ABSTRACT

Gravitational Collapse of Spherical Clouds and Formation of Black Holes in the Background of Dark Energy

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In this thesis, I first review the fundamentals of Einstein's theory of gravity in four-dimensional spacetimes, and then develop the general formulas of thin shells in this theory. Applying these formulas to spherically symmetric thin shells, I study the gravitational collapse of dust clouds in the background of dark energy. To solve the relevant equations, I develop a computer program, and investigate four representative cases, in which one is without dark energy and the others are with. I find that in all the four cases black holes can be formed from the gravitational collapse of the dust cloud.

Gravitational Collapse of Spherical Clouds and Formation of Black Holes in the Background of Dark Energy

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CHAPTER ONE

Fundamentals of Einstein's Theory of General Relativity

1.1 Space-time Manifolds

In this chapter, we shall provide some fundamentals that are to be used in this dissertation. Among these are the definitions of some basic physical quantities (for example, the Riemann tensor, parallel transport, geodesic deviation), and the Einstein field equations with distribution valued tensors. It is not incidental that these introductory sections are lengthy. It was found necessary to present the material in such a way that errors contained in commonly used references can be adequately corrected. From the mathematical point of view the fundamental object of Einstein's theory of general relativity is the space-time manifold $(\Omega,\ g_{\mu\nu})$, where Ω is a *connected four-dimensional Hausdoff* C^{∞} *manifold* and $g_{\mu\nu}$ is a *symmetric Lorentz metric tensor*, or simply the metric, with the signature -2 on Ω (for the study of *Differential Geometry,* please refer to [39]). The points in Ω are labelled by a general non-inertial coordinate system(x^0, x^1, x^2, x^3), often written as $x^{\mu} (\mu = 0, 1, 2, 3)$. We use the convention that Greek indices take the values 0,1,2,3 and repeated Greek indices are to be summed over these values unless specified otherwise. According to the *principle of covariance*, all coordinate systems are equivalent for the description of physical phenomena. Thus the choice of coordinate systems is arbitrary. If we go from one coordinate system, say, x^{μ}, to another, say, $x^{'\mu}$, a *contravariant vector* y^{μ} and a *covariant vector* y_{μ} transform as

$$y^{'\mu} = \frac{\partial x^{'\mu}}{\partial x^{\nu}} y^{\nu}, y^{'}_{\mu} = \frac{\partial x^{\nu}}{\partial x^{'\mu}} y_{\nu}, \tag{1.1}$$

and a mixed tensor such as $y^{\mu}_{\nu\lambda}$ as

$$y^{'\mu}_{\nu\lambda} = \frac{\partial x^{'\mu}}{\partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x^{'\nu}} \frac{\partial x^{\delta}}{\partial x^{'\lambda}} y^{\sigma}_{\rho\delta}, \tag{1.2}$$

etc.

The contravariant tensor, $g^{\mu\nu}$, corresponding to $g_{\mu\nu}$, is defined by

$$g^{\mu\nu}g_{\mu\lambda} = \delta^{\nu}_{\lambda}, \tag{1.3}$$

where δ^{μ}_{ν} is the *Kronecker delta*, which is unity for $\mu = \nu$ (no summation is taken) and zero otherwise. By using $g^{\mu\nu}$ and $g_{\mu\nu}$ we can raise and lower the indices as

$$y^{\mu} = g^{\mu\nu}y_{\nu}, y_{\mu} = g_{\mu\nu}y^{\nu} \tag{1.4}$$

We regard tensors derived by such raising and lowering of indices as representing the same geometric quantity, since by raising an index and subsequently lowering it we recover the orignal tensor.

All the information about the space-time is contained in the metric $g_{\mu\nu}$, which determines the square of the space-time interval ds between infinitesimally separated events or points x^{μ} and $x^{\mu} + dx^{\mu}$ as

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{1.5}$$

The contravariant vector dx^{μ} is said to be *time-like, space-like,* or *null* according to whether ds^2 is positive, negative, or zero, respectively. The space-time manifold Ω has three space-like and one time-like dimensions.

Since the Einstein field equations contain the second derivatives of the metric and the Bianchi identities contain the third derivatives of it, it is necessary to require $g_{\mu\nu}$ to be at least C^3 and $x^{\mu} = x^{\mu}(x^{'\mu})$ to be at least C^4 so that the Einstein field equations are defined everywhere and the Bianchi identities are defined at every point of the space-time manifold.

1.2 Covariant Differentiation, the Riemann Tensor, and the Einstein Field Equations

To generalize the ordinary(partial) differentiation to the Riemann manifold, it is required to introduce an additional structure into the manifold. This additional structure is an *affine connection,* ∇, which assigns to each vector field X on Ω a differential operator, ∇_X, which maps an arbitrary vector field Y into a vector field $\nabla_X Y$.

Associated with each metric, we can endow the manifold with a unique torsion-free connection by requiring

$$\nabla g = 0 \tag{1.6}$$

where the term torsion-free means every element in g has finite order.

In a local coordinate basis$\{\partial_\lambda\}$, Eq.(1.6) can be written in the form

$$\nabla_\lambda g_{\mu\nu} = g_{\mu\nu,\lambda} - g_{\mu\delta}\Gamma^\delta_{\nu\lambda} - g_{\delta\nu}\Gamma^\delta_{\mu\lambda} = 0 \tag{1.7}$$

where a comma denotes partial differentiation with respect to the corresponding variable, and $\partial_\lambda \equiv \partial/\partial x^\lambda$. $\Gamma^\lambda_{\nu\mu}$ are called the *Christoffel connection coefficients*, and the connection itself called the *Christoffel connection*. From Eq.(1.7) by using the symmetry of $\Gamma^\lambda_{\mu\nu}$, we derive

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\delta}[g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta}] \tag{1.8}$$

The covariant differentiation for a contravariant and a covariant vector is defined as

$$A^\mu_{;\nu} = A^\mu_{,\nu} + \Gamma^\mu_{\nu\lambda}A^\lambda, A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^\lambda_{\mu\nu}A_\lambda \tag{1.9}$$

and for a mixed tensor such as $A^\mu_{\nu\lambda}$ as

$$A^\mu_{\nu\lambda;\sigma} = A^\mu_{\nu\lambda,\sigma} + \Gamma^\mu_{\delta\sigma}A^\delta_{\nu\lambda} - \Gamma^\delta_{\nu\sigma}A^\mu_{\delta\lambda} - \Gamma^\delta_{\lambda\sigma}A^\mu_{\nu\delta} \tag{1.10}$$

and so on, where a semicolon denotes the covariant differentiation.

Under a coordinate transformation, say, from x^μ to x'^μ, the connection coefficients, $\Gamma^\lambda_{\mu\nu}$, transform as

$$\Gamma'^\mu_{\nu\lambda} = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\delta}{\partial x'^\lambda}\Gamma^\rho_{\sigma\delta} + \frac{\partial^2 x^\sigma}{\partial x'^\nu \partial'^\lambda}\frac{\partial x'^\mu}{\partial x^\sigma} \tag{1.11}$$

Therefore, the connection coefficients do not form the components of a tensor.

For a covariant vector A_μ it can be shown that

$$A_{\mu;\nu;\lambda} - A_{\mu;\lambda;\nu} = A_\delta R^\delta_{\mu\nu\lambda} \tag{1.12}$$

where $R^\delta_{\mu\nu\lambda}$ is the *Riemann tensor* defined by

$$R^\sigma_{\mu\nu l} = \Gamma^\sigma_{\mu\lambda,\nu} - \Gamma^\sigma_{\mu\nu,\lambda} + \Gamma^\sigma_{\delta\nu}\Gamma^\delta_{\mu\lambda} - \Gamma^\sigma_{\delta\lambda}\Gamma^\delta_{\mu\nu} \tag{1.13}$$

and Eq.(1.12) is the *Ricci identity*.

The Riemann tensor has symmetry properties

$$\begin{aligned} R_{\sigma\mu\nu\lambda} = -R_{\mu\sigma\nu\lambda} &= -R_{\sigma\mu\lambda\nu} \\ R_{\sigma\mu\nu\lambda} &= R_{\nu\lambda\sigma\mu} \\ R_{\sigma\mu\nu\lambda} + R_{\sigma\lambda\mu\nu} + R_{\sigma\nu\lambda\mu} &= 0. \end{aligned} \tag{1.14}$$

and satisfies the *Bianchi identities*

$$R^\sigma_{\mu\nu\lambda;\rho} + R^\sigma_{\mu\rho\nu;\lambda} + R^\sigma_{\mu\lambda\rho;\nu} = 0 \tag{1.15}$$

The *Ricci tensor* $R_{\mu\lambda}$ is defined by

$$R_{\mu\lambda} = g^{\sigma\nu} R_{\sigma\mu\nu\lambda} = R^\delta_{\mu\delta\lambda} \tag{1.16}$$

From Eqs(1.14) and (1.16) it is easy to show that

$$R_{\mu\lambda} = R_{\lambda\mu} \tag{1.17}$$

The *Ricci scalar* is defined by

$$R = g^{\sigma\lambda} R_{\sigma\lambda} = R^\lambda_\lambda \tag{1.18}$$

By contracting the Bianchi identities on the pairs of indices $\mu\nu$ and $\sigma\rho$, we find that

$$(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)_{;\lambda} g^{\lambda\nu} = 0 \tag{1.19}$$

The tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is sometimes called the *Einstein tensor*.

We are now in a position to write down the Einstein field equations which are the fundamental differential equations of GR

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.20}$$

where κ ($\equiv 8\pi G/c^4$) is the *Einstein gravitational constant*, and Λ the *cosmological constant.* In the following we consider only the case where $\Lambda = 0$, unless specified otherwise. $T_{\mu\nu}$ denotes the energy-stress tensor of the source producing the gravitational field. Without loss of generality, we choose units such that $\kappa = 1$.

It must be noted, however, that the form of the Einstein field equations used by Chandrasekhar [7] is not consistent with the requirement that the *energy density* of matter fields must be positive, and the correct one corresponding to the above definitions for the Riemann and Ricci tensors [see Eqs(1.12) and (1.16)] and the signature (-2) of metric (1.5), is Eq.(1.20)(see, for example, [23][22][37][21][9]).

The combination of Eqs.(1.19) and (1.20) gives

$$T^{\mu\nu}_{;\nu} = 0 \tag{1.21}$$

which are the equations for the conservation of energy and stress of the source.

1.3 Curves, Parallel Transport, and Geodesics

A *curve* in a Riemannian space is defined by points $x^\mu(\lambda)$ where x^μ are suitably differentiable functions of the real parameter λ, varying over some interval of the real line. The curve is time-like, space-like, or null according to whether its tangent vector, $(dx^\mu/d\lambda)$, is time-like, space-like, or null.

In *Euclidean geometry*, for an arbitrary vector field X we will say that X is "parallelly transported" along the curve if $X^\mu_{,\nu}(dx^\nu/d\lambda) = 0$. In a general differentiable manifold with a connection, we define analogously that a vector X is *parallelly transported* along the curve if its covariant derivative $X^\mu_{;\nu}(dx^\nu/d\lambda)$ along this curve is zero, that is, if

$$X^{\mu}_{;\nu}\frac{dx^{\nu}}{d\lambda} = X^{\mu}_{,\nu}\frac{dx^{\nu}}{d\lambda} + \Gamma^{\mu}_{\nu\delta}X^{\delta}\frac{dx^{\nu}}{d\lambda}$$
$$= \frac{dX^{\mu}}{d\lambda} + \Gamma^{\mu}_{\nu\delta}X^{\delta}\frac{dx^{\nu}}{d\lambda}$$
$$= 0. \qquad (1.22)$$

A similar definition holds for tensors. Given any curve $x^{\mu}(\lambda)$ with end points $\lambda = \lambda_1$ and $\lambda = \lambda_2$, the theory of solutions of ordinary differential equations shows that if the $\Gamma^{\lambda}_{\mu\nu}$'s are suitably differentiable functions of x^{μ}, we obtain a unique tensor at $\lambda = \lambda_2$ by parallelly transporting it from the point $\lambda = \lambda_1$, along the curve, to the point $\lambda = \lambda_2$.

A particular case is the covariant derivative of the tangent vector itself along the curve $x^{\mu}(\lambda)$. The curve is said to be a *geodesic* if the tangent vector is parallelly transported along this curve, i.e., if

$$\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\nu\delta}\frac{dx^{\nu}}{d\lambda}\frac{dx^{\delta}}{d\lambda} = 0 \qquad (1.23)$$

When the equation for a geodesic is reduced to the form of Eq.(1.23), we say that it is *affinely parameterized.* It should be noted that the freedom of choice we have is the origin and the scalar of λ. Eq.(1.23) also represents the motion of a free particle.

1.4 Geodesic Deviation

A major problem which has to be solved in the study of gravitational radiation is how to identify a gravitational radiation field. The problem arises because of the *principle of equivalence*, which says that the motion of a test particle in a gravitational field is independent of its mass and composition. This implies that mechanical phenomena are the same in an accelerated laboratory as in the earth's gravitational field, if observations are confined to a region over which the variation in the earth's gravitational field is small. Thus, in a local experiment we cannot distinguish an inertial field from a genuine gravitational one. However, if we are allowed to carry out non-local experiments, we can distinguish one of them from another by observing the variation of the field rather than the field itself. In GR this variation is described by the Riemann tensor which specifies the relative acceleration of neighboring free particles.

Let us consider a one-parameter family of geodesics $\Gamma(w)$ specified by the equations

$$x^\mu = x^\mu(\lambda, w) \tag{1.24}$$

where we assume x^μ to be twice continuously differentiable functions of both λ and w. The parameter w varies from one geodesic to another while λ varies along each of geodesics. For fixed w we have the geodesics equations [see Eq.(1.23)]

$$\frac{\partial^2 x^\mu}{\partial \lambda^2} = -\Gamma^\mu_{\nu\delta}\frac{\partial x^\nu}{\partial \lambda}\frac{\partial x^\delta}{\partial \lambda}, x^\mu = x^\mu(\lambda, w) \tag{1.25}$$

We might, in general, identify λ with the arc length on each of the geodesics. We prefer, however, to leave λ to be defined just by Eq.(1.25) so that our following discussion remains also valid for null geodesics.

The family of geodesics gives rise to the vector fields

$$\begin{aligned} t^\mu(\lambda, w) &= \frac{\partial x^\mu(\lambda, w)}{\partial \lambda} \\ \eta^\mu(\lambda, w) &= \frac{\partial x^\mu(\lambda, w)}{\partial w} \end{aligned} \tag{1.26}$$

where $t^\mu(\lambda, w)$ is the tangent vector along each geodesic, and $\eta^\mu(\lambda, w)$ is the vector which describes the deviation of two points on two infinitesimally near geodesics which have the same parameter value λ. η^μ is usually called the *geodesic deviation vector*.

From Eq.(1.26) we find that the covariant differentiation of η^μ along each geodesic is given by

$$\frac{D\eta^\mu}{D\lambda} \equiv \eta^\mu_{;\nu}\frac{\partial x^\nu}{\partial \lambda} = \frac{\partial \eta^\mu}{\partial \lambda} + \Gamma^\mu_{\nu\delta}\eta^\delta\frac{\partial x^\nu}{\partial \lambda} = \frac{\partial t^\mu}{\partial w} + \Gamma^\mu_{\nu\delta}\eta^\delta t^\nu \tag{1.27}$$

The remarkable fact is that the second differentiation of η^μ will bring us directly to the Riemann tensor. Actually we have

$$\begin{aligned}
\frac{D^2\eta^\mu}{D\lambda^2} &= \frac{\partial}{\partial\lambda}\{\frac{D\eta^\mu}{D\lambda}\} + \Gamma^\mu_{\nu\delta}t^\nu\frac{D\eta^\delta}{D\lambda} \\
&= \frac{\partial}{\partial w}\{\frac{\partial^2 x^\mu}{\partial\lambda^2}\} + \Gamma^\mu_{\nu\delta,\rho}t^\rho t^\nu\eta^\delta + \Gamma^\mu_{\nu\delta}\eta^\delta\frac{\partial^2 x^\mu}{\partial\lambda^2} + \Gamma^\mu_{\nu\delta}t^\nu\frac{\partial^2 x^\delta}{\partial\lambda\partial w} \\
&+ \Gamma^\mu_{\nu\delta}t^\mu\frac{\partial^2 x^\delta}{\partial\lambda\partial w} + \Gamma^\mu_{\nu\delta}\Gamma^\delta_{\rho\sigma}t^\nu t^\rho\eta^\sigma
\end{aligned} \tag{1.28}$$

Inserting Eq.(1.25) into Eq.(1.28), we find the well-known geodesic deviation equations

$$\frac{D^2\eta^\mu}{D\lambda^2} = -R^\mu_{\nu\delta\sigma}\eta^\delta t^\nu t^\sigma \tag{1.29}$$

where $R^\mu_{\nu\lambda\sigma}$ is the Riemann tensor given by Eq.(1.13).

To illustrate the physical meaning of the geodesic deviation equations, let us consider a time-like geodesic, say, C. We introduce an orthogonal triad of space-like vectors $\lambda^\mu_{(a)}$ $(a = 1, 2, 3)$. Throughout the following, we use the convention that the indices inside parentheses denote tetrad indices, Roman indices take the values 1,2,3, and repeated Roman indices are to be summed over these values unless some specific statement to the contrary is made. These space-like vectors are assumed orthogonal to each other and to the tangent vector $\lambda^\mu_{(0)} \equiv t^\mu$,

$$\lambda^\mu_{(\alpha)}\lambda^\nu_{(\beta)}g_{\mu\nu} = \lambda^\mu_{(\alpha)}\lambda_{\mu(\beta)} = \eta_{\alpha\beta} \tag{1.30}$$

where $\eta_{\alpha\beta}$ denote the *Minkowiski metric* components given by

$$(\eta_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.31}$$

The tangent vector $\lambda^\mu_{(0)}$ may be interpreted physically as the four-velocity of an observer whose world-line is C, and the space-like vectors $\lambda^\mu_{(a)}$ as rectangular coordinate axes used by this observer. For the sake of convenience, we assume that the orientations of the axes are fixed so that they are non-rotating as determined by local dynamical experiments (for example, see [30]). This means that the vectors $\lambda^{\mu}_{(a)}$ are parallelly transported along C,

$$\lambda^{\mu}_{(a);\nu}t^{\nu} = 0 \tag{1.32}$$

Without loss of generality, we also assume that η^{μ} is orthogonal to $\lambda^{\mu}_{(0)}$. Thus the tetrad components of the deviation vector η^{μ} are

$$\eta^{(a)} = \eta^{(a)(\sigma)}\lambda^{\mu}_{(\sigma)}\eta^{\nu}g_{\mu\nu} = \lambda^{(a)}_{\nu}\eta^{\nu}, \eta^{(0)} = 0 \tag{1.33}$$

The components, $\eta^{(a)}$, represent the position coordinates of a particle which moves near the observer on its own geodesic, say, C'.

Contracting Eq.(1.29) with $\lambda^{(a)}_{\mu}$ and using Eq.(1.32) we find that the acceleration of the particle relative to the observer is given by

$$\frac{d^2\eta^{(a)}}{d\tau^2} = -K^{(a)(b)}\eta_{(b)} \tag{1.34}$$

where

$$K^{(a)(b)} \equiv R_{\mu\nu\rho\sigma}t^{\nu}t^{\sigma}\lambda^{\mu(a)}\lambda^{\rho(b)} \tag{1.35}$$

are some of the tetrad components of the Riemann tensor. In writing Eq.(1.34) we replaced the parameter λ by the proper time τ measured by the observer using his own clock.

On the other hand, let us consider the same question in the framework of Newtonian gravitational theory. To be distinguishable, we use t as the time used by the observer and $\zeta^{\mu}(t)$ as the coordinate position of the particle relative to the observer. The gravitational field is described by the Newton potential ϕ. If $\zeta^{\mu}(t)$ is infinitesimal, then the equation of motion for the observer and the particle are given, respectively, by

$$\frac{d^2x^a}{dt^2} = -\partial^a\phi \tag{1.36}$$

$$\frac{d^2x^a}{dt^2}+\frac{d^2\zeta^a}{dt^2}=-\{\partial^a\phi\}|_{x+\zeta}=-\partial^a\phi-\zeta^b\partial^a\partial^b\phi$$

where the derivative of ϕ are evaluated at the point x^μ. It then follows that

$$\frac{d^2\zeta^a}{dt^2}=-K^{ab}\zeta^b, K^{ab}\equiv\partial^a\partial^b\phi \tag{1.37}$$

The condition for the Laplacian potential $\nabla^2\phi=0$ leads to

$$K^{aa}=0 \tag{1.38}$$

The similarity between Eqs(1.37) and (1.34) is evident. Moreover, we even have $K^{(a)(a)}=0$, wherever the Einstein vacuum field equations are satisfied.

