$ can take any form in terms of $\mathcal{A}$. A particular case is the potential given by Eq. (2.12) with $\gamma_1 = \gamma_2 = \beta_5 = \beta_6 = 0$, for which we have

$$\mathcal{G}(\mathcal{A}) = 2\Lambda + \beta a_i a^i + \frac{1}{\zeta^2}\Big[\beta_1\left(a_i a^i\right)^2 + \beta_2\left({a^i}_i\right)^2 + \beta_3 a_i a^i {a^j}_j + \beta_4 a^{ij} a_{ij}\Big]. \tag{2.74}$$

In this case, Eq. (2.57) reduces to

$$az^2 + bz + c = 0, \tag{2.75}$$

but now with

$$\begin{aligned} a &= -2\beta_1 + \beta_3, \\ b &= 2\beta_2 - \beta_3 + \beta_4, \\ c &= -\alpha^2\beta - \beta_4. \end{aligned} \tag{2.76}$$

Thus, in general there are two solutions,

$$z_{\pm} = \frac{1}{2(2\beta_1 - \beta_3)}\left[(2\beta_2 - \beta_3 + \beta_4) \pm \sqrt{D}\right], \tag{2.77}$$

where $D \equiv (2\beta_2 - \beta_3 + \beta_4)^2 + 4(\alpha^2\beta + \beta_4)(\beta_3 - 2\beta_1)$. Clearly, for $z_{\pm}$ to be real, we must assume that $D \geq 0$.

2.4 Scalar Field in the Lifshitz Spacetime

The action of a scalar field in the HL theory takes the form,

$$S_M = \int dt d^2x N\sqrt{g}\left\{\frac{1}{2N^2}[\dot{\varphi} - N^i\nabla_i\varphi]^2 - V(\varphi) - \mathcal{V}_\phi^{(2)} - \frac{1}{M_*^2}\mathcal{V}_\phi^{(4)}\right\}, \tag{2.78}$$

where $\mathcal{V}_\phi^{(2)}$ and $\mathcal{V}_\phi^{(4)}$ are, respectively, the second and forth order operators, made of R_{ij}, a_i, ∇_i and ϕ, where

$$[R_{ij}] = 2, \quad [a_i] = 1 = [\nabla_i], \quad [\phi] = 0. \tag{2.79}$$

In general, they take the forms [44, 98],

$$\begin{aligned} \mathcal{V}_\phi^{(2)} &= \underline{\frac{1}{2}\left[1 + 2V_1(\varphi)\right](\nabla_i\varphi)^2} + \epsilon_1(\phi)a_i\nabla^i\phi + \epsilon_2(\phi)a_ia^i + \epsilon_3(\phi)R + ..., \\ \mathcal{V}_\phi^{(4)} &= \underline{V_2(\varphi)\left(\nabla^2\varphi\right)^2} + \underline{V_4(\varphi)\nabla^4\varphi} + \delta_1(\phi)R_{ij}\nabla^i\phi\nabla^j\phi \\ &\quad + \delta_2(\phi)\left(a_i\nabla^i\phi\right)^2 + \delta_3(\phi)R^2 + ..., \end{aligned} \tag{2.80}$$

where V_i, ϵ_i and δ_i are arbitrary functions of ϕ only, and the elapsing terms are the mixed ones made of R_{ij}, a_i and $\nabla_i\phi$. When the background is fixed, these terms always give rise to low order operators in terms of the scalar field ϕ. For example, the term $\epsilon_1(\phi)a_i\nabla^i\phi$ appearing in $\mathcal{V}_\phi^{(2)}$ contributes to the equation of motion of the scalar field only with the first-order spatial derivative, $\nabla^i\left[\epsilon_1(\phi)a_i\right]$, while the term $\delta_1(\phi)R_{ij}\nabla^i\phi\nabla^j\phi$ appearing in $\mathcal{V}_\phi^{(4)}$ contributes only with the second-order spatial derivative, $\nabla^j\left[\delta_1(\phi)R_{ij}\nabla^j\phi\right]$. In addition, the term $\delta_3(\phi)R^2$ had contributions of the form, $\delta_3'(\phi)R^2$, which acts as a potential term once the background is fixed. Therefore, when the space-time background is fixed, the dominant terms in the UV are only the V_2 and V_4 terms appearing in Eq. (2.80). In the IR, on the other hand, their contributions must be so that the resulted action is of general covariance, in order to have a consistent theory with observations [45] [2] . Therefore, in this chapter, without loss of the generality, we shall keep only the underlined $V_i(\phi)$ terms appearing in Eq. (2.80) and absorb the factor M_*^{-2} into $V_2(\phi)$ and $V_4(\phi)$. Then, the variation of the action with respect to φ yields,

$$\begin{aligned}\frac{1}{\sqrt{g}}\partial_t\left[\frac{\sqrt{g}}{N}(\dot{\varphi}-N^i\nabla_i\varphi)\right] &= \nabla_i\left[\frac{N^i}{N}(\dot{\varphi}-N^k\nabla_k\varphi)\right]+\nabla^i[N(\nabla_i\varphi)(1+2V_1)]\\ &-\nabla^2[2NV_2(\nabla^2\varphi)]-\nabla^4[NV_4]-N[V'+V_1'(\nabla\varphi)^2+V_2'(\nabla^2\varphi)^2+V_4'(\nabla^4\varphi)]\end{aligned} \tag{2.81}$$

To compare with the results obtained in [26], we first set $L=\ell=1,\ z=2$ and $u=1/r$. Then, the metric (2.49) becomes,

$$ds^2=-\frac{1}{u^4}dt^2+\frac{1}{u^2}(dx^2+du^2). \tag{2.82}$$

In the probe limit, the backreaction of the scalar field is neglected. Hence, taking the above space-time as the background, and choosing

$$V=m^2\varphi^2,\ V_1=a_1,\ V_2=\frac{\hat{a_2}}{M_*^2}\equiv a_2,\ V_4=\frac{\hat{a_4}}{M_*^2}\varphi\equiv a_4\varphi, \tag{2.83}$$

[2] The only possible contributions of these terms are in the intermediate energy scales. However, the study of them in these energy scales in general are very complicated, and are hardly carried out analytically. Thus, in this chapter we shall not consider them.

where a_n are constants, we find that Eq. (2.81) reduces to,

$$
\begin{aligned}
& u^2\partial_t^2\varphi = (1+2a_1)\left(\partial_x^2\varphi + \partial_u^2\varphi - \frac{2}{u}\partial_u\varphi\right) - \frac{2}{u^2}m^2\varphi \\
& -a_4\left[8\partial_x^2\varphi + 16\partial_u^2\varphi - \frac{32}{u}\partial_u\varphi + \frac{36\varphi}{u^2}\right] \\
& -2u^2(a_2+a_4)(\partial_x^4\varphi + 2\partial_x^2\partial_u^2\varphi + \partial_u^4\varphi). \qquad (2.84)
\end{aligned}
$$

At the boundary $u = 0$, the scalar field takes the asymptotical form,

$$
\varphi \sim u^{\triangle}\varphi_1(t,x), \qquad (2.85)
$$

where $\triangle$ is one of the real roots of the equation,

$$
\begin{aligned}
& (1+2a_1)(\triangle^2 - 3\triangle) - 2m^2 - a_4(16\triangle^2 - 48\triangle + 36) \\
& -2(a_2+a_4)\triangle(\triangle - 1)(\triangle - 2)(\triangle - 3) = 0. \qquad (2.86)
\end{aligned}
$$

From the action (2.78), integrating by parts and discarding boundary terms, we find that it takes the form,

$$
\begin{aligned}
S_M &= \int dtd^2xN\sqrt{g}\Bigg\{ - \frac{\varphi}{N\sqrt{g}}\partial_t(\frac{\sqrt{g}\dot{\varphi}}{2N}) - m^2\varphi^2 \\
& + \frac{(1+2a_1)\varphi}{2N}\nabla_i(N\nabla^i\varphi) - \frac{a_2\varphi}{N}\nabla^2\left(N\nabla^2\varphi\right) - a_4\varphi\nabla^4\varphi\Bigg\}. \qquad (2.87)
\end{aligned}
$$

It can be shown that both actions (2.78) and (2.87) are finite for

$$
\triangle > \frac{3}{2} \qquad (2.88)
$$

with the asymptotic condition (2.85).

In the IR, the V_2 and V_4 terms are very small, and can be set to zero safely. In addition, in this limit the scalar field should be relativistic, so $V_1 = 0$. Hence, the above equation reduces to

$$
\triangle^2 - 3\triangle - 2m^2 = 0, \qquad (2.89)
$$

which has the solutions,

$$
\triangle_\pm = \frac{1}{2}\left(3 \pm \sqrt{9+8m^2}\right). \qquad (2.90)
$$

For

$$m^2 > -\frac{9}{8}, \tag{2.91}$$

in contrast to the case considered in [26], now only the solution with $\Delta = \Delta_+$,

$$\varphi(u,t,x) \to u^{\triangle_+}\left(\varphi(t,x) + O(u^2)\right), \tag{2.92}$$

leads to a finite action either in the form of Eq. (2.78) or in the one of Eq. (2.87).

In the UV, on the other hand, the V_2 and V_4 terms dominate, and Eq. (2.86) becomes,

$$\begin{aligned}
&(a_2 + a_4) \triangle^4 - 6(a_2 + a_4) \triangle^3 + (11a_2 + 27a_4) \triangle^2 \\
&- (6a_2 + 54a_4) \triangle + 36a_4 = 0.
\end{aligned} \tag{2.93}$$

In the case $a_4 = 0$, the above equation reduces to

$$\triangle^3 - 6 \triangle^2 + 11 \triangle - 6 = 0, \quad (a_4 = 0), \tag{2.94}$$

which has solutions

$$\triangle_1 = 1, \qquad \triangle_2 = 2, \qquad \triangle_3 = 3, \ (a_4 = 0). \tag{2.95}$$

If we choose $a_2 = -a_4$, Eq. (4.14) has the double root

$$\triangle = 6, \ (a_2 = -a_4). \tag{2.96}$$

From the above analysis, one can see that the scalar field has quite different behaviors at the boundary $u = 0$ in the two limits, IR and UV.

2.5 Two-point Correlation Functions

The bulk field $\varphi(u,t,x)$ can be written in the form

$$\varphi(u,t,x) = \int d^3x' \varphi(0,t',x') G(u,t,x;0,t',x'). \tag{2.97}$$

where $\varphi(0,t,x)$ is the scalar field on the boundary and $G(u,t,x;0,t',x')$ the boundary to bulk propagator. It is easy to work in the Fourier space due to the translational invariance in t and x. In the Fourier space, we have

$$\tilde{\varphi}(u,\omega,k) = \tilde{G}(u,\omega,k)\tilde{\varphi}(0,\omega,k). \tag{2.98}$$

2.5.1 In the IR

In the IR, we set $a_1 = a_2 = a_4 = 0$, Eq. (2.84) reduces to

$$-u^2\partial_\tau^2\varphi = \partial_x^2\varphi + \partial_u^2\varphi - \frac{2}{u}\partial_u\varphi - \frac{2}{u^2}m^2\varphi, \tag{2.99}$$

and $\tilde{G}(u,\omega,k)$ in Fourier space satisfies the equation,

$$\partial_u^2\tilde{G} - \frac{2}{u}\partial_u\tilde{G} - (\omega^2u^2 + |k|^2)\tilde{G} = 0, \tag{2.100}$$

with the boundary conditions,

$$\begin{aligned} &(i)\;\; \tilde{G}(0,\omega,k) = 1, \\ &(ii)\;\; \tilde{G}(\infty,\omega,k) \;\text{ is finite.} \end{aligned} \tag{2.101}$$

Note that in writing down Eq. (2.99), we had set $t = i\tau$. Then, the above conditions uniquely determine the propagator $\tilde{G}(u,\omega,k)$,

$$\tilde{G}(u,\omega,k) \;=\; \frac{2}{\sqrt{\pi}}e^{-|\omega|u^2/2}\Gamma\left(\frac{k^2}{4|\omega|}+\frac{5}{4}\right)U\left(\frac{k^2}{4|\omega|}-\frac{1}{4},-\frac{1}{2},|\omega|u^2\right), \tag{2.102}$$

where $U(a,b,u)$ is the confluent hypergeometric function of the second kind. Near $u=0$, $\tilde{G}$ is given by

$$\tilde{G} = 1 - \frac{k^2}{2}u^2 + \frac{8\Gamma\left(\frac{k^2}{4|\omega|}+\frac{5}{4}\right)|\omega|^{3/2}}{3\Gamma\left(\frac{k^2}{4|\omega|}-\frac{1}{4}\right)}u^3 + O\left(u^4\right). \tag{2.103}$$

In the IR limit and $m = 0$, the action Eq. (2.78) yields

$$\begin{aligned} S_M^* \;&\equiv\; \frac{i}{2}S_M = \frac{1}{2}\int d\tau d^2x N\sqrt{g}\left\{\frac{1}{N^2}\varphi'^2 + (\nabla\varphi)^2\right\} \\ &=\; \frac{1}{2}\int d\tau d^2x\sqrt{{}^{(3)}g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi, \end{aligned} \tag{2.104}$$

where $\varphi' = \frac{\partial\varphi}{\partial\tau}$. Integrating by parts, one can show that the on-shell bulk action is determined by the values of the field on the boundary

$$\begin{aligned} S_M^* &= \int d\tau dx [\sqrt{^{(3)}g} g^{uu} \varphi \partial_u \varphi]_\epsilon^\infty \\ &= \int d\omega dk \tilde{\varphi}(0,k,\omega)\mathcal{F}(k,\omega)\tilde{\varphi}(0,-k,-\omega), \end{aligned} \tag{2.105}$$

where we had cut off the space at $u = \epsilon$ to regulate the bulk action, and the "flux factor" $\mathcal{F}$ is defined as

$$\mathcal{F}(k,\omega) = [\tilde{G}(u,k,\omega)\sqrt{^{(3)}g} g^{uu} \partial_u \tilde{G}(u,-k,-\omega)]_\epsilon^\infty. \tag{2.106}$$

Since the propagator $\tilde{G}$ vanishes at $u = \infty$, $\mathcal{F}$ only receives a contribution from the cutoff at $u = \epsilon$. The momentum space two-point function for the operator $\mathcal{O}_\varphi$ dual to φ is given by differentiating Eq. (2.105) twice with respect to $\varphi(0,k,\omega)$:

$$\langle \mathcal{O}_\varphi(k,\omega)\mathcal{O}_\varphi(-k,-\omega)\rangle = \mathcal{F}(k,\omega). \tag{2.107}$$

Plugging Eq. (2.103) into Eq. (2.106), we pick out the leading non-polynomial piece in either k or ω. This gives the correlation function, after taking the limit $\epsilon \to 0$,

$$\langle \mathcal{O}_\varphi(k,\omega)\mathcal{O}_\varphi(-k,-\omega)\rangle = -\frac{8|\omega|^{3/2}\Gamma(a+\frac{3}{2})}{\Gamma(a)}, \tag{2.108}$$

where $a \equiv \frac{k^2}{4|\omega|} - \frac{1}{4}$. Since $\Gamma(a \simeq 0) \to \infty$, we find that $\langle \mathcal{O}_\varphi(k,\omega)\mathcal{O}_\varphi(-k,-\omega)\rangle \simeq 0$ as $a \to 0$. When $a \gg 1$, on the other hand, we find $\langle \mathcal{O}_\varphi(k,\omega)\mathcal{O}_\varphi(-k,-\omega)\rangle \simeq -8|\omega|^{1/2}(k^2 + |\omega|)$, which gives rise to correlations between points only with temporal separation.

In general, the divergence arising as $\epsilon \to 0$ from the term proportional to u^2 is removed via local boundary terms [26, 46], and the terms $\mathcal{O}(u^4)$ and higher vanish as the cutoff is removed when taking the limit $\epsilon \to 0$.

