ABSTRACT

Conformal Mapping Methods for Spectral Zeta Function Calculations

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We first show how to relate two spectral zeta functions corresponding to conformally equivalent two-dimensional smooth Riemannian manifolds. Next, the functional determinant of the Laplacian on an annulus is used to calculate the functional determinant of the Laplacian on a region bounded by two ellipses. We develop perturbation theory for Hermitian partial differential operators and show how this, combined with a conformal map from a disk to an elliptic region, can be used to derive a perturbative expansion for the spectral zeta function of the Laplacian on an elliptic region that is nearly circular. Finally, this perturbative expansion of the zeta function is used to approximate quantities of interest such as the functional determinant and heat kernel coefficients.

Conformal Mapping Methods for Spectral Zeta Function Calculations

by

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To the memories of my mother,

Pamela Quantz,

and my grandfather,

Chester Quantz

CHAPTER ONE

Introduction

A dynamical system is a fixed rule that describes how a point in some space depends on time. In quantum mechanics, the Hamiltonian is the operator which corresponds to the total energy of a dynamical system, and it often appears in the form $H = T + V$ where T is called the kinetic operator, usually a multiple of the Laplacian, and V is the potential energy function, a function of the spatial configuration of the system and time. The Riemann zeta function, denoted $\zeta(s)$, is the spectral zeta function of a Hamiltonian whose eigenvalues are the natural numbers.

The process by which a finite number is assigned to an otherwise divergent quantity is called regularization. The analytic continuation of ζ evaluated at $s = -1$ can be used to assign the value of $-1/12$ to the sum of the natural numbers. Many other quantities related to the analytic continuation of ζ can be computed explicitly, such as $\zeta'(0)$. However, for other spectral zeta functions, calculating values such as the derivative at zero can prove to be quite difficult since the spectrum of the Hamiltonian may be unknown. In this case, a conformal mapping can sometimes be used to study the spectral zeta function.

Conformal mapping techniques have been employed by mathematical physicists to study the functional determinant, stress tensors, quantum billiards, and zeta functions since as early as 1978 [4, 6, 8–13, 19, 45]. In this work, we use conformal mappings to calculate spectral zeta function quantities associated with $T = -\Delta$ where Δ is the Laplacian and $V = 0$ in two dimensions with various boundary conditions on two particular smooth Riemannian manifolds, a region bounded by two ellipses, $\mathcal{E}^*$, and an ellipse, $\mathcal{E}$.

We begin Chapter Two by focusing on preliminaries. First, the spectral zeta function and functional determinant of an operator are defined. We will see that in order to calculate the functional determinant of an operator, it suffices to calculate the derivative of its associated zeta function at zero. Additionally, a connection between certain types of Hamiltonians and the Riemann hypothesis is described. The final two sections of Chapter Two each include a conformal relation. These conformal relations will be used to say something about an unknown spectral zeta function in terms of a known spectral zeta function.

Chapter Three begins by introducing a conformal map from an annulus, $\mathcal{A}$, to $\mathcal{E}^*$. Then, the derivative at zero of the zeta function associated to $-\Delta$ with Dirichlet and Neummann boundary conditions on $\mathcal{E}^*$ is related to the derivative at zero of the zeta function associated to $-\Delta$, with the same boundary conditions on $\mathcal{A}$. Next, we detour into perturbation theory. Chapter Four begins with the derivation of fundamental results from nondegenerate perturbation theory. Then, these results are extended to a special case of degenerate perturbation theory. Finally, a formula for a perturbed zeta function is given.

Chapter Five starts with details regarding the formula for f, a conformal map from a disk to an ellipse $\mathcal{E}$. A perturbative expansion of this map is derived in terms of the eccentricity of the ellipse, ϵ. This perturbative expansion, together with a conformal relation from Chapter Two, are used to give a perturbative expansion for $\zeta_{\mathcal{E}}'(0)$, where $\zeta_{\mathcal{E}}$ is the spectral zeta function of $-\Delta$ on $\mathcal{E}$ with the Dirichlet boundary condition. Finally, we examine a perturbative expansion of $\zeta_{\mathcal{E}}$, obtained by restricting ϵ, which allows us to compare terms between various calculations for quantities related to $\zeta_{\mathcal{E}}$.

In Chapter Six, we focus on the heat kernel. First, the definition of a heat kernel of a differential operator on a smooth Riemannian manifold is given. Then, we introduce formulas relating heat kernel coefficients to spectral zeta functions. Next, explicit formulas for heat kernel coefficients of $-\Delta$ with the Dirichlet boundary condition on a planar, simply connected, open, smooth manifold is given in terms of geometric quantities. These formulas are used to calculate heat kernel coefficients for $\mathcal{E}$. Finally, we compare expressions for $\zeta_{\mathcal{E}}$ derived via a perturbative expansion of the conformal map to expressions derived via the heat kernel coefficients.

CHAPTER TWO

Preliminaries

2.1 The Riemann Zeta Function

The Riemann zeta function is defined by

$$\zeta\left(s\right)=\sum_{n=1}^{\infty}n^{-s},$$

where $s=\Re s+i\Im s$ for some $\Re s,\Im s\in\mathbb{R}$ and $i^2=-1$. Since $\left|e^{i\theta}\right|=1$ for all $\theta\in\mathbb{R}$ and $n\in\mathbb{N}$, we have

$$\left|n^{-s}\right|=\left|e^{-s\ln n}\right|=\left|e^{-\Re s\ln n}\right|\left|e^{-i\Im s\ln n}\right|=\left|n\right|^{-\Re s}.$$

So, the above series converges for all $s\in\mathbb{C}$ with $\Re s>1$. This series was first studied for $s\in\mathbb{N}$ in 1740 by Euler [14], who discovered

$$\sum_{n=1}^{\infty}n^{-s}=\prod_{p\text{ prime}}\frac{1}{1-p^{-s}},$$

the identity that connects $\zeta\left(s\right)$ to the prime numbers.

In his 1859 article, titled in German that translates to "On the Number of Primes Less Than a Given Number" [35], Riemann introduced the notation $\zeta\left(s\right)$ for the above series, proved $\zeta\left(s\right)$ can be analytically continued to the whole complex s-plane with a simple pole at $s=1$, and discussed the relationship between $\zeta\left(s\right)$ and the distribution of the prime numbers. Among other things, Riemann proved the functional equation

$$\zeta\left(s\right)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(1-s\right)\zeta\left(1-s\right),\tag{2.1.1}$$

where $\Gamma\left(n\right)=\left(n-1\right)!$ for $n\in\mathbb{N}$. On $\Re s>0$, the gamma function can be defined as the convergent improper integral

$$\Gamma\left(s\right)=\int_0^{\infty}x^{s-1}e^{-x}dx.$$

This integral function can be extended by analytic continuation to all complex numbers except the non-positive integers [1,44], where the function has simple poles. So $\Gamma(s)$ is a meromorphic function and (2.1.1) is an equality of meromorphic functions, valid on the whole complex plane.

In (2.1.1), the factor $\sin\left(\frac{\pi s}{2}\right)$ is zero when $s = 2n$, $n \in \mathbb{Z}$. However, the fact that $\Gamma(1-s)$ has simple poles when $1-s$ is a nonpositive integer shows $s = -2n$, $n \in \mathbb{N}$, are zeros of $\zeta(s)$. These are called the trivial zeros of $\zeta(s)$. In what has since been referred to as the Riemann hypothesis, considered one of the greatest unsolved problems in mathematics, Riemann asserted that any non-trivial zero of $\zeta(s)$ has real part equal to 1/2 [35].

2.2 The Spectral Zeta Function

Bra-ket notation was first introduced by Dirac in 1939 as a way to represent the mathematical formulations of quantum mechanics [7]. With this notation, an element of a Hilbert space, $\mathcal{H}$, is called a ket and is denoted $|u\rangle$. Throughout this work, we will view elements of a Hilbert space of functions as either functions or kets.

We define the Hermitian conjugate, or conjugate transpose, of a ket $|u\rangle$ to be the bra

$$|u\rangle^{\dagger} = \langle u| .$$

In bra-ket notation, the inner product of the ket $|u\rangle$ with the ket $|v\rangle$ is denoted $\langle u|v\rangle$ and can be thought of as being obtained by applying the bra $\langle u|$ to the left of the ket $|v\rangle$. A linear operator A on H can be viewed as mapping a ket $|u\rangle$ to a ket $A|u\rangle$ or a bra $\langle u|$ to a bra $\langle u| A$. Given A, a linear operator $A^{\dagger}$ is defined by

$$|v\rangle = A|u\rangle \text{ if and only if } \langle v| = \langle u| A^{\dagger}.$$

Existence and uniqueness of $A^{\dagger}$ follows from the Riesz representation theorem [22]. We say A when Hermitian if $A = A^{\dagger}$. If A is Hermitian and the domain of A equals the domain of $A^\dagger$ then A is said to be self-adjoint. When A is self-adjoint, there is no ambiguity in writing $\langle u|A|v\rangle$, as we will often do. Also, we will have occasion to refer to an operator as Hermitian even when the domain of the operator and its conjugate transpose are equal.

