BAYLOR UNIVERSITY

OSCILLATING SATELLITES ABOUT THE STRAIGHT LINE EQUILIBRIUM POINTS

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Lagrange [1] has shown in the three body problem, where one of them is an infinitesimal, that there are three equilibrium points lying in the straight line through the finite bodies. If the infinitesimal body were placed at one of these points it would remain forever relatively at rest with respect to the finite bodies or rapidly leave the system. These points of equilibrium may be readily obtained by expanding the necessary differential equations in which both the velocity and acceleration of the infinitesimal body are set equal to zero. The infinitesimal body must remain relatively at rest when these conditions are imposed upon its motion.

In what follows we shall adopt the method as set out in Chapter V of Periodic Orbits by Moulton.[2] In this method the differential equations of motion are first set up with respect to the center of gravity of the system. An axis is chosen with the origin at the

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1 Forest Ray Moulton. <u>An Introduction to Celestial Mechanics</u>. The Macmillan Company, New York, 1947. p. 290

2 F. R. Moulton. <u>Periodic Orbits</u>. Carnegie Institution of Washington, 1920. pp. 151-198

center of gravity and revolving at the same rate as the rotation of the finite masses so that the three bodies always remain in a straight line. The sum of the masses is used as the unit of mass and the distance between the finite masses as the unit of distance. The unit of time shall be so chosen that the Gaussian constant will be unity.

The center of the coordinate system will then be translated in succession to each of the equilibrium points. A small displacement, together with a small velocity, will be given to the infinitesimal body in each instance.

The differential equations will then be changed by the introduction of two parameters to be determined later. One involves the coordinates of displacement and the other involves the time. The equations will then be expanded into a power series in terms of the displacement parameter and a solution will be made in terms up to the fifth power of this parameter. At this point the restriction will be made that the infinitesimal body shall be projected at right angles to the axis of the finite bodies and shall have a constant velocity in the direction perpendicular to the plane of revolution of the finite bodies. The restriction will also be made that the motion of the infinitesimal body shall be periodic.

The masses of the two finite bodies will then be assigned and the three equations of motion and the period will be computed in series to the desired accuracy. Finally a complete revolution of the infinitesimal body will be computed with enough data provided to allow a reasonable graph to be made of the path of the body in three dimensions.

The Equations of Motion

The original axes with reference to the center of gravity of the system shall be as follows:

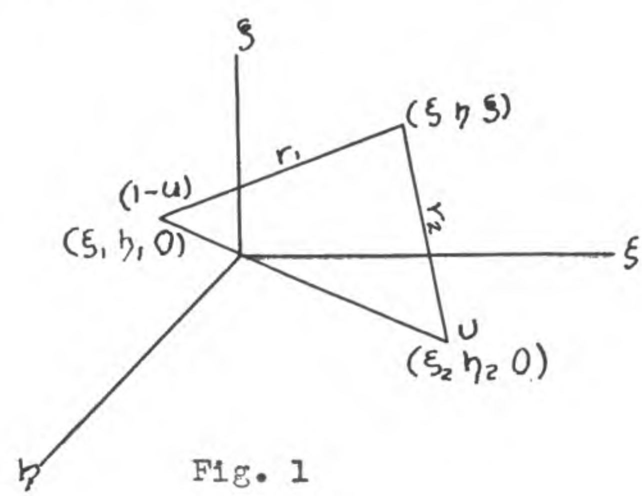


Fig. 1

Let the mass of the major body be 1-u and that of the second body be u. The distance from the infinitesimal body to the major body is designated r_1 and to the second body r_2. The coordinates of the major body, the second body and the infinitesimal body are respectively $(\xi_1 \eta_1 0_1)$ $(\xi_2 \eta_2 0)$ and $(\xi \eta \zeta)$. Hence we have
$r_1 = \sqrt{(\xi - \xi_1)^2 + (\eta - \eta_1)^2 + \zeta^2}$ and $r_2 = \sqrt{(\xi - \xi_2)^2 + (\eta - \eta_2)^2 + \zeta^2}$

The differential equations of motion for the infinitesimal body then become

$$
\left.
\begin{aligned}
\frac{d^2\xi}{dt^2} &= -(1-u) \frac{(\xi - \xi_1)}{r_1^3} - u \frac{(\xi - \xi_2)}{r_2^3} \\
\frac{d^2\eta}{dt^2} &= -(1-u) \frac{(\eta - \eta_1)}{r^3} - u \frac{(\eta - \eta_2)}{r^3} \\
\frac{d^2\zeta}{dt^2} &= -(1-u) \frac{\zeta}{r_1^3} - u \frac{\zeta}{r_2^3}
\end{aligned}
\right] \quad (1)
$$

Let this motion be referred to a new set of axes lying in the same plane with the finite bodies and rotating with uniform angular velocity unity by means of the following equations.

$$
\left.
\begin{aligned}
&\xi = x \cos t - y \sin t \qquad \xi_1 = x_1 \cos t - y_1 \sin t \\
&\xi_2 = x_2 \cos t - y_2 \sin t
\end{aligned}
\right] \quad (2)
$$

$$\left.\begin{aligned}
\eta &= x \sin t + y \cos t \qquad \eta_1 = x_1 \sin t + y_1 \cos t \\
\eta_2 &= x_2 \sin t + y_2 \cos t \\
\xi &= z
\end{aligned}\right\} \quad (2)$$

The equations of relative motion for the infinitesimal body then become

$$\left.\begin{aligned}
\frac{d^2x}{dt^2} - 2\frac{dy}{dt} &= x - (1-u)\frac{(x-x_1)}{r_1^3} - u\frac{(x-x_2)}{r_2^3} \\
\frac{d^2y}{dt^2} + 2\frac{dx}{dt} &= y - (1-u)\frac{(y-y_1)}{r_1^3} - u\frac{(y-y_2)}{r_2^3} \\
\frac{d^2z}{dt^2} &= -(1-u)\frac{z}{r_1^3} - u\frac{z}{r_2^3}
\end{aligned}\right\} \quad (3)$$

If the axes are so rotated that the x axis continually passes through the finite bodies $y_1 = y_2 = 0$ then equations (3) become

$$\left.\begin{aligned}
\frac{d^2x}{dt^2} - 2\frac{dy}{dt} &= x - (1-u)\frac{(x-x_1)}{r_1^3} - u\frac{(x-x_2)}{r_2^3} \\
\frac{d^2y}{dt^2} + 2\frac{dx}{dt} &= y - (1-u)\frac{y}{r_1^3} - u\frac{y}{r_2^3} \\
\frac{d^2z}{dt^2} &= -(1-u)\frac{z}{r_1^3} - u\frac{z}{r_2^3}
\end{aligned}\right\} \quad (4)$$

The Points of Equilibrium

The necessary and sufficient conditions for equilibrium at the three points are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0 \\ \frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0 \end{aligned}\right] \quad (5)$$

The last two equations of (4) are satisfied for all values of x when $y = z = 0$. The first equation of (4) is satisfied when the right member is placed equal to zero. Therefore the conditions for equilibrium may be stated as

$$\left.\begin{aligned} x - (1-u)\frac{(x - x_1)}{[(x - x_1)^2]^{\frac{3}{2}}} - \frac{u(x - x_2)}{[(x - x_2)^2]^{\frac{3}{2}}} &= 0 \\ y &= 0 \\ z &= 0 \end{aligned}\right] \quad (6)$$

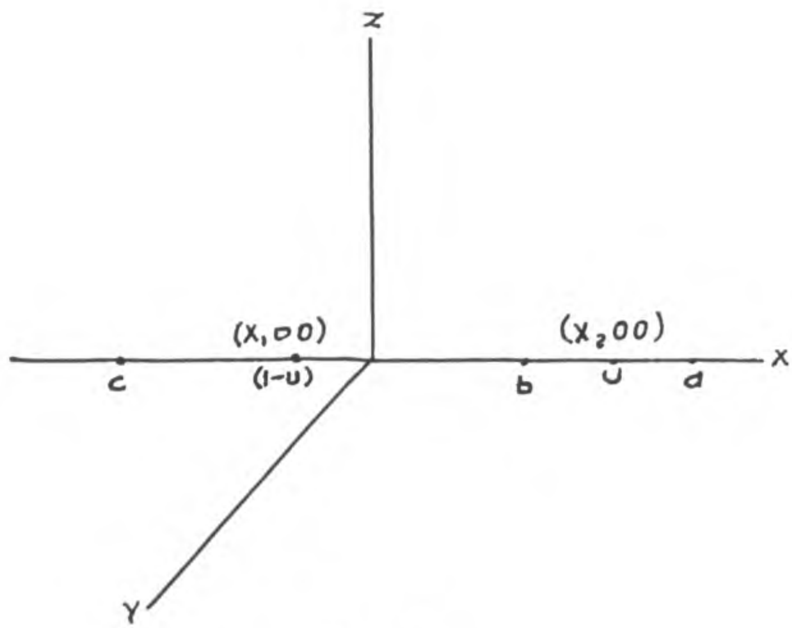


Fig. 2

At the equilibrium point a we have

$$x - x_2 = r_2$$

$$x - x_1 = r_1 = 1 + r_2$$

$$x = 1 - u + r_2$$

Then the first of (6) becomes

$$1 - u + r_2 - (1 - u)\frac{(1 + r_2)}{\left[(1 + r_2)^2\right]^{\frac{3}{2}}} - u\frac{r_2}{\left[(1 + r_2)^2\right]^{\frac{3}{2}}} = 0$$

or $$1 - u + r_2 - \frac{(1 - u)}{(1 + r_2)^2} - \frac{u}{r_2^2} = 0$$

or $$r_2^{\,5} + (3-u)r_2^4 + (3-2u)r_2^3 - ur_2^2 - 2ur_2 - u = 0$$

When this equation is expanded in a power series in $u^{\frac{1}{3}}$ we find

$$r_2 = \left\{\frac{u}{3}\right\}^{\frac{1}{3}} + \frac{1}{3}\left\{\frac{u}{3}\right\}^{\frac{2}{3}} - \frac{1}{9}\left\{\frac{u}{3}\right\}^{\frac{3}{3}} \cdots\cdots \quad . \tag{7}$$

This will give the distance from the smaller finite body to the equilibrium point opposite to the larger finite body.

