ABSTRACT

Existence of Positive Solutions to Right Focal Three Point
Singular Boundary Value Problems

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In this dissertation, we prove the existence of positive solutions to two classes of three point right focal singular boundary value problems of at least fourth order.

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Singular Boundary Value Problems

by

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CHAPTER ONE

History and Introduction

The study of singular boundary value problems for ordinary differential equations is usually traced back to a 1989 paper by Gatika, Oliker, and Waltman [7]. This paper provided a fixed point theorem for operators that are decreasing with respect to a cone as well as an example of a singular boundary value problem for which the existence of a positive solution is proven by use of the fixed point theorem. The history may, however, be instead traced back to 1970 as Gatika, Oliker, and Waltman recognized in their 1989 paper to the study of differential inequalities. A portion of their fixed point theorem, and in fact the portion used in this dissertation, is acknowledged by Walter [15] with regard to a class of differential inequalities.

Traditionally, the existence of solutions to singular boundary value problems is proven with topological arguments, as with Taliaferro [14]. These arguments tend to involve *a priori* estimates on solutions alongside transversality theorems [11], super-linear and sub-linear conditions [12], or upper and lower solutions methods [16]. The adaption and result of Gatika, Oliker, and Waltman allows for additional arguments that may be applied to these problems.

The applications of singular boundary value problems are varied and include reaction-diffusion theory [2], boundary layer theory [3], semi-positone and positone problems [1], and in the study of non-Newtonian and pseudoplastic fluid theory [4]. For many applications, only positive solutions are meaningful, and this dissertation continues the tradition of focusing on positive solutions.

Justified by these various applications, the study of theoretical singular boundary value problems has expanded [5]. Much of the focus, however, has been on two point boundary conditions. That said, positive solutions for singular boundary value problems with three point boundary conditions have been explored somewhat: a nonlocal second order problem by Singh [13], a right focal third order problem by Maroun [10], a right focal fourth order problem Henderson [9], and a nonlocal second order problem on a time scale by DaCunha, Davis, and Singh [6]. This dissertation builds on the three point problems by studying two of the right focal fourth order problems.

We begin in Chapter Two by introducing some definitions and properties of cones and the fixed point theorem by Gatika, Oliker, and Waltman that is used in each subsequent chapter of this dissertation. In Chapter Three, we explore the first of two classes of fourth order problems. In Chapter Four, we extend this result to a class of fifth order problems that can be recasted into similar fourth order problems in order to determine a method of generalization which we apply in Chapter Five for a class of nth order problems. In Chapter Six, we explore the second class of fourth order problems, requiring slightly different tools. In Chapter Seven, we apply the method of generalization utilized in Chapter Five to the equations from Chapter Six to prove the existence of solutions to another class of nth order problems.

Our tools throughout will include the Green's functions and appropriate tent functions for our class of problems, *a priori* bounds on the norms of any potential positive solutions, and sequences of boundary value problems similar but inequivalent to our original problems. For problems of order higher than four, we make use of reducing them to fourth order integro-differential problems, proving the existence of solutions, and integrating the solutions as necessary to produce the desired solution to the original problems.

CHAPTER TWO

The Fixed Point Theorem

2.1 Definitions

In each chapter of this dissertation, we will seek fixed points of operators that are decreasing with respect to partial orderings induced by cones. To this end, we introduce a fixed point theorem due to Gatica, Oliker, and Waltman [7] that we will use in each chapter. We also provide give a few preliminary definitions for terminology used in the statement of the fixed point theorem.

Let $(\mathcal{B}, \|\cdot\|)$ be a real Banach space. A nonempty subset $\mathcal{K} \subset \mathcal{B}$ is a cone provided:

(a) $\mathcal{K}$ is closed;

(b) if $u, v \in \mathcal{K}$, then $\alpha u + \beta v \in \mathcal{K}$ for all $\alpha, \beta \geq 0$; and

(c) if both $u \in \mathcal{K}$ and $-u \in \mathcal{K}$, then u is the zero element of $\mathcal{B}$.

A cone, $\mathcal{K}$, induces a partial ordering, $\leq$, on $\mathcal{B}$ given by $x \leq y$ iff $y - x \in \mathcal{K}$. While this can create some confusion from the ambiguity of "$\leq$" with respect to the cone and with respect to the real numbers, the choices of Banach spaces and cones used in this dissertation lend themselves to a natural connection between the two uses of "$\leq$". For this reason, no alternate notation for the partial ordering induced by $\mathcal{K}$ is used.

A cone is *normal* in $\mathcal{B}$ provided there exists a $\delta > 0$ such that $\|e_1 + e_2\| \geq \delta$ for all $e_1, e_2 \in \mathcal{K}$ with $\|e_1\| = \|e_2\| = 1$. The *closed order interval* between x and y is defined as the set $< x, y >= \{z \in \mathcal{K} | x \leq z \leq y\}$.

2.2 The Gatika, Oliker, Waltman Fixed Point Theorem

Theorem 1 (Gatica, Oliker, Waltman). *Let $\mathcal{B}$ be a Banach space, $\mathcal{K}$ a normal cone in $\mathcal{B}$, D a subset of $\mathcal{K}$ such that $x, y \in D$ with $x \leq y$ implies $< x, y >\subset D$, and $T : D \to \mathcal{K}$ a continuous decreasing mapping which is compact on any closed order interval contained in D. Suppose also that there exists an $x_0 \in D$ such that $T^2x_0 = T(Tx_0)$ is defined and both Tx_0 and T^2x_0 are order comparable to x_0. Then T has a fixed point in D if any of the following hold:*

(a) $Tx_0 \leq x_0$ and $T^2x_0 \leq x_0$;

(b) $Tx_0 \geq x_0$ and $T^2x_0 \geq x_0$; or

(c) the complete sequence of iterates $\{T^n x_0\}_{n=0}^{\infty}$ is defined and there exists a $y_0 \in D$ such that $Ty_0 \in D$ and $y_0 \leq T^n x_0$ for every n.

CHAPTER THREE

The First Fourth Order Problem

3.1 Introduction

We would like to find a positive solution to the fourth order ordinary differential equation

$$y^{(4)} + f(x, y) = 0, 0 < x \leq 1 \tag{3.1}$$

that satisfies the three point right focal boundary conditions

$$y(0) = y'(p) = y''(p) = y'''(1) = 0 \tag{3.2}$$

where $0 < p < 1$ is fixed and $f(x, y)$ is singular at $x = 0$, $y = 0$, and possibly $y = \infty$.

We assume the following hold for $f(x, y)$:

(i) $f(x, y) : (0, 1] \times (0, \infty) \to (0, \infty)$ is continuous and decreasing in y for all $x \in (0, 1]$

(ii) $\lim_{y \to 0^+} f(x, y) = \infty$ and $\lim_{y \to +\infty} f(x, y) = 0$ uniformly on compact subsets of $(0, 1]$.

We start by defining a few things to prepare to use the fixed point theorem. First, we define our Banach space $\mathcal{B} := \{u : [0, 1] \to \mathbb{R} |$ u is continuous$\}$ with the max norm. Next, we define a cone $\mathcal{K} := \{u \in \mathcal{B} |\ u(x) \geq 0$ on $[0, 1]\}$, and note that this cone is normal in B. To see this, we let $e_1, e_2 \in K$ with $\|e_1\| = \|e_2\| = 1$. Since K is a cone, $(e_1 + e_2) \in K$ and so $(e_1 + e_2)(x) \geq 0$ for all $x \in [0, 1]$. Also, since $\|e_1\| = 1$, there exists an $x_1 \in [0, 1]$ such that $e_1(x_1) = 1$. Then, we know $(e_1 + e_2)(x_1) \geq 1$ and so $\|e_1 + e_2\| \geq 1$; hence, K is normal in B. We also define an important function, $g_1 : [0, 1] \to [0, \infty)$ by

$$g_1(x) := \frac{(x - p)^3 + p^3}{3p^2 - 3p + 1}$$

and a family of related functions for $\theta > 0$ on $[0, 1]$ by

$$g_\theta(x) := \theta \cdot g_1(x).$$

Let $D := \{\phi \in K \mid \exists\, \theta(\phi) > 0$ such that $\phi(x) \geq g_\theta(x)$ for all $x \in [0, 1]\}$, and note that this subset of our cone has the desired property regarding closed order intervals that the subset D in the statement of the fixed point theorem has. To see this, let $z_1, z_2 \in D$ with $z_1 < z_2$, and let $z_3 \in\ < z_1, z_2 > \subseteq K$. Then, $z_3(x) \geq z_1(x)$ for all $x \in [0, 1]$, and since $z_1 \in D$, there exists a $\theta_z > 0$ such that $z_1(x) \geq g_\theta(x)$ for all $x \in [0, 1]$. Combining these two statements yields $z_3(x) \geq z_1(x) \geq g_\theta(x)$ for all $x \in [0, 1]$. Hence, $z_3 \in D$; furthermore, $< z_1, z_2 > \subseteq D$ as desired.

We state the Green's function, $G : [0, 1] \times [0, 1] \to [0, \infty)$, for $-y^{(4)} = 0$ satisfying (3.2):

$$G(x,t) = \begin{cases} \frac{t^3}{6} & t \leq p \text{ and } t \leq x, \\ \frac{(x-p)^3 + p^3}{6} & t > p \text{ and } t > x, \\ \frac{(x-t)^3 + t^3}{6} & t \leq p \text{ and } t > x, \\ \frac{p^3 + (t-x)^3 + (x-p)^3}{6} & t > p \text{ and } t \leq x. \end{cases}$$

By direct calculation, we know that $G(x, t)$ is bounded by $\frac{1}{3}$ and positive when x and t are nonzero, and $G(x, t) = 0$ when $x = 0$ or $t = 0$.

Now, as a final assumption regarding $f(x, y)$, we assume

(iii) $\displaystyle\int_0^1 f(x, g_\theta(x))\, dx < \infty$ for all $\theta > 0$,

and we point out that the function $f(x, y) = \frac{1}{\sqrt[5]{xy}}$ satisfies all three of our assumptions on $f(x, y)$. We then define an integral operator, $T : D \to \mathcal{K}$ by

$$T\phi(x) = \int_0^1 G(x, t) f(t, \phi(t))\, dt,$$

and note that due to its domain being restricted to D and from assumption (iii), T is well-defined. Also, we note that T is a decreasing operator. To see this, let

$\phi_1, \phi_2 \in D$ with $\phi_1 < \phi_2$. Since ϕ_1 and ϕ_2 are continuous, there must exist a subinterval, $[a, b]$, of $(0, 1]$ on which $\phi_1(x) < \phi_2(x)$ for all $x \in [a, b]$. Then,

$$\begin{aligned}
(T\phi_1 - T\phi_2)(x) &= \int_0^1 G(x,t) f(t, \phi_1(t))\, dt - \int_0^1 G(x,t) f(t, \phi_2(t))\, dt \\
&= \int_0^1 G(x,t)[f(t, \phi_1(t)) - f(t, \phi_2(t))]\, dt \\
&\geq \int_a^b G(x,t)[f(t, \phi_1(t)) - f(t, \phi_2(t))]\, dt \\
&> 0.
\end{aligned}$$

The last inequality above is due to the nonnegativity of $G(x, t)$ and assumption (ii). The final line is a function whose output is zero when $x = 0$ and whose output is positive elsewhere due to the facts that $G(0, t) = 0$ for all $t \in [0, 1]$ and $G(x, t) > 0$ when both x and t are non-zero.