2.5.2 In the UV

In the UV limit, the last term in Eq. (2.84) dominates, and we find that

$$\partial_\tau^2 \varphi = 2a_{24}(\partial_x^4 \varphi + 2\partial_x^2 \partial_u^2 \varphi + \partial_u^4 \varphi), \tag{2.109}$$

where $a_{24} \equiv a_2 + a_4$. In the Fourier space, this becomes

$$\partial_u^4 \tilde{G} - 2k^2 \partial_u^2 \tilde{G} + \left(k^4 + \frac{\omega^2}{2a_{24}}\right) \tilde{G} = 0, \tag{2.110}$$

with the same boundary condition as in Eq. (2.101). Then, we find that

$$\tilde{G} = c_1 e^{-u\sqrt{\rho}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})} + (1-c_1) e^{-u\sqrt{\rho}(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2})}, \tag{2.111}$$

where c_1 is an integration constant, and

$$\rho\cos\theta = k^2, \quad \rho\sin\theta = \sqrt{\frac{w^2}{2a_{24}}}. \tag{2.112}$$

Thus, with $m = 0$, the action (2.78) gives rise to,

$$\begin{aligned}
iS_M &= \int d\tau d^2x N\sqrt{g}\Big\{\frac{1}{2N^2}\varphi'^2 + a_2(\nabla^2\varphi)^2 + a_4\phi\nabla^4\phi\Big\} \\
&= \int d\omega dk \tilde{\varphi}(0,k,\omega) \int_\epsilon^\infty du \Big\{\frac{\omega^2}{2}\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega) \\
&\quad + a_{24}k^4 \tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega) - 2a_{24}k^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega) \\
&\quad + a_2\partial_u^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega) + a_4\tilde{G}(u,k,\omega)\partial_u^4\tilde{G}(u,-k,-\omega) \\
&\quad + \frac{4a_4}{u}[\tilde{G}(u,k,\omega)\partial_u^3\tilde{G}(u,-k,-\omega) - k^2\tilde{G}(u,k,\omega)\partial_u\tilde{G}(u,-k,-\omega)] \\
&\quad + \frac{2a_4}{u^2}[\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega) - k^2\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega)]\Big\}\tilde{\varphi}(0,-k,-\omega) \\
&= \int d\omega dk \tilde{\varphi}(0,k,\omega)\mathcal{F}(k,\omega)\tilde{\varphi}(0,-k,-\omega),
\end{aligned} \tag{2.113}$$

where

$$\begin{aligned}
\mathcal{F}(k,\omega) &= \int_\epsilon^\infty du\Big\{\frac{\omega^2}{2}\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega) + a_{24}k^4\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega) \\
&\quad - 2a_{24}k^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega) + a_2\partial_u^2\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega) \\
&\quad + a_4\tilde{G}(u,k,\omega)\partial_u^4\tilde{G}(u,-k,-\omega) + \frac{4a_4}{u}[\tilde{G}(u,k,\omega)\partial_u^3\tilde{G}(u,-k,-\omega) \\
&\quad - k^2\tilde{G}(u,k,\omega)\partial_u\tilde{G}(u,-k,-\omega)] + \frac{2a_4}{u^2}[\tilde{G}(u,k,\omega)\partial_u^2\tilde{G}(u,-k,-\omega) \\
&\quad - k^2\tilde{G}(u,k,\omega)\tilde{G}(u,-k,-\omega)]\Big\}.
\end{aligned} \tag{2.114}$$

Plugging Eq. (2.111) into Eq. (2.114), and taking the limit $\epsilon \to 0$, we find that

$$\mathcal{F}(k,\omega) = 4a_2c_1(1-c_1)\rho^{\frac{3}{2}}\sin\theta\sin\frac{\theta}{2}. \tag{2.115}$$

CHAPTER THREE

Charged Einstein-aether Black Holes

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In this chapter, we present two new classes of exact charged black hole solutions in the framework of the Einstein-Maxwell-æther theory, which are asymptotically flat and possess the universal as well as Killing horizons. We also construct the Smarr formulas, and calculate the temperatures of the horizons, using the Smarr mass-area relation. We find that, in contrast to the neutral case, such obtained temperature is not proportional to its surface gravity at any of the two kinds of the horizons.

3.1 Introduction

Lorentz invariance is one of the fundamental principles of Einstein's general relativity (GR) and modern physics. The success of GR to describe all observed gravitational phenomena, together with its intrinsic mathematical elegance is interpreted as a further proof of the importance of the Lorentz invariance [54]. However, Lorentz invariance may not be an exact symmetry at all energies [55]. Any effective description must break down at a certain cutoff scale signaling the emergence of new physical degrees of freedom beyond that scale. For example, the hydrodynamics, Fermi's theory of beta decay [56] and quantization of GR [57] at energies beyond the Planck energy. Astrophysical observations suggest that the high energy cosmic rays above the Greisen-Zatsepin-Kuzmin cutoff is a result of the Lorentz violation [58]. Lorentz invariance also leads to divergences in quantum field theory which can be cured with a short distance of cutoff that breaks it [8]. Often, the late cosmic acceleration is also interpreted as a demand for a modification of GR at cosmological scales [17, 59].

There are several gravitational theories that violate the Lorentz invariance [55], e.g., Hořava-Lifshitz theory [7], ghost condensations [60], warped brane worlds, Einstein-æther theory [9], etc. In Einstein-æther theory, the background tensor fields break the Lorentz symmetry, and were once thought must to be dynamical [9], but more careful investigations recently revealed that it is not necessary [61]. In this theory, the Lorentz symmetry is broken only down to a rotation subgroup by the existence of a preferred time direction at every point of spacetime, i.e., existing a preferred frame of reference established by æther vector u^a. This timelike unit vector field u^a can be interpreted as a velocity four-vector of some medium substratum (æther, vacuum or dark fluid), bringing into consideration of non-uniformly moving continuous media and their interaction with other fields. Meanwhile, this theory can be also considered as a realization of dynamic self-interaction of complex systems moving with a space-time dependant macroscopic velocity. As to an accelerated expansion of the universe, this dynamic self-interaction can produce the same cosmological effects as the dark energy [62].

The introduction of the æther vector allows for some novel effects, e.g., matter fields can travel faster than the speed of light [63], new gravitational wave polarizations can spread at different speeds [64]. It should be noted that the propagation faster than that of the light does not violate causality [65]. In particular, gravitational theories with breaking Lorentz invariance still allow the existence of black holes [17,18,66–68]. However, instead of Killing horizons, now the boundaries of black holes are hypersurfaces, termed universal horizons [17,18], which can trap excitations traveling at arbitrarily high velocities. This universal horizon may radiate thermally at a fixed temperature and strengthen a possible thermodynamic interpretation though there is no universal light cone [69] (See also [70] for a different suggestion.).

It is natural to extend the Einstein-æther theory to include other fields, i.e., the electromagnetic field [71]. As for cosmology, the interaction of electromagnetic waves with a non-uniformly moving æther can change the details of the standard history of the relic photons that could be tested using observational data. As for black holes, the interaction of electromagnetic radiation with a deformed æther will induce new dynamo-optical effects that could be also tested. As for gravitational waves, the Einstein-Maxwell-æther theory is expected to predict new forms for gravitational wave propagations [59, 64]. Our goal here is to extend Einstein-æther theory to include a source-free Maxwell field. For more general formalism of the Einstein-Maxwell-æther theory, see [71].

The rest of the chapter is organized as follows. In Section 3.2 we provide the background for the Einstein-Maxwell-æther theory to be studied in this chapter. In Section 3.3 we construct a Smarr formula for spherically symmetric solutions. In Section 3.4, we first construct two new classes of exact charged solutions, and then use them as examples to study the Smarr formula.

Before proceeding further, we would like to note that the exact charged solutions presented in this chapter can be considered as a generalization of the neutral ($Q = 0$) ones given in [65]. Therefore, in the following there may exist repeating of the materials presented there, in order for the current chapter to be as much independent as possible, although we shall try to limit this to its minimum. For more detail, we refer readers to [65].

3.2 Einstein-Maxwell-aether Theory

The general action for the Einstein-æther theory can be constructed by assuming that: (1) it is general covariant; and (2) it is a functional of only the spacetime metric g_{ab} and a unit timelike vector u^a, and involves no more than two derivatives of them, so that the resulting field equations are second-order differential equations of g_{ab} and u^a. Then, the Einstein-Maxwell-æther theory to be studied in this chapter is described by the action,

$$\mathcal{S} = \int d^4x\sqrt{-g}\Big[\frac{1}{16\pi G_{æ}}(\mathcal{R}+\mathcal{L}_{æ}) + \mathcal{L}_M\Big]\,. \tag{3.1}$$

In terms of the tensor $Z^{ab}{}_{cd}$ defined as [73, 97],

$$Z^{ab}{}_{cd} = c_1 g^{ab}g_{cd} + c_2\delta^a{}_c\delta^b{}_d + c_3\delta^a{}_d\delta^b{}_c - c_4 u^a u^b g_{cd}\,, \tag{3.2}$$

the æther Lagrangian $\mathcal{L}_{æ}$ is given by

$$-\mathcal{L}_{æ} = Z^{ab}{}_{cd}(\nabla_a u^c)(\nabla_b u^d) - \lambda(u^2+1), \tag{3.3}$$

where $c_i (i = 1, 2, 3, 4)$ are coupling constants of the theory. The æther Lagrangian is therefore the sum of all possible terms for the æther field u^a up to mass dimension two, and the constraint term $\lambda(u^2 + 1)$ with the Lagrange multiplier λ implementing the normalization condition $u^2 = -1$. The source-free Maxwell Lagrangian $\mathcal{L}_M$ is given by

$$\mathcal{L}_M = -\frac{1}{16\pi G_{æ}}\mathcal{F}_{ab}\mathcal{F}^{ab},\ \mathcal{F}_{ab} = \nabla_a\mathcal{A}_b - \nabla_b\mathcal{A}_a, \tag{3.4}$$

where $\mathcal{A}_a$ is the electromagnetic potential four-vector.

There are a number of theoretical and observational bounds on the coupling constants c_i [9, 59, 74]. Here, we impose the following constraints[1] ,

$$0 \le c_{14} < 2, \quad 2 + c_{13} + 3c_2 > 0, \quad 0 \le c_{13} < 1, \tag{3.5}$$

where $c_{14} \equiv c_1 + c_4$, and so on. The constant $G_{æ}$ is related to Newton's gravitational constant G_N by $G_{æ} = (1 - c_{14}/2)G_N$, which can be obtained by using the weak field/slow-motion limit of the Einstein-æther theory [58, 97].

The equations of motion, obtained by varying the action (4.2) with respect to g_{ab}, u^a, $\mathcal{A}^a$ and λ are

$$\mathcal{G}_{ab} = \mathcal{T}^{æ}_{ab} + 8\pi G_{æ}\mathcal{T}^{M}_{ab}, \quad Æ_a = 0, \quad \nabla^a\mathcal{F}_{ab} = 0, \quad u^2 = -1, \tag{3.6}$$

[1] Note the slight difference between the constraints imposed here and the ones imposed in [65], as in this chapter we also require that vacuum Cerenkov radiation of gravitons is forbidden [75].

respectively, where the æther and Maxwell energy-momentum stress tensors $\mathcal{T}^{æ}_{ab}$ and $\mathcal{T}^{M}_{ab}$ are given by

$$\mathcal{T}^{æ}_{ab} = \lambda u_a u_b + c_4 a_a a_b - \frac{1}{2} g_{ab} Y^c{}_d \nabla_c u^d + \nabla_c X^c{}_{ab} + c_1[(\nabla_a u_c)(\nabla_b u^c) - (\nabla^c u_a)(\nabla_c u_b)],$$
$$\mathcal{T}^{M}_{ab} = \frac{1}{4\pi G_{æ}}\Big[-\frac{1}{4} g_{ab}\mathcal{F}_{mn}\mathcal{F}^{mn} + \mathcal{F}_{am}\mathcal{F}_b{}^m\Big], \qquad (3.7)$$

with

$$\begin{aligned} Æ_a &= \nabla_b Y^b{}_a + \lambda u_a + c_4(\nabla_a u^b) a_b, \\ Y^a{}_b &= Z^{ac}{}_{bd}\nabla_c u^d, \\ X^c{}_{ab} &= Y^c{}_{(a} u_{b)} - u_{(a} Y_{b)}{}^c + u^c Y_{(ab)}. \end{aligned} \qquad (3.8)$$

The acceleration vector a^a appearing in the expression for the æther energy-momentum stress tensor is defined as the parallel transport of the æther field along itself, $a^a \equiv \nabla_u u^a$, where $\nabla_X \equiv X^b \nabla_b$.

Following [65], we first define a set of basis vectors at every point in the spacetime, so that we can project out various components of the equations of motion. Let us first take the æther field u^a to be the basis vector. Then, pick up two spacelike unit vectors, denoted, respectively, by m^a and n^a, both of which are normalized to unity, mutually orthogonal, and lie on the tangent plane of the two-spheres $\mathcal{B}$ that foliate the hypersurface Σ_U. Finally, let us pick up s^a, a spacelike unit vector that is orthogonal to u^a, m^a, n^a, and points "outwards" along a Σ_U hypersurface, so we have the four tetrad, $e^a_{(b)} \equiv (u^a, s^a, m^a, n^a)$, with

$$g^{ab} = \eta^{cd} e^a_{(c)} e^b_{(d)} = -u^a u^b + s^a s^b + \hat{g}^{ab}, \quad e_{(b)} \cdot e_{(c)} = \eta_{bc}, \qquad (3.9)$$

where $\hat{g}^{ab} \equiv m^a m^b + n^a n^b$. By spherical symmetry, any physical vector A^a has at most two non-vanishing components along, respectively, u^a and s^a, i.e., $A^a = A_1 u^a + A_2 s^a$. In particular, the acceleration a^a has only one component along s^a, namely, $a^a = (a \cdot s) s^a$. Similarly, any rank-two tensor F_{ab} may have components along the directions of the bi-vectors $u_a u_b,\ u_{(a}s_{b)},\ u_{[a}s_{b]},\ s_a s_b,\ \hat{g}_{ab}$, where $\hat{g}_{ab}$ is the projection tensor onto the two-sphere $\mathcal{B}$, bounding a section of a Σ_U hypersurface. In the following, we study the expansion of the Maxwell field $\mathcal{F}^{ab}$, Killing vector χ^a, surface gravity κ, energy-momentum stress tensors $\mathcal{T}^{æ}_{ab}$ and $\mathcal{T}^{M}_{ab}$, and Ricci tensor $\mathcal{R}_{ab}$. The given source-free Maxwell field $\mathcal{F}^{ab}$ can be formulated in terms of four-vectors representing physical fields. They are the electric field E^a and magnetic excitation B^a as,

$$E^a = \mathcal{F}^{ab}u_b, \quad B^a = \frac{e^{abmn}}{2\sqrt{-g}}\mathcal{F}_{mn}u_b, \tag{3.10}$$

where e^{abmn} is the Levi-Civita tensor. From Eq. (3.6) it can be shown $B^a = 0$. Then, we find

$$\mathcal{F}^{ab} = -E^a u^b + E^b u^a. \tag{3.11}$$

On the other hand, the electric field is spacelike, since $E^a u_a = 0$. So, we have $E^a = (E \cdot s)s^a$. Thus, $\mathcal{F}_{ab} = (E\cdot s)(-s_a u_b + s_b u_a)$. After substituting it into (3.6), we can see $(E \cdot s) = Q/r^2$, where Q is an integral constant, representing the total charge of the space-time. Therefore, we have

$$\mathcal{F}_{ab} = \frac{Q}{r^2}(u_a s_b - u_b s_a). \tag{3.12}$$

The Einstein, æther and Maxwell equations of motion (3.6) can be decomposed by using the tetrad defined above. In particular, the æther and electromagnetic energy-momentum stress tensors and the Ricci tensor can be cast, respectively, in the forms,

$$\begin{aligned}
\mathcal{T}^{æ}_{ab} &= \mathcal{T}^{æ}_{uu}u_a u_b - 2\mathcal{T}^{æ}_{us}u_{(a}s_{b)} + \mathcal{T}^{æ}_{ss}s_a s_b + \frac{\hat{\mathcal{T}}_{æ}}{2}\hat{g}_{ab},\\
\mathcal{R}_{ab} &= \mathcal{R}_{uu}u_a u_b - 2\mathcal{R}_{us}u_{(a}s_{b)} + \mathcal{R}_{ss}s_a s_b + \frac{\hat{\mathcal{R}}}{2}\hat{g}_{ab},\\
\mathcal{T}^{M}_{ab} &= \mathcal{T}^{M}_{uu}u_a u_b - 2\mathcal{T}^{M}_{us}u_{(a}s_{b)} + \mathcal{T}^{M}_{ss}s_a s_b + \frac{\hat{\mathcal{T}}_{M}}{2}\hat{g}_{ab}.
\end{aligned} \tag{3.13}$$

The coefficients of $\mathcal{T}^{æ}_{ab}$ and $\mathcal{T}^{M}_{ab}$ in (3.13) can be computed from the general expression (3.7). The corresponding coefficients for $\mathcal{R}_{ab}$, on the other hand, are computed from the definition $[\nabla_a,\ \nabla_b]X^c \equiv -\mathcal{R}^c{}_{abd}X^d$ by choosing $X^a = u^a$ or s^a, and then contracting the resulting expressions again with u^a and/or s^a appropriately. The coefficients for the three (u, s) cross terms are

$$\mathcal{T}^{æ}_{us} = c_{14}\left[\hat{K}(a\cdot s) + \nabla_u(a\cdot s)\right], \quad \mathcal{T}^{M}_{us} = 0, \quad \mathcal{R}_{us} = (K_0 - \hat{K}/2)\hat{k} - \nabla_s\hat{K}, \tag{3.14}$$

where

$$\begin{aligned}\nabla_{[a}s_{b]} &\equiv -K_0 u_{[a}s_{b]},\\ \hat{k} &\equiv \frac{1}{2}g^{ab}\mathcal{L}_s\hat{g}_{ab},\\ \hat{K} &\equiv \frac{1}{2}g^{ab}\mathcal{L}_u\hat{g}_{ab},\end{aligned} \tag{3.15}$$

with K $(\equiv K_0 + \hat{K})$ being the trace of the extrinsic curvature of the hypersurface Σ_U. The æther equation $s\cdot Æ = 0$ and the us-component $\mathcal{R}_{us} = \mathcal{T}^{æ}_{us} + 8\pi G_{æ}\mathcal{T}^{M}_{us}$ yield

$$c_{123}\nabla_s K_0 - (1 - c_{13})(K_0 - \hat{K}/2)\hat{k} + (1 + c_2)\nabla_s\hat{K} = 0, \tag{3.16}$$

$$c_{123}\nabla_s K - (1 - c_{13})\mathcal{T}^{æ}_{us} = 0. \tag{3.17}$$

The uu- and ss-components of the gravitational field equations give

$$\begin{aligned}&\left(1 - \frac{c_{14}}{2}\right)\left[(\hat{k} + \nabla_s)(a\cdot s) + a^2\right] - (1 - c_{13})\left(K_0^2 + \frac{\hat{K}^2}{2}\right)\\ &-\left(1 + c_2 + \frac{c_{123}}{2}\right)\nabla_u K - \frac{c_{123}}{2}K^2 - \frac{Q^2}{r^4} = 0,\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\frac{c_{123}}{2}(K + \nabla_u)(\hat{K} - K_0) + (1 + c_2)KK_0 - \left(1 + \frac{c_{14}}{2}\right)\left[\nabla_s(a\cdot s) + a^2\right]\\ &+c_{14}a^2 - \left[\nabla_s + \frac{\hat{k}}{2} + \frac{c_{14}}{2}(a\cdot s)\right]\hat{k} + \frac{Q^2}{r^4} = 0.\end{aligned} \tag{3.19}$$

In the next sections, we will use these equations to obtain new black holes solutions.