It is known that self-adjoint operators have real eigenvalues [22]. Now, let P be a non-negative self-adjoint linear operator mapping a separable Hilbert space $\mathcal{H}$ to itself. We say P is of trace class if there is some orthonormal basis $\{e_n\}$ such that

$$\sum_{n=1}^{N} \langle e_n|P|e_n\rangle < \infty,$$

where $N \leq \infty$. We call the above sum the trace of P and denote it by $Tr\,(P)$. If P is trace class with eigenvalues λ_n, listed such that if the algebraic multiplicity of λ_n is k then λ_n is repeated k times in the list, then by Lidskii's theorem [25]

$$Tr\,(P) = \sum_{n=1}^{N} \lambda_n.$$

When $\mathcal{H}$ is finite-dimensional, every P is trace class and the definition of $Tr\,(P)$ coincides with the definition of the trace of a matrix. Assume P is of trace class with eigenvalues λ_n, listed with multiplicities. Then P^{-s} has eigenvalues λ_n^{-s} and we define the spectral zeta function of the mapping P to be

$$\zeta_P\,(s) = \sum_{n=1}^{N} \lambda_n^{-s} = Tr\left(P^{-s}\right).$$

It is known that $\zeta_P\,(s)$ can be analytically continued to a function that is analytic at $s = 0$ [20, 38] and λ_n is bounded below for all $n \in \mathbb{N}$ when P is a self-adjoint operator on a separable Hilbert space [22]. So, by employing a suitable shift, we can make $\lambda_n > 0$ for all $n \in \mathbb{N}$. For the remainder of this work, we will assume all eigenvalues are positive. Since, formally,

$$-\zeta_P'\,(0) = -\frac{d}{ds}\sum_{n=1}^{N} \lambda_n^{-s}|_{s=0} = \sum_{n=1}^{N} \ln \lambda_n = \ln \prod_{n=1}^{N} \lambda_n,$$

we define

$$\det P = \prod_{n=1}^{N} \lambda_n = e^{-\zeta_P'(0)} \tag{2.2.1}$$

as in [33]. When $\mathcal{H}$ is finite-dimensional, this definition of $\det P$ coincides with the definition of the determinant of a matrix.

Radiation frequencies of atoms are sets of lines called spectra. Hilbert coined the term "spectrum" for the set of eigenvalues of a self-adjoint operator. The later development of quantum mechanics would reveal this to be a fortuitous choice of name as it was discovered that atomic spectra were actually spectra in the sense of operator theory where the self-adjoint operators are the Hamiltonians of the dynamical systems representing the atoms in question.

The spectra of nuclei are more complicated than the spectra of atoms. A nucleus absorbs a neutron and later undergoes radioactive decay. The energies of such events are again quantized, but of such complexity that only a statistical understanding of their nature is attempted.

The Gaussian Unitary Ensemble (GUE) models random Hamiltonians without time reversal symmetry. The GUE of degree n consists of the set of all $n \times n$ Hermitian matrices whose individual matrix entries are independent random variables, together with the unique probability measure that is invariant under conjugation by unitary matrices.

If the zeros of $\zeta(s)$ are the eigenvalues of $1/2 + iT$, where T is a self-adjoint operator on some Hilbert space then, since the eigenvalues of a self-adjoint operator are real, the Riemann hypothesis follows. This is the Hilbert–Pólya conjecture [30], which offers an idea of how to prove the Riemann hypothesis by means of spectral theory. Montgomery's results [28] led to the conjecture that zeta zeros behave asymptotically like eigenvalues of large random matrices from the GUE, which has been studied extensively by mathematical physicists. Although this conjecture is very speculative, Odlyzko's numerical evidence [31] is overwhelmingly in its favor.

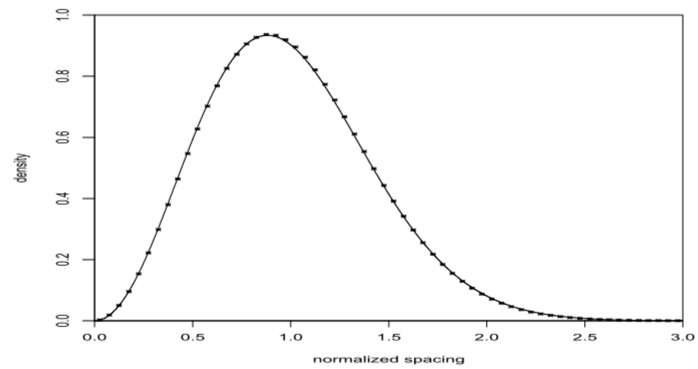

Figure 2.1: Spacing distribution for a billion zeroes of the Riemann zeta function, and the corresponding prediction from random matrix theory [31, 41]

To quote Tao, "But this does not mean that the primes are somehow nuclear-powered, or that atomic physics is somehow driven by the prime numbers; instead, it is evidence that a single law for spectra is so universal that it is the natural end product of any number of different processes, whether it comes from nuclear physics, random matrix models, or number theory." [41]

2.3 A Conformal Relation

Let $\Omega \subset \mathbb{R}^n$ be a connected domain. The Laplacian is the operator on $C^\infty(\Omega)$ defined by

$$\Delta\varphi = \sum_{k=1}^{n} \frac{\partial^2 \varphi}{\partial x_k^2},$$

where x_k are the position variables. More generally, for a Riemannian manifold $(\mathcal{M}, g)$, we may define the Laplacian to be

$$\Delta_g = \operatorname{div}_g \circ \nabla_g$$

where div_g and ∇_g are the divergence and gradient operators, respectively. In all of our considerations the manifolds will be 2-dimensional subsets of the plane, g will be the usual Euclidean metric, and the differential operator will be minus the Laplacian. Since several different manifolds with the same metric will be considered in this work , we will write

$$\Delta_{\mathcal{M}} = \operatorname{div} \circ \nabla$$

to make clear which $C^{\infty}(\mathcal{M})$ the operator is acting on.

Now, consider the eigenvalue problem

$$-\Delta_{\mathcal{M}} \varphi = \lambda \varphi$$

with either the Dirichlet boundary condition

$$\varphi|_{\partial \mathcal{M}} = 0$$

or the Neumann boundary condition

$$\varphi_{;m}|_{\partial \mathcal{M}} = 0,$$

where the subscript $;m$ indicates that the derivative of φ is to be taken with respect to an outward-pointing normal of $\partial\mathcal{M}$. We will show how a known spectral zeta function can be used to study an unknown spectral zeta function.

Let $\mathcal{M}$, $\mathcal{N} \subset \mathbb{C}$ be open. Call $f : \mathcal{M} \to \mathcal{N}$ conformal if f is analytic and $f' \neq 0$ on $\mathcal{M}$. Consider $f = u(x, y) + iv(x, y)$. Let $\mathcal{N} = f(\mathcal{M})$ and $\phi : \mathcal{N} \to \mathbb{R}$ have continuous second-order partial derivatives. Then

$$(\phi \circ f)_x = \phi_u u_x + \phi_v v_x \text{ and } (\phi \circ f)_y = \phi_u u_y + \phi_v v_y$$

imply

$$(\phi \circ f)_{xx} = \phi_{uu}(u_x)^2 + \phi_{uv} u_x v_x + \phi_u u_{xx} + \phi_{vv}(v_x)^2 + \phi_{vu} v_x u_x + \phi_v v_{xx}$$

and

$$(\phi \circ f)_{yy} = \phi_{uu}(u_y)^2 + \phi_{uv} u_y v_y + \phi_u u_{xx} + \phi_{vv}(v_y)^2 + \phi_{vu} v_y u_y + \phi_v v_{yy}$$

By the Cauchy-Riemann equations [1],

$$u_x = v_y \text{ and } v_x = -u_y,$$

and the real and imaginary parts of analytic functions are harmonic. That is,

$$v_{xx} + v_{yy} = u_{xx} + u_{yy} = 0.$$

So

$$\Delta(\phi \circ f) = (\phi \circ f)_{xx} + (\phi \circ f)_{yy} = \left[(u_x)^2 + (u_y)^2\right](\phi_{uu} + \phi_{vv}) = |f'|^2 \left[(\Delta\phi) \circ f\right]. \tag{2.3.4}$$

This implies the problem

$$-\Delta_{\mathcal{N}}\phi = \mu\phi \text{ with } \phi|_{\partial\mathcal{N}} = 0 \tag{2.3.5}$$

has the same eigenvalues as the problem

$$-|f'|^{-2}\Delta_{\mathcal{M}}\psi = \mu\psi \text{ with } \psi|_{\partial\mathcal{M}} = 0, \tag{2.3.6}$$

where $\psi = \phi \circ f$. Since the eigenvalues in (2.3.5) equal the eigenvalues in (2.3.6), we can use our knowledge of f and $\zeta_{-\Delta_{\mathcal{M}}}$ to study $\zeta_{-\Delta_{\mathcal{N}}}$.