Now, we make note of two lemmas by Bo Yang [17] that gives insight into the nature of any solutions to (3.1),(3.2) that may exist.

Lemma 1. *Suppose* $u \in C^{(4)}[0, 1]$ *satisfies boundary conditions* (3.2) *and* $u^{(4)}(x) \leq 0$, *for* $0 \leq x \leq 1$. *Then* $u'(x) \geq 0$, *for* $0 \leq x \leq 1$, *and hence* $0 \leq u(x) \leq u(1)$, *for* $0 \leq x \leq 1$.

Lemma 2. *Suppose* $u \in C^{(4)}[0, 1]$ *satisfies boundary conditions* (3.2) *and* $u^{(4)}(x) \leq 0$, *for* $0 \leq x \leq 1$. *Then* $u(x) \geq g_1(x) u(1)$, *for* $0 \leq x \leq 1$.

We can apply these two lemmas to any positive solution, $y(x)$, of (3.1),(3.2). When we do, we see the following immediate results:

(a) $y(x)$ is nondecreasing;

(b) $\|y\| = y(1) > 0$; and

(c) if $\theta = y(1)$, then $g_\theta(x) \leq y(x)$ for $0 \leq x \leq 1$.

Also, using these lemmas, it can be shown that $y \in D$ is a solution of of (3.1),(3.2) if and only if $Ty = y$.

3.2 A Priori Bounds on Norms of Solutions

In order to use the fixed point theorem to prove the existence of a solution to (3.1),(3.2), we need to establish *a priori* bounds on the norms of any solutions that may exist. We establish these upper and lower bounds with proofs done by contradiction.

Lemma 3. *Suppose $f(x, y)$ satisfies assumptions $(i)-(iii)$, then there exists an $S > 0$ such that $\|\phi\| \leq S$ for any solution, $\phi \in D$ of* (3.1),(3.2).

Proof. To prove this, we assume for a contradiction that no such S exists. Hence, there must exist a sequence of solutions, $\{\phi_k\}_{k=1}^{\infty}$, of (3.1),(3.2) with $\phi_k(x) > 0$ on $x \in (0, 1]$ for all k, $\|\phi_{k+1}\| \geq \|\phi_k\|$ for all k, and $\lim_{k\to\infty} \|\phi_k\| = \infty$.

We define $M := \max\{G(x, t) \mid (x, t) \in [0, 1] \times [0, 1]\}$. Since $G(x, t)$ is bounded above by $\frac{1}{3}$, $M \leq \frac{1}{3}$ for all $0 < p < 1$. From our assumption (ii), there must exist a k_0 such that if $k \geq k_0$, then $f(t, \phi_k(t)) \leq \frac{1}{M(1-p)}$ for $t \in [p, 1]$. For notation convenience, let $\theta = \|\phi_{k_0}\|$. If $k \geq k_0$, from our earlier lemmas,

$$\phi_k(x) \geq g_1(x)\|\phi_k\| \geq g_1(x)\|\phi_{k_0}\| = g_\theta(x).$$

Then for $k \geq k_0$,

$$\begin{aligned}
\phi_k(x) &= T\phi_k(x) \\
&= \int_0^1 G(x, t) f(t, \phi_k(t))\, dt \\
&= \int_0^p G(x, t) f(t, \phi_k(t))\, dt + \int_p^1 G(x, t) f(t, \phi_k(t))\, dt \\
&\leq \int_0^p M \cdot f(t, g_\theta(t))\, dt + \int_p^1 M \cdot \frac{1}{M(1-p)}\, dt \\
&= M \cdot \int_0^p f(t, g_\theta(t))\, dt + 1
\end{aligned}$$

for all $x \in [0, 1]$. As a result of assumption (iii), this is a finite value, and it is a bound on $\|\phi_k\|$ for all k. This directly contradicts what we already know about the limit of the norms of the elements of $\{\phi_k\}_{k=1}^{\infty}$, so we are forced to conclude that such an upper bound, S, exists. $\square$

Lemma 4. *Suppose $f(x,y)$ satisfies assumptions $(i)-(iii)$, then there exists an $R>0$ such that $\|\phi\| \geq R$ for any solution, $\phi \in D$ of* (3.1),(3.2).

Proof. To prove this, we assume for a contradiction that no such R exists. Hence, there must exist a sequence of solutions, $\{\phi_k\}_{k=1}^{\infty}$, of (3.1),(3.2) with $\phi_k(x) > 0$ on $x \in (0,1]$ for all k, $\|\phi_{k+1}\| \leq \|\phi_k\|$ for all k, and $\lim\limits_{k\to\infty} \|\phi_k\| = 0$.

We define $m := \inf\{G(x,t) \mid (x,t) \in [p,1] \times [p,1]\} > 0$. From our assumption (ii), there must exist a $\delta > 0$ such that if $x \in [p,1]$ and $y \in (0,\delta)$, then $f(x,y) > \frac{1}{m(1-p)}$. There also must exist a k_0 such that if $k \geq k_0$, we have $0 < \phi_k(t) < \frac{\delta}{2}$ for all $t \in [p,1]$. So, for $k \geq k_0$ and $x \in [p,1]$,

$$\begin{aligned}
\phi_k(x) &= T\phi_k(x) \\
&= \int_0^1 G(x,t) f(t,\phi_k(t))\, dt \\
&\geq \int_p^1 G(x,t) f(t,\phi_k(t))\, dt \\
&\geq \int_p^1 G(x,t) f\left(t, \frac{\delta}{2}\right) dt \\
&\geq \int_p^1 m \cdot \frac{1}{m(1-p)}\, dt = 1.
\end{aligned}$$

This directly contradicts what we already know about the limit of the norms of the elements of $\{\phi_k\}_{k=1}^{\infty}$, so we are forced to conclude that such a lower bound, R, exists. $\square$

3.3 Existence Result

With the *a priori* bounds on the norms of solutions established, we proceed to the main existence result of this chapter.

Theorem 2. *If $f(x,y)$ satisfies assumptions $(i)-(iii)$, then* (3.1),(3.2) *has a positive solution, $\phi(x)$, in D.*

Proof. To prove this theorem, we define a few families of new functions and operators on which we apply the aforementioned fixed point theorem to gain a sequence of functions that converge to a solution of (3.1),(3.2). First, for all $c \in \mathbb{N}$, define the operator $\psi_c(x) := T(c)$ in the sense that T is applied to the function on $[0,1]$ whose constant output is c. So

$$\psi_c(x) = \int_0^1 G(x,t)f(t,c)\,dt.$$

Also, $0 < \psi_{c+1}(x) \leq \psi_c(x)$ for $x \in (0,1]$ for all c. Furthermore, we have $\lim_{c\to\infty} \psi_c(x) = 0$ uniformly on $[0,1]$ from our assumptions on $f(x,y)$.

Next, for each c, define $f_c : (0,1] \times [0,\infty) \to (0,\infty)$ by

$$f_c(x,y) := f(x, \max\{y, \psi_c(x)\}).$$

Note that each $f_c(x,y)$ is continuous and not singular at $y = 0$, and $f_c(x,y) \leq f(x,y)$ for $(x,y) \in (0,1] \times (0,\infty)$.

Next, for each c, define $T_c : \mathcal{K} \to \mathcal{K}$ by

$$T_c\phi(x) := \int_0^1 G(x,t)f_c(t,\phi(t))\,dt.$$

It is standard that each T_c is a compact mapping on $\mathcal{K}$. Moreover, $T_c(0) \geq 0$ and $T_c^2(0) \geq 0$ for each c. Thus, the fixed point theorem by Gatica, Oliker, and Waltman guarantees us that there exists a fixed point, $\phi_c \in \mathcal{K}$, for each T_c. Now, since $f_c(x,\phi_c(x)) = f(x, \max\{\phi_c(x), \psi_c(x)\}) \leq f(x, \psi_c(x))$, we have

$$\begin{aligned} T_c\phi_c(x) &= \int_0^1 G(x,t)f_c(t,\phi_c(t))\,dt \\ &\leq \int_0^1 G(x,t)f(t,\psi_c(t))\,dt \\ &= T\psi_c(x). \end{aligned}$$

This, along with the fact that each ϕ_c satisfies the boundary conditions (3.2), allows us to make similar arguments as were used to prove the previous two lemmas to show there exist $R, S > 0$ such that $R \leq \|\phi_c\| \leq S$ for all c.

So, $\{\phi_c\}_{c=1}^{\infty} \subseteq < g_R, S^* > \subseteq D$ since $g_R, S^* \in D$, where S^* is the function on $[0,1]$ whose constant output is S. Next, note that since T is a compact mapping, $\phi^* := \lim_{c\to\infty} T\phi_c$ exists.

Now, in order to prove that ϕ^* is our desired solution for (3.1),(3.2), we need to show $\lim_{c\to\infty}(T\phi_c(x) - \phi_c(x)) = 0$ so that we will have $\phi^* \in D$ and

$$T\phi^*(x) = T\left(\lim_{c\to\infty} T\phi_c(x)\right) = T\left(\lim_{c\to\infty} \phi_c(x)\right) = \lim_{c\to\infty} T\phi_c(x) = \phi^*(x).$$

If $\theta = R$, then $g_\theta(x) \le \phi_c(x)$ for $x \in [0,1]$ for all c. Let $\varepsilon > 0$ be given, and choose $\delta \in (0,1)$ such that $\int_0^\delta f(t, g_\theta(t))\, dt < \frac{\varepsilon}{2M}$. Then, there exists a c_0 such that for $c \ge c_0$ and $x \in [\delta, 1]$, $\psi_c(x) \le g_\theta(x) \le \phi_c(x)$. So, for $x \in [\delta, 1]$, $f_c(t, \phi_c(t)) = f(t, \phi_c(t))$. Then, for all $x \in [0,1]$,

$$\begin{aligned}
T\phi_c(x) - \phi_c(x) &= T\phi_c(x) - T_c\phi_c(x)\\
&= \int_0^1 G(x,t) f(t,\phi_c(t))\, dt - \int_0^1 G(x,t) f_c(t,\phi_c(t))\, dt\\
&= \int_0^\delta G(x,t) f(t,\phi_c(t))\, dt - \int_0^\delta G(x,t) f_c(t,\phi_c(t))\, dt,
\end{aligned}$$

and also for all $x \in [0,1]$,

$$\begin{aligned}
|T\phi_c(x) - \phi_c(x)| &\le \int_0^\delta G(x,t) f(t,\phi_c(t))\, dt + \int_0^\delta G(x,t) f_c(t,\phi_c(t))\, dt\\
&\le \int_0^\delta G(x,t) f(t,\phi_c(t))\, dt + \int_0^\delta G(x,t) f(t,\phi_c(t))\, dt\\
&\le 2M \int_0^\delta f(t,\phi_c(t))\, dt\\
&< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon.
\end{aligned}$$

Thus, ϕ^* is our desired solution for (3.1),(3.2). $\square$

CHAPTER FOUR

The First Fifth Order Problem

4.1 Introduction

We would now like to find a positive solution to a similar fifth order problem:

$$y^{(5)} + f(x, y) = 0, 0 < x \leq 1 \tag{4.1}$$

that satisfies the three point right focal boundary conditions

$$y(0) = y'(0) = y''(p) = y'''(p) = y^{(4)}(1) = 0 \tag{4.2}$$

where $0 < p < 1$ is fixed and $f(x, y)$ is singular at $x = 0$, $y = 0$, and possibly $y = \infty$.