3.3 Smarr Formula

The studies of black holes have been one of the main objects both theoretically and observationally over the last half of century [77, 78], and so far there are many solid observational evidences for their existence in our universe. Theoretically, such investigations have been playing a fundamental role in the understanding of the nature of gravity in general, and quantum gravity in particular. They started with the discovery of the laws of black hole mechanics [79] and Hawking radiation [16], and led to the profound recognition of the thermodynamic interpretation of the four laws [13] and the reconstruction of general relativity (GR) as the thermodynamic limit of a more fundamental theory of gravity [80]. More recently, they are essential in understanding the AdS/CFT correspondence [81, 82] and firewalls [83].

To derive the Smarr formula, we first introduce the ADM mass, which is identical to the Komar mass defined in stationary spacetimes with the time translation Killing vector χ^a [65],

$$M_{ADM} = -\frac{1}{4\pi G_{æ}} \int_{\mathcal{B}_\infty} \nabla^a \chi^b d\Sigma_{ab}, \tag{3.20}$$

where $d\Sigma_{ab} \equiv -u_{[a}s_{b]}dA$, with dA ($\equiv r^2 \sin\theta d\theta d\phi$) being the differential area element on the two-sphere $\mathcal{B}$, and $\mathcal{B}_\infty$ is the sphere at infinity. The derivative of the Killing vector $\chi^a = -(u\cdot\chi)u^a + (s\cdot\chi)s^a$ is given by

$$\nabla^a \chi^b = -2\kappa u^{[a}s^{b]}, \tag{3.21}$$

where κ denotes the surface gravity usually defined in GR, and is given by

$$\kappa = \sqrt{-\frac{1}{2}(\nabla_a \chi_b)(\nabla^a \chi^b)} = -(a\cdot s)(u\cdot\chi) + K_0(s\cdot\chi). \tag{3.22}$$

At the infinity, we have $(u\cdot\chi) = -1$ and $(s\cdot\chi) = 0$. Then, Eq. (3.20) yields,

$$M_{ADM} = \lim_{r\to\infty}\left(\frac{r^2(a\cdot s)}{G_{æ}}\right). \tag{3.23}$$

In the studies of black hole physics, the physics of horizons has provided useful information. In particular, at the Killing horizon the first law [84] and Smarr formula [85] for the Reissner-Nordstrom black hole take the forms,

$$\delta M_{ADM} = \frac{\kappa_{KH}\delta A_{KH}}{8\pi G_N} + \frac{V_{KH}\delta Q}{G_N}, \quad M_{ADM} = \frac{\kappa_{KH} A_{KH}}{4\pi G_N} + \frac{V_{KH} Q}{G_N}, \tag{3.24}$$

where M_{ADM} is the ADM mass of the spacetime and, $\kappa_{KH}[\equiv \kappa(r_{KH})]$, A_{KH} and V_{KH} are the surface gravity, cross-sectional area and electromagnetic potential evaluated on the Killing horizon, respectively. Identifying $T_{KH} = \kappa_{KH}/2\pi$ as the temperature of the horizon and, the entropy $S = A_{KH}/4G_N$, one can obtain the analogy with the first law of thermodynamics, $\delta E = T\delta S + V\delta Q$ and $E = 2TS + VQ$.

Any causal boundary in a gravitational theory should have an entropy associated with it. Therefore, in the Einstein-æther theory, the universal horizons are expected to have also their entropy and the first law of black hole mechanics, though whether this entropy is to be proportional to its area or not is still an open issue. Meanwhile, these black holes still have Killing horizons. Then, a question is: How is their thermodynamics?

To obtain some hints, in this section we shall present the Smarr formulas of the universal and Killing horizons for general static and spherically symmetric Einstein-Maxwell-æther black holes. Let us first consider the geometric identity [79],

$$\mathcal{R}_{ab}\chi^b = \nabla^b(\nabla_a \chi_b). \tag{3.25}$$

From the Einstein field equations (3.6), we find that $\mathcal{R}_{ab} = \mathcal{T}^{æ}_{ab} - g_{ab}\mathcal{T}^{æ}/2 + 8\pi G_{æ}\mathcal{T}^{M}_{ab}$, where [56]

$$\begin{aligned} 8\pi G_{æ}\mathcal{T}^{M}_{ab}\chi^b &= -\frac{Q^2}{r^4}\chi_a, \\ \left(\mathcal{T}^{æ}_{ab} - \frac{1}{2}g_{ab}\mathcal{T}^{æ}\right)\chi^b &= \nabla^b\Big\{\big[c_{14}\,(a\cdot s)\,(u\cdot\chi) \\ &\quad + (c_{123}K - 2c_{13}K_0)\,(a\cdot s)\,\big]u_{[a}s_{b]}\Big\}. \end{aligned} \tag{3.26}$$

Setting

$$-\frac{Q^2}{r^4}\chi_a = 2\nabla^b\left(F^Q(r)u_{[a}s_{b]}\right), \tag{3.27}$$

we find that Eq. (3.25) can be cast in the form,

$$\nabla_b F^{ab} = 0, \quad F^{ab} \equiv 2F(r)u^{[a}s^{b]}, \tag{3.28}$$

where

$$F(r) = F^Q(r) + q(r),$$
$$q(r) \equiv -\left(1 - \frac{c_{14}}{2}\right)(a \cdot s)(u \cdot \chi) + \left[(1 - c_{13})K_0 + \frac{c_{123}}{2}K\right](s \cdot \chi). \quad (3.29)$$

On the other hand, comparing Eq. (3.28) with the soucre-free Maxwell equations (3.6), we find that its solution must also take the form (3.12), that is, $F(r) = F_0/r^2$. To determine the integration constant F_0, we note that, for asymptotically flat spacetime, we have [65, 97],

$$u \cdot \chi \simeq -1, \quad s \cdot \chi \simeq 0, \quad a \cdot s = \frac{r_0}{2r^2} + \mathcal{O}(r^{-3}), \quad (3.30)$$

as $r \to \infty$. Then, from Eq. (3.29) we find that $F_0 = r_0(1 - c_{14}/2)/2$. Thus, we have

$$F(r) = \left(1 - \frac{c_{14}}{2}\right)\frac{r_0}{2r^2}. \quad (3.31)$$

Inserting Eq. (3.30) into Eq. (3.23), on the other hand, we find that the ADM mass is given by $M_{ADM} = r_0/2G_{æ}$. Therefore, the total mass M of the spacetime is

$$M \equiv M_{ADM} + M_{æ} = \left(1 - \frac{c_{14}}{2}\right)\frac{r_0}{2G_{æ}} = \frac{1}{4\pi G_{æ}}\int_{\mathcal{B}_\infty} F dA, \quad (3.32)$$

where $M_{æ} = -c_{14}M_{ADM}/2$ is the æther mass or æther contribution to the renormalization of M_{ADM} [97]. On the other hand, using Gauss' law, from Eq. (3.28) we find that

$$\begin{aligned} 0 &= \int_\Sigma \left(\nabla_b F^{ab}\right) d\Sigma_a \\ &= \int_{\mathcal{B}_\infty} F^{ab} d\Sigma_{ab} - \int_{\mathcal{B}_H} F^{ab} d\Sigma_{ab} \\ &= \int_{\mathcal{B}_\infty} F dA - \int_{\mathcal{B}_H} F dA. \end{aligned} \quad (3.33)$$

Here $d\Sigma_a$ is the surface element of a spacelike hypersurface Σ. The boundary $\partial\Sigma$ of Σ consists of the boundary at spatial infinity $\mathcal{B}_\infty$, and the horizon $\mathcal{B}_H$, either the Killing or the universal. Note that Eq. (3.33) is nothing but the conservation law of the flux of F^{ab}. Substituting the above expression into Eq. (3.32) and taking Eq. (3.29) into account, we find the following Smarr formula,

$$MG_{æ} = \frac{q_{UH}A_{UH}}{4\pi} + V_{UH}Q, \quad MG_{æ} = \frac{q_{KH}A_{KH}}{4\pi} + V_{KH}Q, \tag{3.34}$$

where A_{UH} and A_{KH} are, respectively, the area of the universal and Killing horizons, and M is the total mass of an asymptotically flat solution defined in the asymptotic æther rest frame. The potential V_H is defined as $V_H \equiv r_H^2 F^Q(r_H)/Q$. Hence, the first law for the æther black hole may be obtained via a variation of the Smarr relation. In the next section we consider it for two new classes of exact charged æther black hole solutions.

For the surface gravity at the universal horizon, when one considers the peeling behavior of particles moving at any speed, i.e., capturing the role of the æther in the propagation of the physical rays, one finds that the surface gravity at the universal horizon is [66, 74, 76]

$$\kappa_{UH} \equiv \frac{1}{2}\nabla_u(u\cdot\chi) = \frac{1}{2}\left(a\cdot s\right)\left(s\cdot\chi\right)\Big|_{r=r_{UH}}, \tag{3.35}$$

where in the last step we used the fact that χ_a is a Killing vector, $\nabla_{(a}\chi_{b)} = 0$. It must be noted that this is different from the surface gravity defined in GR by Eq. (3.22). In particular, at the universal horizon we have $u\cdot\chi = 0$, and Eq. (3.22) yields,

$$\kappa\left(r_{UH}\right) = K_0(s\cdot\chi)\big|_{r=r_{UH}}. \tag{3.36}$$

3.4 Exact Solutions of Charged Æther Black Holes

To construct exact solutions of charged æther black holes, let us first choose the Eddington-Finklestein coordinate system, in which the metric takes the form

$$ds^2 = -e(r)dv^2 + 2f(r)dvdr + r^2 d\Omega_2^2, \tag{3.37}$$

and the corresponding timelike Killing and æther vectors are

$$\chi^a = (1,0,0,0), \qquad u^a = \big(\alpha,\ \beta,\ 0,\ 0\big),$$
$$u_a dx^a = \big(-e\alpha + f\beta,\ f\alpha,\ 0,\ 0\big)\begin{pmatrix} dv \\ dr \\ d\theta \\ d\phi \end{pmatrix}, \tag{3.38}$$

where $\alpha(r)$ and $\beta(r)$ are functions of r only. Then, the metric can be written as $g_{ab} = -u_a u_b + s_a s_b + \hat{g}_{ab}$, where we have the constraints $u^2 = -1,\ s^2 = 1,\ u \cdot s = 0$. The boundary conditions on the metric coefficients are such that the solution is asymptotically flat, while those for the æther components are such that

$$\lim_{r\to\infty} u^a = \{1,0,0,0\}\ . \tag{3.39}$$

Some quantities that explicitly appear in Eqs. (3.16)-(3.19) are [86]

$$(a\cdot s) = -\frac{(u\cdot\chi)'}{f}, \quad K_0 = -\frac{(s\cdot\chi)'}{f},$$
$$\hat{K} = -\frac{2(s\cdot\chi)}{rf}, \quad \hat{k} = -\frac{2(u\cdot\chi)}{rf}, \tag{3.40}$$

where a prime ($'$) denotes a derivative with respect to r. And $\alpha(r)$, $\beta(r)$ and $e(r)$ are

$$\alpha(r) = \frac{1}{(s\cdot\chi)-(u\cdot\chi)}, \quad \beta(r) = \frac{(s\cdot\chi)}{f},$$
$$e(r) = (u\cdot\chi)^2 - (s\cdot\chi)^2. \tag{3.41}$$

Then, from Eqs. (3.35) and (3.36) we obtain

$$\kappa_{UH} = -\left.\frac{1}{2f}\,(u\cdot\chi)'\,(s\cdot\chi)\right|_{UH},$$
$$\kappa(r_{UH}) = -\left.\frac{1}{f}\,(s\cdot\chi)'\,(s\cdot\chi)\right|_{UH}. \tag{3.42}$$

Clearly, in general $\kappa_{UH} \neq \kappa(r_{UH})$.

From the above expressions one can see that all quantities can be calculated from $(u\cdot\chi)$ and $(s\cdot\chi)$ under the condition $f(r) = 1$. A straightforward calculation of Eq.

(3.14) yields

$$\mathcal{R}_{us} = \frac{2(s\cdot\chi)(u\cdot\chi)f'(r)}{rf^3(r)}. \tag{3.43}$$

In the static spherical symmetric and asymptotically flat spacetime, if we assume that $f = 1$ holds in the whole space-time, we find

$$\mathcal{R}_{us} = \mathcal{T}^{æ}_{us} = 0, \ (f = 1). \tag{3.44}$$

From Eq. (3.27), we also find that

$$F^Q(r) = -\frac{Q^2}{r^2}\int^r \frac{f(r')}{r'^2}dr' = \frac{Q^2}{r^3}, \ (f = 1). \tag{3.45}$$

In the following, we shall use the above expressions first to obtain two classes of exact solutions for the cases $c_{14} = 0, \ c_{123} \neq 0$ and $c_{123} = 0, \ c_{14} \neq 0$, all with $f = 1$. Then, we shall study their main properties by using the Smarr formulas given above.

3.4.1 Exact Solutions for $c_{14} = 0$

When the coupling constant c_{14} is set to zero and $c_{123} \neq 0$, from Eqs. (3.17) and (3.44) one can see the quantity $\nabla_s K$ has to be vanished, i.e., $\nabla_s K = 0$. So, the trace of the extrinsic curvature K of the Σ_U hypersurface is constant. In the infinity, this constant will vanish asymptotically due to the asymptotical flat conditions. Therefore, it must vanish everywhere. Substituting $K = 0$ into Eqs. (3.18) and (3.19), we obtain

$$\begin{aligned}(s\cdot\chi) &= \frac{r_{æ}^2}{r^2},\\ (u\cdot\chi) &= -\sqrt{1-\frac{r_0}{r}+\frac{Q^2}{r^2}+\frac{(1-c_{13})r_{æ}^4}{r^4}},\end{aligned} \tag{3.46}$$

where $r_{æ}$ is another integral constant. Then, using the formula (4.9), we find

$$e(r) = 1-\frac{r_0}{r}+\frac{Q^2}{r^2}-\frac{c_{13}r_{æ}^4}{r^4}, \quad f(r) = 1, \tag{3.47}$$

which reduces to those given in Ref. [65] when $Q = 0$.