2.4 Another Conformal Relation

Consider the family of operators defined by

$$P(\delta) = e^{-2\delta F} P$$

where $\delta \in \mathbb{R}$, F is a smooth function on a Riemannian manifold with boundary and $P = -\Delta_{\mathcal{M}}$ with $\Delta_{\mathcal{M}}$ being the Laplace operator on smooth functions on $\mathcal{M}$ with either the Dirichlet or Neumann boundary condition. Let

$$\zeta'_{P(0)}(0) = \zeta'_0(0) \text{ and } \zeta'_{P(\delta)}(0) = \zeta'_\delta(0).$$

Throughout this work, all manifolds will be two-dimensional. Then, being in dimension two and having the Dirichlet boundary condition implies [2, 26, 32]

$$\zeta'_1(0) - \zeta'_0(0) = \frac{1}{12\pi}\left\{\int_{\mathcal{M}} dx Tr_V\left(F\left[6E + R - \Delta_{\mathcal{M}}F\right]\right)\right.$$

$$+ \int_{\partial\mathcal{M}} dyTr_V \left(F\left[2K + F_{;m}\right] + 3F_{;m}\right)\Bigg\}, \tag{2.4.1}$$

where R and dx, respectively K and dy, are the scalar curvatures and volume elements of the manifolds $\mathcal{M}$ and $\partial\mathcal{M}$, and E is the potential term in the operator $P(0)$. Throughout this work, Tr_V is the identity, $E = 0$, and $R = 0$ since $\mathcal{M}$ will be flat. We will also only consider cases when F is harmonic on $\mathcal{M}$, that is, $\Delta_{\mathcal{M}}F = 0$. So, in this work, (2.4.1) reduces to

$$\zeta_1'(0) - \zeta_0'(0) = \frac{1}{12\pi}\int_{\partial\mathcal{M}} dy\left(F\left[2K + F_{;m}\right] + 3F_{;m}\right). \tag{2.4.2}$$

We will also refer to (2.4.2) as the conformal relation since, as we see in Section 2.5, given a conformal map $f : \mathcal{M} \to \mathcal{N}$, we may define F so that a known spectral zeta function values may be used to study an unknown spectral zeta function. We will focus primarily on the Dirichlet boundary condition; although, in all our considerations

$$\zeta_1'(0) - \zeta_0'(0) = \frac{1}{12\pi}\int_{\partial\mathcal{M}} dy\left(F\left[2K + F_{;m}\right] - 3F_{;m}\right), \tag{2.4.3}$$

will hold for the Neumann boundary condition [20].

We now show $F = \ln|f'(z)|$ is harmonic when f is conformal. To see this, let $\mathcal{M} \subset \mathbb{C}$ be open and note that for any $z \in \mathcal{M}$ there is a simply connected open set U such that $z \in U \subset \mathcal{M}$. Then

$$\ln|f'| = \Re \log f',$$

where log is a branch of the complex logarithm chosen so that $\log f'$ is single-valued and analytic on U. Such a branch of the logarithm exists [1] since f' is analytic and nonzero on U. The real part of an analytic function is harmonic, so F, being a multiple of a harmonic function, is harmonic on U and hence harmonic on M. Throughout the remainder of this work, we will assume $F = \ln|f'(z)|$ where f :

$\mathcal{M} \to \mathcal{N}$ is a conformal map and some information about $\zeta_{-\Delta_{\mathcal{M}}}$ is known. With this definition of F, we have

$$P(0) = -\Delta_{\mathcal{M}} \text{ and } P(1) = -|f'|^{-2}\,\Delta_{\mathcal{M}}.$$

Since $\zeta_0 = \zeta_{-\Delta_{\mathcal{M}}}$ and $\zeta_1 = \zeta_{-\Delta_{\mathcal{N}}}$ in (2.4.1), we can calculate $\zeta'_{-\Delta_{\mathcal{N}}}(0)$ by using f and $\zeta'_{-\Delta_{\mathcal{M}}}(0)$. In this work, we will frequently be concerned with $-\Delta$. So, we will have occasion to write $\zeta_{\mathcal{M}}$ in place of $\zeta_{-\Delta_{\mathcal{M}}}$ and rely on the context to make clear which spectral zeta function is being considered.

CHAPTER THREE

Region Bounded by Two Ellipses Calculation

3.1 The Conformal Map

Consider

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

We see f is conformal on $C - \{0, \pm 1\}$ since

$$f'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right).$$

Let $\rho, \rho_1, \rho_2 \in (1, \infty)$ and $\rho_2 > \rho_1$. Note

$$\mathcal{A}(\rho_1, \rho_2) = \{z \in \mathbb{C} : \rho_1 < |z| < \rho_2\}$$

is an annulus and

$$\mathcal{E}(\rho) = \left\{ u + iv \in \mathbb{C} : \frac{u^2}{\frac{1}{4}\left(\rho + \frac{1}{\rho}\right)^2} + \frac{v^2}{\frac{1}{4}\left(\rho - \frac{1}{\rho}\right)^2} < 1 \right\}$$

is an ellipse. The foci of $\mathcal{E}(\rho)$ are at ± 1 since

$$\frac{1}{4}\left(\rho + \frac{1}{\rho}\right)^2 - \frac{1}{4}\left(\rho - \frac{1}{\rho}\right)^2 = 1.$$

It can be shown [18] that $f : \mathcal{A}(1, \rho) \to \mathcal{E}(\rho) - [-1, 1]$ is bijective and conformal, where

$$[-1, 1] = \{x + iy \in \mathbb{C} : -1 \le x \le 1 \text{ and } y = 0\}.$$

In particular, $f : \mathcal{A}(\rho_1, \rho_2) \to \mathcal{E}^*(\rho_1, \rho_2)$ is bijective and conformal, where

$$\mathcal{E}^*(\rho_1, \rho_2) = \mathcal{E}(\rho_2) - \mathcal{E}(\rho_1).$$

For notational convenience, we now suppress the arguments of $\mathcal{A}$ and $\mathcal{E}^*$. We find

$$|f'(z)|^2 = \frac{1}{4}\left|1 - \frac{1}{z^2}\right|^2 = \frac{1}{4|z|^2}\left|z - \frac{1}{z}\right|^2$$

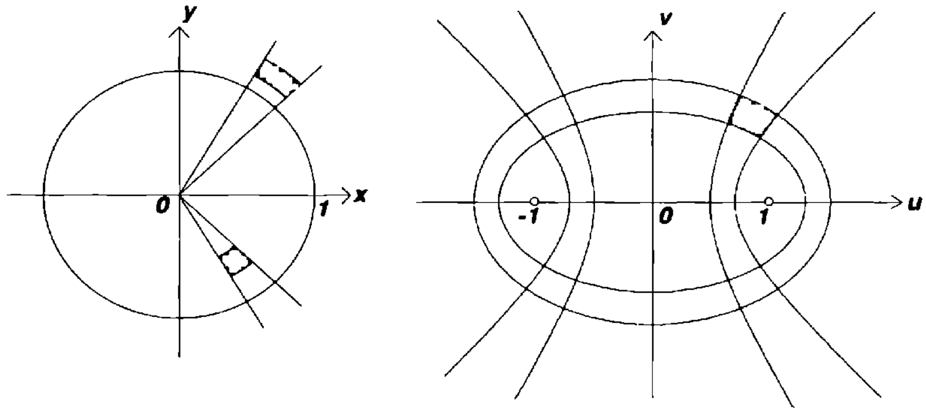

Figure 3.1. Conformally mapping $\mathcal{A}$ to $\mathcal{E}^*$ [18]

$$= \frac{1}{4r^2}\left(z - \frac{1}{z}\right)\left(\overline{z} - \frac{1}{\overline{z}}\right)$$
$$= \frac{1}{4r^2}\left(|z|^2 - \frac{\overline{z}}{z} - \frac{z}{\overline{z}} + \frac{1}{|z|^2}\right).$$

When $z = re^{i\theta}$,

$$\frac{\overline{z}}{z} + \frac{z}{\overline{z}} = \frac{\overline{z}^2 + z^2}{|z|^2} = e^{-2i\theta} + e^{2i\theta} = 2\cos 2\theta.$$

Then

$$\left|f'\left(re^{i\theta}\right)\right|^2 = \frac{1}{4r^2}\left(\frac{1}{r^2} - 2\cos 2\theta + r^2\right) = \frac{1}{4r^4}\left(1 - 2r^2\cos 2\theta + r^4\right)$$

implies

$$F(r,\theta) = \frac{1}{2}\ln\frac{1 - 2r^2\cos 2\theta + r^4}{4r^4}.$$

So

$$F_r(r,\theta) = \frac{2r^2\cos 2\theta - 2}{r - 2r^3\cos 2\theta + r^5}.$$

From (2.4.2) we have

$$\begin{aligned}
&\zeta'_{\mathcal{E}^*}(0) - \zeta'_{\mathcal{A}}(0)\\
&= \frac{1}{12\pi}\left\{\left(\int_{|z|=\rho_1} + \int_{|z|=\rho_2}\right)|dz|\left(F\left[2K + F_{;m}\right] + 3F_{;m}\right)\right\}\\
&= \frac{1}{12\pi}\left[G_-^D(\rho_1) + G_+^D(\rho_2)\right] \qquad (3.1.1)
\end{aligned}$$

for the Dirichlet boundary condition where

$$G_{\pm}^{D}(\rho) = \int_{0}^{2\pi} \rho d\theta \left(F\left[2\rho^{-1} \pm F_r\right] \pm 3F_r\right)(\rho, \theta)$$

and from (2.4.3) we have

$$\zeta_{\mathcal{E}^*}'(0) - \zeta_{\mathcal{A}}'(0) = \frac{1}{12\pi}\left[G_{-}^{N}(\rho_1) + G_{-}^{N}(\rho_2)\right] \tag{3.1.2}$$

where

$$G_{\pm}^{N}(\rho) = \int_{0}^{2\pi} \rho d\theta \left(F\left[2\rho^{-1} \pm F_r\right] \mp 3F_r\right)(\rho, \theta)$$

for the Neumann boundary condition.