The above considerations provide additional support for the choice of the field equation

$$R_{\mu\nu}=0 \tag{1.39}$$

as a description of a free gravitational field.

1.5 Decomposition of the Riemann Tensor

The Riemann tensor $R^\mu_{\nu\lambda\rho}$ defined by Eq.(1.13) has 20 independent components whereas the Ricci tensor $R_{\mu\nu}$ defined by Eq.(1.16) has only 10. Physically, it is convenient to decompose the Riemann tensor into three parts which are irreducible representations of the full *Lorenz group* [11]

$$R_{\mu\nu\lambda\rho}=C_{\mu\nu\lambda\rho}+E_{\mu\nu\lambda\rho}+G_{\mu\nu\lambda\rho} \tag{1.40}$$

where

$$\begin{aligned}E_{\mu\nu\lambda\rho} &\equiv \frac{1}{2}[g_{\mu\lambda}S_{\nu\rho}+g_{\nu\rho}S_{\mu\lambda}-g_{\nu\lambda}S_{\mu\rho}-g_{\mu\rho}S_{\nu\lambda}],\\ G_{\mu\nu\lambda\rho} &\equiv \frac{1}{12}[g_{\nu\rho}g_{\mu\lambda}-g_{\nu\lambda}g_{\mu\rho}]R,\\ S_{\mu\nu} &\equiv R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\end{aligned} \tag{1.41}$$

In Eq.(1.41), $S_{\mu\nu}$ denotes the traceless part of the Ricci tensor. The *Weyl tensor* $C_{\mu\nu\lambda\rho}$, being thought of as representing the free gravitational field [34], has all the symmetries of the Riemann tensor [see Eq.(1.14)], and is traceless

$$C^{\lambda}_{\mu\lambda\nu} = 0 \tag{1.42}$$

Combining the fact that the Weyl tensor has all the symmetries of the Riemann tensor and Eq.(1.42), we can see that the Weyl tensor has 10 independent components. These components are, at any point of the space-time, completely independent of the Ricci tensor components. Globally, however, the Weyl tensor and the Ricci tensor are not independent, as they are connected by the Bianchi identities [see Eq.(1.15)]. These identities can be now written in the form [27]

$$C_{\mu\nu\sigma\rho};^{\rho} = R_{\sigma[\mu;\nu]} - \frac{1}{6} g_{\sigma[\mu} R_{,\nu]} \tag{1.43}$$

where square brackets denote the antisymmetrization

$$A_{[\mu\nu]} \equiv \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) \tag{1.44}$$

The remarkable analogy between the Bianchi identities of Eq.(1.43) and the *Maxwell equations*

$$F^{\mu\nu}_{;\nu} = j^{\mu} \tag{1.45}$$

suggests that the Bianchi identities represent the interaction between the free gravitational field and matter fields.

If we define the tensor $J_{\mu\nu\sigma}$ as

$$J_{\mu\nu\sigma} \equiv R_{\sigma[\mu;\nu]} - \frac{1}{6} g_{\sigma[\mu} R_{,\nu]} \tag{1.46}$$

we have

$$J_{\mu\nu\lambda};^{\lambda} = 0 \tag{1.47}$$

which strongly resembles the equation for the conservation of charge in electrodynamics

$$J^{\lambda}_{;\lambda} = 0 \tag{1.48}$$

Hence, $J_{\mu\nu\lambda}$ defined by Eq.(1.46) can be considered as representing a *matter current*, which consists of those parts of the source that interact with the free gravitational field. These parts are called *gravitationally active*, while the parts of the source that do not contribute to $J_{\mu\nu\lambda}$ are called *gravitationally inert*. The propagation of the free gravitational field is in no way dependent upon the inert parts of the source.

An equivalent form for the decomposition of Eqs(1.40) and (1.41) is given by

$$R_{\mu\nu\lambda\rho} = C_{\mu\nu\lambda\rho} + \frac{1}{2}[g_{\mu\nu}R_{\nu\rho} + g_{\nu\rho}R_{\nu\lambda} - g_{\nu\lambda}R_{\mu\rho} - g_{\mu\rho}R_{\nu\lambda}] + \frac{1}{6}[g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}]R \quad (1.49)$$

When the Weyl tensor $C_{\mu\nu\lambda\delta}$ vanishes, the space-time is said to be *conformally flat*.

1.6 Matter Fields

In this thesis, besides considering exact solutions of the Einstein vacuum equations, we shall also consider solutions of the Einstein field equations for the following physically relevant energy-stress tensors.

(α) A massless scalar field:

The energy-stress tensor for a massless scalar field, ϕ, takes the form

$$T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\lambda}\phi^{;\lambda} \quad (1.50)$$

where ϕ satisfies the *massless Klein-Gordon equation*

$$\phi_{;\mu;\nu}g^{\mu\nu} = 0 \quad (1.51)$$

(β) A pure radiation field:

The energy-stress tensor in this case is given by

$$T_{\mu\nu} = \varepsilon\kappa_{\mu}\kappa_{\nu}, \kappa^{\nu}\kappa_{\nu} = 0 \quad (1.52)$$

where ε is non-negative.

Note that the energy-stress tensor for several matter fields has the same form as Eq.(1.52), for example, a *electromagnetic field*, a *massless scalar field*, or a *neutrino* *field* [11]. For the latter case, however, the corresponding matter field equations must be also satisfied, while for a pure radiation field it is not necessary.

(γ) An electromagnetic field:

For an electromagnetic field $F_{\mu\nu}$, the energy-stress tensor takes the form

$$T_{\mu\nu} = F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\lambda\rho} \tag{1.53}$$

where the antisymmetric tensor $F_{\mu\nu}$ satisfies the *Maxwell equations*

$$F_{[\mu\nu;\lambda]} = 0, F_{\mu\nu;\lambda}g^{\nu\lambda} = 0 \tag{1.54}$$

Introducing the following notation [25]

$$\begin{aligned} \Phi_0 &\equiv F_{(0)(2)} = F_{\mu\nu}l^{\mu}m^{\nu} \\ \Phi_1 &\equiv \frac{1}{2}[F_{(0)(1)} - F_{(2)(3)}] = \frac{1}{2}(F_{\mu\nu}l^{\mu}n^{\nu} - F_{\mu\nu}m^{\mu}\bar{m}^{\nu}) \\ \Phi_2 &\equiv -F_{(1)(3)} = -F_{\mu\nu}n^{\mu}\overline{m}^{\nu} \end{aligned} \tag{1.55}$$

or inversely

$$F_{\mu\nu} = 2\{-\Phi_0 n_{[\mu}\bar{m}_{\nu]} - \bar{\Phi}_0 n_{[\mu}m_{\nu]} + \Phi_2 l_{[\mu}m_{\nu]} + \bar{\Phi}_2 l_{[\mu}\bar{m}_{\nu]}\} - 4Re(\Phi_1)l_{[\mu}n_{\nu]} + 4iIm(\Phi_1)m_{[\mu}\bar{m}_{\nu]}$$

we find that the Ricci tensors are given by

$$\Phi_{mn} = \Phi_m\bar{\Phi}_n, \Lambda = 0, (m, n = 0, 1, 2) \tag{1.56}$$

and that the Maxwell equations read

$$\begin{aligned} D\Phi_1 - \bar{\delta}\Phi_0 &= (\tau - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 \\ D\Phi_2 - \bar{\delta}\Phi_1 &= (\rho - 2\varepsilon)\Phi_2 + 2\pi\Phi_1 - \lambda\Phi_0 \\ \delta\Phi_1 - \Delta\Phi_0 &= (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 \\ \delta\Phi_2 - \Delta\Phi_1 &= (\tau - 2\beta)\Phi_2 + 2\mu\Phi_1 - \nu\Phi_0 \end{aligned} \tag{1.57}$$

(δ) A massless neutrino field

The case for a massless neutrino field, in general, is much more complicated than the previous ones. This is mainly due to the fact that a neutrino field is described by a *two-component spinor* Φ_A, which satisfies the *neutrino Weyl equations*

$$\sigma^{\mu}_{AB}\phi^{A}_{;\mu} = 0 \tag{1.58}$$

where σ^{μ}_{AB} are the *complex Pauli spin matrices* [14], and the spin indices A,B take the values 1,2.

The energy-stress tensor for a massless neutrino field takes the form

$$T_{\mu\nu} = i[\sigma_{\mu A\dot{B}}(\phi^{A}\phi^{\dot{B}}_{;\nu} - \phi^{\dot{B}}\phi^{A}_{;\nu}) + \sigma_{\nu A\dot{B}}(\phi^{A}\phi^{\dot{B}}_{;\mu} - \phi^{\dot{B}}\phi^{A}_{;\mu})] \tag{1.59}$$

In a spinor basis $(o_A \iota_A)$, the neutrino spinor ϕ_A can be written as

$$\phi_A = \Phi_{o_A} + \Psi_{\iota_A} \tag{1.60}$$

where o_A and ι_A are normalized by the conditions

$$o_A \iota^A = -\iota_A o^A = 1 \tag{1.61}$$

In terms of Φ and Ψ and the spin coefficients, Eq.(1.58) takes the form

$$\begin{aligned} D\Phi + \bar{\delta}\Psi &= (\rho - \varepsilon)\Phi + (\alpha - \pi)\Psi \\ \delta\Phi + \Delta\Psi &= (\tau - \beta)\Phi + (\gamma - \mu)\Psi \end{aligned} \tag{1.62}$$

The Ricci scalars are now given by

$$
\begin{aligned}
\Phi_{00} &= i[\Psi D\bar{\Psi} - \bar{\Psi} D\Psi + \kappa\Phi\bar{\Psi} - \bar{\kappa}\Psi\bar{\Phi} + (\varepsilon - \bar{\varepsilon})\Psi\bar{\Psi}] \\
\Phi_{01} &= i\frac{1}{2}[\Psi\delta\bar{\Psi} - \bar{\Psi}\delta\Psi - \Psi D\bar{\Phi} + \bar{\Phi} D\Psi - (\bar{\rho} + \varepsilon + \bar{\varepsilon})\Psi\bar{\Phi} + (\beta - \bar{\alpha} - \bar{\pi})\Psi\bar{\Psi} - \kappa\Phi\bar{\Phi} \\
&+ \sigma\Phi\bar{\Psi}] \\
\Phi_{02} &= -i[\Psi\delta\bar{\Phi} - \bar{\Phi}\delta\Psi + (\bar{\alpha} + \beta)\Psi\bar{\Phi} + \sigma\Phi\bar{\Phi} + \lambda\Psi\bar{\Psi}] \\
\Phi_{11} &= \frac{1}{2}i[\Phi D\bar{\Phi} - \bar{\Phi} D\Phi + \Psi\Delta\bar{\Psi} - \bar{\Psi}\Delta\Psi + (\bar{\varepsilon} - \varepsilon)\Phi\bar{\Phi} + (\tau + \bar{\pi})\bar{\Psi}\Phi - (\bar{\tau} + \pi)\Psi\bar{\Phi} \\
&+ (\gamma - \bar{\gamma})\Psi\bar{\Psi}] \\
\Phi_{12} &= \frac{1}{2}i[\Phi\delta\bar{\Phi} - \bar{\Phi}\delta\Phi - \Psi\Delta\bar{\Phi} + \bar{\Phi}\Delta\Psi + (\bar{\alpha} - \beta - \tau)\Phi\bar{\Phi} - (\mu + \gamma + \bar{\gamma})\Psi\bar{\Phi} - \bar{\nu}\Psi\bar{\Psi} \\
&+ \bar{\lambda}\Phi\bar{\Psi}] \\
\Phi_{22} &= i[\Phi\Delta\bar{\Phi} - \bar{\Phi}\Delta\Phi + (\bar{\gamma} - \gamma)\Phi\bar{\Phi} + \bar{\nu}\Phi\bar{\Psi} - \nu\Psi\bar{\Phi}] \qquad (1.63)
\end{aligned}
$$

Eqs(1.62) and (1.63) are the basic equations for a neutrino field.

(ε) An isotropic perfect fluid

The energy-stress tensor for a perfect fluid takes the form

$$T_{\mu\nu} = (\mu + \rho)u_\mu u_\nu - p g_{\mu\nu}, u_\mu u_\nu g^{\mu\nu} = 1 \qquad (1.64)$$

where u_μ is the four-velocity of the fluid, p the pressure, and μ the energy density. Inserting Eq(1.64) into Eq.(1.21) we obtain

$$
\begin{aligned}
\mu_{;\nu}u^\nu + (\mu + p)u^\nu_{;\nu} &= 0 \\
(\mu + p)u^\mu_{;\nu}u^\nu + (u^\mu u^\nu - g^{\mu\nu})p_{;\nu} &= 0 \qquad (1.65)
\end{aligned}
$$

which are the conditions imposed on a perfect fluid. In order to completely describe a perfect fluid, however, Eq(1.65) has to be supplemented by an *equation of state* [36]. More frequently, the relation $p = p(\mu)$ is prescribed. We call a perfect fluid *isotropic* if the pressure p is a function of the energy density μ only.

The simplest of the isotropic fluids are those with a "*gamma equation of state*"

$$p = (\gamma - 1)\mu \qquad (1.66)$$

where γ is a constant. In all of the above cases, the energy-stress tensor must satisfy some conditions in order to be accepted physically (The energy conditions for a neutrino field are discussed by *Griffiths* [14]). These are either the so-called *weak energy condition*, or the *dominant energy condition, strong energy condition* [15].

(a) The weak energy condition

This condition says that the energy density measured by any observer must be non-negative. Mathematically, it is equivalent to saying that for any time-like vector u_μ we must have

$$T^{\mu\nu}u_\mu u_\nu \geqslant 0 \tag{1.67}$$

Eq.(1.67) is also true even for any null vector k_μ

(b) The dominant energy condition

The dominant energy condition is stronger than the weak energy condition. Besides the requirement of Eq.(1.67), it also requires that for any observer the local energy flow vector $(T^{\mu\nu}u_\mu)$ be non-space-like, i.e.

$$(T^\nu_\mu u_\nu)(T^{\mu\lambda}u_\lambda) \geqslant 0 \tag{1.68}$$

(c) The strong energy condition.

The expansion θ of a timelike geodesic congruence with zero vorticity (which means with zero local angular rate of rotation) will monotonically decrease along a geodesic if $R_{ab}W^aW^b \geqslant 0$ for any timelike vector W. We shall call this the *timelike convergence condition.* By the Einstein equation, this condition will be satisfied if the energy-momentum tensor obeys the inequality,

$$T_{ab}W^aW^b \geqslant W^aW_a(\frac{1}{2}T - \frac{1}{8\pi}\Lambda) \tag{1.69}$$

We shall say that the energy-momentum tensor satisfies the *strong energy condition* if it obeys the above inequality for $\Lambda = 0$.

1.7 Matter Shells

This section is based on the lecture notes of Wang [40].

1.7.1 Notations and Convention

In this section, we shall give a systematic study of a thin shell (a hypersurface across which the metric coefficients are only C^0) in a n-dimensional Riemannian manifold (γ_{AB}, Ω). We shall closely follow notations and convention of d'Inverno [12]. The metric is given by

$$ds^2 = \gamma_{AB}(x^C)dx^A dx^B \tag{1.70}$$

with the signature,

$$sign(\gamma_{AB}) = \{+, -, -, \ldots, -\} \tag{1.71}$$

In this section we shall use uppercase Latin indices, such as, A, B, C, to run from 0 to *n-1*, and the Greek indices, such as, μ, ν, λ, to run from 0 to $n-2$. The Riemann tensor is defined by ,

$$(D_C D_D - D_D D_C)X^A =^{(n)} R^A_{BCD} X^B \tag{1.72}$$

where D_A denotes the covariant derivative with respect to γ_{AB}, and

$${}^{(n)}R^A_{BCD} \equiv {}^{(n)}\Gamma^A_{BD,C} - {}^{(n)}\Gamma^A_{BC,D} + {}^{(n)}\Gamma^{A(n)}_{CE}\Gamma^E_{BD} - {}^{(n)}\Gamma^A_{DE}{}^{(n)}\Gamma^E_{BC} \tag{1.73}$$

with

$${}^{(n)}\Gamma^A_{BC} = \frac{1}{2}\gamma^{AD}(\gamma_{DC,B} + \gamma_{BD,C} - \gamma_{BC,D}) \tag{1.74}$$

and $\gamma_{AB,C} \equiv \partial\gamma_{AB}/\partial x^C$.

The Ricci and Einstein tensors are defined as

$$\begin{aligned} {}^{(n)}R_{AB} &\equiv {}^{(n)}R^C_{ACB} \\ &= {}^{(n)}\Gamma^C_{AB,C} - {}^{(n)}\Gamma^C_{AC,B} + {}^{(n)}\Gamma^C_{CE}{}^{(n)}\Gamma^E_{AB} - {}^{(n)}\Gamma^C_{BE}{}^{(n)}\Gamma^E_{AC} \qquad (1.75) \\ {}^{(n)}G_{AB} &\equiv {}^{(n)}R_{AB} - \frac{1}{2}\gamma_{AB}{}^{(n)}R \qquad (1.76) \end{aligned}$$

1The definition for Riemann tensor adopted here is the same as that used by [19].

2Israel [19] defined the Ricci tensor as $R_{AB} = -R^C_{ACB}$, while the Einstein tensor as that given here. Thus, the Einstein field equations used by Israel are $G_{AB} = -\kappa T_{AB}$.

where

$$ {}^{(n)}R \equiv {}^{(n)}R_{AB}\gamma^{AB} \tag{1.77} $$

The Einstein field equations are given by

$$ {}^{(n)}G_{AB} - \Lambda_n \gamma^{AB} = \kappa_n {}^{(n)}T_{AB} \tag{1.78} $$

where Λ_n and κ_n denote, respectively, the cosmological and Einstein constant, and ${}^{(n)}T_{AB}$ is the energy-momentum tensor.

The Weyl tensor is defined as

$$ \begin{aligned} {}^{(n)}C_{ABCD} &= {}^{(n)}R_{ABCD} \\ &+ \frac{1}{n-2}(\gamma_{AD}{}^{(n)}R_{BC} + \gamma_{BC}{}^{(n)}R_{AD}) \\ &- \gamma_{AC}{}^{(n)}R_{BD} - \gamma_{BD}{}^{(n)}R_{AC} \\ &+ \frac{1}{(n-1)(n-2)}(\gamma_{AC}\gamma_{BD} - \gamma_{AD}\gamma_{BC}){}^{(n)}R \end{aligned} \tag{1.79} $$

Note that the above definitions are simply generalizations to N dimensional spacetimes used in Section 1.2, and when $n = 4$ they reduce to them.

If we make following exchanges, we shall get Israel's expressions from these presented in this section,

$$ \begin{aligned} \gamma_{AB} &= -\bar{\gamma}_{AB} \\ {}^{(n)}\Gamma^{A}_{BC} &= {}^{(n)}\bar{\Gamma}^{A}_{BC} \\ {}^{(n)}R^{A}_{BCD} &= {}^{(n)}\bar{R}^{A}_{BCD} \\ {}^{(n)}R_{ABCD} &= -\bar{R}_{ABCD} \\ {}^{(n)}R_{AB} &= -{}^{(n)}\bar{R}_{AB} \\ {}^{(n)}R &= {}^{(n)}\bar{R} \\ {}^{(n)}G_{AB} &= -{}^{(n)}\bar{G}_{AB} \\ g_{\mu\nu} &= -\bar{g}_{\mu\nu} \end{aligned} $$

$$
\begin{aligned}
\Gamma^{\lambda}_{\mu\nu} &= \bar{\Gamma}^{\lambda}_{\mu\nu} \\
{}^{(n-1)}R^{\lambda}_{\mu\nu\sigma} &= {}^{(n-1)}\bar{R}^{\lambda}_{\mu\nu\sigma} \\
{}^{(n-1)}R_{\lambda\mu\nu\sigma} &= -{}^{(n-1)}\bar{R}_{\lambda\mu\nu\sigma} \\
{}^{(n-1)}R_{\mu\nu} &= -{}^{(n-1)}\bar{R}_{\mu\nu} \\
{}^{(n-1)}R &= {}^{(n-1)}\bar{R} \\
{}^{(n-1)}G_{\mu\nu} &= -{}^{(n-1)}\bar{G}_{\mu\nu} \\
K_{\mu\nu} &= \bar{K}_{\mu\nu} \\
K &= -\bar{K} \\
\varepsilon(n) &= -\bar{\varepsilon}(n)
\end{aligned}
\tag{1.80}
$$

where the quantities with bars denote those used by Israel in [19].

