ABSTRACT

Inverse Limits with Irreducible Set-valued Functions

James Pierre Kelly, Ph.D.

Advisor: David J. Ryden, Ph.D

We define a class of set-valued functions called irreducible functions and show that their inverse limits are indecomposable continua. We go on to further explore this class of inverse limit spaces. This includes a characterization of chainability and a characterization of endpoints of inverse limits of certain irreducible functions. Additionally, we develop multiple tools for determining when two inverse limits of irreducible functions are or are not homeomorphic. This culminates in a topological classification of the inverse limits of four specific families of irreducible functions.

Inverse Limits with Irreducible Set-valued Functions

by

James Pierre Kelly, B.S.

A Dissertation

Approved by the Department of Mathematics

Lance L. Littlejohn, Ph.D., Chairperson

Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

Approved by the Dissertation Committee

David J. Ryden, Ph.D, Chairperson

Jonathan Meddaugh, Ph.D.

Brian E. Raines, D.Phil.

James D. Stamey, Ph.D.

Ronald J. Stanke, Ph.D.

Accepted by the Graduate School
May 2015

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I would like to first thank all of the members of my dissertation committee for taking the time to read this and provide valuable input. I would also like to specifically thank my advisor, Dr. David Ryden, for all of his help and support throughout this process. I appreciate all of the guidance he has given me on both the research and the writing process. Special thanks are also due to Dr. Jonathan Meddaugh for helping me get started in my research and continually being a soundboard for my ideas (both the good ones and the bad ones).

I am also appreciative of my family for their continued support and encouragement. The investments my parents have made in my education have helped me reach this point and made possible my academic achievements.

Finally, I would like to thank my wonderful fiancée, Jessica. Her love and joyful spirit are a constant source of happiness in my life. I am thankful that she puts up with me and does not mind too much when I spend our time together thinking about math.

To the memory of my grandfather, Douglas Scott,
from whom I inherited my love of math (and cats)

CHAPTER ONE

Introduction

The topic of inverse limits has been studied in the context of continuum theory for decades. Inverse limits provide a useful tool both for generating interesting spaces and for studying known spaces. In 2004, Mahavier began the study of a generalized notion of inverse limits which accommodates set-valued functions [40]. In that paper, only set-valued functions on $[0, 1]$ are considered, but in 2006, Ingram and Mahavier generalized the results to set-valued functions on compact Hausdorff spaces [29]. In these papers, it is shown that many of the results concerning inverse limits of continuous, single-valued functions fail to hold in the more general setting without additional assumptions. In the years since, research into generalized inverse limits has grown rapidly and includes a variety of topics [11, 12, 15, 16, 25, 37, 46, 47, 50].

In this dissertation, we focus on the inverse limits of a particular class of set-valued functions called *irreducible functions*. We demonstrate that irreducible functions may be used to generate indecomposable continua as inverse limits, and we explore properties of these inverse limits, including chainability and endpoints. We also establish conditions under which two inverse limits from this family are or are not homeomorphic.

In Chapter Two, we give preliminary definitions and theorems and a review of the relevant literature. Then, in Chapter Three, we define irreducible functions and demonstrate that they may be used to construct indecomposable continua as inverse limits. This generalizes results due to Ingram and Varagona [23, 50, 51].

In the subsequent chapters, we investigate other properties of this class of inverse limits spaces. We give a characterization for chainability in Chapter Four which builds upon the work of Ingram, [26], and in Chapter Five we characterize endpoints of inverse limits of certain set-valued functions. Finally, in Chapter Six, we establish sufficient conditions for two inverse limits from this class to be (or to not be) homeomorphic which generalize results due to Watkins, Varagona, and Smith and Varagona [49, 52, 54]. This culminates in Section 6.3 in a topological classification of certain families of inverse limits of irreducible functions. This dissertation includes results which have previously been published [34–36].

CHAPTER TWO

Preliminary Definitions and Theorems

We begin with some preliminary definitions from continuum theory. For a more in-depth introduction to continuum theory see [44].

A set X is a *continuum* if it is a non-empty, compact, connected metric space. A subset of a space X which is a continuum is called a *subcontinuum* of X. A continuum is called *decomposable* if it is the union of two proper subcontinua. A non-degenerate continuum which is not decomposable is called *indecomposable*.

Let Y be a metric space and $X \subseteq Y$ be a continuum. If $A, B \subseteq Y$ are closed, we say that X is *irreducible between A and B* if X intersects each of A and B but no proper subcontinuum of X does. If X is irreducible between $\{a\}$ and $\{b\}$ for some $a, b \in X$, we will simply say that X is irreducible between a and b. We say that X is *irreducible* if there exist two points between which it is irreducible.

A *chain* in a metric space X is a collection $\{C_1, \ldots, C_n\}$ of open sets such that for $i, j = 1, \ldots, n$, $C_i \cap C_j \neq \emptyset$ if, and only if, $|i - j| \leq 1$. A continuum is called *chainable* if for every $\epsilon > 0$, it can be covered by a chain $\{C_1, \ldots, C_n\}$ such that for all $i = 1, \ldots, n$, $\operatorname{diam} C_i < \epsilon$. Given continua X and Y, a continuous function $f : X \to Y$ is called an *ϵ-map* if for each $y \in Y$, $\operatorname{diam} f^{-1}(y) < \epsilon$. A continuum X is said to be *arc-like* if for every $\epsilon > 0$, there exists an ϵ-map $f : X \to [0, 1]$. Chainability and arc-likeness are equivalent. (See [44, p. 235] for a proof.)

A continuum X is called *unicoherent* if any two subcontinua of X whose union is X have a connected intersection. A continuum is called *hereditarily unicoherent* if each of its subcontinua is unicoherent. Equivalently, a continuum is hereditarily unicoherent if any two of its subcontinua have a connected intersection. In [7], Bing shows that every chainable continuum is hereditarily unicoherent.

Given a continuum X, a point p is called an *endpoint* of X if for any two subcontinua of X, each of which contains p, one of them is a subset of the other.

2.1 Traditional Inverse Limits

Inverse limits have been widely used and studied in the context of continuum theory. The definitions in the traditional setting are as follows: Given a sequence, $\mathbf{X} = (X_i)_{i=1}^{\infty}$, of topological spaces and a sequence, $\mathbf{f} = (f_i)_{i=1}^{\infty}$, of continuous functions such that for each $i \in \mathbb{N}$, $f_i : X_{i+1} \to X_i$, the pair $\{\mathbf{X}, \mathbf{f}\}$ is called an *inverse sequence*. The *inverse limit* of this inverse sequence is defined to be the set

$$\varprojlim \mathbf{f} = \left\{ \mathbf{x} \in \prod_{i=1}^{\infty} X_i : x_i = f_i(x_{i+1}) \text{ for all } i \in \mathbb{N} \right\}.$$

Each space, X_i is called a *factor space*, and each function, f_i, is called a *bonding map*. Given $n \in \mathbb{N}$, we define $\pi_n : \varprojlim \mathbf{f} \to X_n$ to be projection onto the nth factor space.

Capel shows in [10] that if $\{\mathbf{X}, \mathbf{f}\}$ is an inverse sequence where each factor space is a continuum, then $\varprojlim \mathbf{f}$ is a continuum. Another well known result, due to Isbell, [32], is that a continuum X is chainable (or equivalently arc-like) if, and only if, there exists an inverse sequence $\{\mathbf{X}, \mathbf{f}\}$ where for each $i \in \mathbb{N}$, $X_i = [0, 1]$, such that $\varprojlim \mathbf{f}$ is homeomorphic to X.

A property which has been widely studied with regard to inverse limits is indecomposability. Recall that a continuum is indecomposable if it is not the union of two of its proper subcontinua. In the traditional setting, there are many results concerning indecomposability of inverse limits. One particularly well-known result establishes a definition of an indecomposable map, and shows that the inverse limit of indecomposable maps is an indecomposable continuum.

Definition 2.1. Let X and Y be continua. A map $f : X \to Y$ is called *indecomposable* provided that for any two subcontinua A and B of X with $A \cup B = X$, either $f(A) = Y$ or $f(B) = Y$.

Theorem 2.2. *Let $\{\mathbf{X}, \mathbf{f}\}$ be an inverse sequence with indecomposable bonding maps. Then $\varprojlim \mathbf{f}$ is an indecomposable continuum.*

For a proof of this theorem, see [44, p. 21]. Central to the proof is the fact that if K is a subcontinuum of $\varprojlim \mathbf{f}$, and $\pi_n(K) = X_n$ for infinitely many $n \in \mathbb{N}$, then $K = \varprojlim \mathbf{f}$. This is important, because under the generalized notion of an inverse limit which we will introduce in Section 2.2, this does not always hold.

Another sufficient condition for indecomposability was established by Ingram in [21].

Definition 2.3. Let G be a graph. A map $f : G \to G$ is said to be a *two-pass map* provided there exist two non-overlapping subgraphs G_1 and G_2 of G such that $f[G_i] = G$ for $i = 1, 2$.

Theorem 2.4 (Ingram). *If T is a tree with no more than one branch point and $f : T \to T$ is a two-pass map, then $\varprojlim \mathbf{f}$ is indecomposable.*

For additional reading on the topic of traditional inverse limits, see [19, 30].

2.2 Inverse Limits of Set-valued Functions

In 2004, Mahavier, [40], introduced a generalized notion of an inverse limit which allows for the bonding functions to be set-valued. This was further built upon in [29] by Ingram and Mahavier. They show that many of the properties which hold for traditional inverse limits fail to hold for inverse limits with set-valued functions. Since then, there has been considerable research concerning inverse limits of set-valued functions. Much of it has focused on determining what conditions are necessary for the results of traditional inverse limit theory to extend to inverse limits of set-valued functions. Ingram provides a thorough introduction to inverse limits of set-valued functions in [25].

2.2.1 *Definitions*

Given a topological space X, we define the following *hyperspaces* of X:

$$\begin{aligned} 2^X &= \{A \subseteq X : A \text{ is nonempty, closed, and compact}\} \\ C(X) &= \left\{A \in 2^X : A \text{ is connected}\right\}. \end{aligned}$$

Hence, if X is a metric space, then $C(X)$ is the set of all subcontinua of X.

Given spaces X and Y and a point $x_0 \in X$, a function $F : X \to 2^Y$ is *upper semi-continuous at* x_0 provided that if $V \subseteq Y$ is an open set containing $F(x_0)$, then there exists an open set $U \subseteq X$ containing x_0, such that for all $x \in U$, $F(x) \subseteq V$. We say that F is *upper semi-continuous* if it is upper semi-continuous at each $x \in X$.

Given spaces X and Y, and a function $F : X \to 2^Y$, we define the *graph* of F to be the set

$$\Gamma(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

Ingram and Mahavier show in [29] that a set-valued function between compact Hausdorff spaces is upper semi-continuous if, and only if, its graph is closed. Since all of the spaces considered in this dissertation are compact Hausdorff spaces, we will consider this to be the definition of upper semi-continuous.

A set-valued function $F : X \to 2^Y$ is called *surjective* if $\bigcup_{x \in X} F(x) = Y$, and the *inverse* of F is defined to be the set-valued function $F^{-1} : Y \to 2^X$ where

$$F^{-1}(y) = \{x \in X : y \in F(x)\}.$$

Let $\mathbf{X} = (X_i)_{i=1}^{\infty}$ be a sequence of topological spaces and $\mathbf{F} = (F_i)_{i=1}^{\infty}$ be a sequence of upper semi-continuous, set-valued functions such that for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$. The pair $\{\mathbf{X}, \mathbf{F}\}$ is called an *inverse sequence*, the spaces, X_i, are called *factor spaces*, and the set-valued functions, F_i, are called *bonding functions*. The *inverse limit* of such an inverse sequence is defined to be

$$\varprojlim \mathbf{F} = \left\{\mathbf{x} \in \prod_{i=1}^{\infty} X_i : x_i \in F_i(x_{i+1}) \text{ for all } i \in \mathbb{N}\right\}.$$

A sequence of sets which are useful in the study of these inverse limits is $\mathbf{\Gamma} = (\Gamma_n)_{n=1}^{\infty}$, where for each $n \in \mathbb{N}$,

$$\Gamma_n(\mathbf{F}) = \left\{ \mathbf{x} \in \prod_{i=1}^{n} X_i : x_i \in F_i(x_{i+1}) \text{ for } 1 \leq i < n \right\}.$$

For each $n \in \mathbb{N}$, $\Gamma_n(\mathbf{F})$ is called the *n-fold inverse graph* of $\mathbf{F}$. When no ambiguity shall arise, we will simply write Γ_n.

Given a space X, and a set-valued function $F : X \to 2^X$, there is an induced inverse sequence $\{\mathbf{X}, \mathbf{F}\}$ where for each $i \in \mathbb{N}$, $X_i = X$ and $F_i = F$. All of the examples and many of the results in this dissertation concern the inverse limits of inverse sequences induced by a single function.

Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence, and let $j, k \in \mathbb{N}$ with $j < k$. Then we define $\pi_j : \varprojlim \mathbf{F} \to X_j$ to be projection onto the jth factor space, and we define $\pi_{[j,k]} : \varprojlim \mathbf{F} \to \prod_{i=j}^{k} X_i$ by $\pi_{[j,k]}(\mathbf{x}) = (x_j, x_{j+1}, \ldots, x_k)$. At times, we will use the same notation to refer to projection maps whose domain is Γ_n for some $n \in \mathbb{N}$. In context, the intended domain should be clear.

2.2.2 Connectedness

Mahavier, [40], and Ingram and Mahavier, [29], demonstrate that many well-known results concerning traditional inverse limits do not hold in general for inverse limits of set-valued functions. In particular, if $\{\mathbf{X}, \mathbf{F}\}$ is an inverse sequence where for each $i \in \mathbb{N}$, X_i is a continuum, $\varprojlim \mathbf{F}$ need not be connected. It is however true that $\varprojlim \mathbf{F}$ is connected if, and only if, Γ_n is connected for all $n \in \mathbb{N}$. For example, Ingram and Mahavier show that the inverse limit of the set-valued function whose graph is pictured in Figure 2.1 is not connected. They do this by showing that the point $(1/4, 1/4, 3/4)$ is isolated in Γ_3.

They also give the following results concerning connectedness.

Theorem 2.5 (Ingram and Mahavier). *Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence. If for each $i \in \mathbb{N}$, X_i is a continuum and $F_i : X_{i+1} \to C(X_i)$, then $\varprojlim \mathbf{F}$ is a continuum.*

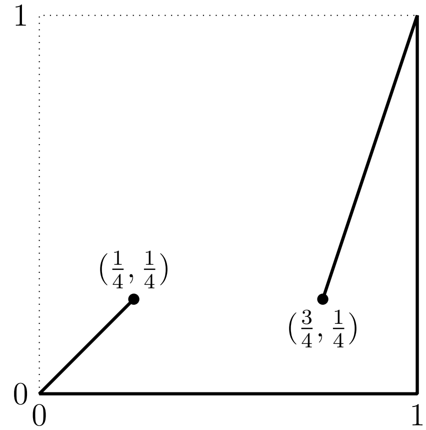

Figure 2.1. Set-valued function with a disconnected inverse limit

Theorem 2.6 (Ingram and Mahavier). *Let* $\{\mathbf{X}, \mathbf{F}\}$ *be an inverse sequence. If for each* $i \in \mathbb{N}$, X_i *is a continuum and* $F_i : X_{i+1} \to 2^{X_i}$ *is surjective such that for each* $x \in X_i$, $F_i^{-1}(x)$ *is connected, then* $\varprojlim \mathbf{F}$ *is a continuum.*

Connectedness of inverse limits has also been studied by Nall. He presents the following results in [46].

Theorem 2.7 (Nall). *Let* X *be a continuum, and let* $\{F_\alpha : X \to C(X) : \alpha \in A\}$ *be a collection of upper semi-continuous continuum-valued functions such that for each* $x \in X$, $\bigcup_{\alpha \in A} F_\alpha(x)$ *is compact. Define* $F : X \to 2^X$ *by* $F(x) = \bigcup_{\alpha \in A} F_\alpha(x)$. *If* F *is surjective and* $\Gamma(F)$ *is connected, then* $\varprojlim \mathbf{F}$ *is a continuum.*

Theorem 2.8 (Nall). *Let* X *be a continuum, and let* $F : X \to 2^X$ *be surjective and upper semi-continuous. Then* $\varprojlim \mathbf{F}$ *is a continuum if, and only if,* $\varprojlim \mathbf{F^{-1}}$ *is a continuum.*

Additional results concerning the connectedness of inverse limits of set-valued functions can be found in [14, 16, 17, 22–24, 31].

2.2.3 Topological Conjugacy

One result which does naturally extend to the generalized theory of inverse limits concerns topological conjugacy. To define topological conjugacy we must first define what is meant by composition of set-valued functions. If X, Y, and Z are sets, $F : X \to 2^Y$ and $G : Y \to 2^Z$, then we define $G \circ F : X \to 2^Z$ by

$$G \circ F(x) = \bigcup_{y \in F(x)} G(y).$$

Let X and Y be topological spaces, and let $F : X \to 2^X$ and $G : Y \to 2^Y$ be upper semi-continuous. We say that F and G are *topologically conjugate* if there exists a homeomorphism $\varphi : X \to Y$ such that $\varphi \circ F = G \circ \varphi$. The following theorem appears in [29].

Theorem 2.9 (Ingram and Mahavier). *Let X and Y be compact Hausdorff spaces, and let $F : X \to 2^X$ and $G : Y \to 2^Y$ be upper semi-continuous. If F and G are topologically conjugate, then $\varprojlim \mathbf{F}$ is homeomorphic to $\varprojlim \mathbf{G}$.*

2.2.4 Full Projection Property and Indecomposability

It was noted that essential to the proof of Theorem 2.2 is the fact that no proper subcontinuum of an inverse limit can have full projection in infinitely many coordinates. This does not hold in general for inverse limits of set-valued functions, and it is an important property in the study of indecomposability of inverse limits.

Definition 2.10. Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence where for each $i \in \mathbb{N}$, X_i is a continuum. We say that $\varprojlim \mathbf{F}$ has the *full projection property* provided that if $K \subseteq \varprojlim \mathbf{F}$ is a continuum such that $\pi_i(K) = X_i$ for infinitely many $i \in \mathbb{N}$, then $K = \varprojlim \mathbf{F}$.

The full projection property seems to be indispensable in obtaining an indecomposable continuum as an inverse limit. While it has not been proven to be a necessary condition for the inverse limit to be indecomposable, there are no results concerning indecomposability known to the author which do not include the full projection property as an assumption. The most notable results concerning the full projection property and indecomposability of inverse limits with set-valued functions are due to Ingram, [23], and Varagona, [50, 51].

Example 2.11 (Varagona). For each $n \in \mathbb{N}$, let L_n be a line segment whose endpoints are

(1) $(2^{-n+1}, 1)$ and $(2^{-n}, 0)$ if n is odd

(2) $(2^{-n+1}, 0)$ and $(2^{-n}, 1)$ if n is even,

and let L_0 be the line segment whose endpoints are $(0, 0)$ and $(0, 1)$. Let $F : [0, 1] \to 2^{[0,1]}$ be the set-valued function whose graph is equal to $\bigcup_{n=0}^{\infty} L_n$ (pictured on the left in Figure 2.2). Then $\varprojlim \mathbf{F}$ is an indecomposable continuum.

Example 2.12 (Ingram). Let $G : [0, 1] \to 2^{[0,1]}$ be the set-valued function whose graph consists of three line segments, the first from $(0, 0)$ to $(1/2, 1)$, the second from $(1/2, 1)$ to $(1/2, 0)$, and the third from $(1/2, 0)$ to $(1, 1)$ (pictured on the right in Figure 2.2). Then $\varprojlim \mathbf{G}$ is an indecomposable continuum.

In each of these examples, the proof includes demonstrating that the inverse limit has the full projection property. The respective authors differ substantially in how they go about proving this which led Ingram to ask in [25, Problem 6.26] whether there was a single theorem which would establish the full projection property for both inverse limits. We give such a result in Theorem 3.15.

2.2.5 Additional Topics

The topics of the previous subsections are of the greatest pertinence to this dissertation, but there are many other interesting topics concerning inverse limits of set-valued functions which have been explored. Illanes, [18], shows that a simple closed curve cannot be obtained as the inverse limit of a single set-valued function on $[0, 1]$, and Nall, [45, 47], extends this to include all finite graphs other than the

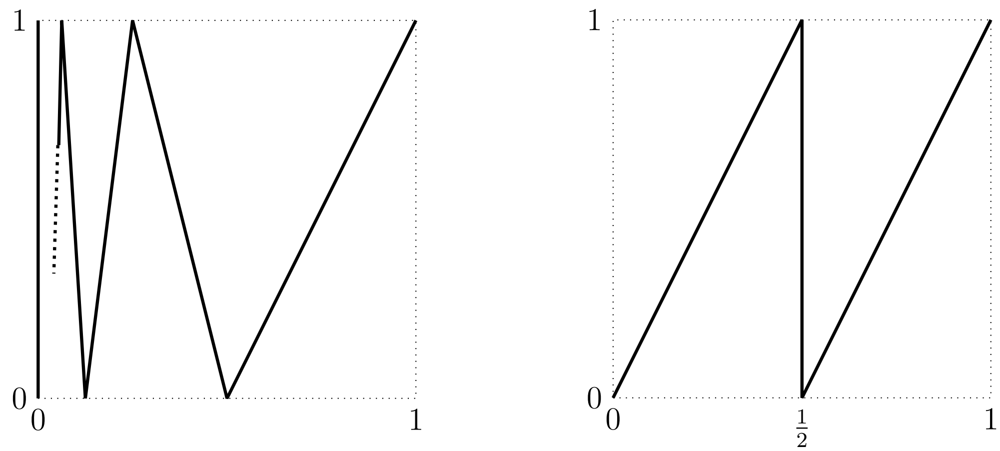

Figure 2.2. Set-valued functions with indecomposable inverse limits

arc. Charatonik, Roe, and Vernon have each explored properties of inverse limits generated by systems indexed by sets other than the natural numbers [12, 13, 53]. Also, Banič, Charatonik and Roe, and Ingram have studied the topic of dimension of inverse limits of set-valued functions, [4, 11, 27, 28].

CHAPTER THREE

Indecomposability

In this chapter, we generalize Theorem 2.2 as well as Examples 2.11 and 2.12. In Section 3.1 we develop a generalized definition for an indecomposable function which can accommodate set-valued functions and demonstrate that such functions have indecomposable inverse limits so long as the inverse limits are connected and have the full projection property. We go on to give a sufficient condition for an inverse limit to have the full projection property.

Then in Section 3.2 we define a class of set-valued functions called irreducible functions, and we prove that they yield inverse limits which have the full projection property and are indecomposable continua. This class of function includes those from Examples 2.11 and 2.12. Hence, the main result of the section, Theorem 3.15, generalizes these examples.

This chapter includes work done in collaboration with Jonathan Meddaugh.

3.1 Indecomposable Set-valued Functions and the Full Projection Property

We begin with the definition of an indecomposable set-valued function.

Definition 3.1. Let X and Y be continua. An upper semi-continuous set-valued function $F : X \to 2^Y$ is *indecomposable* provided that for any two subcontinua A and B of $\Gamma(F)$ with $A \cup B = \Gamma(F)$, then either $\pi_2(A) = Y$ or $\pi_2(B) = Y$.

This is one of several possible ways to generalize the notion of indecomposable from single-valued functions to set-valued functions. It is important to note that an indecomposable set-valued function does not necessarily result in an indecomposable inverse limit without additional conditions (Theorem 3.3). The following lemma verifies that this definition is consistent with the original.

Lemma 3.2. *Let* $f : X \to Y$ *be a continuous, single-valued function, and define* $F : X \to 2^Y$ *by* $F(x) = \{f(x)\}$. *Then* f *is indecomposable as a single-valued function if, and only if,* F *is indecomposable as a set-valued function.*

Proof. Note that since $f : X \to Y$ is continuous and single-valued, X is homeomorphic to $\Gamma(f) = \Gamma(F)$. Thus, if A and B are subcontinua of X whose union is X, then $\Gamma(F|_A)$ and $\Gamma(F|_B)$ are subcontinua of $\Gamma(F)$ whose union is $\Gamma(F)$. Similarly, if C and D are subcontinua of $\Gamma(f)$ whose union is $\Gamma(f)$, then $\pi_1(C)$ and $\pi_1(D)$ are subcontinua of X whose union is X. Moreover, if $A \subseteq X$, then $f(A) = \pi_2(\Gamma(f|_A))$. The result follows. $\square$

This brings us to our generalization of Theorem 2.2.

Theorem 3.3. *Let* $\{\mathbf{X}, \mathbf{F}\}$ *be an inverse sequence for which each* F_i *is indecomposable. If* $\varprojlim \mathbf{F}$ *is connected and has the full projection property, then* $\varprojlim \mathbf{F}$ *is an indecomposable continuum.*

Proof. Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence for which each bonding function is indecomposable. Furthermore, suppose that $\varprojlim \mathbf{F}$ is connected and has the full projection property.

Let A and B be subcontinua of $\varprojlim \mathbf{F}$ with $A \cup B = \varprojlim \mathbf{F}$. Then, for each $i > 1$, the projections $\pi_{[i,i+1]}(A)$ and $\pi_{[i,i+1]}(B)$ are subcontinua of $\Gamma_{[i,i+1]}$ for which $\pi_{[i,i+1]}(A) \cup \pi_{[i,i+1]}(B) = \Gamma_{[i,i+1]}$. As observed earlier, $\Gamma_{[i,i+1]}$ is the graph of F_i^{-1}. Since F_i is indecomposable, it follows that one of $\pi_i(A)$ or $\pi_i(B)$ is equal to X_i.

Since this holds for all $i > 1$, it follows that for some $Z \in \{A, B\}$, $\pi_i(Z) = X_i$ for infinitely many $i \in \mathbb{N}$. Since $\varprojlim \mathbf{F}$ has the full projection property, $Z = \varprojlim \mathbf{F}$. Thus one of A or B is equal to $\varprojlim \mathbf{F}$, and so $\varprojlim \mathbf{F}$ is indecomposable. $\square$

Unfortunately, the indecomposability of the bonding functions alone is not sufficient for the inverse limit to be indecomposable. As was noted in the previous chapter, there is extensive literature on connectedness of inverse limits. There is significantly less on the full projection property. We give a sufficient condition for an inverse limit to have the full projection property here.

Theorem 3.4. *Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence such that $\varprojlim \mathbf{F}$ is a continuum. If for each $n \in \mathbb{N}$, there exist closed sets $A, B \subseteq X_n$, such that Γ_n is irreducible between the sets*

$$\{\mathbf{x} \in \Gamma_n : x_n \in A\} \text{ and } \{\mathbf{x} \in \Gamma_n : x_n \in B\},$$

then $\varprojlim \mathbf{F}$ has the full projection property.

Proof. Let K be a subcontinuum of $\varprojlim \mathbf{F}$ with $\pi_i(K) = X_i$ for infinitely many $i \in \mathbb{N}$. Choose $j \in \mathbb{N}$ such that $\pi_j(K) = X_j$ and choose $A, B \subseteq X_j$ such that Γ_j is irreducible between the sets

$$\{\mathbf{x} \in \Gamma_j : x_j \in A\} \text{ and } \{\mathbf{x} \in \Gamma_j : x_j \in B\}.$$

Since $\pi_j(K) = X_j$, it contains both A and B, so the continuum $\pi_{[1,j]}(K)$ must intersect both

$$\{\mathbf{x} \in \Gamma_j : x_j \in A\} \text{ and } \{\mathbf{x} \in \Gamma_j : x_j \in B\}.$$

Since Γ_j is irreducible between these sets, this means that $\pi_{[1,j]}(K) = \Gamma_j$. It follows that for all $i \in \mathbb{N}$ with $1 \leq i \leq j$, $\pi_{[1,i]}(K) = \Gamma_i$.

Since there are infinitely many such $j \in \mathbb{N}$, it follows that $\pi_{[1,i]}(K) = \Gamma_i$ for all $i \in \mathbb{N}$. Therefore, $K = \varprojlim \mathbf{F}$. □

3.2 Irreducible Set-valued Functions

In this section, we will define a type of upper semi-continuous set-valued function which we call an irreducible function. The purpose of this definition will be realized in Theorem 3.15 where we state that sequences of irreducible functions may be used to yield an inverse limit which has the full projection property and is an indecomposable continuum. Towards this end, we will first show in Lemma 3.14

that irreducible functions are also indecomposable functions. Thus, once it has been established that the inverse limits described in Theorem 3.15 have the full projection property, the fact that they are also indecomposable continua will follow from Theorem 3.3.

3.2.1 Irreducible Collections of Maps

The definition of an irreducible function is given in terms of the function's inverse. Its inverse must be the union of a collection of single-valued maps. The criteria such a collection must meet are outlined in this next definition. In this definition as well as in the rest of this dissertation, given a subset Λ of the real numbers, Λ' refers to the set of limit points of Λ.

Definition 3.5. Let X and Y be irreducible continua, and $\Lambda \subseteq [0, 1]$ be a closed set with $0, 1 \in \Lambda$ and $\overline{\Lambda \setminus \Lambda'} = \Lambda$. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a collection of continuous functions from Y to X. We say that $\{f_\lambda\}_{\lambda\in\Lambda}$ is *irreducible with respect to* $a, b \in X$ *and* $c, d \in Y$ if X is irreducible between a and b, Y is irreducible between c and d, and the following hold:

(1) $a \in f_\lambda(Y)$ if, and only if, $\lambda = 0$, and $b \in f_\lambda(Y)$ if, and only if, $\lambda = 1$.

(2) If $0 \notin \Lambda'$, then $f_0^{-1}(a) = \{c\}$ or $f_0^{-1}(a) = \{d\}$.

(3) If $1 \notin \Lambda'$, then $f_1^{-1}(b) = \{c\}$ or $f_1^{-1}(b) = \{d\}$.

(4) If $\lambda, \mu \in \Lambda$ with $\lambda < \mu$, then $f_\lambda(y) \neq f_\mu(y)$ for all $y \notin \{c, d\}$, and $\Gamma(f_\lambda) \cap \Gamma(f_\mu) \neq \emptyset$ if, and only if, $(\lambda, \mu) \cap \Lambda = \emptyset$.

(5) If $(\lambda_i)_{i\in\mathbb{N}}$ is a sequence of points in Λ and $\lambda_i \to \lambda$ as $i \to \infty$, then $f_{\lambda_i} \to f_\lambda$ uniformly as $i \to \infty$.

When no ambiguity shall arise, or when mention of the points, $a, b \in X$ and $c, d \in Y$ is unnecessary, we will simply say that $\{f_\lambda\}_{\lambda\in\Lambda}$ is an *irreducible collection of maps.*

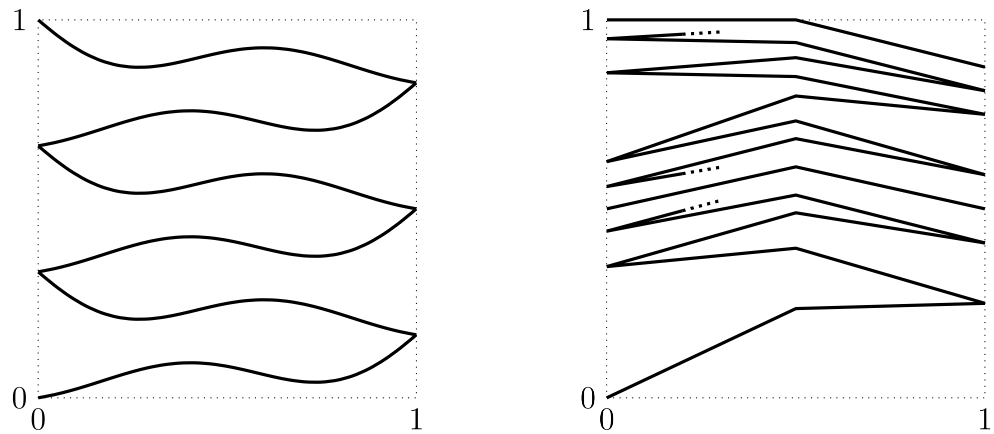

Figure 3.1. Irreducible collections of maps

Figure 3.1 provides examples of the graphs of irreducible collections of maps. On the left is a collection $\{f_\lambda\}_{\lambda\in\Lambda}$ where $\Lambda = \{0, 1/5, 2/5, 3/5, 4/5, 1\}$. f_0 is the bottom function and is the only function which takes on the value of 0; $f_{1/5}$ is the function immediately above f_0 whose graph intersects the graph of f_0 only at 1; and so on. Notice that since Λ has no limit points, no function is a limit of other functions.