For the equilibrium point b we have

$$x - x_2 = -r_2$$

$$x - x_1 = 1 - r_2$$

$$x = 1 - r_2 - u = (1-u) - r_2$$

Then the first of (6) becomes

$$1 - u - r_2 - (1 - u)\ \frac{(1 - r_2)}{\left[(1 - r_2)^2\right]^{\frac{3}{2}}} + u\ \frac{r_2}{\left[(-r_2)^2\right]^{\frac{3}{2}}} = 0$$

or $1 - u - r_2 - \dfrac{(1 - u)}{(1 - r_2)^2} + \dfrac{u}{r_2^2} = 0$

or $r_2^5 + (3-u)r_2^4 + (3-2u)r_2^3 - ur_2^2 + 2ur_2 - u = 0$

When this is expanded into a power series in $u^{\frac{1}{3}}$ we find

$$r_2 = \left\{\frac{u}{3}\right\}^{\frac{1}{3}} - \frac{1}{3}\left\{\frac{u}{3}\right\}^{\frac{2}{3}} - \frac{1}{9}\left\{\frac{u}{3}\right\}^{\frac{3}{3}} \cdots\cdots \quad . \tag{8}$$

This will give the distance from the smaller finite body to the central equilibrium point.

At the equilibrium point c we have

$$x - x_1 = -1 + p$$

$$x - x_2 = -1 - (1-p) = -2 + p$$

$$x = -u - 1 + p$$

Then the first of (6) becomes

$$-(u+1)+p-(1-u)\,\frac{(-1+p)}{\left[(-1+p)^2\right]^{\frac{3}{2}}}-u\,\frac{(-2+p)}{\left[(-2+p)^2\right]^{\frac{3}{2}}}=0$$

or $$-(u+1)+p+\frac{(1-u)}{(1-p)^2}+\frac{u}{(-2+p)^2}=0$$

or $$p^5-(7+u)p^4+(19+6u)p^3-(24+13u)p^2+(12+14u)-7u=0$$

When this is expanded as a power series in u we find

$$p=\frac{7}{12}\,u+\frac{23.\ 7^2u^2}{12^4}\ \ldots\ldots\ldots\ldots$$

or $$r_2=2-\frac{7}{12}\,u-\frac{23.\ 7^2u^2}{12^4}\ \ldots\ldots\ldots\ldots \qquad (9)$$

This will give the distance from the smaller finite body to the equilibrium point opposite to the smaller finite body.

Region of Convergence

Now at the points a, b or c let $x = x_o$, $y = 0$ and $z = 0$, the value of x_o depending upon which point is used. Let us give the infinitesimal a small displacement and a small velocity. We then have

$$\left.\begin{array}{lll}
x = x_0 + x' & y = 0 + y' & z = 0 + z' \\
\frac{dx}{dt} = \frac{dx'}{dt} & \frac{dy}{dt} = \frac{dy'}{dt} & \frac{dz}{dt} = \frac{dz'}{dt} \\
\frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2} & \frac{d^2y}{dt^2} = \frac{d^2y'}{dt^2} & \frac{d^2z}{dt^2} = \frac{d^2z'}{dt^2} \\
\multicolumn{3}{c}{r_1 = \sqrt{(x_0 + x' + u)^2 + y'^2 + z'^2}} \\
\multicolumn{3}{c}{r_2 = \sqrt{(x_0 + x' - 1 + u)^2 + y'^2 + z'^2}}
\end{array}\right] \quad (10)$$

Let $U = \frac{1}{2}(1-u)\ (r_1^2 + \frac{2}{r_1}) + \frac{1}{2}u(r_2^2 + \frac{2}{r_2}) - \frac{1}{2}z'^2 - \frac{1}{2}u(1-u)$

The equations (4) then become

$$\left.\begin{array}{l}
\frac{d^2x'}{dt^2} - 2\frac{dy'}{dt} = \frac{\partial U}{\partial x'} \\
\frac{d^2y'}{dt^2} + 2\frac{dx'}{dt} = \frac{\partial U}{\partial y'} \\
\frac{d^2z'}{dt^2} = \frac{\partial U}{\partial z'}
\end{array}\right] \quad (11)$$

U can be expressed as a power function in x', y'^2, and z'^2. The convergence of the right side of equations (11) will depend upon the convergence of $\frac{1}{r_1}$ and $\frac{1}{r_2}$.

We will expand these expressions and find

$$\frac{1}{r_1} = \left[(x_0 + x' + u)^2 + y'^2 + z'^2\right]^{-\frac{1}{2}} = (x_0 + u)^{-1} - \tfrac{1}{2}(x_0 + u)^{-3} \left[x'^2 + y'^2 + z'^2 + 2x'(x_0 + u)\right] + \frac{3}{8}(x_0 + u)^{-5}\left[x'^2 + y'^2 + z'^2 + 2x'(x_0 + u)\right]^2 - \ldots\ldots\ldots\ldots .$$

Now

$$(x_0 + u)^{-1} - \tfrac{1}{2}(x_0 + u)^{-3}\left[x'^2 + y'^2 + z'^2 + 2x'(x_0 + u)\right] + \frac{3}{8}(x_0 + u)^{-5}\left[x'^2 + y'^2 + z'^2 + 2x'(x_0 + u)\right]^2 \ldots$$

$$< (x_0 + u)^{-1} + (x_0 + u)^{-3}\left[x'^2 + y'^2 + z'^2 + 2x'(x_0 + u)\right] + (x_0 + u)^{-5}\left[x'^2 + y'^2 + z'^2 + 2x'(x_0 + u)\right]^2 \ldots\ldots\ldots .$$

The right side of the inequality is a geometric progression with the ratio $\dfrac{x'^2 + y'^2 + z'^2 + 2x'(x_0 + u)}{(x_0 + u)^2}$ which will converge if the |ratio| is less than unity.

Hence

$$-1 < \frac{x'^2 + y'^2 + z'^2 + 2x'(x_0 + u)}{(x_0 + u)^2} < 1$$

The limits of convergence will occur when the inequalities are replaced by equalities. We have then

$x'^2 + y'^2 + z'^2 + 2x'(x_0 + u) = -(x_0 + u)^2$, which is the equation of a point at (1-u). We have also

$x'^2 + y'^2 + z'^2 + 2x'(x_0 + u) = (x_0 + u)^2$, which is the equation of a sphere with center at (1-u) and radius $\sqrt{2}(x_0 + u)$.

Likewise for $\frac{1}{r_2}$, $x'^2 + y'^2 + z'^2 + 2x'(x_0 - 1 + u)$ $= -(x_0 - 1 + u)^2$, which is the equation of a point at u. We also have $x'^2 + y'^2 + z'^2 + 2x'(x_0 - 1 + u) =$ $(x_0 - 1 + u)^2$, which is the equation of a sphere with center at u and radius $\sqrt{2}\ (x_0 - 1 + u)$.

In figure 2. the distances from (1-u) and u to a are $(x_0 + u)$ and $(x_0 - 1 + u)$ respectively. These are the radii of the above spheres if multiplied by $\sqrt{2}$. Since

$$\sqrt{2}\ (x_0 + u) - 1 > \sqrt{2}\ (x_0 - 1 + u)$$

the sphere with (1-u) as a center completely surrounds the sphere with u as a center. The point a is included in both.

Therefore, within the sphere with u as a center and with radius $\sqrt{2}\ (x_0 - 1 + u)$, the series converges everywhere except at point u. By the same reasoning the series can be shown to converge at the points b and c.