We assume the following hold for $f(x, y)$:

(i) $f(x, y) : (0, 1] \times (0, \infty) \to (0, \infty)$ is continuous and decreasing in y for all $x \in (0, 1]$

(ii) $\lim_{y \to 0^+} f(x, y) = \infty$ and $\lim_{y \to +\infty} f(x, y) = 0$ uniformly on compact subsets of $(0, 1]$.

Our primary strategy for establishing the existence of a solution to (4.1),(4.2) is to make the substitution $u(x) = y'(x)$, apply similar techniques to the fourth order problem to solve the amended problem, then manipulate the acquired solution to solve (4.1),(4.2). To this end, we make the substitution $u(x) = y'(x)$ which recasts our BVP (4.1),(4.2) as

$$u^{(4)} + f\left(x, \int_0^x u(s)\, ds\right) = 0, 0 < x \leq 1 \tag{4.3}$$

$$u(0) = u'(p) = u''(p) = u'''(1) = 0 \tag{4.4}$$

Let $\mathcal{B} := \{u : [0,1] \to \mathbb{R} |$ u is continuous$\}$ with the max norm, and let the cone $\mathcal{K} := \{u \in \mathcal{B} | u(x) \geq 0$ on $[0,1]\}$. We now define an important function, $g_1 : [0,1] \to [0,\infty)$ by

$$g_1(x) := \frac{(x-p)^3 + p^3}{3p^2 - 3p + 1}$$

and a family of related functions for $\theta > 0$ on $[0,1]$ by

$$g_\theta(x) := \theta \cdot g_1(x).$$

Let $D := \{\phi \in K \mid \exists\, \theta(\phi) > 0$ such that $\phi(x) \geq g_\theta(x)$ for all $x \in [0,1]\}$. We state the Green's function, $G : [0,1] \times [0,1] \to [0,\infty)$, for $-y^{(4)} = 0$ satisfying (4.2):

$$G(x,t) = \begin{cases} \frac{t^3}{6} & t \leq p \text{ and } t \leq x, \\ \frac{(x-p)^3+p^3}{6} & t > p \text{ and } t > x, \\ \frac{(x-t)^3+t^3}{6} & t \leq p \text{ and } t > x, \\ \frac{p^3+(t-x)^3+(x-p)^3}{6} & t > p \text{ and } t \leq x. \end{cases}$$

By direct calculation, we know that $G(x,t)$ is bounded by $\frac{1}{3}$ and positive when x and t are nonzero, and $G(x,t) = 0$ when $x = 0$ or $t = 0$.

Now, as a final assumption regarding $f(x,y)$, we assume

(iii) $\displaystyle\int_0^1 f\left(x, \int_0^x g_\theta(s)\, ds\right) dx < \infty$ for all $\theta > 0$.

We point out that the function $f(x,y) = \frac{1}{\sqrt[6]{xy}}$ satisfies all three of our assumptions on $f(x,y)$. We then define an integral operator, $T : D \to \mathcal{K}$ by

$$T\phi(x) = \int_0^1 G(x,t) f\left(t, \int_0^t \phi(s)\, ds\right) dt.$$

It can be shown that T is a decreasing operator. Also, by restricting the domain of T to the subset D and due to assumption (iii), T is well-defined.

Now, we state with adjusted notation the two lemmas by Bo Yang [17] that give insight into the nature of any solutions to (4.3),(4.4) that may exist.

Lemma 5. *Suppose* $u \in C^{(4)}[0,1]$ *satisfies boundary conditions* (4.4) *and* $u^{(4)}(x) \leq 0$, *for* $0 \leq x \leq 1$. *Then* $u'(x) \geq 0$, *for* $0 \leq x \leq 1$, *and hence* $0 \leq u(x) \leq u(1)$, *for* $0 \leq x \leq 1$.

Lemma 6. *Suppose* $u \in C^{(4)}[0,1]$ *satisfies boundary conditions* (4.4) *and* $u^{(4)}(x) \leq 0$, *for* $0 \leq x \leq 1$. *Then* $u(x) \geq g_1(x)u(1)$, *for* $0 \leq x \leq 1$.

We can apply these two lemmas to any positive solution, $u(x)$, of (4.3),(4.4). When we do, we see the following immediate results:

(a) $u(x)$ is nondecreasing;

(b) $\|u\| = u(1) > 0$; and

(c) if $\theta = u(1)$, then $g_\theta(x) \leq u(x)$ for $0 \leq x \leq 1$.

Also, using these lemmas, it can be shown that $u \in D$ is a solution of of (4.3),(4.4) if and only if $Tu = u$.

4.2 A Priori Bounds on Norms of Solutions

In order to use the fixed point theorem to prove the existence of a solution to (4.3),(4.4), we need to establish *a priori* bounds on the norms of any solutions that may exist. We establish these upper and lower bounds with proofs done by contradiction.

Lemma 7. *Suppose* $f(x,y)$ *satisfies assumptions* $(i)-(iii)$, *then there exists an* $S > 0$ *such that* $\|u\| \leq S$ *for any solution,* $\phi \in D$ *of* (4.3),(4.4).

Proof. To prove this, we assume for a contradiction that no such S exists. Hence, there must exist a sequence of solutions, $\{\phi_k\}_{k=1}^{\infty}$, of (4.3),(4.4) with $\phi_k(x) > 0$ on $x \in (0,1]$ for all k, $\|\phi_{k+1}\| \geq \|\phi_k\|$ for all k, and $\lim_{k\to\infty} \|\phi_k\| = \infty$.

We define $M := \max\,\{G(x,t) \mid (x,t) \in [0,1] \times [0,1]\}$, and we recall that $M \leq \frac{1}{3}$ for all $0 < p < 1$. From our assumption (ii), there must exist a k_0 such that if $k \geq k_0$, then $f\left(t, \int_0^t \phi_k(s)\, ds\right) \leq \frac{1}{M(1-p)}$ for $t \in [p, 1]$. For notation convenience, let $\theta = \|\phi_{k_0}\|$. If $k \geq k_0$, then from our earlier lemmas,

$$\phi_k(x) \geq g_1(x)\|\phi_k\| \geq g_1(x)\|\phi_{k_0}\| = g_\theta(x).$$

Then,

$$\begin{aligned}
\phi_k(x) &= T\phi_k(x) \\
&= \int_0^1 G(x,t) f\left(t, \int_0^t \phi_k(s) ds\right) dt \\
&\leq \int_0^p M \cdot f\left(t, \int_0^t g_\theta(s) ds\right) dt + \int_p^1 M \cdot \frac{1}{M(1-p)}\, dt \\
&= M \cdot \int_0^p f\left(t, \int_0^t g_\theta(s) ds\right) dt + 1
\end{aligned}$$

for all $x \in [0, 1]$.

As a result of assumption (iii), this is a finite value, and it is a bound on $\|\phi_k\|$ for all k. This directly contradicts what we already know about the limit of the norms of the elements of $\{\phi_k\}_{k=1}^{\infty}$, so we are forced to conclude that such an upper bound, S, exists. $\square$

Lemma 8. *Suppose $f(x, y)$ satisfies assumptions $(i)-(iii)$, then there exists an $R > 0$ such that $\|u\| \geq R$ for any solution, $\phi \in D$ of* (4.3),(4.4).

Proof. To prove this, we assume for a contradiction that no such R exists. Hence, there must exist a sequence of solutions, $\{\phi_k\}_{k=1}^{\infty}$, of (4.3),(4.4) with $\phi_k(x) > 0$ on $x \in (0, 1]$ for all k, $\|\phi_{k+1}\| \leq \|\phi_k\|$ for all k, and $\lim_{k\to\infty} \|\phi_k\| = 0$.

We define $m := \inf \{G(x, t) \mid (x, t) \in [p, 1] \times [p, 1]\} > 0$. From our assumption (ii), there must exist a $\delta > 0$ such that if $x \in [p, 1]$ and $y \in (0, \delta)$, then $f(x, y) > \frac{1}{m(1-p)}$. There also must exist a k_0 such that if $k \geq k_0$, we have $0 < \phi_k(s) < \frac{\delta}{2}$ for all $s \in [p, 1]$. So for $x \in [p, 1]$,

$$\phi_k(x) = T\phi_k(x)$$

$$
\begin{aligned}
&= \int_0^1 G(x,t) f\left(t, \int_0^t \phi_k(s)ds\right) dt \\
&\geq \int_p^1 G(x,t) f\left(t, \int_0^t \phi_k(s)ds\right) dt \\
&\geq \int_p^1 G(x,t) f\left(t, \frac{\delta}{2}\right) dt \\
&\geq \int_p^1 m \cdot \frac{1}{m(1-p)}\, dt = 1.
\end{aligned}
$$

This directly contradicts what we already know about the limit of the norms of the elements of $\{\phi_k\}_{k=1}^{\infty}$, so we are forced to conclude that such a lower bound, R, exists. $\square$

4.3 Existence Result

With the *a priori* bounds on the norms of solutions established, we proceed to the main existence result of this chapter.

Theorem 3. *If $f(x,y)$ satisfies assumptions $(i)-(iii)$, then* (4.3),(4.4) *has a positive solution, $\phi(x)$, in D that gives rise to a solution, $y(x) = \displaystyle\int_0^x \phi(s)\, ds$, of* (4.1),(4.2).

Proof. To prove this theorem, we define a few families of new functions and operators on which we apply the aforementioned fixed point theorem to gain a sequence of functions that converge to the desired solution of (4.3),(4.4).

First, for all $c \in \mathbb{N}$, define the operator $\psi_c(x) := T(c)$ in the sense that T is applied to the function on $[0,1]$ whose constant output is c. So

$$\psi_c(x) = \int_0^1 G(x,t) f(t, ct)\, dt$$

Also, $0 < \psi_{c+1}(x) \leq \psi_c(x)$ for $x \in (0,1]$ for all c. Furthermore, we have $\lim\limits_{c\to\infty} \psi_c(x) = 0$ uniformly on $[0,1]$ from our assumptions on $f(x,y)$.