On the right is a collection $\{g_\omega\}_{\omega\in\Omega}$. Perhaps the simplest indexing set for this collection would be

$$\Omega = \left\{\frac{2^n-1}{2^{n+1}} : n \in \mathbb{N}\right\} \cup \left\{\frac{2^n+1}{2^{n+1}} : n \in \mathbb{N}\right\} \cup \left\{\frac{2^{n+1}-1}{2^{n+1}} : n \in \mathbb{N}\right\} \cup \left\{\frac{1}{2}, 1\right\},$$

but any closed subset $\Omega \subseteq [0,1]$ could be used so long as $0 \in \Omega$, and Ω has exactly two limit points–one of which is 1, and the other is a two-sided limit point which lies in $(0,1)$.

Another thing worth noting about the collection pictured on the right in Figure 3.1 is that $g_1^{-1}(1) = [0, 1/2]$. This is allowed because 1 is a limit point of Ω. Since 0 is not a limit point of Ω, $g_0^{-1}(0)$ must be a singleton subset of $\{0,1\}$. Specifically, in this case, $g_0^{-1}(0) = \{0\}$.

This definition of an irreducible collection of maps can be generalized to the context of irreducibility with respect to closed sets in the following way.

Definition 3.6. Let X and Y be irreducible continua, and $\Lambda \subseteq [0,1]$ be a closed set with $0,1 \in \Lambda$ and $\overline{\Lambda \setminus \Lambda'} = \Lambda$. Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a collection of continuous functions from Y to X. We say that $\{f_\lambda\}_{\lambda\in\Lambda}$ is *irreducible with respect to* $A, B \subseteq X$ *and* $C, D \subseteq Y$ if X is irreducible between the sets A and B, Y is irreducible between the sets C and D, and the following hold:

(1) $A \cap f_\lambda(Y) \neq \emptyset$ if, and only if, $\lambda = 0$, and $B \cap f_\lambda(Y) \neq \emptyset$ if, and only if, $\lambda = 1$.

(2) If $0 \notin \Lambda'$, then $f_0^{-1}(A) \subseteq C$ or $f_0^{-1}(A) \subseteq D$.

(3) If $1 \notin \Lambda'$, then $f_1^{-1}(B) \subseteq C$ or $f_1^{-1}(B) \subseteq D$.

(4) (a) If $\lambda, \mu \in \Lambda$ with $\lambda < \mu$, then $f_\lambda(y) \neq f_\mu(y)$ for all $y \notin C \cup D$, and $\Gamma(f_\lambda) \cap \Gamma(f_\mu) \neq \emptyset$ if, and only if, $(\lambda, \mu) \cap \Lambda = \emptyset$.

  (b) If $\lambda, \mu \in \Lambda$, and $L \in \{C, D\}$, then $\Gamma(f_\lambda|_L) \cap \Gamma(f_\mu|_L) \neq \emptyset$ implies that $\Gamma(f_\lambda|_L) \cap \Gamma(f_\sigma|_L) = \emptyset$ for all $\sigma \in \Lambda \setminus \{\lambda, \mu\}$.

  (c) If $L \in \{C, D\}$ and $A \cap f_0(L) \neq \emptyset$, then $\Gamma(f_0|_L) \cap \Gamma(f_\lambda|_L) = \emptyset$ for all $\lambda \in \Lambda \setminus \{0\}$; and if $B \cap f_1(L) \neq \emptyset$, then $\Gamma(f_1|_L) \cap \Gamma(f_\lambda|_L) = \emptyset$ for all $\lambda \in \Lambda \setminus \{1\}$.

(5) If $(\lambda_i)_{i\in\mathbb{N}}$ is a sequence of points in Λ and $\lambda_i \to \lambda$ as $i \to \infty$, then $f_{\lambda_i} \to f_\lambda$ uniformly as $i \to \infty$.

When no ambiguity shall arise, or when mention of the sets, $A, B \subseteq X$ and $C, D \subseteq Y$ is unnecessary, we will simply say that $\{f_\lambda\}_{\lambda\in\Lambda}$ is an irreducible collection of maps.

Note that if A, B, C, and D are singleton sets, then Definition 3.6 is equivalent to Definition 3.5.

Figure 3.2 gives an example of an irreducible collection of maps on an irreducible continuum other than $[0,1]$. The continuum X is pictured at the top, and

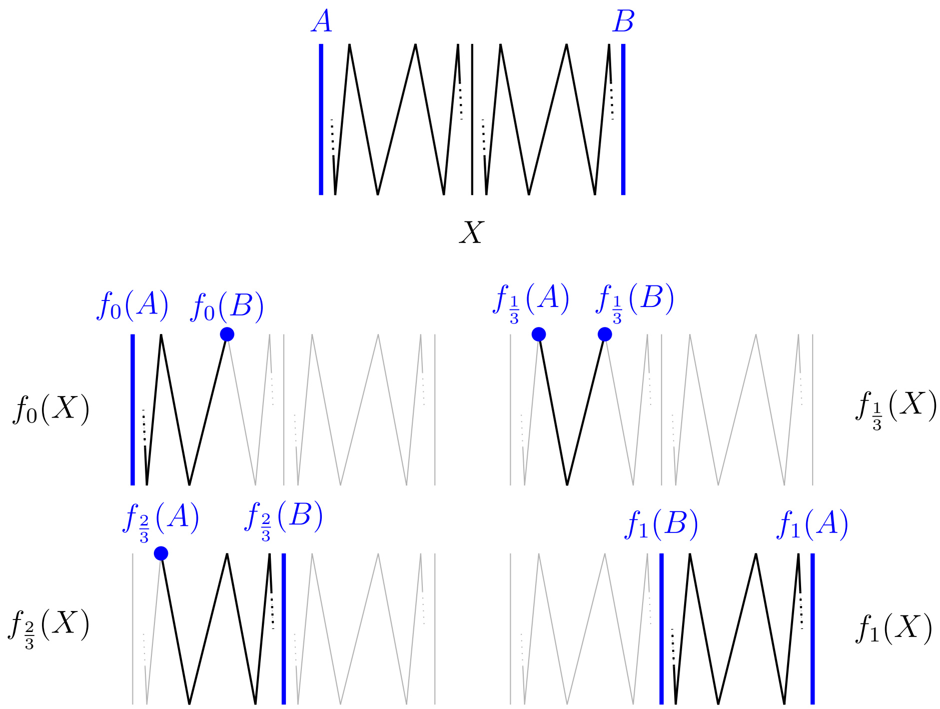

Figure 3.2. Irreducible collection of maps on a continuum other than an interval

below it are the images of four maps $f_0, f_{\frac{1}{3}}, f_{\frac{2}{3}}, f_1 : X \to X$ which could satisfy Definition 3.6. Since it is impossible to show the graphs of these functions with only two dimensions, only their images are shown, but one can easily imagine a collection of functions with the given images that satisfy Definition 3.6.

Lemma 3.7. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be an irreducible collection of maps from Y to X. Then $\bigcup_{\lambda\in\Lambda} \Gamma(f_\lambda)$ is a continuum.*

Proof. Let $K = \bigcup_{\lambda\in\Lambda} \Gamma(f_\lambda)$. First, to show that K is compact, define a function $H : \Lambda \times Y \to Y \times X$ by $H(\lambda, y) = (y, f_\lambda(y))$. From Property 5 of Definition 3.6, it follows that H is continuous. Since $\Lambda \times Y$ is compact, $H(\Lambda \times Y) = K$ is compact.

Next, suppose that K is not connected. Then there exist non-empty, closed, disjoint sets $A, B \subseteq K$ with $K = A \cup B$. Since each f_λ is a continuous function, its graph is connected, so either $\Gamma(f_\lambda) \subseteq A$ or $\Gamma(f_\lambda) \subseteq B$. Let $\mathcal{A} = \{\lambda \in \Lambda : \Gamma(f_\lambda) \subseteq A\}$ and $\mathcal{B} = \{\lambda \in \Lambda : \Gamma(f_\lambda) \subseteq B\}$.

Since A and B are both non-empty, $\mathcal{A}$ and $\mathcal{B}$ are both non-empty. Without loss of generality, suppose that $1 \in \mathcal{B}$, and let $\alpha = \max \mathcal{A}$. Then $[\alpha, 1] \cap \mathcal{B}$ is a closed set, so it has a minimal element β. Since $\alpha \notin \mathcal{B}$, $\beta \neq \alpha$, so $\beta > \alpha$. In particular, β is the smallest element of Λ greater than α. This means that $(\alpha, \beta) \cap \Lambda = \emptyset$, so by Property 4 of Definition 3.6, we have that $\Gamma(f_\alpha) \cap \Gamma(f_\beta) \neq \emptyset$. This is a contradiction since $\Gamma(f_\alpha) \subseteq A$, $\Gamma(f_\beta) \subseteq B$, and A and B are disjoint.

Therefore, K must be connected and is thus a continuum. □

Corollary 3.8. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a collection of maps irreducible with respect to $A, B \subseteq X$ and $C, D \subseteq Y$. Then $\bigcup_{\lambda\in\Lambda} f_\lambda(Y) = X$.*

Proof. Let $K = \bigcup_{\lambda\in\Lambda} \Gamma(f_\lambda)$. Then $\bigcup_{\lambda\in\Lambda} f_\lambda(Y) = \pi_2(K)$ where $\pi_2 : Y \times X \to X$. From Lemma 3.7, K is a continuum, so since π_2 is continuous, $\bigcup_{\lambda\in\Lambda} f_\lambda(Y)$ is a continuum. Also, since $A \cap f_0(Y) \neq \emptyset$ and $B \cap f_1(Y) \neq \emptyset$, $\bigcup_{\lambda\in\Lambda} f_\lambda(Y)$ is a subcontinuum of X which intersects both A and B. Since X is irreducible between A and B, it follows that $\bigcup_{\lambda\in\Lambda} f_\lambda(Y) = X$. □

Lemma 3.10 and Corollary 3.11 below should begin to make apparent the purpose of each element of Definition 3.6 as well as why the word "irreducible" was chosen to describe these collections of maps. This next theorem appears in [44, p. 72] and will be useful in the proof of Lemma 3.10.

Theorem 3.9 (Cut-wire Theorem). *Let X be a compact metric space, and let A and B be closed subsets of X. If no component of X intersects both A and B, then $X = X_1 \cup X_2$ where X_1 and X_2 are disjoint closed subsets of X with $A \subseteq X_1$ and $B \subseteq X_2$.*

Lemma 3.10. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be irreducible with respect to $A, B \subseteq X$ and $C, D \subseteq Y$. Let K be a subcontinuum of $\bigcup_{\lambda\in\Lambda} \Gamma(f_\lambda)$, and let $\Omega_K = \{\lambda \in \Lambda : K \cap \Gamma(f_\lambda) \neq \emptyset\}$. Then Ω_K is the intersection of a closed (possibly degenerate) interval with Λ. Moreover, if $\lambda \in \Omega_K \setminus \{\min \Omega_K, \max \Omega_K\}$, then $\Gamma(f_\lambda) \subseteq K$.*

Proof. First, for each $\lambda \in \Lambda$, let $C_\lambda = \Gamma(f_\lambda|_C)$ and $D_\lambda = \Gamma(f_\lambda|_D)$.

Suppose that there exist $\lambda_0, \mu_0 \in \Omega_K$ with $\lambda_0 < \mu_0$, and $(\lambda_0, \mu_0) \cap \Lambda \neq \emptyset$. We must show that $[\lambda_0, \mu_0] \cap \Lambda \subseteq \Omega_K$. In fact, to prove the latter part of the lemma, we must show that for all $\omega \in (\lambda_0, \mu_0) \cap \Lambda$, $\Gamma(f_\omega) \subseteq K$. Towards this end, choose $\omega \in (\lambda_0, \mu_0) \cap \Lambda$.

Case 1: Suppose that ω is isolated in Λ. We will first show that C_ω and D_ω each intersects K.

Let $\underline{\omega}$ be the element of Λ immediately preceding ω, and $\overline{\omega}$ the element of Λ immediately succeeding ω. From Definition 3.6, either

(1) $C_\omega \cap C_{\underline{\omega}} \neq \emptyset$ and $D_\omega \cap D_{\overline{\omega}} \neq \emptyset$, or

(2) $C_\omega \cap C_{\overline{\omega}} \neq \emptyset$ and $D_\omega \cap D_{\underline{\omega}} \neq \emptyset$.

If (1) holds, then the sets

$$U_1 = \bigcup_{\lambda \in [0, \underline{\omega}] \cap \Lambda} \Gamma(f_\lambda) \setminus C_\omega \text{ and } V_1 = \bigcup_{\lambda \in [\omega, 1] \cap \Lambda} \Gamma(f_\lambda) \setminus C_\omega$$

are mutually separated, as are the sets

$$U_2 = \bigcup_{\lambda \in [0, \omega] \cap \Lambda} \Gamma(f_\lambda) \setminus D_\omega \text{ and } V_2 = \bigcup_{\lambda \in [\overline{\omega}, 1] \cap \Lambda} \Gamma(f_\lambda) \setminus D_\omega.$$

Since $\lambda_0 < \omega$, and $\lambda_0 \in \Omega_K$, one of two things holds. First, it's possible that $\lambda_0 = \underline{\omega}$ and that $\Gamma(f_{\underline{\omega}}) \cap K \subseteq C_{\underline{\omega}} \cap C_\omega$. If this is the case, then we have that $K \cap C_\omega \neq \emptyset$.

Second, it could be the case that either $\lambda_0 < \underline{\omega}$ or that $\lambda_0 = \underline{\omega}$, but $K \cap \Gamma(f_{\lambda_0})$ is not a subset of $C_{\underline{\omega}} \cap C_\omega$. In either case we have that K must intersect U_1. Also, since $\mu_0 > \omega$, K intersects V_1. Thus since K is connected, it cannot be contained in $U_1 \cup V_1$, so $K \cap C_\omega \neq \emptyset$. Similarly, we may use U_2 and V_2 to demonstrate that $K \cap D_\omega \neq \emptyset$.

If (2) holds, then we will proceed in a nearly identical manner except that we will define

$$\widetilde{U}_1 = \bigcup_{\lambda \in [0,\underline{\omega}] \cap \Lambda} \Gamma(f_\lambda) \setminus D_\omega \text{ and } \widetilde{V}_1 = \bigcup_{\lambda \in [\omega,1] \cap \Lambda} \Gamma(f_\lambda) \setminus D_\omega,$$

and we will define

$$\widetilde{U}_2 = \bigcup_{\lambda \in [0,\omega] \cap \Lambda} \Gamma(f_\lambda) \setminus C_\omega \text{ and } \widetilde{V}_2 = \bigcup_{\lambda \in [\overline{\omega},1] \cap \Lambda} \Gamma(f_\lambda) \setminus C_\omega.$$

In either case, it holds that C_ω and D_ω each intersects K. In particular, we have established that $\omega \in \Omega_K$. To go further and show that $\Gamma(f_\omega) \subseteq K$, we will consider $K \cap \Gamma(f_\omega)$.

If there exists a component L of $K \cap \Gamma(f_\omega)$ which intersects both C_ω and D_ω, then since $\Gamma(f_\omega)$ is irreducible between these sets, we must have that $L = \Gamma(f_\omega)$. Hence, $\Gamma(f_\omega) \subseteq K$.

If no component of $K \cap \Gamma(f_\omega)$ intersects both C_ω and D_ω, then by the Cut-wire Theorem, there exist mutually separated sets A and B with $C_\omega \subseteq A$, $D_\omega \subseteq B$, and $A \cup B \supseteq (K \cap \Gamma(f_\omega))$. Again using possibilities (1) and (2) above, if (1) holds, then K is separated by

$$\bigcup_{\lambda \in [0,\underline{\omega}] \cap \Lambda} \Gamma(f_\lambda) \cup A \text{ and } \bigcup_{\lambda \in [\overline{\omega},1] \cap \Lambda} \Gamma(f_\lambda) \cup B.$$

If (2) holds, then K is separated by

$$\bigcup_{\lambda \in [0,\underline{\omega}] \cap \Lambda} \Gamma(f_\lambda) \cup B \text{ and } \bigcup_{\lambda \in [\overline{\omega},1] \cap \Lambda} \Gamma(f_\lambda) \cup A.$$

In either case, we have a contradiction, so it follows that $\Gamma(f_\omega) \subseteq K$. This concludes Case 1.

Case 2: Suppose that $\omega \in \Lambda'$. Since $\Lambda \setminus \Lambda'$ is dense in Λ, there exists a sequence of isolated points $(\omega_i)_{i \in \mathbb{N}}$ in $(\lambda_0, \mu_0) \cap \Lambda$ limiting to ω. From Case 1, we have that for each $i \in \mathbb{N}$, $\Gamma(f_{\omega_i}) \subseteq K$, and from Definition 3.5 Property (5), we have that for each $y \in Y$, the sequence $(y, f_{\omega_i}(y))_{i\in\mathbb{N}}$ converges to $(y, f_\omega(y))$. Therefore, since K is closed, we have that $\Gamma(f_\omega) \subseteq K$. $\square$

Corollary 3.11. *Suppose $\{f_\lambda\}_{\lambda\in\Lambda}$ is irreducible with respect to $A, B \subseteq X$ and $C, D \subseteq Y$. Then the continuum $\bigcup_{\lambda\in\Lambda} \Gamma(f_\lambda)$ is irreducible between the sets $Y \times A$ and $Y \times B$.*

Proof. Let K be a subcontinuum of $\bigcup_{\lambda\in\Lambda} \Gamma(f_\lambda)$ which intersects both $Y \times A$ and $Y \times B$, and let $\Omega_K = \{\lambda \in \Lambda : \Gamma(f_\lambda) \cap K \neq \emptyset\}$. Then, from Definition 3.6, it follows that $0, 1 \in \Omega_K$. Hence, by Lemma 3.10, for all $\lambda \in (0, 1) \cap \Lambda$, $\Gamma(\lambda) \subseteq K$, so to show that $K = \bigcup_{\lambda\in\Lambda} \Gamma(f_\lambda)$, it suffices to show that $\Gamma(f_0)$ and $\Gamma(f_1)$ are contained in K.

Suppose that 0 is a limit point of Λ, then there is a sequence of functions whose graphs are contained in K which converge to f_0. It follows that $\Gamma(f_0) \subseteq K$.

If 0 is isolated in Λ, then let $\lambda = \min(0, 1] \cap \Lambda$. Since K intersects $\Gamma(f_0)$ and $\Gamma(f_\lambda)$ it must intersect their intersection. Since K also intersects $Y \times A$, it follows that K intersects both $\Gamma(f_0|_C)$ and $\Gamma(f_0|_D)$. Then, just as in the proof of the previous lemma, it follows that $\Gamma(f_0) \subseteq K$.

Similarly, $\Gamma(f_1) \subseteq K$. $\square$

3.2.2 Irreducible Functions

We are now ready to define the term *irreducible function.*

Definition 3.12. A function $F : X \to 2^Y$ is called *irreducible with respect to $A, B \subseteq X$ and $C, D \subseteq Y$*, if there exists a collection of maps $\{f_\lambda\}_{\lambda\in\Lambda}$ which is irreducible with respect to $A, B \subseteq X$ and $C, D \subseteq Y$ such that for all $x \in X$,

$$F(x) = \bigcup_{\lambda\in\Lambda} f_\lambda^{-1}(x).$$

When no ambiguity shall arise, or when mention of the sets, $A, B \subseteq X$ and $C, D \subseteq Y$ is unnecessary, we will simply say that F is an *irreducible function.*

Figure 3.3 shows an irreducible collection of maps (left) and its corresponding irreducible function (right).

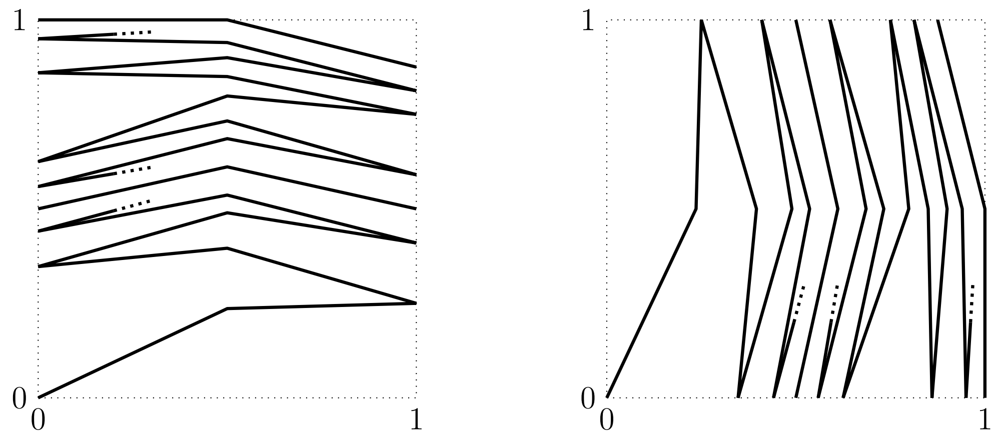

Figure 3.3. Irreducible collection of maps and its corresponding irreducible function

Corollary 3.8 stated that given an irreducible collection of maps $\{f_\lambda\}_{\lambda\in\Lambda}$ from Y to X, the union of their images, $\bigcup_{\lambda\in\Lambda} f_\lambda(Y)$, is equal to X. Thus, the maps in the collection can be inverted to yield an irreducible function $F : X \to 2^Y$. Thus, any of the irreducible collections from Figures 3.1 or 3.2 correspond to irreducible functions.

Lemma 3.13. *If $F : X \to 2^Y$ is an irreducible function, then F is upper-semi continuous, and $\Gamma(F)$ is a continuum.*

Proof. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be the irreducible collection corresponding to F. then $\Gamma(F) = \bigcup_{\lambda\in\Lambda} \Gamma(f_\lambda^{-1})$ which is homeomorphic to $\bigcup_{\lambda\in\Lambda} \Gamma(f_\lambda)$. Therefore, by Lemma 3.7, $\Gamma(F)$ is a continuum. In particular, $\Gamma(F)$ is also closed, so F is upper semi-continuous. $\square$

Lemma 3.14. *Every irreducible set-valued function is an indecomposable function.*

Proof. Let $F : X \to 2^Y$ be an irreducible function with the corresponding collection $\{f_\lambda\}_{\lambda\in\Lambda}$ irreducible with respect to $A, B \subseteq X$ and $C, D \subseteq Y$. Suppose that K and L are subcontinua of $\Gamma(F)$ with $K \cup L = \Gamma(F)$.

Case 1: Suppose that there exists $\lambda \in \Lambda$ such that $\Gamma(f_\lambda^{-1}) \cap K = \emptyset$. Then $\Gamma(f_\lambda^{-1}) \subseteq L$, so $\pi_2(L) \supseteq \pi_2[\Gamma(f_\lambda^{-1})] = Y$. Similarly, if there exists $\lambda \in \Lambda$ such that $\Gamma(f_\lambda^{-1}) \cap L = \emptyset$, then $\pi_2(K) = Y$.

Case 2: Suppose that for all $\lambda \in \Lambda$, $\Gamma(f_\lambda^{-1}) \cap K$ and $\Gamma(f_\lambda^{-1}) \cap L$ are both nonempty. Let λ_0, λ_1 be adjacent elements of Λ. Without loss of generality, suppose that $\Gamma(f_{\lambda_0}|_C) \cap \Gamma(f_{\lambda_1}|_C) \neq \emptyset$.

In a way similar to what was done in the proof of Lemma 3.10, $\Gamma(F) \setminus \Gamma[(f_{\lambda_0}|_C)^{-1}]$ is not connected, and it can be separated by the sets U and V, where if $\lambda_1 < \lambda_0$, U and V are defined by

$$\begin{aligned} U &= \bigcup_{\lambda \in [0,\lambda_1] \cap \Lambda} \Gamma(f_\lambda^{-1}) \setminus \Gamma[(f_{\lambda_0}|_C)^{-1}], \text{ and} \\ V &= \bigcup_{\lambda \in [\lambda_0,1] \cap \Lambda} \Gamma(f_\lambda^{-1}) \setminus \Gamma[(f_{\lambda_0}|_C)^{-1}], \end{aligned}$$

and if $\lambda_0 < \lambda_1$,

$$\begin{aligned} U &= \bigcup_{\lambda \in [\lambda_1,1] \cap \Lambda} \Gamma(f_\lambda^{-1}) \setminus \Gamma[(f_{\lambda_0}|_C)^{-1}], \text{ and} \\ V &= \bigcup_{\lambda \in [0,\lambda_0] \cap \Lambda} \Gamma(f_\lambda^{-1}) \setminus \Gamma[(f_{\lambda_0}|_C)^{-1}]. \end{aligned}$$

In either case, since for all $\lambda \in \Lambda$, $\Gamma(f_\lambda^{-1}) \cap K$ and $\Gamma(f_\lambda^{-1}) \cap L$ are each nonempty, they each intersect U and V. Thus, since they are connected, they must both intersect $\Gamma[(f_{\lambda_0}|_C)^{-1}]$. Now, since $\Gamma(F) = K \cup L$, at least one of K and L must intersect $\Gamma[(f_{\lambda_0}|_D)^{-1}]$. Without loss of generality, we will say that K does. Then, we may apply the Cut-wire Theorem (Theorem 3.9) in the same way as in the proof of Lemma 3.10 to say that $\Gamma(f_{\lambda_0}^{-1}) \subseteq K$. Thus, $\pi_2(K) = Y$. □

We are now ready to prove the main result of this section.

Theorem 3.15. *Let* $\{\mathbf{X}, \mathbf{F}\}$ *be an inverse sequence where for each* $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ *is irreducible with respect to* $A_{i+1}, B_{i+1} \subseteq X_{i+1}$ *and* $A_i, B_i \subseteq X_i$. *Then* $\varprojlim \mathbf{F}$ *has the full projection property and is an indecomposable continuum.*

Proof. Since each F_i is an irreducible function, for each $i \in \mathbb{N}$ there is a corresponding collection $\{f_\lambda^{(i)} : X_i \to X_{i+1}\}_{\lambda \in \Lambda_i}$ which is irreducible with respect to $A_{i+1}, B_{i+1} \subseteq$

X_{i+1} and $A_i, B_i \subseteq X_i$. From Lemma 3.7, we have that $\Gamma_2 = \bigcup_{\lambda \in \Lambda_1} \Gamma(f_\lambda^{(1)})$ is connected, and from Corollary 3.11, Γ_2 is irreducible between the sets $X_1 \times A_2$ and $X_1 \times B_2$.

Now suppose that for some $n \in \mathbb{N}$, Γ_n is a continuum and is irreducible between the sets

$$\mathcal{A} = \{\mathbf{x} \in \Gamma_n : x_n \in A_n\} \text{ and } \mathcal{B} = \{\mathbf{x} \in \Gamma_n : x_n \in B_n\}.$$

For each $\lambda \in \Lambda_n$, define a function $\mathfrak{f}_\lambda^{(n)} : \Gamma_n \to X_{n+1}$ by $\mathfrak{f}_\lambda^{(n)}(x_1, \ldots, x_n) = f_\lambda^{(n)}(x_n)$. Then the collection of maps $\{\mathfrak{f}_\lambda^{(n)}\}_{\lambda \in \Lambda_n}$ is irreducible with respect to $A_{n+1}, B_{n+1} \subseteq X_{n+1}$ and $\mathcal{A}, \mathcal{B} \subseteq \Gamma_n$. Also $\Gamma_{n+1} = \bigcup_{\lambda \in \Lambda} \Gamma(\mathfrak{f}_\lambda^{(n)})$. By Lemma 3.7, $\bigcup_{\lambda \in \Lambda} \Gamma(h_\lambda)$ is a continuum, and by Corollary 3.11, it is irreducible between the sets $\Gamma_n \times A_{n+1}$ and $\Gamma_n \times B_{n+1}$.

By induction we can say that for each $n \in \mathbb{N}$, Γ_n is a continuum which is irreducible between the sets $\Gamma_{n-1} \times A_n$ and $\Gamma_{n-1} \times B_n$. Therefore, $\varprojlim \mathbf{F}$ is a continuum, and by Theorem 3.4, $\varprojlim \mathbf{F}$ has the full projection property.

Finally, by Lemma 3.14, each F_i is an indecomposable function, so by Theorem 3.3, $\varprojlim \mathbf{F}$ is an indecomposable continuum. $\square$

CHAPTER FOUR

Chainability of Inverse Limits of Irreducible Set-valued Functions on $[0, 1]$

Now that we have defined this class of functions, it is natural to further study the properties of their inverse limits. In this chapter, we give a characterization for chainability of inverse limits with irreducible functions on $[0, 1]$. Ingram considers chainability of inverse limits in [26] and gives the following theorem.

Theorem 4.1 (Ingram). *Suppose* $\mathbf{X}$ *is a sequence of continua and* $F_n : X_{n+1} \to 2^{X_n}$ *is an upper semi-continuous function for each positive integer* n*. If* Γ_n *is a chainable continuum for each* $n \in \mathbb{N}$*, then* $\varprojlim \mathbf{F}$ *is a chainable continuum.*

In some circumstances, each Γ_n is homeomorphic to a subcontinuum of $\varprojlim \mathbf{F}$. This is significant when discussing chainability because if a continuum is chainable, so are all of its subcontinua. Hence, in these circumstances, each Γ_n being a chainable continuum would be both sufficient and necessary for $\varprojlim \mathbf{F}$ to be chainable. In [41], Marsh gives a specific condition which implies that for each (or for some) $n \in \mathbb{N}$, Γ_n is homeomorphic to a subcontinuum of $\varprojlim \mathbf{F}$.

Theorem 4.2 (Marsh). *Let* $\{\mathbf{X}, \mathbf{F}\}$ *be an inverse sequence. Suppose there exists an* $n \in \mathbb{N}$ *such that for every* $i \geq n$*, there exists a continuous single-valued map* $f_i : X_i \to X_{i+1}$ *with* $\Gamma(f_i) \subseteq \Gamma(F_i^{-1})$*. Then for each* $i \geq n$*, there exists a subcontinuum* $A(i)$ *of* $\varprojlim \mathbf{F}$ *which is homeomorphic to* Γ_i *through the projection map* $\pi_{[1,i]}|_{A(i)} : A(i) \to \Gamma_i$*.*

From Definition 3.12, we have that if $F : X \to 2^Y$ is an irreducible function, then there is a collection $\{f_\lambda\}_{\lambda \in \Lambda}$ of maps such that for any $\lambda \in \Lambda$, $\Gamma(f_\lambda) \subseteq \Gamma(F^{-1})$. Hence, we have the following corollary.

Corollary 4.3. *Suppose* $\{\mathbf{X}, \mathbf{F}\}$ *is an inverse sequence where for each* $i \in \mathbb{N}$, $F_i \to X_{i+1} \to 2^{X_i}$ *is an irreducible function. Then for each* $n \in \mathbb{N}$, Γ_n *is homeomorphic to a subcontinuum of* $\varprojlim \mathbf{F}$.

Hence, we can restate Theorem 4.1 in the context of irreducible functions.

Corollary 4.4. *Suppose* $\{\mathbf{X}, \mathbf{F}\}$ *is an inverse sequence where for each* $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ *is an irreducible function. Then* $\varprojlim \mathbf{F}$ *is chainable if, and only if,* Γ_n *is chainable for all* $n \in \mathbb{N}$.