Introduction of Parameters

If in equations (11) the following substitutions are made

$$x' = x\varepsilon \qquad y' = y\varepsilon \qquad z' = z\varepsilon \qquad t - t_0 = (1 + \delta)T$$

where ε and δ are constants at present undetermined, we have

$$\frac{dx'}{dt} = \varepsilon \frac{dx}{dT}\frac{dT}{dt} = \frac{\varepsilon}{(1+\delta)}\frac{dx}{dT} \qquad \frac{d^2x'}{dt^2} = \frac{\varepsilon}{(1+\delta)^2}\frac{d^2x}{dT^2}$$

$$\frac{dy'}{dt} = \varepsilon \frac{dy}{dT}\frac{dT}{dt} = \frac{\varepsilon}{(1+\delta)}\frac{dy}{dT} \qquad \frac{d^2y'}{dt^2} = \frac{\varepsilon}{(1+\delta)^2}\frac{d^2y}{dT^2}$$

$$\frac{dz'}{dt} = \varepsilon \frac{dz}{dT}\frac{dT}{dt} = \frac{\varepsilon}{(1+\delta)}\frac{dz}{dT} \qquad \frac{d^2z'}{dt^2} = \frac{\varepsilon}{(1+\delta)^2}\frac{d^2z}{dT^2}$$

Then

$$\left.\begin{aligned}
\frac{d^2x'}{dt^2} - 2\frac{dy'}{dt} &= \frac{\varepsilon}{(1+\delta)^2}\frac{d^2x}{dT^2} - 2\,\frac{\varepsilon}{(1+\delta)}\frac{dy}{dT} \\
\frac{d^2y'}{dt^2} + 2\frac{dx'}{dt} &= \frac{\varepsilon}{(1+\delta)^2}\frac{d^2y}{dT^2} + 2\,\frac{\varepsilon}{(1+\delta)}\frac{dx}{dT} \\
\frac{d^2z'}{dt^2} &= \frac{\varepsilon}{(1+\delta)^2}\frac{d^2z}{dT^2}
\end{aligned}\right] \quad (12)$$

$$
\begin{aligned}
U = {} & \tfrac{1}{2}(1-u)\Big\{ (x_o + x' + u)^2 + y'^2 + z'^2 + 2\Big[\frac{1}{x_o + u} - \\
& \tfrac{1}{2}\,\frac{x'^2 + y'^2 + z'^2 + 2x'(x_o + u)}{(x_o + u)^3} + \frac{3}{8}\,\frac{[x'^2 + y'^2 + z'^2 + 2x'(x_o + u)]^2}{(x_o + u)^5} \\
& - \frac{5}{16}\,\frac{[x'^2 + y'^2 + z'^2 + 2x'(x_o + u)]^3}{(x_o + u)^7} \\
& + \frac{35}{128}[\cdot\cdot\ \cdot\cdot]\cdot\cdot\Big]\Big\} + \tfrac{1}{2}u\Big\{(x_o - 1 + x' + u)^2 + y'^2 + z'^2 \\
& + 2\Big[\frac{1}{x_o - 1 + u} - \tfrac{1}{2}\,\frac{x'^2 + y'^2 + z'^2 + 2x'(x_o - 1 + u)}{(x_o - 1 + u)^3} \\
& + \frac{3}{8}\,\frac{[x'^2 + y'^2 + z'^2 + 2x'(x_o - 1 + u)]^2}{(x_o - 1 + u)^5} - \frac{5}{16}\,\frac{[x'^2 + y'^2 + z'^2 + 2x'(x_o - 1 + u)]^3}{(x_o - 1 + u)^7} \\
& + \frac{35}{128}[\cdot\cdot\cdot\cdot]\cdot\cdot\cdot\cdot\cdot\Big]\Big\} \\
& + \tfrac{1}{2}z'^2 - \tfrac{1}{2}u(1 - u).
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\frac{\partial U}{\partial x'} = {} & x' + \frac{(1-u)\,2x'}{(x_o + u)^3} + \frac{2ux'}{(x_o - 1 + u)^3} + \frac{3}{2}(1-u)\,\frac{(x'^2 + y'^2 + z'^2)(x_o + x' + u)}{(x_o + u)^5} + 3\,\frac{(1-u)\,x'^2(x_o + u)}{(x_o + u)^5} \\
& - (1-u)\,\frac{15}{8}\,\frac{[x'^2 + y'^2 + z'^2 + 2x'(x_o + u)]^2(x_o + x' + u)}{(x_o + u)^7}
\end{aligned}
$$

$$+ \frac{35}{16}\left[\cdots\cdot\right] \cdots\cdots \quad + \frac{3}{2}\, \frac{u\,(x'^2 + y'^2 + z'^2)(x' + x_0}{(x_0 - 1 + u)^5}$$

$$\underline{- 1 + u)} + 3\, \frac{u\, x'^2 (x_0 - 1 + u)}{(x_0 - 1 + u)^5} - \frac{15}{8}\, u\, \underline{\left[x'^2 + y'^2 + z'^2\right.}$$

$$\frac{\left. + 2x'(x_0 - 1 + u)\right]^2 \quad (x' + x_0 - 1 + u)}{(x_0 - 1 + u)^7} + \frac{35}{16}\left[\cdots\right]\cdot \Big\}.$$

$$\frac{\partial U}{\partial y'} = y' \Big\{ 1 - \frac{1 - u}{(x_0 + u)^3} - \frac{u}{(x_0 - 1 + u)^3} + \frac{3}{2}\, \underline{(1 - u)}$$

$$\frac{\left[x'^2 + y'^2 + z'^2 + 2x'(x_0 + u)\right]}{(x_0 + u)^5} - \frac{15}{8}\, \underline{(1 - u)\left[x'^2 + y'^2\right.} \qquad (13)$$

$$\frac{\left. + z'^2 + 2x'(x_0 + u)\right]^2}{(x_0 + u)^7} + \frac{35}{16}\left[\cdots\cdots\right]\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{3}{2}\, \frac{u\left[x'^2 + y'^2 + z'^2 + 2x'(x_0 - 1 + u)\right]}{(x_0 - 1 + u)^5} - \frac{15}{8}\, u$$

$$\frac{\left[x'^2 + y'^2 + z'^2 + 2x'(x_0 - 1 + u)\right]^2}{(x_0 - 1 + u)^7} + \frac{35}{16}\left[\cdots\right]\cdots \Big\}$$

$$\frac{\partial U}{\partial z'} = z' \Big\{ - \frac{1 - u}{(x_0 + u)^3} - \frac{u}{(x_0 - 1 + u)^3} + \frac{3}{2}\, \underline{(1 - u)}$$

$$\frac{\left[x'^2 + y'^2 + z'^2 + 2x'(x_0 + u)\right]}{(x_0 + u)^5} - \frac{15}{8}\, \underline{(1 - u)\left[x'^2 + \right.}$$

$$\frac{y'^2 + z'^2 + 2x'(x_o + u)\Big]^2}{(x_o + u)^7} + \frac{35}{16}[\cdots\cdots]\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{3}{2}\,\frac{u\left[x'^2 + y'^2 + z'^2 + 2x'(x_o - 1 + u)\right]}{(x_o - 1 + u)^5} - \frac{15}{8}\;u \qquad (13)$$

$$\left.\frac{\left[x'^2 + y'^2 + z'^2 + 2x'(x_o - 1 + u)\right]^2}{(x_o - 1 + u)^7} + \frac{35}{16}[\cdots]\cdots\cdots\right\}$$

When we substitute $x' = x\varepsilon$, $y' = y\varepsilon$ and $z' = z\varepsilon$ in (13) and combine with (12) we have

$$\frac{d^2x}{dT^2} - 2(1 + \delta)\,\frac{dy}{dT} = (1 + \delta)^2 \;\Big\{ (1 + 2A)x \; +$$

$$\frac{3}{2}\frac{(1 - u)\left[(\varepsilon x^2 + \varepsilon y^2 + \varepsilon z^2)(\varepsilon x + x_o + u) + 2\varepsilon x^2(x_o + u)\right]}{(x_o + u)^5}$$

$$- \frac{15}{8}\,(1 - u)\,\frac{\left[\varepsilon x^2 + \varepsilon y^2 + \varepsilon z^2 + 2x(x_o + u)\right]^2 \varepsilon\,(\varepsilon x + x_o + u)}{(x_o - 1 + u)^7} \qquad (14)$$

$$+ \frac{35}{16}\;[\cdots]\cdots\cdots\cdots \;+ \frac{3}{2}\,u\frac{(\varepsilon x^2 + \varepsilon y^2 + \varepsilon z^2)(\varepsilon x + x_o}{(x_o - 1 + u)^5}$$

$$\underline{- 1 + u)} \; + 3\,\frac{u\,\varepsilon x^2\,(x_o - 1 + u)}{(x_o - 1 + u)^5} - \frac{15}{8}\,u\,\big[\underline{\varepsilon x^2 + \varepsilon y^2 + \varepsilon z^2}$$

$$\frac{+ 2x\,(x_o - 1 + u)\Big]^2\cdot(\varepsilon x + x_o - 1 + u)\varepsilon}{x_o - 1 + u)^7} + \frac{35}{16}[\cdots]. \Big\}$$

$$\frac{d^2y}{dT^2} + 2(1+\delta)\frac{dx}{dT} = (1+\delta)^2\Big\{(1-A)\,y +$$

$$+ \frac{3}{2}\frac{(1-u)\left[\varepsilon^2x^2+\varepsilon^2y^2+\varepsilon^2z^2+2\varepsilon x(x_o+u)\right]y}{(x_o+u)^5} -$$