Next, for each c, define $f_c : (0,1] \times [0,\infty) \to (0,\infty)$ by

$$f_c(x,y) := f\left(x, \max\{y, \int_0^x \psi_c(s)\, ds\}\right)$$

Note that each $f_c(x,y)$ is continuous and not singular at $y=0$, and $f_c(x,y) \le f(x,y)$ for $(x,y) \in (0,1] \times (0,\infty)$. Hence for $\phi \in K$,

$$f_c\left(x, \int_0^x \phi_c(s)\,ds\right) \le f\left(x, \int_0^x \psi_c(s)\,ds\right).$$

Next, for each c, define $T_c : \mathcal{K} \to \mathcal{K}$ by

$$T_c\phi(x) := \int_0^1 G(x,t) f_c\left(t, \int_0^t \phi(s)\,ds\right) dt.$$

It is standard that each T_c is a compact mapping on $\mathcal{K}$. Moreover, $T_c(0) \ge 0$ and $T_c^2(0) \ge 0$ for each c. Thus, the fixed point theorem by Gatica, Oliker, and Waltman guarantees us that there exists a fixed point, $\phi_c \in \mathcal{K}$, for each T_c. Now,

$$\begin{aligned} T_c\phi_c(x) &= \int_0^1 G(x,t) f_c\left(t, \int_0^t \phi_c(s)\,ds\right) dt \\ &\le \int_0^1 G(x,t) f\left(t, \int_0^t \psi_c(s)\,ds\right) dt \\ &= T\psi_c(x). \end{aligned}$$

This, along with the fact that each ϕ_c satisfies the boundary conditions (4.4), allows us to make similar arguments as were used to prove the previous two lemmas to show there exist $R, S > 0$ such that $R \le \|\phi_c\| \le S$ for all c.

So, $\{\phi_c\}_{c=1}^{\infty} \subseteq < g_R, S^* > \subseteq D$ since $g_R, S^* \in D$, where S^* is the function on $[0,1]$ whose constant output is S. Then, since T is a compact mapping, $\phi^* := \lim\limits_{c\to\infty} T\phi_c$ exists.

Now, in order to prove that ϕ^* is our desired solution for (4.3),(4.4), we need to show $\lim\limits_{c\to\infty}(T\phi_c(x) - \phi_c(x)) = 0$ so that we will have $\phi^* \in D$ and

$$\begin{aligned} T\phi^*(x) &= T\left(\lim_{c\to\infty} T\phi_c(x)\right) = T\left(\lim_{c\to\infty} \phi_c(x)\right) \\ &= \lim_{c\to\infty} T\phi_c(x) = \phi^*(x). \end{aligned}$$

If $\theta = R$, then $g_\theta(x) \le \phi_c(x)$ for $x \in [0,1]$ for all c, so

$$\int_0^x g_\theta(s)\,ds \le \int_0^x \phi_c(s)\,ds \text{ for all } c.$$

Let $\varepsilon > 0$ be given, and choose $\delta \in (0,1)$ such that

$$\int_0^\delta f\left(t, \int_0^t g_\theta(s)\,ds\right) dt < \frac{\varepsilon}{2M}.$$

Then, there exists an c_0 such that for $c \geq c_0$ and $x \in [\delta, 1]$

$$\int_0^x \psi_c(s)\,ds \leq \int_0^x g_\theta(s)\,ds \leq \int_0^x \phi_c(s)\,ds.$$

So, for $x \in [\delta, 1]$,

$$f_c\left(t, \int_0^t \phi_c(s)\,ds\right) = f\left(t, \int_0^t \phi_c(s)\,ds\right)$$

Then, for $0 \leq x \leq 1$,

$$\begin{aligned} T\phi_c(x) - \phi_c(x) &= T\phi_c(x) - T_c\phi_c(x) \\ &= \int_0^1 G(x,t) f\left(t, \int_0^t \phi_c(s)ds\right) dt \\ &- \int_0^1 G(x,t) f_c\left(t, \int_0^t \phi_c(s)ds\right) dt \\ &= \int_0^\delta G(x,t) f\left(t, \int_0^t \phi_c(s)ds\right) dt \\ &- \int_0^\delta G(x,t) f_c\left(t, \int_0^t \phi_c(s)ds\right) dt, \end{aligned}$$

and for $0 \leq x \leq 1$,

$$\begin{aligned} |T\phi_c(x) - \phi_c(x)| &\leq \int_0^\delta G(x,t) f\left(t, \int_0^t \phi_c(s)ds\right) dt \\ &+ \int_0^\delta G(x,t) f_c\left(t, \int_0^t \phi_c(s)ds\right) dt \\ &\leq \int_0^\delta G(x,t) f\left(t, \int_0^t \phi_c(s)ds\right) dt \\ &+ \int_0^\delta G(x,t) f\left(t, \int_0^t \phi_c(s)ds\right) dt \\ &\leq 2M \int_0^\delta f\left(t, \int_0^t \phi_c(s)ds\right) dt \\ &< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon. \end{aligned}$$

Thus, ϕ^* is our desired solution for (4.3),(4.4). An application of the Fundamental Theorem of Calculus yields that $y(x) = \int_0^x \phi^*(s)\,ds$ is a positive solution of (4.1),(4.2). $\square$

CHAPTER FIVE

The First Nth Order Problem

5.1 Introduction

We would like next to extend our results for the fifth order problem and find a positive solution to the nth order ordinary differential equation

$$y^{(n)} + f(x, y) = 0, 0 < x \leq 1 \tag{5.1}$$

that satisfies the three point right focal boundary conditions

$$y(0) = \cdots = y^{(n-4)}(0) = y^{(n-3)}(p) = y^{(n-2)}(p) = y^{(n-1)}(1) = 0 \tag{5.2}$$

where $0 < p < 1$ is fixed and $f(x, y)$ is singular at $x = 0$, $y = 0$, and possibly $y = \infty$.

We assume the following hold for $f(x, y)$:

(i) $f(x, y) : (0, 1] \times (0, \infty) \to (0, \infty)$ is continuous and decreasing in y for all $x \in (0, 1]$

(ii) $\lim_{y \to 0^+} f(x, y) = \infty$ and $\lim_{y \to +\infty} f(x, y) = 0$ uniformly on compact subsets of $(0, 1]$.

Our primary strategy for establishing the existence of a solution to (5.1),(5.2) is to recast the problem as a fourth order BVP, find a solution to the recasted BVP, and then manipulate the solution to solve (5.1),(5.2). To this end, we make the substitution $u(x) = y^{(n-4)}(x)$ which recasts our BVP (5.1),(5.2) as

$$u^{(4)} + f\left(x, \int_0^x \frac{(x-s)^{n-5}}{(n-5)!} u(s)\, ds\right) = 0, 0 < x \leq 1 \tag{5.3}$$

$$u(0) = u'(p) = u''(p) = u'''(1) = 0 \tag{5.4}$$

Let $\mathcal{B} := \{u : [0,1] \to \mathbb{R}|$ u is continuous$\}$ with the max norm, and let the cone $\mathcal{K} := \{u \in \mathcal{B}|u(x) \geq 0$ on $[0,1]\}$. We now define an important function, $g_1 : [0,1] \to [0,\infty)$ by

$$g_1(x) := \frac{(x-p)^3 + p^3}{3p^2 - 3p + 1}$$

and three related functions for $\theta > 0$ on $[0,1]$:

$$\begin{aligned}
h_1(x) &:= \int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\, g_1(s)ds \\
g_\theta(x) &:= \theta \cdot g_1(x) \\
h_\theta(x) &:= \theta \cdot h_1(x) = \int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\, g_\theta(s)ds.
\end{aligned}$$

Let $D := \{\phi \in K \mid \exists\, \theta(\phi) > 0$ such that $\phi(x) \geq g_\theta(x)$ for all $x \in [0,1]\}$. We state the Green's function, $G : [0,1] \times [0,1] \to [0,\infty)$, for $-y^{(4)} = 0$ satisfying (5.4):

$$G(x,t) = \begin{cases} \frac{t^3}{6} & t \leq p \text{ and } t \leq x, \\ \frac{(x-p)^3+p^3}{6} & t > p \text{ and } t > x, \\ \frac{(x-t)^3+t^3}{6} & t \leq p \text{ and } t > x, \\ \frac{p^3+(t-x)^3+(x-p)^3}{6} & t > p \text{ and } t \leq x. \end{cases}$$

By direct calculation, we know that $G(x,t)$ is bounded by $\frac{1}{3}$ and positive when x and t are nonzero, and $G(x,t) = 0$ when $x = 0$ or $t = 0$.

Now, as a final assumption regarding $f(x,y)$, we assume

(iii) $\displaystyle\int_0^1 f(x, h_\theta(x))\, dx < \infty$ for all $\theta > 0$.

We point out that the function $f(x,y) = \frac{1}{\sqrt[n+1]{xy}}$ satisfies all three of our assumptions on $f(x,y)$. We then define an integral operator, $T : D \to \mathcal{K}$ by

$$T\phi(x) = \int_0^1 G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\phi(s)\, ds\right) dt.$$

It can be shown that T is a decreasing operator. Also, by restricting the domain of T to the subset D and due to assumption (iii), T is well-defined.

Next, we reiterate with adjusted notation the two lemmas by Bo Yang [17] that give insight into the nature of any solutions to (5.3),(5.4) that may exist.

Lemma 9. *Suppose* $u \in C^{(4)}[0,1]$ *satisfies boundary conditions* (5.4) *and* $u^{(4)}(x) \leq 0$, *for* $0 \leq x \leq 1$. *Then* $u'(x) \geq 0$, *for* $0 \leq x \leq 1$, *and hence* $0 \leq u(x) \leq u(1)$, *for* $0 \leq x \leq 1$.

Lemma 10. *Suppose* $u \in C^{(4)}[0,1]$ *satisfies boundary conditions* (5.4) *and* $u^{(4)}(x) \leq 0$, *for* $0 \leq x \leq 1$. *Then* $u(x) \geq g_1(x)u(1)$, *for* $0 \leq x \leq 1$.

We can apply these two lemmas to any positive solution, $u(x)$, of (5.3),(5.4). When we do, we see the following immediate results:

(a) $u(x)$ is nondecreasing;

(b) $\|u\| = u(1) > 0$; and

(c) if $\theta = u(1)$, then $g_\theta(x) \leq u(x)$ for $0 \leq x \leq 1$.

Also, using these lemmas, it can be shown that $u \in D$ is a solution of of (5.3),(5.4) if and only if $Tu = u$.

5.2 A Priori Bounds on Norms of Solutions

In order to use the fixed point theorem to prove the existence of a solution to (5.3),(5.4), we need to establish *a priori* bounds on the norms of any solutions that may exist. We establish these upper and lower bounds with proofs done by contradiction.

Lemma 11. *Suppose* $f(x, y)$ *satisfies assumptions* $(i) - (iii)$, *then there exists an* $S > 0$ *such that* $\|u\| \leq S$ *for any solution,* $\phi \in D$ *of* (5.3),(5.4).

Proof. To prove this, we assume for a contradiction that no such S exists. Hence, there must exist a sequence of solutions, $\{\phi_k\}_{k=1}^{\infty}$, of (5.3),(5.4) with $\phi_k(x) > 0$ on $x \in (0,1]$ for all k, $\|\phi_{k+1}\| \geq \|\phi_k\|$ for all k, and $\lim_{k\to\infty} \|\phi_k\| = \infty$.