4.1 Structure of Γ_n

Some of what makes inverse limits of irreducible functions nice to work with is that they lend themselves naturally to proofs by induction on the sequence $\mathbf{\Gamma}$. Ingram pointed out in [25, p. 53] that if $F : [0,1] \to 2^{[0,1]}$ is an upper semi-continuous function, then for each $n \in \mathbb{N}$,

$$\Gamma_{n+1} = \{\mathbf{x} \in [0,1]^{n+1} : (x_1, \ldots, x_n) \in \Gamma_n \text{ and } x_{n+1} \in F^{-1}(x_n)\}.$$

By looking at these sets in this way, we can see that if $F : [0,1] \to 2^{[0,1]}$ is an irreducible function with the corresponding irreducible collection of maps $\{f_\lambda\}_{\lambda \in \Lambda}$, that for each $n \in \mathbb{N}$,

$$\begin{aligned} \Gamma_{n+1} &= \left\{\mathbf{x} \in [0,1]^{n+1} : (x_1, \ldots, x_n) \in \Gamma_n \text{ and } x_{n+1} \in F^{-1}(x_n)\right\} \\ &= \bigcup_{\lambda \in \Lambda} \left\{\mathbf{x} \in [0,1]^{n+1} : (x_1, \ldots, x_n) \in \Gamma_n \text{ and } x_{n+1} = f_\lambda(x_n)\right\}. \end{aligned}$$

Note that for each $\lambda \in \Lambda$, the set $\{\mathbf{x} \in [0,1]^{n+1} : (x_1, \ldots, x_n) \in \Gamma_n \text{ and } x_{n+1} = f_\lambda(x_n)\}$ is homeomorphic to Γ_n. Thus, Γ_{n+1} is a union of homeomorphic copies of Γ_n. Looking at these sets in this way is crucial to the discussion of their structure in this section.

Recall that in the proof of Theorem 3.15, it was shown that if $F : [0,1] \to 2^{[0,1]}$ is an irreducible function, then for all $n \in \mathbb{N}$, Γ_n is irreducible between the sets

$\{\mathbf{x} \in \Gamma_n : x_n = 0\}$ and $\{\mathbf{x} \in \Gamma_n : x_n = 1\}$. These sets are discussed extensively in this section and the next, so we establish the following notation.

Notation 4.5. Let $F : [0,1] \to 2^{[0,1]}$ be an irreducible function. For each $n \in \mathbb{N}$, let

$$\begin{aligned} \mathcal{A}_n &= \{\mathbf{x} \in \Gamma_n : x_n = 0\} \text{ and} \\ \mathcal{B}_n &= \{\mathbf{x} \in \Gamma_n : x_n = 1\}. \end{aligned}$$

Lemma 4.6. *Let $F : [0,1] \to 2^{[0,1]}$ be an irreducible function. If $F(0), F(1) \in \{\{0\}, \{1\}, [0,1]\}$, then $\mathcal{A}_n$ and $\mathcal{B}_n$ are continua for each $n \in \mathbb{N}$.*

Proof. First, if $F(0)$ and $F(1)$ are both singleton sets, then $\mathcal{A}_n$ and $\mathcal{B}_n$ are degenerate for all $n \in \mathbb{N}$ and are hence continua. Similarly, if $F(0) = \{0\}$, then $\mathcal{A}_n$ is a degenerate continuum for all $n \in \mathbb{N}$, and if $F(1) = \{1\}$, then $\mathcal{B}_n$ is a degenerate continuum for all $n \in \mathbb{N}$.

Next, if $F(0) = [0,1]$, then for all $n \in \mathbb{N}$,

$$\mathcal{A}_n = \{(x_1, \ldots, x_{n-1}, 0) : (x_1, \ldots, x_{n-1}) \in \Gamma_{n-1}\}$$

which is homeomorphic to Γ_{n-1}. Then since Γ_{n-1} is a continuum by Lemma 3.11, it follows that $\mathcal{A}_n$ is a continuum. Similarly, if $F(1) = [0,1]$, then $\mathcal{B}_n$ is homeomorphic to Γ_{n-1} for all $n \in \mathbb{N}$, and hence is a continuum.

This leaves only two cases to check. We must verify that if $F(0) = \{1\}$ and $F(1) = [0,1]$, then $\mathcal{A}_n$ is a continuum, and that if $F(1) = \{0\}$ and $F(0) = [0,1]$, then $\mathcal{B}_n$ is a continuum. In the first case, if $F(0) = \{1\}$, and $F(1) = [0,1]$, note that as we have already observed, $\mathcal{B}_n$ is a continuum for all $n \in \mathbb{N}$. Also, for all $n \in \mathbb{N}$,

$$\mathcal{A}_{n+1} = \{\mathbf{x} \in \Gamma_{n+1} : (x_1, \ldots, x_n) \in \mathcal{B}_n \text{ and } x_{n+1} = f_0(x_n)\}$$

which is homeomorphic to $\mathcal{B}_n$. Hence $\mathcal{A}_{n+1}$ is a continuum.

Similarly, if $F(1) = \{0\}$, and $F(0) = [0,1]$, then for each $n \in \mathbb{N}$, $\mathcal{B}_{n+1}$ is homeomorphic to $\mathcal{A}_n$ and is thus a continuum. $\square$

In [39, Section 48], Kuratowski defines the notion of "layers" of an irreducible continuum. More specifically, if X is an hereditarily decomposable, irreducible continuum, there exists a decomposition $\mathcal{D}$ of X into continua such that $X/\mathcal{D}$ is an arc, and for any other decomposition $\mathcal{E}$ of X into continua such that $X/\mathcal{E}$ is an arc, each element of $\mathcal{E}$ is a union of elements of $\mathcal{D}$. The elements of this decomposition $\mathcal{D}$ are called the *Kuratowski layers* of X.

Let $g : X \to X/\mathcal{D}$ be the decomposition map (called the *Kuratowski function*). If a and b are the endpoints of $X/\mathcal{D}$, then the layers $g^{-1}(a)$ and $g^{-1}(b)$ are called the *end layers* of X. Another way to view the end layers of an irreducible continuum is to say that a point $p \in X$ is in an end layer of X if, and only if, X is irreducible between p and some other point. Thus, one can say that the end layers of an hereditarily decomposable, irreducible continuum X are the two maximal continua between which X is irreducible. In [42], Minc and Transue prove the following result concerning ϵ-maps and end layers of a continuum.

Theorem 4.7 (Minc and Transue). *Suppose X is a chainable, hereditarily decomposable continuum, A and B are end layers of X, $\epsilon > 0$, $f : A \cup B \to \mathbb{R}$ is an ϵ-map with $f(A) = [a_1, a_2]$, $f(B) = [b_1, b_2]$ and $a_2 < b_1$. Then there exists an ϵ-map $\overline{f} : X \to [a_1, b_2]$ with $\overline{f}|_{A \cup B} = f$.*

Corollary 4.8. *Suppose X is a chainable, hereditarily decomposable continuum, A is an end layer of X, $\epsilon > 0$, and $f : A \to \mathbb{R}$ is an ϵ-map with $f(A) = [a_1, a_2]$. Then for any $b > a_2$, there exists an ϵ-map $\overline{f} : X \to [a_1, b]$ with $\overline{f}|_A = f$.*

Proof. Let B be the end layer of X other than A, and let $b_1 \in \mathbb{R}$ with $a_2 < b_1 < b$. Since B is a subcontinuum of a chainable continuum, there exists an ϵ-map $g : B \to \mathbb{R}$ such that $g(B) = [b_1, b]$. Then the function $h : A \cup B \to \mathbb{R}$ defined piecewise by $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$ is an ϵ-map. Thus by Theorem 4.7, h can be extended to an ϵ-map on X whose image is $[a_1, b]$. $\square$

The following corollary follows from Theorem 4.7 and iterative applications of Corollary 4.8.

Corollary 4.9. *Suppose that X is a chainable, hereditarily decomposable continuum with end layers A and B. Suppose also that there exist sequences $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = A$ and $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_m = B$, $n, m \geq 1$, such that for each $i = 1, \ldots, n-1$, A_i is an end layer of A_{i+1}, and for each $j = 1, \ldots, m-1$, B_j is an end layer of B_{j+1}. If $\epsilon > 0$, $f : A_1 \cup B_1 \to \mathbb{R}$ is an ϵ-map with $f(A_1) = [a_1, a_2]$, $f(B_1) = [b_1, b_2]$, and $a_2 < b_1$, then there exists an ϵ-map $\overline{f} : X \to [a_1, b_2]$ with $\overline{f}|_{A_1 \cup B_1} = f$.*

One of our main goals of this section is to show that under the right conditions, any ϵ-map defined on $\mathcal{A}_n \cup \mathcal{B}_n$ may be extended to an ϵ-map on Γ_n. To do this we will show that $\mathcal{A}_n$ and $\mathcal{B}_n$ either are end layers of Γ_n, or that there exist sequences $\mathcal{A}_n = A_1 \subseteq \cdots \subseteq A_k = \Gamma_n$ and $\mathcal{B}_n = B_1 \subseteq \cdots \subseteq B_m = \Gamma_n$ as in Corollary 4.9. We do this in the following two lemmas. In these lemmas, we use the following notation.

Definition 4.10. Let $F : [0,1] \to 2^{[0,1]}$ be an irreducible function with the corresponding irreducible collection $\{f_\lambda\}_{\lambda \in \Lambda}$. For each $n \in \mathbb{N}$ and $\lambda \in \Lambda$, define $\mathfrak{f}_\lambda^{(n)} : \Gamma_n \to [0,1]$ by

$$\mathfrak{f}_\lambda^{(n)}(x_1, x_2, \ldots, x_n) = f_\lambda(x_n).$$

Remark 4.11. It can be easily verified that for each $n \in \mathbb{N}$, the collection of maps $\{\mathfrak{f}_\lambda^{(n)} : \Gamma_n \to [0,1]\}_{\lambda \in \Lambda}$ is irreducible with respect to $\{0\}, \{1\} \subseteq [0,1]$ and $\mathcal{A}_n, \mathcal{B}_n \subseteq \Gamma_n$. Thus, Lemma 3.11 and Lemma 3.10 apply to this collection. Moreover, for each $n \in \mathbb{N}$,

$$\Gamma_{n+1} = \bigcup_{\lambda \in \Lambda} \Gamma(\mathfrak{f}_\lambda^{(n)}).$$

Lemma 4.12. *Let $F : [0,1] \to 2^{[0,1]}$ be an irreducible function with the associated irreducible collection $\{f_\lambda\}_{\lambda \in \Lambda}$. If $0 \in \Lambda'$, then for all $n \in \mathbb{N}$, $\Gamma(\mathfrak{f}_0^{(n)})$ is an end layer of Γ_{n+1}. Likewise, if $1 \in \Lambda'$, then for all $n \in \mathbb{N}$, $\Gamma(\mathfrak{f}_1^{(n)})$ is an end layer of Γ_{n+1}.*

Proof. Suppose that $0 \in \Lambda'$. We will show that for any $n \in \mathbb{N}$, Γ_{n+1} is irreducible between $\Gamma(\mathfrak{g}_0^{(n)})$ and $\mathcal{B}_{n+1}$. Note that we already have from Lemma 3.11 that for all $n \in \mathbb{N}$, Γ_{n+1} is irreducible between $\mathcal{A}_{n+1}$ and $\mathcal{B}_{n+1}$. We also have from Definition 3.5, Property (1), that $\mathcal{A}_{n+1} \subseteq \Gamma(\mathfrak{f}_0^{(n)})$ and $\mathcal{B}_{n+1} \subseteq \Gamma(\mathfrak{f}_1^{(n)})$.

Let $n \in \mathbb{N}$ and suppose that K is a subcontinuum of Γ_{n+1} which intersects $\Gamma(\mathfrak{f}_0^{(n)})$ and $\mathcal{B}_{n+1}$. Let $\Omega_K = \{\lambda \in \Lambda : \Gamma(\mathfrak{f}_\lambda^{(n)}) \cap K \neq \emptyset\}$. Since K intersects $\Gamma(\mathfrak{f}_0^{(n)})$ and $\mathcal{B}_{n+1}$ (which is a subset of $\Gamma(\mathfrak{f}_1^{(n)})$), we have that $0, 1 \in \Omega_K$. Thus, by Lemma 3.10, $\Omega_K = \Lambda$. Lemma 3.10 also gives us that for all $\lambda \in \Lambda \setminus \{0,1\}$, $\Gamma(\mathfrak{f}_\lambda^{(n)}) \subseteq K$.

Since $0 \in \Lambda'$, we may choose a sequence $(\lambda_i)_{i\in\mathbb{N}}$ in $\Lambda \setminus \{0,1\}$ which converges to 0. Then by Definition 3.6, Property (5), for each $\mathbf{x} \in \Gamma_n$, $(\mathbf{x}, \mathfrak{f}_{\lambda_i}^{(n)}(\mathbf{x})) \to (\mathbf{x}, \mathfrak{f}_0^{(n)}(\mathbf{x}))$ as $i \to \infty$. Since K is closed, this implies that $(\mathbf{x}, \mathfrak{f}_0^{(n)}(\mathbf{x})) \in K$ for all $\mathbf{x} \in \Gamma_n$, so $\Gamma(\mathfrak{f}_0^{(n)}) \subseteq K$.

In particular then, $\mathcal{A}_{n+1} \subseteq K$. Since Γ_{n+1} is irreducible between $\mathcal{A}_{n+1}$ and $\mathcal{B}_{n+1}$, it follows that $K = \Gamma_{n+1}$. This shows that Γ_{n+1} is irreducible between $\Gamma(\mathfrak{f}_0^{(n)})$ and $\mathcal{B}_{n+1}$, so $\Gamma(\mathfrak{f}_0^{(n)})$ must be contained in an end layer.

To show that $\Gamma(\mathfrak{f}_0^{(n)})$ must in fact be equal to one of the end layers, we will show that for any $\lambda \in \Lambda \setminus \{0,1\}$ and any point $\mathbf{x} \in \Gamma(\mathfrak{f}_\lambda^{(n)})$, Γ_{n+1} is not irreducible between $\mathbf{x}$ and any other point. To show this, fix $\lambda_0 \in \Lambda \setminus \{0,1\}$ and a point $\mathbf{x} \in \Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$. Let $\mathbf{y} \in \Gamma_{n+1}$, and choose $\mu \in \Lambda$ so that $\mathbf{y} \in \Gamma(\mathfrak{f}_\mu^{(n)})$. Let J be the closed interval whose endpoints are μ and λ_0. Since $\lambda_0 \in (0,1)$, J is a proper subset of $[0,1]$, so $J \cap \Lambda$ is a proper subset of Λ. Thus, $\bigcup_{\lambda \in J\cap\Lambda} \Gamma(\mathfrak{f}_\lambda^{(n)})$ is a proper subcontinuum of Γ_{n+1}. Moreover, since $\mu, \lambda_0 \in J \cap \Lambda$, $\mathbf{x}, \mathbf{y} \in \bigcup_{\lambda \in J\cap\Lambda} \Gamma(\mathfrak{f}_\lambda^{(n)})$. So Γ_{n+1} is not irreducible between $\mathbf{x}$ and any other point, so $\mathbf{x}$ is not in an end layer. This means that in general, the end layers of Γ_{n+1} are contained in $\Gamma(\mathfrak{f}_0^{(n)})$ and $\Gamma(\mathfrak{f}_1^{(n)})$. Therefore, in this case, we have that $\Gamma(\mathfrak{f}_0^{(n)})$ is an end layer.

In the same way, if $1 \in \Lambda'$, then for all natural numbers n, $\Gamma(\mathfrak{f}_1^{(n)})$ is an end layer of Γ_{n+1}. $\square$

We give one more lemma before we establish the main result of this section.

Lemma 4.13. *Let $F : [0,1] \to 2^{[0,1]}$ be an irreducible function such that $F(0), F(1) \in \{\{0\},\{1\},[0,1]\}$. For all $n \in \mathbb{N}$, there exists a sequence of continua $\mathcal{A}_n = A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k = \Gamma_n$ with $2 \le k \le n$ such that for each $i = 1,\ldots,k-1$, A_i is an end layer of A_{i+1}. Likewise, there exists a sequence of continua $\mathcal{B}_n = B_1 \subseteq B_2 \subseteq \cdots \subseteq B_m = \Gamma_n$ with $2 \le m \le n$ such that for each $i = 1,\ldots,m-1$, B_i is an end layer of B_{i+1}.*

Proof. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be the irreducible collection of maps corresponding to F.

Since $\Gamma_1 = [0,1]$, we have that $\mathcal{A}_1 = \{0\}$ and $\mathcal{B}_1 = \{1\}$ are end layers of Γ_1. For $n \ge 2$ notice first that if we have that $F(0) = [0,1]$ (so $f_0^{-1}(0) = [0,1]$), then, by Definition 3.5, Property (2), $0 \in \Lambda'$, and for all $n \ge 2$, $\Gamma(\mathfrak{f}_0^{(n-1)}) = \mathcal{A}_n$. Thus by Lemma 4.12, $\mathcal{A}_n$ is an end layer of Γ_n. Similarly, if $F(1) = [0,1]$, then for all $n \ge 2$, $\mathcal{B}_n$ is an end layer of Γ_n

For the other cases we will use induction, so suppose that for some $n \in \mathbb{N}$, we have sequences of continua $\mathcal{A}_n = A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k = \Gamma_n$ and $\mathcal{B}_n = B_1 \subseteq B_2 \subseteq \cdots \subseteq B_m = \Gamma_n$ such that for each $i = 1,\ldots,k-1$, A_i is an end layer of A_{i+1}, and for each $i = 1,\ldots,m-1$, B_i is an end layer of B_{i+1}.

First we will show that we can create a new sequence $\mathcal{A}_{n+1} = \widetilde{A}_1 \subseteq \cdots \subseteq \widetilde{A}_{\widetilde{k}} = \Gamma_{n+1}$ such that for $i = 1,\ldots,\widetilde{k}-1$, $\widetilde{A}_i$ is an end layer of $\widetilde{A}_{i+1}$.

Case 1: Suppose that 0 is not a limit point of Λ. Then there exists a smallest element of $(0,1]\cap\Lambda$. Call this element λ_0. Then either $f_0(0) = 0$ and $f_0(1) = f_{\lambda_0}(1)$, or $f_0(1) = 0$ and $f_0(0) = f_{\lambda_0}(0)$.

Sub-case 1(a): Suppose that $f_0(0) = 0$ and $f_0(1) = f_{\lambda_0}(1)$. Then we have that $\mathcal{A}_{n+1} = \Gamma(\mathfrak{f}_0^{(n)}|_{\mathcal{A}_n})$ and $\Gamma(\mathfrak{f}_0^{(n)}|_{\mathcal{B}_n}) = \Gamma(\mathfrak{f}_{\lambda_0}^{(n)}|_{\mathcal{B}_n})$. By assumption, A_{k-1} and B_{m-1} are the end layers of Γ_n, and Γ_n is homeomorphic to $\Gamma(\mathfrak{f}_0^{(n)})$ through the homeomorphism $\mathbf{x} \mapsto (\mathbf{x}, \mathfrak{f}_0^{(n)}(\mathbf{x}))$. Thus $\Gamma(\mathfrak{f}_0^{(n)}|_{A_{k-1}})$ and $\Gamma(\mathfrak{f}_0^{(n)}|_{B_{m-1}})$ are the end layers of $\Gamma(\mathfrak{f}_0^{(n)})$. Then since it is within the set $\Gamma(\mathfrak{f}_0^{(n)}|_{B_{m-1}})$ that $\Gamma(\mathfrak{f}_0^{(n)})$ intersects $\Gamma_{n+1} \setminus \Gamma(\mathfrak{f}_0^{(n)})$, we have that $\Gamma(\mathfrak{f}^{(n)}|_{A_{k-1}})$ must be an end layer of Γ_{n+1}. Thus for each $i = 1,\ldots,k-1$, let $\widetilde{A} = \Gamma(\mathfrak{f}_0^{(n)}|_{A_i})$, and let $\widetilde{A}_k = \Gamma_{n+1}$. Then we have that $\widetilde{A}_i$ is an end layer of $\widetilde{A}_{i+1}$ for $i = 1, \ldots, k-1$, and $\widetilde{A}_{k-1}$ is an end layer of $\widetilde{A}_k = \Gamma_{n+1}$.

Sub-case 1(b): Suppose that $f_0(1) = 0$ and $f_0(0) = f_{\lambda_0}(0)$. Just as in Sub-case 1(a), we have that $\Gamma(\mathfrak{f}_0^{(n)}|_{A_{k-1}})$ and $\Gamma(\mathfrak{f}_0^{(n)}|_{B_{m-1}})$ are the end layers of $\Gamma(\mathfrak{f}_0^{(n)})$, but this time it is within $\Gamma(\mathfrak{f}_0^{(n)}|_{A_{k-1}})$ that $\Gamma(\mathfrak{f}_0^{(n)})$ intersects $\Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$. Therefore, we will define for each $i = 1, \ldots, m-1$, $\widetilde{A}_i = \Gamma(\mathfrak{f}_0^{(n)}|_{B_i})$ and $\widetilde{A}_m = \Gamma_{n+1}$. This will give us our desired sequence.

Case 2: Suppose that 0 is a limit point of Λ and that $F(0) \in \{\{0\}, \{1\}\}$. Since $0 \in \Lambda'$, we have by Lemma 4.12 that $\Gamma(\mathfrak{f}_0^{(n)})$ is an end layer of Γ_n.

Sub-case 2(a): Suppose that $f_0(0) = 0$. Then $\mathcal{A}_{n+1} = \Gamma(\mathfrak{f}_0^{(n)}|_{\mathcal{A}_n})$. Just as before, $\Gamma(\mathfrak{f}_0^{(n)})$ is homeomorphic to Γ_n by the homeomorphism $\mathbf{x} \mapsto (\mathbf{x}, \mathfrak{f}_0^{(n)}(\mathbf{x}))$. Thus, for $i = 1, \ldots, k$, let $\widetilde{A}_i = \Gamma(\mathfrak{f}_0^{(n)}|_{A_i})$, and let $\widetilde{A}_{k+1} = \Gamma_{n+1}$. Then we have a sequence $\mathcal{A}_{n+1} = \widetilde{A}_1 \subseteq \cdots \subseteq \widetilde{A}_k \subseteq \widetilde{A}_{k+1} = \Gamma_{n+1}$. Through the homeomorphism between $\Gamma(\mathfrak{f}^{(n)})$ and Γ_n, we have that for all $i = 1, \ldots, k-1$ that $\widetilde{A}_i$ is an end layer of $\widetilde{A}_{i+1}$. Then $\widetilde{A}_k = \Gamma(\mathfrak{f}_0^{(n)})$ which is an end layer of $\Gamma_{n+1} = \widetilde{A}_{k+1}$.

Sub-case 2(b): Suppose that $f_0(1) = 0$. Then $\mathcal{A}_{n+1} = \Gamma(\mathfrak{f}_0^{(n)}|_{\mathcal{B}_n})$ so we similarly define for $i = 1, \ldots, m$ $\widetilde{A}_i = \Gamma(\mathfrak{f}_0^{(n)}|_{B_i})$ and $\widetilde{A}_{m+1} = \Gamma'_{n+1}$. This yields the desired sequence.

In either case, we have a sequence $\mathcal{A}_{n+1} = \widetilde{A}_1 \subseteq \cdots \subseteq \widetilde{A}_{\widetilde{k}} = \Gamma_{n+1}$ such that for all $i = 1, \ldots, \widetilde{k}$, $\widetilde{A}_i$ is an end layer of $\widetilde{A}_{i+1}$. In a similar way we may construct a sequence $\mathcal{B}_{n+1} = \widetilde{B}_1 \subseteq \cdots \subseteq \widetilde{B}_{\widetilde{m}} = \Gamma_{n+1}$ such that for $i = 1, \ldots, \widetilde{m} - 1$, $\widetilde{B}_i$ is an end layer of $\widetilde{B}_{i+1}$.

This concludes the induction step. Thus we have shown that for all $n \in \mathbb{N}$, there exist ascending sequences beginning at $\mathcal{A}_n$ and $\mathcal{B}_n$ respectively and ending at Γ_n such that each term of the sequence is an end layer of the next. $\square$

Lemma 4.13 in conjunction with Corollary 4.9 yields the following result which is the main result of this section.

Corollary 4.14. *Let* $F : [0,1] \to 2^{[0,1]}$ *be an irreducible function such that* $F(0), F(1) \in \{\{0\},\{1\},[0,1]\}$. *For all* $n \in \mathbb{N}$ *such that* Γ_n *is chainable and all* $\epsilon > 0$, *if* $f : \mathcal{A}_n \cup \mathcal{B}_n \to \mathbb{R}$ *is an* ϵ*-map with* $f(\mathcal{A}_n) = [a_1, a_2]$, $f(\mathcal{B}_n) = [b_1, b_2]$, *and* $a_2 < b_1$, *then there exists an* ϵ*-map* $h : \Gamma_n \to [a_1, b_2]$ *with* $h|_{\mathcal{A}_n \cup \mathcal{B}_n} = f$.

4.2 Chainability

In this section we put the results of previous section to use as we give a characterization of chainability for inverse limits with irreducible functions on arcs. We begin with Theorem 4.16 where we demonstrate various conditions under which $\varprojlim \mathbf{F}$ is not chainable. Each of these conditions in fact yields not only that $\varprojlim \mathbf{F}$ is not chainable, but more specifically that Γ_3 is not chainable.

Additionally, given an irreducible function $F : [0,1] \to 2^{[0,1]}$, if the graph of F contains a simple closed curve, then by Corollary 4.3, so does $\varprojlim \mathbf{F}$, so $\varprojlim \mathbf{F}$ is not chainable. (For an example of an irreducible function whose graph contains a simple closed curve, see Figure 4.2.)

The rest of the section builds towards Theorem 4.20 where we demonstrate that if $F : [0,1] \to 2^{[0,1]}$ is an irreducible function whose graph does not contain a simple closed curve, and if F meets none of the criteria of Theorem 4.16, then $\varprojlim \mathbf{F}$ is chainable. Theorems 4.16 and 4.20 together give a characterization of chainability of $\varprojlim \mathbf{F}$ which requires one only to check whether $\Gamma(F)$ contains a simple closed curve and to examine the sets $F(0)$ and $F(1)$. This characterization is stated in Theorem 4.21.

Additionally, each of the conditions in Theorem 4.16 leads to the continuum Γ_3 not being chainable. Thus, we conclude in Theorem 4.21 that given an irreducible function $F : [0,1] \to 2^{[0,1]}$, $\varprojlim \mathbf{F}$ is chainable if, and only if, Γ_3 is chainable.

We define the following terms which will be used in the statement and proof of Theorem 4.16.

Definition 4.15. An *arc* is a continuum which is homeomorphic to a closed interval. A *simple closed curve* is a continuum which is homeomorphic to a circle. For $n \geq 3$, a *simple n-od* is a continuum which is the union of n arcs, each sharing a common endpoint which is the only point common to any two of them. A simple 3-od is also called a *simple triod.*

Note that neither a simple closed curve nor a simple n-od, $n \geq 3$, is chainable nor is any continuum which contains a simple closed curve or a simple n-od.

Theorem 4.16. *Suppose $F : [0,1] \to 2^{[0,1]}$ is an irreducible function, and let $\alpha \in \{0,1\}$.*

(1) If $F(\alpha)$ is not connected, then $\varprojlim \mathbf{F}$ contains a simple closed curve.

(2) If $F(\alpha)$ is a non-degenerate proper sub-interval of $[0,1]$, then $\varprojlim \mathbf{F}$ contains a simple triod.

(3) If $F(\alpha) = \{y_0\}$ where $y_0 \in (0,1)$, then $\varprojlim \mathbf{F}$ contains a simple four-od.

Proof. First, let $\{f_\lambda\}_{\lambda \in \Lambda}$ be the irreducible collection of maps associated with F. Note that if any of the above conditions hold, then α is a limit point of Λ, so in particular, Λ contains infinitely many points, so we may choose $\lambda \in \Lambda \setminus \Lambda', \lambda \neq 0, 1$. Then there exist, $\mu_1, \mu_2 \in \Lambda$ such that $\mu_1 < \lambda < \mu_2$, and $(\mu_1, \lambda) \cap \Lambda = (\lambda, \mu_2) \cap \Lambda = \emptyset$. Then the graph of f_λ must meet the graph of f_{μ_1} at either 0 or 1, and it must meet the graph of f_{μ_2} at the other. Let $\mu \in \{\mu_1, \mu_2\}$ such that $f_\mu(\alpha) = f_\lambda(\alpha)$, and let $p = f_\mu(\alpha) = f_\lambda(\alpha)$.

Case 1: Suppose that $F(\alpha)$ is not connected. Choose $a, b \in F(\alpha)$, $a < b$, so that $F(\alpha) \subseteq [0,a] \cup [b,1]$, and consider the following two sets:

$$\begin{aligned} M_1 &= \{(x_1, x_2, x_3) : x_1 \in [a,b], x_2 = f_\alpha(x_1), x_3 = f_\lambda(x_2)\} \\ M_2 &= \{(x_1, x_2, x_3) : x_1 \in [a,b], x_2 = f_\alpha(x_1), x_3 = f_\mu(x_2)\}. \end{aligned}$$

Since f_α, f_λ, and f_μ are continuous, M_1 and M_2 are arcs, and they share their endpoints $(a,\alpha,p),(b,\alpha,p)$. Since $f_\alpha(x) \in (0,1)$ for all $x \in (a,b)$ and $f_\lambda(x) \neq f_\mu(x)$ for all $x \in (0,1)$, it follows that the only points common to M_1 and M_2 are their endpoints. Therefore $M_1 \cup M_2$ is a simple closed curve. By construction, $M_1, M_2 \subseteq \Gamma_3$, and by Corollary 4.3, Γ_3 is homeomorphic to a subcontinuum of $\varprojlim \mathbf{F}$. Therefore, $\varprojlim \mathbf{F}$ contains a simple closed curve.

Case 2: Let $F(\alpha) = [a,b]$ be a proper sub-interval of $[0,1]$. Then either $a \neq 0$ or $b \neq 1$. Suppose that $a \neq 0$, and consider the following three sets:

$$\begin{aligned} M_1 &= [a,b] \times \{\alpha\} \times \{p\} \\ M_2 &= \{(x_1,x_2,x_3) : x_1 \in [0,a], x_2 = f_\alpha(x_1), x_3 = f_\lambda(x_2)\} \\ M_3 &= \{(x_1,x_2,x_3) : x_1 \in [0,a], x_2 = f_\alpha(x_1), x_3 = f_\mu(x_2)\}. \end{aligned}$$

Again, since f_α, f_λ, and f_μ are continuous, each of these sets is an arc, and they all share the endpoint (a,α,p). In fact, this is only point any two of these arcs share. To see this, notice that if $(x_1,x_2,x_3) \in M_1 \cap M_2$, then $x_1 \in [a,b] \cap [0,a]$, so $x_1 = a$, which means that $x_2 = \alpha$ and $x_3 = p$. Likewise if $(x_1,x_2,x_3) \in M_1 \cap M_3$.

Then if $(x_1,x_2,x_3) \in M_2 \cap M_3$, it follows that $f_\lambda(x_2) = f_\mu(x_2)$. This only holds when $x_2 = \alpha$. Hence, $f_\alpha(x_1)$ must equal α, and since $x_1 \in [0,a]$, $f_\alpha(x_1) = \alpha$ if, and only if, $x_1 = a$.

If $b \neq 1$, then we use a similar construction and define

$$\begin{aligned} M_1 &= [a,b] \times \{\alpha\} \times \{p\} \\ M_2 &= \{(x_1,x_2,x_3) : x_1 \in [b,1], x_2 = f_\alpha(x_1), x_3 = f_\lambda(x_2)\} \\ M_3 &= \{(x_1,x_2,x_3) : x_1 \in [b,1], x_2 = f_\alpha(x_1), x_3 = f_\mu(x_2)\}. \end{aligned}$$

In either case $M_1 \cup M_2 \cup M_3$ is a simple triod. Then, just as in Case 1, $M_1 \cup M_2 \cup M_3$ is a subcontinuum of Γ_3 which is homeomorphic to a subcontinuum of $\varprojlim \mathbf{F}$. Therefore $\varprojlim \mathbf{F}$ contains a simple triod.

Case 3: Suppose $F(\alpha) = \{y_0\}$ where $y_0 \in (0,1)$. Consider the following four sets:

$$
\begin{aligned}
M_1 &= \{(x_1,x_2,x_3) : x_1 \in [0,y_0], x_2 = f_\alpha(x_1), x_3 = f_\lambda(x_2)\} \\
M_2 &= \{(x_1,x_2,x_3) : x_1 \in [y_0,1], x_2 = f_\alpha(x_1), x_3 = f_\lambda(x_2)\} \\
M_3 &= \{(x_1,x_2,x_3) : x_1 \in [0,y_0], x_2 = f_\alpha(x_1), x_3 = f_\mu(x_2)\} \\
M_4 &= \{(x_1,x_2,x_3) : x_1 \in [y_0,1], x_2 = f_\alpha(x_1), x_3 = f_\mu(x_2)\}\,.
\end{aligned}
$$

Each of these is an arc, and the only point common to any two of them is (y_0, α, p). Therefore $M_1 \cup M_2 \cup M_3 \cup M_4$ is a simple four-od which is a subcontinuum of Γ_3 and is therefore homeomorphic to a subcontinuum of $\varprojlim \mathbf{F}$. $\square$

Before we are able to give the statement of Theorem 4.20, we will need three more lemmas.