The location of the universal horizon r_{UH} is the largest root of equation $u\cdot\chi = 0$. Meanwhile, $u\cdot\chi$ is a physical component of the æther, and should be regular and real everywhere. However, from Eq. (4.12) one can see that in the region $r_- < r < r_{UH}$, this term becomes purely imaginary, where r_- is another root of $u \cdot \chi = 0$, unless the two real roots coincide. Then, $r_æ$ becomes a function of r_0. That is, the global existence of the æther reduces the number of three independent constants $(r_0, r_æ, Q)$ to two, (r_0, Q), the same as in GR. Thus, from $(u \cdot \chi)^2 = 0$ and $d(u \cdot \chi)^2/dr = 0$ [66], we find

$$
\begin{aligned}
r_{UH} &= \frac{r_0}{2}\left(\frac{3}{4} + \sqrt{\frac{9}{16} - 2\frac{Q^2}{r_0^2}}\right), \\
r_æ^4 &= \frac{1}{1 - c_{13}}\left(r_{UH}^4 - \frac{1}{2} r_0 r_{UH}^3\right),
\end{aligned}
\tag{3.48}
$$

which is showed in Figure 3.1. One can see that the charge Q is subjected to the

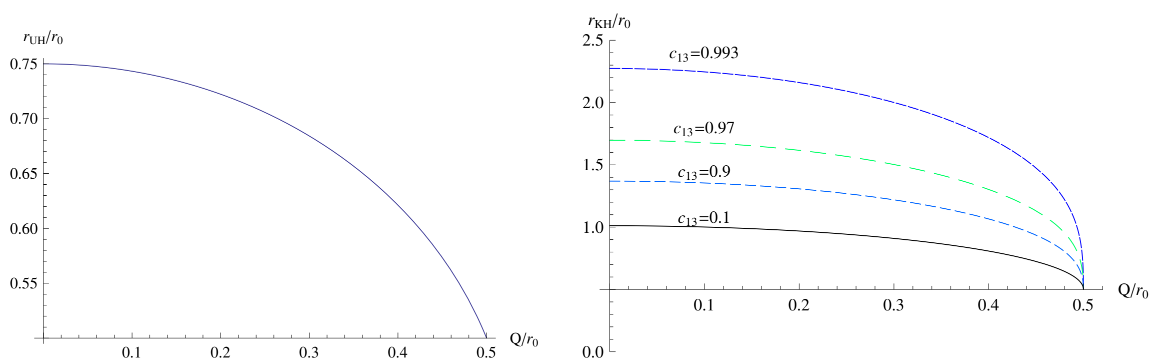

Figure 3.1: The universal and Killing horizons of the charged æther black hole with different c_{13} in the case of $c_{14} = 0$, $c_{123} \neq 0$. The presence of the charge Q makes both horizons smaller. The universal horizon does not depend on c_{13}, while the Killing horizon becomes bigger with the increasing of c_{13}. When $c_{13} = 0$, the Killing horizon reduces to that of the Reissner-Nordstrom black hole.

condition $Q \leq 3r_0/4\sqrt{2}$, in order to have r_{UH} real. When $Q = r_0/2$, we find $r_æ = 0$ and $r_{UH} = r_{KH} = r_0/2$. When $Q > r_0/2$, we have $r_æ^4 < 0$. Thus, in order to have the æther be regular everywhere, the charge should be,

$$
Q \leq \frac{r_0}{2}, \tag{3.49}
$$

which is the same as that given in the Reissner-Nordstrom black hole.

Now let us derive the Smarr relation. Using Eq. (3.35), the surface gravity at the universal horizon can be computed and is given by

$$
\begin{aligned}
\kappa_{UH} &= \frac{1}{2}\nabla_u(u\cdot\chi) \\
&= \frac{1}{2r_{UH}}\sqrt{\frac{2}{3(1-c_{13})}\left(1-\frac{Q^2}{2r_{UH}^2}\right)\left(1-\frac{Q^2}{r_{UH}^2}\right)},
\end{aligned}
\tag{3.50}
$$

which is showed in Figure 3.2. When $Q=0$, we find that $r_{UH}=3r_0/4$ and $\kappa_{UH}=\frac{2}{3r_0}\sqrt{\frac{2}{3(1-c_{13})}}$, which is the same as those given in [69, 76]. The Smarr formula at the

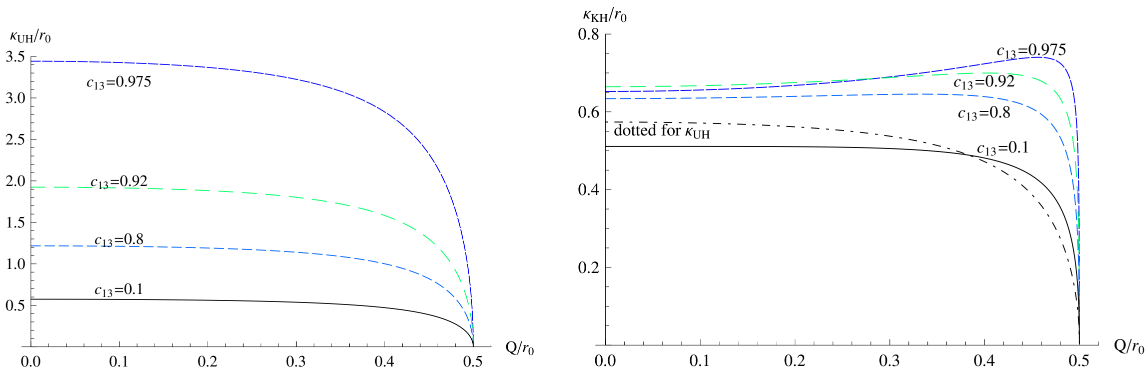

Figure 3.2: The surface gravity at the universal and Killing horizons of the charged æther black hole in the case $c_{14}=0$, $c_{123}\neq 0$. To compare with the results given in Ref. [66], we shift the κ_{UH} line with $c_{13}=0.1$ from the left-hand side as a dot-dashed line to the right. One can see that when c_{13} is small, κ_{UH} is larger than κ_{KH} in the low charge region, while lower in the large charge region, similar to that given in Ref. [66] for the Khronometric theory.

universal horizon is

$$
\begin{aligned}
MG_{æ} &= \frac{q_{UH}A_{UH}}{4\pi} + V_{UH}Q, \\
q_{UH} &= \frac{2}{3}\left(\frac{1}{r_{UH}} - \frac{Q^2}{r_{UH}^3}\right), \\
V_{UH} &= \frac{Q}{r_{UH}},
\end{aligned}
\tag{3.51}
$$

which don't depend explicitly on the coupling constants c_i's, because now we have $c_{14}=0$ and $M_{æ}=0$. It is easy to see that q_{UH} isn't proportional to κ_{UH} given by Eq. (3.50). On the other hand, at the universal horizon we find

$$
G_{æ}\delta M = \frac{1}{8\pi}\left(\frac{2}{3r_{UH}} - \frac{Q^2}{3r_{UH}^3}\right)\delta A_{UH} + \frac{2}{3}V_{UH}\delta Q. \tag{3.52}
$$

Why is there the factor 2/3 in the front of V_{UH}? For a better understanding, let us use the method proposed in [84], i.e., using M's expression from $A = 4\pi r_{UH}^2$,

$$G_{æ}M = \frac{1}{3}\sqrt{\frac{A}{\pi} + 4Q^2 + \frac{4\pi}{A}Q^4}, \tag{3.53}$$

and writing the variation of M as $G_{æ}\delta M = T\delta A + V\delta Q$, we obtain,

$$\begin{aligned} T &\equiv \frac{\partial (G_{æ}M)}{\partial A} = \frac{1}{8\pi}\left(\frac{2}{3r_{UH}} - \frac{Q^2}{3r_{UH}^3}\right), \\ V &\equiv \frac{\partial (G_{æ}M)}{\partial Q} = \frac{2Q}{3r_{UH}} = \frac{2}{3}V_{UH}, \end{aligned} \tag{3.54}$$

which are the same as those given in Eq. (3.52). However, such defined temperature T is also not proportional to κ_{UH} given by Eq. (3.50).

On the other hand, the location of the Killing horizon is the largest root of $e(r) = 0$. Using Eq. (3.47), we find

$$\begin{aligned} r_{KH} &= \frac{r_0}{2}\left(\frac{1}{2} + L + \sqrt{N - P + \frac{1 - 4Q^2/r_0^2}{4L}}\right), \\ L &= \sqrt{\frac{N}{2} + P}, \\ N &= \frac{1}{2} - \frac{4Q^2}{3r_0^2}, \\ P &= \frac{2^{1/3}(12I + Q^4/r_0^4)}{3H} + \frac{H}{3 \cdot 2^{1/3}}, \\ I &= -\frac{c_{13}}{1 - c_{13}}\left(\frac{r_{UH}^4}{r_0^4} - \frac{r_{UH}^3}{2r_0^3}\right), \\ H &= \left(J + \sqrt{-4(12I + Q^4/r_0^4)^3 + J^2}\right)^{1/3}, \\ J &= 27I - 72IQ^2/r_0^2 + 2Q^6/r_0^6, \end{aligned} \tag{3.55}$$

which is showed in Figure 3.1. When $c_{13} = 0$, we find that $r_{KH} = r_+$, that is, it coincides with the Reissner-Nordstrom black hole Killing horizon (here after we denote $r_\pm = (1 \pm \sqrt{1 - 4Q^2/r_0^2})r_0/2.$). Then, the Smarr formula and surface gravity (using Eq. (3.22)) at the Killing horizon are

$$\begin{aligned}
MG_{æ} &= \frac{q_{KH}A_{KH}}{4\pi} + V_{KH}Q, \\
q_{KH} &= \left(\frac{r_0}{2r_{KH}^2} - \frac{Q^2}{r_{KH}^3}\right), \\
\kappa_{KH} &= \frac{2}{r_{KH}} - \frac{3r_0}{2r_{KH}^2} + \frac{Q^2}{r_{KH}^3},
\end{aligned} \tag{3.56}$$

which is showed in Figure 3.2. Note again that the q_{KH} is still not proportional to the κ_{KH}. When $c_{13} = 0$, both q_{KH} and κ_{KH} reduce to those given in the Reissner-Nordstrom black hole, $q_{KH} = \kappa_{KH} = (r_+ - r_-)/2r_+^2$. The first law at the Killing horizon cannot be obtained via the variation method, although it may be obtained via Smarr's method [84]. However, due to its complexity, we shall not consider this possibility, as even we do it, we do not expect to get much from such complicated expressions.

Note that, when $c_{13} \ll 1$, from Eq. (3.47) we find that the solution reduces to the usual Reissner-Nordstrom black hole with a universal horizon given by (4.15) that is always inside its Killing horizon $r_{EH} = r_+$, which is the same as that derived in the Khronometric theory [66].

Finally, let us turn to Figure 3.2, from which we can see that the presence of the charge Q always makes the surface gravity κ_{UH} lower, while the presence of the constant c_{13} always makes it bigger, after the constraints (3.5) are taken into account. For the κ_{KH}, the situation becomes more complicated. In particular, when both c_{13} and Q are small, the effects of them is similar to that presented in κ_{UH} as shown in the figure. But for large c_{13}, e.g. $c_{13} = 0.92,\ 0.975$, the presence of the charge increases the temperature at the beginning and then decreases it when the charge becomes very large.

3.4.2 Exact Solutions for $c_{123} = 0$

In this case, from Eq. (3.32) we find that the total mass is

$$MG_{æ} = \left(1 - \frac{c_{14}}{2}\right)\frac{r_0}{2}. \tag{3.57}$$

There exists a range of the coupling constants that passes all the current observational tests in the one-parameter family of the Einstein-æther theories [87]. Setting $c_{123} = 0$, from Eqs. (3.18), (3.19) and (3.40) we obtain

$$\begin{aligned} &(u \cdot \chi) = -1 + \frac{r_0}{2r}, \quad (s \cdot \chi) = \frac{r_0 + 2r_u}{2r}, \\ &r_u = \frac{r_0}{2}\left(\sqrt{\frac{2 - c_{14}}{2(1 - c_{13})} - \frac{4Q^2}{(1 - c_{13})r_0^2}} - 1\right). \end{aligned} \tag{3.58}$$

Then, we find that

$$e(r) = 1 - \frac{r_0}{r} - \frac{r_u(r_0 + r_u)}{r^2}, \quad f(r) = 1, \tag{3.59}$$

which again reduces to that given in Ref. [65] when $Q = 0$. From the above expressions, we find

$$\alpha(r) = \frac{1}{(s \cdot \chi) - (u \cdot \chi)} = \frac{1}{1 + \frac{r_u}{r}}. \tag{3.60}$$

Since it is one of the component of u^a, it should be regular everywhere (possibly except at the singular point $r = 0$), we must have

$$\begin{aligned} r_u &\geq 0 \Rightarrow Q \leq \sqrt{\frac{2c_{13} - c_{14}}{2}}\frac{r_0}{2} = \sqrt{1 - \frac{c_{14}}{2} - (1 - c_{13})}\frac{r_0}{2}, \\ c_{13} &\geq \frac{c_{14}}{2}. \end{aligned} \tag{3.61}$$

The position of the universal horizon r_{UH} and its surface gravity (using Eq. 3.35) are

$$r_{UH} = \frac{r_0}{2}, \quad \kappa_{UH} = \frac{1}{2\sqrt{(1 - c_{13})}r_{UH}}\sqrt{1 - \frac{c_{14}}{2} - \frac{Q^2}{r_{UH}^2}}. \tag{3.62}$$

And the κ_{UH} is showed in Figure 3.3. Also, when $Q = 0$ it reduces to the one obtained

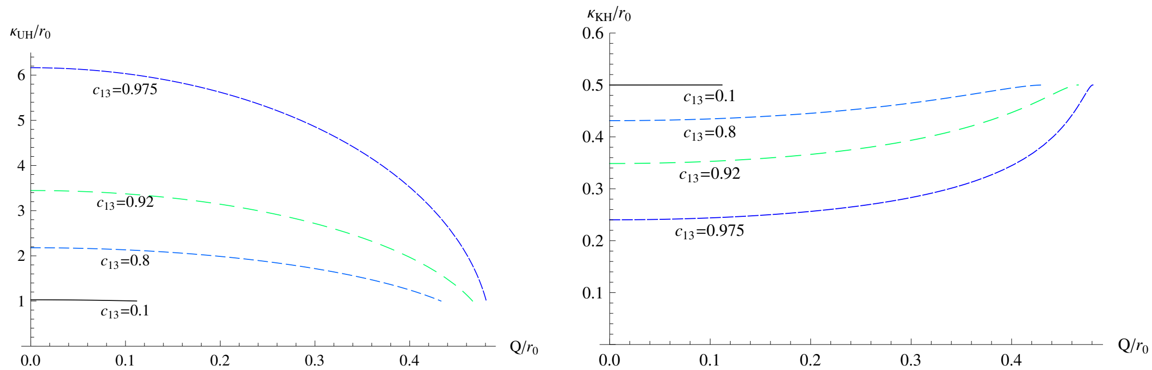

Figure 3.3: The surface gravity at the universal horizon and Killing horizon of charged Einstein-æther black holes in the case of $c_{14} = 0.1,\ c_{123} = 0$. The restrictions are $c_{13} \geq 0.05,\ Q/r_0 \leq \sqrt{(2c_{13} - 0.1)/8}$.

in [69, 76]. One can see that r_{UH} does not depend on the charge Q, but κ_{UH} depends on it. In other words, to the same universal horizon $r_{UH} = r_0/2$, there are different thermal temperatures of the horizon with different charge Q, if we assume that T is still somehow proportional to κ_{UH}.

The Smarr formula new reads,

$$\begin{aligned}
G_{æ}M &= \frac{q_{UH}A_{UH}}{4\pi} + V_{UH}Q, \\
q_{UH} &= \frac{1}{r_{UH}}\left(1 - \frac{c_{14}}{2} - \frac{Q^2}{r_{UH}^2}\right), \\
G_{æ}\delta M &= \left(1 - \frac{c_{14}}{2}\right)\frac{1}{r_{UH}}\frac{\delta A_{UH}}{8\pi},
\end{aligned} \tag{3.63}$$

in which the term proportional to δQ is absent. To see this more clearly, let us consider the Smarr method, from which we find that

$$G_{æ}M = \left(1 - \frac{c_{14}}{2}\right)\sqrt{\frac{A}{4\pi}}. \tag{3.64}$$

Then, writing the variation of M as $G_{æ}\delta M = T\delta A + V\delta Q$, we obtain,

$$T = \frac{\partial\left(G_{æ}M\right)}{\partial A} = \left(1 - \frac{c_{14}}{2}\right)\frac{1}{8\pi r_{UH}}, \quad V = \frac{\partial\left(G_{æ}M\right)}{\partial Q} = 0, \tag{3.65}$$

which are the same as those given in the second line of Eq. (3.63). Once again, q_{UH} and T aren't proportional to the κ_{UH} given by Eq. (3.62).