In Appendix A we show

$$\int_{0}^{2\pi} d\theta F(r, \theta) = -\pi \ln 4$$

for $r \geq 1$ and

$$\int_{0}^{2\pi} d\theta F_r(r, \theta) = 0$$

for $r > 1$. Then

$$G_{\pm}^{D}(\rho) = -2\pi \ln 4 \pm \rho I(\rho) = G_{\pm}^{N}(\rho), \tag{3.1.3}$$

where

$$I(\rho) = \int_{0}^{2\pi} d\theta F_r F(\rho, \theta). \tag{3.1.4}$$

So, to rewrite $\zeta_{\mathcal{E}^*}'(0) - \zeta_{\mathcal{A}}'(0)$ without integral terms, all that remains is to rewrite $I(\rho)$. For certain values of ρ, Mathematica can be used to calculate

$$I(2) = -\pi \ln\left(\frac{16}{15}\right) \approx -0.203 \text{ and } I(3) = -\frac{2\pi}{3} \ln\left(\frac{81}{80}\right) \approx -0.026;$$

however, Mathematica cannot find an antiderivative for $F_r F$. The numerical results above will be useful when it comes time to check our rewritten $I(\rho)$ for correctness.

3.2 Rewriting $I(\rho)$

The purpose of this section is to rewrite $I(\rho)$ without integral terms, thereby giving a rewriting of $\zeta'_{\mathcal{E}^*}(0)$ without integral terms. We have

$$\begin{aligned} I(\rho) &= \int_0^{2\pi} d\theta F_r F(\rho,\theta) = \int_0^{2\pi} d\theta \frac{1}{2}\frac{\partial}{\partial r}|_{r=\rho} F^2(r,\theta) \\ &= \frac{1}{2}\frac{\partial}{\partial r}|_{r=\rho}\int_0^{2\pi} d\theta F^2(r,\theta) = \frac{1}{8}\frac{\partial}{\partial r}|_{r=\rho}\int_0^{2\pi} d\theta \ln^2 \frac{4r^4}{r^4-2r^2\cos 2\theta+1}. \end{aligned}$$

Note

$$\begin{aligned} 2\pi\ln 4 &= -2\int_0^{2\pi} d\theta F(\rho,\theta) = \int_0^{2\pi} d\theta \ln 4\rho^4 - \int_0^{2\pi} d\theta \ln\left(\rho^4-\rho^2\cos 2\theta+1\right) \\ &\Rightarrow \int_0^{2\pi} d\theta \ln\left(\rho^4-2\rho^2\cos 2\theta+1\right) = 2\pi\left(\ln 4\rho^4-\ln 4\right) = 2\pi\ln\rho^4 = 8\pi\ln\rho. \end{aligned}$$

and

$$\begin{aligned} \ln^2\frac{4r^4}{r^4-2r^2\cos 2\theta+1} &= \left[\ln 4r^4-\ln\left(r^4-2r^2\cos 2\theta+1\right)\right]^2 \\ &= \ln^2 4r^4 - 2\ln 4r^4\ln\left(r^4-2r^2\cos 2\theta+1\right) \\ &\quad +\ln^2\left(r^4-2r^2\cos 2\theta+1\right). \end{aligned}$$

Let

$$J(r) = \int_0^{2\pi} d\theta \ln^2\left(r^4-2r^2\cos 2\theta+1\right).$$

Then

$$\begin{aligned} & I(\rho) \\ = & \frac{1}{8}\frac{\partial}{\partial r}|_{r=\rho}\int_0^{2\pi} d\theta[\ln^2 4r^4 - 2\ln 4r^4\ln\left(r^4-2r^2\cos 2\theta+1\right) \\ & +\ln^2\left(r^4-2r^2\cos 2\theta+1\right)] \\ = & \frac{1}{8}\frac{\partial}{\partial r}|_{r=\rho}\left[\ln^2 4r^4\int_0^{2\pi} d\theta - 2\ln 4r^4\int_0^{2\pi} d\theta\ln\left(r^4-2r^2\cos 2\theta+1\right) + J(r)\right] \\ = & \frac{1}{8}\frac{\partial}{\partial r}|_{r=\rho}\left[2\pi\ln^2 4r^4 - 16\pi\ln 4r^4\ln r + J(r)\right]. \end{aligned}$$

We find

$$\frac{1}{8}\frac{\partial}{\partial r}|_{r=\rho}\left(2\pi\ln^2 4r^4 - 16\pi\ln 4r^4\ln r\right)$$

$$
\begin{aligned}
&= \frac{2\pi}{\rho}\ln 4\rho^4 - 2\pi\left(\frac{4}{\rho}\ln\rho + \frac{1}{\rho}\ln 4\rho^4\right) \\
&= \frac{2\pi}{\rho}\ln 4\rho^4 - 2\pi\left(\frac{1}{\rho}\ln\rho^4 + \frac{1}{\rho}\ln 4\rho^4\right) \\
&= \frac{2\pi}{\rho}\ln 4\rho^4 - \frac{2\pi}{\rho}\ln 4\rho^8 = \frac{2\pi}{\rho}\ln\rho^{-4} \\
&= -\frac{8\pi}{\rho}\ln\rho.
\end{aligned}
$$

Therefore

$$
I(\rho) = -\frac{8\pi}{\rho}\ln\rho + \frac{1}{8}J_r(\rho). \tag{3.2.1}
$$

So, to rewrite $I(\rho)$, all that remains is to rewrite $J_r(\rho)$. We will show that $J_r(\rho)$ can be expressed as a sum.

3.3 Expressing $J_r(\rho)$ as a Sum

Recall

$$
\ln(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k} \text{ for } |x| < 1.
$$

Note

$$
r > 1 \Rightarrow 0 < r^4 - 2r^2 + 1 = \left(r^2 - 1\right)^2 \Rightarrow 2r^2 < r^4 + 1 \Rightarrow 1 > \frac{2r^2}{r^4+1} \geq \left|\frac{2r^2\cos\theta}{r^4+1}\right|.
$$

So we write

$$
\begin{aligned}
& J(r) \\
&= \int_0^{2\pi} d\theta \ln^2\left(r^4 - 2r^2\cos\theta + 1\right) = \int_0^{2\pi} d\theta \ln^2\left[\left(r^4+1\right)\left(1 - \frac{2r^2\cos\theta}{r^4+1}\right)\right] \\
&= \int_0^{2\pi} d\theta \left[\ln\left(r^4+1\right) + \ln\left(1 - \frac{2r^2\cos\theta}{r^4+1}\right)\right]^2 \\
&= \int_0^{2\pi} d\theta \left[\ln^2\left(r^4+1\right) + 2\ln\left(r^4+1\right)\ln\left(1 - \frac{2r^2\cos\theta}{r^4+1}\right) + \ln^2\left(1 - \frac{2r^2\cos\theta}{r^4+1}\right)\right] \\
&= 2\pi\ln^2\left(r^4+1\right) + 2\ln\left(r^4+1\right)\int_0^{2\pi} d\theta \ln\left(1 - \frac{2r^2\cos\theta}{r^4+1}\right) \\
&\quad + \int_0^{2\pi} d\theta \ln^2\left(1 - \frac{2r^2\cos\theta}{r^4+1}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_0^{2\pi} d\theta \ln\left(1 - \frac{2r^2\cos\theta}{r^4+1}\right) \\
= &-\int_0^{2\pi} d\theta \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{2r^2\cos\theta}{r^4+1}\right)^k = -\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{2r^2}{r^4+1}\right)^k \int_0^{2\pi} d\theta \cos^k\theta \\
= &-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{2r^2}{r^4+1}\right)^k \int_0^{2\pi} d\theta \cos^k\theta \\
= &-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{2r^2}{r^4+1}\right)^k \oint_{|z|=1}\frac{dz}{iz}\left(\frac{z+\frac{1}{z}}{2}\right)^k \\
= &\; i\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{2r^2}{r^4+1}\right)^k 2^{-k}\oint_{|z|=1}\frac{dz}{z}\sum_{j=0}^{k}\binom{k}{j} z^j\left(\frac{1}{z}\right)^{k-j} \\
= &\; i\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{r^2}{r^4+1}\right)^k \oint_{|z|=1} dz\sum_{j=0}^{k}\binom{k}{j} z^{2j-k-1}.
\end{aligned}
$$

Note

$$
\binom{k}{k/2} = \frac{k!}{(k-k/2)!\,(k/2)!} = \frac{k!}{(k/2)!^2}
$$

and

$$
2j-k-1 = -1 \Leftrightarrow j = k/2.
$$

Since $j \in \mathbb{Z}$ implies k must be even, by Cauchy's Residue Theorem [1],

$$
\begin{aligned}
\int_0^{2\pi} d\theta \ln\left(1 - \frac{2r^2\cos\theta}{r^4+1}\right) &= i\sum_{k \text{ even},\, k>0}\frac{1}{k}\left(\frac{r^2}{r^4+1}\right)^k 2\pi i\binom{k}{k/2} \\
&= -\pi\sum_{k=1}^{\infty}\left(\frac{r^2}{r^4+1}\right)^{2k}\frac{(2k)!}{k\,(k!)^2}.
\end{aligned}
$$