Lemma 4.17. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be an irreducible collection of maps such that $\bigcup_{\lambda\in\Lambda} \Gamma(f_\lambda)$ does not contain a simple closed curve. For any $\epsilon > 0$, there exists a finite covering $\{[\lambda_1,\mu_1],\ldots,[\lambda_n,\mu_n]\}$ of Λ by mutually disjoint closed intervals of length less than ϵ such that for all $i = 1,\ldots,n$ the following hold:*

(1) $0 = \lambda_1 \le \mu_1 < \lambda_2 \le \mu_2 < \cdots \le \mu_{n-1} < \lambda_n \le \mu_n = 1.$

(2) For each $i = 1,\ldots,n-1$, μ_i and λ_{i+1} are adjacent in Λ.

(3) If $f_{\mu_{i-1}}(\alpha) = f_{\lambda_i}(\alpha)$, then $f_{\mu_i}(1-\alpha) = f_{\lambda_{i+1}}(1-\alpha)$.

Proof. The existence of a finite covering of Λ by mutually disjoint closed intervals $\{U_1,\ldots,U_s\}$ of length less than ϵ follows from the fact that Λ is a compact totally disconnected set (Definition 3.6). Without loss of generality suppose that the sets U_i are in ascending order (i.e. for all $i,j \in \{1,\ldots,s\}$ with $i < j$, $\lambda < \mu$ for all $\lambda \in U_i$ and $\mu \in U_j$).

This collection $\{U_1, \ldots, U_s\}$ meets Criteria 1 and 2, but not necessarily Criterion 3. We will construct from this collection, a new collection of closed intervals $\mathcal{W} = \{W_1, \ldots, W_n\}$ where for each $i = 1, \ldots, s$, either U_i is equal to a member of $\mathcal{W}$, or U_i is the union of two members of $\mathcal{W}$. This new collection $\mathcal{W}$ will be our desired partition.

Let $W_1 = U_1$, and let $\alpha_1 \in \{0, 1\}$ be the element such that $f_{\max U_1}(\alpha_1) = f_{\min U_2}(\alpha_1)$. Then suppose that for some k, W_j and α_j have been defined for all $1 \leq j \leq k$ so that $\bigcup_{i=1}^{k} W_i = \bigcup_{i=1}^{m} U_i$ for some $m \leq s$, and $f_{\max W_k}(\alpha_k) = f_{\min U_{m+1}}(\alpha_k)$.

If $f_{\max U_{m+1}}(1-\alpha_k) = f_{\min U_{m+2}}(1-\alpha_k)$, then let $W_{k+1} = U_{m+1}$, and let $\alpha_{k+1} = 1-\alpha_k$. Note that if U_{m+1} is degenerate, then this will be the case. If $f_{\max U_{m+1}}(\alpha_k) = f_{\min U_{m+2}}(\alpha_k)$, then U_{m+1} is non-degenerate, so there exists an isolated point $\omega \in U_{m+1}$. Let $\underline{\omega}$ be the immediate predecessor of ω in Λ, and $\overline{\omega}$ the immediate successor of ω in Λ.

If $f_{\underline{\omega}}(\alpha_k) = f_\omega(\alpha_k)$, then $f_\omega(1-\alpha_k) = f_{\overline{\omega}}(1-\alpha_k)$. In this case, let $W_{k+1} = [\min U_{m+1}, \omega]$, $W_{k+2} = [\overline{\omega}, \max U_{m+1}]$, $\alpha_{k+1} = 1 - \alpha_k$, and $\alpha_{k+2} = \alpha_k$.

If $f_{\underline{\omega}}(1-\alpha_k) = f_\omega(1-\alpha_k)$, then we let $W_{k+1} = [\min U_{m+1}, \underline{\omega}]$, $W_{k+2} = [\omega, \max U_{m+1}]$, $\alpha_{k+1} = 1 - \alpha_k$, and $\alpha_{k+2} = \alpha_k$.

In this way, the desired covering of Λ is defined. $\square$

Lemma 4.18. *Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be an irreducible collection of maps from a continuum Y to a continuum X, and let d_Y be the metric on Y and $d_{Y \times X}$ be the metric on $Y \times X$. Suppose $\epsilon > 0$, $\Omega \subseteq \Lambda$ is a closed set such that for $\lambda, \mu \in \Omega$, $d_{Y\times X}[(y, f_\lambda(y)), (y, f_\mu(y))] < \epsilon$ for all $y \in Y$, and $W = \bigcup_{\omega \in \Omega} \Gamma(f_\omega)$. Then there exists $\delta > 0$ such that if $y_1, y_2 \in Y$ with $d_Y(y_1, y_2) < \delta$ and $\omega \in \Omega$, it follows that $d_{Y\times X}[(y_1, f_\omega(y_1)), (y_2, f_\omega(y_2))] < \epsilon$. Moreover, if $h : Y \to Z$ is a δ-map, then $h \circ \pi_1|_W : W \to Z$ is a 2ϵ-map.*

Proof. First, let d_X be the metric on X. Now to establish the existence of such a δ, note that for each $\omega \in \Omega$, f_ω is a continuous function with a compact domain, so it is uniformly continuous. Thus, for each $\omega \in \Omega$, there exists $\delta_\omega > 0$ such that when $a, b \in Y$ with $d_Y(a,b) < \delta_\omega$, it follows that $d_X(f_\omega(a), f_\omega(b)) < \epsilon$. From Definition 3.6, Property (5), we have that δ_ω is a continuous function of ω, so since Ω is a compact set, the collection $\{\delta_\omega : \omega \in \Omega\}$ has a minimum element. Choose δ to be this minimum.

Now let $h : Y \to Z$ be a δ-map. To show that $h \circ \pi_1|_W$ is a 2ϵ-map, let $z \in Z$, and let $(y_1, x_1), (y_2, x_2) \in (h \circ \pi_1|_W)^{-1}(z)$. Choose $\lambda, \mu \in \Omega$ such that $(y_1, x_1) \in \Gamma(f_\lambda)$ and $(y_2, x_2) \in \Gamma(f_\mu)$. Then in particular, $y_1, y_2 \in h^{-1}(z)$, so $d_Y(y_1, y_2) < \delta$. Thus, if $\lambda = \mu$, then by the choice of δ, $d_{Y\times X}[(y_1, x_1), (y_2, x_2)] < \epsilon$.

If λ is not equal to μ, then consider the point $(y_1, f_\mu(y_1))$. This point is an element of $\Gamma(f_\mu)$ as is (y_2, x_2), so $d_{Y\times X}[(y_1, f_\mu(y_1)), (y_2, x_2)] < \epsilon$. Also, from the construction of the set Ω, we have that $d_{Y\times X}[(y_1, f_\mu(y_1)), (y_1, x_1)] < \epsilon$. Hence, by the triangle inequality, $d_{Y\times X}[(y_1, x_1), (y_2, x_2)] < 2\epsilon$. □

Lemma 4.19. *Let (X, d) and (Z, d') be metric spaces, and let $A_1, \ldots, A_n \subseteq X$ such that for $i, j \in \{1, \ldots, n\}$, $A_i \cap A_j = \emptyset$ if $|i - j| > 1$. Suppose that $\epsilon > 0$, and for each $i \in \{1, \ldots, n\}$, $f_i : A_i \to Z$ is an ϵ-map such that the following hold.*

(1) For each $i, j \in \{1, \ldots, n\}$, $f_i(A_i) \cap f_j(A_j) = \emptyset$ if $|i - j| > 1$.

(2) For each $i \in \{1, \ldots, n-1\}$, $f_i|_{A_i \cap A_{i+1}} = f_{i+1}|_{A_i \cap A_{i+1}}$.

(3) For each $i \in \{1, \ldots, n-1\}$, $f_i(A_i \cap A_{i+1}) = f_i(A_i) \cap f_{i+1}(A_{i+1})$.

Then the function $h : X \to Z$ defined piecewise by $h(x) = f_i(x)$ if $x \in A_i$ is a 2ϵ-map.

Proof. First, from Condition 2, we may apply an extension of the Pasting Lemma, [43, p. 108], to get that h is continuous.

Now to check that h is a 2ϵ-map, let $z \in Z$. If there exists $j \in \{1, \ldots, n\}$ such that $z \in f_j(A_j)$ and $z \notin f_i(A_i)$ for $i \neq j$, then $h^{-1}(z) = f_j^{-1}(z)$, so since f_j is an ϵ-map, the diameter of $h^{-1}(z)$ is less than ϵ which is less than 2ϵ.

Now, suppose that $z \in f_j(A_j) \cap f_{j+1}(A_{j+1})$ for some $j \in \{1, \dots, n-1\}$, and let $x, y \in h^{-1}(z)$. If x and y are both in A_j, then $x, y \in f_j^{-1}(z)$, so $d(x, y) < \epsilon$. Likewise, if x and y are both in A_{j+1}, then $x, y \in f_{j+1}^{-1}(z)$, and $d(x, y) < \epsilon$. If however, $x \in A_j$ and $y \in A_{j+1}$, then we use the assumption that $f_j(A_j) \cap f_{j+1}(A_{j+1}) = f_j(A_j \cap A_{j+1})$, and we choose $a \in f_j^{-1}(z) \cap A_j \cap A_{j+1}$. Then $a, x \in f_j^{-1}(z)$, so $d(a, x) < \epsilon$, and $a, y \in f_{j+1}^{-1}(z)$, so $d(a, y) < \epsilon$. By the triangle inequality, it follows that $d(x, y) < 2\epsilon$. $\square$

We are now ready to state conditions under which the inverse limit of an irreducible function on $[0, 1]$ will be chainable.

Theorem 4.20. *Let $F : [0, 1] \to 2^{[0,1]}$ be an irreducible function. If $\Gamma(F)$ does not contain a simple closed curve and $F(0), F(1) \in \{\{0\}, \{1\}, [0, 1]\}$, then $\varprojlim \mathbf{F}$ is chainable.*

Proof. First, let $\{f_\lambda\}_{\lambda \in \Lambda}$ be the irreducible collection of maps corresponding to F. Since $\Gamma(F)$ does not contain any simple closed curves, for $\lambda, \mu \in \Lambda$ with λ and μ adjacent in Λ, $\Gamma(f_\lambda) \cap \Gamma(f_\mu)$ is degenerate. In other words, $f_\lambda(0) = f_\mu(0)$ or $f_\lambda(1) = f_\mu(1)$ but not both.

For each $n \in \mathbb{N}$, let ρ_n be the metric on Γ_n given by $\rho_n(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : 1 \le i \le n\}$.

By Theorem 4.1, we must only show that Γ_n is chainable for all $n \in \mathbb{N}$. We have that $\Gamma_1 = [0, 1]$ is chainable, so proceeding by induction, suppose that for some $n \in \mathbb{N}$, Γ_n is chainable, and let $\epsilon > 0$.

Define a function $H : \Lambda \times \Gamma_n \to \Gamma_{n+1}$ by

$$H(\lambda, x_1, \dots, x_n) = (x_1, \dots, x_n, f_\lambda(x_n)).$$

H is continuous by Definition 3.5, Property (5), and since $\Lambda \times \Gamma_n$ is compact, H is uniformly continuous. Hence there exists $\delta > 0$ such that for $\lambda, \mu \in \Lambda$ and $\mathbf{x}, \mathbf{y} \in \Gamma_n$, whenever $|\lambda - \mu| < \delta$ and $\rho_n(\mathbf{x}, \mathbf{y}) < \delta$, it follows that $\rho_{n+1}(H(\lambda, \mathbf{x}), H(\mu, \mathbf{y})) < \epsilon$.

Then, let $\{[\lambda_1, \mu_1], \ldots [\lambda_m, \mu_m]\}$ be the covering of Λ given by Lemma 4.17 (with respect to δ). Let $\alpha = (\alpha_i)_{i=1}^{m-1}$ be a sequence of zeros and ones such that $f_{\mu_i}(\alpha_i) = f_{\lambda_{i+1}}(\alpha_i)$. From Lemma 4.17, we have that for each $i = 1, \ldots, m-1$, $\alpha_i = 1 - \alpha_{i+1}$. For each $i = 1, \ldots, m$, let

$$W_i = H\left[([\lambda_i, \mu_i] \cap \Lambda) \times \Gamma_n\right].$$

Note that the collection $\{W_i\}_{i=1}^m$ covers Γ_{n+1}.

Case 1: Suppose that $\mathcal{A}_n$ and $\mathcal{B}_n$ are degenerate. Let $\mathcal{A}_n = \{\mathbf{a}\}$ and $\mathcal{B}_n = \{\mathbf{b}\}$. From Corollary 4.14, there exists a δ-map $h : \Gamma_n \to [0,1]$ such that $h(\mathbf{a}) = 0$ and $h(\mathbf{b}) = 1$. Define a function $\widehat{h} : \Gamma_{n+1} \to [0,1]$ by $\widehat{h} = h \circ \pi_{[1,n]}$. By Lemma 4.18, for each $i = 1, \ldots, m$, $\widehat{h}|_{W_i}$ is a 2ϵ-map. We will use this function $\widehat{h}$ to define a function on all of $\Gamma_{n+1} = \bigcup_{i=1}^m W_i$ which will be a 4ϵ-map onto an arc.

Sub-case 1(a): Suppose that $\alpha_1 = 1$ (so $f_{\mu_1}(1) = f_{\lambda_2}(1)$). From Lemma 4.17, we have that the α_i alternate between 0 and 1, so it must be the case that $\alpha_i = 1$ for all odd i, and $\alpha_i = 0$ for all even i.

Define for each $i = 1, \ldots, m$ a function $\Phi_i : W_i \to [i-1, i]$ as follows:

(1) If i is odd, $\Phi_i = i - 1 + \widehat{h}|_{W_i}$.

(2) If i is even, $\Phi_i = i - \widehat{h}|_{W_i}$.

Claim: If $\mathbf{x} \in W_i \cap W_{i+1}$, then $\Phi_i(\mathbf{x}) = \Phi_{i+1}(\mathbf{x})$.

First, note that if $\mathbf{x} \in W_i \cap W_{i+1}$, then $f_{\mu_i}(x_n) = x_{n+1} = f_{\lambda_{i+1}}(x_n)$, so $x_n = \alpha_i$. Now if i is even, then we have that $x_n = \alpha_i = 0$, so $\pi_{[1,n]}(\mathbf{x}) = \mathbf{a}$, and thus $\widehat{h}(\mathbf{x}) = 0$. Then, $\Phi_i(\mathbf{x}) = i - \widehat{h}(\mathbf{x}) = i$, and $\Phi_{i+1}(\mathbf{x}) = i + \widehat{h}(\mathbf{x}) = i$. Hence $\Phi_i(\mathbf{x}) = \Phi_{i+1}(\mathbf{x})$.

Similarly, if i is odd, then $x_n = \alpha_i = 1$, so $\pi_{[1,n]}(\mathbf{x}) = \mathbf{b}$, and hence $\widehat{h}(\mathbf{x}) = 1$. Then $\Phi_i(\mathbf{x}) = i - 1 + \widehat{h}(\mathbf{x}) = i - 1 + 1 = i$, and $\Phi_{i+1}(\mathbf{x}) = i + 1 - \widehat{h}(\mathbf{x}) = i + 1 - 1 = i$, so $\Phi_i(\mathbf{x}) = \Phi_{i+1}(\mathbf{x})$. This proves the claim.

Additionally, these sets W_i and the 2ϵ-maps Φ_i meet the criteria of Lemma 4.19, so the function $\Phi : \Gamma_{n+1} \to [0, m]$, defined by $\Phi(\mathbf{x}) = \Phi_i(\mathbf{x})$ if $\mathbf{x} \in W_i$, is a 4ϵ-map.

Sub-case 1(b): Suppose that $\alpha_1 = 0$ (so $f_{\mu_1}(0) = f_{\lambda_2}(0)$). Then $\alpha_i = 0$ for all odd i, and $\alpha_i = 1$ for all even i. We proceed almost identically as in Sub-case 1(a), but instead, we define $\Phi_i : W_i \to [i-1, i]$ as follows:

(1) If i is odd, $\Phi_i(\mathbf{x}) = i - \widehat{h}(\mathbf{x})$.

(2) If i is even, $\Phi_i(\mathbf{x}) = i - 1 + \widehat{h}(\mathbf{x})$.

Just as in Sub-case 1(a), these functions agree where their domains intersect, so we can define $\Phi : \Gamma_{n+1} \to [0, m]$ by $\Phi(\mathbf{x}) = \Phi_i(\mathbf{x})$ if $x \in W_i$, and Φ will be a 4ϵ-map. This concludes Case 1.

Case 2: Suppose that $\mathcal{A}_n$ and $\mathcal{B}_n$ are both non-degenerate. Let $\varphi_1 : \mathcal{A}_n \cup \mathcal{B}_n \to [0, 3]$ be a δ-map such that $\varphi_1(\mathcal{A}_n) = [0, 1]$ and $\varphi_1(\mathcal{B}_n) = [2, 3]$. Define another δ-map $\varphi_2 : \mathcal{A}_n \cup \mathcal{B}_n \to [0, 3]$ so that $\varphi_2|_{\mathcal{A}_n} = 1 - \varphi_1|_{\mathcal{A}_n}$ and $\varphi_2|_{\mathcal{B}_n} = 5 - \varphi_1|_{\mathcal{B}_n}$. By Corollary 4.14, there exist δ-maps $h_1, h_2 : \Gamma_n \to [0, 3]$ such that $h_1|_{\mathcal{A}_n \cup \mathcal{B}_n} = \varphi_1$ and $h_2|_{\mathcal{A}_n \cup \mathcal{B}_n} = \varphi_2$.

Define two functions $\widehat{h}_1, \widehat{h}_2 : \Gamma_{n+1} \to [0, 3]$ by $\widehat{h}_1 = h_1 \circ \pi_{[1,n]}$ and $\widehat{h}_2 = h_2 \circ \pi_{[1,n]}$. Just as in Case 1, from Lemma 4.18, we have that for each $i = 1, \ldots, m$, $\widehat{h}_1|_{W_i}$ and $\widehat{h}_2|_{W_i}$ are each 2ϵ-maps. Also, as was done in Case 1, for each $i = 1, \ldots, m$, we will define a 2ϵ-map $\Phi_i : W_i \to [2(i-1), 2i+1]$.

Sub-case 2(a): Suppose that $\alpha_i = 1$ for odd i, and $\alpha_i = 0$ for even i. Then define Φ_i as follows:

(1) If i is odd, $\Phi_i = \widehat{h}_1|_{W_i} + 2(i-1)$.

(2) If i is even, $\Phi_i = 5 - \widehat{h}_2|_{W_i} + 2(i-2)$.

Claim: If $\mathbf{x} \in W_i \cap W_{i+1}$, then $\Phi_i(\mathbf{x}) = \Phi_{i+1}(\mathbf{x})$.

First, note that if $\mathbf{x} \in W_i \cap W_{i+1}$, then $f_{\mu_i}(x_n) = x_{n+1} = f_{\lambda_{i+1}}(x_n)$, so $x_n = \alpha_i$. In particular, either $\pi_{[1,n]}(\mathbf{x}) \in \mathcal{A}_n$ or $\pi_{[1,n]}(\mathbf{x}) \in \mathcal{B}_n$.

Now, if i is even, then we have that $x_n = \alpha_i = 0$, so $\pi_{[1,n]}(\mathbf{x}) \in \mathcal{A}_n$. Recalling that $\varphi_2|_{\mathcal{A}_n} = 1 - \varphi_1|_{\mathcal{A}_n}$, we have that

$$\begin{aligned}
\Phi_i(\mathbf{x}) &= 5 - \widehat{h}_2(\mathbf{x}) + 2(i-2) \\
&= 5 - \varphi_2 \circ \pi_{[1,n]}(\mathbf{x}) + 2(i-2) \\
&= 5 - (1 - \varphi_1 \circ \pi_{[1,n]}(\mathbf{x})) + 2(i-2) \\
&= \varphi_1 \circ \pi_{[1,n]}(\mathbf{x}) + 2i.
\end{aligned}$$

Then $i+1$ is odd, so

$$\begin{aligned}
\Phi_{i+1}(\mathbf{x}) &= \widehat{h}_1(\mathbf{x}) + 2((i+1)-1) \\
&= \varphi_1 \circ \pi_{[1,n]}(\mathbf{x}) + 2i.
\end{aligned}$$

Thus $\Phi_i(\mathbf{x}) = \Phi_{i+1}(\mathbf{x})$.

If i is odd, then we have that $x_n = \alpha_i = 1$, so $\pi_{[1,n]}(\mathbf{x}) \in \mathcal{B}_n$, and

$$\begin{aligned}
\Phi_i(\mathbf{x}) &= \widehat{h}_1(\mathbf{x}) + 2(i-1) \\
&= \varphi_1 \circ \pi_{[1,n]}(\mathbf{x}) + 2(i-1).
\end{aligned}$$

Then $i+1$ is even, so, recalling that $\varphi_2|_{\mathcal{B}_n} = 5 - \varphi_1|_{\mathcal{B}_n}$,

$$\begin{aligned}
\Phi_{i+1}(\mathbf{x}) &= 5 - \widehat{h}_2 + 2((i+1)-2) \\
&= 5 - \varphi_2 \circ \pi_{[1,n]}(\mathbf{x}) + 2(i-1) \\
&= 5 - (5 - \varphi_1 \circ \pi_{[1,n]}(\mathbf{x})) + 2(i-1) \\
&= \varphi_1 \circ \pi_{[1,n]}(\mathbf{x}) + 2(i-1),
\end{aligned}$$

so again we get that $\Phi_i(\mathbf{x}) = \Phi_{i+1}(\mathbf{x})$. This proves the claim.

Now, observe that $\Phi_i(W_i) = [2(i-1), 2i+1]$, so for any $i = 1, \ldots, m-1$,

$$\Phi_i(W_i) \cap \Phi_{i+1}(W_{i+1}) = [2(i-1), 2i+1] \cap [2i, 2(i+1)+1] = [2i, 2i+1],$$

and this is also equal to $\Phi_i(W_i \cap W_{i+1})$ and $\Phi_{i+1}(W_i \cap W_{i+1})$. Thus, since each Φ_i is a 2ϵ-map, by Lemma 4.19, we have that the function $\Phi : \Gamma_{n+1} \to [0, 2m+1]$, defined by $\Phi(\mathbf{x}) = \Phi_i(\mathbf{x})$ if $\mathbf{x} \in W_i$, is a 4ϵ-map.

Sub-case 2(b): If $\alpha_i = 0$ for odd i, and $\alpha_i = 1$ for even i, then we will define $\Phi_i : W_i \to [2(i-1), 2i+1]$ as follows:

(1) If i is odd, $\Phi_i = 5 - \widehat{h}_1 + (i-1)$.

(2) If i is even, $\Phi_i = \widehat{h}_2 + (i-2)$.

Just as in Sub-case 2(a), it works out that for $i = 1, \ldots, m-1$, Φ_i and Φ_{i+1} agree on $W_i \cap W_{i+1}$. Then since each of these is a 2ϵ-map, by Lemma 4.19, the function $\Phi : \Gamma_{n+1} \to [0, 2m+1]$ defined by $\Phi(\mathbf{x}) = \Phi_i(\mathbf{x})$ when $\mathbf{x} \in W_i$ is a 4ϵ-map. This concludes Case 2.

Case 3: For our final case, suppose that one of $\mathcal{A}_n$ and $\mathcal{B}_n$ is degenerate and the other not. For simplicity, we suppose that $\mathcal{A}_n = \{\mathbf{a}\}$ is degenerate, and $\mathcal{B}_n$ is non-degenerate. The case where these roles are reversed is not meaningfully different.

Let $\varphi_1, \varphi_2 : \{\mathbf{a}\} \cup \mathcal{B}_n \to [0, 2]$ be δ-maps such $\varphi_1(\mathbf{a}) = \varphi_2(\mathbf{a}) = 0$, $\varphi_1(\mathcal{B}_n) = \varphi_2(\mathcal{B}_n) = [1, 2]$, and $\varphi_2|_{\mathcal{B}_n} = 3 - \varphi_1|_{\mathcal{B}_n}$. Then by Corollary 4.14, there exist δ-maps $h_1, h_2 : \Gamma_n \to [0, 2]$ such that $h_1|_{\mathcal{B}_n} = \varphi_1$, $h_2|_{\mathcal{B}_n} = \varphi_2$, and $h_1(\mathbf{a}) = h_2(\mathbf{a}) = 0$. Then define $\widehat{h}_1, \widehat{h}_2 : \Gamma_{n+1} \to [0, 2]$ by $\widehat{h}_1 = h_1 \circ \pi_{[1,n]}$ and $\widehat{h}_2 = h_2 \circ \pi_{[1,n]}$. Again by Lemma 4.18, $\widehat{h}_1|_{W_i}$ and $\widehat{h}_2|_{W_i}$ are 2ϵ-maps for each $i = 1, \ldots, m$.

Sub-case 3(a): Suppose that $\alpha_i = 1$ for odd i, and $\alpha_i = 0$ for even i. Then we will define for each $i = 1, \ldots, m$ a 2ϵ-map $\Phi_i : W_i \to \mathbb{R}$ as follows:

(1) If i is odd, $\Phi_i = \widehat{h}_1|_{W_i} + \frac{3}{2}(i-1)$.

(2) If i is even, $\Phi_i = 3 - \widehat{h}_2|_{W_i} + \frac{3}{2}(i-2)$.

Just as before, we claim that with this definition for each $i = 1, \ldots, m-1$, $\Phi_i|_{W_i \cap W_{i+1}} = \Phi_{i+1}|_{W_i \cap W_{i+1}}$. To prove this, suppose that $\mathbf{x} \in W_i \cap W_{i+1}$. Then $f_{\mu_i}(x_n) = x_{n+1} = f_{\lambda_{i+1}}(x_n)$, so either $\pi_{[1,n]}(\mathbf{x}) = \mathbf{a}$ or $\mathbf{x} \in \mathcal{B}_n$.

If i is even, then $x_n = \alpha_i = 0$, so $\pi_{[1,n]}(\mathbf{x}) = \{\mathbf{a}\}$. Hence $\widehat{h}_1(\mathbf{x}) = \widehat{h}_2(\mathbf{x}) = 0$. Then since i is even,

$$\Phi_i(\mathbf{x}) = 3 - \widehat{h}_2(\mathbf{x}) + \frac{3}{2}(i-2) = \frac{3}{2}i,$$

and since $i+1$ is odd,

$$\Phi_{i+1}(\mathbf{x}) = \widehat{h}_1(\mathbf{x}) + \frac{3}{2}((i+1)-1) = \frac{3}{2}i.$$

Then if i is odd, then $x_n = \alpha_i = 1$, so $\pi_{[1,n]}(\mathbf{x}) \in \mathcal{B}_n$, so $\widehat{h}_1(\mathbf{x}) = \varphi_1 \circ \pi_{[1,n]}(\mathbf{x})$, and $\widehat{h}_2(\mathbf{x}) = \varphi_2 \circ \pi_{[1,n]}(\mathbf{x})$. Then

$$\begin{aligned} \Phi_i(\mathbf{x}) &= \widehat{h}_1(\mathbf{x}) + \frac{3}{2}(i-1) \\ &= \varphi_1 \circ \pi_{[1,n]}(\mathbf{x}) + \frac{3}{2}(i-1), \end{aligned}$$

and $i+1$ will be odd, so (recalling that $\varphi_2|_{\mathcal{B}_n} = 3 - \varphi_1|_{\mathcal{B}_n}$)

$$\begin{aligned} \Phi_{i+1}(\mathbf{x}) &= 3 - \widehat{h}_2(\mathbf{x}) + \frac{3}{2}((i+1)-2) \\ &= 3 - \varphi_2 \circ \pi_{[1,n]}(\mathbf{x}) + \frac{3}{2}(i-1) \\ &= 3 - [3 - \varphi_1 \circ \pi_{[1,n]}(\mathbf{x})] + \frac{3}{2}(i-1) \\ &= \varphi_1 \circ \pi_{[1,n]}(\mathbf{x}) + \frac{3}{2}(i-1). \end{aligned}$$

This proves the claim. Then by Lemma 4.19, the function $\Phi : \Gamma_{n+1} \to \mathbb{R}$ defined by $\Phi(\mathbf{x}) = \Phi_i(\mathbf{x})$ when $x \in W_i$ is a 4ϵ-map whose image is an arc.

Sub-case 3(b): Suppose that $\alpha_i = 0$ for odd i, and $\alpha_i = 1$ for even i. Then we will define for each $i = 1, \ldots, m$ a function $\Phi_i : W_i \to \mathbb{R}$ as follows:

(1) If i is odd, $\Phi_i = 2 - \widehat{h}_1|_{W_i} + \frac{3}{2}(i-1)$;

(2) If i is even, $\Phi_i = \widehat{h}_2|_{W_i} + \frac{3}{2}i - 1$.

Similarly, these definitions work out so that the function $\Phi : \Gamma_{n+1} \to \mathbb{R}$ defined by $\Phi(\mathbf{x}) = \Phi_i(\mathbf{x})$ if $\mathbf{x} \in W_i$ is well-defined and a 4ϵ-map whose image is an arc.

Thus, in every case, there exists a 4ϵ-map from Γ_{n+1} to an arc. Therefore Γ_{n+1} is chainable, and by induction Γ_j is chainable for all $j \in \mathbb{N}$. Therefore, by Theorem 4.1, $\varprojlim \mathbf{F}$ is chainable. $\square$

Notice that if $F : [0,1] \to 2^{[0,1]}$ is an irreducible function, then if $F(0)$ (or $F(1)$) is not an element of $\{\{0\},\{1\},[0,1]\}$, then one of the three conditions of Theorem 4.16 applies. In addition, for each condition in Theorem 4.16, we have not only that $\varprojlim \mathbf{F}$ is not chainable, but that Γ_3 in particular is not chainable. This is also the case if the graph of F contains a simple closed curve. Thus, we have the following characterization of chainability for inverse limits of irreducible functions on $[0,1]$.

Theorem 4.21. *Let $F : [0,1] \to 2^{[0,1]}$ be an irreducible function. Then the following are equivalent.*

(1) $\varprojlim \mathbf{F}$ is chainable.

(2) Γ_3 is chainable.

(3) The graph of F does not contain a simple closed curve, and

$$F(0), F(1) \in \{\{0\},\{1\},[0,1]\}.$$

Finally, if $F : [0,1] \to 2^{[0,1]}$ is an irreducible function, while we have not discussed anything about $\varprojlim \mathbf{F^{-1}}$ (where $\mathbf{F^{-1}}$ is the constant sequence $(F^{-1})_{n\in\mathbb{N}}$), we can state the following corollary.

Corollary 4.22. *Let $F : [0,1] \to 2^{[0,1]}$ be an irreducible function. If $\Gamma(F)$ does not contain a simple closed curve and $F(0), F(1) \in \{\{0\},\{1\},[0,1]\}$, then $\varprojlim \mathbf{F^{-1}}$ is chainable.*

Proof. This follows from the fact that for each $n \in \mathbb{N}$, $\Gamma_n(F)$ and $\Gamma_n(F^{-1})$ are homeomorphic under the homeomorphism $h : \Gamma_n(F) \to \Gamma_n(F^{-1})$ where

$$h(x_1, \ldots, x_n) = (x_n, \ldots, x_1).$$

Then if $\Gamma(F)$ does not contain any simple closed curves and $F(0), F(1) \in \{\{0\}, \{1\}, [0,1]\}$, we have from Theorem 4.20 that $\varprojlim \mathbf{F}$ is chainable and hence, by Corollary 4.4, so is $\Gamma_n(F)$ for each $n \in \mathbb{N}$. Therefore it follows that $\Gamma_n(F^{-1})$ is a chainable continuum for each $n \in \mathbb{N}$, so $\varprojlim \mathbf{F^{-1}}$ is chainable. $\square$

4.3 Examples

Example 4.23. Let $F : [0,1] \to 2^{[0,1]}$ be the irreducible function pictured on the left in Figure 4.1. Then $\varprojlim \mathbf{F}$ contains a simple closed curve.