On the other hand, the Killing horizon and its surface gravity (using Eq. 3.22) are

$$r_{KH} = r_0 + r_u, \quad \kappa_{KH} = \frac{2r_u + r_0}{2r_{KH}^2}, \tag{3.66}$$

which are showed in Figure 3.3. In order for them to be real we must assume that $Q \leq \sqrt{1 - \frac{c_{14}}{2}}\frac{r_0}{2}$. Comparing Eq. (3.61) with (3.66), one can see that the latter condition on Q is contained in the former, i.e., in the charged æther black hole, the electric charge is subjected to more stringent restrictions. Again when $c_{13} = c_{14} = 0$, κ_{KH} reduces to $(r_+ - r_-)/2r_+^2$. The Smarr formula at the Killing horizon is

$$\begin{aligned} G_{æ}M &= \frac{q_{KH}A_{KH}}{4\pi} + V_{KH}Q, \\ q_{KH} &= \left[\left(1 - \frac{c_{14}}{2}\right)\frac{r_0}{2r_{KH}^2} - \frac{Q^2}{r_{KH}^3}\right], \end{aligned} \tag{3.67}$$

which depends on the coupling constants c_{13} and c_{14}. q_{KH} approaches to $(r_+ - r_-)/2r_+^2$, if $c_{13} = c_{14} = 0$. We find that taking variation with respect to each term cannot obtain the first law, so instead we use Smarr's method [84], and find that

$$\begin{aligned} &G_{æ}\delta M = \frac{\partial M}{\partial A}\delta A_{KH} + \frac{\partial M}{\partial Q}\delta Q, \\ &\frac{\partial M}{\partial Q} = \frac{c_a c_b Q}{\sqrt{c_a(c_a c_b - 1)Q^2 + c_b r_{KH}^2}}, \\ &c_a \equiv \frac{1}{1 - c_{13}}, \qquad c_b \equiv 1 - \frac{c_{14}}{2}, \\ &T = \frac{\partial M}{\partial A} = \frac{c_b}{c_a c_b - 1}\left(\frac{c_a c_b}{\sqrt{c_a(c_a c_b - 1)Q^2 + c_a c_b r_{KH}^2}} - \frac{1}{r_{KH}}\right). \end{aligned} \tag{3.68}$$

Note that, similar to the previous case, now q_{KH} and T aren't proportional to the κ_{KH} given by Eq. (3.66), either.

Finally, from Figure 3.3 we note that the dependence of κ_{UH} on Q is similar to the former case. In particular, its presence always makes the temperature lower, while the presence of c_{13} increases it. The surface gravity κ_{UH} is always larger than κ_{KH}. At the Killing horizon, the effects of the charge and c_{13} on κ_{KH} are just opposite.

CHAPTER FOUR

Hawking Radiation of Charged Einstein-aether Black Holes

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In this chapter, we study analytically quantum tunneling of relativistic and non-relativistic particles at both Killing and universal horizons of Einstein-Maxwell-æther black holes, after high-order curvature corrections are taken into account, for which the dispersion relation of the particles becomes nonlinear. Our results at the Killing horizons confirm the previous ones, i.e., at high frequencies the corresponding radiation remains thermal and the nonlinearity of the dispersion does not alter the Hawking radiation significantly. On the contrary, non-relativistic particles are created at universal horizons and are radiated out to infinity. Although the radiation is also thermal spectrum, different species of particles, characterized by a parameter z, which denotes the power of the leading term in the nonlinear dispersion relation, in general experience different temperatures, $T^z_{UH} = 2\kappa_{UH}(z-1)/(2\pi z)$, where κ_{UH} is the surface gravity of the universal horizon, defined by peeling behavior of ray trajectories at the universal horizon. We also study the Smarr formula by assuming that: (a) the entropy is proportional to the area of the universal horizon, and (b) the first law of black hole thermodynamics holds, whereby we derive the Smarr mass, which in general is different from the total mass obtained at infinity. This indicates that one or both of these assumptions must be modified.

4.1 Introduction

In the Einstein-æther theory, a timelike æther vector field is introduced to describe extra degrees of the gravitational sector, in addition to the spin-2 ones found in general relativity that move with the speed of light [8, 9]. In fact, due to the presence of the æther field, spin-0 and spin-1 particles are also present, and all move at different speeds [64]. Moreover, due to Cherenkov effects they must move with speeds no less than that of light [75]. It should be noted that here the propagations faster than that of light do not violate causality [63]. In particular, gravitational theories with breaking Lorentz invariance (LI) still allow the existence of black holes [17, 18, 65–68, 88]. However, instead of Killing horizons, now the boundaries of black holes are hypersurfaces, termed as *universal horizons*, which are always inside Killing horizons and trap excitations traveling at arbitrarily high velocities. The crucial ingredient for the existence of a universal horizon is the presence of a globally timelike foliation of the spacetime [17]. Such a preferred foliation, for example, naturally rises in the Hořava theory [7]. But in the Einstein-æther theory this is true only when the æther is hypersurface-orthorgonal [10, 88]. This is always the case in spherically symmetric spacetimes, although in other spacetimes, such as the ones with rotation, the æther is generically not hypersurface-orthorgonal [10, 88]. With the above in mind, a slightly modified first law of black hole mechanics was found to exist for the neutral Einstein-æther black holes [65], but for the charged Einstein-æther black holes, such a law is still absent [22].

Berglund *et al* [69] used tunneling method to study the corresponding Hawking radiation at the universal horizon for a scalar field that violates the local LI, and found that the universal horizon radiates as a blackbody at a fixed temperature. Using a collapsing null shell, on the other hand, Michel and Parentani [70] computed the late time radiation and found that the mode pasting across the shell is adiabatic at late time. This implies that large black holes emit a thermal flux with a temperature fixed by the surface gravity of the Killing horizon. This, in turn, suggests that the universal horizon should play no role in the thermodynamical properties of these black holes. However, it should be noted that in such a setting, the khronon field is not continuous across the collapsing null shell [89]. Normally, it is expected that such discontinuities should not affect the final results [70]. However, the khronon field here plays a special role, and in particular it defines the causality of the spacetime. So far, it is not clear whether the results presented in [70] will remain the same or not, after the continuity of the æther field across the collapsing surface is assumed.

Another different approach was taken by Cropp *et al* [76], in which ray trajectories in such black hole backgrounds were studied, and evidence was found which shows that Hawking radiation is associated with the universal horizon, while the "lingering" of low-energy ray trajectories near the Killing horizon hints at reprocessing there.

In this chapter, we have no intention to resolve the above discrepancy, but rather study the Hawking radiation at both universal and Killing horizons of the charged Einstein-æther black holes found in [22]. Although we also use the tunneling approach, we shall give up the null geodesic method [90]. Instead, we shall adopt the Hamilton-Jacobi method [91–94], and show that particles with $z \geq 2$ are indeed created at the universal horizon, and the corresponding Hawking radiation is thermal, where z characterizes the nonlinearity of the dispersion relation, appearing in Eq. (4.31) given below. Although for any given $z \geq 2$ the universal horizon radiates thermally, particles with different z will feel different temperatures, given by

$$T^z_{UH} = \left(2 - \frac{2}{z}\right)\frac{\kappa_{UH}}{2\pi}, \tag{4.1}$$

where κ_{UH} is the surface gravity of the universal horizon, defined by peeling behavior of ray trajectories at the universal horizon [22, 66, 76]. On the other hand, in high frequencies only relativistic particles are created at the Killing horizon, and the corresponding Hawking radiation is the same as that obtained in general relativity [77]. This is consistent with previous findings [95] [1] .

Specifically, the chapter is organized as follows. In Section 4.2 we give a brief review of the Einstein-æther theory and the charged black holes obtained in [22], while

[1] It should be noted that in low frequencies the Hawking radiation is sensitive to high-order corrections. For detail, see, for example, [96].

in Section 4.3 we study the tunneling of spin-0 particles with a nonlinear dispersion. In Section 4.4 we study the Smarr formula by assuming that the first law of black hole mechanics holds at the universal horizon, and find the corresponding Smarr mass, which in general is quite different from the Arnowitt-Deser-Misner (ADM) mass at infinity.

4.2 Einstein-Maxwell-aether Theory and Charged Black Holes

The Einstein-Maxwell-æther theory considered in [22] is described by the action,

$$\mathcal{S} = \int d^4x\sqrt{-g}\Big[\frac{1}{16\pi G_{æ}}(\mathcal{R} + \mathcal{L}_{æ}) + \mathcal{L}_M\Big], \tag{4.2}$$

where $G_{æ}$ is a coupling constant of the theory, and is related to Newton's gravitational constant G_N by $G_{æ} = (1-c_{14}/2)G_N$ [97]. $\mathcal{R}$ is the four-dimensional (4D) Ricci scalar, $\mathcal{L}_M$ denotes the matter Lagrangian density, and $\mathcal{L}_{æ}$ the æther Lagrangian density, defined as

$$-\mathcal{L}_{æ} = Z^{ab}{}_{cd}(\nabla_a u^c)(\nabla_b u^d) - \lambda(u^2+1), \tag{4.3}$$

where ∇_μ denotes the covariant derivative with respect to the 4D metric g_{ab}, which has the signatures $(-,+,+,+)$. u_a is the four-velocity of the æther, λ a Lagrangian multiplier that guarantees u_a to be timelike, and $Z^{ab}{}_{cd}$ is defined as [73,97],

$$Z^{ab}{}_{cd} = c_1 g^{ab}g_{cd} + c_2\delta^a{}_c\delta^b{}_d + c_3\delta^a{}_d\delta^b{}_c - c_4 u^a u^b g_{cd}, \tag{4.4}$$

where c_i's are coupling constants of the theory. There are a number of theoretical and observational bounds on the coupling constants c_i [74]. Here, we impose the following constraints [22], $0 \le c_{14} < 2,\ 2 + c_{13} + 3c_2 > 0,\ 0 \le c_{13} < 1$, where $c_{14} \equiv c_1 + c_4$, and so on. The source-free Maxwell Lagrangian $\mathcal{L}_M$ is given by

$$\mathcal{L}_M = -\frac{1}{16\pi G_{æ}}\mathcal{F}_{ab}\mathcal{F}^{ab},\ \mathcal{F}_{ab} = \nabla_a\mathcal{A}_b - \nabla_b\mathcal{A}_a, \tag{4.5}$$

where $\mathcal{A}_a$ is the four-vector of the electromagnetic field.

The static spherically symmetric spacetimes in the Eddington-Finklestein coordinates are described by the metric [56],

$$ds^2 = -e(r)dv^2 + 2dvdr + r^2 d\Omega^2, \tag{4.6}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. The corresponding time-translation Killing and æther vectors are given, respectively, by

$$\chi^a = \delta^a_v, \quad u^a = \alpha\delta^a_v + \beta\delta^a_r, \tag{4.7}$$

where α, β are functions of r only, and the constrain is $u^2 = -1$. Introducing the spacelike unit vector s_a via the relations $u^a s_a = 0$, $s^2 = 1$, we find that the metric can be written as

$$g_{ab} = -u_a u_b + s_a s_b + \hat{g}_{ab}, \tag{4.8}$$

where $\hat{g}_{ab} \equiv \mathrm{diag}\,(0, 0, r^2, r^2 \sin\theta^2)$, and that,

$$\begin{aligned} \alpha(r) &= \frac{1}{(s\cdot\chi) - (u\cdot\chi)}, \quad \beta(r) = -(s\cdot\chi), \\ e(r) &= (u\cdot\chi)^2 - (s\cdot\chi)^2. \end{aligned} \tag{4.9}$$

The Killing horizon is the location where χ^a becomes null, i.e., $e(r_{KH}) = 0$.

The universal horizon, on the other hand, is the located at $(u\cdot\chi) = 0$ [17, 18], that is,

$$\left(e\alpha^2 + 1\right)\Big|_{UH} = 0. \tag{4.10}$$

The surface gravity at the universal horizon is defined as [76],

$$\begin{aligned} \kappa_{UH} &\equiv \frac{1}{2}\nabla_u (u\cdot\chi)\Big|_{UH} \\ &= \frac{1}{2}(a\cdot s)(s\cdot\chi)\Big|_{UH}, \end{aligned} \tag{4.11}$$

which is precisely the one obtained from the peeling behavior of rays propagating with infinite group velocity with respect to the æther as shown explicitly in [66, 76].

In [22], two classes of the charged Einstein-æther black hole solutions were found in closed forms, for particular choices of the coupling constants c_i's. They are given as follows.

4.2.1 Exact Charged Einstein-aether Solutions for $c_{14} = 0$

When $c_{14} = 0$, which corresponds to the case in which the spin-0 particle of the khronon field has an infinitely large velocity, the charged Einstein-æther black hole solutions are given by [22],

$$
\begin{aligned}
(s\cdot\chi) &= \frac{r_{æ}^2}{r^2},\\
(u\cdot\chi) &= -\sqrt{1-\frac{r_0}{r}+\frac{Q^2}{r^2}+\frac{(1-c_{13})r_{æ}^4}{r^4}},\\
e(r) &= 1-\frac{r_0}{r}+\frac{Q^2}{r^2}-\frac{c_{13}r_{æ}^4}{r^4},
\end{aligned}
\tag{4.12}
$$

where $r_0, r_{æ}$ and Q are the integration constants, and Q is related to the Maxwell field via the relation,

$$
\mathcal{F}_{ab} = \frac{Q}{r^2}(u_a s_b - u_b s_a). \tag{4.13}
$$

In order for the khronon field to be well-defined in the whole spacetime, the integration constant $r_{æ}$ must be given by [22],

$$
r_{æ}^4 = \frac{1}{1-c_{13}}\left(r_{UH}^4 - \frac{1}{2}r_0 r_{UH}^3\right), \tag{4.14}
$$

where r_{UH} is the location of the universal horizon, given by

$$
r_{UH} = \frac{r_0}{2}\left(\frac{3}{4}+\sqrt{\frac{9}{16}-2\frac{Q^2}{r_0^2}}\right). \tag{4.15}
$$

The location of the Killing horizon is at $r = r_{KH}$, given by,

$$
r_{KH} = \frac{r_0}{2}\left(\frac{1}{2}+L+\sqrt{N-P+\frac{1-4Q^2/r_0^2}{4L}}\right), \tag{4.16}
$$

where

$$
\begin{aligned}
L &= \sqrt{\frac{N}{2}+P}, \quad N = \frac{1}{2}-\frac{4Q^2}{3r_0^2},\\
P &= \frac{2^{1/3}(12I+Q^4/r_0^4)}{3H}+\frac{H}{3\cdot 2^{1/3}},\\
I &= -\frac{c_{13}}{1-c_{13}}\left(\frac{r_{UH}^4}{r_0^4}-\frac{r_{UH}^3}{2r_0^3}\right),\\
H &= \left(J+\sqrt{-4(12I+Q^4/r_0^4)^3+J^2}\right)^{1/3},\\
J &= 27I-72IQ^2/r_0^2+2Q^6/r_0^6.
\end{aligned}
\tag{4.17}
$$

4.2.2 Exact Charged Einstein-aether Solutions for $c_{123} = 0$

When $c_{123} = 0$, the velocity of the spin-0 particle of the khronon field is zero, and the solutions are given by,

$$(u \cdot \chi) = -1 + \frac{r_0}{2r}, \quad (s \cdot \chi) = \frac{r_0 + 2r_u}{2r},$$
$$e(r) = 1 - \frac{r_0}{r} - \frac{r_u(r_0 + r_u)}{r^2}, \tag{4.18}$$

where r_0 is a non-negative integration constant, and r_u is given by,

$$r_u = \frac{r_0}{2}\left(\sqrt{\frac{\mathrm{p}}{\mathrm{g}} - \frac{4Q^2}{\mathrm{g}r_0^2}} - 1\right), \tag{4.19}$$

where

$$\mathrm{g} \equiv 1 - c_{13}, \ \mathrm{p} \equiv 1 - \frac{c_{14}}{2}. \tag{4.20}$$

The locations of the universal and Killing horizons are given, respectively, by

$$r_{UH} = \frac{r_0}{2}, \quad r_{KH} = r_0 + r_u. \tag{4.21}$$

It should be noted that, in order to have the khronon field well-defined in the whole spacetime, in the present case we must assume that

$$|Q| \leq \frac{1}{2}\sqrt{\mathrm{p} - \mathrm{g}}\, r_0, \quad \mathrm{p} \geq \mathrm{g}. \tag{4.22}$$

4.3 Hawking Radiation with Nonlinear Dispersion Relation

The semi-classical tunneling approximations that model the Hawking radiation usually follow two approaches, the null geodesics (NG) method explored by Parikh and Wilczek [90], and the Hamilton-Jacobi (HJ) method used by Agheben *et al* [91–94]. Since the final results should not depend on the methods to be used, in this chapter we choose the HJ method. In each method, particles with positive (negative) energy just inside (outside) of the horizon are assumed to escape (fall into) it. Both of the processes are forbidden classically, so the radiation is quantum mechanical in nature.