$$\frac{15}{8}\frac{(1-u)\left[\varepsilon^2x^2+\varepsilon^2y^2+\varepsilon^2z^2+2\varepsilon x(x_o+u)\right]^2 y}{(x_o+u)^7} + \frac{35}{16}[\cdot]$$

$$+ \frac{3}{2}\,u\frac{\left[\varepsilon^2x^2+\varepsilon^2y^2+\varepsilon^2z^2+2\varepsilon x(x_o-1+u)\right]y}{(x_o-1+u)^5} - \frac{15}{8}\,u$$

$$\frac{\left[\varepsilon^2x^2+\varepsilon^2y^2+\varepsilon^2z^2+2\varepsilon x(x_o-1+u)\right]^2}{(x_o-1+u)^7}\,y + \frac{35}{16}[\cdot\cdot]\cdots\Big\} \tag{14}$$

$$\frac{d^2z}{dT^2} = (1+\delta)^2\Big\{-Az + \frac{3}{2}\frac{(1-u)\left[\varepsilon^2x^2+\varepsilon^2y^2+\varepsilon^2z^2\right.}{(x_o+u)^5}$$

$$\frac{\left.+2\varepsilon x(x_o+u)\right]z}{} - \frac{15}{8}\frac{(1-u)\left[\varepsilon^2x^2+\varepsilon^2y^2+\varepsilon^2z^2+\right.}{(x_o+u)^7}$$

$$\frac{\left.2\varepsilon x(x_o+u)\right]^2 z}{} + \frac{35}{16}[\cdot\ \cdot]\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{3}{2}\,u\frac{\left[\varepsilon^2x^2+\varepsilon^2y^2+\varepsilon^2z^2+2\varepsilon x(x_o-1+u)\right]}{(x_o-1+u)^5}\,z - \frac{15}{8}\,u$$

$$\frac{\left[\varepsilon^2x^2+\varepsilon^2y^2+\varepsilon^2z^2+2\varepsilon x(x_o-1+u)\right]^2}{(x_o-1+u)^7}\,z + \frac{35}{16}[\cdot]\cdots\Big\}$$

Where $A = \dfrac{(1-u)}{(x_o+u)^3} + \dfrac{u}{(x_o-1+u)^3}$

When $\varepsilon = \delta = 0$, equations (14) become

$$
\left.
\begin{aligned}
\frac{d^2x}{dT^2} - 2\frac{dy}{dT} &= (1+2A)x \\
\frac{d^2y}{dT^2} + 2\frac{dx}{dT} &= (1-A)y \\
\frac{d^2z}{dT^2} &= -Az
\end{aligned}
\right] \qquad (15)
$$

For $\varepsilon \neq 0, \delta \neq 0$ equations (14) may be written as

$$
\left.
\begin{aligned}
\frac{d^2x}{dT^2} - 2(1+\delta)\frac{dy}{dT} &= (1+\delta)^2\left[X_1 + X_2\varepsilon + X_3\varepsilon^2 + X_4\varepsilon^3 + \cdots\right. \\
\frac{d^2y}{dT^2} + 2(1+\delta)\frac{dx}{dT} &= (1+\delta)^2\left[Y_1 + Y_2\varepsilon + Y_3\varepsilon^2 + Y_4\varepsilon^3 + \cdots\right. \\
\frac{d^2z}{dT^2} &= (1+\delta)^2\left[Z_1 + Z_2\varepsilon + Z_3\varepsilon^2 + Z_4\varepsilon^3 + \cdots\right.
\end{aligned}
\right] \qquad (16)
$$

Where

$$
\left.
\begin{aligned}
X_1 &= (1+2A)x & X_2 &= \tfrac{3}{2}B(-2x^2+y^2+z^2) \\
X_3 &= 2C(2x^3-3xy^2-3xz^2) & X_4 &= \tfrac{1}{8}D(-40x^4-15y^4-15z^4 \\
& & &\quad +120x^2y^2+120x^2z^2-30y^2z^2)
\end{aligned}
\right| \qquad (17)
$$

$$Y_1 = (1-A)y \qquad Y_2 = 3Bxy$$

$$Y_3 = \frac{3C}{2}(-4x^2y + y^3 + yz^2) \qquad Y_4 = \frac{5D}{2}(4x^3y - 3xy^3 - 3xyz^2)$$

$$Z_1 = -Az \qquad Z_2 = 3Bxz \tag{17}$$

$$Z_3 = \frac{3C}{2}(-4x^2z + y^2z + z^3) \qquad Z_4 = \frac{5D}{2}(4x^3z - 3xy^2z - 3xz^3)$$

and where

$$A = \frac{1-u}{{}^{(o)}r_1^{3}} + \frac{u}{{}^{(o)}r_2^{3}} \qquad B = \begin{smallmatrix}+\\+\\-\end{smallmatrix}\frac{1-u}{{}^{(o)}r_1^{4}} \begin{smallmatrix}+\\-\\-\end{smallmatrix} \frac{u}{{}^{(o)}r_2^{4}}$$

$$C = \frac{1-u}{{}^{(o)}r_1^{5}} + \frac{u}{{}^{(o)}r_2^{5}} \qquad D = \begin{smallmatrix}+\\+\\-\end{smallmatrix}\frac{1-u}{{}^{(o)}r_1^{6}} \begin{smallmatrix}+\\-\\-\end{smallmatrix} \frac{u}{{}^{(o)}r_2^{6}} \tag{18}$$

The top, center and bottom signs are for points (a), (b) and (c) respectively. The symbol ${}^{(o)}r_1$ designates the distances from the major finite body to the points (a), (b) and (c) respectively and the symbol ${}^{(o)}r_2$ represents the same distances from the minor finite body.

Expansion in Series

Let x, y, z and δ be expanded in a power series in ε such that they are periodic with period $\frac{2\pi}{\sqrt{A}}$. Let

$$\left.\begin{aligned}
x &= x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + x_4\varepsilon^4 + \dots\dots \\
y &= y_1\varepsilon + y_2\varepsilon^2 + y_3\varepsilon^3 + y_4\varepsilon^4 + \dots\dots \\
z &= \frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + z_1\varepsilon + z_2\varepsilon^2 + z_3\varepsilon^3 + z_4\varepsilon^4 + \dots\dots \\
(1+\delta) &= 1 + \delta_2\varepsilon^2 + \delta_3\varepsilon^3 + \delta_4\varepsilon^4 + \dots\dots
\end{aligned}\right] \quad (19)$$

Then equations (14) become

$$\left.\begin{aligned}
&\left[\frac{d^2x_1}{dT^2} - 2\frac{dy_1}{dT}\right]\varepsilon + \left[\frac{d^2x_2}{dT^2} - 2\frac{dy_2}{dT}\right]\varepsilon^2 + \left[\frac{d^2x_3}{dT^2} - 2\left(\frac{dy_3}{dT} + \delta_2\frac{dy_1}{dT}\right)\right]\varepsilon^3 + \\
&\left[\frac{d^2x_4}{dT^2} - 2\left(\frac{dy_4}{dT} + \delta_2\frac{dy_2}{dT} + \delta_3\frac{dy_1}{dT}\right)\right]\varepsilon^4 + \dots\dots = \\
&\left[(1+2A)x_1 + \frac{3B}{2}\frac{c^2}{A}\sin^2\sqrt{A}\,T\right]\varepsilon + \left[(1+2A)x_2 + 3Bz_1\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T\right]\varepsilon^2 \\
&+ \left[(1+2A)x_3 + \frac{3}{2}B(-2x_1^2 + y_1^2 + z_1^2) + 3Bz_2\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T \right. -
\end{aligned}\right| \quad (20)$$

$$6Cx_1\frac{c^2}{A}\sin^2\sqrt{A}\,T + 2\delta_2(1+2A)x_1 + 3\delta_2 B\frac{c^2}{A}\sin^2\sqrt{A}\,T -$$

$$\frac{15}{8}D\frac{c^4}{A^2}\sin^4\sqrt{A}\,T\Big]\varepsilon^3 + \Big[(1+2A)x_4 + \frac{3}{2}B(-4x_1x_2 + 2y_1y_2$$

$$+ 2z_3\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + 2z_1z_2) - 6C(2x_1z_1\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T +$$

$$x_2\frac{c^2}{A}\sin^2\sqrt{A}\,T) + 2\delta_2(1+2A)x_2 + 6B\delta_2 z_1\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T +$$

$$2\delta_3(1+2A)x_1 + 3\delta_3 B\frac{c^2}{A}\sin^2\sqrt{A}\,T - \frac{15}{2}Dz_1\frac{c^3}{A^{3/2}}\sin^3\sqrt{A}\,T\Big]\varepsilon^4$$

$$+ \ldots\ldots\ldots\ldots\ldots$$

(20)

$$\Big[\frac{d^2y_1}{dT^2} + 2\frac{dx_1}{dT}\Big]\varepsilon + \Big[\frac{d^2y_2}{dT^2} + 2\frac{dx_2}{dT}\Big]\varepsilon^2 + \Big[\frac{d^2y_3}{dT^2} + 2(\frac{dx_3}{dT} + \delta_2\frac{dx_1}{dT})\Big]\varepsilon^3$$