We define $M := \max\{G(x,t) \mid (x,t) \in [0,1]\times[0,1]\}$, and we recall that $M \leq \frac{1}{3}$ for all $0 < p < 1$. From our assumption (ii), there must exist a k_0 such that if $k \geq k_0$, then $f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\phi_k(s)\,ds\right) \leq \frac{1}{M(1-p)}$ for $t \in [p,1]$. For notation convenience, let $\theta = \|\phi_{k_0}\|$. If $k \geq k_0$, then from our earlier lemmas,

$$\phi_k(x) \geq g_1(x)\|\phi_k\| \geq g_1(x)\|\phi_{k_0}\| = g_\theta(x).$$

Then,

$$\begin{aligned}
\phi_k(x) &= T\phi_k(x) \\
&= \int_0^1 G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\phi_k(s)\,ds\right) dt \\
&\leq \int_0^p M \cdot f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!} g_\theta(s)\,ds\right) dt + \int_p^1 M \cdot \frac{1}{M(1-p)}\,dt \\
&= M \cdot \int_0^p f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!} g_\theta(s)\,ds\right) dt + 1
\end{aligned}$$

for all $x \in [0,1]$.

As a result of assumption (iii), this is a finite value, and it is a bound on $\|\phi_k\|$ for all k. This directly contradicts what we already know about the limit of the norms of the elements of $\{\phi_k\}_{k=1}^{\infty}$, so we are forced to conclude that such an upper bound, S, exists. $\square$

Lemma 12. *Suppose $f(x,y)$ satisfies assumptions $(i)-(iii)$, then there exists an $R > 0$ such that $\|u\| \geq R$ for any solution, $\phi \in D$ of* (5.3),(5.4).

Proof. To prove this, we assume for a contradiction that no such R exists. Hence, there must exist a sequence of solutions, $\{\phi_k\}_{k=1}^{\infty}$, of (5.3),(5.4) with $\phi_k(x) > 0$ on $x \in (0,1]$ for all k, $\|\phi_{k+1}\| \leq \|\phi_k\|$ for all k, and $\lim_{k\to\infty} \|\phi_k\| = 0$.

We define $m := \inf\{G(x,t) \mid (x,t) \in [p,1]\times[p,1]\} > 0$. From our assumption (ii), there must exist a $\delta > 0$ such that if $x \in [p,1]$ and $y \in (0,\delta)$, then $f(x,y) > \frac{1}{m(1-p)}$. There also must exist a k_0 such that if $k \geq k_0$, we have $0 < \phi_k(s) < \frac{\delta}{2}$ for all

$s \in [p, 1]$. So for $x \in [p, 1]$,

$$
\begin{aligned}
\phi_k(x) &= T\phi_k(x) \\
&= \int_0^1 G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\phi_k(s)\, ds\right) dt \\
&\geq \int_p^1 G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\phi_k(s)\, ds\right) dt \\
&\geq \int_p^1 G(x,t) f\left(t, \frac{\delta}{2}\right) dt \\
&\geq \int_p^1 m \cdot \frac{1}{m(1-p)}\, dt = 1
\end{aligned}
$$

This directly contradicts what we already know about the limit of the norms of the elements of $\{\phi_k\}_{k=1}^{\infty}$, so we are forced to conclude that such a lower bound, R, exists. $\square$

5.3 Existence Result

With the *a priori* bounds on the norms of solutions established, we proceed to the main existence result of this chapter.

Theorem 4. *If $f(x,y)$ satisfies assumptions $(i)-(iii)$, then* (5.3),(5.4) *has a positive solution, $\phi(x)$, in D that gives rise to a solution, $y(x) = \displaystyle\int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\phi(s)\, ds$, of* (5.1),(5.2).

Proof. To prove this theorem, we define a few families of new functions and operators on which we apply the aforementioned fixed point theorem to gain a sequence of functions that converge to the desired solution of (5.3),(5.4).

First, for all $c \in \mathbb{N}$, define the operator $\psi_c(x) := T(c)$ in the sense that T is applied to the function on $[0, 1]$ whose constant output is c. So,

$$
\begin{aligned}
\psi_c(x) &= \int_0^1 G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\, c\, ds\right) dt \\
&= \int_0^1 G(x,t) f\left(t, \frac{ct^{n-4}}{(n-4)!}\right) dt.
\end{aligned}
$$

Also, $0 < \psi_{c+1}(x) \leq \psi_c(x)$ for $x \in (0,1]$ for all c. Furthermore, we have $\lim_{c\to\infty} \psi_c(x) = 0$ uniformly on $[0,1]$ from our assumptions on $f(x,y)$.

Next, for each c, define $f_c : (0,1] \times [0,\infty) \to (0,\infty)$ by

$$f_c(x,y) := f\left(x, \max\{y, \int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\,\psi_c(s)\,ds\}\right).$$

Note that each $f_c(x,y)$ is continuous and not singular at $y = 0$, and $f_c(x,y) \leq f(x,y)$ for $(x,y) \in (0,1] \times (0,\infty)$. Also note that for $\phi \in K$,

$$\begin{aligned}
&f_c\left(x, \int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right)\\
&\quad = f\left(x, \min\left\{\int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds, \int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\,\psi_c(s)\,ds\right\}\right)\\
&\quad \leq f\left(x, \int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\,\psi_c(s)\,ds\right).
\end{aligned}$$

Next, for each c, define $T_c : \mathcal{K} \to \mathcal{K}$ by

$$T_c\phi(x) := \int_0^1 G(x,t) f_c\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi(s)\,ds\right) dt.$$

It is standard that each T_c is a compact mapping on $\mathcal{K}$. Moreover, $T_c(0) \geq 0$ and $T_c^2(0) \geq 0$ for each c. Thus, the fixed point theorem by Gatica, Oliker, and Waltman guarantees us that there exists a fixed point, $\phi_c \in \mathcal{K}$, for each T_c. Now,

$$\begin{aligned}
T_c\phi_c(x) &= \int_0^1 G(x,t) f_c\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) dt\\
&\leq \int_0^1 G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\psi_c(s)\,ds\right) dt\\
&= T\psi_c(x).
\end{aligned}$$

This, along with the fact that each ϕ_c satisfies the boundary conditions (5.4), allows us to make similar arguments as were used to prove the previous two lemmas to show there exist $R, S > 0$ such that $R \leq \|\phi_c\| \leq S$ for all c.

So, $\{\phi_c\}_{c=1}^{\infty} \subseteq \, < g_R, S^* > \, \subseteq D$ since $g_R, S^* \in D$, where S^* is the function on $[0,1]$ whose constant output is S. Then, since T is a compact mapping, $\phi^* := \lim_{c\to\infty} T\phi_c$ exists.

Now, in order to prove that ϕ^* is our desired solution for (5.3),(5.4), we need to show $\lim_{c\to\infty}(T\phi_c(x) - \phi_c(x)) = 0$ so that we will have $\phi^* \in D$ and

$$\begin{aligned} T\phi^*(x) &= T\left(\lim_{c\to\infty} T\phi_c(x)\right) = T\left(\lim_{c\to\infty} \phi_c(x)\right) \\ &= \lim_{c\to\infty} T\phi_c(x) = \phi^*(x). \end{aligned}$$

If $\theta = R$, then $g_\theta(x) \leq \phi_c(x)$ for $x \in [0,1]$ for all c, so

$$h_\theta(x) \leq \int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds \text{ for all } c.$$

Let $\varepsilon > 0$ be given, and choose $\delta \in (0,1)$ such that

$$\int_0^\delta f(t, h_\theta(t))\,dt < \frac{\varepsilon}{2M}.$$

Then, there exists an c_0 such that for $c \geq c_0$ and $x \in [\delta, 1]$

$$\int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\,\psi_c(s)\,ds \leq h_\theta(x) \leq \int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds.$$

So, for $x \in [\delta, 1]$,

$$f_c\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) = f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right)$$

Then, for $0 \leq x \leq 1$,

$$\begin{aligned} T\phi_c(x) - \phi_c(x) &= T\phi_c(x) - T_c\phi_c(x) \\ &= \int_0^1 G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) dt \\ &\quad - \int_0^1 G(x,t) f_c\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) dt \\ &= \int_0^\delta G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) dt \\ &\quad - \int_0^\delta G(x,t) f_c\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) dt, \end{aligned}$$

and for $0 \leq x \leq 1$,

$$\begin{aligned}
|T\phi_c(x) - \phi_c(x)| &\leq \int_0^\delta G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) dt \\
&\quad + \int_0^\delta G(x,t) f_c\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) dt \\
&\leq \int_0^\delta G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) dt \\
&\quad + \int_0^\delta G(x,t) f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) dt \\
&\leq 2M \int_0^\delta f\left(t, \int_0^t \frac{(t-s)^{n-5}}{(n-5)!}\,\phi_c(s)\,ds\right) dt \\
&< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon.
\end{aligned}$$

Thus, ϕ^* is our desired solution for (5.3),(5.4).

Doing $n-4$ applications of the Fundamental Theorem of Calculus yields that $y(x) = \displaystyle\int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\,\phi^*(s)ds$ is a positive solution of (5.1),(5.2). $\square$

CHAPTER SIX

The Second Fourth Order Problem

6.1 Introduction

We now look at the remaining choice for local three point right focal boundary conditions. To that end, we would like to find a positive solution to the fourth order ordinary differential equation

$$y^{(4)} + f(x,y) = 0, 0 < x \leq 1 \tag{6.1}$$

that satisfies the three point right focal boundary conditions

$$y(0) = y'(p) = y''(1) = y'''(1) = 0 \tag{6.2}$$

where $1 - \frac{\sqrt{3}}{3} < p \leq 1$ is fixed and $f(x,y)$ is singular at $x = 0$, $y = 0$, and possibly $y = \infty$.

We assume the following hold for $f(x,y)$:

(i) $f(x,y) : (0,1] \times (0,\infty) \to (0,\infty)$ is continuous and decreasing in y for all $x \in (0,1]$

(ii) $\lim_{y\to 0^+} f(x,y) = \infty$ and $\lim_{y\to +\infty} f(x,y) = 0$ uniformly on compact subsets of $(0,1]$.

Let $\mathcal{B} := \{u : [0,1] \to \mathbb{R} |$ u is continuous$\}$ with the max norm, and let the cone $\mathcal{K} := \{u \in \mathcal{B} | u(x) \geq 0$ on $[0,1]\}$. We also define an important function, $a_1 : [0,1] \to [0,\infty)$ by

$$a_1(x) := \frac{3p(2-p)x - 3x^2 + x^3}{p^2(3-2p)}$$

and a family of related functions for $\theta > 0$ on $[0,1]$ by:

$$a_\theta(x) := \theta \cdot a_1(x).$$

Before continuing, we mention a few properties of a_θ when $p \leq 1$. First, it is a cubic function centered at $x = 1$. It passes through the points $(0,0)$ and $(p,1)$ and is concave downward on [0,1]. Hence, $a_\theta(1) > 0$ precisely when $p > 1 - \dfrac{\sqrt{3}}{3}$.