In the semi-classical approximation, the charged massless scalar field $\phi(x)$ can be written as $\phi(x) = \phi_0 \exp[i\mathcal{S}(\phi)]$ in terms of its action $\mathcal{S}(\phi)$. Then, the four-momentum of such an excitation is given by

$$k_a = \frac{1}{i\phi}(\nabla_a \mp iqA_a)\phi, \tag{4.23}$$

where $\mp q$ is the electric charge of the positive/negative energy excitation, respectively, and $A_a = (-Q/r, 0, 0, 0)$ is the 4-potential of the electromagnetic field. Then, within the WKB approximation let us consider the ansatz

$$\mathcal{S}(\phi) = \mp\omega v + \int^r dr' k_r(r'), \tag{4.24}$$

for the phase of the field configuration, where the top and bottom sign $\mp$ refer, respectively, to positive and negative energy excitations. Plugging it into (4.23), the wave four-vector takes the form,

$$\begin{aligned} k_a dx^a &= \mp(\omega - q\varphi)dv + k_r dr \\ &= \big[\pm(\omega - q\varphi)\ell_{-a} + k_r \rho_a\big]dx^a, \end{aligned} \tag{4.25}$$

where $\varphi = Q/r$ is the electric potential, $\ell_{-a} = (-1, 0)$ is the radial null vector, and $\rho_a = (0, 1)$ is the redshift vector. The radial momentum k_r can be solved from the dispersion relation

$$e(r)k_r^2 \mp 2(\omega - q\varphi)k_r = k^2, \tag{4.26}$$

once $k^2(\omega)$ is given. Clearly, in general the above equation has four solutions: $k^{\pm}_{r(I)}$ and $k^{\pm}_{r(O)}$, where $\pm$ refer, respectively, to the positive and negative energy, I (O) means in-going (out-going) particles. Due to the time reversal invariance, we have $k^+_{r(O)} = -k^-_{r(I)}$ and $k^-_{r(O)} = -k^+_{r(I)}$. From the standard results in quantum mechanics, the emission rate Γ is given by $\Gamma \sim \exp[-2\mathrm{Im}\mathcal{S}]$. From Eq. (4.24) we can see that only the singular parts of $k_r(r)$ have contributions to Im$\mathcal{S}$. In particular, we have

$$\begin{aligned}
\mathrm{Im}\mathcal{S} &= \mathrm{Im}\lim_{\epsilon\to 0}\int_{r_H-\epsilon}^{r_H+\epsilon} k^{+}_{r(O)}(r')dr' \\
&= -\mathrm{Im}\lim_{\epsilon\to 0}\int_{r_H+\epsilon}^{r_H-\epsilon} k^{-}_{r(I)}(r')dr' \\
&= \mathrm{Im}\lim_{\epsilon\to 0}\int_{r_H+\epsilon}^{r_H-\epsilon} k^{+}_{r(O)}(r')dr', \qquad (4.27)
\end{aligned}$$

where r_H is the location of the singularity of $k^{+}_{r(O)}(r)$. Deforming the contour into the low half complex plane of the singularity located at $r = r_H$ for the first integral and the upper half complex plane for the last one, we find

$$\begin{aligned}
2\mathrm{Im}\mathcal{S} &= \mathrm{Im}\lim_{\epsilon\to 0}\left\{\int_{r_H-\epsilon}^{r_H+\epsilon} k^{+}_{r(O)}(r')dr' + \int_{r_H+\epsilon}^{r_H-\epsilon} k^{+}_{r(O)}(r')dr'\right\} \\
&= \mathrm{Im}\oint dr k^{+}_{r(O)}(r), \qquad (4.28)
\end{aligned}$$

where the closed circuit is always anticlockwise. Therefore, to calculate the emission rate we need only consider the out-going positive energy particles.

On the other hand, in the frame comoving with the æther, k_a can be written as

$$k_a = -k_u u_a + k_s s_a, \qquad (4.29)$$

where $k_u \equiv (u \cdot k)$ and $k_s \equiv (s \cdot k)$ are corresponding to, respectively, the energy and momentum, measured by observers that are comoving with the æther, and are given by

$$\begin{aligned}
k_u(r) &= \frac{\pm(\omega - q\varphi)}{(u\cdot\chi) - (s\cdot\chi)} - k_r(s\cdot\chi), \\
k_s(r) &= \frac{\pm(\omega - q\varphi)}{(u\cdot\chi) - (s\cdot\chi)} - k_r(r)(u\cdot\chi). \qquad (4.30)
\end{aligned}$$

Then, we have $k^2 = -k_u^2 + k_s^2$, which is a function of k_r. In this chapter, we consider the non-relativistic dispersion relation, given by [95, 98],

$$k_u^2 = k_0^2 \sum_{n=1}^{z} a_n \left(\frac{k_s}{k_0}\right)^{2n}, \qquad (4.31)$$

where a_n's are dimensionless constants, which will be considered as order of unit in the following discussions [98], and z is an integer [2] . Lorentz symmetry requires $(a_1, z) = (1, 1)$. Therefore, in this chapter we shall set $a_1 = 1$. In the Horava theory of gravity [7], the power-counting renormalizability requires $z \geq 3$. The constant k_0 is the UV Lorentz-violating (LV) energy scale for the matter [76] or the suppression mass scale [98]. The experimental viable range for the k_0 is rather broad and its value shows the size of LV of the given field. When $k_s/k_0 \to 0$, the field becomes relativistic and one recovers the standard dispersion relation $k_u^2 = k_s^2$.

To study the effects of high-order corrections, characterized by the critical exponent z, in the following we shall study the Hawking radiation for various choices of z at both of the universal and Killing horizons.

To see clearly the difference between relativistic and non-relativistic particles, in the following we first consider the relativistic case ($z = 1$), and re-obtain the well-known results of the Hawking radiation at the Killing horizons [77, 95, 96]. However, we find that at universal horizons relativistic particles are not created. Then, we move onto the non-relativistic ones ($z \geq 2$), and show that such particles are indeed created at universal horizons. It should be noted that in doing so we implicitly assume that both of these two kinds of horizons have an associated temperature. However, this is not well grounded [99], and is closely related to the theory of Hawking radiation at high energies. We shall come back to this issue at the end of Section V. In addition, in high frequencies non-relativistic particles ($z \geq 2$) are not created at Killing horizons, which confirms the earlier findings [95, 96].

[2] A more general expression for the nonlinear dispersion relation in a curved background was given in [56]. However, to make the problem tractable, in this chapter we restrict ourselves to the cases defined by Eq. (4.31). For a further justification of the use of this form at the universal horiozn, see [56].

4.3.1 Hawking Radiation for $z = 1$

When $z = 1$ or $k_s \ll k_0$, the dispersion relation reduces to the relativistic one, $k^2 = -k_u^2 + k_s^2 = 0$, or $k_u = \pm k_s$. From Eq. (4.30), one can see that at both of the Killing and universal horizons, the solution $k_u = k_s$ will all lead to $k_r = 0$. For the outgoing positive energy or ingoing negative energy particles, the relation $k_u = -k_s$ together with Eq. (4.30) leads to

$$\begin{aligned} k^+_{r(O)}(r) &= -\frac{2(\omega - q\varphi)}{(s\cdot\chi) - (u\cdot\chi)}\frac{1}{(s\cdot\chi) + (u\cdot\chi)} \\ &= \frac{2(\omega - q\varphi)}{e(r)}, \end{aligned} \tag{4.32}$$

which is finite at the universal horizon $(u \cdot \chi) = 0$, but singular at the Killing horizon $e(r) = 0$. This implies that *relativistic particles cannot escape from the universal horizons even quantum mechanically, as their velocity is finite and the horizon serves as an infinitely large barrier to them.* However, they can be created at the Killing horizon with the standard results [77],

$$\begin{aligned} 2\mathrm{Im}S &= \frac{\omega - \mu_0}{T_{KH}}, \\ T_{KH} &= \frac{e'(r_{KH})}{4\pi} = \frac{\kappa^{GR}_{KH}}{2\pi}, \end{aligned} \tag{4.33}$$

where $\mu_0 = q\varphi_{KH}$ and $\varphi_{KH} \equiv Q/r_{KH}$, a prime denotes the derivative with respect to r, and κ^{GR}_{KH} denotes the surface gravity defined as

$$\kappa^{GR} \equiv \sqrt{-\frac{1}{2}\left(\nabla_a\chi_b\right)\left(\nabla^a\chi^b\right)}\,. \tag{4.34}$$

It should be noted that, in Ref. [70] by using collapsing shell method, the authors showed that at the Killing horizon, with a given k_0 there exists an effective temperature $T_\omega(k_0)$. When k_0 is increasing, T_ω approaches to the Hawking temperature T_{KH}. In Ref. [76], on the other hand, it was shown that energetic particles simply pass the Killing horizon, while low-energy particles linger and eventually escape to infinity.

4.3.2 Hawking Radiation for $z > 1$

When $z > 1$, from Eq. (4.30) we find that,

$$k_u(r) = \frac{1}{(u \cdot \chi)} \big[\pm (\omega - q\varphi) + k_s(r)(s \cdot \chi) \big],$$
$$k_r(r) = -\frac{1}{(u \cdot \chi)} \left[\frac{\mp(\omega - q\varphi)}{(u \cdot \chi) - (s \cdot \chi)} + k_s(r) \right]. \tag{4.35}$$

At the Killing horizon we have $(s \cdot \chi) = -(u \cdot \chi)$, and $(u \cdot \chi)$ is finite, so one can see that the momentum k_r is always regular, indicating that non-relativistic particles may not created at the Killing horizon, as they can escape the Killing horizon even classically. This is consistent with the results obtained in [95, 96]. The reason is simply the following: To have terms with $z > 1$ be leading, we implicitly assume that $k > k_0$, as one can see from Eq. (4.31). Therefore, our above claim is actually valid only for modes with $k > k_0$, i.e., the high frequency modes [95, 96]. For modes with $k < k_0$, the quadratic term k^2 is important, and we must consider it together with high-order corrections. In the latter, it was shown that the spectrum of the corresponding Hawking radiation is modified [95, 96]. So, in the rest of this section we shall focus ourselves only at the universal horizon.

For the outgoing modes with positive Killing energy [the top sign in Eqs. (4.35)], $k_s(r)$ has a singularity at the universal horizon. In review of Eqs. (4.26), (4.30) and (4.31), we assume that it takes the form

$$k_s(r) = \frac{k_0 b(\omega, r)}{|u \cdot \chi|^m}, \quad m > 0, \tag{4.36}$$

where $b\,(\omega, r_{UH}) \neq 0$, and m is the smallest positive real number such that $|u \cdot \chi|^m\, k_s(r)$ is finite at the horizon. Combining Eq. (4.36) with Eqs. (4.31) and (4.35), we find that $m = 1/(z-1)$. Then, the outgoing positive energy mode is given by,

$$k^+_{r(O)}(r) \quad = \quad \frac{1}{(-u \cdot \chi)} \left[\frac{\omega - q\varphi}{(s \cdot \chi - u \cdot \chi)} + \frac{k_0 b}{|u \cdot \chi|^{\frac{1}{z-1}}} \right], \tag{4.37}$$

where b satisfies the relation

$$b\left[\sqrt{a_z}b^{z-1}-(s\cdot\chi)\right]=\frac{\omega-q\varphi}{k_0}|u\cdot\chi|^{\frac{1}{z-1}}. \tag{4.38}$$

In the following, let us consider the three cases, $z=2$, $z=3$ and $z\geq 4$, separately.

Hawking radiation with $z=2$. This case was studied in some detail in [56], and results for $Q=0$ were reported in [69]. To show how to generalize such studies to the cases with $z>2$, in the following let us first study this case in more details. In particular, when $z=2$, we have $m=1/(z-1)=1$. It can be shown that this is the only case in which m is an integer. Then, Eqs. (4.37) and (4.38) become

$$k^{+}_{r(O)}(r)=\frac{\omega-q\varphi}{(-u\cdot\chi)(s\cdot\chi-u\cdot\chi)}+\frac{k_0 b}{(-u\cdot\chi)^2}, \tag{4.39}$$

$$b\left[\sqrt{a_2}b-(s\cdot\chi)\right]=\frac{\omega-q\varphi}{k_0}(-u\cdot\chi). \tag{4.40}$$

Denoting $\epsilon\equiv r-r_{UH}$, we find that near the universal horizon $r=r_{UH}$ we have

$$\begin{aligned}(-u\cdot\chi)&=\epsilon\left[\alpha_1+\alpha_2\epsilon+\mathcal{O}(\epsilon^2)\right],\\ (s\cdot\chi)&=s_0+s_1\epsilon+\mathcal{O}(\epsilon^2),\end{aligned} \tag{4.41}$$

where

$$\begin{aligned}&\alpha_1\equiv(-u\cdot\chi)'|_{UH}>0,\quad \alpha_2\equiv\frac{1}{2}(-u\cdot\chi)''|_{UH}<0,\\ &s_0\equiv(s\cdot\chi)|_{UH},\quad s_1\equiv(s\cdot\chi)'|_{UH}.\end{aligned} \tag{4.42}$$

Setting

$$b=b_0+b_1\epsilon+\mathcal{O}(\epsilon^2), \tag{4.43}$$

from Eq. (4.40), we obtain

$$b_0=\frac{s_0}{\sqrt{a_2}},\quad b_1=\frac{\omega-q\varphi}{s_0k_0}\alpha_1+\frac{s_1}{\sqrt{a_2}}. \tag{4.44}$$

On the other hand, we also have,

$$
\begin{aligned}
(-u\cdot\chi)^{-2} &= \frac{1}{\epsilon^2}\left(\frac{1}{\alpha_1+\alpha_2\epsilon+\mathcal{O}(\epsilon^2)}\right)^2\\
&= \frac{1}{\epsilon^2}\left(\frac{1}{\alpha_1}-\frac{\alpha_2}{\alpha_1^2}\epsilon+\mathcal{O}(\epsilon^2)\right)^2\\
&= \frac{1}{\epsilon^2}\left(\frac{1}{\alpha_1^2}-2\frac{\alpha_2}{\alpha_1^3}\epsilon+\mathcal{O}(\epsilon^2)\right). \qquad (4.45)
\end{aligned}
$$

Substituting it together with Eq. (4.43) into Eq. (4.39), we find,

$$
\begin{aligned}
& k^{+}_{r(O)}(r) \simeq 2\frac{\omega-q\varphi-\mu}{s_0\alpha_1}\frac{1}{\epsilon}+\frac{k_0 b_0}{(\alpha_1\epsilon)^2},\\
& \mu = -\frac{k_0}{2}\left(\frac{s\cdot\chi}{a\cdot s}\right)\left[\frac{(s\cdot\chi)'}{\sqrt{a_2}}+\frac{(s\cdot\chi)(u\cdot\chi)''}{\sqrt{a_2}(a\cdot s)}\right]_{UH}. \qquad (4.46)
\end{aligned}
$$

Inserting the above expressions into Eq. (4.28), and using the residual theorem, we finally obtain the Boltzman factor

$$
2\mathrm{Im}S = \frac{\omega-\mu_0}{T^{z=2}_{UH}}, \qquad (4.47)
$$

where $\mu_0 = (q\varphi+\mu)_{UH}$ is the chemical potential of the scalar field, and

$$
T^{z=2}_{UH} = \left.\frac{(a\cdot s)(s\cdot\chi)}{4\pi}\right|_{UH} = \frac{\kappa_{UH}}{2\pi}, \qquad (4.48)
$$

where $\kappa_{UH} = s_0\alpha_1/2$ denotes the surface gravity defined by Eq. (4.11). Clearly, $T^{z=2}_{UH}$ and κ_{UH} satisfy the standard relation $T = \kappa/2\pi$ [76]. However, as to be shown below, this is no longer the case for a general z, although T^{z}_{UH} is still proportional to κ_{UH}.

Applying the above general formula (4.48) to the two particular solutions given in the last section, we find that

$$
T^{z=2}_{UH} = \begin{cases} \frac{1}{4\pi r_{UH}\sqrt{3\mathrm{g}}}\sqrt{(1-\frac{Q^2}{r^2_{UH}})(2-\frac{Q^2}{r^2_{UH}})}, & c_{14}=0,\\ \frac{1}{4\sqrt{\mathrm{g}}\pi r_{UH}}\sqrt{\mathrm{p}-\frac{Q^2}{r^2_{UH}}}, & c_{123}=0. \end{cases} \qquad (4.49)
$$

When $Q=0$, it reduces to the one obtained in [69], calculated in the PG coordinates. However, it is interesting to note that such obtained temperature is different from that obtained by the Smarr relation, by simply adopting the mass defined in [97]. We shall come back to this issue in the next section.