1.7.2 Gauss and Codacci Equations

Assume that Σ is a hypersurface in Ω by

$$\Sigma := \{x^A : \Phi(x^C) = 0\} \tag{1.81}$$

If we choose the intrinsic coordinates of Σ as

$$\{\xi^\mu\} = \{\xi^0, \xi^1, \ldots, \xi^{n-2}\} \tag{1.82}$$

we find that the hypersurface Σ can be also written in the form

$$x^A = x^A(\xi^\mu) \tag{1.83}$$

Then, we have

$$d\Phi(x^C) = \frac{\partial \Phi(x^C)}{\partial x^A}\frac{\partial x^A(\xi^\nu)}{\partial \xi^\lambda} d\xi^\lambda = 0 \tag{1.84}$$

Since $d\xi^\lambda$'s are linearly independent, we must have

$$N_A e^A_{(\mu)} = 0 \tag{1.85}$$

where

$$\begin{aligned} N_A &\equiv \frac{\partial \Phi(x^C)}{\partial x^A} \\ e^A_{(\mu)} &\equiv \frac{\partial x^A(\xi^\nu)}{\partial \xi^\mu} \end{aligned} \tag{1.86}$$

where N_A denotes the normal vector to the hypersurface $\Phi(x^C) = 0$, and $e^A_{(\mu)}$'s are the tangent vectors.

When $N_A N^A \neq 0$, a condition that we shall assume in the rest of the thesis, we define the unit normal vector n_A as

$$n_A = \frac{N_A}{|N_C N^C|^{\frac{1}{2}}} \tag{1.87}$$

with

$$n_A n_B \gamma^{AB} = \varepsilon(n) \tag{1.88}$$

where $\varepsilon(n) = \pm 1$. When $\varepsilon(n) = +1$ the normal vector n_A is timelike, and the corresponding hypersurface Σ is spacelike; and when $\varepsilon(n) = -1$ the normal vector n_A is spacelike, and the corresponding hypersurface Σ is timelike.

On the hypersurface Σ, the metric (1.70) reduces to

$$ds^2|_\Sigma = \gamma_{AB}(x^C(\xi^\lambda))\frac{\partial x^A(\xi^\rho)}{\partial \xi^\mu}\frac{\partial x^B(\xi^\sigma)}{\partial \xi^\nu} d\xi^\mu d\xi^\nu = g_{\mu\nu} d\xi^\mu d\xi^\nu \tag{1.89}$$

where $g_{\mu\nu}$ is the reduced metric on Σ and defined as

$$g_{\mu\nu}(\xi^\lambda) \equiv \gamma_{AB}(x^C(\xi^\lambda))\frac{\partial x^A(\xi^\rho)}{\partial \xi^\mu}\frac{\partial x^B(\xi^\sigma)}{\partial \xi^\nu} \tag{1.90}$$

On the other hand, introducing the projection operator,h_{AB}, by

$$h_{AB} = \gamma_{AB} - \varepsilon(n) n_A n_B \tag{1.91}$$

we find the following useful relations

$$\begin{aligned} \gamma^{AB} &= g^{\mu\nu} e^A_{(\mu)} e^B_{(\nu)} + \varepsilon(n) n^A n^B \\ g_{\mu\nu} &= \gamma_{AB} e^A_{(\mu)} e^B_{(\nu)} \\ h_{\mu\nu} &= \gamma_{AB} - \varepsilon(n) n_A n_B = g^{\mu\nu} e_{(\mu)A} e_{(\nu)B} \end{aligned} \tag{1.92}$$

where

$$e_{(\mu)A} \equiv \gamma_{AB} e^B_{(\mu)}, g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu, \gamma^{AC}\gamma_{CB} = \delta^A_B \tag{1.93}$$

For a tangent vector $\mathbf{A}$, we have

$$A_\mu = \mathbf{e}_{(m)} \cdot \mathbf{A} = e^C_{(\mu)} A_C, \mathbf{A} = A^\mu \mathbf{e}_{(\mu)}, \tag{1.94}$$

with $\mathbf{A} \cdot \mathbf{n} = 0$, and

$$A^\mu = g^{\mu\nu} A_\nu \tag{1.95}$$

The intrinsic covariant derivative of $\mathbf{A}$ with respect to ξ^μ is the projection of the vector $D\mathbf{A}/D\xi^\mu$ onto Σ,

$$\begin{aligned} A_{\mu;\nu} &\equiv \mathbf{e}_{(\mu)} \cdot \frac{D\mathbf{A}}{D\xi^\nu} = e^C_{(\mu)} \frac{\partial x^B}{\partial \xi^\nu} D_B A_C \\ &= \frac{\partial x^B}{\partial \xi^\nu} [D_B(e^C_{(\mu)} A_C) - A_C D_B(e^C_{(\mu)})] \\ &= \frac{\partial x^B}{\partial \xi^\nu} D_B(e^C_{(\mu)} A_C - \mathbf{A} \cdot \frac{D}{D\xi^\nu}(\mathbf{e}_{(\mu)}) \end{aligned} \tag{1.96}$$

Since

$$\begin{aligned} \frac{D}{D\xi^\nu}(\mathbf{e}_{(\mu)} \cdot \mathbf{A}) &= \frac{\partial x^C}{\partial \xi^\nu} D_C(A_\mu) = \frac{\partial A_\mu}{\partial \xi^\nu} \\ \mathbf{A} \cdot \frac{D}{D\xi^\nu}(\mathbf{e}_{(\mu)}) &= A^\sigma \mathbf{e}_{(\sigma)} \cdot \frac{D}{D\xi^\nu}(\mathbf{e}_{(\mu)}) \end{aligned} \tag{1.97}$$

we find that Eq.(1.96) can be written as

$$A_{\mu;\nu} = \mathbf{e}_{(\mu)} \cdot \frac{D\mathbf{A}}{D\xi^\nu} = A_{\mu,\nu} - A_\lambda \Gamma^\lambda_{\mu\nu} \tag{1.98}$$

where

$$\Gamma^\lambda_{\mu\nu} \equiv g^{\lambda\sigma} \mathbf{e}_{(\sigma)} \cdot \frac{D\mathbf{e}_{(\mu)}}{D\xi^\nu}. \tag{1.99}$$

After tedious but simple calculations, we find that

$$\Gamma^\lambda_{\mu\nu} \equiv g^{\lambda\sigma} \mathbf{e}_{(\sigma)} \cdot \frac{D\mathbf{e}_{(\mu)}}{D\xi^\nu} = \frac{1}{2} g^{\lambda\sigma}(g_{\sigma\nu,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}) \tag{1.100}$$

Properties of a non-intrinsic character enter when we consider the way in which Σ bends in Ω. This is measured by the variations of $Dn_A/D\xi^\mu$ of the normal vector. Since each of these $(n-1)$ vectors is perpendicular to n_A, we can write

$$\frac{Dn^A}{D\xi^\nu} = K^\lambda_\nu e^A_{(\lambda)} \tag{1.101}$$

thus defining the extrinsic curvature $K_{\mu\nu}$ of the hypersurface Σ. From Eqs.(1.92) and (1.101) we obtain that

$$K_{\mu\nu} = g_{\mu\lambda}K^\lambda_\nu = e_{(\mu)A}K^\lambda_\nu = e_{(\mu)A}\frac{Dn^A}{D\xi^\nu} = e^A_{(\mu)}e^B_{(\nu)}D_B n_A \tag{1.102}$$

Since we have $n_A e^A_{(\mu)} = 0$, we find that

$$\begin{aligned}
K_{\mu\nu} &= e^A_{(\mu)}e^B_{(\nu)}D_B n_A = -n_A e^B_{(\nu)}D_B(e^A_{(\mu)}) \\
&= -n_A e^B_{(\nu)}(e^A_{(\mu),B} +^{(n)} \Gamma^A_{BC}e^C_{(\mu)}) \\
&= -n_A(\frac{\partial^2 x^A}{\partial\xi^\mu\partial\xi^\nu} +^{(n)} \Gamma^A_{BC}\frac{\partial x^B}{\partial\xi^\nu}\frac{\partial x^C}{\partial\xi^\mu}) \\
&= K_{\nu\mu}
\end{aligned} \tag{1.103}$$

On the other hand, assuming

$$\frac{D\mathbf{e}_{(\mu)}}{D\xi^\nu} = \alpha_{\mu\nu}\mathbf{n} + \beta^\sigma_{\mu\nu}\mathbf{e}_{(\sigma)} \tag{1.104}$$

we find that

$$\begin{aligned}
\mathbf{n}\cdot\frac{D\mathbf{e}_{(\mu)}}{D\xi^\nu} &= \alpha_{\mu\nu}\varepsilon(n) = -K_{\mu\nu} \\
\mathbf{e}_{(\lambda)}\cdot\frac{D\mathbf{e}_{(\mu)}}{D\xi^\nu} &= \beta^\sigma_{\mu\nu}g_{\lambda\sigma} = \Gamma_{\lambda\mu\nu}
\end{aligned} \tag{1.105}$$

namely

$$\alpha_{\mu\nu} = -\varepsilon(n)K_{\mu\nu}, \beta^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} \tag{1.106}$$

Inserting Eq.(1.106) into Eq.(1.104), we find that

$$\frac{D\mathbf{e}_{(\mu)}}{D\xi^\nu} = -\varepsilon(n)K_{\mu\nu}\mathbf{n} + \Gamma^\sigma_{\mu\nu}\mathbf{e}(\sigma) \tag{1.107}$$

which is usually called the *Gauss-Weingarten equation.* Thus, for any vector $\mathbf{A}$ that is tangent to Σ, we find that

$$\begin{aligned}
\frac{D\mathbf{A}}{D\xi^\nu} &= \frac{D}{D\xi^\nu}(A^\mu \mathbf{e}_{(\mu)}) \\
&= \frac{DA^\mu}{D\xi^\nu}\mathbf{e}_{(\mu)} + A^\mu \frac{D\mathbf{e}_{(\mu)}}{D\xi^\nu} \\
&= \frac{\partial A^\mu}{\partial x^\nu}\mathbf{e}_{(\mu)} + A^\mu(-\varepsilon(n)K_{\mu\nu}\mathbf{n} + \Gamma^\sigma_{\mu\nu}\mathbf{e}_{(\sigma)}) \\
&= A^\mu_{;\nu}\mathbf{e}_{(\mu)} - \varepsilon(n)A^\mu K_{\mu\nu}\mathbf{n} \qquad (1.108)
\end{aligned}$$

that is,

$$\frac{D\mathbf{A}}{D\xi^\nu} = A^\mu_{;\nu}\mathbf{e}_{(\mu)} - \varepsilon(n)A^\mu K_{\mu\nu}\mathbf{n} \qquad (1.109)$$

Operating on Eq.(1.107) with $D/D\xi^\lambda$ and using Eq.(1.101), we find that

$$\begin{aligned}
\frac{D}{D\xi^\lambda}(\frac{De^A_{(\mu)}}{D\xi^\nu}) &= \frac{D}{D\xi^\lambda}(-\varepsilon(n)K_{\mu\nu}n^A + \Gamma^\sigma_{\mu\nu}e^A_{(\sigma)}) \\
&= -\varepsilon(n)\frac{DK_{\mu\nu}}{D\xi^\lambda}n^A - \varepsilon(n)K_{\mu\nu}\frac{Dn^A}{D\xi^\lambda} + \frac{D\Gamma^\sigma_{\mu\nu}}{D\xi^\lambda}e^A_{(\sigma)} + \Gamma^\delta_{\mu\nu}\frac{De^A_{(\delta)}}{D\xi^\lambda} \\
&= -\varepsilon(n)K_{\mu\nu,\lambda}n^A - \varepsilon(n)K_{\mu\nu}K^\sigma_\lambda e^A_{(\sigma)} + \Gamma^\sigma_{\mu\nu,\lambda}e^A_{(\sigma)} + \Gamma^\delta_{\mu\nu}(-\varepsilon(n)K_{\delta\lambda}n^A + \Gamma^\sigma_{\delta\lambda}e^A_{(\sigma)}) \\
&= (\Gamma^\sigma_{\mu\nu,\lambda} + \Gamma^\delta_{\mu\nu}\Gamma^\sigma_{\delta\lambda} - \varepsilon(n)K_{\mu\nu}K^\sigma_\lambda)e^A_{(\sigma)} - \varepsilon(n)(K_{\mu\nu,\lambda} + \Gamma^\delta_{\mu\nu}K_{\delta\lambda})n^A \qquad (1.110)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
(\frac{D^2}{D\xi^\lambda D\xi^\nu} - \frac{D^2}{D\xi^\nu D\xi^\lambda})e^A_{(\mu)} &= {}^{(n-1)}R^\sigma_{\mu\lambda\nu}e^A_{(\sigma)} + \varepsilon(n)(K_{\mu\lambda}K^\sigma_\nu - K_{\mu\nu}K^\sigma_\lambda)e^A_{(\sigma)} \\
&+ \varepsilon(n)(K_{\mu\lambda;\nu} - K_{\mu\nu;\lambda})n^A, \qquad (1.111)
\end{aligned}$$

where

$$ {}^{(n-1)}R^\sigma_{\mu\lambda\nu} \equiv \Gamma^\sigma_{\mu\nu,\lambda} - \Gamma^\sigma_{\mu\lambda,\nu} + \Gamma^\delta_{\mu\nu}\Gamma^\sigma_{\delta\lambda} - \Gamma^\delta_{\mu\lambda}\Gamma^\sigma_{\delta\nu} \qquad (1.112)$$

On the other hand, we have

$$\begin{aligned}
\frac{D^2 e^A_{(\mu)}}{D\xi^\lambda D\xi^\nu} &= \frac{D}{D\xi^\lambda}(\frac{De^A_{(\mu)}}{D\xi^\nu}) = e^C_{(\lambda)}D_C(e^B_{(\nu)}D_B e^A_{(\mu)}) \\
&= e^C_{(\lambda)}e^B_{(\nu)}(D_C D_B e^A_{(\mu)}) + e^C_{(\lambda)}(D_C e^B_{(\nu)})(D_B e^A_{(\mu)}) \\
&= e^C_{(\lambda)}e^B_{(\nu)}(D_C D_B e^A_{(\mu)}) + (D_B e^A_{(\mu)})(\frac{\partial^2 x^B}{\partial\xi^\lambda\partial\xi^\nu} \\
&+ {}^{(n)}\Gamma^B_{CD}\frac{\partial x^C}{\partial\xi^\lambda}\frac{\partial x^D}{\partial\xi^\nu}) \qquad (1.113)
\end{aligned}$$

$$
\begin{aligned}
(\frac{D^2}{D\xi^\lambda D\xi^\nu} - \frac{D^2}{D\xi^\nu D\xi^\lambda})e^A_{(\mu)} &= [(D_C D_B - D_B D_C)e^A_{(\mu)}]e^C_{(\lambda)}e^B_{(\nu)} \\
&= {}^{(n)}R^A_{DCB}e^D_{(\mu)}e^C_{(\lambda)}e^B_{(\nu)} \qquad (1.114)
\end{aligned}
$$

Then, the combination of Eqns.(1.111) and (1.114) yields,

$$
\begin{aligned}
{}^{(n)}R^A_{DCB}e^D_{(\mu)}e^C_{(\lambda)}e^B_{(\nu)} &= {}^{(n-1)}R^\sigma_{\mu\lambda\nu}e^A_{(\sigma)} + \varepsilon(n)(K_{\mu\lambda}K^\sigma_\nu - K_{\mu\nu}K^\sigma_\lambda)e^A_{(\sigma)} \\
&+ \varepsilon(n)(K_{\mu\lambda;\nu} - K_{\mu\nu;\lambda})n^A \qquad (1.115)
\end{aligned}
$$

Multiplying Eq.(1.115) by $e_{(\rho)A}$ we obtain the *Gauss equation*,

$$
{}^{(n)}R_{ABCD}e^A_{(\rho)}e^B_{(\mu)}e^C_{(\lambda)}e^D_{(\nu)} ={}^{(n-1)} R_{\rho\mu\lambda\nu} + \varepsilon(n)(K_{\mu\lambda}K_{\nu\rho} - K_{\mu\nu}K_{\lambda\rho}) \qquad (1.116)
$$

Similarly, multiplying Eq.(1.115) with n_A we obtain the *Codacci equation*,

$$
{}^{(n)}R_{ABCD}n^A e^B_{(\mu)}e^C_{(\lambda)}e^D_{(\nu)} = K_{\mu\lambda;\nu} - K_{\mu\nu;\lambda} \qquad (1.117)
$$

Thus, the *Gauss and Codacci equations* are given by

$$
{}^{(n)}R_{ABCD}e^A_{(\rho)}e^B_{(\mu)}e^C_{(\lambda)}e^D_{(\nu)} ={}^{(n-1)} R_{\rho\mu\lambda\nu} + \varepsilon(n)(K_{\mu\lambda}K_{\nu\rho} - K_{\mu\nu}K_{\lambda\rho}) \qquad (1.118)
$$

$$
{}^{(n)}R_{ABCD}n^A e^B_{(\mu)}e^C_{(\lambda)}e^D_{(\nu)} = (K_{\mu\lambda;\nu} - K_{\mu\nu;\lambda})g^{\mu\nu} \qquad (1.119)
$$

which are exactly the expressions of Eqs.(12) and (13) obtained by Israel in [19], after considering the fact of Eq.(1.80).

Multiplying Eq.(1.116) by $g^{\rho\lambda}g^{\mu\nu}$, and noting

$$
g^{\mu\nu}e^A_{(\mu)}e^B_{(\nu)} = \gamma^{AB} - \varepsilon(n)n^A n^B \qquad (1.120)
$$

we find that

$$
\begin{aligned}
&{}^{(n)}R_{ABCD}e^A_{(\rho)}e^B_{(\mu)}e^C_{(\lambda)}e^D_{(\nu)}g^{\rho\lambda}g^{\mu\nu} \\
&= {}^{(n)}R_{ABCD}(\gamma^{AC} - \varepsilon(n)n^A n^C)(\gamma^{BD} - \varepsilon(n)n^B n^D) \\
&= {}^{(n)}R_{ABCD}(\gamma^{AC}\gamma^{BD} - \varepsilon(n)\gamma^{AC}n^B n^D - \varepsilon(n)\gamma^{BD}n^A n^C) \\
&= {}^{(n)}R - 2\varepsilon(n){}^{(n)}R_{AB}n^A n^B \\
&= -2\varepsilon(n){}^{(n)}G_{AB}n^A n^B \\
&= {}^{(n-1)}R + \varepsilon(n)(K^\lambda_\sigma K^\sigma_\lambda - K^2) \qquad (1.121)
\end{aligned}
$$

this is,

$$-2\varepsilon(n)^{(n)}G_{AB}n^A n^B = {}^{(n-1)}R + \varepsilon(n)(K^\lambda_\sigma K^\sigma_\lambda - K^2) \tag{1.122}$$

where $K = g^{\mu\nu}K_{\mu\nu}$

On the other hand, multiplying Eq.(1.117) by $g^{\mu\nu}$, we find that

$$\begin{aligned} {}^{(n)}R_{ABCD}n^A e^B_{(\mu)} e^C_{(\lambda)} e^D_{(\nu)} g^{\mu\nu} &= {}^{(n)}R_{ABCD}n^A e^C_{(\lambda)}(\gamma^{BD} - \varepsilon(n)n^B n^D) \\ &= {}^{(n)}R_{AC}n^A e^C_{(\lambda)} ={}^{(n)} G_{AC}n^A e^C_{(\lambda)} \\ &= (K^\sigma_\lambda - \delta^\sigma_\lambda)_{;\sigma} \end{aligned} \tag{1.123}$$

or

$${}^{(n)}G_{AC}n^A e^C_{(\lambda)} = (K^\sigma_\lambda - \delta^\sigma_\lambda)_{;\sigma} \tag{1.124}$$

In summary, we have

$$-2\varepsilon(n)^{(n)}G_{AB}n^A n^B ={}^{(n-1)} R + \varepsilon(n)(K^\lambda_\sigma K^\sigma_\lambda - K^2) \tag{1.125}$$

$${}^{(n)}G_{AC}n^A e^C_{(\lambda)} = (K^\sigma_\lambda - \delta^\sigma_\lambda K)_{;\sigma} \tag{1.126}$$

which are exactly Eqs.(14) and (15) obtained by Israel in [19], after some corresponding changes are made due to different definitions of some quantities [see Eq.(1.80)].