Let $D := \{\phi \in K \mid \exists\ \theta(\phi) > 0 \text{ such that } \phi(x) \geq a_\theta(x) \text{ for all } x \in [0,1]\}$, and note that this subset of our cone has the desired property regarding closed order intervals that the subset D in the statement of the fixed point theorem has. To see this, let $z_1, z_2 \in D$ with $z_1 < z_2$, and let $z_3 \in \, < z_1, z_2 > \, \subseteq K$. Then, $z_3(x) \geq z_1(x)$ for all $x \in [0,1]$, and since $z_1 \in D$, there exists a $\theta_z > 0$ such that $z_1(x) \geq a_\theta(x)$ for all $x \in [0,1]$. Combining these two statements yields $z_3(x) \geq z_1(x) \geq a_\theta(x)$ for all $x \in [0,1]$. Hence, $z_3 \in D$; furthermore, $< z_1, z_2 > \, \subseteq D$ as desired.

We state the Green's function, $G : [0,1] \times [0,1] \to [0,\infty)$, for $-y^{(4)} = 0$ satisfying (3.2):

$$G(x,s) = -x\left(\frac{p^2}{2} - ps - \frac{(p-s)^2}{2}H(p-s)\right) - \frac{x^2 s}{2} + \frac{x^3}{6} - \frac{(x-s)^3}{6}H(x-s),$$

where $H(t) = 1$ if $t \geq 0$ and $H(t) = 0$ otherwise.

It is worth noting a few things concerning our Green's function, $G(x,s)$. First, it is continuous. Also, $G(p,0) = 0$ and $G(p,s) > 0$ for all $0 < s \leq 1$. A few direct calculations yield that $G(x,s) \leq G(p,s)$ for all $0 \leq x, s \leq 1$, so $G(x,s) \leq \frac{1}{6}$ for all $x, s \in [0,1]$. Also, $G(x,s) \geq a(x)G(p,s)$ for all $0 \leq x, s \leq 1$, so $G(x,s) > 0$ unless $x = 0$ or $s = 0$. Hence our Green's function is non-negative and bounded by $\frac{1}{6}$.

Now, as a final assumption regarding $f(x,y)$, we assume

(iii) $\displaystyle\int_0^1 f(s, a_\theta(s))\,ds < \infty$ for all $\theta > 0$.

We next define an integral operator, $T : D \to \mathcal{K}$ by

$$T\phi(x) = \int_0^1 G(x,s) f(s,\phi(s))\,ds$$

and note that due to assumption (iii) in conjunction with our restriction of the domain of T to the subset D, T is a well-defined operator.

In order to fulfill the requirements for the fixed point theorem by Gatica, Oliker, and Waltman we intend to use, we now show that T is a decreasing operator. To that end, let $z_1, z_2 \in D$ such that $z_1 > z_2$ with regards to the partial ordering induced by the cone, K. Hence, for all $x \in [0,1]$, $z_1(x) - z_2(x) \geq 0$ and

$$\begin{aligned}
(Tz_2 - Tz_1)(x) &= (Tz_2)(x) - (Tz_1)(x) \\
&= \int_0^1 G(x,s))f(s,z_2(s))\,ds - \int_0^1 G(x,s)f(s,z_1(s))\,ds \\
&= \int_0^1 G(x,s)[f(s,z_2(s)) - f(s,z_1(s))]\,ds > 0.
\end{aligned}$$

In the final inequality above, the nonnegativity of $(Tz_2 - Tz_1)(x)$ follows from the decreasing property of $f(x,y)$ in its second argument. However, in order to argue $(Tz_2 - Tz_1)(x) \not\equiv 0$ we invoke the continuity of $f(x,y)$ with respect to y and assumption (ii) to ensure the existence of a subinterval of $[0,1]$ on which $z_1(s) > z_2(s)$ and hence $f(s,z_2(s)) - f(s,z_1(s)) > 0$ for all s in that subinterval. As a result, we have $(Tz_2 - Tz_1)(x) > 0$ for all $x \in (0,1]$, and so T is a decreasing operator.

Next, we make note of a lemma that follows directly from two results by John Graef [8] that give insight into the nature of any solutions to (6.1),(6.2) that may exist. It should be noted now that these lemmas are attributed to Graef since they are inspired by his results, are proven identically to his results, and have assumptions that parallel his results, though the assumptions in the following lemmas are slightly stronger to fit the singular case of (6.1),(6.2) on which we are focused.

Lemma 13. *Suppose* $1 - \dfrac{\sqrt{3}}{3} < p \leq 1$ *is fixed and that assumptions* $(i) - (iii)$ *hold. If* $y \in C^4(0,1] \bigcap C^3[0,1]$ *satisfies the boundary conditions* (6.2) *and* $y^{(4)}(x) \leq 0$ *for all* $x \in (0,1]$*, then*

(a) $\|y\| = y(p)$*;*

(b) $y(x) \geq 0$*; and*

(c) $y(x) \geq a_1(x)y(p)$.

We can apply this lemma to any positive solution, $y(x)$, of (6.1),(6.2). When we do, a direct consequence of this lemma is that if $\theta = y(p)$, then $a_\theta(x) \leq y(x)$ for all $x \in [0,1]$. As a result, $y \in D$. Also, $y(x)$ is a solution of (6.1),(6.2) if and only if $Ty = y$.

6.2 A Priori Bounds on Norms of Solutions

In order to use the fixed point theorem to prove the existence of a solution to (6.1),(6.2), we need to establish *a priori* bounds on the norms of any solutions that may exist. We establish these upper and lower bounds with proofs done by contradiction.

Lemma 14. *Suppose* $1 - \dfrac{\sqrt{3}}{3} < p \leq 1$ *is fixed and that assumptions* $(i) - (iii)$ *hold. Then there exists an* $S > 0$ *such that* $\|\phi\| \leq S$ *for any solution,* $\phi \in D$ *of* (6.1),(6.2).

Proof. To prove this, we assume for a contradiction that no such S exists. Hence, there must exist a sequence of solutions, $\{\phi_k\}_{k=1}^{\infty}$, of (6.1),(6.2) with $\phi_k(x) > 0$ on $x \in (0,1]$ for all k, $\|\phi_{k+1}\| \geq \|\phi_k\|$ for all k, and $\lim_{k\to\infty} \|\phi_k\| = \infty$.

We define $M := \max\{G(x,t) \mid (x,t) \in [0,1]\times[0,1]\}$. Since $G(x,t)$ is bounded above by $\frac{1}{6}$, we know $M \leq \frac{1}{6}$ for all $1 - \frac{\sqrt{3}}{3} < p \leq 1$. Then, from assumption (ii), there must exist a k_0 such that if $k \geq k_0$, then $f(s,\phi_k(s)) \leq \frac{1}{M(1-p)}$ for all $s \in [p,1]$. For notation convenience, let $\theta = \|\phi_{k_0}\|$. If $k \geq k_0$, then from our earlier lemmas, $\phi_k(x) \geq a_1(x)\|\phi_k\| \geq a_1(x)\|\phi_{k_0}\| = a_\theta(x)$. Then, for all $x \in [0,1]$,

$$\begin{aligned}
\phi_k(x) &= T\phi_k(x) \\
&= \int_0^1 G(x,s) f(s,\phi_k(s))\, ds \\
&= \int_0^p G(x,s) f(s,\phi_k(s))\, ds + \int_p^1 G(x,s) f(s,\phi_k(s))\, ds \\
&\leq \int_0^p M \cdot f(s,a_\theta(s))\, ds + \int_p^1 M \cdot \frac{1}{M(1-p)}\, ds \\
&= M \cdot \int_0^p f(s,a_\theta(s))\, ds + 1.
\end{aligned}$$

As a result of assumption (iii), this is a finite value, and it is a bound on $\|\phi_k\|$ for all k. This directly contradicts what we already know about the limit of the norms of the elements of $\{\phi_k\}_{k=1}^{\infty}$, so we are forced to conclude that such an upper bound, S, exists. $\square$

Lemma 15. *Suppose* $1 - \dfrac{\sqrt{3}}{3} < p \leq 1$ *is fixed and that assumptions* $(i) - (iii)$ *hold. Then there exists an* $R > 0$ *such that* $\|\phi\| \geq R$ *for any solution,* $\phi \in D$ *of* (6.1),(6.2).

Proof. To prove this, we assume for a contradiction that no such R exists. Hence, there must exist a sequence of solutions, $\{\phi_k\}_{k=1}^{\infty}$, of (6.1),(6.2) with $\phi_k(x) > 0$ on $x \in (0,1]$ for all k, $\|\phi_{k+1}\| \leq \|\phi_k\|$ for all k, and $\lim\limits_{k\to\infty} \|\phi_k\| = 0$.

We define $m := \inf\{G(x,s) \mid (x,s) \in [p,1]\times[p,1]\} > 0$. From assumption (ii), there must exist a $\delta > 0$ such that if $x \in [p,1]$ and $y \in (0,\delta)$, then $f(x,y) > \frac{1}{m(1-p)}$. There also must exist a k_0 such that if $k \geq k_0$, we have $0 < \phi_k(s) < \frac{\delta}{2}$ for all $s \in [p,1]$. So, for $k \geq k_0$ and $x \in [p,1]$,

$$\begin{aligned}
\phi_k(x) &= T\phi_k(x) \\
&= \int_0^1 G(x,s) f(s,\phi_k(s))\, ds \\
&\geq \int_p^1 G(x,s) f(s,\phi_k(s))\, ds \\
&\geq \int_p^1 G(x,s) f\left(s, \frac{\delta}{2}\right) ds \\
&\geq \int_p^1 m \cdot \frac{1}{m(1-p)}\, ds \\
&= 1.
\end{aligned}$$

This directly contradicts what we already know about the limit of the norms of the elements of $\{\phi_k\}_{k=1}^{\infty}$, so we are forced to conclude that such a lower bound, R, exists. $\square$

6.3 Existence Result

With the *a priori* bounds on the norms of solutions established, we proceed to the main existence result of this chapter.

Theorem 5. *If $f(x, y)$ satisfies assumptions $(i) - (iii)$, then* (6.1),(6.2) *has a positive solution, $\phi(x)$, in D.*

Proof. To prove this theorem, we define a few families of new functions and operators on which we apply the aforementioned fixed point theorem to gain a sequence of functions that converge to a solution of (6.1),(6.2).

First, for all $c \in \mathbb{N}$, define the operator $\psi_c(x) := T(c)$ in the sense that T is applied to the function on $[0, 1]$ whose constant output is c. So,

$$\psi_c(x) = \int_0^1 G(x, s) f(s, c)\, ds.$$

Also, $0 < \psi_{c+1}(x) \leq \psi_c(x)$ for $x \in (0, 1]$ for all c. Furthermore, we have that $\lim_{c\to\infty} \psi_c(x) = 0$ uniformly on $[0, 1]$ from our assumptions on $f(x, y)$ and the fact that $1 - \frac{\sqrt{3}}{3} < p \leq 1$.

Next, for each c, define $f_c : (0, 1] \times [0, \infty) \to (0, \infty)$ by

$$f_c(x, y) := f(x, \max\{y, \psi_c(x)\}).$$

Note that each $f_c(x, y)$ is continuous and not singular at $y = 0$, and $f_c(x, y) \leq f(x, y)$ for $(x, y) \in (0, 1] \times (0, \infty)$.