Proof. Since $F(0) = \{0,1\}$ is not connected, F meets Condition 1 of Theorem 4.16.

To make this more explicit, let $\{f_\lambda\}_{\lambda\in\Lambda}$ be the irreducible collection of maps corresponding to F (pictured on the right of Figure 4.1) where

$$\Lambda = \left\{ \ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots \right\} \cup \{0,1\}.$$

Let $f_{3/4}$ be the function whose graph is the line segment from $(1, 1/2)$ to $(0, 3/4)$, and let $f_{7/8}$ be the function whose graph is the line segment from $(0, 3/4)$ to $(1, 7/8)$. Then $f_{3/4}(0) = f_{7/8}(0) = 3/4$. Thus, we may use $\lambda = 3/4$ and $\mu = 7/8$ for the construction of the arcs M_1 and M_2 from Case 1 of the proof of Theorem 4.16. $\square$

Example 4.24. Let $F : [0,1] \to 2^{[0,1]}$ and $G : [0,1] \to 2^{[0,1]}$ be the irreducible functions pictured in Figure 4.2 on the left and right respectively. Then $\varprojlim \mathbf{F}$ contains a simple triod, and $\varprojlim \mathbf{G}$ contains a simple closed curve.

Proof. $F(1) = [0, 1/2]$ which is a proper sub-interval of $[0,1]$. Therefore, by Theorem 4.16 Condition 2, $\varprojlim \mathbf{F}$ contains a simple triod.

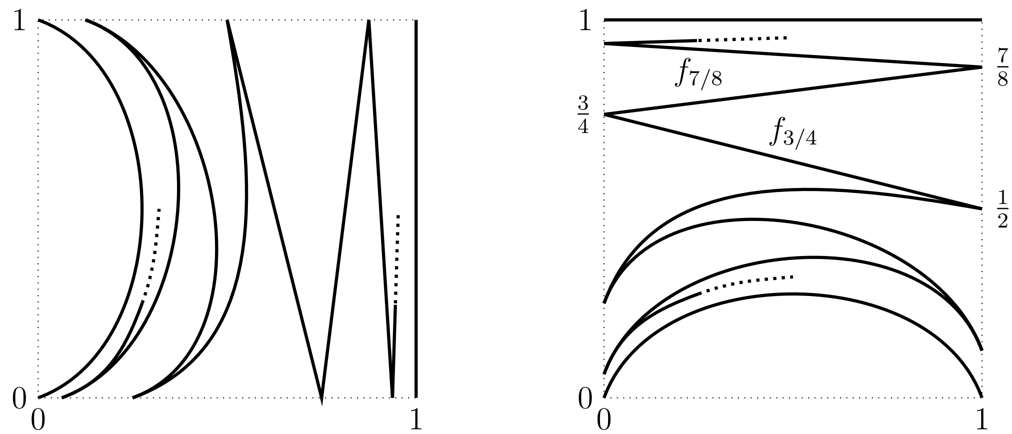

Figure 4.1. Irreducible function whose inverse limit contains a simple closed curve

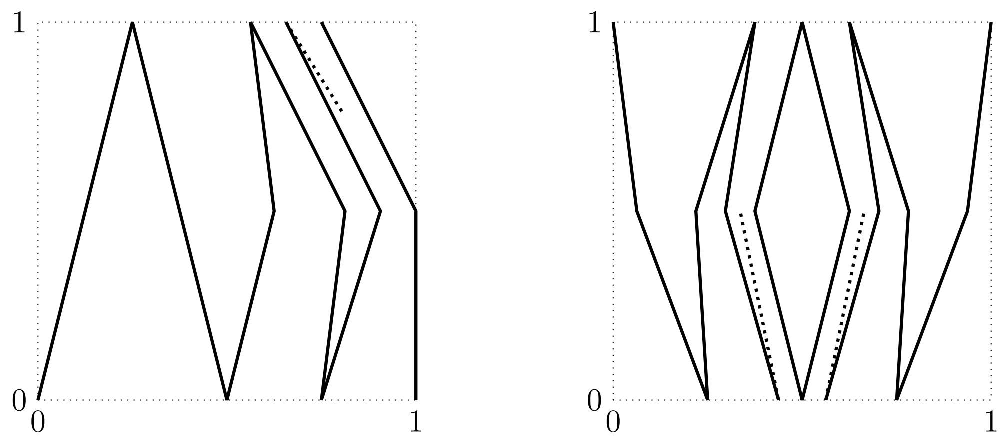

Figure 4.2. Irreducible functions with non-chainable inverse limits

The graph of G contains a simple closed curve, and $\Gamma(G)$ is homeomorphic to $\Gamma_2(G)$. Thus, $\varprojlim \mathbf{G}$ contains a simple closed curve. $\square$

Example 4.25. Let C be the Cantor middle thirds set, and let A be the set consisting of the midpoint of each removed interval. Then let $\Omega = C \cup A$. For each $\omega \in \Omega$ we define a continuous function $h_\omega : [0,1] \to [0,1]$. If $\omega \in C$, let $h_\omega(x) = \omega$ for all $x \in [0,1]$. If $\omega \in A$, then let $\underline{\omega}$ and $\overline{\omega}$ be the endpoints of the deleted interval of which ω is the midpoint. Then let h_ω be the function whose graph is the straight line from $(1, \overline{\omega})$ to $(0, \underline{\omega})$.

The resulting collection, $\{h_\omega\}_{\omega \in \Omega}$, is irreducible, and if $H : [0,1] \to 2^{[0,1]}$ is the corresponding irreducible function (pictured in Figure 4.3), then $\varprojlim \mathbf{H}$ is chainable.

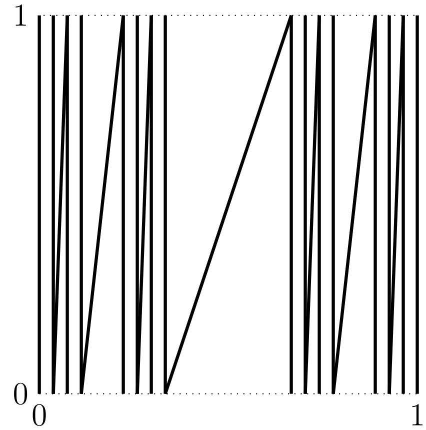

Figure 4.3. Irreducible function with chainable inverse limit

Proof. The graph of H does not contain a simple closed curve. Also, since $0, 1 \in C$, $f_0([0,1]) = \{0\}$, and $f_1([0,1]) = \{1\}$, so $H(0) = H(1) = [0,1]$. Therefore, by Theorem 4.20, $\varprojlim \mathbf{H}$ is chainable. $\square$

One last thing worth noting is that if $F : [0,1] \to 2^{[0,1]}$ is an irreducible function whose corresponding irreducible collection of maps $\{f_\lambda\}_{\lambda \in \Lambda}$ is finite, then $\varprojlim \mathbf{F}$ is chainable. To see this, note that since Λ is finite, it has no limit points. In particular, 0 and 1 are not limit points, so from Definitions 3.5, $F(0) = f_0^{-1}(0)$ and $F(1) = f_1^{-1}(1)$ are each either $\{0\}$ or $\{1\}$. Also, because there are no limit points of Λ, there can be no simple closed curves in the graph of F. Thus, by Theorem 4.20, $\varprojlim \mathbf{F}$ is chainable.

We will show in Chapter Six that, more specifically, if $F : [0,1] \to 2^{[0,1]}$ is an irreducible function whose inverse is the union of finitely many single-valued maps, then $\varprojlim \mathbf{F}$ is a particular type of chainable continuum known as a Knaster continuum.

CHAPTER FIVE

Endpoints of Inverse Limits with Set-valued Functions

We now investigate the topic of endpoints of inverse limits of set-valued functions. Given a compact metric space X, a point $p \in X$ is called an *endpoint* of X if given any two continua, $H, K \subseteq X$, each containing p, either $H \subseteq K$ or $K \subseteq H$.

Much has been written concerning endpoints of traditional inverse limits. In [6], Barge and Martin give a characterization of endpoints of inverse limits with a single continuous bonding function on $[0, 1]$. They also show that the study of endpoints of the inverse limit can be related to the study of the dynamics of the function. Since then, there have been many more results concerning endpoints and other characterizations (see [1–3, 8, 9, 20, 33]). All of these results have been in the case of a single bonding function on $[0, 1]$, and most of them focus on unimodal functions.

One of the main reasons endpoints of inverse limit spaces are studied is that endpoints are a topological invariant, so they can be used to show that two inverse limit spaces are not homeomorphic. Watkins uses this in his classification of the inverse limits of certain piecewise linear open functions in [54], and the study of endpoints played a large role in the work leading to the proof of the Ingram Conjecture which ultimately proven in [5].

5.1 A Characterization of Endpoints of Certain Inverse Limits

In this section, we give a characterization of endpoints of inverse limits of set-valued functions whose inverse is the union of mappings. We begin with the following lemma.

Lemma 5.1. *Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence. Let H and K be closed sets in $\varprojlim \mathbf{F}$. If for all $n \in \mathbb{N}$, $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$, then $H \subseteq K$.*

Proof. Let $\mathbf{x} \in H$. Then, for each $n \in \mathbb{N}$, $\pi_{[1,n]}(\mathbf{x}) \in \pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$. Therefore, for each $n \in \mathbb{N}$, there exists a point $\mathbf{y}(n) \in K$ such that $\pi_{[1,n]}(\mathbf{y}(n)) = \pi_{[1,n]}(\mathbf{x})$. It follows that $\mathbf{y}(n) \to \mathbf{x}$ as $n \to \infty$, so since K is closed, and each $\mathbf{y}(n) \in K$, we have that $\mathbf{x} \in K$. $\square$

This brings us to the following result which gives a sufficient condition for a point of the inverse limit space to be an endpoint.

Theorem 5.2. *Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence. For any point $\mathbf{p} \in \varprojlim \mathbf{F}$, if $\pi_{[1,n]}(\mathbf{p})$ is an endpoint of Γ_n for infinitely many $n \in \mathbb{N}$, then $\mathbf{p}$ is an endpoint of $\varprojlim \mathbf{F}$.*

Proof. Let $H, K \subseteq \varprojlim \mathbf{F}$ be two continua, each containing $\mathbf{p}$. We will show that either $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$ or $\pi_{[1,n]}(K) \subseteq \pi_{[1,n]}(H)$ will hold for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ such that $\pi_{[1,n]}(\mathbf{p})$ is an endpoint of Γ_n. Note that each of $\pi_{[1,n]}(H)$ and $\pi_{[1,n]}(K)$ is a subcontinuum of Γ_n containing $\pi_{[1,n]}(\mathbf{p})$, so either $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$ or $\pi_{[1,n]}(K) \subseteq \pi_{[1,n]}(H)$.

Hence, for all $n \in \mathbb{N}$ for which $\pi_{[1,n]}(\mathbf{p})$ is an endpoint of Γ_n, we have that the continua $\pi_{[1,n]}(H)$ and $\pi_{[1,n]}(K)$ are nested. Since there are infinitely many such $n \in \mathbb{N}$, it follows that either $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$ for infinitely many $n \in \mathbb{N}$, or $\pi_{[1,n]}(K) \subseteq \pi_{[1,n]}(H)$ for infinitely many $n \in \mathbb{N}$.

Now, note that, if for some $N \in \mathbb{N}$, $\pi_{[1,N]}(H) \subseteq \pi_{[1,N]}(K)$, then $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$ for all $n \leq N$. Therefore, if $\pi_{[1,n]}(H) \subseteq \pi_{[1,n]}(K)$ holds for infinitely many $n \in \mathbb{N}$, then it holds for all $n \in \mathbb{N}$. Likewise, if $\pi_{[1,n]}(K) \subseteq \pi_{[1,n]}(H)$ holds for infinitely many $n \in \mathbb{N}$, then it holds for all $n \in \mathbb{N}$.

It follows then from Lemma 5.1 that either $H \subseteq K$ or $K \subseteq H$. Therefore, $\mathbf{p}$ is an endpoint of $\varprojlim \mathbf{F}$. $\square$

The main result of this section deals with the special case where each bonding function is the inverse of a union of maps. In this case, we have a characterization of the endpoints of the inverse limit.

Theorem 5.3. *Let $\{\mathbf{X}, \mathbf{F}\}$ be an inverse sequence. Suppose that for each $i \in \mathbb{N}$ there exists a collection $\{f_\alpha^{(i)} : X_i \to X_{i+1}\}_{\alpha \in A_i}$ of continuous functions such that*

$$\Gamma(F_i^{-1}) = \bigcup_{\alpha \in A_i} \Gamma(f_\alpha^{(i)}).$$

Then for every $\mathbf{p} \in \varprojlim \mathbf{F}$, the following are equivalent.

(1) $\mathbf{p}$ is an endpoint of $\varprojlim \mathbf{F}$.

(2) $\pi_{[1,n]}(\mathbf{p})$ is an endpoint of Γ_n for infinitely many $n \in \mathbb{N}$.

(3) $\pi_{[1,n]}(\mathbf{p})$ is an endpoint of Γ_n for all $n \in \mathbb{N}$.

Proof. Clearly (3) implies (2), and by Theorem 5.2, (2) implies (1). Thus, we must only show that (1) implies (3).

We will show that the negation of (3) implies the negation of (1). Suppose that $\mathbf{p} \in \varprojlim \mathbf{F}$, and there exists an $n \in \mathbb{N}$ such that $\pi_{[1,n]}(\mathbf{p})$ is not an endpoint of Γ_n. Then there exist two continua, $H, K \subseteq \Gamma_n$ such that $\pi_{[1,n]}(\mathbf{p}) \in H \cap K$, and neither H nor K is contained in the other.

By assumption, for each $i \in \mathbb{N}$, and each $x \in X_i$,

$$F_i^{-1}(x) = \bigcup_{\alpha \in A_i} f_\alpha^{(i)}(x).$$

Thus, since for each $i \in \mathbb{N}$, $p_{i+1} \in F_i^{-1}(p_i)$, there exists a sequence $(\alpha_i)_{i=1}^{\infty}$ with $\alpha_i \in A_i$ such that $p_{i+1} = f_{\alpha_i}^{(i)}(p_i)$ for all $i \in \mathbb{N}$. Define two sets, $\widetilde{H}$ and $\widetilde{K}$, by

$$\begin{aligned}
\widetilde{H} &= \left\{\mathbf{x} : (x_i)_{i=1}^n \in H, \text{ and } x_{i+1} = f_{\alpha_i}^{(i)}(x_i) \text{ for } i \geq n\right\}. \\
\widetilde{K} &= \left\{\mathbf{x} : (x_i)_{i=1}^n \in K, \text{ and } x_{i+1} = f_{\alpha_i}^{(i)}(x_i) \text{ for } i \geq n\right\}.
\end{aligned}$$

Then each of $\widetilde{H}$ and $\widetilde{K}$ is a subcontinuum of $\varprojlim \mathbf{F}$, each contains $\mathbf{p}$, and neither is contained in the other. Therefore $\mathbf{p}$ is not an endpoint of $\varprojlim \mathbf{F}$. $\square$

5.2 *Endpoints of Inverse Limits with Irreducible Functions*

Notice that, in particular, irreducible functions satisfy the hypotheses of Theorem 5.3. Not every inverse limit of an irreducible function has endpoints, and for those that do, the set of endpoints is not necessarily simple to determine. However, there are two restrictions we will place on the irreducible functions we consider which will ensure that the endpoints of the inverse limit are precisely the points of the inverse limit consisting only of zeros and ones.

Recall that if $F : [0,1] \to 2^{[0,1]}$ is an irreducible function with the corresponding irreducible collection $\{f_\lambda\}_{\lambda\in\Lambda}$, then for each $n \in \mathbb{N}$ and $\lambda \in \Lambda$, $\mathfrak{f}_\lambda^{(n)} : \Gamma_n \to [0,1]$ is defined by $\mathfrak{f}_\lambda^{(n)}(\mathbf{x}) = f_\lambda(x_n)$.

Lemma 5.4. *Let $F : [0,1] \to 2^{[0,1]}$ be an irreducible function with the associated irreducible collection $\{f_\lambda\}_{\lambda\in\Lambda}$. Suppose that*

(1) $\Gamma(F)$ does not contain any simple closed curves,

(2) if 0 is a limit point of Λ, then $F(0) = [0,1]$, and

(3) if 1 is a limit point of Λ, then $F(1) = [0,1]$.

For each $n \in \mathbb{N}$, if E_n is the set of endpoints of Γ_n, then $E_n \subseteq \Gamma_n \cap \{0,1\}^n$.

Proof. Clearly, this holds for $\Gamma_1 = [0,1]$, so, suppose that for some $n \in \mathbb{N}$, $E_n \subseteq \Gamma_n \cap \{0,1\}^n$.

We will show that $\Gamma_{n+1} \setminus \{0,1\}^{n+1} \subseteq \Gamma_{n+1} \setminus E_{n+1}$. Let $\mathbf{x} \in \Gamma_{n+1} \setminus \{0,1\}^{n+1}$.

Case 1: Suppose that $(x_1, \ldots, x_n) \in \{0,1\}^n$. Then $x_{n+1} \notin \{0,1\}$.

Sub-case (a): Suppose there are two elements $\lambda, \mu \in \Lambda$ such that $x_{n+1} = f_\mu(x_n) = f_\lambda(x_n)$. Then each of $\Gamma(\mathfrak{f}_\lambda^{(n)})$ and $\Gamma(\mathfrak{f}_\mu^{(n)})$ is a continuum containing $\mathbf{x}$, so $\mathbf{x}$ is not an endpoint of Γ_{n+1}.

Sub-case (b): Suppose there is a unique $\lambda_0 \in \Lambda$ such that $x_{n+1} = f_{\lambda_0}(x_n)$. We will show that $\lambda_0 \neq 0, 1$. Suppose that $\lambda_0 = 0$. Then, since $x_{n+1} \neq 0$, it follows from Property (2) that 0 is not a limit point of Λ. Hence, there is a unique smallest element, λ_1, of $\Lambda \setminus \{0\}$. Then either $f_0(0) = 0$ and $f_0(1) = f_{\lambda_1}(1)$ or $f_0(1) = 0$ and $f_0(0) = f_{\lambda_1}(0)$. However, since $x_n \in \{0, 1\}$, this would mean that either $x_{n+1} = 0$ or $x_{n+1} = f_0(x_n) = f_{\lambda_1}(x_n)$. The former contradicts the assumption of Case 1, and the latter contradicts the assumption of Sub-case (b), so λ_0 cannot equal 0. Similarly, λ_0 cannot equal 1. Thus, $\lambda_0 \in \Lambda \setminus \{0, 1\}$, and we may define two subcontinua of Γ_{n+1} as follows:

$$\begin{aligned} H &= \bigcup_{\lambda \in \Lambda \cap [0, \lambda_0]} \Gamma(\mathfrak{f}_\lambda^{(n)}), \text{ and} \\ K &= \bigcup_{\lambda \in \Lambda \cap [\lambda_0, 1]} \Gamma(\mathfrak{f}_\lambda^{(n)}). \end{aligned}$$

Each of H and K contains $\mathbf{x}$, and neither is contained in the other. Therefore, $\mathbf{x}$ is not an endpoint of Γ_{n+1}.

Case 2: Suppose that $(x_1, \ldots, x_n) \notin \{0, 1\}^n$. Then by the induction hypothesis, $(x_1, \ldots, x_n)$ is not an endpoint of Γ_n, so there are two continua L and M such that $(x_1, \ldots, x_n) \in L \cap M$, and neither L nor M is contained in the other. Choose $\lambda \in \Lambda$ such that $x_{n+1} = f_\lambda(x_n)$, and define two subcontinua of Γ_{n+1}:

$$\begin{aligned} \widetilde{L} &= \Gamma(\mathfrak{f}_\lambda^{(n)}|_L), \text{ and} \\ \widetilde{M} &= \Gamma(\mathfrak{f}_\lambda^{(n)}|_M). \end{aligned}$$

Each of $\widetilde{L}$ and $\widetilde{M}$ contains $\mathbf{x}$, and neither is contained in the other. Therefore, $\mathbf{x}$ is not an endpoint of Γ_{n+1}. $\square$

Recall from Chapter Two that every chainable continuum is hereditarily unicoherent. Hence, for a chainable continuum, any two subcontinua have a connected intersection. This will be useful in the proof of the following lemma where we show that for certain irreducible functions, every point of $\{0, 1\}^n \cap \Gamma_n$ is an endpoint of Γ_n.

Lemma 5.5. *Let* $F : [0,1] \to 2^{[0,1]}$ *be an irreducible function with the associated irreducible collection* $\{f_\lambda\}_{\lambda\in\Lambda}$. *Suppose that*

(1) $\Gamma(F)$ *does not contain any simple closed curves, and*

(2) if $\alpha = 0, 1$ *is a limit point of* Λ, *then* $F(\alpha) = [0,1]$.

For each $n \in \mathbb{N}$, *if* E_n *is the set of endpoints of* Γ_n, *then* $E_n \supseteq \Gamma_n \cap \{0,1\}^n$.

Proof. This clearly holds for $\Gamma_1 = [0,1]$, so, proceeding by induction, suppose that for some $n \in \mathbb{N}$, $E_n \supseteq \Gamma_n \cap \{0,1\}^n$.

First note that since $\Gamma(F)$ does not contain any simple closed curves, and $F(0)$ and $F(1)$ are each either degenerate or equal to $[0,1]$, it follows from Theorem 4.21 that Γ_{n+1} is chainable and thus is hereditarily unicoherent. In particular, given a subcontinuum L of Γ_{n+1} and $\lambda \in \Lambda$, we have that $L \cap \Gamma(\mathfrak{f}_\lambda^{(n)})$ is a continuum.

Also note that, as it was shown in Corollary 6.25, if any subcontinuum $L \subseteq \Gamma_{n+1}$ intersects both $\mathcal{A}_{n+1}$ and $\Gamma(\mathfrak{f}_\lambda^{(n)})$ for some $\lambda \neq 0$, then $L \supseteq \Gamma(\mathfrak{f}_0^{(n)})$.

Now, let $\mathbf{x} \in \Gamma_{n+1} \cap \{0,1\}^{n+1}$, and let H and K be subcontinua of Γ_{n+1}, each containing $\mathbf{x}$. For simplicity, we will suppose that $x_{n+1} = 0$. (The proof is not different for $x_{n+1} = 1$.) Since $x_{n+1} = 0$, we have that $\mathbf{x} \in \mathcal{A}_{n+1} \subseteq \Gamma(\mathfrak{f}_0^{(n)})$.

Case 1: Suppose that H and K are each subsets of $\Gamma(\mathfrak{f}_0^{(n)})$. Since $\Gamma(\mathfrak{f}_0^{(n)})$ is homeomorphic to Γ_n, and since $(x_1, \ldots, x_n)$ is an endpoint of Γ_n, it follows that H and K are nested.

Case 2: Suppose that H is a subset of $\Gamma(\mathfrak{f}_0^{(n)})$, but K is not. Then, K must intersect $\Gamma(\mathfrak{f}_\lambda^{(n)})$ for some $\lambda \neq 0$, so, as was previously noted, this implies that $K \supseteq \Gamma(\mathfrak{f}_0^{(n)}) \supseteq H$. Similarly, if K is a subset of $\Gamma(\mathfrak{f}_0^{(n)})$ while H is not, then $H \supseteq K$.

Case 3: Suppose that neither H nor K is contained in $\Gamma(\mathfrak{f}_0^{(n)})$. Let

$$\begin{aligned} \Lambda_H &= \left\{\lambda \in \Lambda : H \cap \Gamma(\mathfrak{f}_\lambda^{(n)}) \neq \emptyset\right\}, \text{ and} \\ \Lambda_K &= \left\{\lambda \in \Lambda : K \cap \Gamma(\mathfrak{f}_\lambda^{(n)}) \neq \emptyset\right\}. \end{aligned}$$

Let $\lambda_1 = \max \Lambda_H$, and let $\lambda_2 = \max \Lambda_K$. If $\lambda_1 < \lambda_2$, then by Lemma 3.10,

$$K \supseteq \bigcup_{\lambda \in \Lambda \cap [0,\lambda_1]} \Gamma(\mathfrak{f}_\lambda^{(n)}) \supseteq H.$$

Similarly, if $\lambda_2 < \lambda_1$, then $H \supseteq K$.

Finally, suppose that $\lambda_1 = \lambda_2$. Then by Lemma 3.10, for all $\lambda \in \Lambda \cap [0, \lambda_1)$, H and K each contain $\Gamma(\mathfrak{f}_\lambda^{(n)})$. Hence, to show that H and K are nested, it suffices to show that $H \cap \Gamma(\mathfrak{f}_{\lambda_1}^{(n)})$ and $K \cap \Gamma(\mathfrak{f}_{\lambda_1}^{(n)})$ are nested.

Sub-case (a): Suppose that $\lambda_1 = \lambda_2$ is a limit point of $\Lambda \cap [0, \lambda_1]$. Then by Lemma 6.24, $\Gamma(\mathfrak{f}_{\lambda_1}^{(n)})$ is a C-set in

$$\bigcup_{\lambda \in \Lambda \cap [0,\lambda_1]} \Gamma(\mathfrak{f}_\lambda^{(n)}),$$

so H and K must both contain it. Hence $H = K$.

Sub-case (b): Suppose that $\lambda_1 = \lambda_2$ is isolated in $\Lambda \cap [0, \lambda_1]$. Then let μ be the point of Λ immediately preceding λ_1. Suppose that $f_{\lambda_1}(0) = f_\mu(0)$. (It is not meaningfully different if $f_{\lambda_1}(1) = f_\mu(1)$ instead.) Since

$$F(0), F(1) \in \{\{0\}, \{1\}, [0,1]\},$$

it follows that there is a point $\mathbf{p} \in \mathcal{A}_n \cap \{0,1\}^n$ which, by the induction hypothesis, is a subset of $\mathcal{A}_n \cap E_n$.

By Lemma 3.10, H and K each contain $\Gamma(\mathfrak{f}_\mu^{(n)})$, and since $\mathbf{p} \in \mathcal{A}_n$, we have that $\mathfrak{f}_\mu^{(n)}(\mathbf{p}) = \mathfrak{f}_{\lambda_1}^{(n)}(\mathbf{p})$. Therefore, both H and K contain the point $(p_1, \ldots, p_n, f_{\lambda_1}(p_n))$ which is an endpoint of $\Gamma(\mathfrak{f}_{\lambda_1}^{(n)})$.

As was noted previously, $H \cap \Gamma(\mathfrak{f}_{\lambda_1}^{(n)})$ and $K \cap \Gamma(\mathfrak{f}_{\lambda_1}^{(n)})$ are both continua, so since they contain a common endpoint, they must be nested. Therefore, H and K must be nested, and hence, $\mathbf{x}$ is an endpoint of Γ_{n+1}. $\square$

The following theorem follows immediately from Lemma 5.4, Lemma 5.5, and Theorem 5.3.

Theorem 5.6. *Let* $F : [0,1] \to 2^{[0,1]}$ *be an irreducible function with the associated irreducible collection* $\{f_\lambda\}_{\lambda \in \Lambda}$. *Suppose that*

(1) $\Gamma(F)$ *does not contain any simple closed curves,*

(2) if 0 *is a limit point of* Λ, *then* $F(0) = [0,1]$, *and*

(3) if 1 *is a limit point of* Λ, *then* $F(1) = [0,1]$.

If E *is the set of endpoints of* $\varprojlim \mathbf{F}$, *then* $E = \varprojlim \mathbf{F} \cap \{0,1\}^{\mathbb{N}}$.

CHAPTER SIX

Towards a Classification of Inverse Limits with Irreducible Set-valued Functions

We now turn our attention the question of when two inverse limits are or are not homeomorphic. We already have two tools for distinguishing between inverse limits, chainability and endpoints. In this chapter we develop an additional tool for demonstrating two inverse limits are topologically distinct. We also establish a sufficient condition for two inverse limits to be homeomorphic. This culminates in Section 6.3 with a topological classification of four families of inverse limits with irreducible functions.

6.1 Homeomorphisms between Inverse Limits of Irreducible Functions

In Subsection 6.1.1 we will establish sufficient conditions for two inverse sequences of irreducible functions to have homeomorphic inverse limits. We will establish our conditions first for sequences of functions which are irreducible with respect to points. This will lead to one of our main results, Theorem 6.6.

Next, in the context of irreducibility with respect to sets, the conditions will be more restrictive, but we will be able to establish conditions under which two inverse sequences of irreducible functions will have homeomorphic inverse limits. This result will be stated in Theorem 6.11.

In Subsection 6.1.2, we will discuss some applications of Theorem 6.6. Specifically, we will focus on the case where all of our factor spaces are $[0, 1]$, and all of the bonding functions are the same irreducible function whose corresponding irreducible collection is finite. We show that the inverse limit of such a function is homeomorphic to the inverse limit of an open mapping on $[0, 1]$. Thus we may use the existing classification of open mappings on the interval to classify the inverse limits of these irreducible functions.

6.1.1 Consistent Irreducible Functions

We begin with the following definition and lemma which will be applied extensively in this section.

Definition 6.1. Let $\{\mathbf{X},\mathbf{F}\}$ be an inverse sequence where for each $i\in\mathbb{N}$, $F_i : X_{i+1}\to 2^{X_i}$ is an irreducible function with the associated irreducible collection $\{f_\lambda^{(i)} : X_i \to X_{i+1}\}_{\lambda\in\Lambda_i}$. Define the *itinerary map* for $\{\mathbf{X},\mathbf{F}\}$ to be the function $\mathcal{F} : X_1 \times \prod_{i=1}^{\infty}\Lambda_i \to \varprojlim \mathbf{F}$ given by $\mathcal{F}(x,\lambda_1,\lambda_2,\ldots) = \mathbf{y}$ where $y_1 = x$ and $y_{i+1} = f_{\lambda_i}^{(i)}(y_i)$ for $i\in\mathbb{N}$.

Lemma 6.2. *Let $\{\mathbf{X},\mathbf{F}\}$ be an inverse sequence where for each $i\in\mathbb{N}$, $F_i : X_{i+1}\to 2^{X_i}$ is an irreducible function with the associated irreducible collection $\{f_\lambda^{(i)}\}_{\lambda\in\Lambda_i}$. Then the itinerary map $\mathcal{F}$ for this inverse sequence is continuous and a closed map.*

Proof. $\mathcal{F}$ is clearly continuous in its first coordinate, and its continuity in all other coordinates follows from Property 5 of Definition 3.6. Then, since its domain is compact and its range is Hausdorff, $\mathcal{F}$ is a closed map. $\square$

Definition 6.3. Let X and Y be irreducible continua, and let $\{f_\lambda\}_{\lambda\in\Lambda}$ and $\{g_\lambda\}_{\lambda\in\Lambda}$ each be irreducible with respect to $a,b\in X$ and $c,d\in Y$. Let these collections have the additional property that each of $f_0^{-1}(a), f_1^{-1}(b), g_0^{-1}(a)$, and $g_1^{-1}(b)$ is either a subset of $\{c,d\}$ or is equal to Y. We say that $\{f_\lambda\}_{\lambda\in\Lambda}$ and $\{g_\lambda\}_{\lambda\in\Lambda}$ are *consistent* if the following hold:

(1) $f_0^{-1}(a) = g_0^{-1}(a)$ and $f_1^{-1}(b) = g_1^{-1}(b)$, and

(2) for each $\lambda,\mu\in\Lambda$,

$$\{y\in Y : f_\lambda(y) = f_\mu(y)\} = \{y\in Y : g_\lambda(y) = g_\mu(y)\}.$$

Two irreducible functions are said to be *consistent* if their corresponding irreducible collections are consistent.

The following terminology will be useful in the proof of Lemma 6.5. If $\{f_\lambda\}_{\lambda\in\Lambda}$ and $\{g_\lambda\}_{\lambda\in\Lambda}$ are consistent, then given a pair $(\alpha, l) \in \{(0, a), (1, b)\}$, we say that (α, l) is *Type I* if $f_\alpha^{-1}(l) = g_\alpha^{-1}(l) \subseteq \{c, d\}$, and we say that (α, l) is *Type II* if $f_\alpha^{-1}(l) = g_\alpha^{-1}(l) = Y$.

Example 6.4. The irreducible collections pictured in Figure 6.1 are consistent.

Proof. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be the irreducible collection pictured on the left and let $\{g_\omega\}_{\omega\in\Omega}$ be the collection on the right.