Next, using the Cauchy product [36], we find

$$
\begin{aligned}
&\int_0^{2\pi} d\theta \ln^2\left(1 - \frac{2r^2\cos\theta}{r^4+1}\right) \\
= &\int_0^{2\pi} d\theta\left[-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{2r^2\cos\theta}{r^4+1}\right)^k\right]^2 \\
= &\int_0^{2\pi} d\theta\left[\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{2r^2\cos\theta}{r^4+1}\right)^{k+1}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{2\pi} d\theta \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k+1}\left(\frac{2r^2\cos\theta}{r^4+1}\right)^{k+1} \frac{1}{n-k+1}\left(\frac{2r^2\cos\theta}{r^4+1}\right)^{n-k+1} \\
&= \int_0^{2\pi} d\theta \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)}\left(\frac{2r^2\cos\theta}{r^4+1}\right)^{n+2} \\
&= \sum_{n=0}^{\infty}\left(\frac{2r^2}{r^4+1}\right)^{n+2}\sum_{k=0}^{n}\frac{1}{(k+1)(n-k+1)}\int_0^{2\pi} d\theta \cos^{n+2}\theta \\
&= \sum_{n=0}^{\infty}\left(\frac{2r^2}{r^4+1}\right)^{n+2}\sum_{k=0}^{n}\frac{1}{(k+1)(n-k+1)} \\
&\quad \times \oint_{|z|=1}\frac{dz}{iz}\left(\frac{z+\frac{1}{z}}{2}\right)^{n+2} \\
&= -i\sum_{n=0}^{\infty}\left(\frac{2r^2}{r^4+1}\right)^{n+2} 2^{-n-2}\sum_{k=0}^{n}\frac{1}{(k+1)(n-k+1)} \\
&\quad \times \oint_{|z|=1}\frac{dz}{z}\sum_{j=0}^{n+2}\binom{n+2}{j} z^j \left(\frac{1}{z}\right)^{n+2-j} \\
&= -i\sum_{n=0}^{\infty}\left(\frac{r^2}{r^4+1}\right)^{n+2}\sum_{k=0}^{n}\frac{1}{(k+1)(n-k+1)} \\
&\quad \times \oint_{|z|=1} dz \sum_{j=0}^{n+2}\binom{n+2}{j} z^{2j-n-3}.
\end{aligned}
$$

Note

$$
2j-n-3=-1 \Leftrightarrow 2j-n=2 \Leftrightarrow j=\frac{n+2}{2}
$$

and

$$
\binom{n+2}{\frac{n+2}{2}} = \frac{(n+2)\,!}{\left(\frac{n+2}{2}\right)!\left(n+2-\frac{n}{2}-1\right)!} = \frac{(n+2)\,!}{\left(\frac{n}{2}+1\right)!\left(\frac{n}{2}+1\right)!} = \frac{(n+2)\,!}{\left(\frac{n}{2}+1\right)!^2}.
$$

Since $j \in \mathbb{Z}$ implies k must be even, by Cauchy's Residue Theorem,

$$
\begin{aligned}
&\int_0^{2\pi} d\theta \ln^2\left(1-\frac{2r^2\cos\theta}{r^4+1}\right) \\
&= -i\sum_{n \text{ even},\, n\geq 0}\left(\frac{r^2}{r^4+1}\right)^{n+2}\sum_{k=0}^{n}\frac{1}{(k+1)(n-k+1)} 2\pi i \binom{n+2}{\frac{n+2}{2}} \\
&= 2\pi\sum_{n=0}^{\infty}\left(\frac{r^2}{r^4+1}\right)^{2n+2}\frac{(2n+2)\,!}{(n+1)\,!^2}\sum_{k=0}^{2n}\frac{1}{(k+1)(2n-k+1)}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{1}{(k+1)(2n-k+1)} &= \frac{A}{k+1} + \frac{B}{2n-k+1} \\
\Rightarrow \quad A(2n-k+1) + B(k+1) &= 2An - Ak + A + Bk + B = 1 \\
\Rightarrow \quad (B-A)k &= 0 \text{ and } 2An + A + B = 1 \\
\Rightarrow \quad B &= A \\
\Rightarrow \quad 2Bn + 2B &= 1 \\
\Rightarrow \quad B &= \frac{1}{2(n+1)} \\
\Rightarrow \quad \frac{1}{(k+1)(2n-k+1)} &= \frac{1}{2(n+1)}\left(\frac{1}{k+1} + \frac{1}{2n-k+1}\right).
\end{aligned}
$$

So

$$
\begin{aligned}
&\sum_{k=0}^{2n} \frac{1}{(k+1)(2n-k+1)} \\
&= \frac{1}{2(n+1)}\left(\sum_{k=0}^{2n} \frac{1}{k+1} + \sum_{k=0}^{2n} \frac{1}{2n-k+1}\right) \\
&= \frac{1}{2(n+1)}\left(\sum_{k=0}^{2n} \frac{1}{k+1} + \sum_{j=0}^{2n} \frac{1}{j+1}\right) \\
&= \frac{1}{n+1} \sum_{k=0}^{2n} \frac{1}{k+1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
J(r) = {}& 2\pi \ln^2\left(r^4+1\right) - 2\pi \ln\left(r^4+1\right) \sum_{k=1}^{\infty} \left(\frac{r^2}{r^4+1}\right)^{2k} \frac{(2k)!}{k!^2\, k} \\
&+ 2\pi \sum_{n=0}^{\infty} \left(\frac{r^2}{r^4+1}\right)^{2n+2} \frac{(2n+2)!}{(n+1)!^2\,(n+1)} \sum_{k=0}^{2n} \frac{1}{k+1} \\
= {}& 2\pi \ln^2\left(r^4+1\right) - 2\pi \ln\left(r^4+1\right) \sum_{n=0}^{\infty} \left(\frac{r^2}{r^4+1}\right)^{2n+2} \frac{(2n+2)!}{(n+1)!^2\,(n+1)} \\
&+ 2\pi \sum_{n=0}^{\infty} \left(\frac{r^2}{r^4+1}\right)^{2n+2} \frac{(2n+2)!}{(n+1)!^2\,(n+1)} \sum_{k=0}^{2n} \frac{1}{k+1} \\
= {}& 2\pi \ln^2\left(r^4+1\right) \\
&- 2\pi \sum_{n=0}^{\infty} \frac{(2n+2)!}{(n+1)!^2\,(n+1)} \left(\frac{r^2}{r^4+1}\right)^{2n+2} \left\{\ln\left(r^4+1\right) - \sum_{k=0}^{2n} \frac{1}{k+1}\right\}.
\end{aligned}
\tag{3.3.1}
$$

Since

$$\begin{aligned}
&\frac{(2n+2)\,!}{(n+1)\,!^2\,(n+1)}\frac{d}{dr}\left[\left(\frac{r^2}{r^4+1}\right)^{2n+2}\right] \\
=\ & \frac{(2n+2)\,!\,(2n+2)}{(n+1)\,!^2\,(n+1)}\left(\frac{r^2}{r^4+1}\right)^{2n+1}\left(\frac{2r\left(r^4+1\right)-r^2\left(4r^3\right)}{\left(r^4+1\right)^2}\right) \\
=\ & -4r\frac{(2n+2)\,!}{(n+1)\,!^2}\left(\frac{r^2}{r^4+1}\right)^{2n+1}\frac{r^4-1}{\left(r^4+1\right)^2},
\end{aligned}$$

Applying $\frac{d}{dr}|_{r=\rho}$ to (3.3.1) yields

$$\begin{aligned}
&J_r\left(\rho\right) \\
&= \frac{16\pi\rho^3}{\rho^4+1}\ln\left(\rho^4+1\right) \\
&- 2\pi\sum_{n=0}^{\infty}\frac{(2n+2)\,!}{(n+1)\,!^2\,(n+1)}\left[\ln\left(\rho^4+1\right)-\sum_{k=0}^{2n}\frac{1}{k+1}\right]\frac{d}{dr}|_{r=\rho}\left[\left(\frac{r^2}{r^4+1}\right)^{2n+2}\right] \\
&- 2\pi\sum_{n=0}^{\infty}\left(\frac{\rho^2}{\rho^4+1}\right)^{2n+2}\frac{(2n+2)\,!}{(n+1)\,!^2\,(n+1)}\frac{4\rho^3}{\rho^4+1} \\
&= \frac{8\pi\rho^3}{\rho^4+1}\left[2\ln\left(\rho^4+1\right)-\sum_{n=0}^{\infty}\left(\frac{\rho^2}{\rho^4+1}\right)^{2n+2}\frac{(2n+2)\,!}{(n+1)\,!^2\,(n+1)}\right] \\
&+ 8\pi\rho\frac{\rho^4-1}{\left(\rho^4+1\right)^2}\sum_{n=0}^{\infty}\frac{(2n+2)\,!}{(n+1)\,!^2}\left(\frac{\rho^2}{\rho^4+1}\right)^{2n+1}\left[\ln\left(\rho^4+1\right)-\sum_{k=0}^{2n}\frac{1}{k+1}\right]. \qquad (3.3.2)
\end{aligned}$$

3.4 The Functional Determinant

Numerical approximations using Mathematica show the rewriting of $I(\rho)$ given by (3.2.1) and (3.3.2) agrees with the integral form in (3.1.4) for $\rho = 2, 3$. Moreover, for fixed ρ, it has been observed that Mathematica numerically calculates $I\left(\rho\right)$ much faster when using the above rewriting. However, no effort is made to make this claim rigorous.