Next, for each c, define $T_c : \mathcal{K} \to \mathcal{K}$ by

$$T_c\phi(x) := \int_0^1 G(x, s) f_c(s, \phi(s))\, ds.$$

It is standard that each T_c is a compact mapping on $\mathcal{K}$. Moreover, $T_c(0) \geq 0$ and $T_c^2(0) \geq 0$ for each c. Thus, the fixed point theorem by Gatica, Oliker, and Waltman guarantees us that there exists a fixed point, $\phi_c \in \mathcal{K}$, for each T_c. Now, since $f_c(x, \phi_c(x)) = f(x, \max\{\phi_c(x), \psi_c(x)\}) \leq f(x, \psi_c(x))$, we have

$$
\begin{aligned}
T_c\phi_c(x) &= \int_0^1 G(x,s) f_c(s,\phi_c(s))\, ds \\
&\le \int_0^1 G(x,s) f(s,\psi_c(s))\, ds \\
&= T\psi_c(x).
\end{aligned}
$$

This, along with the fact that each ϕ_c satisfies the boundary conditions (6.2), allows us to make similar arguments as were used to prove the previous two lemmas to show there exist $R, S > 0$ such that $R \le \|\phi_c\| \le S$ for all c.

So, $\{\phi_c\}_{c=1}^{\infty} \subseteq < a_R, S^* > \subseteq D$ since $a_R, S^* \in D$, where S^* is the function on $[0,1]$ whose constant output is S. Next, note that since T is a compact mapping, $\phi^* := \lim\limits_{c\to\infty} T\phi_c$ exists.

Now, in order to prove that ϕ^* is our desired solution for (6.1),(6.2), we need to show $\lim\limits_{c\to\infty}(T\phi_c(x) - \phi_c(x)) = 0$ so that we will have $\phi^* \in D$ and

$$
\begin{aligned}
T\phi^*(x) = T\left(\lim_{c\to\infty} T\phi_c(x)\right) &= T\left(\lim_{c\to\infty} \phi_c(x)\right) \\
&= \lim_{c\to\infty} T\phi_c(x) = \phi^*(x).
\end{aligned}
$$

If $\theta = R$, then $a_\theta(x) \le \phi_c(x)$ for $x \in [0,1]$ for all c. Let $\varepsilon > 0$ be given, and choose $\delta \in (0,1)$ such that $\int_0^\delta f(s, a_\theta(s))\, ds < \frac{\varepsilon}{2M}$. Then, there exists a c_0 such that for $c \ge c_0$ and $x \in [\delta, 1]$, $\psi_c(x) \le a_\theta(x) \le \phi_c(x)$. So, for $x \in [\delta, 1]$, $f_c(s, \phi_c(s)) = f(s, \phi_c(s))$. Then, for all $x \in [0,1]$,

$$
\begin{aligned}
T\phi_c(x) - \phi_c(x) &= T\phi_c(x) - T_c\phi_c(x) \\
&= \int_0^1 G(x,s) f(s,\phi_c(s))\, ds \\
&\quad - \int_0^1 G(x,s) f_c(s,\phi_c(s))\, ds \\
&= \int_0^\delta G(x,s) f(s,\phi_c(s))\, ds \\
&\quad - \int_0^\delta G(x,s) f_c(s,\phi_c(t))\, ds,
\end{aligned}
$$

and also for all $x \in [0,1]$,

$$
\begin{aligned}
|T\phi_c(x) - \phi_c(x)| &\leq \int_0^\delta G(x,s) f(s,\phi_c(s))\, ds \\
&\quad + \int_0^\delta G(x,s) f_c(s,\phi_c(s))\, ds \\
&\leq \int_0^\delta G(x,s) f(s,\phi_c(s))\, ds \\
&\quad + \int_0^\delta G(x,s) f(s,\phi_c(s))\, ds \\
&\leq 2M \int_0^\delta f(s,\phi_c(s))\, ds \\
&\leq 2M \int_0^\delta f(s,a_\theta(s))\, ds \\
&< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon.
\end{aligned}
$$

Thus, ϕ^* is our desired solution for (6.1),(6.2). $\square$

CHAPTER SEVEN

The Second Nth Order Problem

7.1 Introduction

We end by extending the result for the BVP from Chapter 6 to the nth order case using the same techniques we used in Chapter 5. So, we would like to find a positive solution to the nth order ordinary differential equation

$$y^{(n)} + f(x,y) = 0, 0 < x \leq 1 \tag{7.1}$$

that satisfies the three point right focal boundary conditions

$$y(0) = \cdots = y^{(n-4)}(0) = y^{(n-3)}(p) = y^{(n-2)}(1) = y^{(n-1)}(1) = 0 \tag{7.2}$$

where $1 - \frac{\sqrt{3}}{3} < p \leq 1$ is fixed and $f(x,y)$ is singular at $x = 0$, $y = 0$, and possibly $y = \infty$.

We assume the following hold for $f(x,y)$:

(i) $f(x,y) : (0,1] \times (0,\infty) \to (0,\infty)$ is continuous and decreasing in y for all $x \in (0,1]$

(ii) $\lim_{y \to 0^+} f(x,y) = \infty$ and $\lim_{y \to +\infty} f(x,y) = 0$ uniformly on compact subsets of $(0,1]$.

Our primary strategy for establishing the existence of a solution to (7.1),(7.2) is to recast the problem as a fourth order BVP, find a solution to the recasted BVP, and then manipulate the solution to solve (7.1),(7.2). To this end, we make the substitution $u(x) = y^{(n-4)}(x)$ which recasts our BVP (7.1),(7.2) as

$$u^{(4)} + f\left(x, \int_0^x \frac{(x-s)^{n-5}}{(n-5)!} u(s)\, ds\right) = 0, 0 < x \leq 1 \tag{7.3}$$

$$u(0) = u'(p) = u''(1) = u'''(1) = 0. \tag{7.4}$$

Let $\mathcal{B} := \{u : [0,1] \to \mathbb{R} |$ u is continuous$\}$ with the max norm, and let the cone $\mathcal{K} := \{u \in \mathcal{B} | u(x) \geq 0$ on $[0,1]\}$. We also define an important function, $a_1 : [0,1] \to [0,\infty)$ by

$$a_1(x) := \frac{3p(2-p)x - 3x^2 + x^3}{p^2(3-2p)}$$

and two families of related functions for $\theta > 0$ on $[0,1]$:

$$a_\theta(x) := \theta \cdot a_1(x).$$
$$h_\theta(x) := \int_0^x \frac{(x-s)^{n-5}}{(n-5)!} a_\theta(s)ds = \theta \cdot h_1(x).$$

Let $D := \{\phi \in K \mid \exists\, \theta(\phi) > 0$ such that $\phi(x) \geq a_\theta(x)$ for all $x \in [0,1]\}$. We state the Green's function, $G : [0,1] \times [0,1] \to [0,\infty)$, for $-y^{(4)} = 0$ satisfying (6.2):

$$G(x,s) = -x\left(\frac{p^2}{2} - ps - \frac{(p-s)^2}{2}H(p-s)\right) - \frac{x^2 s}{2} + \frac{x^3}{6} - \frac{(x-s)^3}{6}H(x-s),$$

where $H(t) = 1$ if $t \geq 0$ and $H(t) = 0$ otherwise.

Now, as a final assumption regarding $f(x,y)$, we assume

(iii) $\int_0^1 f(s, h_\theta(s))\, ds < \infty$ for all $\theta > 0$.

We next define an integral operator, $T : D \to \mathcal{K}$ by

$$T\phi(x) = \int_0^1 G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\phi(r)\, dr\right) ds$$

and note that due to assumption (iii) in conjunction with our restriction of the domain of T to the subset D, T is a well-defined operator.

In order to fulfill the requirements for the fixed point theorem by Gatica, Oliker, and Waltman we intend to use, we now show that T is a decreasing operator. To that end, let $z_1, z_2 \in D$ such that $z_1 > z_2$ with regards to the partial ordering induced by the cone, K. Hence, $z_1(x) - z_2(x) \geq 0$ for all $x \in [0,1]$. Then for all $x \in [0,1]$,

$$(Tz_2 - Tz_1)(x) = (Tz_2)(x) - (Tz_1)(x)$$

$$
\begin{aligned}
&= \int_0^1 G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\, z_2(r)\, dr\right) ds \\
&- \int_0^1 G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\, z_1(r)\, dr\right) ds \\
&= \int_0^1 G(x,s) \left[f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\, z_2(r)\, dr\right)\right. \\
&\qquad\qquad \left. - f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\, z_1(r)\, dr\right)\right] ds > 0.
\end{aligned}
$$

In the final inequality above, the nonnegativity of $(Tz_2 - Tz_1)(x)$ follows from the decreasing property of $f(x,y)$ in its second argument. However, in order to argue $(Tz_2 - Tz_1)(x) \not\equiv 0$ we invoke the continuity of $f(x,y)$ with respect to y and assumption (ii) to ensure the existence of a subinterval of $[0,1]$ on which $z_1(s) > z_2(s)$ and hence

$$
f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\, z_2(r)\, dr\right) - f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\, z_1(r)\, dr\right) > 0
$$

for all s in that subinterval. As a result, we have $(Tz_2 - Tz_1)(x) > 0$ for all $x \in (0,1]$, and so T is a decreasing operator.

Now, we state with adjusted notation the lemma we attributed to John Graef [8] that gives insight into the nature of any solutions to (7.3),(7.4) that may exist.

Lemma 16. *Suppose* $1 - \dfrac{\sqrt{3}}{3} < p \leq 1$ *is fixed and that assumptions* $(i) - (iii)$ *hold. If* $u \in C^4(0,1] \bigcap C^3[0,1]$ *satisfies the boundary conditions* (7.4) *and* $u^{(4)}(x) \leq 0$ *for all* $x \in (0,1]$*, then*

(a) $\|u\| = u(p)$*;*

(b) $u(x) \geq 0$*; and*

(c) $u(x) \geq a_1(x) u(p)$

We can apply this lemma to any positive solution, $u(x)$, of (7.3),(7.4). When we do, a direct consequence of this lemma is that if $\theta = u(p)$, then $a_\theta(x) \leq u(x)$ for all $x \in [0,1]$. As a result, $u \in D$. Also, $u(x)$ is a solution of (7.3),(7.4) if and only if $Tu = u$.

7.2 A Priori Bounds on Norms of Solutions

In order to use the fixed point theorem to prove the existence of a solution to (6.1),(6.2), we need to establish *a priori* bounds on the norms of any solutions that may exist. We establish these upper and lower bounds with proofs done by contradiction.

Lemma 17. *Suppose* $1 - \frac{\sqrt{3}}{3} < p \leq 1$ *is fixed and that assumptions* $(i) - (iii)$ *hold. Then there exists an* $S > 0$ *such that* $\|\phi\| \leq S$ *for any solution,* $\phi \in D$ *of* (6.1),(6.2).