$$+ \Big[\frac{d^2y_4}{dT^2} + 2(\frac{dx_4}{dT} + \delta_2\frac{dx_2}{dT} + \delta_3\frac{dx_1}{dT})\Big]\varepsilon^4 + \ldots\ldots = (1-A)y_1\varepsilon$$

$$+ (1-A)y_2\varepsilon^2 + \Big[(1-A)y_3 + 3Bx_1y_1 + \frac{3}{2}C\,y_1\frac{c^2}{A}\sin^2\sqrt{A}\,T +$$

$$2\delta_2(1-A)y_1\Big]\varepsilon^3 + \Big[(1-A)y_4 + 3B(x_1y_2 + x_2y_1) + \frac{3}{2}C(2y_1z_1\frac{c}{\sqrt{A}}$$

$$\sin\sqrt{A}\,T + y_2\frac{c^2}{A}\sin^2\sqrt{A}\,T\,) + 2\,\delta_2(1-A)y_2 + 2\,\delta_3(1-A)y_1\Big]\varepsilon^4$$

$$+ \cdots\cdots\cdots\cdots$$

$$\begin{aligned}
&\frac{d^2z_1}{dT^2}\varepsilon + \frac{d^2z_2}{dT^2}\varepsilon^2 + \frac{d^2z_3}{dT^2}\varepsilon^3 + \frac{d^2z_4}{dT^2}\varepsilon^4 + \cdots\cdots\cdots = -Az_1\varepsilon + \\
&\Big[-Az_2 + 3Bx_1\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + \frac{3}{2}C\,\frac{c^3}{A^{3/2}}\sin^3\sqrt{A}\,T \;-2\delta_2\sqrt{A}c\,\sin\sqrt{A}\,T\Big]\varepsilon^2 \\
&+ \Big[-Az_3 + 3B(x_2\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + x_1z_1) + \frac{9}{2}Cz_1\frac{c^2}{A}\sin^2\sqrt{A}\,T - 2\delta_2 Az_1 \\
&-2\delta_3\sqrt{A}c\,\sin\sqrt{A}\,T\Big]\varepsilon^3 + \Big[-Az_4 + 3B(x_3\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + x_1z_2 + x_2z_1) \\
&+ \frac{3}{2}C(-4x_1^2\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + 3z_2\frac{c^2}{A}\sin^2\sqrt{A}\,T + 3z_1^2\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T) \\
&+ \frac{3}{2}Cy_1^2\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T - \frac{15}{2}Dx_1\frac{c^3}{A^{3/2}}\sin^3\sqrt{A}\,T - 2\delta_2 Az_2 + \\
&6\,\delta_2 Bx_1\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + 3\,\delta_2 C\,\frac{c^3}{A^{3/2}}\sin^3\sqrt{A}\,T - 2\,\delta_3 Az_1 - \\
&(2\delta_4 + \delta_2^2)\sqrt{A}\,c\,\sin\sqrt{A}\,T\Big]\varepsilon^4 + \cdots\cdots\cdots\cdots
\end{aligned} \qquad (20)$$

Initial Conditions of Motion

Let the infinitesimal body be given a constant velocity c at the time $T = 0$. Since

$$z = \frac{c}{\sqrt{A}} \sin \sqrt{A}\, T + z_1\epsilon + z_2\epsilon^2 + z_3\epsilon^3 + z_4\epsilon^4 + \cdots$$

$$z' = c \cos \sqrt{A}\, T + z_1'\epsilon + z_2'\epsilon^2 + z_3'\epsilon^3 + z_4'\epsilon^4 + \cdots$$

we have for $T = 0$

$$\left.\begin{aligned} &z = 0; \quad z_1 = z_2 = z_3 = z_4 = \ldots\ldots = 0 \\ &z' = c; \quad z_1' = z_2' = z_3' = z_4' = \ldots\ldots = 0 \end{aligned}\right] \qquad (21)$$

Coefficients of ϵ

From equations (20) we have

$$\left.\begin{aligned} &\frac{d^2x_1}{dT^2} - 2\frac{dy_1}{dT} - (1+2A)x_1 = \frac{3}{2}B\,\frac{c^2}{A}\sin^2\sqrt{A}\,T \\ &\frac{d^2y_1}{dT^2} + 2\frac{dx_1}{dT} - (1-A)y_1 = 0 \\ &\frac{d^2z_1}{dT^2} + Az_1 = 0 \end{aligned}\right] \qquad (22)$$

On solving the first two of these equations simultaneously we find that

$$x_1 = C_1 e^{\sqrt{\frac{(A-2) + \sqrt{9A^2-8A}}{2}}\,T} + C_2 e^{-\sqrt{\frac{(A-2) + \sqrt{9A^2-8A}}{2}}\,T}$$

$$+ C_3 \sin \sqrt{\frac{(A-2)-\sqrt{9A^2-8A}}{2}}\,T + C_4 \sin \sqrt{\frac{(A-2)-\sqrt{9A^2-8A}}{2}}\,T$$

$$- \frac{3Bc^2}{4A(1+2A)} + \frac{3(3A+1)Bc^2}{4A(1-7A+18A^2)} \cos 2\sqrt{A}\,T.$$

The first and second of these terms are rejected because they are not periodic. The third and fourth terms must also be rejected because they are not of the period $\frac{2\pi}{\sqrt{A}}$ as restricted on page 20. Therefore,

$$C_1 = C_2 = C_3 = C_4 = 0 \tag{23}$$

When this value of x_1 is substituted in (22) we find y_1.

On solving the last of equations (20) we find

$$z_1 = C_5 \sin \sqrt{A}\,T + C_6 \cos \sqrt{A}\,T \quad \text{and}$$

$$z_1' = C_5 \sqrt{A} \cos \sqrt{A}\,T - C_6 \sqrt{A} \sin \sqrt{A}\,T$$

Using equations (21) we find $C_5 = C_6 = 0$. Hence

$$\left.\begin{aligned}
x_1 &= -\frac{3Bc^2}{4A(1+2A)} + \frac{3(3A+1)Bc^2}{4A(1-7A+18A^2)} \cos 2\sqrt{A}\,T \\
y_1 &= -\frac{3Bc^2}{\sqrt{A}\,(1-7A+18A^2)} \sin 2\sqrt{A}\,T \\
z_1 &= 0
\end{aligned}\right] \tag{24}$$

Coefficients of ϵ^2

From equations (20) we have

$$\left.\begin{aligned}
&\frac{d^2x_2}{dT^2} - 2\frac{dy_2}{dT} - (1+2A)x_2 = 3Bz_1\,\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T \\
&\frac{d^2y_2}{dT^2} + 2\frac{dx_2}{dT} - (1-A)y_2 = 0 \\
&\frac{d^2z_2}{dT^2} + Az_2 = 3Bx_1\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + \frac{3}{2}C\,\frac{c^3}{A^{3/2}}\sin^3\sqrt{A}\,T - 2\sqrt{A}c\delta_2\sin\sqrt{A}\,T
\end{aligned}\right] \quad (25)$$

When equations (24) are substituted in (25) we obtain

$$\left.\begin{aligned}
&\frac{d^2x_2}{dT^2} - 2\frac{dy_2}{dT} - (1+2A)x_2 = 0 \\
&\frac{d^2y_2}{dT^2} + 2\frac{dx_2}{dT} - (1-A)y_2 = 0 \\
&\frac{d^2z_2}{dT^2} + Az_2 = \left[-2\delta_2 c\sqrt{A} - \frac{27B^2c^3(1-3A+14A^2)}{8A^{3/2}(1+2A)(1-7A+18A^2)} + \frac{9Cc^3}{8A^{3/2}}\right]\sin\sqrt{A}\,T \\
&\qquad\qquad + \left[\frac{9B^2c^3(3A+1)}{8A^{3/2}(1-7A+18A^2)} - \frac{3Cc^3}{8A^{3/2}}\right]\sin 3\sqrt{A}\,T
\end{aligned}\right] \quad (26)$$

The solutions of the first two of these equations are $x_2 = y_2 = 0$ since, as in (23), we are to reject all terms which are neither periodic nor of period $\frac{2\pi}{\sqrt{A}}$.