1.7.3 Surface Layers

Assume that the hypersurface Σ divides the whole spacetime Ω into two regions $\Omega^\pm$, where

$$\Omega^+ := \{x^A, \Phi \geqslant 0\}, \Omega^- := \{x^A, \Phi \leqslant 0\} \tag{1.127}$$

If we choose the systems of coordinates differently, say, x^{+A} in region Ω^+ and x^{-A} in region Ω^-. Then, the hypersurface Σ are given by

$$x^{+A} = x^{+A}(\xi^\mu), x^{-A} = x^{-A}(\xi^\mu) \tag{1.128}$$

or equivalently

$$\Phi^+(x^{+B}) = 0, \Phi^-(x^{-B}) = 0 \tag{1.129}$$

From the above equations we find that

$$\begin{aligned} n_A^+ &= \frac{N_A^+}{|N_C^+ N^{+C}|^{\frac{1}{2}}}, N_A^+ = \frac{\partial \Phi^+(x^{+C})}{\partial x^{+A}}, e_{(\mu)}^{+A} \equiv \frac{\partial x^{+A}(\xi^\lambda)}{d\xi^\mu} \\ n_A^- &= \frac{N_A^-}{|N_C^- N^{-C}|^{\frac{1}{2}}}, N_A^- = \frac{\partial \Phi^-(x^{-C})}{\partial x^{-A}}, e_{(\mu)}^{-A} \equiv \frac{\partial x^{-A}(\xi^\lambda)}{\partial \xi^\mu} \end{aligned} \tag{1.130}$$

Then, it is easy to see that in each of the two regions, the Gauss and Codacci equations take the form of Eqs.(1.118) and (1.119), from which Eqs.(1.125) and (1.126) result. On the hypersurface Σ, the reduced metric from each side of Σ should be the same, so we must have

$$g_{\mu\nu}^+(\xi^\mu)|_{\Sigma^+} = g_{\mu\nu}^-(\xi^\mu)|_{\Sigma^-} \tag{1.131}$$

On the other hand, by the Lanczos equations [20],

$$[K_{\mu\nu}]^- - g_{\mu\nu}[K]^- = \kappa S_{\mu\nu} \tag{1.132}$$

one defines the symmetric tensor $S_{\mu\nu}$ as the *surface energy-momentum tensor*, where

$$[K_{\mu\nu}]^- \equiv \lim_{\Phi\to 0^+} K_{\mu\nu}^+ - \lim_{\Phi\to 0^-} K_{\mu\nu}^- \tag{1.133}$$

and $[K]^- \equiv g^{\mu\nu}[K_{\mu\nu}]^-$. The above definition for $S_{\mu\nu}$ can be further justified by considering the integral of ${}^{(n)}T_{AB}e_{(\mu)}^A e_{(\nu)}^B$ with respect to the proper distance τ, measured perpendicular to Σ [10],

$$S_{\mu\nu} = \int_{-\varepsilon}^{+\varepsilon} {}^{(n)}T_{AB}e_{(\mu)}^A e_{(\nu)}^B d\tau \tag{1.134}$$

From Eq.(1.126) we find that

$$[{}^{(n)}G_{AC}n^A e_{(\mu)}^C]^- = -\kappa S_{\mu;\lambda}^\lambda \tag{1.135}$$

which serves as the conservation law for the surface EMT.

In analyzing surface layers, one usually uses the first junction conditions (1.131), the Israel's junction conditions (1.132), the conservation law (1.135), and the n-dimensional Einstein field equations applied on each side of the hypersurface Σ.

The n-dimensional Einstein field equations,

$$^{(n)}R_{AB} - \frac{1}{2}\gamma^{(n)}_{AB}R = \kappa^2_n {}^{(n)}T_{AB} \tag{1.136}$$

can be projected to the base $\{e^A_{(\mu)}, n^A\}$ as

$$^{(n)}G_{AB}n^A n^B = -\frac{\varepsilon(n)}{2}({}^{(n-1)}R + \varepsilon(n)(K^\lambda_\sigma K^\sigma_\lambda - K^2)) \tag{1.137}$$

$$^{(n)}G_{AC}n^A e^C_{(\lambda)} = (K^\sigma_\lambda - \delta^\sigma_\lambda)_{;\sigma} \tag{1.138}$$

$$^{(n)}G_{AB}e^A_{(\mu)}e^B_{(\nu)} = -(K^\sigma_\lambda - \delta^\sigma_\lambda)_{,A}n^A + f(K_{\alpha\beta}, K^2_{\alpha\beta}) \tag{1.139}$$

where $f(K_{\alpha\beta}, K^2_{\alpha\beta})$ is known only in the case where *Gaussian normal coordinates* are used [10]. Clearly, Eqs.(1.137) and (1.138) are, respectively, the Gauss and Codacci equation, while the integral of Eq.(1.139) across Σ gives Israel's junction conditions (1.132).

Since Eq.(1.139) cannot be obtained by only using the Gauss equation (1.118), it is clear that any equations obtained from it should not include (at least totally) the Israel's junction conditions (1.132). This observation is very important when we consider the effective Einstein field equations on Σ by following Shiromizo, Maeda, and Sasaki [35].

1.7.4 Applications to Brane Worlds

From the Gauss equation Eq.(1.116), we find that

$$^{(n-1)}R_{\rho\mu\lambda\nu} = {}^{(n)}R_{ABCD}e^A_{(\rho)}e^B_{(\mu)}e^C_{(\lambda)}e^D_{(\nu)} - \varepsilon(n)(K_{\mu\lambda}K_{\nu\rho} - K_{\mu\nu}K_{\lambda\rho}) \tag{1.140}$$

from which we obtain

$$\begin{aligned} ^{(n-1)}R_{\mu\nu} &= {}^{(n)}R_{AB}e^A_{(\mu)}e^B_{(\nu)} - \varepsilon(n){}^{(n)}R_{ABCD}n^A e^B_{(\mu)}n^C e^D_{(\nu)} - \varepsilon(n)(K_{\mu\sigma}K^\sigma_\nu - KK_{\mu\nu}) \\ ^{(n-1)}R &= {}^{(n)}R - 2\varepsilon(n){}^{(n)}R_{AB}n^A n^B - \varepsilon(n)(K_{\alpha\beta}K^{\alpha\beta} - K^2) \end{aligned} \tag{1.141}$$

On the other hand, from Eq.(1.7.1), we find that

$$^{(n)}R_{ABCD}n^A e^B_{(\mu)}n^C e^D_{(\nu)} = \frac{\varepsilon(n)}{n-2}{}^{(n)}R_{AB}e^A_{(\mu)}e^B_{(\nu)} + \frac{1}{n-2}\{{}^{(n)}R_{AB}n^A n^B - \frac{\varepsilon(n)}{n-1}{}^{(n)}R\}g_{\mu\nu} + {}^{(n)}E_{\mu\nu} \tag{1.142}$$

where

$$^{(n)}E_{\mu\nu} \equiv^{(n)} C_{ABCD} n^A e^B_{(\mu)} n^C e^D_{(\nu)} \tag{1.143}$$

with $^{(n)}C_{ABCD}$ being the Weyl tensor, defined by Eq.(1.7.1).

From the n-dimensional Einstein field equations (1.136), we find that

$$\begin{aligned} ^{(n)}R_{AB} &= \kappa_n^2 \{^{(n)}T_{AB} - \frac{1}{n-2}\gamma_{AB}{}^{(n)}T\} \\ ^{(n)}R &= -\kappa_n^2 \frac{2}{n-2}{}^{(n)}T \end{aligned} \tag{1.144}$$

Then, combining Eqs.(1.144) - (1.144), we obtain

$$\begin{aligned} ^{(n-1)}G_{\mu\nu} &= \kappa_n^2 \frac{n-3}{n-2}\{^{(n)}T_{AB} e^A_{(\mu)} e^B_{(\nu)} + \varepsilon(n)[^{(n)}T_{AB} n^A n^B - \frac{\varepsilon(n)}{n-1}{}^{(n)}T] g_{\mu\nu}\} \\ &- \varepsilon(n)\{K_{\mu\sigma}K^\sigma_\nu - KK_{\mu\nu} - \frac{1}{2}(K_{\alpha\beta}K^{\alpha\beta} - K^2) g_{\mu\nu}\} \\ &- \varepsilon(n)^{(n)}E_{\mu\nu} \end{aligned} \tag{1.145}$$

CHAPTER TWO

Gravitational Collapse of Spherically Symmetric Shells

2.1 Introduction to Dark Energy

Over the past decade, one of the most remarkable discoveries is that our universe is currently accelerating. This was first observed from high red shift supernova Ia [1], and confirmed later by cross checks from the cosmic microwave background radiation [2] and large scale structure [3].

In Einstein's general relativity, in order to have such an acceleration, one needs to introduce a component to the matter distribution of the universe with a large negative pressure. This component is usually referred to as dark energy. Astronomical observations indicate that our universe is flat and currently consists of approximately $\frac{2}{3}$ dark energy and $\frac{1}{3}$ dark matter. The nature of dark energy as well as dark matter is unknown, and many radically different models have been proposed, such as, a tiny positive cosmological constant, quintessence, phantoms, Chaplygin gas, and dark energy in brane worlds, among many others [See the review articles [33] [6] [29] [28] [31] [32], and references therein].

On the other hand, another very important issue in gravitational physics is black holes and their formation in our universe. Although it is generally believed that on scales much smaller than the horizon size the fluctuations of dark energy itself are unimportant [8], their effects on the evolution of matter overdensities may be significant [4]. Then, a natural question is how dark energy affects the process of the gravitational collapse of a star. It is known that dark energy exerts a repulsive force on its surrounding, and this repulsive force may prevent the star from collapse. Another related issue is how dark energy affects already-formed black holes (if they indeed exist in our universe). Recently, it was shown that the mass of a black hole decreases due to phantom energy accretion and tends to zero when the Big Rip approaches [13].

In this Chapter, we shall study the formation of black holes from the gravitational collapse of a dust cloud in the background of dark energy, here "dust cloud" means a cloud made of matter with zero pressure. Thus, it includes the dark matter as a particular case. In section 2 we discuss the basic properties of spherically symmetrical thin shells and derive the extrinsic curverture from the metrics, and in section 3 we study the special solution of the general case called McVittie solution, and also the corresponding metric and the extrinsic curvature from McVittie solution. In section 4 we'll show how to use numerical method to solve the equations of extrinsic curverture so that we can see if it can form black holes in the background of dark energy.

2.2 Spherically Symmetric Spacetimes, Horizons and Black Holes

The general metric for spherically symmetric spacetimes can be cast in the form

$$ds^2 = g_{ab}(x^c)dx^a dx^b - \mathcal{R}^2(x^c)d\Omega^2 \tag{2.1}$$

where $a, b, c = 0 or 1$, $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$, and θ and φ are the usual spherical angular coordinates, with $0 \leqslant \theta \leqslant \pi$, and $0 \leqslant \varphi \leqslant 2\pi$. Clearly, the metric is invariant under the coordinate transformations

$$x^a = x^a(x^{'b}), (a, b = 0, 1) \tag{2.2}$$

Using one of the two degree of the freedom, we can always set $g_{01}(x^c) = 0$, so the metric can be written

$$ds^2 = A^2(T, R)dT^2 - B^2(T, R)dR^2 - \mathcal{R}^2(T, R)d\Omega^2 \tag{2.3}$$

Introducing two null coordinates u and v via the relations

$$\begin{aligned} du &= F[A(T,R)dT - B(T,R)dR] \\ dv &= G[A(T,R)dT + B(T,R)dR] \end{aligned} \tag{2.4}$$

where F and G satisfy the integrability conditions for u and v

$$\begin{aligned} \frac{\partial^2 u}{\partial T \partial R} &= \frac{\partial^2 u}{\partial R \partial T} \\ \frac{\partial^2 v}{\partial T \partial R} &= \frac{\partial^2 v}{\partial R \partial T} \end{aligned} \tag{2.5}$$

the metric (2.3) takes the form

$$ds^2 = 2e^{\sigma(u,v)}dudv - \mathcal{R}^2(u,v)d\Omega^2 \tag{2.6}$$

where

$$\sigma(u,v) \equiv -\frac{1}{2}\ln(2FG) \tag{2.7}$$

Without loss of generality, we shall assume that

$$F > 0, G > 0 \tag{2.8}$$

The coordinates u and v are two null coordinates, with $-\infty < u, v < \infty$. Note that the metric remains the same under the transformations

$$u = u(\bar{u}), v = v(\bar{v}) \tag{2.9}$$

Using this gauge freedom, we can always make the metric coefficients $\sigma(u,v)$ and $\mathcal{R}(u,v)$ non-singular, except for points where the spacetime is singular. In the following we assume that this is always the case.

In addition, the roles of u and v can be interchanged. To fix this particular freedom, we choose coordinates such that along the lines of constant u the radial coordinate $\mathcal{R}$ increases towards the future, while along the line of constant v the coordinate $\mathcal{R}$ decreases towards the future. This, of course, just defines u as outgoing and v as ingoing null coordinates.

Defining the two null vectors, $l_\mu and n_\mu$, along each of the two rays by [5]

$$l_\mu \equiv \frac{\partial u}{\partial x^\mu} = \delta^u_\mu, n_\mu \equiv \frac{\partial v}{\partial x^\mu} = \delta^v_\mu \tag{2.10}$$

we find that

$$l_{\mu;\nu} l^\nu = 0 = n_{\mu;\nu} n^\nu \tag{2.11}$$

i.e these null rays are affinely parameterized null geodesics, where a semicolon denotes the covariant derivative. The expansion for each is defined by

$$\begin{aligned} \theta_+ &\equiv l_{\mu;\nu} g^{\mu\nu} = 2e^{-2\sigma}\frac{\mathcal{R}_{,v}}{\mathcal{R}} \\ \theta_- &\equiv n_{\mu;\nu} g^{\mu\nu} = 2e^{-2\sigma}\frac{\mathcal{R}_{,u}}{\mathcal{R}} \end{aligned} \tag{2.12}$$

where $()_{,\mu} \equiv \frac{\partial()}{\partial x^\mu}$.

It should be noted that the two null vectors l_μ and n_μ are uniquely defined only up to a factor [5]. In fact,

$$\bar{l}_\mu = f(u) l_\mu, \bar{n}_\mu = g(v) n_\mu \tag{2.13}$$

represent another set of null vectors that also define affinely parameterized null geodesic

$$\bar{l}_{\mu;\nu} \bar{l}^\nu = 0 = \bar{n}_{\mu;\nu} \bar{n}^\nu \tag{2.14}$$

and the corresponding expansions are given by

$$\begin{aligned} \bar{\theta}_+ &\equiv \bar{l}_{\mu;\nu} g^{\mu\nu} = f(u)\theta_+ \\ \bar{\theta}_- &\equiv \bar{n}_{\mu;\nu} g^{\mu\nu} = g(v)\theta_- \end{aligned} \tag{2.15}$$

However, since along each geodesic $u = Const.(v = Const.) f(u)(g(v))$ is constant, this does not affect the definition of trapped surfaces in terms of the expansions. Thus without loss of generality, in the following we consider only the expressions given by Eq.(2.12).

Definition [16], [5] The spatial two-surface $\mathcal{S}$ of constant T and R is said trapped, marginally trapped, or untrapped, according to whether $\theta_+\theta_-|_{\mathcal{S}} > 0, \theta_+\theta_-|_{\mathcal{S}} <= 0, or \theta_+\theta_-|_{\mathcal{S}} < 0$.

Assuming that on the marginally trapped surfaces $\mathcal{S}$ we have $\theta_+|_{\mathcal{S}} = 0$, then an *apparent horizon* is the closure $\tilde{\Sigma}$ of a three-surface Σ foliated by the trapped surfaces $\mathcal{S}$ on which $\theta_-|_\Sigma \neq 0$. It is said *outer, degenerate, or inner*, according to whether $\mathcal{L}_-\theta_+|_\Sigma < 0$, $\mathcal{L}_-\theta_+|_\Sigma = 0$, or $\mathcal{L}_-\theta_+|_\Sigma > 0$, where $\mathcal{L}_- \equiv \mathcal{L}_n$ denotes the Lie derivative along n^μ. In addition, if $\theta_-|_\Sigma < 0$ then the apparent horizon is said *future*, and if $\theta_-|_\Sigma > 0$ it is said *past.*

Black holes are usually defined by the existence of *future outer apparent horizons* [15] [16] [18] [5]. However, in a definition given by Tipler [38] the degenerate case was also included [16].

Finally we note that

$$\begin{aligned} \frac{\partial T}{\partial u} &= \frac{1}{2FA}, \frac{\partial T}{\partial v} = \frac{1}{2GA} \\ \frac{\partial R}{\partial u} &= -\frac{1}{2FB}, \frac{\partial R}{\partial v} = \frac{1}{2GB} \end{aligned} \tag{2.16}$$

Then, we find that

$$\begin{aligned} \mathcal{R}_{,u} &= \frac{\partial T}{\partial u}\mathcal{R}_{,T} + \frac{\partial R}{\partial u}\mathcal{R}_{,R} \\ &= \frac{1}{2FAB}(B\mathcal{R}_{,T} - A\mathcal{R}_{,R}) \\ \mathcal{R}_{,v} &= \frac{\partial T}{\partial v}\mathcal{R}_{,T} + \frac{\partial R}{\partial v}\mathcal{R}_{,R} \\ &= \frac{1}{2GAB}(B\mathcal{R}_{,T} + A\mathcal{R}_{,R}) \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \theta_+ &= 2e^{-2\sigma}\frac{\mathcal{R}_{,v}}{\mathcal{R}} \\ &= \frac{2F}{AB\mathcal{R}}(B\mathcal{R}_{,T} + A\mathcal{R}_{,R}) \\ \theta_- &= 2e^{-2\sigma}\frac{\mathcal{R}_{,u}}{\mathcal{R}} \\ &= \frac{2G}{AB\mathcal{R}}(B\mathcal{R}_{,T} - A\mathcal{R}_{,R}) \end{aligned} \tag{2.18}$$

2.3 Spherically Symmetric Thin Shells

In this section we consider the case of spherically symmetric thin shell of a spherical symmetric star with finite thickness, which is made of a dust cloud in the background of dark energy. Let's divide the spaces into three different regions: Σ and $\Omega^{\pm}$, where Σ denotes the surface of the star, $\Omega^{\pm}$ denotes the inside and outside of the shell, where + means outside and - means inside. In this section we'll only study the properties of the Ω^{+} case, the Ω^{-} situation will be discussed in section 2.3.4. Let's use ds_{+} to denote the metric outside the dust cloud, and in general it can be cast in the form:

$$ds_{+}^{2} = A^{2}(T,R)dT^{2} - B^{2}(T,R)[d^{2}R + R^{2}(d^{2}\theta + \sin^{2}\theta d\phi^{2})] \tag{2.19}$$

where $x^{+\mu} \equiv \{T, R, \theta, \phi\}$ denotes the coordinates outside the collapsing of the dust cloud. The surface of the cloud can be expressed in the $x^{+\mu}$ coordinates as:

$$\Sigma_{+} : R = R_{0}(T) \tag{2.20}$$

So substituting $R = R_0(T)$ into the metric ds_{+}^{2} we find,

$$ds_{+}^{2}|_{R=R_{0}(T)} = (A^{2} - B^{2}\dot{R}_{0}^{2})dT^{2} - B^{2}R_{0}^{2}(d^{2}\theta + \sin^{2}\theta d^{2}\phi) \tag{2.21}$$

where $\dot{R}_0 = \frac{dR_0(T)}{dT}$.