We will use the set

$$\Lambda = \{\ldots, 1/16, 1/8, 1/4, 1/2, 3/4, 7/8, 15/16\ldots\} \cup \{0, 1\}$$

as the indexing set. $f_0^{-1}(0) = \{0, 1\} = g_0^{-1}(0)$, and $f_1^{-1}(1) = [0, 1] = g_1^{-1}(1)$, so in particular, $f_0^{-1}(0) = g_0^{-1}(0)$ and $f_1^{-1}(1) = g_1^{-1}(1)$ which satisfies Property 1 of Definition 6.3. In addition, we have that the pair $(0, 0)$ is Type I, and $(1, 1)$ is Type II. For Property 2 of the definition to be met, we must have that for $\lambda, \mu \in \Lambda$, $f_\lambda(x) = f_\mu(x)$ if, and only if, $g_\lambda(x) = g_\mu(x)$.

Note that there are infinitely many ways that Λ can be used to index these collections of maps. Specifically, we may say that $f_{3/4}$ is the function which goes from $(0, 3/4)$ to $(1, 7/8)$ and that $g_{3/4}$ is the function which goes from $(0, 7/8)$ to $(1, 15/16)$. We may then index the rest of the functions accordingly. This insures that $f_\lambda(x) = f_\mu(x)$ if, and only if, $g_\lambda(x) = g_\mu(x)$.

Thus, these irreducible collections meet the conditions of Definition 6.3, so they are consistent. $\square$

Lemma 6.5. *Let $\{\mathbf{X}, \mathbf{F}\}$ and $\{\mathbf{X}, \mathbf{G}\}$ be inverse sequences such that for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ and $G_i : X_{i+1} \to 2^{X_i}$ are irreducible with respect to $a_{i+1}, b_{i+1} \in X_{i+1}$ and $a_i, b_i \in X_i$. Let $\mathcal{F}$ be an itinerary map for $\{\mathbf{X}, \mathbf{F}\}$, and let $\mathcal{G}$ be an itinerary map for $\{\mathbf{X}, \mathbf{G}\}$. If for each $i \in \mathbb{N}$, F_i and G_i are consistent, then the composition $\mathcal{G} \circ \mathcal{F}^{-1}$ is a well-defined function from $\varprojlim \mathbf{F}$ to $\varprojlim \mathbf{G}$.*

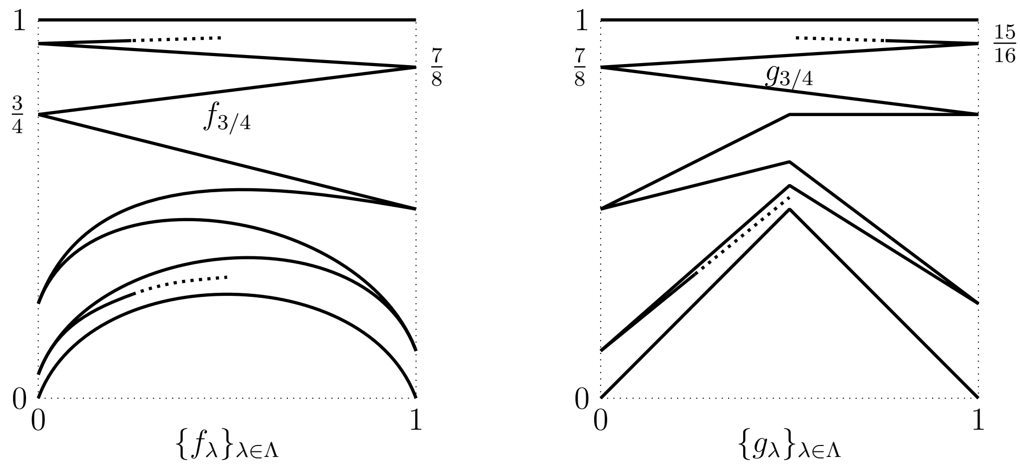

Figure 6.1. Consistent irreducible collections of maps

Proof. For each $i \in \mathbb{N}$, let $\{f_\lambda^{(i)} : X_i \to X_{i+1}\}_{\lambda\in\Lambda_i}$ and $\{g_\lambda^{(i)} : X_i \to X_{i+1}\}_{\lambda\in\Lambda_i}$ be the irreducible collections associated with F_i and G_i.

Let $\mathbf{x} \in \varprojlim \mathbf{F}$, let $(x_1, \lambda_1, \lambda_2, \ldots), (x_1, \mu_1, \mu_2, \ldots) \in \mathcal{F}^{-1}(\mathbf{x})$, and let $\mathbf{y} = \mathcal{G}(x_1, \lambda_1, \lambda_2, \ldots)$ and $\mathbf{z} = \mathcal{G}(x_1, \mu_1, \mu_2, \ldots)$. To show that $\mathcal{G} \circ \mathcal{F}^{-1}$ is well-defined, we must show that $\mathbf{y} = \mathbf{z}$. By the definitions of $\mathcal{F}$ and $\mathcal{G}$, $z_1 = y_1 = x_1$.

Proceeding by induction, suppose that for some $n_0 \in \mathbb{N}$, $y_i = z_i$ for all $i \leq n_0$. If $\lambda_{n_0} = \mu_{n_0}$, then $g_{\lambda_{n_0}}^{(n_0)}(y_{n_0}) = g_{\mu_{n_0}}^{(n_0)}(z_{n_0})$, so $y_{n_0+1} = z_{n_0+1}$. If $\lambda_{n_0} \neq \mu_{n_0}$, we will show that $x_{n_0} = y_{n_0} = z_{n_0}$. Then since $f_{\lambda_{n_0}}^{(n_0)}(x_{n_0}) = x_{n_0+1} = f_{\mu_{n_0}}^{(n_0)}(x_{n_0})$, it follows from Property 2 of Definition 6.3 that $g_{\lambda_{n_0}}^{(n_0)}(y_{n_0}) = g_{\mu_{n_0}}^{(n_0)}(z_{n_0})$, and therefore $y_{n_0+1} = z_{n_0+1}$.

Towards this end, note that since $f_{\lambda_{n_0}}^{(n_0)}(x_{n_0}) = f_{\mu_{n_0}}^{(n_0)}(x_{n_0})$, it follows from the fact that $\{f_\lambda^{(n_0)} : X_{n_0} \to X_{n_0+1}\}_{\lambda\in\Lambda_{n_0}}$ is an irreducible collection that $x_{n_0} \in \{a_{n_0}, b_{n_0}\}$. This means $\lambda_{n_0-1} = \mu_{n_0-1} \in \{0, 1\}$. If $(\lambda_{n_0-1}, x_{n_0}) \in \{(0, a_{n_0}), (1, b_{n_0})\}$ is Type I, then we also have that $x_{n_0-1} \in \{a_{n_0-1}, b_{n_0-1}\}$, and thus $\lambda_{n_0-2} = \mu_{n_0-2} \in \{0, 1\}$. Then if we supposed that $(\lambda_{n_0-2}, x_{n_0-1})$ was also Type I, then we could continue on in this manner. This leads us to Case 1.

Case 1: Suppose that for all $j \in \mathbb{N}$ with $1 < j \leq n_0$, $\lambda_{j-1} = \mu_{j-1} \in \{0, 1\}$, $x_j \in \{a_j, b_j\}$, and the pair (λ_{j-1}, x_j) is Type I. Then since $y_1 = z_1 = x_1$, it follows that $y_j = z_j = x_j$ for all $j \leq n_0$, and in particular, $y_{n_0} = z_{n_0} = x_{n_0}$.

Case 2: Suppose that for some $1 < j \leq n_0$, $\lambda_{j-1} = \mu_{j-1} \in \{0, 1\}$, $x_j \in \{a_j, b_j\}$, and the pair (λ_{j-1}, x_j) is Type II. Then let k be the largest integer less than or equal to n_0 such that (λ_{k-1}, x_k) is Type II. Then from the definition of Type II, $(f^{(k-1)}_{\lambda_{k-1}})^{-1}(x_k) = (g^{(k-1)}_{\lambda_{k-1}})^{-1}(x_k) = X_{k-1}$ which means that $y_k = f^{(k-1)}_{\lambda_{k-1}}(y_{k-1}) = x_k$. Since $\mu_{k-1} = \lambda_{k-1}$, we can similarly show that $z_k = x_k$. Thus, we in fact have that $z_k = y_k = x_k$. If $k = n_0$, then we have our result. If not, then by assumption, (λ_{j-1}, x_j) is Type I for all $k < j \leq n_0$, so from the same argument used in Case 1, it follows that $x_{n_0} = y_{n_0} = z_{n_0}$.

In either case, we have that $x_{n_0} = y_{n_0} = z_{n_0}$, so as already noted, this implies that $y_{n_0+1} = z_{n_0+1}$. Therefore, by induction, $y_i = z_i$ for all $i \in \mathbb{N}$, so $\mathbf{y} = \mathbf{z}$. Therefore $\mathcal{G} \circ \mathcal{F}^{-1}$ is well-defined. $\square$

Theorem 6.6. *Let $\{\mathbf{X}, \mathbf{F}\}$ and $\{\mathbf{X}, \mathbf{G}\}$ be inverse sequences such that for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ and $G_i : X_{i+1} \to 2^{X_i}$ are irreducible with respect to $a_{i+1}, b_{i+1} \in X_{i+1}$ and $a_i, b_i \in X_i$. If for each $i \in \mathbb{N}$, F_i and G_i are consistent, then $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are homeomorphic.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be itinerary maps for $\{\mathbf{X}, \mathbf{F}\}$ and $\{\mathbf{X}, \mathbf{G}\}$ respectively. By Lemma 6.5, the composition $\mathcal{G} \circ \mathcal{F}^{-1}$ is a well-defined function from $\varprojlim \mathbf{F}$ to $\varprojlim \mathbf{G}$. Let $\Phi = \mathcal{G} \circ \mathcal{F}^{-1}$. We may also apply Lemma 6.5 to state that $\mathcal{F} \circ \mathcal{G}^{-1}$ is well-defined. This implies that Φ is invertible and, hence, bijective. Therefore, since $\varprojlim \mathbf{F}$ is compact and $\varprojlim \mathbf{G}$ is Hausdorff, to show that Φ is a homeomorphism, we need only show that Φ is continuous.

From Lemma 6.2, we have that $\mathcal{G}$ is continuous and $\mathcal{F}$ is a closed map. Therefore given a closed set $A \subseteq \varprojlim \mathbf{G}$, $\Phi^{-1}(A) = \mathcal{F}(\mathcal{G}^{-1}(A))$ is closed. Hence Φ continuous and thus a homeomorphism between $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$. $\square$

Example 6.7. Let F and G be the irreducible functions pictured in Figure 6.2 (on the left and right respectively). Then $\varprojlim \mathbf{F}$ is homeomorphic to $\varprojlim \mathbf{G}$.

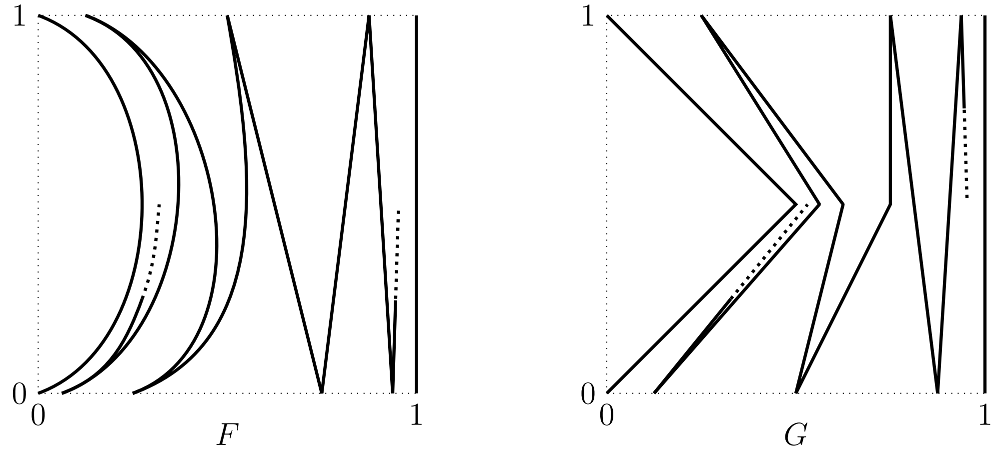

Figure 6.2. Consistent irreducible functions

Proof. These functions correspond to the irreducible collections of maps pictured in Figure 6.1. From Example 6.4, we have that those collections of maps are consistent, so by definition, F and G are consistent. Therefore, by Theorem 6.6, their inverse limits are homeomorphic. $\square$

Example 6.8. Let Λ be the set consisting of the standard Cantor set along with one point from each removed interval. Let $F, G, \widetilde{F}$, and $\widetilde{G}$ each be irreducible functions, as pictured in Figure 6.3, where each of their corresponding irreducible collections is indexed by Λ. Then $\varprojlim \mathbf{F}$, $\varprojlim \mathbf{G}$, $\varprojlim \widetilde{\mathbf{F}}$, and $\varprojlim \widetilde{\mathbf{G}}$ are all homeomorphic.

Proof. First note that F and G are consistent, as are $\widetilde{F}$ and $\widetilde{G}$. Thus Theorem 6.6 gives us that $\varprojlim \mathbf{F}$ is homeomorphic to $\varprojlim \mathbf{G}$, and $\varprojlim \widetilde{\mathbf{F}}$ is homeomorphic to $\varprojlim \widetilde{\mathbf{G}}$.

Hence, to show that they are all homeomorphic it suffices to show that $\varprojlim \mathbf{G}$ is homeomorphic to $\varprojlim \widetilde{\mathbf{G}}$. Towards this end, note that for each $x \in [0,1]$, $G(1-x) = 1 - G(x) = \widetilde{G}(x)$ (where by $1 - G(x)$ we mean the set $\{1 - y : y \in G(x)\}$).

Claim: This property implies that $\varprojlim \mathbf{G}$ and $\varprojlim \widetilde{\mathbf{G}}$ are homeomorphic.

To prove this claim, define a function $\varphi : \varprojlim \mathbf{G} \to \prod_{i=1}^{\infty}[0,1]$ by

$$\varphi(x_1, x_2, x_3, \ldots) = (x_1, 1 - x_2, x_3, 1 - x_4, \ldots).$$

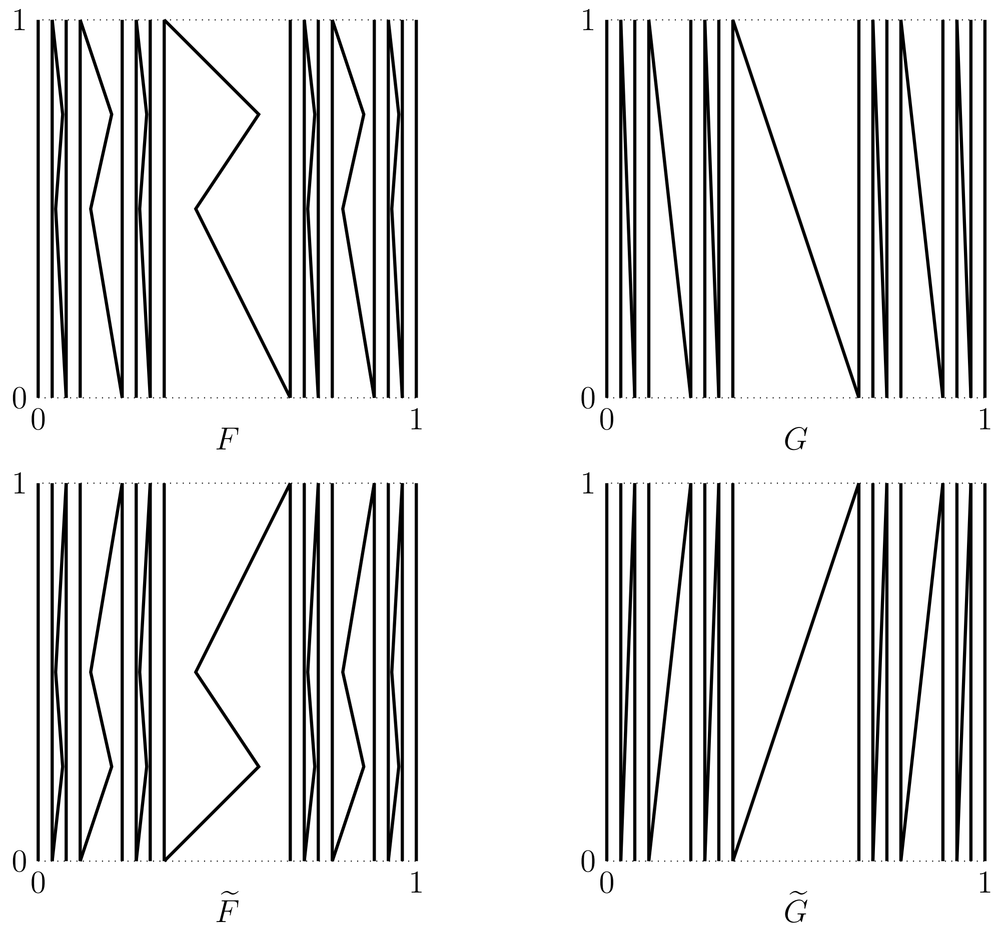

Figure 6.3. Irreducible functions with homeomorphic inverse limits

This function is continuous and injective, so we need only check that $\varphi(\varprojlim \mathbf{G}) = \varprojlim \widetilde{\mathbf{G}}$. First, to show that $\varphi(\varprojlim \mathbf{G}) \subseteq \varprojlim \widetilde{\mathbf{G}}$, let $\mathbf{x} \in \varprojlim \mathbf{G}$ and $\mathbf{y} = \varphi(\mathbf{x})$. For even n, $y_n = 1 - x_n$, and for odd n, $y_n = x_n$. Let $n \in \mathbb{N}$ be even. Then $\widetilde{G}(y_n) = \widetilde{G}(1 - x_n) = G(x_n) \ni x_{n-1} = y_{n-1}$, so $y_{n-1} \in \widetilde{G}(y_n)$. Now let $n \in \mathbb{N}$ be odd. Then $\widetilde{G}(y_n) = \widetilde{G}(x_n) = (1 - G(x_n)) \ni (1 - x_{n-1}) = y_{n-1}$, so again, $y_{n-1} \in \widetilde{G}(y_n)$.

Therefore $\mathbf{y} \in \varprojlim \widetilde{\mathbf{G}}$, so $\varphi(\varprojlim \mathbf{G}) \subseteq \varprojlim \widetilde{\mathbf{G}}$. Now, given $\mathbf{y} \in \varprojlim \widetilde{\mathbf{G}}$, the same argument which was just presented will show that the point

$$(y_1, 1 - y_2, y_3, 1 - y_4, \ldots) \in \varprojlim \mathbf{G}$$

and $\mathbf{y}$ is the image of this point. This means that $\varphi(\varprojlim \mathbf{G}) \supseteq \varprojlim \widetilde{\mathbf{G}}$, and φ is a homeomorphism. $\square$

This example is interesting because F and $\widetilde{F}$ do not satisfy the property that $F(1-x) = 1 - F(x) = \widetilde{F}(x)$, nor are they consistent. Thus it would not be immediately clear that $\varprojlim \mathbf{F}$ and $\varprojlim \widetilde{\mathbf{F}}$ were homeomorphic if it were not for G and $\widetilde{G}$ acting as intermediaries.

Obtaining a result such as Theorem 6.6 for functions which are irreducible with respect to sets is a bit more complicated. We would like to define the term "consistent" in the context of irreducibility with respect to sets, and we would like to do it in a way so that the inverse limits of consistent functions are homeomorphic (as in Theorem 6.6).

In order to do this, we will have to make the definition in this context more stringent than in Definition 6.3.

Definition 6.9. Let $\{\mathbf{X}, \mathbf{F}\}$ and $\{\mathbf{X}, \mathbf{G}\}$ be inverse sequences where for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ and $G_i : X_{i+1} \to 2^{X_i}$ are irreducible with respect to $A_{i+1}, B_{i+1} \subseteq X_{i+1}$ and $A_i, B_i \subseteq X_i$. Let $\{f_\lambda^{(i)}\}_{\lambda \in \Lambda_i}$ and $\{g_\lambda^{(i)}\}_{\lambda \in \Lambda_i}$ be the irreducible collections corresponding to F_i and G_i respectively. We say that these inverse sequences are *consistent* if for each $i \in \mathbb{N}$ and $\lambda, \mu \in \Lambda_i$,

$$\left\{y \in X_i : f_\lambda^{(i)}(y) = f_\mu^{(i)}(y)\right\} = \left\{y \in X_i : g_\lambda^{(i)}(y) = g_\mu^{(i)}(y)\right\}$$

and either of the following hold:

(1) For all $i \in \mathbb{N}$, if $(\alpha, L_{i+1}) \in \{(0, A_{i+1}), (1, B_{i+1})\}$, then

$$(f_\alpha^{(i)})^{-1}(L_{i+1}) = (g_\alpha^{(i)})^{-1}(L_{i+1}) \subseteq A_i \cup B_i,$$

and $f_\alpha^{(i)}|_{(f_\alpha^{(i)})^{-1}(L_{i+1})} = g_\alpha^{(i)}|_{(g_\alpha^{(i)})^{-1}(L_{i+1})}$;

(2) For all $i \in \mathbb{N}$,

$$(f_0^{(i)})^{-1}(A_{i+1}) = (g_0^{(i)})^{-1}(A_{i+1}) = (f_1^{(i)})^{-1}(B_{i+1}) = (g_1^{(i)})^{-1}(B_{i+1}) = X_i,$$

and if $L_i \in \{A_i, B_i\}$, then whenever $\lambda, \mu \in \Lambda_i$ with $\Gamma(f_\lambda^{(i)}|_{L_i}) \cap \Gamma(f_\mu^{(i)}|_{L_i}) \neq \emptyset$, it follows that $f_\lambda^{(i)}|_{L_i} = f_\mu^{(i)}|_{L_i}$.

Lemma 6.10. *Let* $\{\mathbf{X}, \mathbf{F}\}$ *and* $\{\mathbf{X}, \mathbf{G}\}$ *be inverse sequences such that for each* $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ *and* $G_i : X_{i+1} \to 2^{X_i}$ *are irreducible with respect to* $A_{i+1}, B_{i+1} \subseteq X_{i+1}$ *and* $A_i, B_i \subseteq X_i$*. Let* $\mathcal{F}$ *be an itinerary map for* $\{\mathbf{X}, \mathbf{F}\}$*, and let* $\mathcal{G}$ *be an itinerary map for* $\{\mathbf{X}, \mathbf{G}\}$*. If* $\{\mathbf{X}, \mathbf{F}\}$ *and* $\{\mathbf{X}, \mathbf{G}\}$ *are consistent, then the composition* $\mathcal{G} \circ \mathcal{F}^{-1}$ *is a well-defined function from* $\varprojlim \mathbf{F}$ *to* $\varprojlim \mathbf{G}$*.*

Proof. Let $\mathbf{x} \in \varprojlim \mathbf{F}$, let $(x_1, \lambda_1, \lambda_2, \ldots), (x_1, \mu_1, \mu_2, \ldots) \in \mathcal{F}^{-1}(\mathbf{x})$, and let $\mathbf{y} = \mathcal{G}(x_1, \lambda_1, \lambda_2, \ldots)$ and $\mathbf{z} = \mathcal{G}(x_1, \mu_1, \mu_2, \ldots)$. We must show that $\mathbf{y} = \mathbf{z}$. This will be done by induction.

By the definition of $\mathcal{G}$, $y_1 = z_1 = x_1$. Now suppose that for some $n_0 \in \mathbb{N}$, $y_i = z_i$ for all $i \leq n_0$. We want to show that this implies that $y_{n_0+1} = z_{n_0+1}$. If $\lambda_{n_0} = \mu_{n_0}$, then this will clearly hold, since $y_{n_0+1} = g_{\lambda_{n_0}}(y_{n_0})$ and $z_{n_0+1} = g_{\mu_{n_0}}(z_{n_0})$. Suppose then that $\lambda_{n_0} \neq \mu_{n_0}$.

Case 1: Suppose that $\{\mathbf{X}, \mathbf{F}\}$ and $\{\mathbf{X}, \mathbf{G}\}$ satisfy Property (1) of Definition 6.9. Since $f^{(n_0)}_{\lambda_{n_0}}(x_{n_0}) = f^{(n_0)}_{\mu_{n_0}}(x_{n_0})$, we must have that $x_{n_0} \in A_{n_0} \cup B_{n_0}$. We have that $(f^{(i)}_0)^{-1}(A_{i+1}) \subseteq A_i \cup B_i$ and $(f^{(i)}_1)^{-1}(B_{i+1}) \subseteq A_i \cup B_i$ for all $i \in \mathbb{N}$. Another way of saying this is to say that $F_i(A_{i+1})$ and $F_i(B_{i+1})$ are subsets of $A_i \cup B_i$ for all $i \in \mathbb{N}$. Thus, $x_i \in A_i \cup B_i$ for all $i \leq n_0$.

This also means then that $\lambda_i = \mu_i \in \{0, 1\}$ for all $i < n_0$ and that $x_i \in (f^{(i)}_0)^{-1}(A_{i+1}) \cup (f^{(i)}_1)^{-1}(B_{i+1})$ for all $i < n_0$. Now specifically, we know that $x_1 = y_1 = z_1$ and that this point is either an element of $(f^{(1)}_0)^{-1}(A_2)$ or of $(f^{(1)}_1)^{-1}(B_2)$. Because we are supposing that Property (1) holds, we know that f_0 and g_0 are equal when restricted to $(f^{(1)}_0)^{-1}(A_2)$, and we know that f_1 and g_1 are equal when restricted to $(f^{(1)}_1)^{-1}(B_2)$. This means that $x_2 = y_2 = z_2$. Continuing on in this manner, we conclude that $x_i = y_i = z_i$ for all $i \leq n_0$

Therefore, since $f^{(n_0)}_{\lambda_{n_0}}(x_{n_0}) = f^{(n_0)}_{\mu_{n_0}}(x_{n_0})$, it follows that $g^{(n_0)}_{\lambda_{n_0}}(y_{n_0}) = g^{(n_0)}_{\mu_{n_0}}(z_{n_0})$, so $y_{n_0+1} = z_{n_0+1}$.

Case 2: Suppose that $\{\mathbf{X}, \mathbf{F}\}$ and $\{\mathbf{X}, \mathbf{G}\}$ satisfy Property (2) of Definition 6.9. Again, since $f^{(n_0)}_{\lambda_{n_0}}(x_{n_0}) = f^{(n_0)}_{\mu_{n_0}}(x_{n_0})$, we must have that $x_{n_0} \in A_{n_0} \cup B_{n_0}$. In this case, we may only infer that $\lambda_{n_0-1} = \mu_{n_0-1} \in \{0, 1\}$. Let $L_{n_0} \in \{A_{n_0}, B_{n_0}\}$ be the set containing x_{n_0}. Because there is some value of L_{n_0} at which $f_{\lambda_{n_0}}$ and $f_{\mu_{n_0}}$ are equal (specifically x_{n_0}), we have that the equality $f^{(n_0)}_{\lambda_{n_0}}(x) = f^{(n_0)}_{\mu_{n_0}}(x)$ must hold for all $x \in L_{n_0}$. It follows that $g^{(n_0)}_{\lambda_{n_0}}(x) = g^{(n_0)}_{\mu_{n_0}}(x)$ for all $x \in L_{n_0}$. Also, since we are supposing that Property (2) holds, we know that

$$(g^{(n_0-1)}_{\lambda_{n_0-1}})^{-1}(L_{n_0}) = (g^{(n_0-1)}_{\mu_{n_0-1}})^{-1}(L_{n_0}) = (f^{(n_0-1)}_{\lambda_{n_0-1}})^{-1}(L_{n_0}) = X_{n_0-1}.$$

This means that $y_{n_0} = z_{n_0} \in L_{n_0}$. Then, since it has already been established that $g^{(n_0)}_{\lambda_{n_0}}$ and $g^{(n_0)}_{\mu_{n_0}}$ are equal when restricted to L_{n_0}, it follows that $g^{(n_0)}_{\lambda_{n_0}}(y_{n_0}) = g^{(n_0)}_{\mu_{n_0}}(z_{n_0})$.

In either case, $y_{n_0+1} = z_{n_0+1}$, and hence, by induction, $z_i = y_i$ for all $i \in \mathbb{N}$, and $\mathbf{y} = \mathbf{z}$. $\square$

Just as in Theorem 6.6, once it has been established that $\mathcal{G} \circ \mathcal{F}^{-1}$ is well-defined, it follows easily that it is in fact a homeomorphism.

Theorem 6.11. *Let $\{\mathbf{X}, \mathbf{F}\}$ and $\{\mathbf{X}, \mathbf{G}\}$ be inverse sequences such that for each $i \in \mathbb{N}$, $F_i : X_{i+1} \to 2^{X_i}$ and $G_i : X_{i+1} \to 2^{X_i}$ are irreducible with respect to $A_{i+1}, B_{i+1} \subseteq X_{i+1}$ and $A_i, B_i \subseteq X_i$. If $\{\mathbf{X}, \mathbf{F}\}$ and $\{\mathbf{X}, \mathbf{G}\}$ are consistent, then $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are homeomorphic.*

6.1.2 Applications

Below is a corollary to Theorem 6.6, and it deals with the case of an irreducible function on $[0, 1]$ whose corresponding irreducible collection is finite. Recall that two set-valued functions $F : X \to 2^X$ and $G : Y \to 2^Y$ are topologically conjugate if there exists a homeomorphism $\varphi : X \to Y$ such that $\varphi \circ F = G \circ \varphi$.

Corollary 6.12. *Let $F, G : [0,1] \to 2^{[0,1]}$ be irreducible functions. If their corresponding irreducible collections are each finite and contain the same number of maps, then $\varprojlim \mathbf{F}$ is homeomorphic to $\varprojlim \mathbf{G}$.*

Proof. Let k be the cardinality of the irreducible collections corresponding to F and G. Let $h : [0,1] \to [0,1]$ be the map consisting of k straight lines–the first from $(0,0)$ to $(1/k, 1)$, the second from $(1/k, 1)$ to $(2/k, 0)$, and so on. Notice that h and $1-h$ are both irreducible functions.

Claim: $\varprojlim \mathbf{h}$ is homeomorphic to $\varprojlim (\mathbf{1}-\mathbf{h})$.

To see that this is true, notice that if k is odd, then $h(1-x) = 1-h(x)$. Thus, in this case, just as in Example 6.8, $\varprojlim \mathbf{h}$ is homeomorphic to $\varprojlim (\mathbf{1}-\mathbf{h})$. If k is even, then if $\varphi : [0,1] \to [0,1]$ is defined by $\varphi(x) = 1-x$ we have that $\varphi \circ h = (1-h) \circ \varphi$. Therefore, in this case, h and $1-h$ are topologically conjugate, so by Theorem 2.9, $\varprojlim \mathbf{h}$ is homeomorphic to $\varprojlim (\mathbf{1}-\mathbf{h})$.

Since F^{-1} is the union of k maps, it will be consistent with either h or $1-h$. Since $\varprojlim \mathbf{h}$ and $\varprojlim (\mathbf{1}-\mathbf{h})$ are homeomorphic though, in either case, $\varprojlim \mathbf{F}$ is homeomorphic to $\varprojlim \mathbf{h}$. Similarly, $\varprojlim \mathbf{G}$ is homeomorphic to $\varprojlim \mathbf{h}$, so $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are homeomorphic to each other. $\square$

In [54], Watkins discusses functions such as h from the above proof. He would call h the kth degree hat function. More specifically, given $n \in \mathbb{N}$, the *nth degree hat function* is an open mapping on $[0,1]$, such that $f(0) = 0$, and for each $i = 1, \ldots, n$, f restricted to $[(i-1)/n, i/n]$ is linear and onto $[0,1]$. The inverse limits of these functions are the class of continua known as the Knaster continua. The main theorem of [54] is the following:

Theorem 6.13 (Watkins). *Let $n, m \in \mathbb{N}$. If $f : [0,1] \to [0,1]$ is the nth degree hat function and $g : [0,1] \to [0,1]$ is the mth degree hat function, then $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$ if, and only if, n and m have the same prime factors.*

In light of Corollary 6.12, we may generalize Watkins's classification in the following way.