On solving the last of equations (26) we get

$$z_2 = C_1 \sin\sqrt{A}\, T + C_2 \cos\sqrt{A}\, T + \left[\frac{27B^2c^3(1-3A+14A^2)}{16A^2(1+2A)(1-7A+18A^2)} - \frac{9Cc^3}{16A^2} + 2\delta_2 c \right] T \cos\sqrt{A}\, T + \left[\frac{3Cc^3}{64A^{\frac{5}{2}}} - \frac{9(3A+1)B^2c^3}{64A^{\frac{5}{2}}(1-7A+18A^2)} \right] \sin 3\sqrt{A}\, T.$$

The third term of this expression is non periodic and must vanish. Hence

$$\delta_2 = - \frac{27B^2c^2(1-3A+14A^2)}{16A^2(1+2A)(1-7A+18A^2)} + \frac{9Cc^2}{16A^2} \qquad (27)$$

By (21) $C_2 = 0$ and since

$$z_2' = C_1\sqrt{A} \cos\sqrt{A}\, T + 3\sqrt{A}\left[\frac{3Cc^3}{64A^{\frac{5}{2}}} - \frac{9(3A+1)B^2c^3}{64A^{\frac{5}{2}}(1-7A+18A^2)} \right] \cos 3\sqrt{A}\, T.$$

$$C_1 = -3\left[\frac{3Cc^3}{64A^{\frac{5}{2}}} - \frac{9(3A+1)B^2c^3}{64A^{\frac{5}{2}}(1-7A+18A^2)} \right].$$ Hence

$$\left.\begin{aligned} x_2 &= 0 \\ y_2 &= 0 \\ z_2 &= \left[\frac{9(3A+1)B^2c^3}{64A^{\frac{5}{2}}(1-7A+18A^2)} - \frac{3Cc^3}{64A^{\frac{5}{2}}} \right] \left[3\sin\sqrt{A}\, T - \sin 3\sqrt{A}\, T \right] \end{aligned}\right] \qquad (28)$$

Coefficients of ε^3

From equations (20) we have

$$
\left.
\begin{aligned}
&\frac{d^2x_3}{dT^2}-2\frac{dy_3}{dT}-(1+2A)x_3=\frac{3}{2}B\,(-2x_1^{\,2}+y_1^{\,2}+z_1^{\,2})+\\
&3Bz_2\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T-6Cx_1\frac{c^2}{A}\sin^2\sqrt{A}\,T+2\,\delta_2(1+2A)x_1+\\
&3\,\delta_2 B\,\frac{c^2}{A}\sin^2\sqrt{A}\,T-\frac{15}{8}D\,\frac{c^4}{A^2}\sin^4\sqrt{A}\,T+2\,\delta_2\frac{dy_1}{dT}\\[2ex]
&\frac{d^2y_3}{dT^2}+2\frac{dx_3}{dT}-(1-A)y_3=3Bx_1y_1+\frac{3}{2}Cy_1\frac{c^2}{A}\sin^2\sqrt{A}\,T+\\
&2\,\delta_2(1-A)y_1\;-2\,\delta_2\frac{dx_1}{dT}\\[2ex]
&\frac{d^2z_3}{dT^2}+Az_3=3Bx_2\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T+3Bx_1z_1+\frac{9}{2}Cz_1\frac{c^2}{A}\sin^2\sqrt{A}\,T\\
&-2\,\delta_2Az_1\;-2\,\delta_3\sqrt{A}\,c\,\sin\sqrt{A}\,T.
\end{aligned}
\right]\quad(29)
$$

When equations (24), (27), and (28) are substituted in (29) we obtain

$$
\left.
\begin{aligned}
&\frac{d^2x_3}{dT^2} - 2\frac{dy_3}{dT} - (1 + 2A)x_3 = (M + N\cos 2\sqrt{A}\,T + P\cos 4\sqrt{A}\,T)c^4 \\
&\frac{d^2y_3}{dT^2} + 2\frac{dx_3}{dT} - (1 - A)y_3 = (Q\sin 2\sqrt{A}\,T + R\sin 4\sqrt{A}\,T)c^4 \\
&\frac{d^2z_3}{dT^2} + Az_3 = -2\delta_3\sqrt{A}\,c\sin\sqrt{A}\,T
\end{aligned}
\right\rbrack \qquad (30)
$$

where

$$
\begin{aligned}
&H = \frac{-3B}{4A(1+2A)} \qquad J = \frac{-3B}{\sqrt{A}(1-7A+18A^2)} \\
&I = \frac{3B(1+3A)}{4A(1-7A+18A^2)} \qquad K = \frac{-27B^2(1-3A+14A^2)}{16A^2(1+2A)(1-7A+18A^2)} + \frac{9C}{16A^2} \\
&L = \frac{3}{64A^{5/2}}\left[\frac{3B^2(1+3A)}{1-7A+18A^2} - C\right] \\
&M = -3BH^2 - \frac{3}{2}BI^2 + \frac{9BL}{2\sqrt{A}} - \frac{3CH}{A} + \frac{3CI}{2A} - \frac{45D}{64A^2} + 2KH(1+2A) + \\
&\qquad \frac{3}{4}BJ2 + \frac{3KB}{2A} \\
&N = 4JK\sqrt{A} - 6BHI - \frac{6BL}{\sqrt{A}} + \frac{3CH}{A} - \frac{3CI}{A} + \frac{15D}{16A^2} - \frac{3KB}{2A}
\end{aligned}
\qquad (31)
$$

$$+ 2KI(1+2A)$$

$$P = -\frac{3BJ^2}{4} - \frac{3BI^2}{2} + \frac{3BL}{2\sqrt{A}} + \frac{3CI}{2A} - \frac{15D}{64A^2}$$

$$Q = 4\sqrt{A}IK + 3BHJ + \frac{3CJ}{4A} + 2KJ(1-A) \tag{31}$$

$$R = \frac{3}{2}BIJ - \frac{3CJ}{8A}$$

When we solve the third equation in (30) we get

$$z_3 = C_1 \sin\sqrt{A}\,T + C_2 \cos\sqrt{A}\,T + c\delta_3 T \cos\sqrt{A}\,T.$$

The third term is non periodic and therefore

$$\delta_3 = 0 \tag{32}$$

C_1 and C_2 = 0 by (21), and therefore $z_3 = 0$.

On solving the first two equations of (30) and making use of (23) we have

$$x_3 = \frac{-Mc^4}{(1+2A)} + \frac{4Q\sqrt{A} - 3AN - N}{1-7A + 18A^2}c^4 \cos 2\sqrt{A}\,T + \frac{8R\sqrt{A}-15AP-P}{1-31A+270A^2}c^4$$

$$\cos 4\sqrt{A}\,T.$$

$$y_3 = \frac{4N\sqrt{A}-Q-6AQ}{1-7A+18A^2}c^4 \sin 2\sqrt{A}\,T - \frac{R-8P\sqrt{A}+18RA}{1-31A+270A^2}c^4 \sin 4\sqrt{A}\,T \tag{33}$$

$$z_3 = 0$$

Coefficients of ε^4

From equations (20) we have

$$\frac{d^2x_4}{dT^2} - 2\frac{dy_4}{dT} - (1+2A)x_4 = \frac{3}{2}B(-4x_1x_2 + y_1y_2 + 2z_3\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T$$

$$+ 2z_1z_2) - 6C(2x_1z_1\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + x_2\frac{c^2}{A}\sin^2\sqrt{A}\,T) +$$

$$2\delta_2(1+2A)x_2 + 6B\delta_2 z_1\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + 2\delta_3(1+2A)x_1 +$$

$$3\delta_3 B\frac{c^2}{A}\sin^2\sqrt{A}\,T - \frac{15}{2}Dz_1\frac{c^3}{A^{3/2}}\sin^3\sqrt{A}\,T + 2\delta_2\frac{dy_2}{dT} + 2\delta_3\frac{dy_1}{dT}$$

$$\frac{d^2y_4}{dT^2} + 2\frac{dx_4}{dT} - (1-A)y_4 = 3B(x_1y_2 + x_2y_1) + \frac{3}{2}C(2y_1z_1\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T \qquad (34)$$

$$+ y_2\frac{c^2}{A}\sin^2\sqrt{A}\,T) + 2\delta_2(1-A)y_2 + 2\delta_3(1-A)y_1 - 2\delta_2\frac{dx_2}{dT} -$$

$$2\delta_3\frac{dx_1}{dT}\,.$$

$$\frac{d^2z_4}{dT^2} + Az_4 = 3B(x_3\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + x_1z_2 + x_2z_1) + \frac{3}{2}C(-4x_1^2\frac{c}{\sqrt{A}}$$

$$
\left.
\begin{aligned}
&\sin\sqrt{A}\,T + 3z_2\frac{c^2}{A}\sin^2\sqrt{A}\,T + 3z_1^2\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T) + \\
&\frac{3}{2}Cy_1^2\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T - \frac{15}{2}Dx_1\frac{c^3}{A^{\frac{3}{2}}}\sin^3\sqrt{A}\,T - 2\delta_2 Az_2 + \\
&6\delta_2 Bx_1\frac{c}{\sqrt{A}}\sin\sqrt{A}\,T + 3\delta_2 C\frac{c^3}{A^{\frac{3}{2}}}\sin^3\sqrt{A}\,T - 2\delta_3 Az_1 - \\
&(2\delta_4 + \delta_2^{\,2})\sqrt{A}\,c\sin\sqrt{A}\,T\,.
\end{aligned}
\right\rbrack \quad (34)
$$