Proof. To prove this, we assume for a contradiction that no such S exists. Hence, there must exist a sequence of solutions, $\{\phi_k\}_{k=1}^{\infty}$, of (6.1),(6.2) with $\phi_k(x) > 0$ on $x \in (0, 1]$ for all k, $\|\phi_{k+1}\| \geq \|\phi_k\|$ for all k, and $\lim_{k\to\infty} \|\phi_k\| = \infty$.

We define $M := \max\{G(x, s) \mid (x, s) \in [0, 1] \times [0, 1]\}$, and we recall that $M \leq \frac{1}{6}$ for all $1 - \frac{\sqrt{3}}{3} < p \leq 1$. Then, from assumption (ii), there must exist a k_0 such that if $k \geq k_0$, then

$$f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_k(r)\,dr\right) \leq \frac{1}{M(1-p)}$$

for all $s \in [p, 1]$. For notation convenience, let $\theta = \|\phi_{k_0}\|$. If $k \geq k_0$, then from our earlier lemmas, $\phi_k(x) \geq a_1(x)\|\phi_k\| \geq a_1(x)\|\phi_{k_0}\| = a_\theta(x)$. Then,

$$\begin{aligned}
\phi_k(x) &= T\phi_k(x) \\
&= \int_0^1 G(x, s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_k(r)\,dr\right) ds \\
&\leq \int_0^p M \cdot f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,a_\theta(r)\,dr\right) ds + \int_p^1 M \cdot \frac{1}{M(1-p)}\,ds \\
&= M \cdot \int_0^p f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,a_\theta(r)\,dr\right) ds + 1
\end{aligned}$$

for all $x \in [0, 1]$.

As a result of assumption (iii), this is a finite value, and it is a bound on $\|\phi_k\|$ for all k. This directly contradicts what we already know about the limit of the norms of the elements of $\{\phi_k\}_{k=1}^{\infty}$, so we are forced to conclude that such an upper bound, S, exists. $\square$

Lemma 18. *Suppose* $1 - \frac{\sqrt{3}}{3} < p \leq 1$ *is fixed and that assumptions* $(i) - (iii)$ *hold. Then there exists an* $R > 0$ *such that* $\|\phi\| \geq R$ *for any solution,* $\phi \in D$ *of* (6.1),(6.2).

Proof. To prove this, we assume for a contradiction that no such R exists. Hence, there must exist a sequence of solutions, $\{\phi_k\}_{k=1}^{\infty}$, of (6.1),(6.2) with $\phi_k(x) > 0$ on $x \in (0,1]$ for all k, $\|\phi_{k+1}\| \leq \|\phi_k\|$ for all k, and $\lim_{k\to\infty} \|\phi_k\| = 0$.

We define $m := \inf\{G(x,s) \mid (x,s) \in [p,1] \times [p,1]\} > 0$. From assumption (ii), there must exist a $\delta > 0$ such that if $x \in [p,1]$ and $y \in (0,\delta)$, then $f(x,y) > \frac{1}{m(1-p)}$. There also must exist a k_0 such that if $k \geq k_0$, we have $0 < \phi_k(r) < \frac{\delta}{2}$ for all $r \in [p,1]$. So, for $k \geq k_0$ and $x \in [p,1]$,

$$\begin{aligned}
\phi_k(x) &= T\phi_k(x) \\
&= \int_0^1 G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_k(r)\,dr\right) ds \\
&\geq \int_p^1 G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_k(r)\,dr\right) ds \\
&\geq \int_p^1 G(x,s) f\left(s, \frac{\delta}{2}\right) ds \\
&\geq \int_p^1 m \cdot \frac{1}{m(1-p)}\,ds = 1.
\end{aligned}$$

This directly contradicts what we already know about the limit of the norms of the elements of $\{\phi_k\}_{k=1}^{\infty}$, so we are forced to conclude that such a lower bound, R, exists. □

7.3 Existence Result

With the *a priori* bounds on the norms of solutions established, we proceed to the main existence result of this chapter.

Theorem 6. *If* $f(x,y)$ *satisfies assumptions* $(i)-(iii)$*, then* (7.3),(7.4) *has a positive solution,* $\phi(x)$*, in* D *that gives rise to a solution,* $y(x) = \int_0^x \frac{(x-r)^{n-5}}{(n-5)!}\,\phi(r)\,dr$*, of* (7.1),(7.2).

Proof. To prove this theorem, we define a few families of new functions and operators on which we apply the aforementioned fixed point theorem to gain a sequence of functions that converge to a solution of (7.3),(7.4). First, for all $c \in \mathbb{N}$, define the operator $\psi_c(x) := T(c)$ in the sense that T is applied to the function on $[0,1]$ whose constant output is c. So,

$$\begin{aligned}\psi_c(x) &= \int_0^1 G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\, c\, dr\right) ds. \\ &= \int_0^1 G(x,s) f\left(s, \frac{cs^{n-4}}{(n-4)!}\right) ds.\end{aligned}$$

Also, $0 < \psi_{c+1}(x) \le \psi_c(x)$ for $x \in (0,1]$ for all c. Furthermore, we have $\lim_{c\to\infty} \psi_c(x) = 0$ uniformly on $[0,1]$ from our assumptions on $f(x,y)$.

Next, for each c, define $f_c : (0,1] \times [0,\infty) \to (0,\infty)$ by

$$f_c(x,y) := f\left(x, \max\{y, \int_0^x \frac{(x-s)^{n-5}}{(n-5)!}\, \psi_c(s)\, ds\}\right)$$

Note that each $f_c(x,y)$ is continuous and not singular at $y = 0$, and $f_c(x,y) \le f(x,y)$ for $(x,y) \in (0,1] \times (0,\infty)$. Also note that for $\phi \in K$,

$$\begin{aligned}&f_c\left(x, \int_0^x \frac{(x-r)^{n-5}}{(n-5)!}\, \phi_c(r)\, dr\right) \\ &\quad = f\left(x, \min\left\{\int_0^x \frac{(x-r)^{n-5}}{(n-5)!}\, \phi_c(r)\, dr, \int_0^x \frac{(x-r)^{n-5}}{(n-5)!}\, \psi_c(r)\, dr\right\}\right). \\ &\quad \le f\left(x, \int_0^x \frac{(x-r)^{n-5}}{(n-5)!}\, \psi_c(r)\, dr\right).\end{aligned}$$

Next, for each c, define $T_c : \mathcal{K} \to \mathcal{K}$ by

$$T_c\phi(x) := \int_0^1 G(x,s) f_c\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\, \phi(r)\, dr\right) ds.$$

It is standard that each T_c is a compact mapping on $\mathcal{K}$. Moreover, $T_c(0) \ge 0$ and $T_c^2(0) \ge 0$ for each c. Thus, the fixed point theorem by Gatica, Oliker, and Waltman guarantees us that there exists a fixed point, $\phi_c \in \mathcal{K}$, for each T_c. Now,

$$T_c\phi_c(x) = \int_0^1 G(x,s) f_c\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\, \phi_c(r)\, dr\right) ds$$

$$\leq \int_0^1 G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\psi_c(r)\,dr\right) ds$$
$$= T\psi_c(x).$$

This, along with the fact that each ϕ_c satisfies the boundary conditions (7.4), allows us to make similar arguments as were used to prove the previous two lemmas to show there exist $R, S > 0$ such that $R \leq \|\phi_c\| \leq S$ for all c.

So, $\{\phi_c\}_{c=1}^{\infty} \subseteq\ < a_R, S^* > \subseteq D$ since $a_R, S^* \in D$, where S^* is the function on $[0,1]$ whose constant output is S. Then, since T is a compact mapping, $\phi^* := \lim_{c\to\infty} T\phi_c$ exists.

Now, in order to prove that ϕ^* is our desired solution for (7.3),(7.4), we need to show $\lim_{c\to\infty}(T\phi_c(x) - \phi_c(x)) = 0$ so that we will have $\phi^* \in D$ and

$$T\phi^*(x) = T\left(\lim_{c\to\infty} T\phi_c(x)\right) = T\left(\lim_{c\to\infty} \phi_c(x)\right)$$
$$= \lim_{c\to\infty} T\phi_c(x) = \phi^*(x).$$

If $\theta = R$, then $a_\theta(x) \leq \phi_c(x)$ for $x \in [0,1]$ for all c, so

$$h_\theta(x) \leq \int_0^x \frac{(x-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr \text{ for all } c.$$

Let $\varepsilon > 0$ be given, and choose $\delta \in (0,1)$ such that $\displaystyle\int_0^\delta f(s, h_\theta(s))\,ds < \frac{\varepsilon}{2M}$. Then, there exists a c_0 such that for $c \geq c_0$ and $x \in [\delta, 1]$,

$$\int_0^x \frac{(x-r)^{n-5}}{(n-5)!}\,\psi_c(r)\,dr \leq h_\theta(x) \leq \int_0^x \frac{(x-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr.$$

So, for $x \in [\delta, 1]$,

$$f_c\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right) = f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right).$$

Then, for $0 \leq x \leq 1$,

$$T\phi_c(x) - \phi_c(x) = T\phi_c(x) - T_c\phi_c(x)$$

$$
\begin{aligned}
&= \int_0^1 G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right) ds \\
&\quad - \int_0^1 G(x,s) f_c\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right) ds \\
&= \int_0^\delta G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right) ds \\
&\quad - \int_0^\delta G(x,s) f_c\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right) ds,
\end{aligned}
$$

and for $0 \le x \le 1$,

$$
\begin{aligned}
|T\phi_c(x) - \phi_c(x)| &\le \int_0^\delta G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right) ds \\
&\quad + \int_0^\delta G(x,s) f_c\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right) ds \\
&\le \int_0^\delta G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right) ds \\
&\quad + \int_0^\delta G(x,s) f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right) ds \\
&\le \ 2M \int_0^\delta f\left(s, \int_0^s \frac{(s-r)^{n-5}}{(n-5)!}\,\phi_c(r)\,dr\right) ds \\
&< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon.
\end{aligned}
$$

Thus, ϕ^* is our desired solution for (7.3),(7.4).

Doing $n-4$ applications of the Fundamental Theorem of Calculus yields that $y(x) = \displaystyle\int_0^x \frac{(x-r)^{n-5}}{(n-5)!}\,\phi^*(r)\,dr$ is a positive solution of (7.1),(7.2). $\square$

CHAPTER EIGHT

Conclusion and Ideas for Extension

The results from Chapters 3 and 6, alongside the 4th order extension due to Maroun [10], prove the existence of positive solutions to our stated class of differential equations under all three right focal boundary condition options (given suitable restrictions on the location of the central boundary condition). The remaining results are extensions of these results based on adding boundary conditions at the left end of the interval. Comparing the results in this dissertation with the earlier results by Singh [13] and Maroun [10] suggests that extensions including the other boundary conditions with require individually prepared setup for each new set of boundary conditions rather than being done similarly to the ones provided here.

These results may be able to be extended further to the paradigm of time scales as was done for a non-local problem [6]. However, it would be prudent to attempt first such an extension to the third order case explored by Maroun. Another extension that may be attainable, with a suitable function akin to g_1 and a_1, is the problem on the real numbers with four point right focal boundary conditions. Optimistically, it is hoped an algorithm to generate functions akin to g_1 and a_1 could be found for each of the aforementioned extensions.

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