From (3.1.3) we have

$$G_-^D\left(\rho_1\right)+G_+^D\left(\rho_2\right) = -4\pi\ln 4-\rho_1 I\left(\rho_1\right)+\rho_2 I\left(\rho_2\right) = G_-^N\left(\rho_1\right)+G_+^N\left(\rho_2\right).$$

Then, (3.1.1) implies

$$\zeta'_{\mathcal{E}^*}\left(0\right) = \zeta'_{\mathcal{A}}\left(0\right)-\frac{1}{3}\ln 4+S\left(\rho_1,\rho_2\right)$$

for the Dirichlet boundary condition, where (3.2.1) gives

$$S\left(\rho_1,\rho_2\right)=\frac{1}{3}\left[\frac{2}{3}\ln\rho_1-\frac{\rho_1}{8}J_r\left(\rho_1\right)-\frac{2}{3}\ln\rho_2+\frac{\rho_2}{8}J_r\left(\rho_2\right)\right].$$

Therefore

$$\det\left(-\Delta_{\mathcal{E}^*}\right)=e^{-\zeta'_{\mathcal{E}^*}(0)}=\det\left(-\Delta_{\mathcal{A}}\right)e^{-S(\rho_1,\rho_2)}.$$

From [43], we know

$$\det\left(-\Delta_{\mathcal{A}}\right)=\pi^{-1}\left(\frac{\rho_1}{\rho_2}\right)^{1/3}\ln\left(\frac{\rho_2}{\rho_2}\right)\left\{H\left(\left[\frac{\rho_1}{\rho_2}\right]^2\right)\right\}^{-2},$$

where

$$H\left(x\right)=\prod_{j=1}^{\infty}\left(1-x^n\right)^{-1}.$$

Therefore

$$\det\left(-\Delta_{\mathcal{E}^*}\right)=\pi^{-1}\left(\frac{\rho_1}{\rho_2}\right)^{1/3}\ln\left(\frac{\rho_2}{\rho_2}\right)\left\{H\left(\left[\frac{\rho_1}{\rho_2}\right]^2\right)\right\}^{-2}4^{1/3}e^{-S(\rho_1,\rho_2)}$$

for the Dirichlet boundary condition. Similarly, from [43] we know

$$\det{}_*\left(-\Delta_{\mathcal{A}}\right)=2\pi\left(\frac{\rho_1}{\rho_2}\right)^{1/3}\left[\left(\rho_2\right)^2-\left(\rho_1\right)^2\right]\left\{H\left(\left[\frac{\rho_1}{\rho_2}\right]^2\right)\right\}^{-2}$$

for the Neumann boundary condition, where $\det_*$ indicates that the zero mode should be omitted. Then, by (3.1.2), we have

$$\det{}_*\left(-\Delta_{\mathcal{E}^*}\right)=\pi\left(\frac{\rho_1}{\rho_2}\right)^{1/3}\left[\left(\rho_2\right)^2-\left(\rho_1\right)^2\right]\left\{H\left(\left[\frac{\rho_1}{\rho_2}\right]^2\right)\right\}^{-2}2^{5/3}e^{-S(\rho_1,\rho_2)}$$

for the Neumann boundary condition.

CHAPTER FOUR

Perturbation Theory

4.1 Results for Nondegenerate Eigenvalues

When the formula for the conformal map of interest is complicated, we can use an approximation to f to approximate the operator $H = -|f'|^{-2}\Delta_{\mathcal{M}}$. In this chapter, we show how knowledge of $H_0 = -\Delta_{\mathcal{M}}$ can be used to study an approximation of H. The results of this chapter do not involve a conformal map f and apply to all Hermitian operators for which a certain perturbative expansion exists.

Now, suppose E_n and E_n^0 are eigenvalues with corresponding eigenkets $|u_n\rangle$ and $|u_n^0\rangle$, respectively. That is,

$$H|u_n\rangle = E_n|u_n\rangle \text{ and } H_0\left|u_n^0\right\rangle = E_n^0\left|u_n^0\right\rangle.$$

Suppose E_n^0 is nondegenerate for all n. That is, $E_n^0 \neq E_k^0$ for $k \neq n$. Further, suppose the perturbative expansions

$$H = H_0 + H_1\epsilon + H_2\epsilon^2 + \cdots, \tag{4.1.1}$$

$$E_n = E_n^0 + E_n^1\epsilon + E_n^2\epsilon^2 + \cdots, \tag{4.1.2}$$

and

$$|u_n\rangle = \left|u_n^0\right\rangle + \left|u_n^1\right\rangle\epsilon + \left|u_n^2\right\rangle\epsilon^2 + \cdots \tag{4.1.3}$$

exist for some $\epsilon \in \mathbb{R}$, where H and H_j are Hermitian for all $j \geq 0$. Given a Hermitian operator, there exists an orthonormal basis of eigenkets for the underlying Hilbert space $\mathcal{H}$ [22]. So, we will assume any collection of eigenkets corresponding to the same Hermitian operator is an orthonormal basis for $\mathcal{H}$. Since the collection of eigenkets $|u_k^0\rangle$ forms a basis for $\mathcal{H}$, we have

$$\left|u_n^1\right\rangle = \sum_k a_{k,n}^1\left|u_k^0\right\rangle \text{ and } \left|u_n^2\right\rangle = \sum_k a_{k,n}^2\left|u_k^0\right\rangle \tag{4.1.4}$$

for some scalars $a_{k,n}^{j}$. For each eigenket $|u_n\rangle$, we may choose $\alpha_n \in \mathbb{R}$ so that $\Im e^{i\alpha_n} a_{nn}^1 = 0$. Note $e^{i\alpha_n} |u_n\rangle$ is also an eigenket of H. So, by rescaling, we may assume $\Im a_{nn}^1 = 0$ for each n. Under this assumption, we will show $a_{nn}^1 = 0$. Now, let E_n be the eigenvalue associated with the rescaled $|u_n\rangle$. The remainder of this section will be dedicated to showing

$$E_n^1 = \langle u_n^0 | H_1 | u_n^0 \rangle , \tag{4.1.5}$$

$$a_{k,n}^1 = \begin{cases} \frac{\langle u_k^0 | H_1 | u_n^0 \rangle}{E_n^0 - E_k^0} & \text{if } k \neq n \\ 0 & \text{if } k = n \end{cases} , \tag{4.1.6}$$

and

$$E_n^2 = \langle u_n^0 | H_2 | u_n^0 \rangle + \sum_{k \neq n} \frac{|\langle u_k^0 | H_1 | u_n^0 \rangle|^2}{E_n^0 - E_k^0}, \tag{4.1.7}$$

which agrees with [24].

Now, substituting (4.1.1) and (4.1.3) into $H|u_n\rangle = E_n |u_n\rangle$ gives

$$\begin{aligned} & H|u_n\rangle \\ = \; & \left(H_0 + \epsilon H_1 + \epsilon^2 H_2 + \cdots\right)\left(|u_n^0\rangle + \epsilon |u_n^1\rangle + \epsilon^2 |u_n^2\rangle + \cdots\right) \\ = \; & E_n^0 |u_n^0\rangle + \left(H_0 |u_n^1\rangle + H_1 |u_n^0\rangle\right)\epsilon + \left(H_2 |u_n^0\rangle + H_1 |u_n^1\rangle + H_0 |u_n^2\rangle\right)\epsilon^2 + \cdots \end{aligned}$$

On the other hand, (4.1.2) gives

$$\begin{aligned} & E_n |u_n\rangle \\ = \; & \left(E_n^0 + E_n^1 \epsilon + E_n^2 \epsilon^2 + \cdots\right)\left(|u_n^0\rangle + |u_n^1\rangle \epsilon + |u_n^2\rangle \epsilon^2 + \cdots\right) \\ = \; & E_n^0 |u_n^0\rangle + \left(E_n^0 |u_n^1\rangle \epsilon + E_n^1 |u_n^0\rangle\right) + \left(E_n^0 |u_n^2\rangle + E_n^1 |u_n^1\rangle + E_n^2 |u_n^2\rangle\right)\epsilon^2 + \cdots . \end{aligned}$$

Thus

$$H_0 |u_n^1\rangle + H_1 |u_n^0\rangle = E_n^0 |u_n^1\rangle + E_n^1 |u_n^0\rangle , \tag{4.1.8}$$

and

$$H_2 |u_n^0\rangle + H_1 |u_n^1\rangle + H_0 |u_n^2\rangle = E_n^0 |u_n^2\rangle + E_n^1 |u_n^1\rangle + E_n^2 |u_n^0\rangle . \tag{4.1.9}$$

Let

$$\delta(n,k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}.$$

Since the eigenkets $|u_n\rangle$ are orthonormal,

$$\begin{aligned}
\delta(n,k) &= \langle u_n | u_k \rangle \\
&= \left(\langle u_n^0 | + \langle u_n^1 | \epsilon + \langle u_n^2 | \epsilon^2 + \cdots \right) \left(|u_k^0\rangle + |u_k^1\rangle \epsilon + |u_k^2\rangle \epsilon^2 + \cdots \right) \\
&= \langle u_n^0 | u_k^0 \rangle + \left(\langle u_n^1 | u_k^0 \rangle + \langle u_n^0 | u_k^1 \rangle \right) \epsilon \\
&\quad + \left(\langle u_n^0 | u_k^2 \rangle + \langle u_n^1 | u_k^1 \rangle + \langle u_n^2 | u_k^0 \rangle \right) \epsilon^2 + \cdots .
\end{aligned}$$