When equations (24), (27), (28), (31), (32), and (33) are substituted in equations (34) we obtain

$$
\left.
\begin{aligned}
&\frac{d^2x_4}{dT^2} - 2\frac{dy_4}{dT} - (1+2A)x_4 = \quad 0 \\
&\frac{d^2y_4}{dT^2} + 2\frac{dx_4}{dT} - (1-A)y_4 = \quad 0 \\
&\frac{d^2z_4}{dT^2} + Az_4 = S\sin\sqrt{A}\,T + V\,c^5\sin 3\sqrt{A}\,T + W\,c^5\sin 5\sqrt{A}\,T
\end{aligned}
\right\rbrack \quad (35)
$$

where

$$
S = \left[\frac{-3BM}{\sqrt{A}(1+2A)} + 9BHL - \frac{6CH^2}{\sqrt{A}} - 6KAL + \frac{6KBH}{\sqrt{A}} - K^2\sqrt{A} - \right. \quad (36)
$$

$$\frac{3B(4Q\sqrt{A} - 3AN - N)}{2\sqrt{A}(18A^2 - 7A + 1)} - 6BIL - \frac{6CHI}{\sqrt{A}} - \frac{3KBI}{\sqrt{A}} + \frac{45CL}{4A} + \frac{3CJ^2}{4\sqrt{A}}$$

$$\left. - 3\frac{CI^2}{\sqrt{A}} - \frac{45DH}{8A^{5/2}} + \frac{15DI}{4A^{3/2}} + \frac{9KC}{4A^{3/2}} \right] c^5 - 2\,\delta_4\sqrt{A}\ c$$

$$V = \frac{3B(4Q\sqrt{A} - 3AN - N)}{2\sqrt{A}(18A^2 - 7A + 1)} + \frac{9BIL}{2} - \frac{6CHI}{\sqrt{A}} + \frac{3KBI}{\sqrt{A}} - 3BHL +$$

$$2KAL - \frac{45CL}{8A} + \frac{3CJ^2}{8\sqrt{A}} + \frac{3CI^2}{2\sqrt{A}} + \frac{15DH}{8A^{3/2}} - \frac{45DI}{16A^{3/2}} - \frac{3KC}{4A^{3/2}} \qquad (36)$$

$$- \frac{3B(8R\sqrt{A} - 15AP - P)}{2\sqrt{A}(270A^2 - 31A + 1)}$$

$$W = - \frac{3CI^2}{2\sqrt{A}} - \frac{3BIL}{2} + \frac{9CL}{8A} + \frac{15DI}{16A^{3/2}} + \frac{3B(8R\sqrt{A} - 15AP - P)}{2\sqrt{A}(270A^2 - 31A + 1)} - \frac{3CJ^2}{8\sqrt{A}}$$

The solutions of the first two equations of (35) are by (23) $x_4 = y_4 = 0$. The solution of the last equation in (35) is

$$z_4 = C_1 \sin\sqrt{A}\ T + C_2 \cos\sqrt{A}\ T - \frac{S}{2\sqrt{A}}\ T \cos\sqrt{A}\ T - \frac{Vc^5}{8A} \sin 3\sqrt{A}\ T$$

$$- \frac{Wc^5}{24A} \sin 5\sqrt{A}\ T.$$

The coefficient of $T \cos\sqrt{A}\, T$ must be zero since this is not a periodic term. Therefore $S = 0$, or δ_4 must be such that

$$\delta_4 = \left[-\frac{3BM}{2A(1+2A)} + \frac{9BHL}{2\sqrt{A}} - \frac{3KAL}{\sqrt{A}} + \frac{3KBH}{A} - \frac{K^2}{2} - \frac{6BIL}{2\sqrt{A}} - \frac{3CH^2}{A} - \frac{3B(4Q\sqrt{A} - 3AN - N)}{4A(18A^2 - 7A + 1)} - \frac{3CHI}{A} - \frac{3KBI}{2A} - \frac{45CL}{8A^{3/2}} + \frac{3CJ^2}{8A} - \frac{3CI^2}{2A} - \frac{45DH}{16A^2} + \frac{15DI}{8A^2} + \frac{9KC}{8A^2} \right] c^4 \qquad (37)$$

The constant $C_2 = 0$ by (21). Hence

$$z_4 = C_1 \sin\sqrt{A}\, T - \frac{Vc^5}{8A} \sin 3\sqrt{A}\, T - \frac{Wc^5}{24A} \sin 5\sqrt{A}\, T .$$

$$z_4' = C_1 \sqrt{A} \cos\sqrt{A}\, T - \frac{3V\sqrt{A}c^5}{8A} \cos 3\sqrt{A}\, T - \frac{5W\sqrt{A}c^5}{24A} \cos 5\sqrt{A}\, T$$

Therefore, by (21), $C_1 = \frac{9V + 5W}{24A} c^5$ and we have

$$\left.\begin{aligned} x_4 &= 0 \\ y_4 &= 0 \\ z_4 &= \frac{(9V + 5W)}{24A} c^5 \sin\sqrt{A}\, T - \frac{Vc^5}{8A} \sin 3\sqrt{A}\, T - \frac{Wc^5}{24A} \sin 5\sqrt{A}\, T \end{aligned}\right] \qquad (38)$$

Orbit around Point (a)

Let us assume the two major bodies to be the Sun and Jupiter. μ will then be approximately .001 and $1-\mu$ will be .999. The constants will then be as follows for the point (a).

Constant	Value
$r_2^{(o)}$	.070902
$r_1^{(o)}$	1.070902
A	3.619065
B	40.330485
C	558.81970
D	7872.2828
H	-1.014541
I	.468734
J	-.300817
K	3.122813
L	-.536456
M	-30.347163
N	24.464192
P	-65.393526

Constant	Value
Q	18.147951
R	8.888348
$\frac{-M}{1+2A}$	3.683745
$\frac{4\sqrt{A}Q-3AN-M}{18A^2-7A+1}$	-.718837
$\frac{4\sqrt{A}N-Q-6AQ}{18A^2-7A+1}$	-1.069219
$\frac{R-8P\sqrt{A}+18RA}{1-31A+270A^2}$	.462206
$\frac{8\sqrt{A}R-15AP-P}{1-31A+270A^2}$	1.095017
δ_4	$1752.0296c^4$
V	-2549.90818
W	352.53022
$\frac{9V+5W}{24A}$	-243.92264
$\frac{V}{8A}$	-88.072075
$\frac{W}{24A}$	4.058717

These values are substituted in equations (24), (28), (33) and (38) and the results substituted in equations (19). At the same time we make use of the fact that $x' = \frac{x}{\varepsilon}$, $y' = \frac{y}{\varepsilon}$, and $z' = \frac{z}{\varepsilon}$. Let us write $c\varepsilon = k$ since c and ε appear in each term in the same degree. We then have

$$\left.\begin{aligned}
x' &= -1.014541k^2 + .468734k^2\cos 2\sqrt{AT} + 3.683745k^4 \\
&\quad -.718837k^4\cos 2\sqrt{AT} + 1.095017k^4\cos 4\sqrt{AT} + \dots \\
y' &= -.300817k^2\sin 2\sqrt{AT} - 1.069219k^4\sin 2\sqrt{AT} \\
&\quad -.462206k^4\sin 4\sqrt{AT} + \dots \\
z' &= .525656k\sin\sqrt{AT} - 1.609368k^3\sin\sqrt{AT} + .536456k^3 \\
&\quad \sin 3\sqrt{AT} - 243.92264k^5\sin\sqrt{AT} + 88.072075k^5\sin 3\sqrt{AT} \\
&\quad -4.058717k^5\sin 5\sqrt{AT} + \dots \\
\text{Period} &= 3.302796(1 + 3.122813k^2 + 1752.0296k^4 + \dots
\end{aligned}\right] \quad (39)$$

Orbits around Point (b)

Let us now take the same two major bodies, the Sun and Jupiter, and with the same masses as before. We find the following values for point (b).

Constant	Value
$r_2^{(o)}$	.067697
$r_1^{(o)}$	.932303
A	4.456094
B	-46.291469
C	704.75932
D	-10388.0854
H	.786028
I	-.342106
J	.201045
K	5.050581
L	-.472459
M	52.887090
N	-39.024429
P	66.526244
Q	-19.706580

Constant	Value
R	-7.147895
$\frac{-M}{1 + 2A}$	-5.335561
$\frac{4Q\sqrt{A} - 3AN - N}{1 - 7A + 18A^2}$	1.205014
$\frac{4\sqrt{A}N - Q - 6AQ}{1 - 7A + 18A^2}$	.663373
$\frac{8R\sqrt{A} - 15AP - P}{1 - 31A + 270A^2}$	-.887088
$\frac{R - 8P\sqrt{A} + 18RA}{1 - 31A + 270A^2}$	-3.261652
δ_4	$1295.2523c^4$
V	2012.966297
W	246.859163
$\frac{9V + 5W}{24A}$	-157.85873
$\frac{V}{8A}$	-56.466690
$\frac{W}{24A}$	2.308254

When these values are substituted in equations (24), (28), (33) and (38) and the results substituted in equations (19) in the same manner as was done for point (a), we have, letting $c\varepsilon = k$ as before

$$\left.\begin{aligned}
x' &= .786028k^2 - .342106k^2\cos 2\sqrt{AT} - 5.335561k^4 \\
&\quad + 1.205014k^4\cos 2\sqrt{AT} - .887088k^4\cos 4\sqrt{AT} + \cdots \\
y' &= .201045k^2\sin 2\sqrt{AT} + .663373k^4\sin 2\sqrt{AT} \\
&\quad + 3.261652k^4\sin 4\sqrt{AT} + \cdots \\
z' &= .473721k\sin\sqrt{AT} - 1.417376k^3\sin\sqrt{AT} + .472459k^3 \\
&\quad \sin 3\sqrt{AT} - 157.85873k^5\sin\sqrt{AT} + 56.46669k^5\sin 3\sqrt{AT} \\
&\quad - 2.308254k^5\sin 5\sqrt{AT} + \cdots \\
\text{Period} &= 2.976478(1 + 5.050581k^2 + 1295.2523k^4 + \cdots
\end{aligned}\right] \quad (40)$$