Theorem 6.14. *Let $n, m \in \mathbb{N}$. Suppose $F : [0,1] \to 2^{[0,1]}$ is an irreducible function whose inverse is the union of n maps, and $G : [0,1] \to 2^{[0,1]}$ is an irreducible function whose inverse is the union of m maps. Then $\varprojlim \mathbf{F}$ is homeomorphic to $\varprojlim \mathbf{G}$ if, and only if, n and m have the same prime factors.*

6.2 Additional Method for Demonstrating Inverse Limits are not Homeomorphic

We now consider the question of when two inverse limits are not homeomorphic. We develop an additional tool beyond chainability and endpoints for distinguishing topologically between two inverse limits.

6.2.1 Proper Subcontinua of the Inverse Limit

In this subsection, we consider proper subcontinua of the inverse limits of irreducible functions. First, we discuss a method for defining proper subcontinua of inverse limits, then we move toward the main result of this subsection, Theorem 6.17, which says that if $F : X \to 2^X$ is irreducible with respect to $a, b \in X$, and $F(a), F(b) \in \{\{a\}, \{b\}, \{a, b\}, X\}$, then every proper subcontinuum of $\varprojlim \mathbf{F}$ is homeomorphic to a subcontinuum of Γ_n for some $n \in \mathbb{N}$. This fact will be instrumental for the results concerning distinguishing between inverse limits in Subsection 6.2.2.

Recall that Corollary 4.3 stated that if $F : X \to 2^X$ is an irreducible function, then for each $n \in \mathbb{N}$, Γ_n is homeomorphic to a subcontinuum of $\varprojlim \mathbf{F}$. This provides a method for defining specific subcontinua of an inverse limit. We now move towards the primary result of this section where we show that, with one added restriction, every proper subcontinuum of the inverse limit is homeomorphic to a subcontinuum of Γ_n for some $n \in \mathbb{N}$.

We now introduce the following notation which will be utilized for the remainder of the subsection.

Notation 6.15. Suppose $F : X \to 2^X$ is an upper semi-continuous function and K is a closed subset of $\varprojlim \mathbf{F}$. For each $n \in \mathbb{N}$, let $K_n = \pi_n(K)$, and $K_{[n,n+1]} = \pi_{[n,n+1]}(K)$.

Lemma 6.16. *Let $F : X \to 2^X$ be irreducible with respect to $a, b \in X$ with the corresponding irreducible collection of maps $\{f_\lambda\}_{\lambda \in \Lambda}$. If K is a subcontinuum of $\varprojlim \mathbf{F}$, then the following hold.*

(1) If for some $n \in \mathbb{N}$, either $a \notin K_n$ or $b \notin K_n$, then there exist $\lambda, \mu \in \Lambda$ such that

$$K_{[n,n+1]} \subseteq \Gamma(f_\lambda|_{K_n}) \cup \Gamma(f_\mu|_{K_n}).$$

(2) If for some $n \in \mathbb{N}$, K_n contains neither a nor b, then there exists $\lambda \in \Lambda_n$ such that $K_{n+1} = f_\lambda(K_n)$.

(3) If for some $n \in \mathbb{N}$, either $a \notin K_n$ or $b \notin K_n$, and $a \in K_{n+1}$ ($b \in K_{n+1}$), then $K_{n+1} = f_0(K_n)$ ($K_{n+1} = f_1(K_n)$).

Proof. First, note that for any $n \in \mathbb{N}$,

$$K_{[n,n+1]} \subseteq \Gamma(F^{-1}) = \bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda).$$

More specifically,

$$K_{[n,n+1]} \subseteq \bigcup_{\lambda \in \Lambda} \Gamma(f_\lambda|_{K_n}).$$

By Definition 3.5, if λ, μ, and ω, are consecutive elements of Λ with $\lambda < \mu < \omega$, then $\Gamma(f_\mu)$ intersects both $\Gamma(f_\lambda)$ and $\Gamma(f_\omega)$, but $\Gamma(f_\lambda)$ and $\Gamma(f_\omega)$ are disjoint. Thus, either $f_\lambda(a) = f_\mu(a)$, and $f_\mu(b) = f_\omega(b)$, or vice versa. In either case, if for some $n \in \mathbb{N}$, either $a \notin K_n$ or $b \notin K_n$, then one of $\Gamma(f_\lambda|_{K_n})$ and $\Gamma(f_\omega|_{K_n})$ is disjoint from the other and from $\Gamma(f_\mu|_{K_n})$. This establishes (1).

If for some $n \in \mathbb{N}$, K_n contains neither a nor b, then for any $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, the graphs of $f_\lambda|_{K_n}$ and $f_\mu|_{K_n}$ would be disjoint. Thus, there must exists a single element $\lambda \in \Lambda$ such that $K_{[n,n+1]} \subseteq \Gamma(f_\lambda)$. It then follows that $K_{[n,n+1]} = \Gamma(f_\lambda|_{K_n})$, and hence, $K_{n+1} = f_\lambda(K_n)$. This establishes (2).

Finally, to establish (3), note if $a \in K_{n+1}$, then $K_{[n,n+1]}$ intersects $\Gamma(f_0|_{K_n})$.

Case 1: Suppose 0 is a limit point of Λ. If K_{n+1} did not equal $f_0(K_n)$, that would imply that $K_{[n,n+1]}$ intersected $\Gamma(f_\lambda|_{K_n})$ for infinitely many $\lambda \in \Lambda$. From (1), we know that this cannot happen, so $K_{n+1} = f_0(K_n)$.

Case 2: Suppose 0 is not a limit point of Λ. Then, by Definition 3.5, $f_0^{-1}(a) = \{a\}$ or $f_0^{-1}(a) = \{b\}$. Moreover, if λ_0 is the smallest element of Λ larger than 0, then either $f_0(a) = a$ and $f_0(b) = f_{\lambda_0}(b)$, or $f_0(b) = a$ and $f_0(a) = f_{\lambda_0}(a)$. Thus, for a to be an element of K_{n+1}, $\Gamma(f_0|_{K_n})$ must be disjoint from $\Gamma(f_{\lambda_0}|_{K_n})$. Hence $K_{n+1} = f_0(K_n)$. □

Theorem 6.17. *Let $F : X \to 2^X$ be irreducible with respect to $a, b \in X$ with the corresponding irreducible collection of maps $\{f_\lambda\}_{\lambda\in\Lambda}$. If*

$$F(a), F(b) \in \{\{a\}, \{b\}, \{a, b\}, X\},$$

then every proper subcontinuum of $\varprojlim \mathbf{F}$ is homeomorphic to a subcontinuum of Γ_n for some $n \in \mathbb{N}$.

Proof. Let K be a proper subcontinuum of $\varprojlim \mathbf{F}$. By Theorem 3.15, $\varprojlim \mathbf{F}$ has the full projection property, so there exists $N \in \mathbb{N}$ such that for all $n \geq N$, K_n is a proper subcontinuum of X. Hence, for each $n \geq N$, either $a \notin K_n$ or $b \notin K_n$.

Case 1: Suppose that for all $n \geq N$, $K_n \cap \{a, b\} \neq \emptyset$. Then by Lemma 6.16 Part (3), there exists a sequence $(\alpha_n)_{n=N}^{\infty}$, where for each $n \geq N$, $\alpha_n \in \{0, 1\}$, and $K_{n+1} = f_{\alpha_n}(K_n)$. It follows that K is homeomorphic to $\pi_{[1,N]}(K) \subseteq \Gamma_N$.

Case 2: Suppose there exists $n_0 \geq N$ such that for all $n \geq n_0$, K_n contains neither a nor b. Then by Lemma 6.16 Part (2), for each $n \geq n_0$, there exists $\lambda_n \in \Lambda$ such that $K_{n+1} = f_{\lambda_n}(K_n)$. Hence, K is homeomorphic to $\pi_{[1,n_0]}(K) \subseteq \Gamma_{n_0}$.

Case 3: Suppose there exists $n_0 \geq N$ such that K_{n_0} contains neither a nor b, but K_{n_0+1} contains a. By Lemma 6.16 Part (3), it follows that $K_{n_0+1} = f_0(K_{n_0})$. Moreover, since K_{n_0} does not contain a or b, this implies that $f_0^{-1}(a) = F(a)$ is not a subset of $\{a, b\}$. Thus, by assumption, $f_0^{-1}(a) = F(a) = X$.

Hence, we have, in fact, that $K_{n_0+1} = f_0(K_{n_0}) = \{a\}$. Then, for all $n \geq n_0+1$, we have that K_n is degenerate, so K is homeomorphic to $\pi_{[1,n_0]}(K) \subseteq \Gamma_{n_0}$.

Similarly, if there exists $n_0 \geq N$ such that K_{n_0} contains neither a nor b, but K_{n_0+1} contains b, then for all $n \geq n_0 + 1$, K_n is degenerate, and K is homeomorphic to $\pi_{[1,n_0]}(K) \subseteq \Gamma_{n_0}$. □

Another way to state Theorem 6.17 is in terms of composants of the inverse limit. The *composant* of a point x in a continuum X is the union of all proper subcontinua of X which contain x.

Corollary 6.18. *Let $F : X \to 2^X$ be irreducible with respect to $a, b \in X$ with the corresponding irreducible collection of maps $\{f_\lambda\}_{\lambda\in\Lambda}$. Suppose further that*

$$F(a), F(b) \in \{\{a\}, \{b\}, \{a, b\}, X\}.$$

Then, for $\mathbf{x} \in \varprojlim \mathbf{F}$, the composant of $\mathbf{x}$ in $\varprojlim \mathbf{F}$ is the set of all $\mathbf{y} \in \varprojlim \mathbf{F}$ such that there exists a natural number N and a sequence $(\lambda_i)_{i=N}^{\infty} \in \Lambda^{\mathbb{N}}$ such that for $i \geq N$, $x_{i+1} = f_{\lambda_i}(x_i)$, and $y_{i+1} = f_{\lambda_i}(y_i)$.

6.2.2 Some Classification Results

For the remainder of the chapter, we will only consider irreducible functions on $[0, 1]$. Hence, we introduce the following notation.

Notation 6.19. Let $\mathcal{I}$ represent the set of all pairs $(F, \{f_\lambda\}_{\lambda\in\Lambda})$ such that $F : [0, 1] \to 2^{[0,1]}$ is an irreducible function and $\{f_\lambda\}_{\lambda\in\Lambda}$ is its corresponding irreducible collection of maps. If $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I}$, we say that $(F, \{f_\lambda\}_{\lambda\in\Lambda})$ is an *irreducible pair*.

Remark 6.20. Open mappings on $[0,1]$ are irreducible functions, and their corresponding irreducible collections are finite. Moreover, continua which are inverse limits of open mappings on $[0,1]$ are collectively called *Knaster continua*, and they have the property that every proper subcontinuum is an arc. It follows from Corollary 6.12 that for any $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I}$ such that Λ is finite, $\varprojlim \mathbf{F}$ is a Knaster continuum.

Once again, recall that for $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I}$, $n \in \mathbb{N}$, and $\lambda \in \Lambda$, $\mathfrak{f}_\lambda^{(n)} : \Gamma_n \to [0,1]$ is defined by $\mathfrak{f}_\lambda^{(n)}(\mathbf{x}) = f_\lambda(x_n)$.

Theorem 6.21. *Let $(F, \{f_\lambda\}_{\lambda\in\Lambda}), (G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{I}$. If Λ is finite and Ω is infinite, then $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are not homeomorphic.*

Proof. By Remark 6.20, $\varprojlim \mathbf{F}$ is a Knaster continuum, so every proper subcontinuum of $\varprojlim \mathbf{F}$ is an arc.

Since $\Gamma_2(G) = \Gamma(G^{-1})$ is the union of infinitely many mappings, each one having $[0,1]$ as its domain, $\Gamma_2(G)$ is not an arc. Then, by Corollary 4.3, $\Gamma_2(G)$ is homeomorphic to a proper subcontinuum of $\varprojlim \mathbf{G}$. Hence $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are not homeomorphic. $\square$

Recall that if $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I}$, then for each $n \in \mathbb{N}$ and $\lambda \in \Lambda$, the function $\mathfrak{f}_\lambda^{(n)} : \Gamma_n \to [0,1]$ is defined by

$$\mathfrak{f}_\lambda^{(n)}(x_1, \ldots, x_n) = f_\lambda(x_n).$$

Note that for each $n \in \mathbb{N}$,

$$\Gamma_{n+1} = \bigcup_{\lambda\in\Lambda} \Gamma(\mathfrak{f}_\lambda^{(n)}),$$

and that for each $\lambda \in \Lambda$, $\Gamma(\mathfrak{f}_\lambda^{(n)})$ is homeomorphic to Γ_n.

Another definition which will be utilized in this section is the following.

Definition 6.22. Given a continuum X, a set $A \subseteq X$ is called a *C-set* in X if every subcontinuum of X which intersects both A and $X \setminus A$ contains A.

The following remark will be useful in the proof of Lemma 6.24 below. It is a specific case of Lemma 3.10

Remark 6.23. Let $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I}$. Let $n \in \mathbb{N}$, let K be a subcontinuum of Γ_{n+1}, and let $\Omega_K = \{\lambda \in \Lambda : K \cap \Gamma(\mathfrak{f}_\lambda^{(n)}) \neq \emptyset\}$. Then Ω_K is the intersection of a closed (possibly degenerate) interval with Λ. Moreover, if $\lambda \in \Omega_K \setminus \{\min \Omega_K, \max \Omega_K\}$, then $\Gamma(\mathfrak{f}_\lambda^{(n)}) \subseteq K$.

Lemma 6.24. *Let $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I}$. If λ_0 is a limit point of $[\lambda_0, 1] \cap \Lambda$, then for each $n \in \mathbb{N}$, $\Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$ is a C-set in*

$$\bigcup_{\lambda\in\Lambda\cap[\lambda_0,1]} \Gamma(\mathfrak{f}_\lambda^{(n)}).$$

Likewise, if λ_0 is a limit point of $[0, \lambda_0] \cap \Lambda$, then for each $n \in \mathbb{N}$, $\Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$ is a C-set in

$$\bigcup_{\lambda\in\Lambda\cap[0,\lambda_0]} \Gamma(\mathfrak{f}_\lambda^{(n)}).$$

Proof. Suppose that λ_0 is a limit point of $[\lambda_0, 1] \cap \Lambda$. Fix $n \in \mathbb{N}$, and let

$$X = \bigcup_{\lambda\in\Lambda\cap[0,\lambda_0]} \Gamma(\mathfrak{f}_\lambda^{(n)}).$$

Let K be a subcontinuum of X which intersects both $\Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$ and $X \setminus \Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$, and let $\Omega_K = \{\lambda \in \Lambda : K \cap \Gamma(\mathfrak{f}_\lambda^{(n)}) \neq \emptyset\}$. By Remark 6.23, Ω_K is an interval in Λ. Moreover, since K intersects $\Gamma(\mathfrak{f}_\lambda^{(n)})$ and $X \setminus \Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$, Ω_K is non-degenerate.

Therefore, λ_0 is a limit point of Ω_K, so there is a sequence, $(\lambda_i)_{i=1}^\infty$, in $\Omega_K \setminus \{\min \Omega_K, \max \Omega_K\}$ converging to λ_0. By Lemma 6.23, we have that $\Gamma(\mathfrak{f}_{\lambda_i}^{(n)}) \subseteq K$ for all $i \in \mathbb{N}$. It then follows from Property (5) of Definition 3.5 that $\Gamma(\mathfrak{f}_{\lambda_0}^{(n)}) \subseteq K$. □

Recall that, given a pair $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I}$, we define for each $n \in \mathbb{N}$ the sets $\mathcal{A}_n = \{\mathbf{x} \in \Gamma_n : x_n = 0\}$, and $\mathcal{B}_n = \{\mathbf{x} \in \Gamma_n : x_n = 1\}$.

From Definition 3.5, for any two $\lambda, \mu \in \Lambda$, if the graphs of f_λ and f_μ intersect, they do so at either 0 or 1. Hence, if for some $n \in \mathbb{N}$, the graphs of $\mathfrak{f}_\lambda^{(n)}$ and $\mathfrak{f}_\mu^{(n)}$

intersect, they do so over the set $\mathcal{A}_n$ or over $\mathcal{B}_n$. This makes these two sets crucial to the structure of Γ_{n+1}. We elaborate on this in the following two results.

Corollary 6.25. *Let $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I}$. If for some $n \geq 2$, $\mathcal{A}_n$ ($\mathcal{B}_n$) is a C-set in $\Gamma(\mathfrak{f}_0^{(n-1)})$ ($\Gamma(\mathfrak{f}_1^{(n-1)})$), then $\mathcal{A}_n$ ($\mathcal{B}_n$) is a C-set in Γ_n*

Proof. Let $n \geq 2$, and suppose that $\mathcal{A}_n$ is a C-set in $\Gamma(\mathfrak{f}_0^{(n-1)})$. If 0 is a limit point of Λ, then by Lemma 6.24, $\Gamma(\mathfrak{f}_0^{(n-1)})$ is a C-set in Γ_n. It follows that $\mathcal{A}_n$ is a C-set in Γ_n.

If 0 is not a limit point of Λ, then suppose that K is a subcontinuum of Γ_n intersecting $\mathcal{A}_n$ and its complement. If K is a subset of $\Gamma(\mathfrak{f}_0^{(n-1)})$, then since $\mathcal{A}_n$ is a C-set in $\Gamma(\mathfrak{f}_0^{(n-1)})$, K must contain $\mathcal{A}_n$. If K is not a subset of $\Gamma(\mathfrak{f}_0^{(n-1)})$, and λ_0 is the smallest element of Λ larger than 0, then K must intersect $\Gamma(\mathfrak{f}_{\lambda_0}^{(n-1)})$. Note that the graphs of $\mathfrak{f}_0^{(n-1)}$ and $\mathfrak{f}_{\lambda_0}^{(n-1)}$ intersect either over $\mathcal{A}_{n-1}$ or $\mathcal{B}_{n-1}$, and $\mathfrak{f}_0^{(n-1)}$ maps the other to 0.

Hence, for K to intersect $\mathcal{A}_n$ and $\Gamma(\mathfrak{f}_{\lambda_0}^{(n-1)})$, it must intersect $\Gamma(\mathfrak{f}_0^{(n-1)}|_{\mathcal{A}_{n-1}})$ and $\Gamma(\mathfrak{f}_0^{(n-1)}|_{\mathcal{B}_{n-1}})$. It was shown in the proof of Lemma 3.10, that this implies that K contains $\Gamma(\mathfrak{f}_0^{(n-1)})$. In particular then, K contains $\mathcal{A}_n$. Therefore, $\mathcal{A}_n$ is a C-set in Γ_n. □

Before we may prove the main result of this section, Theorem 6.27, we must prove one more lemma.

Lemma 6.26. *Let $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I}$. Let $n \in \mathbb{N}$, and let $\lambda_0 \in \Lambda$. Suppose that $\{K_i\}_{i=1}^{\infty}$ is a collection of subcontinua of Γ_{n+1}, each of which intersects both $\Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$ and $\Gamma_{n+1} \setminus \Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$. If*

$$F(0), F(1) \in \{\{0\}, \{1\}, \{0,1\}, [0,1]\},$$

then there is an infinite subset $A \subseteq \mathbb{N}$ such that for all $i, j \in A$, K_i intersects K_j.

Proof. First, if λ_0 is a limit from the right, and for infinitely many $i \in \mathbb{N}$, there is $\lambda_i > \lambda_0$ such that K_i intersects $\Gamma(\mathfrak{f}_{\lambda_i}^{(n)})$, then by Lemma 6.24, K_i contains $\Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$ for all such $i \in \mathbb{N}$. Hence, any two such members of $\{K_i\}_{i=1}^{\infty}$ must intersect each other. The same holds if λ_0 is a limit from the left, and for infinitely many $i \in \mathbb{N}$, there is $\lambda_i < \lambda_0$ such that K_i intersects $\Gamma(\mathfrak{f}_{\lambda_i}^{(n)})$.

Thus it suffices to show that if $\mu \in \Lambda$ is adjacent to λ_0 in Λ, and K_i intersects $\Gamma(\mathfrak{f}_{\mu}^{(n)}) \setminus \Gamma(\mathfrak{f}_{\lambda_0}^{(n)})$ for infinitely many $i \in \mathbb{N}$, then the result holds.

To do this, we will demonstrate that for each $n \in \mathbb{N}$, $\mathcal{A}_n$ and $\mathcal{B}_n$ are each either C-sets in Γ_n or a finite union of C-sets in Γ_n. Since these are the sets over which the graphs of $\mathfrak{f}_{\lambda_0}^{(n)}$ and $\mathfrak{f}_{\mu}^{(n)}$ intersect, the result will follow. This will be shown in five cases.

Before beginning these cases, note that $\Gamma_1 = [0,1]$, $\mathcal{A}_1 = \{0\}$, and $\mathcal{B}_1 = \{1\}$. Hence, since singleton sets are always C-sets, we have that $\mathcal{A}_1$ and $\mathcal{B}_1$ are C-sets in Γ_1. For the following cases, we will suppose that $n \geq 2$.

Case 1: Suppose that both $F(0)$ and $F(1)$ are finite (i.e. either $\{0\}$ $\{1\}$, or $\{0,1\}$). Then for all $n \geq 2$, $\mathcal{A}_n$ and $\mathcal{B}_n$ are both finite sets. Since singleton sets are C-sets, it follows that for each $n \geq 2$, $\mathcal{A}_n$ and $\mathcal{B}_n$ are each a finite union of C-sets in Γ_n.

Case 2: Suppose that $F(0) = \{0\}$ and $F(1) = [0,1]$. Then for all $n \geq 2$, $\mathcal{A}_n$ is a singleton set, so it is a C-set. Also, since $F(1) = [0,1]$, it must be the case that 1 is a limit point of Λ. Thus, by Lemma 6.24, it follows that for all $n \geq 2$, $\mathcal{B}_n = \Gamma(\mathfrak{f}_1^{(n-1)})$ is a C-set in Γ_n. (The same argument applies if $F(1) = \{1\}$ and $F(0) = [0,1]$.)

Case 3: Suppose that $F(0) = \{1\}$ and $F(1) = [0,1]$. Just as in Case 2, the fact that $F(1) = [0,1]$ implies that for all $n \geq 2$, $\mathcal{B}_n$ is a C-set in Γ_n. Then, note that for all $n \geq 2$, $\Gamma(\mathfrak{f}_0^{(n-1)})$ is homeomorphic to Γ_{n-1}. Also, since $F(0) = \{1\}$, we have that $\mathcal{A}_n = \Gamma(\mathfrak{f}_0^{(n-1)}|_{\mathcal{B}_{n-1}})$. Therefore, $\mathcal{A}_n$ is a C-set in $\Gamma(\mathfrak{f}_0^{(n-1)})$, so by Corollary 6.25, $\mathcal{A}_n$ is a C-set in Γ_n. (Again, the same argument applies if $F(1) = \{0\}$ and $F(0) = [0,1]$.)

Case 4: Suppose that $F(0) = F(1) = [0,1]$. Then, just as in the previous two cases, this implies that for all $n \geq 2$, $\mathcal{A}_n$ and $\mathcal{B}_n$ are each C-sets in Γ_n.

Case 5: Suppose that $F(0) = [0,1]$, and $F(1) = \{0,1\}$. Once again, since $F(0) = [0,1]$, for all $n \geq 2$, $\mathcal{A}_n$ is a C-set in Γ_n.

We will show that for each $n \geq 2$, $\mathcal{B}_n$ is a finite union of C-sets in Γ_n using induction. As has already been established, $\mathcal{B}_1 = \{1\}$ is a C-set in Γ_1. Now suppose that for some $n \in \mathbb{N}$, $\mathcal{B}_n = B_1 \cup \cdots \cup B_k$ where for each $i = 1, \ldots, k$, B_i is a C-set in Γ_n.

Since $F(1) = \{0,1\}$,

$$\mathcal{B}_{n+1} = \Gamma(\mathfrak{f}_1^{(n)}|_{\mathcal{B}_n}) \cup \Gamma(\mathfrak{f}_1^{(n)}|_{\mathcal{A}_n}).$$

Thus, for each $i = 1, \ldots, k$, define $\widetilde{B}_i$ to be $\Gamma(\mathfrak{f}_1^{(n)}|_{B_i})$, and define $\widetilde{B}_{k+1}$ to be $\Gamma(\mathfrak{f}_1^{(n)}|_{\mathcal{A}_n})$. Then $\mathcal{B}_{n+1} = \widetilde{B}_1 \cup \cdots \cup \widetilde{B}_{k+1}$. Moreover, since each of $B_1, \ldots, B_k$, $\mathcal{A}_n$ is a C-set in Γ_n, it follows that for each $i = 1, \ldots, k+1$, $\widetilde{B}_i$ is a C-set in $\Gamma(\mathfrak{f}_1^{(n)})$. Then by Corollary 6.25, for each $i = 1, \ldots, k+1$, $\widetilde{B}_i$ is a C-set in Γ_{n+1}. Therefore, $\mathcal{B}_{n+1}$ is finite union of C-sets in Γ_{n+1}. (The same argument holds if $F(1) = [0,1]$ and $F(0) = \{0,1\}$.) □

Given $\Lambda, \Omega \subseteq [0,1]$, Λ' and Ω' refer to the sets of limit points of Λ and Ω respectively.

Theorem 6.27. *Let $(F, \{f_\lambda\}_{\lambda \in \Lambda}), (G, \{g_\omega\}_{\omega \in \Omega}) \in \mathcal{I}$. If $0 < \operatorname{card} \Lambda' < \operatorname{card} \Omega' < \infty$, and*

$$G(0), G(1) \in \{\{0\}, \{1\}, \{0,1\}, [0,1]\},$$

then $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are not homeomorphic.

Proof. To prove that $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are not homeomorphic, we will construct a proper subcontinuum of $\varprojlim \mathbf{F}$ and demonstrate that it is not homeomorphic to any proper subcontinuum of $\varprojlim \mathbf{G}$.

Fix a point $\lambda_0 \in \Lambda'$, and let $(\lambda_i)_{i\in\mathbb{N}}$ be a monotonic sequence in $\Lambda \setminus \Lambda'$ such that λ_i and λ_{i+1} are adjacent for all $i \in \mathbb{N}$, and $\lambda_i \to \lambda_0$ as $i \to \infty$. (The fact that $\operatorname{card} \Lambda' < \infty$ guarantees that this is possible.)

Claim: For each $i \geq 0$, let $L_i = \Gamma(\mathfrak{f}^{(2)}_{\lambda_i})$. The collection $\{L_i\}_{i=0}^{\infty}$ has the following properties:

(1) For each $i \geq 0$, L_i is homeomorphic to $\Gamma(F)$.

(2) $L_0 = \lim_{i\to\infty} L_i$.

(3) For each $i, j \geq 1$, $L_i \cap L_j \neq \emptyset$ if, and only if, $|i - j| \leq 1$.

(4) For each $i \geq 1$, $L_i \cap L_0 = \emptyset$.

(5) For any $N \geq 1$, $\bigcup_{i=0}^{\infty} L_i$ is homeomorphic to either $L_0 \cup \bigcup_{i=N}^{\infty} L_i$ or to $L_0 \cup \bigcup_{i=N+1}^{\infty} L_i$.

(6) For any $N \geq 1$, $\bigcup_{i=N}^{\infty} L_i$ is connected.

Let $L = \bigcup_{i=0}^{\infty} L_i$. Since L is a subcontinuum of $\Gamma_3(F)$, by Corollary 4.3, $\varprojlim \mathbf{F}$ contains a subcontinuum homeomorphic to L. We will show that $\varprojlim \mathbf{G}$ does not contain a subcontinuum homeomorphic to L by showing that $\Gamma_n(G)$ does not for any $n \in \mathbb{N}$ and then appealing to Theorem 6.17.

First, $\Gamma_1(G) = [0, 1]$, so it does not contain a subcontinuum homeomorphic to L. Proceeding by induction, suppose that for some $n \in \mathbb{N}$, $\Gamma_n(G)$ does not contain a subcontinuum homeomorphic to L.

Suppose that $\Gamma_{n+1}(G)$ does contain a subcontinuum homeomorphic to L. Then $\Gamma_{n+1}(G)$ contains a sequence, $(K_i)_{i=1}^{\infty}$, of continua and a continuum K_0 that satisfy all the properties above which are satisfied by $(L_i)_{i=0}^{\infty}$ such that $\bigcup_{i=0}^{\infty} K_i$ is homeomorphic to L. Specifically, we have that the collection $(K_i)_{i=0}^{\infty}$ satisfies all of the following:

(1) For each $i \geq 0$, K_i is homeomorphic to $\Gamma(F)$.

(2) $K_0 = \lim_{i\to\infty} K_i$.

(3) For each $i, j \geq 1$, $K_i \cap K_j \neq \emptyset$ if, and only if, $|i - j| \leq 1$.

(4) For each $i \geq 1$, $K_i \cap K_0 = \emptyset$.

(5) For any $N \geq 1$, $\bigcup_{i=0}^{\infty} K_i$ is homeomorphic to either $K_0 \cup \bigcup_{i=N}^{\infty} K_i$ or to $K_0 \cup \bigcup_{i=N+1}^{\infty} K_i$.

(6) For any $N \geq 1$, $\bigcup_{i=N}^{\infty} K_i$ is connected.

We will show that this leads to a contradiction.

Case 1: Suppose that there exists $\omega_0 \in \Omega$ such that $K_0 \subseteq \Gamma(\mathfrak{g}_{\omega_0}^{(n)})$. Then since $\Gamma(\mathfrak{g}_{\omega_0}^{(n)})$ is homeomorphic to $\Gamma_n(G)$, $\Gamma_n(G)$ contains a subcontinuum $\widetilde{K}$ which is homeomorphic to K_0 and hence to $\Gamma(F)$. If $n = 1$, then this is already a contradiction. If $n \geq 2$, then since Ω contains more limit points than Λ, $\widetilde{K}$ will necessarily be a proper subcontinuum of $\Gamma_n(G)$. Thus, from Corollary 3.11, $\widetilde{K}$ is disjoint either from $\{\mathbf{x} \in \Gamma_n(G) : x_n = 0\}$ or from $\{\mathbf{x} \in \Gamma_n(G) : x_n = 1\}$. It follows that K_0 is disjoint either from $\{\mathbf{x} \in \Gamma(\mathfrak{g}_{\omega_0}^{(n)}) : x_n = 0\}$ or from $\{\mathbf{x} \in \Gamma(\mathfrak{g}_{\omega_0}^{(n)}) : x_n = 1\}$.

Sub-case (a): Suppose that for all but finitely many $i \in \mathbb{N}$, K_i is disjoint from $\Gamma(\mathfrak{g}_{\omega_0}^{(n)})$. Then, by Property (5) without loss of generality, we may suppose that this holds for all $i \in \mathbb{N}$.

As was previously noted, for either $\alpha = 0$ or $\alpha = 1$, K_0 is disjoint from the set $\{\mathbf{x} \in \Gamma(\mathfrak{g}_{\omega_0}^{(n)}) : x_n = \alpha\}$. Since $K_0 \subseteq \Gamma(\mathfrak{g}_{\omega_0}^{(n)})$, we may say more generally that K_0 is disjoint from the set $\{\mathbf{x} \in \Gamma_{n+1} : x_n = \alpha\}$. Therefore, since $K_i \to K_0$ as $i \to \infty$, we have that there exists $N \in \mathbb{N}$ such that for $i \geq N$, K_i is disjoint from the set $\{\mathbf{x} \in \Gamma_{n+1} : x_n = \alpha\}$.

Now, let

$$\Sigma = \left\{\omega \in \Omega : \Gamma(\mathfrak{g}_{\omega}^{(n)}) \cap K_i \neq \emptyset \text{ for some } i \geq N\right\}.$$

Since K_i is disjoint from $\Gamma(\mathfrak{g}_{\omega_0}^{(n)})$ for all $i \in \mathbb{N}$, we have that ω_0 is not in Σ. However, since $K_i \to K_0$ as $i \to \infty$, it follows that ω_0 is a limit point of Σ, so Σ is infinite.

However, $\bigcup_{i=N}^{\infty} K_i$ is disjoint from $\{\mathbf{x} \in \Gamma_{n+1} : x_n = \alpha\}$, and it follows from the definition of an irreducible collection of functions (Definition 3.5) that a connected subset of Γ_{n+1} which is disjoint from $\{\mathbf{x} \in \Gamma_{n+1} : x_n = \alpha\}$ can intersect the graphs of at most two members of $\{\mathfrak{g}_{\omega}^{(n)}\}_{\omega \in \Omega}$. Thus, we have a contradiction.