Hawking radiation with $z = 3$. In the Hořava theory [7], the power-counting renormalizability condition requires $z \geq 3$, as mentioned above. Therefore, the case $z = 3$ has particular interest, as far as the Hořava theory is concerned.

When $z \geq 3$ the parameter $m[\equiv 1/(z-1)]$ introduced in Eq. (4.36) can no longer be an integer, and the nature of the singularity at $u \cdot \chi = 0$ becomes a branch point, instead of a single pole. To handle this case carefully, we shall use two different methods. One is the more "traditional" one, and the other is the so-called fractional derivative, a branch of mathematics, which has already been well-established [100] and applied to physics in similar situations in various occasions [101]. We shall show that both methods yield the same results, as it should be expected.

Let us first consider the quantity $|u \cdot \chi|^m$, for which we find that it is easier to consider the regions $r > r_{UH}$ and $r < r_{UH}$, separately. In particular, in the region $r > r_{UH}$ we have $(u \cdot \chi) < 0$. Then, Eqs. (4.37) and (4.38) become

$$k^{+}_{r(O)}(r) = \frac{\omega - q\varphi}{(-u \cdot \chi)(s \cdot \chi - u \cdot \chi)} + \frac{k_0 b}{(-u \cdot \chi)^{3/2}}, \tag{4.50}$$

$$b\left[\sqrt{a_3}b^2 - (s \cdot \chi)\right] = \frac{\omega - q\varphi}{k_0}(-u \cdot \chi)^{1/2}. \tag{4.51}$$

At the universal horizon, we have $(-u \cdot \chi) \propto \epsilon$ to the leading order of ϵ. Then, the leading term of the right-hand side of Eq. (4.51) is proportional to $\epsilon^{1/2}$. This implies that the function $b(r)$ must be expanded in terms of $\epsilon^{1/2}$, instead of ϵ as done in the last case with $z = 2$. So, setting

$$\begin{aligned} b = {} & b_0 + b_1\epsilon^{1/2} + b_2\epsilon + b_3\epsilon^{3/2} \\ & + b_4\epsilon^2 + b_5\epsilon^{5/2} + \mathcal{O}(\epsilon^3), \end{aligned} \tag{4.52}$$

we can determine the coefficients b_i's from the relation,

$$b^2\left[\sqrt{a_3}b^2 - (s \cdot \chi)\right]^2 = \frac{(\omega - q\varphi)^2}{k_0^2}(-u \cdot \chi), \tag{4.53}$$

which yields,

$$
\begin{aligned}
b_0 &= \left(\frac{s_0}{\sqrt{a_3}}\right)^{1/2}, \quad b_1 = \frac{\sqrt{\alpha_1}(\omega - q\varphi)}{2s_0 k_0}, \\
b_2 &= \frac{4{s_0}^2{k_0}^2 s_1 - 3\alpha_1\sqrt{a_3}(\omega - q\varphi)^2}{8\ {a_3}^{\frac{1}{4}}{k_0}^2{s_0}^{\frac{5}{2}}}, \\
b_3 &= \frac{\omega - q\varphi}{4\sqrt{\alpha_1}{k_0}^3{s_0}^4}[{k_0}^2{s_0}^2(\alpha_2 s_0 - 2s_1\alpha_1) \\
&\qquad + 2\sqrt{a_3}\alpha_1^2(\omega - q\varphi)^2].
\end{aligned}
\tag{4.54}
$$

From the above derivation, it is easy to see that, if the term $b_1\epsilon^{1/2}$ were not present, Eq. (4.51) would not hold.

To calculate the last term appearing in the right-hand side of Eq. (4.50), as mentioned above, we use two different methods. Let us first consider the fractional derivative. Since $\lim_{\epsilon\to 0}\int \epsilon^\delta d\epsilon = 0$ for any $\delta > -1$, we need to consider the fractional expansion only up to $\epsilon^{-3/2}$, which is sufficient for the calculation of $2\mathrm{Im}\mathcal{S}$ given by Eq. (4.28). Then, from Eq. (4.41) we find that, after taking $\alpha = 1/(z-1) = 1/2$, $(-u\cdot\chi)^{-3/2}$ is given by

$$
(-u\cdot\chi)^{-3/2} = \epsilon^{-3/2}\left(\alpha_1^{-3/2} + \mathcal{O}(\epsilon)\right). \tag{4.55}
$$

This can be also obtained from the following considerations. First, from Eq. (4.51) we have

$$
(-u\cdot\chi)^{3/2} = \left(\frac{k_0}{\omega - q\varphi}\right)^3 b^3\left[\sqrt{a_3}b^2 - (s\cdot\chi)\right]^3. \tag{4.56}
$$

Substituting Eqs. (4.52)-(4.54) into the right-hand side of the above expression, we obtain

$$
(-u\cdot\chi)^{3/2} = \epsilon^{3/2}\left(\alpha_1^{3/2} + \mathcal{O}(\epsilon)\right). \tag{4.57}
$$

Assuming that $(-u\cdot\chi)^{-3/2}$ takes the form, $(-u\cdot\chi)^{-3/2} = \hat{a}_1\epsilon^{-3/2} + \mathcal{O}\left(\epsilon^{-1/2}\right)$, then, using the identity $(-u\cdot\chi)^{3/2}\cdot(-u\cdot\chi)^{-3/2} = 1$, we find that $(-u\cdot\chi)^{-3/2}$ is precisely given by Eq. (4.55).

Substituting Eqs. (4.52) and (4.55) into Eq. (4.50), we find,

$$\begin{aligned} k^{+}_{r(O)} &= \frac{\omega - q\varphi}{s_0}\frac{1}{\epsilon[\alpha_1 + \mathcal{O}(\epsilon)]} + \frac{k_0[b_0 + b_1\epsilon^{1/2} + \mathcal{O}(\epsilon)]}{\epsilon^{3/2}[\alpha_1 + \mathcal{O}(\epsilon)]^{3/2}} \\ &\simeq \frac{3}{2}\frac{\omega - q\varphi}{s_0\alpha_1}\frac{1}{\epsilon} + \frac{k_0 b_0}{(\epsilon\alpha_1)^{3/2}}. \end{aligned} \tag{4.58}$$

In the region $r < r_{UH}$ we have $(u \cdot \chi) > 0$, and Eqs. (4.37) and (4.38) become

$$\begin{aligned} k^{+}_{r(O)} &= \frac{\omega - q\varphi}{s_0}\frac{1}{-\epsilon[\alpha_1 + \mathcal{O}(\epsilon)]} - \frac{k_0[b_0 + b_1\epsilon^{1/2} + \mathcal{O}(\epsilon)]}{\epsilon^{3/2}[\alpha_1 + \mathcal{O}(\epsilon)]^{3/2}} \\ &\simeq \frac{3}{2}\frac{\omega - q\varphi}{s_0\alpha_1}\left(-\frac{1}{\epsilon}\right) - \frac{k_0 b_0}{(\epsilon\alpha_1)^{3/2}}. \end{aligned} \tag{4.59}$$

We set $\epsilon \equiv r_{UH} - r$ and following a similar procedure, it can be shown that

$$\begin{aligned} k^{+}_{r(O)} &= \frac{\omega - q\varphi}{s_0}\frac{1}{-\epsilon[\alpha_1 + \mathcal{O}(\epsilon)]} - \frac{k_0[b_0 + b_1\epsilon^{1/2} + \mathcal{O}(\epsilon)]}{\epsilon^{3/2}[\alpha_1 + \mathcal{O}(\epsilon)]^{3/2}} \\ &\simeq \frac{3}{2}\frac{\omega - q\varphi}{s_0\alpha_1}\left(-\frac{1}{\epsilon}\right) - \frac{k_0 b_0}{(\epsilon\alpha_1)^{3/2}}. \end{aligned} \tag{4.60}$$

Setting $r = r_{UH} + \epsilon e^{i\theta}$, we find

$$k^{+}_{r(O)} \simeq \frac{3}{2}\frac{\omega - q\varphi}{s_0\alpha_1}\frac{1}{\epsilon e^{i\theta}} + \frac{k_0 b_0}{(\epsilon e^{i\theta}\alpha_1)^{3/2}}. \tag{4.61}$$

Inserting the above expression into Eq. (4.28), we find

$$2\mathrm{Im}\mathcal{S} = \frac{\omega - q\varphi - \mu}{T^{z=3}_{UH}}, \tag{4.62}$$

where

$$\begin{aligned} T^{z=3}_{UH} &= \left.\frac{(a \cdot s)(s \cdot \chi)}{3\pi}\right|_{UH} = \frac{2\kappa_{UH}}{3\pi}, \\ \mu &\equiv -T^{z=3}_{UH}\mathcal{I}, \end{aligned} \tag{4.63}$$

with $dr = i\epsilon e^{i\theta}d\theta$, and

$$\mathcal{I} \equiv \mathrm{Im}\lim_{\epsilon\to 0}\oint dr\frac{k_0 b_0}{(\epsilon e^{i\theta}\alpha_1)^{3/2}}. \tag{4.64}$$

To calculate $\mathcal{I}$, we first note that

$$\left(e^{i\theta}\right)^n = e^{in\theta}, \quad \left(e^{i\theta}\right)^{1/n} = e^{i(\theta + 2m\pi)/n}, \tag{4.65}$$

where n is an integer, and $m = 0, 1, 2, ..., n-1$. Then, we find that

$$\begin{aligned}
\mathcal{I} &= \operatorname{Im} \lim_{\epsilon \to 0} \int_0^{2\pi} \frac{i k_0 b_0 \epsilon e^{i\theta}}{(\epsilon e^{i\theta} \alpha_1)^{3/2}} d\theta \\
&= \operatorname{Im} \lim_{\epsilon \to 0} \left(\frac{i k_0 b_0}{\sqrt{\epsilon} \alpha_1^{3/2}} \int_0^{2\pi} e^{-i(\theta + 6m\pi)/2} d\theta \right) \\
&= \operatorname{Im} \lim_{\epsilon \to 0} \left((-1)^m \frac{4 k_0 b_0}{\sqrt{\epsilon} \alpha_1^{3/2}} \right) = 0. \qquad (4.66)
\end{aligned}$$

Thus, finally we obtain

$$2\mathrm{Im}\mathcal{S} = \frac{\omega - q\varphi}{T_{UH}^{z=3}}. \qquad (4.67)$$

It is interesting to note that $T_{UH}^{z=3}$ given above is larger than $T_{UH}^{z=2}$ by a factor 4/3, although both of them are proportional to the surface gravity κ_{UH} defined by Eq. (4.11). In addition, the real part of $\mathcal{I}$ diverges, although its imaginary part vanishes. This is similar to the extremal black holes [102], which are considered to be able in thermal equilibrium at any finite temperature [103].

Hawking radiation with $z \geq 4$. With the above preparations, we are ready to consider the general case with any given $z \geq 4$. Similar to the case $z = 3$, let us first consider the region $r > r_{UH}$, in which we have $(u \cdot \chi) < 0$, and Eqs. (4.37) and (4.38) become

$$\begin{aligned}
k_{r(O)}^{+}(r) &= \frac{\omega - q\varphi}{(-u \cdot \chi)(s \cdot \chi - u \cdot \chi)} + \frac{k_0 b}{(-u \cdot \chi)^{\frac{z}{z-1}}}, \\
b \left[\sqrt{a_z} b^{z-1} - (s \cdot \chi) \right] &= \frac{\omega - q\varphi}{k_0} (-u \cdot \chi)^{\frac{1}{z-1}}. \qquad (4.68)
\end{aligned}$$

To obtain the function $b(\omega, r)$, we need to expand $(-u \cdot \chi)$ only to the first order of ϵ. So, from Eq. (4.68) we find

$$\begin{aligned}
(-u \cdot \chi)^{\frac{1}{z-1}} &= \left[\alpha_1 \epsilon + \mathcal{O}\left(\epsilon^2\right) \right]^{\frac{1}{z-1}} \\
&= (\alpha_1 \epsilon)^{\frac{1}{z-1}} + \mathcal{O}\left(\epsilon^{\frac{2}{z-1}} \right). \qquad (4.69)
\end{aligned}$$

Therefore, for any given z, the following expansion must be performed,

$$b = b_0 + b_1 \epsilon^{\frac{1}{z-1}} + \mathcal{O}\left(\epsilon^{\frac{2}{z-1}} \right). \qquad (4.70)$$

Substituting Eqs. (4.69) and (4.70) into Eq. (4.68), we get

$$b_0 = \left(\frac{s_0}{\sqrt{a_z}}\right)^{\frac{1}{z-1}}, \quad b_1 = \frac{1}{z-1}\frac{\omega - q\varphi}{s_0 k_0}\alpha_1^{\frac{1}{z-1}}. \tag{4.71}$$

Hence, we obtain

$$k^+_{r(O)}(r) \simeq \left(\frac{z}{z-1}\right)\frac{\omega - q\varphi}{s_0\alpha_1}\frac{1}{\epsilon} + \frac{k_0 b_0}{(\epsilon\alpha_1)^{\frac{z}{z-1}}}. \tag{4.72}$$

It is interesting to note the z-dependence of $k^+_{r(O)}(r)$. In addition, as in the last case, the above expression for $k^+_{r(O)}(r)$ can be obtained by either the fractional derivative with $\alpha = 1/(z-1)$ or the more traditional method, illustrated above.

In the region $r < r_{UH}$, we have $(u\cdot\chi) > 0$, and Eqs. (4.37) and (4.38) become

$$\begin{aligned} k^+_{r(O)}(r) &= \frac{\omega - q\varphi}{(-u\cdot\chi)(s\cdot\chi - u\cdot\chi)} - \frac{k_0 b}{(u\cdot\chi)^{\frac{z}{z-1}}}, \\ b\left[\sqrt{a_z}b^{z-1} - (s\cdot\chi)\right] &= \frac{\omega - q\varphi}{k_0}(u\cdot\chi)^{\frac{1}{z-1}}. \end{aligned} \tag{4.73}$$

Following the same steps as given in the region $r > r_{UH}$ we find that,

$$k^+_{r(O)}(r) \simeq \left(\frac{z}{z-1}\right)\frac{\omega - q\varphi}{s_0\alpha_1}\left(-\frac{1}{\epsilon}\right) - \frac{k_0 b_0}{(\epsilon\alpha_1)^{\frac{z}{z-1}}}. \tag{4.74}$$

Combining Eqs. (4.72) and (4.74), and let $r = r_{UH} + \epsilon e^{i\theta}$, there has

$$k^+_{r(O)} \simeq \frac{z}{z-1}\frac{\omega - q\varphi}{s_0\alpha_1}\frac{1}{\epsilon e^{i\theta}} + \frac{k_0 b_0}{(\epsilon e^{i\theta}\alpha_1)^{\frac{z}{z-1}}}. \tag{4.75}$$

Considering Eq. (4.28), we find that

$$2\mathrm{Im}\mathcal{S} = \frac{\omega - q\varphi - \mu}{T^{z\geq 4}_{UH}}, \tag{4.76}$$

where

$$\begin{aligned} T^{z\geq 4}_{UH} &= \frac{(z-1)s_0\alpha_1}{2\pi z} = \frac{2(z-1)}{z}T^{z=2}_{UH}, \\ \mu &= -T^{z\geq 4}_{UH}\,\mathcal{I}_z, \end{aligned} \tag{4.77}$$

with

$$
\begin{aligned}
\mathcal{I}_z &\equiv \text{Im}\lim_{\epsilon\to 0}\int_0^{2\pi}\frac{ik_0b_0\epsilon e^{i\theta}}{(\epsilon e^{i\theta}\alpha_1)^{\frac{z}{z-1}}}d\theta \\
&= \text{Im}\lim_{\epsilon\to 0}\left(\frac{ik_0b_0}{(\alpha_1^z\epsilon)^{\frac{1}{z-1}}}\int_0^{2\pi}e^{-i(\theta+2zm\pi)/(z-1)}d\theta\right) \\
&= \text{Im}\lim_{\epsilon\to 0}\left[\frac{(1-z)k_0b_0}{(\alpha_1^z\epsilon)^{\frac{1}{z-1}}}e^{-i2\pi\frac{mz}{z-1}}\left(e^{-\frac{i2\pi}{z-1}}-1\right)\right] \\
&= \lim_{\epsilon\to 0}\left\{\frac{2(z-1)k_0b_0}{(\alpha_1^z\epsilon)^{\frac{1}{z-1}}}\sin\frac{\pi}{z-1}\cos\frac{(2m+1)\pi}{z-1}\right\} \\
&= \begin{cases} 0, & z=\infty, \\ \pm\infty, & 4\le z<\infty, \end{cases}
\end{aligned}
\tag{4.78}
$$

where $m = 0, 1, ..., z-2$, and

$$
\pm = \text{Sign}\left\{\cos\left(\frac{(2m+1)\pi}{z-1}\right)\right\}. \tag{4.79}
$$

Thus, the chemical potential for $4 \le z < \infty$ is always unbounded, unless $z = \infty$. In the latter, similar to the cases $z = 2$ and $z = 3$, it vanishes. It is interesting to note that the signs of $\mathcal{I}_z$ depends not only on z but also on m. In particular, when $m = 0$ and $m = z-2$, $\cos[(2m+1)\pi/(z-1)]$ is always positive, so that $\mu \propto -\mathcal{I}_z$ always approaches to $-\infty$. Therefore, for any given z there always exists an intermediate region in which μ always approaches to $+\infty$. One may consider this range as physically not realizable, as the corresponding chemical potential becomes infinitely large.