Orbit around Point (c)

Again let us take the Sun and Jupiter to be the two major bodies with the same masses as for points (a) and (b). The constants determined for point (c) will then be as follows

Constant	Value
$r_1^{(o)}$	.999417
$r_2^{(o)}$	1.999417
A	1.000875
B	-1.001397
C	1.001950
D	-1.002519
H	.249984
I	-.249766
J	.249711
K	.000018
L	-.000029
M	-.187389
N	.187308

Constant	Value
P	.000099
Q	−.000066
R	.000068
$\frac{-M}{1 + 2A}$	.062427
$\frac{4Q\sqrt{A} - 3AN - N}{1 - 7A + 18A^2}$	−.062367
$\frac{4\sqrt{A}N - Q - 6AQ}{1 - 7A + 18A^2}$	.062370
$\frac{8R\sqrt{A} - 15AP - P}{1 - 31A + 270A^2}$	−.000004
$\frac{R - 8P\sqrt{A} + 18RA}{1 - 31A + 270A^2}$	.000002
δ_4	$.961289c^4$
V	−.586854
W	.117288
$\frac{9V + 5W}{24A}$	−.195464
$\frac{V}{8A}$	−.073293
$\frac{W}{24A}$	.004883

When these values are substituted in equations (24), (28), (33) and (38) and the results substituted in equations (19) as was done for point (a), we have, after again letting $c\varepsilon = k$

$$
\left.
\begin{aligned}
x' &= .249984k^2 - .249766k^2\cos 2\sqrt{A}T + .062427k^2 - \\
&\quad .062367k^4\cos 2\sqrt{A}T - .000004k^4\cos 4\sqrt{A}T + \ldots \\
y' &= .249711k^2\sin 2\sqrt{A}T + .062370k^4\sin 2\sqrt{A}T \\
&\quad - .000002k^4\sin 4\sqrt{A}T + \ldots \\
z' &= .999563k\sin\sqrt{A}T - .000086k^3\sin\sqrt{A}T + .000029k^3 \\
&\quad \sin 3\sqrt{A}T - .195464k^5\sin\sqrt{A}T + .073293k^5\sin 3\sqrt{A}T \\
&\quad - .004883k^5\sin 5\sqrt{A}T + \ldots \\
\text{Period} &= 6.280437(\,1 + .000018k^2 + .961289k^4 + ..
\end{aligned}
\right] \qquad (41)
$$

Orbits for $k = .1$

By assigning $k = .1$ in equations (39), (40) and (41) we obtain the three equations for the orbits about (a), (b) and (c)

Point (a)

$$\left.\begin{aligned}
x' &= -.009777 + .004615\cos 2\sqrt{A}T + .000110\cos 4\sqrt{A}T + \ldots \\
y' &= -.003115\sin 2\sqrt{A}T - .000046\sin 4\sqrt{A}T + \ldots \\
z' &= .049517\sin\sqrt{A}T + .001417\sin 3\sqrt{A}T - .000041\sin 5\sqrt{A}T + \ldots \\
\text{Period} &= 3.302796(1 + .031228 + .175203 + \ldots
\end{aligned}\right] \quad (42)$$

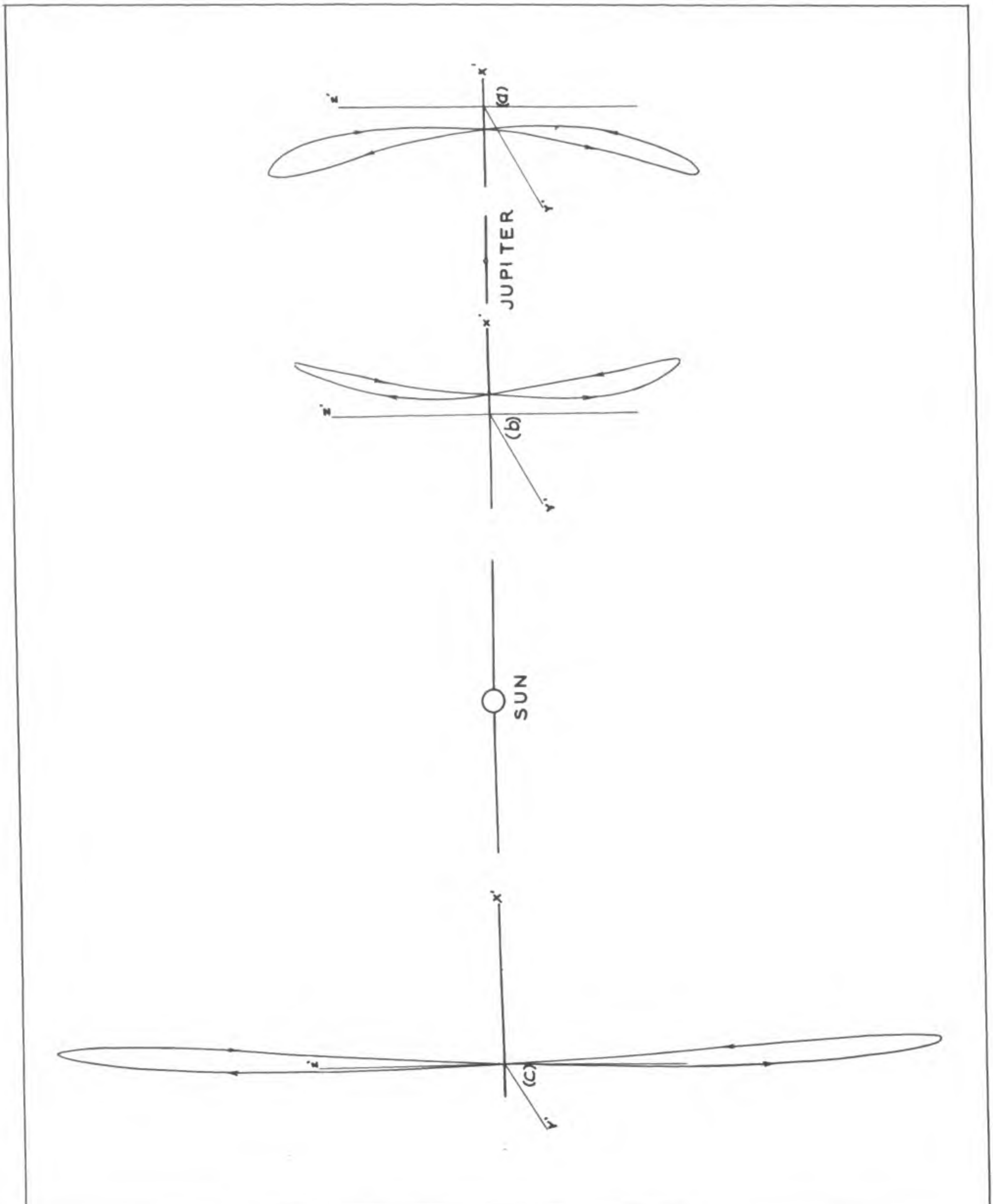
Point (b)

$$\left.\begin{aligned}
x' &= .007327 - .003301\cos 2\sqrt{A}T - .000089\cos 4\sqrt{A}T + \ldots \\
y' &= .002077\sin 2\sqrt{A}T + .000326\sin 4\sqrt{A}T + \ldots \\
z' &= .044376\sin\sqrt{A}T + .001037\sin 3\sqrt{A}T - .000023\sin 5\sqrt{A}T + \ldots \\
\text{Period} &= 2.976478(1 + .050506 + .129525 + \ldots
\end{aligned}\right] \quad (43)$$

Point (c)

$$\left.\begin{aligned}
x' &= .002500 - .002504\cos 2\sqrt{A}T + .000000\cos 4\sqrt{A}T + \ldots \\
y' &= .002503\sin 2\sqrt{A}T + .000000\sin 4\sqrt{A}T + \ldots \\
z' &= .099958\sin\sqrt{A}T + .000000\sin 3\sqrt{A}T + \ldots \\
\text{Period} &= 6.280437(1 + .000000 + .000096 + \ldots
\end{aligned}\right] \quad (44)$$

We assign values to T to obtain the orbits about the respective points which are shown on the graph on the following page.



JUPITER
SUN
(a)
(b)
(c)
x'
y'
z'

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OSCILLATING SATELLITES ABOUT THE STRAIGHT LINE EQUILIBRIUM POINTS

ABSTRACT

The problem as studied in this thesis is the behavior of a satellite at the three equilibrium points. When a satellite is displaced from rest at one of these points it will either oscillate or rapidly leave the system. The equations of motion were first set up between the satellite and two other rotating bodies.

These equations were then expanded in series and certain restrictions placed upon them. With these restrictions, constants of integration were found and the integrated equations of motion were also found.

An application was then made to the Solar System in which the Sun and Jupiter were taken as the major bodies. The masses of these two bodies were then substituted into the equations of motion and equations were found for the three points of equilibrium.

The orbits were then plotted at the three points.

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