This equals to the metric of the general space $ds^{2} = d\tau^{2} - R^{2}(\tau)(d^{2}\theta + \sin^{2}\theta d\phi^{2})$, so we could find the following relations

$$\begin{aligned} [A^{2}(T,R_{0}) - B^{2}(T,R_{0})\dot{R}_{0}^{2}(T)]^{\frac{1}{2}}dT &= d\tau \\ R(\tau) &= B(R_{0},T)R_{0}(T) \end{aligned} \tag{2.22}$$

So, on the surface

$$\Phi = R - R_{0}(T) = 0 \tag{2.23}$$

If we write the function T in terms of the proper time τ we then have

$$R = R_{0}(T) = R_{0}(\tau), T = T(\tau), \theta = \theta, \phi = \phi \tag{2.24}$$

Then from the definition of the normal vector [Eqs.(1.86)] we know that

$$N_\alpha^+ = \frac{\partial\Phi^+}{\partial x^\alpha} = \frac{\partial(R-R_0)}{\partial x^\alpha} = \delta_\alpha^R - \frac{\partial R_0}{\partial T}\cdot\delta_\alpha^T = \delta_\alpha^R - \dot{R}_0\delta_\alpha^T \tag{2.25}$$

Using the same way to find N_β^+ we could have the following

$$N_\alpha^+ N_\beta^+ g^{\alpha\beta} = -\frac{1}{A^2B^2}(-B^2\dot{R}_0^2 + A^2) \tag{2.26}$$

Thus we find the unit normal vector is given by

$$n_\alpha^+ = \frac{N_\alpha^+}{|N^+|} = \frac{AB}{\sqrt{A^2 - B^2\dot{R}_0^2}}[\delta_\alpha^R - \dot{R}_0(T)\delta_\alpha^T] \tag{2.27}$$

Also from the definition of the extrinsic curvature [Eqs.(1.103)] we find

$$\begin{aligned} K_{\tau\tau}^+ &= -n_\nu^+(\frac{\partial^2 x^\nu}{\partial\tau^2} + \Gamma_{\mu\lambda}^\nu\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\lambda}{\partial\tau}) \\ &= -\frac{AB}{\sqrt{A^2 - B^2\dot{R}_0^2}}[\delta_\nu^R - \dot{R}_0(T)\delta_\nu^T](\frac{\partial^2 x^\nu}{\partial\tau^2} + \Gamma_{\mu\lambda}^\nu\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\lambda}{\partial\tau}) \\ &= -\frac{AB}{\sqrt{A^2 - B^2\dot{R}_0^2}}[\frac{\partial^2 R}{\partial\tau^2} + \Gamma_{\mu\lambda}^R\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\lambda}{\partial\tau} - R_0\dot{(T)}\frac{\partial^2 T}{\partial\tau^2} \\ &- \dot{R}_0(T)\Gamma_{\mu\lambda}^T\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\lambda}{\partial\tau}] \end{aligned} \tag{2.28}$$

First from the expression Eq.(2.22) we find that

$$\frac{dT}{d\tau} = (A^2 - B^2\dot{R}_0^2)^{-\frac{1}{2}} \tag{2.29}$$

Taking derivative respect to τ again we find

$$\frac{d^2T}{d\tau^2} = (A^2 - B^2\dot{R}_0^2)^{-2}[B\dot{R}^2\dot{B} + B\dot{R}^3B_{,R} + \dot{R}B^2\ddot{R}_0 - A\dot{A} - AA_{,R}\dot{R}_0] \tag{2.30}$$

where $A_{,R}$ and $B_{,R}$ mean the derivatives of A and B with respect to R. Also from the expression of $\frac{dR}{d\tau}$ we have

$$\frac{d^2R}{d\tau^2} = \dot{R}_0\frac{d^2T}{d\tau^2} + \frac{dT}{d\tau}\ddot{R}_0\frac{dT}{d\tau} \tag{2.31}$$

On the other hand

$$\Gamma_{\mu\lambda}^R\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\lambda}{\partial\tau} = \Gamma_{RR}^R(\frac{\partial R}{\partial\tau})^2 + \Gamma_{TT}^R(\frac{\partial T}{\partial\tau})^2 + 2\Gamma_{RT}^R\frac{\partial R}{\partial\tau}\frac{\partial T}{\partial\tau} \tag{2.32}$$

where

$$\Gamma^R_{RR} = \frac{1}{B}B_{,R}, \qquad \Gamma^R_{TT} = \frac{A}{B^2}A_{,R}, \qquad \Gamma^R_{RT} = \frac{\dot{B}}{B}.$$

Thus from Eq.(2.32) we find

$$\Gamma^R_{\mu\lambda}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\lambda}{\partial \tau} = (\frac{B_{,R}}{B}\dot{R}^2 + \frac{AA_{,R}}{B^2} + 2\frac{\dot{B}}{B}\dot{R}_0)(A^2 - B^2\dot{R}_0^2)^{-1} \tag{2.33}$$

Also we have

$$\Gamma^T_{\mu\lambda}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\lambda}{\partial \tau} = (\frac{B\dot{B}}{A^2}\dot{R}_0^2 + \frac{\dot{A}}{A} + \frac{2}{A}\frac{\partial A}{\partial R}\dot{R}_0)(A^2 - B^2\dot{R}_0^2)^{-1} \tag{2.34}$$

using the relations

$$\Gamma^T_{RR} = \frac{B\dot{B}}{A^2}, \qquad \Gamma^T_{TT} = \frac{\dot{A}}{A}, \qquad \Gamma^T_{RT} = \frac{1}{A}\frac{\partial A}{\partial R}.$$

So from Eq.(2.28) we obtain

$$K^+_{\tau\tau} = (A^2-B^2\dot{R}_0^2)^{-\frac{3}{2}}[-AB\ddot{R}_0+\frac{B^2\dot{B}}{A}\dot{R}^3+(2BA_{,R}-AB_{,R})\dot{R}^2+(B\dot{A}-2A\dot{B})\dot{R}_0-\frac{A^2A_{,R}}{B}] \tag{2.35}$$

On the other hand, from the definition of the extrinsic curverture [Eqs.(1.103)] we find

$$K^+_{\theta\theta} = -\frac{AB}{\sqrt{A^2 - B^2\dot{R}_0^2}}[\Gamma^R_{\mu\lambda}\frac{\partial x^\mu}{\partial \theta}\frac{\partial x^\lambda}{\partial \theta} - \dot{R}_0\Gamma^T_{\mu\lambda}\frac{\partial x^\mu}{\partial \theta}\frac{\partial x^\lambda}{\partial \theta}] \tag{2.36}$$

Also we have the value of $\Gamma^R_{\mu\lambda}\frac{\partial x^\mu}{\partial \theta}\frac{\partial x^\lambda}{\partial \theta}$ and $\Gamma^T_{\mu\lambda}\frac{\partial x^\mu}{\partial \theta}\frac{\partial x^\lambda}{\partial \theta}$ using

$$\begin{aligned}\Gamma^R_{\theta\theta} &= -\frac{1}{B}B_{,R}R^2 - R\\ \Gamma^T_{\theta\theta} &= \frac{B\dot{B}}{A^2}R^2\end{aligned}$$

Then the extrinsic curvature of $\theta\theta$ component is

$$K^+_{\theta\theta} = (A^2 - B^2\dot{R}_0^2)^{-\frac{1}{2}}(AB_{,R}R_0^2 + ABR_0 + \dot{R}_0R_0^2\frac{B^2\dot{B}}{A}) \tag{2.37}$$

The $\phi\phi$ component of the extrinsic curverture can be found from the properties of spherical symmetry

$$K^+_{\phi\phi} = \sin^2\theta \cdot K^+_{\theta\theta} \tag{2.38}$$

So that

$$K^{+}_{\phi\phi} = (A^2 - B^2\dot{R}_0^2)^{-\frac{1}{2}}(AB_{,R}R^2 + ABR + \frac{1}{A}B^2\dot{B}R^2)\sin^2\theta \tag{2.39}$$

2.4 McVittie Solution

2.4.1 The General Properties of McVittie Solution

In 1933, McVittie [24] showed how to embed the Schwarzshild field of a massive particle in the cosmological background given by the Robertson-Walker line element. His solutions can be written [17] (the units in which the velocity of light *c=1* and the gravitational constant *G=1*)

$$ds^2 = -(\frac{1-\frac{m}{2w(R)}}{1+\frac{m}{2w(R)}})^2 dT^2 + e^{\beta}(1+\frac{m}{2w(R)})^4\{dR^2 + h^2(R)(d\theta^2 + \sin^2\theta d\phi^2)\} \tag{2.40}$$

where

$$m = m(T), \beta = \beta(T), \dot{\beta} = -2\frac{\dot{m}}{m} \tag{2.41}$$

and a dot indicates partial derivative with respect to T. The functions *h(R),w(R)* depend on a choice of *k(=-1,0,+1)*, the Riemannian curvature of the surfaces of homogeneity *T=const* in the background Robertson-Walker universe;

$$h(R) = \begin{cases} \sinh R, & k=-1 \\ R, & k=0 \\ \sin R, & k=+1 \end{cases} \tag{2.42}$$

and

$$w(R) = \begin{cases} 2\sinh\frac{R}{2}, & k=-1 \\ R, & k=0 \\ 2\sin\frac{R}{2}, & k=+1 \end{cases} \tag{2.43}$$

In this thesis, we shall consider only the case $k=0$.

When $k=0$, this spacetime is McVittie's solution [24] of Einstein's equation, for which the line element may be written as

$$ds^2 = -(\frac{1-\frac{m}{2u}}{1+\frac{m}{2u}})^2 dt^2 + e^{\beta(T)}(1+\frac{m}{2u})^4(dr^2 + r^2 d\Omega^2) \tag{2.44}$$

where $u = Re^{\frac{\beta}{2}}$. It is convenient for our purposes to introduce a metric radial coordinate rather than isotropic coordinates. This has the advantage of being a covariantly defined geometric object. Thus we define

$$\mathcal{R} = Re^{\frac{\beta}{2}}(1+\frac{m}{2u})^2 = u(1+\frac{m}{2u})^2 \tag{2.45}$$

The resulting coordinate transformation $(R,T) \rightarrow (\mathcal{R},\mathrm{T})$ is a diffeomorphis only if we restrict the range of u to either $(0,\frac{m}{2})$ or $(\frac{m}{2},+\infty)$. Notice that each interval is diffeomorphic via (2.45) to $\mathcal{R} \in (2m,+\infty)$, so that McVittie's solution does not cover the region inside the Schwarzschild radius, $\mathcal{R} \leqslant 2m$. With $u \in (\frac{m}{2},+\infty)$, the transformation (2.45) puts the line element (2.44) into the form

$$ds^2 = -(1-\frac{2m}{\mathcal{R}}-\frac{1}{4}\dot{\beta}^2\mathcal{R}^2)dT^2-\dot{\beta}\mathcal{R}(1-\frac{2m}{\mathcal{R}})^{-\frac{1}{2}}d\mathcal{R}dT+(1-\frac{2m}{\mathcal{R}})^{-1}d\mathcal{R}^2+\mathcal{R}^2dw^2 \tag{2.46}$$

while for $u \in (0,\frac{m}{2})$ we have

$$ds^2 = -(1-\frac{2m}{\mathcal{R}}-\frac{1}{4}\dot{\beta}^2\mathcal{R}^2)dT^2+\dot{\beta}\mathcal{R}(1-\frac{2m}{\mathcal{R}})^{-\frac{1}{2}}d\mathcal{R}dT+(1-\frac{2m}{\mathcal{R}})^{-1}d\mathcal{R}^2+\mathcal{R}^2dw^2 \tag{2.47}$$

These are related by the exchange $\beta(T) \rightarrow -\beta(T)$. The region $u > \frac{m}{2}$ is the region $\mathcal{R} > 2m$ of the spacetime representing a point mass embedded in a spatially flat RW universe with scale factor $e^{\frac{\beta}{2}}$, and the region $u < \frac{m}{2}$ is the region $\mathcal{R} > 2m$ of the spacetime representing a point mass embedded in a spatially flat RW universe with scale factor $e^{-\frac{\beta}{2}}$. There is a scalar curvature singularity at $u = \frac{m}{2}$, i.e. at $\mathcal{R} = 2m$, which is a strong curvature singularity [38], and prevents any extension into the region $R < 2m$.

For the remainder of this section, we focus on the spacetime with line element (2.46). The energy density and isotropic pressure calculated via Einstein's equation satisfy

$$8\pi\rho = \frac{3}{4}\dot{\beta}^2, 8\pi p = -\frac{3}{4}\dot{\beta}^2 - \ddot{\beta}(1-2m\mathcal{R}^{-1})^{-\frac{1}{2}} \tag{2.48}$$

We can immediately see an intriguing aspect of this spacetime: there is an intrinsic curvature singularity at the gravitational radius, $\mathcal{R} = 2m$. The energy density and pressure of the RW background are found by setting $m = 0$ in (2.48); they are given by

$$8\pi\rho_0 = \frac{3}{4}\dot{\beta}^2, 8\pi p_0 = -\frac{3}{4}\dot{\beta}^2 - \ddot{\beta} \tag{2.49}$$

The expansion of the fluid flow lines for the spacetime (2.46) is the same as that of the RW background, and is given by

$$\theta = \frac{3}{2}\dot{\beta}(T) \tag{2.50}$$

2.4.2 The Extrinsic Curvature of the Exterior

From the Eq.(2.19) we know that the line element of the exterior can be written as

$$ds_+^2 = A^2(T,R)dT^2 - B^2(T,R)[d^2R + R^2(d^2\theta + \sin^2\theta d\phi^2)] \tag{2.51}$$

and from the McVittie solution Eq.(2.40) the metric is

$$ds^2 = (\frac{1 - \frac{m}{2w(R)}}{1 + \frac{m}{2w(R)}})^2 dT^2 - e^{\beta}(1 + \frac{m}{2w(R)})^4\{dR^2 + h^2(R)(d\theta^2 + \sin^2\theta d\phi^2)\} \tag{2.52}$$

So by comparing the above two line elements we find that

$$\begin{aligned} A(T,R) &= \frac{1 - \frac{m(T)}{2w(R)}}{1 + \frac{m(T)}{2w(R)}} \\ B(T,R) &= e^{\frac{\beta}{2}}(1 + \frac{m(T)}{2w(R)})^2 \end{aligned} \tag{2.53}$$

and by solving the differential equation of β in Eq.(2.41) we have

$$B(T,R) = \frac{m_0}{m(T)}(1 + \frac{m(T)}{2w(R)})^2 \tag{2.54}$$

where m_0 is the initial value of m.

Having functions A and B written in terms of the function m and w we can rewrite the extrinsic curvature of the exterior in terms of m and w also. Then, combing the extrinsic curverture of Eqs.(2.35), (2.37) and the expression of the functions A and B (Eqs.(2.53)) we can calculate the extrinsic curverture of the exterior. We will leave the exact form of them in Chapter 3 in which we shall discuss how to solve the junction conditions by using numerical method.

2.4.3 The Extrinsic Curvature of the Interior

To simplify the problem, we assume that the spacetime inside the star is homogeneous and isotropic, similar to the Oppenheimer-Synyder (OS) model [26], historically the first model for gravitational collapse. Then the spacetime inside the star is described by the metric

$$ds_-^2 = dt^2 - a^2(t)(dr^2 + r^2 d\Omega^2) \tag{2.55}$$

where *a(t)* is an arbitrary function of t only.

Compare this with the metric (Eq.(2.19)), we find that

$$A(t, r) = 1, B(t, r) = a(t) \tag{2.56}$$

So to calculate the extrinsic curverture we just need to replace the function A and B according to Eq.(2.56) in the explicit form of extrinsic curverture (Eqs.(2.35), (2.37)).

Then, we find

$$\begin{aligned} K^-_{\tau\tau} &= (1 - a^2\dot{r}^2)^{-\frac{3}{2}}(-a\ddot{r} + a^2\dot{a}\dot{r}^3 - 2\dot{a}\dot{r}) \\ K^-_{\theta\theta} &= (1 - a^2\dot{r}^2)^{-\frac{1}{2}}(ar + \dot{r}r^2a^2\dot{a}) \end{aligned} \tag{2.57}$$

And due to the property of spherical symmetry we automatically have

$$K^-_{\phi\phi} = \sin^2\theta K^-_{\theta\theta} = (1 - a^2\dot{r}^2)^{-\frac{1}{2}}(ar^2\theta + \dot{r}r^2a^2\dot{a})\sin^2\theta \tag{2.58}$$

2.4.4 Dynamical of the Thin Shells

In order to connect the interior of the star with the outside of the star we need some junction conditions.

First, the extrinsic curverture of interior and the exterior should be equal to each other, that is

$$K^-_{\tau\tau} = K^+_{\tau\tau}$$

$$
\begin{aligned}
K^-_{\theta\theta} &= K^+_{\theta\theta} \\
K^-_{\phi\phi} &= K^+_{\phi\phi}
\end{aligned}
\tag{2.59}
$$

Because $K_{\theta\theta}$ and $K_{\phi\phi}$ are linearly dependent ($K_{\theta\theta} = \sin^2\theta K_{\phi\phi}$), we actually only have two equations that are linearly independent.

Second, the metric of the outside and inside should be equal to each other around the junction surface, that is

$$
ds^2_+\big|_{\Sigma_+} = ds^2_-\big|_{\Sigma_-} \tag{2.60}
$$

And from Eqs.(2.51), (2.55), we have

$$
A^2(T,R)dT^2 - B^2(T,R)[d^2R + R^2(d^2\theta + \sin^2\theta d\phi^2)]\big|_{\Sigma_+} = dt^2 - a^2(t)(dr^2 + r^2 d\Omega^2)\big|_{\Sigma_-} \tag{2.61}
$$

Rearrange the equation above we have

$$
(A^2 - B^2\dot{R}^2)dT^2 - B^2R^2d\Omega^2 = (1 - a^2\dot{r}^2)dt^2 - a^2r^2d\Omega^2 \tag{2.62}
$$

Comparing term by term we find

$$
\begin{aligned}
(A^2 - B^2\dot{R}^2)^{\frac{1}{2}}dT &= (1 - a^2\dot{r}^2)^{\frac{1}{2}}dt \\
BR &= ar
\end{aligned}
\tag{2.63}
$$

So totally we have four equations (Eqs.(2.59), (2.63)) to determine the dynamical situation of our problem. From the McVittie solution we know the functions A and B can be written in terms of m and w, where w is a function of R (Eq.(2.43)). So actually we total have four unknowns: $m(t), T(t), R(t)$ and $r(t)$, all of which can be considered as functions of t only. Then we have all the equations we need to solve the problem.

2.5 Numerical Solutions

2.5.1 The Equation $BR_0 = ar_0$

For the case $k = 0$, from Eqs.(2.43) we find that $w(R) = R$. So the function B (Eqs.(2.53)) is given by

$$B(T,R) = e^{\frac{\beta}{2}}(1 + \frac{m(T)}{2R})^2 \tag{2.64}$$

Also from the equation of β (Eq.(2.41)), $\dot{\beta} = -2\frac{\dot{m}}{m}$, we find $e^{\frac{\beta}{2}} = \frac{m_0}{m}$, where m_0 is the initial value of m. Then we have

$$B = \frac{m_0}{m}(1 + \frac{m}{2R})^2 \tag{2.65}$$

From Eqs.(2.63) we know $BR = ar$, this equation means the variables R and r are already on the surface, so we could rewrite it to be

$$BR_0 = ar_0 \tag{2.66}$$

where both of R_0 and r_0 are a function of t.