Since the eigenkets $|u_n^0\rangle$ are orthonormal, we have

$$\langle u_n^1 | u_k^0 \rangle + \langle u_n^0 | u_k^1 \rangle = 0, \tag{4.1.10}$$

and

$$\langle u_n^0 | u_k^2 \rangle + \langle u_n^1 | u_k^1 \rangle + \langle u_n^2 | u_k^0 \rangle = 0, \tag{4.1.11}$$

for all k and n. We first use (4.1.10) to show (4.1.5) and (4.1.6), then use (4.1.11) to show (4.1.7).

$$H_0 |u_n^1\rangle + H_1 |u_n^0\rangle = E_n^0 |u_n^1\rangle + E_n^1 |u_n^0\rangle . \tag{4.1.8}$$

In (4.1.8), we use (4.1.4) to expand $|u_n^1\rangle$ in terms of the unperturbed problem, which gives

$$\sum_k a_{k,n}^1 E_k^0 |u_k^0\rangle + H_1 |u_n^0\rangle = E_n^0 \sum_k a_{k,n}^1 |u_k^0\rangle + E_n^1 |u_n^0\rangle . \tag{4.1.12}$$

Applying the bra $\langle u_j^0 |$ to the left of both sides above yields

$$a_{j,n}^1 E_j^0 + \langle u_j^0 | H_1 | u_n^0 \rangle = E_n^0 a_{j,n}^1 + E_j^1 \delta(n,j) .$$

For $j \neq n$ we have

$$a_{j,n}^1 \left(E_j^0 - E_n^0 \right) = - \langle u_j^0 | H_1 | u_n^0 \rangle .$$

Thus

$$a^1_{j,n} = \frac{\langle u^0_j | H_1 | u^0_n \rangle}{E^0_n - E^0_j}$$

for $j \neq n$. For $j = n$ we have

$$E^1_n = \langle u^0_n | H_1 | u^0_n \rangle .$$

Using (4.1.4) in (4.1.10) yields

$$0 = \overline{\left\langle \sum_k a^1_{k,n} u^0_k | u^1_n \right\rangle} + \left\langle u^0_n | \sum_k a^1_{k,k} | u^1_k \right\rangle = \overline{a^1_{k,n}} + a^1_{n,k}.$$

For $k = n$ we have

$$\overline{a^1_{n,n}} + a^1_{n,n} = 0.$$

That is, $\Re a^1_{n,n} = 0$. Since $\Im a^1_{n,n}$ by assumption, we conclude $a^1_{n,n} = 0$, and (4.1.6) follows.

Next, we use (4.1.11) to show (4.1.7). From (4.1.4) and (4.1.11), we have

$$\begin{aligned}
& H_2 \left| u^0_n \right\rangle + H_1 \sum_k a^1_{k,n} \left| u^0_k \right\rangle + H_0 \sum_k a^2_{k,n} \left| u^0_k \right\rangle \\
= \; & H_2 \left| u^0_n \right\rangle + \sum_k a^1_{k,n} H_1 \left| u^0_k \right\rangle + \sum_k a^2_{k,n} E^0_k \left| u^0_k \right\rangle \\
= \; & E^0_n \left| u^2_n \right\rangle + E^1_n \left| u^1_n \right\rangle + E^2_n \left| u^0_n \right\rangle \\
= \; & E^0_n \sum_k a^2_{k,n} \left| u^0_k \right\rangle + E^1_n \sum_k a^1_{k,n} \left| u^0_k \right\rangle + E^2_n \left| u^0_n \right\rangle .
\end{aligned}$$

Applying $\left\langle u^0_j \right|$ and using (4.1.6) shows

$$\begin{aligned}
& \langle u^0_j | H_2 | u^0_n \rangle + \sum_k a^1_{k,n} \langle u^0_j | H_1 | u^0_k \rangle + \sum_k a^2_{k,n} E^0_k \delta(j,k) \\
= \; & \langle u^0_j | H_2 | u^0_n \rangle + \sum_{k \neq n} \frac{\langle u^0_k | H_1 | u^0_n \rangle}{E^0_n - E^0_k} \langle u^0_j | H_1 | u^0_k \rangle + a^2_{j,n} E^0_j \\
= \; & E^0_n a^2_{j,n} + E^1_n a^1_{j,n} + E^2_n \delta(n,j) .
\end{aligned}$$

For $j = n$ we use that H_1 is Hermitian to find

$$\begin{aligned}
E^2_n &= \langle u^0_n | H_2 | u^0_n \rangle + \sum_{k \neq n} \frac{\langle u^0_k | H_1 | u^0_n \rangle}{E^0_n - E^0_k} \langle u^0_n | H_1 | u^0_k \rangle + a^2_{n,n} E^0_n - E^0_n a^2_{n,n} - E^1_n a^1_{n,n} \\
&= \langle u^0_n | H_2 u^0_n \rangle + \sum_{k \neq n} \frac{\left| \langle u^0_n | H_1 | u^0_k \rangle \right|^2}{E^0_n - E^0_k}.
\end{aligned}$$

4.2 Special Case Degenerate Theory

We will be interested in the case where there exists some m such that E_m^0 is degenerate. That is, $E_m^0 = E_j^0$ for some $j \neq m$. Our interest stems from the fact that all but one of the eigenvalues of minus the Dirichlet Laplacian on the disk has degeneracy two [3].

When an eigenvalue is degenerate, the denominators in (4.1.6) and (4.1.7) could be zero. However, we will see that if the eigenkets $|u_n^0\rangle$ are chosen appropriately, then slight modifications of equations (4.1.5), (4.1.6) and (4.1.7) will hold. To this end, suppose

$$M = \left\{ j : E_j^0 = E_m^0 \right\}.$$

Suppose further that all the eigenkets $|u_n^0\rangle$ are orthonormal, including those with the same eigenvalue. Applying $\langle u_l^0|$ to both sides of (4.1.12) with $n = m$ gives

$$\sum_k a_{k,m}^1 E_k^0 \delta(l,k) + \left\langle u_l^0 | H_1 | u_m^0 \right\rangle = E_m^0 \sum_k a_{k,m}^1 \delta(l,k) + E_m^1 \delta(l,m).$$

If $l \neq m$ then

$$\left\langle u_l^0 | H_1 | u_m^0 \right\rangle = \sum_k \left(E_m^0 - E_k^0 \right) a_{k,m}^1 \delta(l,k).$$

Note $E_m^0 = E_k^0$ if and only if $k \in M$. So

$$\left\langle u_l^0 | H_1 | u_m^0 \right\rangle = \sum_{k \notin M} \left(E_m^0 - E_k^0 \right) a_{k,m}^1 \delta(l,k).$$

If $l \in M$ then $\langle u_l^0 | H_1 | u_m^0 \rangle = 0$. If $l \notin M$ then

$$a_{l,m}^1 = \frac{\langle u_l^0 | H_1 | u_m^0 \rangle}{E_m^0 - E_l^0}. \tag{4.2.1}$$

If $l = m$ then

$$E_m^1 = \left\langle u_m^0 | H_1 | u_m^0 \right\rangle. \tag{4.2.2}$$

Next, applying $\langle u_l^0|$ to both sides of (4.1.12) gives

$$\left\langle u_l^0 | H_2 | u_m^0 \right\rangle + \sum_k a_{k,m}^1 \left\langle u_l^0 | H_1 | u_k^0 \right\rangle + \sum_k a_{k,m}^2 E_k^0 \delta(l,k)$$

$$= E_n^0 \sum_k a_{k,m}^2 \delta(l,k) + E_m^1 \sum_k a_{k,m}^1 \left\langle u_l^0 | H_1 | u_k^0 \right\rangle + E_m^2 \left\langle u_l^0 | u_k^0 \right\rangle .$$

If $l = m$ then

$$\begin{aligned} E_m^2 &= \left\langle u_l^0 | H_2 | u_m^0 \right\rangle + \sum_k a_{k,m}^1 \left\langle u_m^0 | H_1 | u_k^0 \right\rangle + \sum_k a_{k,m}^2 E_k^0 \delta(l,k) \\ &\quad - E_n^0 \sum_k a_{k,m}^2 \delta(l,k) - E_m^1 \sum_k a_{k,m}^1 \left\langle u_l^0 | H_1 | u_k^0 \right\rangle \\ &= \left\langle u_m^0 | H_2 | u_m^0 \right\rangle + \sum_k a_{k,m}^1 \left\langle u_m^0 | H_1 | u_k^0 \right\rangle - E_m^1 a_{m,m}^1 \left\langle u_m^0 | H_1 | u_m^0 \right\rangle . \end{aligned}$$

If $l \in M$ then $\langle u_l^0 | H_1 | u_m^0 \rangle = 0$, so that H_1 Hermitian and (4.2.1) imply

$$\sum_k a_{k,m}^1 \left\langle u_m^0 | H_1 u_k^0 \right\rangle = \sum_{k \notin M} \frac{\left| \langle u_k^0 | H_1 | u_m^0 \rangle \right|^2}{E_m^0 - E_k^0}.$$

Therefore, if $l \in M$ then

$$E_m^2 = \left\langle u_m^0 | H_2 | u_m^0 \right\rangle + \sum_{k \notin M} \frac{\left| \langle u_k^0 | H_1 | u_m^0 \rangle \right|^2}{E_m^0 - E_k^0}. \tag{4.2.3}$$