Sub-case (b): Suppose that K_i intersects $\Gamma(\mathfrak{g}_{\omega_0}^{(n)})$ for infinitely many $i \in \mathbb{N}$. From Lemma 6.26 and Property (3) above, we have that K_i can intersect both $\Gamma(\mathfrak{g}_{\omega_0}^{(n)})$ and $\Gamma_{n+1} \setminus \Gamma(\mathfrak{g}_{\omega_0}^{(n)})$ for at most finitely many $i \in \mathbb{N}$. Hence, $K_i \subseteq \Gamma(\mathfrak{g}_{\omega_0}^{(n)})$ for all but finitely many $i \in \mathbb{N}$, so there exists $n_0 \in \mathbb{N}$ such that for $i \geq n_0$, $K_i \subseteq \Gamma(\mathfrak{g}_{\omega_0}^{(n)})$.

Then $\bigcup_{i=n_0}^{\infty} K_i$ is a subset of $\Gamma(\mathfrak{g}_{\omega_0}^{(n)})$, and by Property 5, either $K_0 \cup \bigcup_{i=n_0}^{\infty} K_i$ or $K_0 \cup \bigcup_{i=n_0+1}^{\infty} K_i$ is homeomorphic to L. However, $\Gamma(\mathfrak{g}_{\omega_0}^{(n)})$ is homeomorphic to $\Gamma_n(G)$, so this contradicts the assumption that $\Gamma_n(G)$ does not contain a subcontinuum homeomorphic to L.

Case 2: Suppose that K_0 is not contained in the graph of any member of $\{\mathfrak{g}_{\omega}^{(n)}\}_{\omega \in \Omega}$. Just as before, K_0, cannot contain any of these graphs, so by Lemma 6.23, K_0 must be contained in $\Gamma(\mathfrak{g}_{\omega_1}^{(n)}) \cup \Gamma(\mathfrak{g}_{\omega_2}^{(n)})$ for some $\omega_1, \omega_2 \in \Omega$. Moreover, by assumption, K_0 intersects both $\Gamma(\mathfrak{g}_{\omega_1}^{(n)}) \setminus \Gamma(\mathfrak{g}_{\omega_2}^{(n)})$ and $\Gamma(\mathfrak{g}_{\omega_2}^{(n)}) \setminus \Gamma(\mathfrak{g}_{\omega_1}^{(n)})$.

It follows from Definition 3.5 that the only way that the sequence$(K_i)_{i=1}^{\infty}$ could converge to K_0 is if for infinitely many $i \in \mathbb{N}$, K_i intersects both $\Gamma(\mathfrak{g}_{\omega_1}^{(n)}) \setminus \Gamma(\mathfrak{g}_{\omega_2}^{(n)})$ and $\Gamma(\mathfrak{g}_{\omega_2}^{(n)}) \setminus \Gamma(\mathfrak{g}_{\omega_1}^{(n)})$. By Lemma 6.26, the sequence $(K_i)_{i=0}^{\infty}$ cannot satisfy this requirement while also satisfying Property (3) above. Thus, once again, we have a contradiction.

Hence, $\Gamma_{n+1}(G)$ does not contain a subcontinuum homeomorphic to L, so by induction, for all $n \in \mathbb{N}$, $\Gamma_n(G)$ fails to contain a subcontinuum homeomorphic to L. Hence, by Theorem 6.17, $\varprojlim \mathbf{G}$ also fails to contain a subcontinuum homeomorphic to L, so $\varprojlim \mathbf{G}$ and $\varprojlim \mathbf{F}$ are not homeomorphic. $\square$

We conclude this section with one final theorem. This theorem is nearly identical to Theorem 6.27, and its proof too is almost exactly the same as the proof of Theorem 6.27.

Theorem 6.28. *Let* $(F, \{f_\lambda\}_{\lambda\in\Lambda}), (G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{I}$. *If*

$$0 < \operatorname{card}[\Lambda' \setminus \{0,1\}] < \operatorname{card}[\Omega' \setminus \{0,1\}] < \infty,$$

and

$$G(0), G(1) \in \{\{0\}, \{1\}, \{0,1\}, [0,1]\},$$

then $\varprojlim \mathbf{F}$ *and* $\varprojlim \mathbf{G}$ *are not homeomorphic.*

6.3 Topological Classification of Four Families of Inverse Limits

In this final section, we define four specific families of irreducible pairs. We show that (topologically speaking) the sets of inverse limits which come from these respective families are mutually exclusive. Moreover, within each family, we give a full classification of the inverse limits which arise. The proofs in this section use the results from Subsection 6.2.2 as well as the results concerning endpoints from Chapter Five.

We now define the four families of irreducible pairs whose inverse limits we classify in this section. We define each family by first defining a family of subsets of $[0,1]$ which may be used to index irreducible collections of maps. We then use that family of indexing sets to define a corresponding family of irreducible pairs. The inverse limits which arise from these families are classified in Theorems 6.30, 6.32, 6.33, 6.34, and 6.35.

Definition 6.29.

(1) $\mathcal{K}$ is the set of all finite subsets of $[0,1]$ which include both 0 and 1.

$$\mathcal{F}_\mathcal{K} = \{(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I} : \Lambda \in \mathcal{K}\}$$

(2) $\mathcal{L}$ is the set of all closed subsets $\Lambda \subseteq [0,1]$ in which the isolated points are dense, and such that

(a) $0 < \operatorname{card} \Lambda' < \infty$,

(b) 0 and 1 are isolated in Λ, and

(c) every limit point of Λ is a two-sided limit.

$$\mathcal{F}_{\mathcal{L}} = \{(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I} : \Lambda \in \mathcal{L}\}$$

(3) $\mathcal{M}$ is the set of all closed subsets $\Lambda \subseteq [0,1]$ in which the isolated points are dense, and such that

(a) $0 < \operatorname{card} \Lambda' < \infty$,

(b) either 0 or 1 is isolated in Λ while the other is a limit point of Λ, and

(c) every other limit point of Λ is a two-sided limit.

$$\begin{aligned}\mathcal{F}_{\mathcal{M}} = \{(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I} : \Lambda \in \mathcal{M},\ &\text{for } \alpha = 0, 1,\\ &\text{if } \alpha \in \Lambda',\ \text{then } F(\alpha) = [0,1]\}\end{aligned}$$

(4) $\mathcal{N}$ is the set of all closed subsets $\Lambda \subseteq [0,1]$ in which the isolated points are dense, and such that

(a) $0 < \operatorname{card} \Lambda' < \infty$,

(b) both 0 and 1 are limit points of Λ, and

(c) every other limit point of Λ is a two-sided limit.

$$\mathcal{F}_{\mathcal{N}} = \{(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{I} : \Lambda \in \mathcal{N},\ \text{and } F(0) = F(1) = [0,1]\}$$

A representative of each of these families is pictured in Figure 6.4.

The inverse limits which arise from the family $\mathcal{F}_{\mathcal{K}}$ have already been classified in Theorem 6.14. We restate this theorem below for completeness.

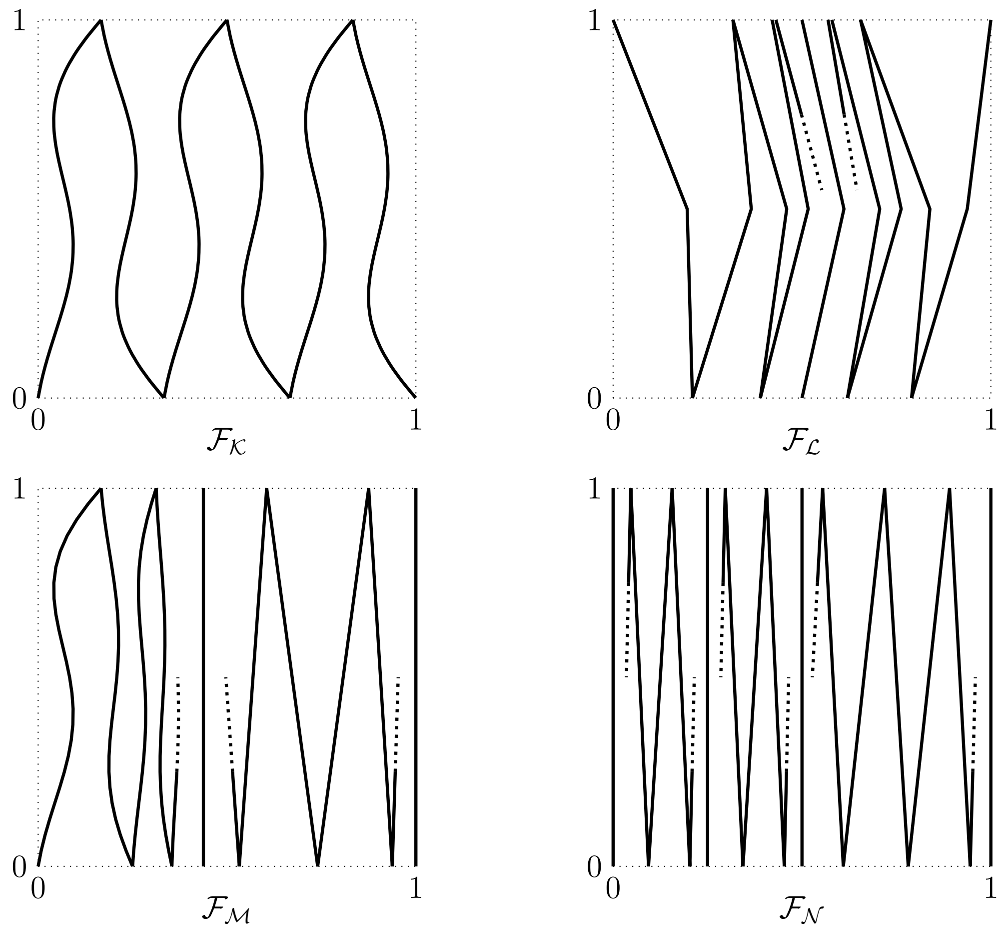

Figure 6.4. Four families of irreducible functions

Theorem 6.30. *Let* $(F, \{f_\lambda\}_{\lambda\in\Lambda}), (G, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{F}_{\mathcal{K}}$. *Suppose that* $\operatorname{card}\Lambda = m$ *and* $\operatorname{card}\Omega = n$. *Then* $\varprojlim \mathbf{F}$ *and* $\varprojlim \mathbf{G}$ *are homeomorphic if, and only if,* m *and* n *have the same prime factors.*

We will now give a classification for each of the remaining families. The inverse limits arising from $\mathcal{F}_{\mathcal{L}}$ are classified in Theorem 6.32, the inverse limits from $\mathcal{F}_{\mathcal{M}}$ are classified in Theorem 6.33, and those arising from $\mathcal{F}_{\mathcal{N}}$ are classified in Theorem 6.34.

Finally, we show in Theorem 6.35 that if $(F, \{f_\lambda\}_{\lambda\in\Lambda})$ is an irreducible pair from one of these families, and $(G, \{g_\omega\}_{\omega\in\Omega})$ is a pair from a different family, then $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are not homeomorphic. All together these theorems give a full classification of the inverse limits arising from $\mathcal{F}_{\mathcal{K}} \cup \mathcal{F}_{\mathcal{L}} \cup \mathcal{F}_{\mathcal{M}} \cup \mathcal{F}_{\mathcal{N}}$.

Remark 6.31. For any $(F, \{f_\lambda\}_{\lambda\in\Lambda})$ from any of the families defined in Definition 6.29, each limit point of Λ other than 0 and 1 is a two-sided limit point, so $\Gamma(F)$ contains no simple closed curves. Hence Theorem 5.6 applies to $(F, \{f_\lambda\}_{\lambda\in\Lambda})$.

Theorem 6.32. *Let*

$$\begin{aligned}\mathcal{F}_{\mathcal{L},1} &= \{(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{F}_{\mathcal{L}} : F(0) = F(1)\}, \text{ and} \\ \mathcal{F}_{\mathcal{L},2} &= \{(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{F}_{\mathcal{L}} : F(0) \neq F(1)\}.\end{aligned}$$

(1) If $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{F}_{\mathcal{L},1}$ and $(G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{F}_{\mathcal{L},2}$, then $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are not homeomorphic.

(2) Suppose $(F, \{f_\lambda\}_{\lambda\in\Lambda}), (G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{F}_{\mathcal{L},i}$ for some $i = 1, 2$. Then $\varprojlim \mathbf{F}$ is homeomorphic to $\varprojlim \mathbf{G}$ if, and only if, Λ and Ω have the same number of limit points.

Proof. First, to see that (1) holds, let $(F, \{f_\lambda\}_{\lambda\in\Lambda})$ be in $\mathcal{F}_{\mathcal{L},1}$, and let $(G, \{g_\omega\}_{\omega\in\Omega})$ be in $\mathcal{F}_{\mathcal{L},2}$.

By Theorem 5.6, the endpoints of $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are precisely the points consisting only of 0s and 1s. Since $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{F}_{\mathcal{L},1}$, we have that $F(0) = F(1)$. Also, since 0 and 1 are not limit points of Λ, either $F(0) = F(1) = \{0\}$ or $F(0) = F(1) = \{1\}$. Thus, $\varprojlim \mathbf{F}$ will have exactly one endpoint, either the point $(0, 0, 0, \ldots)$ or the point $(1, 1, 1, \ldots)$.

Since $(G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{F}_{\mathcal{L},2}$, we have that $G(0) \neq G(1)$. Thus, one of them equals $\{0\}$ while the other equals $\{1\}$. If $G(0) = \{0\}$ and $G(1) = \{1\}$, then the points $(0, 0, 0, \ldots)$ and $(1, 1, 1, \ldots)$ will both be in $\varprojlim \mathbf{G}$, and if $G(0) = \{1\}$ and $G(1) = \{0\}$, then $(0, 1, 0, 1, \ldots)$ and $(1, 0, 1, 0, \ldots)$ will both be in $\varprojlim \mathbf{G}$.

In either case, $\varprojlim \mathbf{G}$ has exactly two endpoints, so $\varprojlim \mathbf{F}$ is not homeomorphic to $\varprojlim \mathbf{G}$. This establishes (1).

Next, to see that (2) holds, suppose that $(F, \{f_\lambda\}_{\lambda\in\Lambda}), (G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{F}_{\mathcal{L},1}$. Theorem 6.27 gives us that if Λ and Ω have different numbers of limit points then $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are not homeomorphic.

Suppose then that Λ and Ω have the same number of limit points.

Case 1: Suppose that $F(0) = F(1) = G(0) = G(1)$. We will show that F and G are consistent as defined in Definition 6.3.

Since Λ and Ω have the same number of limit points, there is an order preserving homeomorphism from Ω onto Λ. This means that the collection $\{g_\omega\}_{\omega\in\Omega}$ could also be indexed by Λ. Since every limit point of Λ is a two-sided limit point, if λ_1 and λ_2 are consecutive limit points of Λ, then the elements of Λ between λ_1 and λ_2 form a bi-infinite sequence. In other words, if Λ were to be used to index $\{g_\omega\}_{\omega\in\Omega}$, there would be no choice as to which functions would be designated g_{λ_1} and g_{λ_2}, but for the functions situated between these two, there would be infinitely many ways that they could be indexed by the elements of $(\lambda_1, \lambda_2) \cap \Lambda$.

Hence there is an indexing of $\{g_\omega\}_{\omega\in\Omega}$ by Λ such that for each $\lambda, \mu \in \Lambda$,

$$\{y \in [0,1] : f_\lambda(y) = f_\mu(y)\} = \{y \in [0,1] : g_\lambda(y) = g_\mu(y)\}.$$

It then follows from the assumption that $F(0) = F(1) = G(0) = G(1)$ that $(F, \{f_\lambda\}_{\lambda\in\Lambda})$ and $(G, \{g_\lambda\}_{\lambda\in\Lambda})$ are consistent, so by Theorem 6.6, $\varprojlim \mathbf{F}$ is homeomorphic to $\varprojlim \mathbf{G}$.

Case 2: Suppose that $F(0) = F(1) \neq G(0) = G(1)$. We will show that G is conjugate to an irreducible function which is consistent with F.

Define $\psi : [0,1] \to [0,1]$ by $\psi(x) = 1 - x$. Since neither 0 nor 1 is a limit point of Λ, and every limit point is a two-sided limit, there is an order reversing homeomorphism $\varphi : \Lambda \to \Omega$ such that for each $\lambda, \mu \in \Lambda$,

$$\{y \in [0,1] : f_\lambda(y) = f_\mu(y)\}$$

$$= \left\{y \in [0,1] : \psi \circ g_{\varphi(\lambda)} \circ \psi^{-1}(y) = \psi \circ g_{\varphi(\mu)} \circ \psi^{-1}(y)\right\}.$$

Therefore, if we define $\widetilde{G} = \psi \circ G \circ \psi^{-1}$, and for each $\lambda \in \Lambda$ we define, $\widetilde{g}_\lambda = \psi \circ g_{\varphi(\lambda)} \circ \psi^{-1}$, then $(F, \{f_\lambda\}_{\lambda\in\Lambda})$ and $(\widetilde{G}, \{\widetilde{g}_\lambda\}_{\lambda\in\Lambda})$ are consistent. Thus, by Theorem 6.6, $\varprojlim \mathbf{F}$ and $\varprojlim \widetilde{\mathbf{G}}$ are homeomorphic, and since $\widetilde{G}$ is conjugate to G, it follows that $\varprojlim \mathbf{F}$ is homeomorphic to $\varprojlim \mathbf{G}$.

A similar argument holds for $(F, \{f_\lambda\}_{\lambda\in\Lambda}), (G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{F}_{\mathcal{L},2}$. $\square$

Theorem 6.33. *Let*

$$\begin{aligned} \mathcal{F}_{\mathcal{M},1} &= \{(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{F}_\mathcal{M} : \text{ either } F(0) = \{0\} \text{ or } F(1) = \{1\}\}, \text{ and} \\ \mathcal{F}_{\mathcal{M},2} &= \{(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{F}_\mathcal{M} : \text{ either } F(0) = \{1\} \text{ or } F(1) = \{0\}\}. \end{aligned}$$

(1) If $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{F}_{\mathcal{M},1}$ and $(G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{F}_{\mathcal{M},2}$, then $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are not homeomorphic.

(2) Suppose $(F, \{f_\lambda\}_{\lambda\in\Lambda}), (G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{F}_{\mathcal{M},i}$ for some $i = 1, 2$. Then $\varprojlim \mathbf{F}$ is homeomorphic to $\varprojlim \mathbf{G}$ if, and only if, Λ and Ω have the same number of limit points.

Proof. First, to see that (1) holds, let $(F, \{f_\lambda\}_{\lambda\in\Lambda})$ be in $\mathcal{F}_{\mathcal{M},1}$, and let $(G, \{g_\omega\}_{\omega\in\Omega})$ be in $\mathcal{F}_{\mathcal{M},2}$.

By Theorem 5.6, the set of endpoints of each of these inverse limits is the intersection of that inverse limit with the set $\{0, 1\}^\mathbb{N}$. We will show that $\varprojlim \mathbf{F}$ has countably many endpoints while $\varprojlim \mathbf{G}$ has uncountably many.

By the definition of $\mathcal{F}_{\mathcal{M},1}$, either $F(0) = \{0\}$ and $F(1) = [0, 1]$, or $F(1) = \{1\}$ and $F(0) = [0, 1]$. Let $\alpha \in \{0, 1\}$ be the point which is fixed by F, and let $\beta \in \{0, 1\}$ be the point whose image is $[0, 1]$. Then $\{0, 1\}^\mathbb{N} \cap \varprojlim \mathbf{F}$ contains the point $(\alpha, \alpha, \ldots)$, the point $(\beta, \beta, \ldots)$, and any point $\mathbf{x}$ such that there exists $N \in \mathbb{N}$ with $x_i = \alpha$ for $x_i \leq N$, and $x_i = \beta$ for $i > N$. These are the only points which are endpoints of $\varprojlim \mathbf{F}$, and there are countably many.

By the definition of the sub-family $\mathcal{F}_{\mathcal{M},2}$, we have that either $G(0) = \{1\}$ and $G(1) = [0,1]$, or that $G(1) = \{0\}$ and $G(0) = [0,1]$. As before, we label 0 and 1 as α and β in such a way that $G(\alpha) = \{\beta\}$, and $G(\beta) = [0,1]$. Then

$$\{0,1\}^{\mathbb{N}} \cap \varprojlim \mathbf{G} = \left\{ \mathbf{x} \in \{0,1\}^{\mathbb{N}} : \text{ if } x_i = \beta \text{ for some } i \in \mathbb{N}, \text{ then } x_{i+1} = \alpha \right\}.$$

We may define an injection $h : \mathbb{N}^{\mathbb{N}} \to \varprojlim \mathbf{G} \cap \{0,1\}^{\mathbb{N}}$ by setting $h(n_1, n_2, n_3, \ldots)$ equal to the sequence which begins with a β, followed by n_1 many αs which are followed by a β which is followed by n_2 many αs which are followed by a β and so on. It follows that $\varprojlim \mathbf{G}$ has at least as many endpoints as the cardinality of the set $\mathbb{N}^{\mathbb{N}}$ which is uncountable.

Next, a similar argument to the one used for Part (2) of Theorem 6.32 will show that Part (2) of this theorem holds. □

Theorem 6.34. *Suppose* $(F, \{f_\lambda\}_{\lambda\in\Lambda}), (G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{F}_{\mathcal{N}}$. *Then* $\varprojlim \mathbf{F}$ *is homeomorphic to* $\varprojlim \mathbf{G}$ *if, and only if,* Λ *and* Ω *have the same number of limit points.*

Proof. In this case, since $F(0) = F(1) = G(0) = G(1) = [0,1]$, if Λ and Ω have the same number of limit points, then $(F, \{f_\lambda\}_{\lambda\in\Lambda})$ and $(G, \{g_\omega\}_{\omega\in\Omega})$ are consistent. The result follows. □

The inverse limits arising from each individual family have been classified. We now show that the inverse limits arising from each of these families are topologically distinct from those arising from any other family.

Theorem 6.35. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be distinct members of* $\{\mathcal{F}_{\mathcal{K}}, \mathcal{F}_{\mathcal{L}}, \mathcal{F}_{\mathcal{M}}, \mathcal{F}_{\mathcal{N}}\}$, *and suppose that* $(F, \{f_\lambda\}_{\lambda\in\Lambda}) \in \mathcal{F}$, *and* $(G, \{g_\omega\}_{\omega\in\Omega}) \in \mathcal{G}$. *Then* $\varprojlim \mathbf{F}$ *and* $\varprojlim \mathbf{G}$ *are not homeomorphic.*

Proof. We have from Theorem 6.21 that if either $\mathcal{F}$ or $\mathcal{G}$ is $\mathcal{F}_{\mathcal{K}}$, then $\varprojlim \mathbf{F}$ is not homeomorphic to $\varprojlim \mathbf{G}$.

If $\mathcal{F} = \mathcal{F}_{\mathcal{L}}$, and $\mathcal{G}$ is either $\mathcal{F}_{\mathcal{M}}$ or $\mathcal{F}_{\mathcal{N}}$, then it follows from Theorem 5.6 that $\varprojlim \mathbf{F}$ has either one or two endpoints, while $\varprojlim \mathbf{G}$ has either countably many or uncountably many endpoints, so these inverse limits would not be homeomorphic.

Finally, suppose that $\mathcal{F} = \mathcal{F}_{\mathcal{M}}$ and $\mathcal{G} = \mathcal{F}_{\mathcal{N}}$. If Λ and Ω have different numbers of limit points, then by Theorem 6.27, $\varprojlim \mathbf{F}$ and $\varprojlim \mathbf{G}$ are not homeomorphic. If Λ and Ω have the same number of limit points, then by the definitions $\mathcal{M}$ and $\mathcal{N}$, 0 and 1 are both limit points of Ω, while only one of them is a limit point of Λ. Hence

$$\operatorname{card}\left[\Omega' \setminus \{0,1\}\right] < \operatorname{card}\left[\Lambda' \setminus \{0,1\}\right],$$

so by Theorem 6.28, $\varprojlim \mathbf{F}$ is not homeomorphic to $\varprojlim \mathbf{G}$. $\square$

BIBLIOGRAPHY

[1] Lori Alvin. Hofbauer towers and inverse limit spaces. *Proc. Amer. Math. Soc.*, 141(11):4039–4048, 2013.

[2] Lori Alvin and Karen Brucks. Adding machines, endpoints, and inverse limit spaces. *Fund. Math.*, 209(1):81–93, 2010.

[3] Lori Alvin and Karen Brucks. Adding machines, kneading maps, and endpoints. *Topology Appl.*, 158(3):542–550, 2011.

[4] Iztok Banič. On dimension of inverse limits with upper semicontinuous set-valued bonding functions. *Topology Appl.*, 154(15):2771–2778, 2007.

[5] Marcy Barge, Henk Bruin, and Sonja Štimac. The Ingram conjecture. *Geom. Topol.*, 16(4):2481–2516, 2012.

[6] Marcy Barge and Joe Martin. Endpoints of inverse limit spaces and dynamics. In *Continua (Cincinnati, OH, 1994)*, volume 170 of *Lecture Notes in Pure and Appl. Math.*, pages 165–182. Dekker, New York, 1995.

[7] R. H. Bing. Snake-like continua. *Duke Math. J.*, 18:653–663, 1951.

[8] Louis Block, James Keesling, Brian Raines, and Sonja Štimac. Homeomorphisms of unimodal inverse limit spaces with a non-recurrent critical point. *Topology Appl.*, 156(15):2417–2425, 2009.

[9] Henk Bruin. Planar embeddings of inverse limit spaces of unimodal maps. *Topology Appl.*, 96(3):191–208, 1999.

[10] C. E. Capel. Inverse limit spaces. *Duke Math. J.*, 21:233–245, 1954.

[11] Włodzimierz J. Charatonik and Robert P. Roe. Inverse limits of continua having trivial shape. *Houston J. Math.*, 38(4):1307–1312, 2012.

[12] Włodzimierz J. Charatonik and Robert P. Roe. Mappings between inverse limits of continua with multivalued bonding functions. *Topology Appl.*, 159(1):233–235, 2012.

[13] Włodzimierz J. Charatonik and Robert P. Roe. On Mahavier products. *Topology Appl.*, 166:92–97, 2014.

[14] Sina Greenwood and Judy Kennedy. Connected generalized inverse limits. *Topology Appl.*, 159(1):57–68, 2012.

[15] Sina Greenwood and Judy Kennedy. Generic generalized inverse limits. *Houston J. Math.*, 38(4):1369–1384, 2012.

[16] Sina Greenwood and Judy Kennedy. Connectedness and ingrammahavier products. *Topology and its Applications*, 166(0):1 – 9, 2014.

[17] Sina Greenwood and Michael Lockyer. Connected generalised inverse limits over Hausdorff continua. *Topology Appl.*, 160(3):513–523, 2013.

[18] Alejandro Illanes. A circle is not the generalized inverse limit of a subset of $[0,1]^2$. *Proc. Amer. Math. Soc.*, 139(8):2987–2993, 2011.

[19] W. T. Ingram. *Inverse limits*, volume 15 of *Aportaciones Matemáticas: Investigación [Mathematical Contributions: Research]*. Sociedad Matemática Mexicana, México, 2000.

[20] W. T. Ingram. Invariant sets and inverse limits. In *Proceedings of the First International Meeting on Continuum Theory (Puebla City, 2000)*, volume 126, pages 393–408, 2002.

[21] W. T. Ingram. Two-pass maps and indecomposability of inverse limits of graphs. *Topology Proc.*, 29(1):175–183, 2005. Spring Topology and Dynamical Systems Conference.

[22] W. T. Ingram. Inverse limits of upper semi-continuous functions that are unions of mappings. *Topology Proc.*, 34:17–26, 2009.

[23] W. T. Ingram. Inverse limits with upper semi-continuous bonding functions: problems and some partial solutions. *Topology Proc.*, 36:353–373, 2010.

[24] W. T. Ingram. Concerning nonconnected inverse limits with upper semi-continuous set-valued functions. *Topology Proc.*, 40:203–214, 2012.

[25] W. T. Ingram. *An introduction to inverse limits with set-valued functions*. Springer Briefs in Mathematics. Springer, New York, 2012.

[26] W. T. Ingram. Concerning chainability of inverse limits on $[0,1]$ with set-valued functions. *Topology Proc.*, 42:327–340, 2013.

[27] W. T. Ingram. Concerning dimension and tree-likeness of inverse limits with set-valued functions. *Houston J. Math.*, 40(2):621–631, 2014.

[28] W. T. Ingram. One-dimensional inverse limits with set-valued functions. *Topology Proc.*, 46:243–253, 2015.

[29] W. T. Ingram and William S. Mahavier. Inverse limits of upper semi-continuous set valued functions. *Houston J. Math.*, 32(1):119–130, 2006.

[30] W. T. Ingram and William S. Mahavier. *Inverse limits*, volume 25 of *Developments in Mathematics*. Springer, New York, 2012. From continua to chaos.

[31] W. T. Ingram and M. M. Marsh. Nonconnectedness of inverse limit sequences. *Topology Proc.*, 41:333–336, 2013.

[32] J. R. Isbell. Embeddings of inverse limits. *Ann. of Math. (2)*, 70:73–84, 1959.

[33] Leslie Braziel Jones. Adding machines and endpoints. *Topology Appl.*, 156(17):2899–2905, 2009.

[34] James P. Kelly. Chainability of inverse limits with a single irreducible function on. *Topology Appl.*, 176(0):57 – 75, 2014.

[35] James P. Kelly. Inverse limits with irreducible set-valued functions. *Topology Appl.*, 166(0):15–31, 2014.

[36] James P. Kelly and Jonathan Meddaugh. Indecomposability in inverse limits with set-valued functions. *Topology Appl.*, 160(13):1720–1731, 2013.

[37] James P. Kelly and Jonathan Meddaugh. Convergence of sequences of inverse limits. *Topology and its Applications*, 184(0):29 – 40, 2015.

[38] K. Kuratowski. *Topology. Vol. I.* New edition, revised and augmented. Translated from the French by J. Jaworowski. Academic Press, New York, 1966.

[39] K. Kuratowski. *Topology. Vol. II.* New edition, revised and augmented. Translated from the French by A. Kirkor. Academic Press, New York, 1968.

[40] William S. Mahavier. Inverse limits with subsets of $[0,1] \times [0,1]$. *Topology Appl.*, 141(1-3):225–231, 2004.

[41] M. M. Marsh. Some structure theorems for inverse limits with set-valued functions. *Topology Proc.*, 42:237–258, 2013.

[42] Piotr Minc and W. R. R. Transue. Accessible points of hereditarily decomposable chainable continua. *Trans. Amer. Math. Soc.*, 332(2):711–727, 1992.

[43] James R. Munkres. *Topology: a first course.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[44] Sam B. Nadler, Jr. *Continuum theory*, volume 158 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker Inc., New York, 1992. An introduction.

[45] Van Nall. Finite graphs that are inverse limits with a set valued function on $[0,1]$. *Topology Appl.*, 158(10):1226–1233, 2011.

[46] Van Nall. Connected inverse limits with a set-valued function. *Topology Proc.*, 40:167–177, 2012.

[47] Van Nall. The only finite graph that is an inverse limit with a set valued function on $[0,1]$ is an arc. *Topology Appl.*, 159(3):733–736, 2012.

[48] Douglas E. Scott. $\sqrt{2}$ as a sequence of rational numbers. *The Mathematics Teacher*, 68(2):164–165, 1975.

[49] Michel Smith and Scott Varagona. Generalized inverse limits with N-type bonding functions. *Houston J. Math*, to appear.

[50] Scott Varagona. Inverse limits with upper semi-continuous bonding functions and indecomposability. *Houston J. Math.*, 37(3):1017–1034, 2011.

[51] Scott Varagona. *Simple Techniques for Detecting Decomposability or Indecomposability of Generalized Inverse Limits*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–Auburn University.

[52] Scott Varagona. Homeomorphisms between inverse limits with N-shaped set-valued functions. *Topology Proc.*, 44:233–248, 2014.

[53] R. Patrick Vernon. Inverse limits of set-valued functions indexed by the integers. *Topology Appl.*, 171:35–40, 2014.

[54] William Thomas Watkins. Homeomorphic classification of certain inverse limit spaces with open bonding maps. *Pacific J. Math.*, 103(2):589–601, 1982.