To find a numerical solution we choose to adopt the Runge-Kutta method(see section 2.5.4), so first we have to transform Eqs.(2.59), (2.63) into the second order ordinary differential equations. In order to achieve this goal we take derivative of Eq.(2.66) with respect to t twice, that is

$$\frac{d^2}{dt^2}(BR_0) = \frac{d^2}{dt^2}(ar_0) \tag{2.67}$$

then we find

$$\begin{aligned} &\left(\frac{m_0}{m} - \frac{m_0 m}{4R_0^2}\right)\ddot{R}_0 + \left(\frac{m_0}{4R_0} - \frac{m_0 R_0}{m^2}\right)\ddot{m} + \frac{m_0 m \dot{R}_0^2}{2R_0^3} + \frac{2m_0\dot{m}^2 R_0}{m^3} \\ &- \frac{2\dot{m}\dot{R}_0 m_0}{m^2} - \frac{m_0\dot{m}\dot{R}_0}{2R_0^2} - 2\dot{a}\dot{r}_0 - r_0\ddot{a} - a\ddot{r}_0 = 0 \end{aligned} \tag{2.68}$$

For the sake of convenience, when doing computer coding we put the above equation in the form

$$P_1\ddot{m} + P_2\ddot{R}_0 + P_3\ddot{r}_0 + C_1 = 0 \tag{2.69}$$

where

$$
\begin{aligned}
P_1 &= \frac{m_0}{4R_0} - \frac{m_0 R_0}{m^2} \\
P_2 &= \frac{m_0}{m} - \frac{m_0 m}{4R_0^2} \\
P_3 &= -a \\
C_1 &= \frac{m_0 m \dot{R}_0^2}{2R_0^3} + \frac{2m_0 \dot{m}^2 R_0}{m^3} - \frac{2\dot{m}\dot{R}_0 m_0}{m^2} - \frac{m_0 \dot{m} \dot{R}_0}{2R_0^2} - 2\dot{a}\dot{r} - r\ddot{a}
\end{aligned} \tag{2.70}
$$

For convenience, from now on we define $(\dot{\ })$ means time derivative respect to t, while $(\ )'$ means time derivative with respect to T.

2.5.2 *The Equation $K_{\tau\tau}^{+} = K_{\tau\tau}^{-}$*

From Eqs.(2.59) we know $K_{\tau\tau}^{+} = K_{\tau\tau}^{-}$, but the equal sign only holds after we take the limit on Σ^{+} and Σ^{-}. So actually we should write $K_{\tau\tau}^{+} = K_{\tau\tau}^{-}$ to be $\lim\limits_{R\to R_0, r\to r_0} K_{\tau\tau}^{+} = \lim\limits_{R\to R_0, r\to r_0} K_{\tau\tau}^{-}$.

Writing the above equation explicitly we get

$$
\begin{aligned}
(& A^2 - B^2 R_0^{'2})^{-\frac{3}{2}}[-ABR_0^{''} + \frac{B^2 \bar{B}^{'}}{A} R_0^{'3} + (2B\bar{A}_{,R} - A\bar{B}_{,R})R_0^{'2} + (B\bar{A}^{'} - 2A\bar{B}^{'})R_0^{'} - \frac{A^2 \bar{A}_{,R}}{B}] \\
= & (1 - a^2 \dot{r}_0^2)^{-\frac{3}{2}}(-a\ddot{r}_0 + a^2 \dot{a} \dot{r}_0^3 - 2\dot{a}\dot{r}_0)
\end{aligned} \tag{2.71}
$$

where $\overline{()}$ means $\lim\limits_{R\to R_0, r\to r_0} ()$.

Then we find

$$
R_0^{'} = \frac{dR_0(T)}{dT} = \frac{dR_0(T)}{dt} \cdot \frac{dt}{dT} = \dot{R}_0 \cdot \dot{T}^{-1} \tag{2.72}
$$

Taking derivative respect to T again, we have

$$
R_0^{''} = \ddot{R}_0 \dot{T}^{-2} - \dot{T}^{-3} \dot{R}_0 \ddot{T} \tag{2.73}
$$

Also from Eqs.(2.63) we find on the surface

$$
\dot{T} = \frac{(1 - a^2 \dot{r}_0^2)^{\frac{1}{2}}}{(A^2 - B^2 R_0^{'2})^{\frac{1}{2}}} \tag{2.74}
$$

then substituting the above relationships into Eq.(2.71) we have

$$-AB(\ddot{R}_0\dot{T}^{-2}-\dot{T}^{-3}\dot{R}_0\ddot{T})+\Delta_1+\dot{T}^{-3}(a\ddot{r}_0-a^2\dot{a}\dot{r}_0^3+2\dot{a}\dot{r}_0)=0 \tag{2.75}$$

where

$$\Delta_1\equiv\frac{B^2\bar{B}^{'}}{A}R_0^{'3}+(2B\bar{A}_{,R}-A\bar{B}_{,R})R_0^{'2}+(B\bar{A}^{'}-2A\bar{B}^{'})R_0^{'}-\frac{A^2\bar{A}_{,R}}{B}],$$

and

$$\bar{B}^{'}=\lim_{R\to R_0(T)}\frac{d}{dT}[\frac{m_0}{m}(1+\frac{m}{2R})^2]=\frac{m_0}{4}(\frac{1}{R_0^2}-\frac{4}{m^2})m^{'}=\frac{m_0}{4}(\frac{1}{R_0^2}-\frac{4}{m^2})\dot{m}\dot{T}^{-1} \tag{2.76}$$

Similarly, we find

$$\begin{aligned}\bar{A}_{,R} &= \frac{4m}{(2R_0+m)^2}\\ \bar{B}_{,R} &= -\frac{m_0(2R_0+m)}{2R_0^3}\\ \bar{A}^{'} &= -\frac{4R_0\dot{m}\dot{T}^{-1}}{(2R_0+m)^2}\end{aligned} \tag{2.77}$$

Now Eq.(2.75) can be written as

$$S_2\ddot{R}_0+S_3\ddot{r}_0+S_4\ddot{T}+C_2=0 \tag{2.78}$$

On the other hand, combining Eq.(2.74) and Eq.(2.72) we have

$$\dot{T}=\frac{(1-a^2\dot{r}_0^2)^{\frac{1}{2}}}{(A^2-B^2\dot{R}_0^2\dot{T}^{-2})^{\frac{1}{2}}} \tag{2.79}$$

so we can solve it for $\dot{T}$ to get

$$\dot{T}=\frac{1}{A}(1-a^2\dot{r}_0^2+B^2\dot{R}_0^2)^{\frac{1}{2}} \tag{2.80}$$

Taking derivative with respect to t again we get

$$\ddot{T}=\frac{1}{2A}(1-a^2\dot{r}_0^2+B^2\dot{R}_0^2)^{-\frac{1}{2}}(-2a\dot{a}\dot{r}_0^2-2a^2\dot{r}_0\ddot{r}_0+2B\dot{B}\dot{R}_0^2+2B^2\dot{R}_0\ddot{R}_0)-\frac{\dot{A}}{A^2}(1-a^2\dot{r}_0^2+B^2\dot{R}_0^2)^{\frac{1}{2}} \tag{2.81}$$

which can be cast in the form

$$\ddot{T} = T_1\ddot{R}_0 + T_2\ddot{r}_0 + \Delta_2 \tag{2.82}$$

the substitution of Eq.(2.82) back into Eq.(2.78) results in

$$(S_2 + S_4T_1)\ddot{R}_0 + (S_3 + S_4T_2)\ddot{r}_0 + S_4\Delta_2 + C_2 = 0 \tag{2.83}$$

which have the form

$$H_1\ddot{R}_0 + H_2\ddot{r}_0 + H_3 = 0 \tag{2.84}$$

2.5.3 The Equation $K^{+}_{\theta\theta} = K^{-}_{\theta\theta}$

Writing the equation $K^{+}_{\theta\theta} = K^{-}_{\theta\theta}$ out explicitly we find

$$A\bar{B}_{,R}R_0^2 + ABR_0 + \dot{R}_0\dot{T}^{-1}R_0^2\frac{B^2\bar{B}'}{A} = \dot{T}^{-1}(ar_0 + \dot{r}_0r_0^2a^2\dot{a}) \tag{2.85}$$

Now taking derivative respect to t on both sides we find

$$\begin{aligned}
\dot{A} \quad & \bar{B}_{,R}R_0^2 + A\bar{B}^{\cdot}_{,R}R_0^2 + 2A\bar{B}_{,R}R_0\dot{R}_0 + \dot{A}BR_0 + A\dot{B}R_0 + AB\dot{R}_0 + \ddot{R}_0\dot{T}^{-1}R_0^2\frac{B^2\bar{B}'}{A} \\
- \quad & \dot{R}_0\dot{T}^{-2}\ddot{T}R_0^2\frac{B^2\bar{B}'}{A} + 2\dot{T}^{-1}R_0\dot{R}_0^2\frac{B^2\bar{B}'}{A} + \dot{R}_0\dot{T}^{-1}R_0^2\frac{2B\dot{B}\bar{B}'}{A} + \dot{R}_0\dot{T}^{-1}R_0^2\frac{B^2\bar{B}'^{\cdot}}{A} \\
- \quad & \dot{R}_0\dot{T}^{-1}R_0^2\frac{B^2\bar{B}'\dot{A}}{A^2} = -\dot{T}^{-2}\ddot{T}(ar_0 + \dot{r}_0r_0a^2\dot{a}) + \dot{T}^{-1}(\dot{a}r_0 + a\dot{r}_0 + \ddot{r}_0r_0^2a^2\dot{a} \\
+ \quad & 2r_0\dot{r}_0^2a^2\dot{a} + 2\dot{r}_0r_0^2a\dot{a}^2 + \dot{r}_0r_0^2a^2\ddot{a})
\end{aligned} \tag{2.86}$$

where

$$\begin{aligned}
\dot{A} &= \frac{4(-R_0\dot{m} + m\dot{R}_0)}{(m+2R_0)^2} \\
\bar{B}^{\cdot}_{,R} &= \frac{m_0[-R_0(\dot{m} - 4\dot{R}_0) + 3m\dot{R}_0]}{2R_0^4} \\
\dot{B} &= \frac{m_0R_0(m^2 - 4R_0^2)\dot{m} - 3m_0m^2(m+2R_0)\dot{R}_0}{4m^2R_0^3}
\end{aligned} \tag{2.87}$$

and

$$\begin{aligned}
\bar{B}'^{\cdot} &= \frac{d}{dt}\bar{B}' = \frac{d}{dt}[\frac{m_0}{4}(\frac{1}{R_0^2} - \frac{4}{m^2})\dot{m}\dot{T}^{-1}] \\
&= \frac{m_0}{4}(-2R_0^{-3}\dot{R}_0 + 8m^{-3}\dot{m})\dot{m}\dot{T}^{-1} + \frac{m_0}{4}(R_0^{-2} - 4m^{-2})\ddot{m}\dot{T}^{-1} \\
&- \frac{m_0}{4}(R_0^{-2} - 4m^{-2})\dot{m}\dot{T}^{-2}\ddot{T}
\end{aligned} \tag{2.88}$$

which can be recast in the form

$$\bar{B}'^{\cdot} = M_1\ddot{m} + M_2(T_1\ddot{R}_0 + T_2\ddot{r}_0 + \Delta_2) + \Delta_4 \tag{2.89}$$

Combining Eq.(2.86) with Eq.(2.82) and Eq.(2.89) and after careful arrangement we can put Eq.(2.86) in the form

$$\begin{aligned} Q \quad & _5M_1\ddot{m} + (Q_2 + Q_4T_1 + Q_5M_2T_1)\ddot{R}_0 + (Q_3 + Q_4T_2 + Q_5M_2T_2)\ddot{r}_0 + Q_4\Delta_2 + Q_5\Delta_4 \\ + \quad & Q_5M_2\Delta_2 + C3 = 0 \end{aligned} \tag{2.90}$$

Rewriting the above equation, we have

$$O_1\ddot{m} + O_2\ddot{R}_0 + O_3\ddot{r}_0 + O_4 = 0 \tag{2.91}$$

2.5.4 The Runge-Kutta Method

Now we have three useful equations (Eqs.(2.69), (2.84), (2.91)), they are

$$\begin{aligned} P_1\ddot{m} + P_2\ddot{R}_0 + P_3\ddot{r}_0 + C_1 &= 0 \\ H_1\ddot{R}_0 + H_2\ddot{r}_0 + H_3 &= 0 \\ O_1\ddot{m} + O_2\ddot{R}_0 + O_3\ddot{r}_0 + O_4 &= 0 \end{aligned} \tag{2.92}$$

To use the Runge-Kutta method we have to find the explicit forms of $\ddot{m}, \ddot{R}_0, \ddot{r}_0$, so we rewrite these equations in the form,

$$\begin{pmatrix} P_1 & P_2 & P_3 \\ 0 & H_1 & H_2 \\ O_1 & O_2 & O_3 \end{pmatrix} \cdot \begin{pmatrix} \ddot{m} \\ \ddot{R}_0 \\ \ddot{r}_0 \end{pmatrix} = \begin{pmatrix} -C_1 \\ -H_3 \\ -O_4 \end{pmatrix} \tag{2.93}$$

Then, the solution of $\ddot{m}, \ddot{R}_0, \ddot{r}_0$ is

$$\begin{pmatrix} \ddot{m} \\ \ddot{R}_0 \\ \ddot{r}_0 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 & P_3 \\ 0 & H_1 & H_2 \\ O_1 & O_2 & O_3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -C_1 \\ -H_3 \\ -O_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{2.94}$$

where $()^{-1}$ means the inverse matrix. Now we define

$$\begin{cases} \dot{m} = W \\ \dot{R}_0 = G \\ \dot{r}_0 = F \end{cases} \tag{2.95}$$

then we have a set of first order ordinary differential equations, they are

$$\begin{cases} \dot{m} = W \\ \dot{W} = f_1 \\ \dot{R}_0 = G \\ \dot{G} = f_2 \\ \dot{r}_0 = F \\ \dot{F} = f_3 \end{cases} \tag{2.96}$$

The next step is to find the initial conditions for the above equations.

The function a is a given function in our research, so the initial value of a is know to be a_0. From Eq.(2.66) we know

$$\frac{m_0}{m}(1+\frac{m}{2R_0})^2 R_0 = a_0 r_0 \tag{2.97}$$

so we can solve it for $m(t)$, which has two solutions,

$$\begin{aligned} m_1 &= \frac{2R_0}{m_0}\{a_0 r_0 - m_0 + [(m_0 - a_0 r_0)^2 - m_0^2]^{\frac{1}{2}}\} \\ m_2 &= \frac{2R_0}{m_0}\{a_0 r_0 - m_0 - [(m_0 - a_0 r_0)^2 - m_0^2]^{\frac{1}{2}}\} \end{aligned} \tag{2.98}$$

On the other hand, the function A in the McVittie's solution must be positive, so from the expression of A ($A = \frac{1-\frac{m}{R_0}}{1+\frac{m}{R_0}}$) we know

$$m < 2R_0 \tag{2.99}$$

Now go back to the first solution in Eq.(2.98), combining it with the above condition we have

$$\frac{2R_0}{m_0}\{a_0 r_0 - m_0 + [(m_0 - a_0 r_0)^2 - m_0^2]^{\frac{1}{2}}\} < 2R_0. \tag{2.100}$$

Solving the equation we find

$$m > \frac{a_0 r_0}{2} \tag{2.101}$$

Also from section 2.4.1 we know that $\mathcal{R} > 2m$, where $\mathcal{R} = a_0 r_0$ on the hypersurface, so we can have

$$m < \frac{a_0 r_0}{2} \tag{2.102}$$

which obviously contradicts with the solution (2.101). Then only the second solution in Eqs.(2.98) is correct. Combining it with the conditions Eqs.(2.99) and (2.101) we can find that the value of m has a range

$$0 < m < \frac{a_0 r_0}{2} \tag{2.103}$$

So we shall choose the initial value of m that falls into this range.

Now we already know the solution of m is

$$m = \frac{2R_0}{m_0}\{a_0 r_0 - m_0 - [(m_0 - a_0 r_0)^2 - m_0^2]^{\frac{1}{2}}\} \tag{2.104}$$

Taking derivatives of the above equation with respect to t we can find the initial value of $\dot{m}$. The initial value of $R_0, \dot{R}_0, r_0, \dot{r}_0$ are *20, -0.1, 5, 0.*

Now we have everything to put into the Runge-Kutta method, the program of it is in appendix.

To make sure the code is correct we used the fourth-order Runge-Kutta method and the second-order Runge-Kutta method, the result curves are just the same. Also, I double checked the code with the result of the variable "m". We used the equation (2.104) to calculate the the value of m and on the other hand we can also get the result of m from the result value of the Runge-Kutta method, after comparing the two result we find they are exactly the same, so we are sure our code is correct.

2.6 Discussion and Conclusions

In this section we'll discuss the result of the calculation of numerical method of different cases, but first we need to introduce some quantities that are important to study the formation of black holes. The first is the expansions (Eqs.(2.18)), from the definition 2.2 we know that if $\theta_+\theta_- > 0$ then a black hole is formed. One can show that

$$\begin{aligned}\theta_+ &= (1-\frac{2m_0}{\mathcal{R}})^{\frac{1}{2}}[\dot{\beta}+\frac{2}{\mathcal{R}}(1-\frac{2m_0}{\mathcal{R}})^{\frac{1}{2}}] \\ \theta_- &= (1-\frac{2m_0}{\mathcal{R}})^{\frac{1}{2}}[\dot{\beta}-\frac{2}{\mathcal{R}}(1-\frac{2m_0}{\mathcal{R}})^{\frac{1}{2}}] \end{aligned} \tag{2.105}$$

where $\mathcal{R} = B(T,R)R$, the geometry radius of the black hole.

The second is that, we can use energy conditions to determine the state of a black hole. They are

$$\begin{aligned} C_1 &= \rho_0 + p_0 \\ C_2 &= \rho_0 - p_0 \\ C_3 &= \rho_0 + 3p_0. \end{aligned} \tag{2.106}$$

From Eqs.(2.48) we find that $8\pi\rho_0 = \frac{3}{4}\dot{\beta}^2, 8\pi\rho_0 = -\frac{3}{4}\dot{\beta}^2 - \ddot{\beta}$. Neglecting the constant 8π we have

$$\begin{aligned} C_1 &= -\ddot{\beta} \\ C_2 &= \frac{1}{2}(3\dot{\beta}^2 + 2\ddot{\beta}) \\ C_3 &= -\frac{3}{2}(\dot{\beta}^2 + 2\ddot{\beta}). \end{aligned} \tag{2.107}$$

The last important quantity is the total mass, and it's defined by

$$M(T) = \frac{1}{2}\mathcal{R}(1 + \nabla_\alpha \mathcal{R}\nabla^\alpha \mathcal{R}). \tag{2.108}$$

One can show that

$$\nabla_\alpha \mathcal{R}\nabla^\alpha \mathcal{R} = \frac{\mathcal{R}^2}{4}[\dot{\beta}^2 - \frac{4}{\mathcal{R}^2}(1-\frac{2m_0}{\mathcal{R}})], \tag{2.109}$$

so the total mass as a function of T is

$$M(T) = \frac{1}{2}\mathcal{R}\{1 + \frac{\mathcal{R}^2}{4}[\dot{\beta}^2 - \frac{4}{\mathcal{R}^2}(1 - \frac{2m_0}{\mathcal{R}})]\}, \tag{2.110}$$

where $\mathcal{R}$ has the same definition of Eq.(2.105).

A. Gravitational Collapse of a Dust Cloud

The figures for this situation are shown below as figures 2.1, 2.2 and 2.3. In this case the function a is chosen to be $a(t) = a_0(t_0 - t)^{\frac{2}{3}}$, and other initial conditions are

- $r_0 = 5, \dot{r_0} = 0, R_0 = 20, \dot{R_0} = -0.1, t_0 = 10, m_0 = \frac{a_{(t_i)} r_0}{31}, a_0 = 1, t_i = -70$

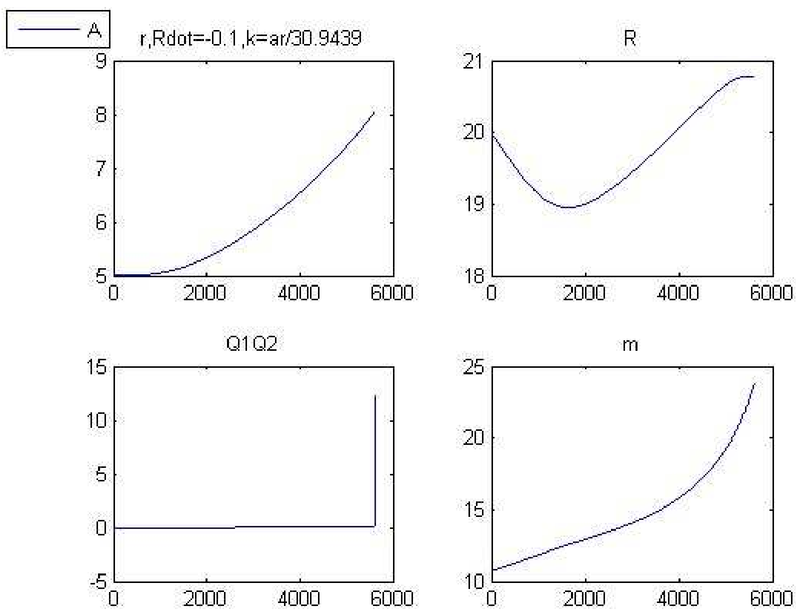

Figure 2.1: Gravitational Collapse of a dust cloud for the functions $m \equiv m(T), Q1Q2 \equiv \theta_+\theta_-$ defined, respectively, by Eqs.(2.104) and (2.105)

From the figures we can see that the geometry radius (bigR in figure 2.2) is getting smaller, and also from the prompt jump of the curve of $\theta_+\theta_-$ (Q1Q2 in figure 2.1) we can see that there is formation of black hole at that moment.