As noted previously, the temperature of the universal horizon is always finite and depends on z explicitly, which characterizes another feature of the nonlinear dispersion relation. Therefore, although, to the leading order, the Hawking radiation is thermal for any given species with a fixed z, the temperature of such a species depends explicitly on z, and increases as z increases. In particular, as $z \to \infty$, a particular case considered also in [56], it approaches to its maximum $T_{UH}^{z=\infty} = 2T_{UH}^{z=2}$.

4.4 Modified Smarr Formula and Mass of a Black Hole

From the above sections one can see that the Hawking radiation of non-relativistic particles can occur at the universal horizon. Then, a natural question is whether the first law of black hole mechanics also holds there? In the neutral case, Berglund *et al* [69] found that a slightly modified first law indeed exists. But, recently Ding *et al* found that a simple generalization of such a formula to the charged case is not possible [22]. A fundamental question is how to define the entropy at the universal horizon, although it is quite reasonable to assume that such an entropy exists. Indeed, from Wald's entropy formula [104], it was shown that the entropy S of the universal horizon is still proportional to its area $S = A_{UH}/4$ [105], since none of the terms $\mathcal{L}_{æ}$ and $\mathcal{L}_M$ appearing in Eq. (4.2) depends on the curvature $R_{\mu\nu\alpha\beta}$.

In this section, we shall flip the logics, and assume that the entropy is proportional to the area of the universal horizon, then study the implications of the first law of black hole mechanics. In particular, we would like to find the mass of the black hole, and then compare it with the well-known one [73,97]. The inconsistency of these two different masses imply that at least one of our assumptions needs to be modified [3], that is, either the entropy is not proportional to the area of the universal horizon, or the first law of black hole mechanics at the universal horizon must be generalized, or both.

With the temperature T_{UH} of the black hole at the universal horizon calculated in the last section, and the assumption that the entropy S of the universal horizon is still proportional to its area $S = A_{UH}/4$ [105], we can uniquely determine the mass of the black hole, by assuming that the first law of the black hole thermodynamics,

$$dM = TdS + VdQ, \tag{4.80}$$

holds at the universal horizon $r = r_{UH}$. To this purpose, let us first note that $M = M(S,Q)$, $T = T(S,Q)$ and $V = V(S,Q)$, where $S = \pi r_{UH}^2$. Then, from the

[3] It is also possible that the masses obtained in [73,97] need to be modified.

integrability condition

$$\frac{\partial V(S,Q)}{\partial S} = \frac{\partial T(S,Q)}{\partial Q}, \tag{4.81}$$

we find

$$V = \int \frac{\partial T(S,Q)}{\partial Q} dS + V_o(Q), \tag{4.82}$$

where $V_o(Q)$ is a function of Q, and will be determined by the integrability condition (4.81). When $Q = 0$, we must have $V(S,0) = 0$. Once V is known, from Eq. (4.80) we can calculate the mass of the black hole,

$$M(S,Q) = \int_0^S T(S',0)dS' + \int_0^Q V(S,Q')dQ'. \tag{4.83}$$

Applying the above formulas to the two particular cases, $c_{123} = 0$ and $c_{14} = 0$, we shall obtain the mass of the black hole in each case. For the sake of simplicity, let us consider only the case with $z = 2$.

4.4.1 Mass of the Black Hole for $c_{123} = 0$

When $c_{123} = 0$, from Eqs. (4.49) and (4.82) we find that

$$V = \frac{1}{2\sqrt{1-c_{13}}} \arctan\left(\frac{Q}{2\sqrt{1-c_{13}}r_{UH}\mathcal{S}}\right), \tag{4.84}$$

where

$$\mathcal{S} \equiv \sqrt{1 - \frac{c_{14}}{2} - \frac{Q^2}{r_{UH}^2}}. \tag{4.85}$$

Then, Eq. (4.83) yields,

$$M = r_{UH}\mathcal{S} + VQ, \tag{4.86}$$

which takes precisely the Smarr form,

$$M = 2T_{UH}S + VQ, \tag{4.87}$$

where T_{UH} is given by Eq. (4.49). It is interesting to note that the above Smarr mass is quite different from the total mass, calculated at spatial infinity [22, 73, 97],

$$M_{tot} = \left(1 - \frac{c_{14}}{2}\right) r_{UH}. \tag{4.88}$$

4.4.2 Mass of the Black Hole for $c_{14} = 0$

In this case, we find that

$$V = \frac{1}{\sqrt{3(1-c_{13})}} \left[E\left(\phi, \frac{1}{2}\right) - \frac{1}{4} F\left(\phi, \frac{1}{2}\right) \right], \tag{4.89}$$

where $\phi = \arcsin(Q/r_{UH})$, and F and E are, respectively, the first and second kind of the elliptic functions. Then, from Eq. (4.83) we obtain

$$M = \mathcal{S} r_{UH} + VQ, \tag{4.90}$$

but now with

$$\mathcal{S} \equiv \frac{1}{\sqrt{3(1-c_{13})}} \sqrt{\left(1 - \frac{Q^2}{r_{UH}^2}\right)\left(1 - \frac{Q^2}{2r_{UH}^2}\right)}. \tag{4.91}$$

Again, such obtained mass satisfies the Smarr formula (4.87). Note that in the present case the total mass is given by [22, 73, 97],

$$M_{tot} = \frac{2}{3} r_{UH} + \frac{Q^2}{3r_{UH}}, \tag{4.92}$$

which is also different from that given by Eq. (4.90).

CHAPTER FIVE

Conclusions and Outlook

In this dissertation, we have studied two gravitational theories which break Lorentz symmetry, HL gravity and Einstein-æther theory, and the extension of relativistic gauge/gravity duality, nonrelativistic holography. We have investigated the holographic duality between nonrelativistic quantum field theories and gravitational theories which break Lorentz symmetry.

In Chapter Two, we have investigated the effects of high-order operators on the non-relativistic Lifshitz holography in the framework of the Hořava-Lifshitz (HL) theory of gravity [7], which contains all the required high-order spatial operators in order to be power-counting renormalizble. The unitarity of the theory is also preserved, because of the absence of the high-order time operators. In this sense, the HL gravity is an ideal place to study the effects of high-order operators on the non-relativistic gauge/gravity duality.

In particular, we have first shown that the Lifshitz spacetime (2.49) is not only a solution of the HL gravity in the IR, as first shown in [34] and later rederived in [35], but also a solution of the full theory. The effects of the high-order operators on the Lifshitz dynamical exponent z is simply to shift it to different values, as these high-order operators become more and more important, as shown explicitly in Section 2.3. This is similar to the case studied in [37].

In Section 2.4, we have studied a scalar field that has the same symmetry in the UV as the HL gravity, the foliation-preserving diffeomorphism described by Eq. (1.5). While in the IR the asymptotic behavior of the scalar field near the boundary is similar to that given in 4-dimensional spacetimes [26], its asymptotic behavior in the UV is dramatically changed, as is the corresponding two-point correlation function, as shown in Section 2.5. This is expected, because the high-order operators dominate the behavior of the scalar field in the UV. Then, according to the holographic correspondence, this in turn affects the two-point correlation functions.

It would be important to study the effects of high-order operators on other properties of the non-relativistic Lifshitz holography, including phase transitions and superconductivity of the corresponding non-relativistic quantum field theories defined on the boundary. In particular, it has been suggested that inflation may be described holographically by means of a dual field theory at the future boundary [47]. This might provide deep insights to the Planckian physics in the very early universe, where (non-perturbative) quantum gravitational effects are expected to play an important role. Recently, a powerful analytical approximation method, the so-called *uniform asymptotic approximation*, was developed [48,49], which is specially designed to study such effects in the very early universe. With the arrival of the era of precision cosmology [50, 51], such effects might be within the range of detection of the forthcoming generation of experiments [52].

Another possible application of these high-order effects might be to Hawking radiation, where quantum gravitational effects also become important. Previous studies of such effects showed that Hawking radiation is robust with respect to the UV corrections [53]. To study them in detail, one can equally apply the uniform asymptotic approximation method developed in [48] to the studies of Hawking radiation. In particular, in the spherical background, one can simply identify the radial coordinate r in Hawking radiation with the time variable η used in the inflationary models. In the inflationary models, the initial conditions are normally the Bunch-Davies vacuum, but here in the studies of Hawking radiation they should be the Unruh vacuum.

In Chapter Three, we have studied the Einstein-Maxwell-æther theory, and found two new classes of charged black hole solutions for special choices of the coupling constants: (1) $c_{14} = 0$, $c_{123} \neq 0$, and (2) $c_{14} \neq 0$, $c_{123} = 0$. In the first case, the universal horizon depends on its electric charge Q, while it doesn't in the second case.

In both cases, the universal horizons are independent of the coupling constants c_i, while the Killing horizons depend on them. When c_{13} ($\equiv c_1 + c_3$) is very small and approaches zero, the solutions in the case $c_{14} = 0,\ c_{123} \neq 0$ reduce to the usual Reissner-Nordstrom black hole solution. The corresponding properties at the universal horizons are the same as those presented in [66] via Khronometric theory.

To study the solutions further, we have considered their surface gravity and constructed the Smarr formula at each of the horizons, universal and Killing. We have shown that there is no problem for such constructions, but when trying to construct the corresponding first law of black hole mechanics, they are all different from the usual one. In particular, we have shown that the temperature obtained from the Smarr mass-area relation is not proportional to its corresponding surface gravity when both the charge and æther are present, in contrast to the case without æther ($c_i = 0$) [66], or the case without charge [65]. In particular, in [65] it was found that in the neutral case, q_{UH} is always proportional to the surface gravity κ_{UH} at the universal horizons, even when the æther is present. From Eqs. (3.56), (3.66) and (3.67) we can see that, when $Q = 0$, q_{KH} is also proportional to κ_{KH}. Then, one can also construct a slightly modified first law of black hole mechanics at the Killing horizons. However, when the charge Q is different from zero, comparing (q_{UH}, q_{KH}) with $(\kappa_{UH}, \kappa_{KH})$, one can see that these proportional relations no longer hold. Therefore, it is not clear how to build the first law for these charged æther black holes before we have a better understanding of the entropy of the universal and/or Killing horizons.

The solutions presented in this chapter can be generalized to the case coupled with the cosmological constant Λ, which are given by

$$e(r) = \begin{cases} 1 - \frac{r_0}{r} + \frac{Q^2}{r^2} - c_{13}\left(\frac{r_{æ}^4}{r^4} + \frac{r^2}{l_s^2}\right) - \frac{2r_{æ}^2}{l_s r} - \frac{1}{3}\Lambda r^2, & (c_{14} = 0,\ c_{123} \neq 0), \\ 1 - \frac{r_0}{r} + \frac{c_{14} - 2c_{13}}{2(1-c_{13})}\frac{r_0^2}{4r^2} + \frac{1}{1-c_{13}}\left(\frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2\right) + \frac{r_s}{r}, & (c_{14} \neq 0,\ c_{123} = 0), \end{cases} \tag{5.1}$$

where $r_0,\ r_{æ},\ l_s,\ Q$ and r_s are integration constants. It can be shown that in the presence of the cosmological constant [86], q_{UH} in general is also not proportional to κ_{UH}.

In addition, from these solutions, one can also construct topological charged Einstein-æther (anti) de Sitter black holes, which are

$$ds^2 = -\big[e(r) - 1\big]dv^2 + 2dvdr + r^2\left(d\hat{\theta}^2 + d\hat{\phi}^2\right), \tag{5.2}$$

where $e(r)$ is given by Eq. (5.1). The studies of the properties of the above solutions are out of the scope of this chapter, and we hope to report them on another occasion.

In Chapter Four, we have studied the quantum tunneling of both relativistic and non-relativistic particles at the Killing and universal horizons of the Einstein-Maxwell-æther black holes found recently in [22], by using the Hamilton-Jacobi method [91,92, 94]. Assuming that the dispersion relation in general takes the form (4.31) [95, 98], we have found that in high frequencies only relativistic particles ($z = 1$) can be created at the Killing horizons. The radiation at the Killing horizons is thermal with a temperature given by $T^{z=1}_{KH} = \kappa^{GR}_{KH}/2\pi$ [77]. This is consistent with previous results [95,96]. To leading order, these results are also consistent with those obtained from ray trajectories [76], in which it was shown that κ^{GR} receives corrections starting from the order of $(\ell\Omega)^{2/3}$, where Ω denotes the Killing energy at infinity, and ℓ is the UV Lorentz-violating scale.

On the other hand, particles with $z \geq 2$ cannot be created at Killing horizons (for high frequency modes). If they exist immediately inside a Killing horizon, they simply pass through it and escape to infinity even classically. On the other hand, the Hawking radiation is purely quantum mechanical. It should be noted that in [76] it was found that low-energy particles linger close to the Killing horizon before escaping to infinity, which cannot be seen from the current calculations of quantum tunneling.

At the universal horizon, the situation is different: only non-relativistic particles (with $k > k_0$) are created quantum mechanically at the universal horizons and radiated to infinity. The corresponding Hawking radiation is thermal, but different species of particles, characterized by the parameter z, experience different temperatures, given by

$$T_{UH}^{z\geq 2} = \left(2 - \frac{2}{z}\right)\frac{\kappa_{UH}}{2\pi}, \tag{5.3}$$

where κ_{UH} is the surface gravity defined in Eq. (4.11). When $z = 2$ it reduces to that obtained in [22], and in the neutral case ($Q = 0$) it further reduces to that obtained in [69]. It is clear that $T_{UH}^{z\geq 2}$ increases as z becomes larger and larger, and finally reaches its maximum, $T_{UH}^{z=\infty}$, which is twice as large as $T_{UH}^{z=2}$, a limiting case that was also considered in [56] without the presence of the electromagnetic field. It should be noted that the corresponding chemical potential always becomes unbounded at the universal horizons, except for the three cases $z = 2, 3, \infty$, in which the chemical potential always vanishes.

As mentioned previously, to arrive at the above conclusions, we have implicitly assumed that each horizon, Killing or universal, is associated with a temperature. One cannot take this for granted, as the system can be well approximated as thermal only in a certain energy regime, but not in an equilibrium state [99]. This relies heavily on the full structure of horizon thermodynamics, and is closely related to the underlying theory at high energies. With this in mind, we note that recently the Hořava theory was shown to be perturbatively renormalizable [106]. In particular, its quantization in 2D spacetimes reduces to that of a simple harmonic oscillator [107]. Therefore, it would be very interesting to study this important issue in a concrete framework, the Hořava theory of quantum gravity.

In addition, we have also studied the Smarr mass function formula, by assuming that: (a) the entropy is proportional to the area of the universal horizon, and (b) the first law of black hole thermodynamics holds at the universal horizon. Together with the temperatures we have just obtained by the Hamilton-Jacobi method, these assumptions uniquely determine the Smarr mass, given by Eq. (4.83). Applying it to the two particular black hole solutions of Eqs. (4.12) and (4.18), we have found that the corresponding Smarr masses are given, respectively, by Eqs. (4.86) and (4.90), which are quite different from the well-known ones obtained in [73, 97]. These differences imply that either the masses given in [73, 97] are incorrect, or at least one of our above two assumptions must be modified.

It would be extremely interesting to see if our results can be also obtained when other methods are used [77, 95, 96, 102].

Besides nonrelativistic holography and black holes and their thermodynamics in the framework of gravitational theories without Lorentz symmetry, including HL gravity, it is worthy to pursue the following aspects of HL gravity:

Infrared limit. One should use the renormalization group flow to study HL gravity at low energies in details.

Quantization and Renormalizability. HL gravity is power-counting renormalizable, but there is no proof of its full renormalizability. Quantization of HL gravity has been accomplished only for some particular cases in [107, 111]. It would be very interesting to generalize to other cases.

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$$\phi(v,r) = \begin{cases} v - r - M\ln|r-M| + \phi_+, & v > v_0, \\ v - r + \phi_-, & v < v_0, \end{cases}$$

where $\phi_\pm$ are two integration constants, $v = v_0$ is the location of the collapsing null shell, and (v,r) denote the Eddington-Finklestein coordinates [cf. Eq. (4.6)]. For more detail, see [70].

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