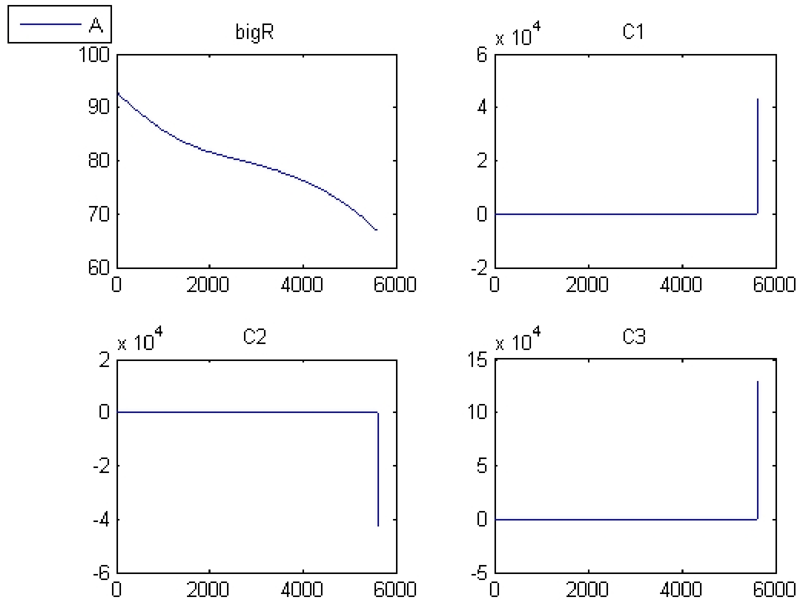

Figure 2.2: Gravitational Collapse of a dust cloud for the functions $bigR \equiv \mathcal{R}(T), C_1, C_2 and C_3$ defined, respectively, by Eqs. (2.105) and (2.107)

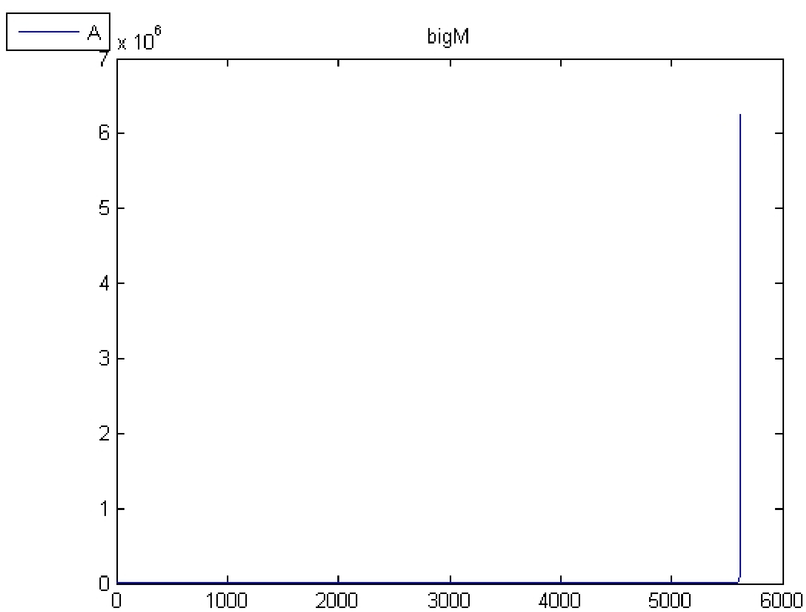

Figure 2.3: Gravitational Collapse of a dust cloud for the function $bigM \equiv M(T)$, defined by Eq.(2.110).

B. Gravitational Collapse of Dark Energy: $w > -1$

The figures for this situation are shown below as figures 2.4, 2.5 and 2.6. To study the effects of dark energy on gravitational collapse, we first consider the case where $\rho_{DM} = 0, p = w, \rho \neq 0$, where w is a non-zero constant. When $w < -\frac{1}{3}$ the strong energy condition is not satisfied [15], and the fluid is said to be made of dark energy. It can be shown that the solution in this case is given by

$$a(t) = a_0(t_0 - t)^{\frac{2}{3(1+w)}}$$

for $w > -1$.

And other initial conditions are

- $r_0 = 5, \dot{r_0} = 0, R_0 = 20, \dot{R_0} = -0.1, t_0 = 10, m_0 = \frac{a_{(t_i)} r_0}{3.852}, a_0 = 1, t_i = -70, w = -0.35$

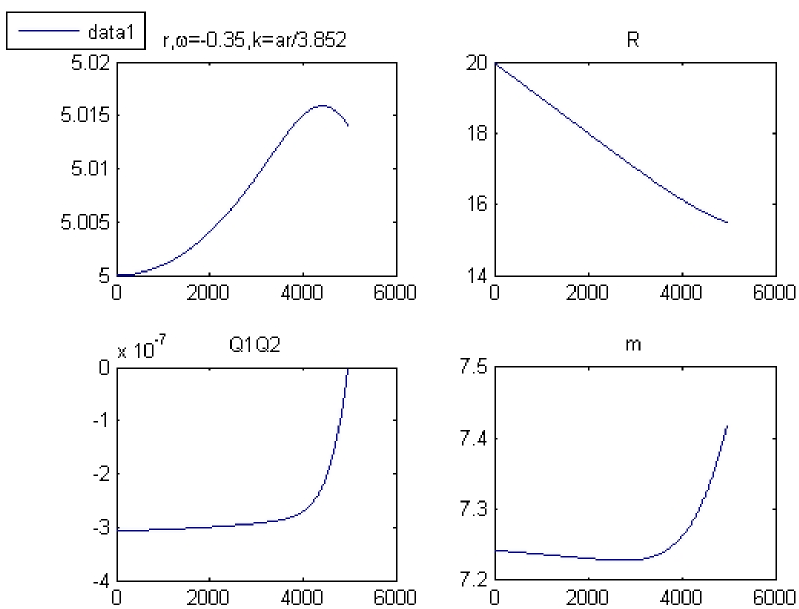

Figure 2.4: Gravitational Collapse of Dark Energy with $w > -1$ for the functions $m \equiv m(T), Q1Q2 \equiv \theta_+\theta_-$ defined, respectively, by Eqs.(2.104) and (2.105).

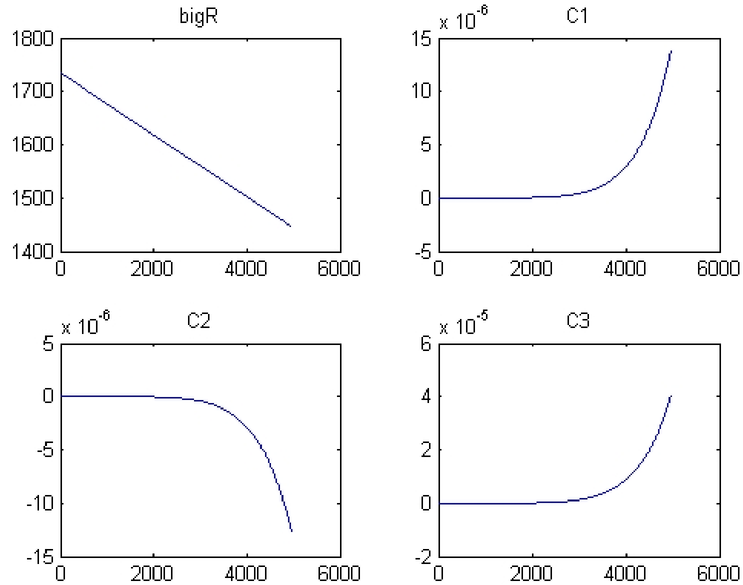

Figure 2.5: Gravitational Collapse of Dark Energy with $w > -1$ for the functions $bigR \equiv \mathcal{R}(T), C_1, C_2 and C_3$ defined, respectively, by Eqs. (2.105) and (2.107).

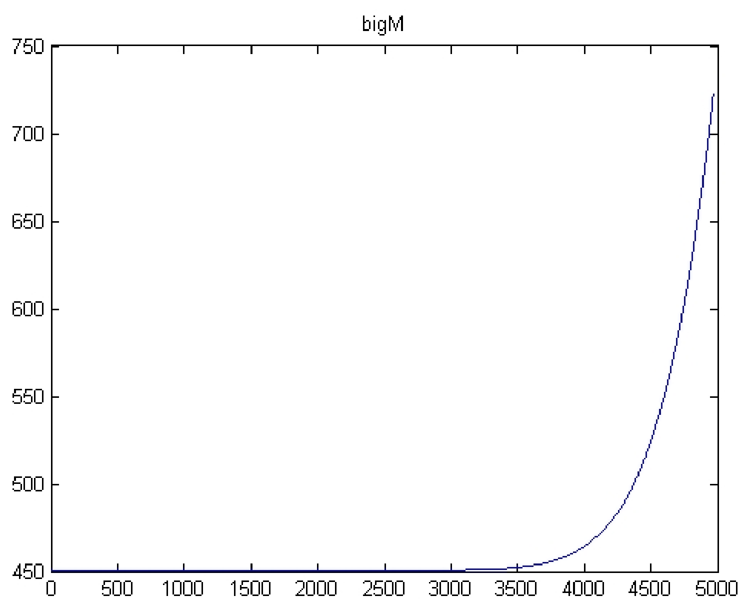

Figure 2.6: Gravitational Collapse of Dark Energy with $w > -1$ for the function $bigM \equiv M(T)$, defined by Eq.(2.110).

We can see that the geometry radius is getting smaller linearly and there should be formation of black hole because the value of $\theta_+\theta_-$ is changing from negative to positive.

C. Gravitational Collapse of Dark Energy: $w < -1$

The figures for this situation are shown below as figures 2.7, 2.8 and 2.9. In the paragraph above we discussed the case when $w > -1$, there is another case that is $w < -1$, the solution of function a when $w < -1$ is given by

$$a(t) = a_0(t - t_0)^{\frac{2}{3(1+w)}} \tag{2.111}$$

while other initial conditions are

- $r_0 = 5, \dot{r_0} = 0.5, R_0 = 20, \dot{R_0} = 0.5, t_0 = 10, m_0 = \frac{a_{(t_i)} r_0}{10}, a_0 = 1, t_i = -70, w = -1.3$

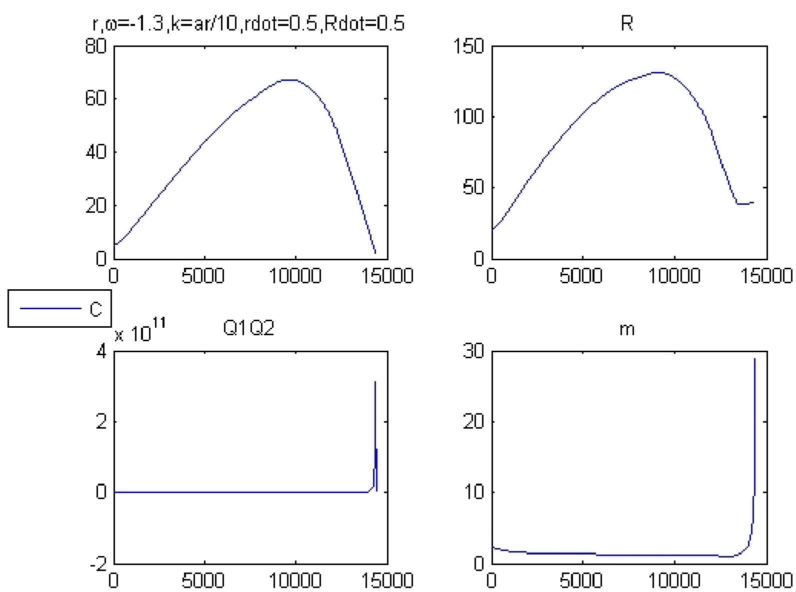

Figure 2.7: Gravitational Collapse of Dark Energy with w¡-1 for the functions $m \equiv m(T), Q1Q2 \equiv \theta_+\theta_-$ defined, respectively, by Eqs.(2.104) and (2.105).

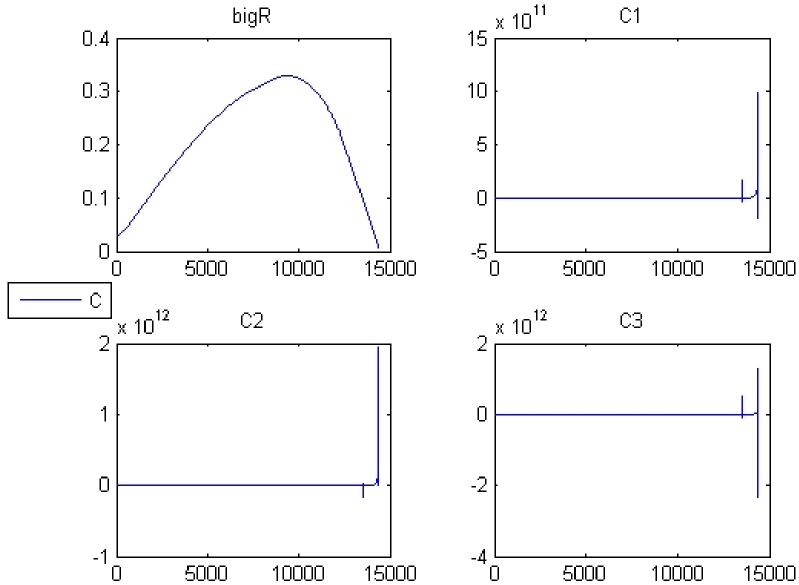

Figure 2.8: Gravitational Collapse of Dark Energy with w¡-1 for the functions $bigR \equiv \mathcal{R}(T), C_1, C_2 and C_3$ defined, respectively, by Eqs. (2.105) and (2.107).

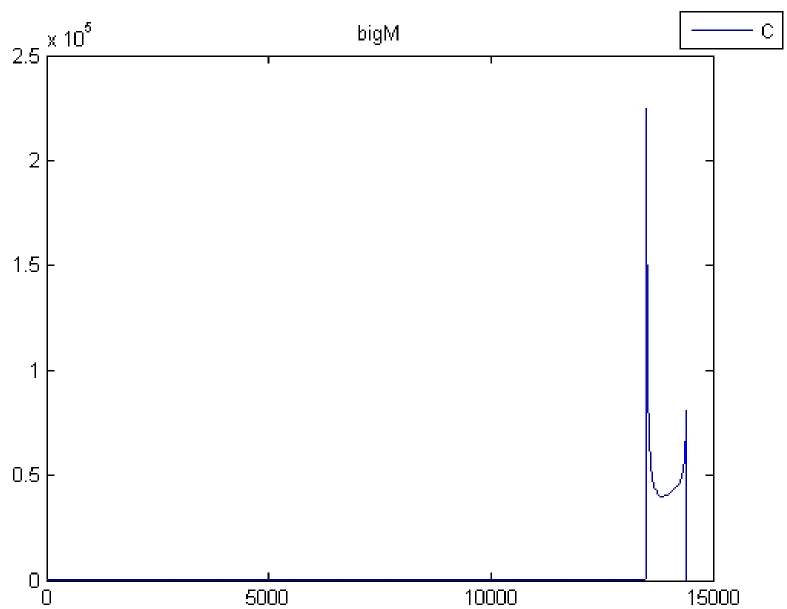

Figure 2.9: Gravitational Collapse of Dark Energy with w¡-1 for the function $bigM \equiv M(T)$, defined by Eq.(2.110).

We can find that despite of $\dot{R} > 0$, we can always find a black hole formation when $w < -1$

D. Gravitational Collapse of a Dust Cloud and Dark Energy when there is no Interaction

The figures for this situation are shown below as figures 2.10, 2.11 and 2.12. When there is no interaction if $w = -\frac{1}{2}$, the function a has the solution

$$a(t) = a_0[(t_0 - t)^2 - A^2]^{\frac{2}{3}} \tag{2.112}$$

where A is a constant, for convenience we choose it to be *1* in calculation.

Other initial conditions are

- $r_0 = 5, \dot{r_0} = 0.5, R_0 = 20, \dot{R_0} = -0.1, t_0 = 10, m_0 = \frac{a_{(t_i)} r_0}{25}, a_0 = 1, t_i = -70$

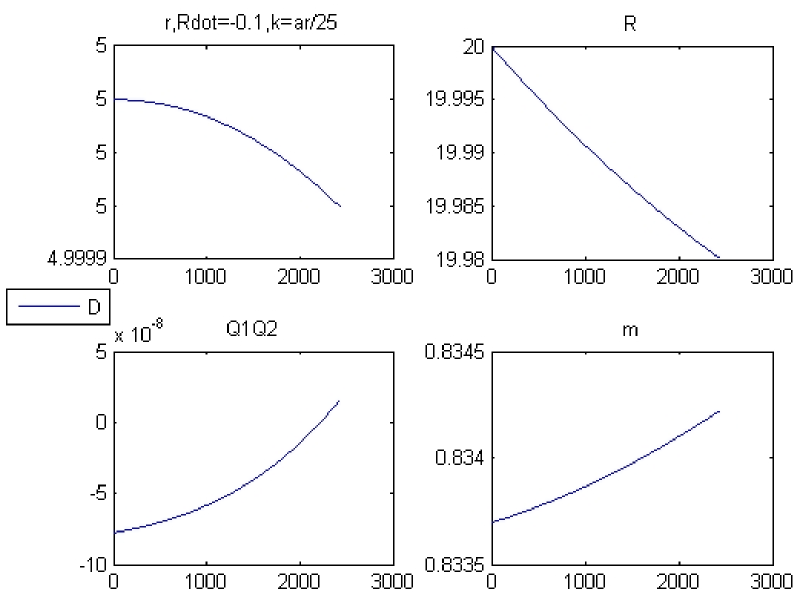

Figure 2.10: Gravitational collapse of a dust cloud and dark energy when there is no interaction for the functions $m \equiv m(T), Q1Q2 \equiv \theta_+\theta_-$ defined, respectively, by Eqs.(2.104) and (2.105).

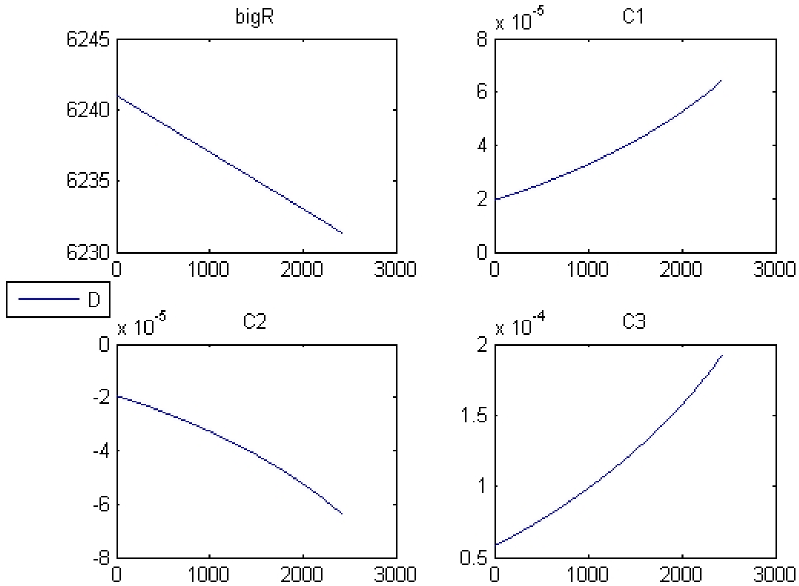

Figure 2.11: Gravitational collapse of a dust cloud and dark energy when there is no interaction for the functions $bigR \equiv \mathcal{R}(T), C_1, C_2 and C_3$ defined, respectively, by Eqs. (2.105) and (2.107).

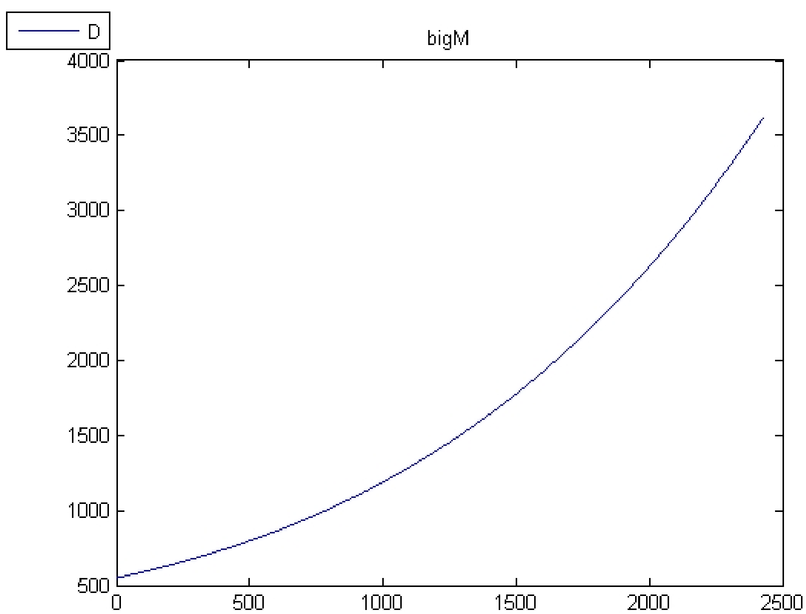

Figure 2.12: Gravitational collapse of a dust cloud and dark energy when there is no interaction for the function $bigM \equiv M(T)$, defined by Eq.(2.110).

The variable bigR means the geometric radius, so from the figure above we can see that the radius is getting smaller linearly and from the change of signs of value $\theta_+\theta_-$ we can find that a black hole is formed.

2.6.1 Conclusion

In summery, in all of the four cases:

A: Gravitational Collapse of a dust cloud

B: Gravitational Collapse of Dark Energy: $w > -1$

C: Gravitational Collapse of Dark Energy: $w < -1$

D: Gravitational Collapse of a dust cloud and dark energy when there is no interaction

No matter there is dark energy or not the black hole is always formed. This means the repulsive force result from the dark energy can not balance the gravitational force. But there is another question, there should be a minimal mass for the black hole to collapse, otherwise the repulsive force of the dark energy should stop the black hole from collapsing. This problem should in the next step of the research of this problem. And also, we can use anisotropic models of the black hole but not a sphere and add some rotation to the black hole in the next research.

CHAPTER THREE

The Numerical Program

The following is the code list of our program whose function is to solve the differential equation group of Eqs.(2.59) and Eqs.(2.63).

```
%the main program=====================
clear all
r(1)=5; %set the initial value of
min = 20; %set the minimum value of
max = 100; %the maximum value of
h = 0.0001; %set the date interval of
t = min:h:max; %initialize the array of
R(1) = 20; %set the initial value of
omega=-1.3; %set the value of
a0 = aa(t(1),omega); %call the function  to
%find the initial value of
kk=10;
k=a0*r/kk; %calculate the value of
h1=((k-a0*r(1))^2-k^2)^(1/2);
m(1)=2*R(1)/k*(a0*r(1)-k-h1); %find the initial value of
G(1)=0.5; %initial value of
F(1)=0.5; %initial value of
a0dott=aadott(t(1),omega); %initial value of
h2=4*m(1)^2*R(1)^2*(a0dott*r(1)+a0*F(1));
h3=k*(m(1)^2-4*R(1)^2);
W(1)=(h2/h3+m(1)*G(1))/R(1);
```

```
%initial value of  betadott1(1)=\
%betadott(k,m(1),R(1),G(1),W(1),F(1),aa(t(1),omega));
%the first %value point of
%the following loop is the core part of the Runge-Kutta method
for i=1:(max-min)/h mm=m(i);
RR=R(i);
rr=r(i);
WW=W(i);
GG=G(i);
FF=F(i);
tt=t(i);
% mprime is for check use,because there are two methods to find the value of ,
%one can directly differentiate the function  to find  or he can also find
%it as an output of the Runge-Kutta method. So after the calculation of the
%Runge-Kutta method he can compare the two result to see if the numerical
%solution is correct.
mprime(i)=2*RR/k*(aa(t(i),omega)*r(i)-k-((k-aa(t(i),omega)\
*r(i))^2-k^2)^(1/2));
%fourth order Runge-Kutta method
[k11 k13 k15]=solution(k,mm,RR,GG,WW,FF,aa(tt,omega),aadott(tt,omega),\
aadoubledott(tt,omega),rr);
k12=WW; k14=GG; k16=FF;
[k21 k23 k25]=solution(k,mm+h/2*k12,RR+h/2*k14,GG+h/2*k13,WW+h/2*k11,\
FF+h/2*k15,aa(tt+h/2,omega),aadott(tt+h/2,omega),aadoubledott(tt+h/2,omega)\
,rr+h/2*k16);
k22=WW+h/2*k11; k24=GG+h/2*k13; k26=FF+h/2*k15;
[k31 k33 k35]=solution(k,mm+h/2*k22,RR+h/2*k24,GG+h/2*k23,WW+h/2*k21,\
```

```
FF+h/2*k25,aa(tt+h/2,omega),aadott(tt+h/2,omega),aadoubledott(tt+h/2,omega)\
,rr+h/2*k26);
k32=WW+h/2*k21; k34=GG+h/2*k23; k36=FF+h/2*k25;
[k41 k43 k45]=solution(k,mm+h*k32,RR+h*k34,GG+h*k33,WW+h*k31,\
FF+h*k35,aa(tt+h,omega),aadott(tt+h,omega),aadoubledott(tt+h,omega)\
,rr+h*k36);
k42=WW+h*k31; k44=GG+h*k33; k46=FF+h*k35;
ttt(i)=t(i);
%if anything is calculated to be imaginary then break the loop and
%set the last
%value of calculation to be NaN;
if imag(G(i))~=0
m(i)=NaN;
R(i)=NaN;
r(i)=NaN;
W(i)=NaN;
G(i)=NaN;
F(i)=NaN;
m(i-1)=NaN;
R(i-1)=NaN;
r(i-1)=NaN;
W(i-1)=NaN;
G(i-1)=NaN;
F(i-1)=NaN;
mprime(i)=NaN;
mprime(i-1)=NaN;
Q1Q2(i)=NaN;
```

```
bigR(i)=NaN;
C1(i)=NaN;
C2(i)=NaN;
C3(i)=NaN;
r(i-2)=NaN;
R(i-2)=NaN;
m(i-2)=NaN;
break
end
%calculate the value of next point
m(i+1)=m(i)+h/6*(k12+2*k22+2*k32+k42);
R(i+1)=R(i)+h/6*(k14+2*k24+2*k34+k44);
r(i+1)=r(i)+h/6*(k16+2*k26+2*k36+k46);
W(i+1)=W(i)+h/6*(k11+2*k21+2*k31+k41);
G(i+1)=G(i)+h/6*(k13+2*k23+2*k33+k43);
F(i+1)=F(i)+h/6*(k15+2*k25+2*k35+k45);
%find the quantities of
betadott1(i+1)=betadott(k,m(i+1),R(i+1),G(i+1),W(i+1),F(i+1),\
aa(t(i+1),omega)); betadoubledott1(i)=\
(betadott1(i+1)-betadott1(i))/h/Tdott(k,m(i),R(i),G(i),F(i),\
aa(t(i),omega));
C1(i)=-betadoubledott1(i);
C2(i)=1/2*(3*betadott1(i)^2+2*betadoubledott1(i));
C3(i)=-3/2*(betadott1(i)^2+2*betadoubledott1(i));
bigR(i)=B(k,m(i),R(i))*R(i); I(i)=(1-2*k/bigR(i))*f2(k,m(i),\
R(i),W(i),betadoubledott1(i)); bigM1(i)=bigM(k,m(i),R(i),\
betadott1(i)); Tdott1(i)=Tdott(k,m(i),R(i),G(i),F(i),\
```

```
aa(t(i),omega));
bigR1(i)=bigR(i)-2*k; bigR2(i)=aa(t(i),omega)*r(i);
bigR3(i)=bigR(i)-bigR2(i); bigRdott(i)=Bdott(k,m(i),R(i),W(i),\
G(i))*R(i)+B(k,m(i),R(i))*G(i);
Q1Q2(i)=(1-2*k/bigR(i))*f1(k,m(i),R(i),G(i),W(i),F(i),\
aa(t(i),omega));
end
%plot everything we need, and present the parameters automatically subplot(2,2,1);
num2str(kk),',rdot=',num2str(F(1)),',Rdot=',num2str(G(1))]);
subplot(2,2,2);plot(R);title('R');
subplot(2,2,3);plot(Q1Q2);title('Q1Q2');
subplot(2,2,4);plot(m);title('m');
figure;
subplot(2,2,1);plot(bigR);title('bigR');
subplot(2,2,2);plot(C1);title('C1');
subplot(2,2,3);plot(C2);title('C2');
subplot(2,2,4);plot(C3);title('C3');
figure; plot(bigM1);title('bigM');
%======================================
%following are the functions I used,
%you can put them into seperate ``.m'' files %to rebuid the
%program yourself
%function A
function out = A(m,R)
H1 = 1-m/(2*R);
H2 = 1+m/(2*R);
out = H1/H2;
```

```
end
%function aa
function out = aa(t,omega)
out = (t-10)^(2/3/(1+omega));
end
%function aadott
function out = aadott(t,omega)
out = 2/3/(1+omega)*(t-10)^(2/3/(1+omega)-1);
end
%function aadoubledott
function out = aadoubledott(t,omega)
out = 2/3/(1+omega)*(2/3/(1+omega)-1)*\
(t-10)^(2/3/(1+omega)-2);
end
%function Adott
function out = Adott(m,R,W,G)
out = 4*(-R*W+m*G)/(m+2*R)^2;
end
%function AstardotR
function out = AstardotR(m,R)
out = 4*m/(m+2*R)^2;
end
%function AstardotT
function out = AstardotT(k,m,R,W,G,F,a)
Tdott1=Tdott(k,m,R,G,F,a);
out = -4*R*W/Tdott1/(m+2*R)^2;
end
```

```
%function B

function out = B(k,m,R)

out = k/m*(1+m/(2*R))^2;

end

%function Bdott

function out = Bdott(k,m,R,W,G)

h1=k*R*(m^2-4*R^2)*W;

h2=-2*k*m^2*(m+2*R)*G;

out = (h1+h2)/(4*m^2*R^3);

end

%function betadott

function out = betadott(k,m,R,G,W,F,a)

out = -2*W/m/Tdott(k,m,R,G,F,a);

end

%function bigM

function out = bigM(k,m,R,betadott)

bigR=B(k,m,R)*R;

h1 = betadott^2-4/bigR^2*(1-2*k/bigR);

out = 1/2*bigR*(1+bigR^2/4*h1);

end

%function BstardotR

function out = BstardotR(k,m,R)

out = -k*(m+2*R)/(2*R^3);

end

%function BstardotRt

function out = BstardotRt(k,m,R,W,G)

out = -k*(2*G+W)/(2*R^3)+k/2*(2*R+m)*3*G/R^4;
```

```
end
%function BstardotT
function out = BstardotT(k,m,R,W,G,F,a)
Tdott1=Tdott(k,m,R,G,F,a);
out = k/4*(R^(-2)-4*m^(-2))*W/Tdott1;
end
%function C1
function out = C1(k,m,R,r,G,W,F,adott,adoubledott)
h1 = k*m*G^2/(2*R^3)+2*k*W^2*R/m^3;
h2 = -2*W*G*k/m^2-k*W*G/(2*R^2)-r*adoubledott-\
2*adott*F;
out = h1+h2;
end
%function C2
function out = C2(k,m,R,G,W,F,a,adott)
delta11=delta1(k,m,R,G,W,F,a);
Tdott1=Tdott(k,m,R,G,F,a);
out = delta11+(-a^2*adott*F^3+2*adott*F)/Tdott1^3;
end
%function C3
function out = C3(k,m,R,G,W,F,a,adott,adoubledott,r)
Adott1=Adott(m,R,W,G);
BstardotR1=BstardotR(k,m,R);
A1=A(m,R);
BstardotRt1=BstardotRt(k,m,R,W,G);
h1 = Adott1*BstardotR1*R^2+A1*BstardotRt1*R^2+\
A1*BstardotR1*2*R*G;
```

```
B1=B(k,m,R);
Bdott1=Bdott(k,m,R,W,G);
h2 = Adott1*B1*R+A1*Bdott1*R+A1*B1*G;
Tdott1=Tdott(k,m,R,G,F,a);
BstardotT1=BstardotT(k,m,R,W,G,F,a);
h3 = G/Tdott1*2*R*G*B1^2*BstardotT1/A1+\
G/Tdott1*R^2*2*B1*Bdott1*BstardotT1/A1;
h4 = -G/Tdott1*R^2*B1^2*BstardotT1*Adott1/A1^2;
h5 = -(adott*r+a*F+F*2*r*F*a^2*adott)/Tdott1;
h6 = -(F*r^2*2*a*adott^2+F*r^2*a^2*adoubledott)/Tdott1;
out = h1+h2+h3+h4+h5+h6;
end
%function delta1
function out = delta1(k,m,R,G,W,F,a)
A1=A(m,R);
B1=B(k,m,R);
Tdott1=Tdott(k,m,R,G,F,a);
BstardotT1=BstardotT(k,m,R,W,G,F,a);
AstardotR1=AstardotR(m,R);
BstardotR1=BstardotR(k,m,R);
AstardotT1=AstardotT(k,m,R,W,G,F,a);
h1 = B1^2*BstardotT1*G^3/A1/Tdott1^3;
h2 = (2*B1*AstardotR1-A1*BstardotR1)*G^2/Tdott1^2;
h3 = (B1*AstardotT1-2*A1*BstardotT1)*W/Tdott1;
h4 = -A1^2*AstardotR1/B1;
out = h1+h2+h3+h4;
end
```

```
%function delta2
function out = delta2(k,m,R,G,W,F,a,adott)
A1=A(m,R);
B1=B(k,m,R);
Adott1=Adott(m,R,W,G);
Bdott1=Bdott(k,m,R,W,G);
h1=1/(2*A1)*(1-a^2*F^2+B1^2*G^2)^(-1/2);
h2=-Adott1/A1^2*(1-a^2*F^2+B1^2*G^2)^(1/2);
out = h1*(-2*a*adott*F^2+2*B1*Bdott1*G^2)+h2;
end
%function delta4
function out = delta4(k,m,R,G,W,F,a)
Tdott1=Tdott(k,m,R,G,F,a);
out = k/4*(-2*R^(-3)*G+8*m^(-3)*W)*W/Tdott1;
end
%function f1
function out = f1(k,m,R,G,W,F,a)
betadott1=betadott(k,m,R,G,W,F,a);
bigR=B(k,m,R)*R;
out = betadott1^2-4/bigR^2*(1-2*k/bigR);
end
%function f2
function out = f2(k,m,R,W,betadoubledott)
bigR=B(k,m,R)*R;
out = betadoubledott+\
4/bigR^2*(1-2*m/bigR)^(1/2)*(1-3*k/bigR);
end
```

```
%function M1
function out = M1(k,m,R,G,F,a)
Tdott1=Tdott(k,m,R,G,F,a);
out = k/4*(R^(-2)-4*m^(-2))/Tdott1;
end
%function M2
function out = M2(k,m,R,G,W,F,a)
Tdott1=Tdott(k,m,R,G,F,a);
out = -k/4*(R^(-2)-4*m^(-2))*W/Tdott1^2;
end
%function P1
function out = P1(k,m,R)
out = k/(4*R)-k*R/m^2;
end
%function P2
function out = P2(k,m,R)
out = k/m-k*m/(4*R^2);
end
%function P3
function out = P3(a)
out = -a;
end
%function Q2
function out = Q2(k,m,R,G,W,F,a)
Tdott1=Tdott(k,m,R,G,F,a);
A1=A(m,R);
B1=B(k,m,R);
```

```
BstardotT1=BstardotT(k,m,R,W,G,F,a);
out = R^2*B1^2*BstardotT1/A1/Tdott1;
end
%function Q3
function out = Q3(k,m,R,G,F,a,adott,r)
A1=A(m,R);
B1=B(k,m,R);
Tdott1=Tdott(k,m,R,G,F,a);
out = -r^2*a^2*adott/Tdott1;
end
%function Q4
function out = Q4(k,m,R,G,W,F,a,adott,r)
A1=A(m,R);
B1=B(k,m,R);
Tdott1=Tdott(k,m,R,G,F,a);
BstardotT1=BstardotT(k,m,R,W,G,F,a);
h1=-G/Tdott1^2*R^2*B1^2*BstardotT1/A1;
h2=(a*r+F*r*a^2*adott)/Tdott1^2;
out = h1+h2;
end
%function Q5
function out = Q5(k,m,R,G,F,a)
Tdott1=Tdott(k,m,R,G,F,a);
A1=A(m,R);
B1=B(k,m,R);
out = G/Tdott1*R^2*B1^2/A1;
end
```

```
%function S2
function out = S2(k,m,R,G,F,a)
Tdott1=Tdott(k,m,R,G,F,a);
A1=A(m,R);
B1=B(k,m,R);
out = -A1*B1/Tdott1^2;
end
%function S3
function out = S3(k,m,R,G,F,a)
Tdott1=Tdott(k,m,R,G,F,a);
out = a/Tdott1^3;
end
%function S4
function out = S4(k,m,R,G,F,a)
A1=A(m,R);
B1=B(k,m,R);
Tdott1=Tdott(k,m,R,G,F,a);
out = A1*B1*G/Tdott1^3;
end
%function solution
function [out1 out2 out3] = \
solution(k,m,R,G,W,F,a,adott,adoubledott,r) P11=P1(k,m,R);
P21=P2(k,m,R);
P31=P3(a);
C11=C1(k,m,R,r,G,W,F,adott,adoubledott);
S21=S2(k,m,R,G,F,a);
S31=S3(k,m,R,G,F,a);
```

```
S41=S4(k,m,R,G,F,a);
C21=C2(k,m,R,G,W,F,a,adott);
T11=T1(k,m,R,G,F,a,adott);
T21=T2(k,m,R,G,F,a,adott);
delta21=delta2(k,m,R,G,W,F,a,adott);
H11=S21+S41*T11;
H21=S31+S41*T21;
H31=S41*delta21+C21;
Q21=Q2(k,m,R,G,W,F,a);
Q31=Q3(k,m,R,G,F,a,adott,r);
Q41=Q4(k,m,R,G,W,F,a,adott,r);
Q51=Q5(k,m,R,G,F,a);
C31=C3(k,m,R,G,W,F,a,adott,adoubledott,r);
M11=M1(k,m,R,G,F,a);
M21=M2(k,m,R,G,W,F,a);
delta41=delta4(k,m,R,G,W,F,a);
O11=Q51*M11;
O21=Q21+Q41*T11+Q51*M21*T11;
O31=Q31+Q41*T21+Q51*M21*T21;
O41=Q41*delta21+Q51*delta41+Q51*M21*delta21+C31;
bigA=[P11 P21 P31;0 H11 H21;O11 O21 O31];
bigB=[-C11;-H31;-O41];
bigX=bigAbigB;
out1=bigX(1);
out2=bigX(2);
out3=bigX(3);
end
```

```
%function T1

function out = T1(k,m,R,G,F,a,adott)

A1=A(m,R);

B1=B(k,m,R);

h1=1/(2*A1)*(1-a^2*F^2+B1^2*G^2)^(-1/2);

out = h1*B1^2*2*G;

end

%function T2

function out = T2(k,m,R,G,F,a,adott)

A1=A(m,R);

B1=B(k,m,R);

h1=1/(2*A1)*(1-a^2*F^2+B1^2*G^2)^(-1/2);

out = h1*(-a^2*2*F);

end

%function Tdott

function out = Tdott(k,m,R,G,F,a)

A1=A(m,R);

B1=B(k,m,R);

out = 1/A1*(1-a^2*F^2+B1^2*G^2)^(1/2);

end
```

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