4.3 Spectral Zeta Function Perturbation

Since

$$E_n = \sum_{j=0}^{n} E_n^j \epsilon^j + O\left(\epsilon^{n+1}\right)$$

and

$$(1+x)^r = \sum_{j=0}^{\infty} \frac{\Gamma(r+1)}{\Gamma(r-j+1)\, j!} x^j,$$

we have

$$\begin{aligned} \zeta_H(s) &= \sum_k (E_k)^{-s} = \sum_k \left[\sum_{j=0}^{n} E_k^j \epsilon^j + O\left(\epsilon^{n+1}\right) \right]^{-s} \\ &= \sum_k \left(E_k^0\right)^{-s} \left[1 + \sum_{j=1}^{n} \frac{E_k^j}{E_k^0} \epsilon^j \right]^{-s} + O\left(\epsilon^{n+1}\right) \\ &= \sum_k \left(E_k^0\right)^{-s} \sum_{j=0}^{\infty} \frac{\Gamma(-s+1)}{\Gamma(-s-j+1)\, j!} \left(\sum_{i=1}^{n} \frac{E_k^i}{E_k^0} \epsilon^i \right)^j + O\left(\epsilon^{n+1}\right), \end{aligned}$$

Note that

$$\frac{\Gamma\left(1-s\right)}{\Gamma\left(-s\right)}=-s$$

implies

$$\frac{\Gamma\left(1-s\right)}{\Gamma\left(-s-1\right)}=\frac{-\Gamma\left(1-s\right)\left(s+1\right)}{\Gamma\left(-s\right)}=s\left(s+1\right).$$

For $n=2$ we find

$$\zeta_{H}\left(s\right)=\zeta_{H_0}\left(s\right)+\zeta_{H}^{1}\left(s\right)\epsilon+\zeta_{H}^{2}\left(s\right)\epsilon^{2}+O\left(\epsilon^{3}\right), \tag{4.3.1}$$

where

$$\zeta_{H}^{1}\left(s\right)=-s\sum_{k}\left(E_{k}^{0}\right)^{-s}\frac{E_{k}^{1}}{E_{k}^{0}} \tag{4.3.2}$$

and

$$\zeta_{H}^{2}\left(s\right)=\sum_{k}\left(E_{k}^{0}\right)^{-s}\left[\frac{\left(s^{2}+s\right)}{2}\left(\frac{E_{k}^{1}}{E_{k}^{0}}\right)^{2}-s\frac{E_{k}^{2}}{E_{k}^{0}}\right]. \tag{4.3.3}$$

CHAPTER FIVE

Ellipse Zeta Function Calculations

5.1 The Conformal Map

Let $a > b > 0$ and consider the ellipse

$$\mathcal{E}_\epsilon = \left\{ w \in \mathbb{C} : \left(\frac{\Re w}{a}\right)^2 + \left(\frac{\Im w}{b}\right)^2 < 1 \right\},$$

where the eccentricity, which is a measure of how much an ellipse deviates from a disk, is given by

$$\epsilon = \sqrt{1 - \left(\frac{b}{a}\right)^2}. \tag{5.1.1}$$

Let

$$\mathcal{D}_R = \{z : |z| < R\}$$

where $R = \sqrt{ab}$. Note $\mathcal{E}_\epsilon \to \mathcal{D}_a$ and $\epsilon \to 0$ as $a \to b$. That is, when ϵ is close to zero the ellipse $\mathcal{E}_\epsilon$ is close to the disk $\mathcal{D}_a$.

A consequence of the Riemann mapping theorem says that if U is a proper, nonempty, simply connected open subset of $\mathbb{C}$ then there exists a biholomorphic mapping from $\mathcal{D}_R$ to U [1, 34]. Since this mapping is invertible and analytic with analytic inverse, it is conformal. So, theoretically, if we know values of $\zeta_{\mathcal{D}_R}$ then we can calculate the corresponding values of ζ_U where U is a set for which the Riemann mapping theorem applies. However, actually calculating the values ζ_U can prove to be quite difficult since the formula for the mapping can be rather complicated. For instance, a biholomorphic map $g : \mathcal{E}_\epsilon \to \mathcal{D}_R$ is given by [21, 27, 29, 37, 40, 45]

$$g(w) = R\sqrt{k}sn\left(\frac{2K(k^2)}{\pi}\sin^{-1}\frac{w}{\sqrt{a^2 - b^2}}, k^2\right), \tag{5.1.2}$$

where

$$sn^{-1}(u, m) = \int_0^u \frac{dt}{\sqrt{(1 - t^2)(1 - mt^2)}}$$

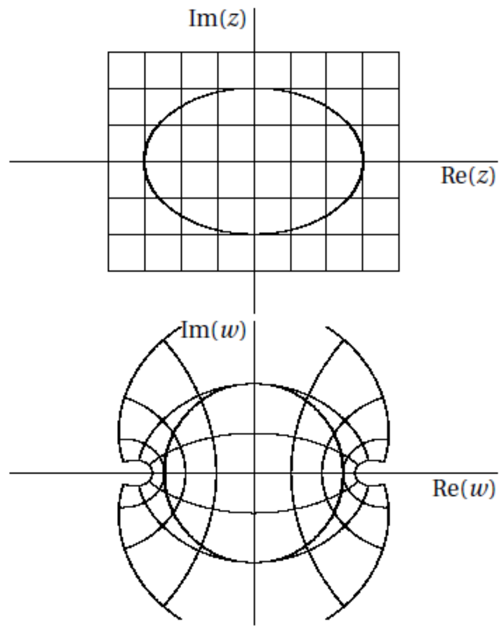

Figure 5.1. Conformal Map from $\mathcal{E}_\epsilon$ to $\mathcal{D}_R$ [21]

is the inverse elliptic sine function with $\sqrt{\cdot}$ chosen so that $\sqrt{1} = 1$, $K(m) = sn^{-1}(1, m)$ is the complete elliptic integral of the first kind,

$$h = \left(\frac{a-b}{a+b}\right)^2 \text{ and } k = \left[\frac{\theta_2(0,h)}{\theta_3(0,h)}\right]^2,$$

where θ_j are the elliptic functions of the jth kind defined by

$$\theta_2(\tau, q) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2} e^{2ni\tau},$$

and

$$\theta_3(\tau, q) = \sum_{n\in\mathbb{Z}} q^{n^2} e^{(2n+1)i\tau}.$$

Since $f = g^{-1}$ is a complicated map, we will use a perturbative expansion of f in terms of ϵ, where $\epsilon \approx 0$, to study $\zeta_{\mathcal{E}_\epsilon}$.

5.2 *Perturbing the Conformal Map*

We will derive an expansion of g in terms of $\epsilon \approx 0$, accurate up to ϵ^4. In doing so, we will use O notation. We will write

$$u(\epsilon) = O(\epsilon^n)$$

when there exists $M \in (0, \infty)$ such that

$$u(\epsilon) \leq \epsilon^n M$$

for $\epsilon \approx 0$. In this work, we will not use the above definition of O notation directly. Rather, when a function is expanded in terms of powers of ϵ, we will use $O(\epsilon^n)$ to absorb terms of order ϵ^n or greater. We will also have occasion to differentiate, with respect to a variable other than ϵ, the coefficients in the expansion of a function in terms of powers of ϵ. Although the order with respect to ϵ of coefficients will not change after such differentiation, it should be understood that the constant M in the definition of O notation above will change.

Now, since $a > 0$, we have

$$\epsilon^2 = 1 - \left(\frac{b}{a}\right)^2 \Rightarrow b = a\sqrt{1-\epsilon^2}.$$

A Taylor expansion about $\epsilon = 0$ shows

$$h = \left(\frac{a-b}{a+b}\right)^2 = \left(\frac{1-\sqrt{1-\epsilon^2}}{1+\sqrt{1-\epsilon^2}}\right)^2 = \frac{\epsilon^4}{16} + \frac{\epsilon^6}{16} + \frac{7\epsilon^8}{128} + O\left(\epsilon^{10}\right).$$

The Taylor expansions

$$\sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \frac{z^3}{16} - \frac{5z^4}{128} + O\left(z^5\right) \text{ and } \frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 + O\left(z^5\right)$$

show

$$\begin{aligned} k &= \left(\frac{\sum_{n\in\mathbb{Z}} h^{(n+1/2)^2}}{\sum_{n\in\mathbb{Z}} h^{n^2}}\right)^2 \\ &= \left(\frac{2h^{1/4} + 2\sum_{n\in\mathbb{N}} h^{n^2+n+1/4}}{1 + 2\sum_{n\in\mathbb{N}} h^{n^2}}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= 4h^{1/2}\left(\frac{1+\sum_{n\in\mathbb{N}} h^{n^2+n}}{1+2\sum_{n\in\mathbb{N}} h^{n^2}}\right)^2 \\
&= 4\left[\frac{\epsilon^4}{16}+\frac{\epsilon^6}{16}+\frac{7\epsilon^8}{128}+O\left(\epsilon^{10}\right)\right]^{1/2}\left\{\frac{1+\left[\frac{\epsilon^4}{16}+\frac{\epsilon^6}{16}+\frac{7\epsilon^8}{128}+O\left(\epsilon^{10}\right)\right]^2}{1+2\left[\frac{\epsilon^4}{16}+\frac{\epsilon^6}{16}+\frac{7\epsilon^8}{128}+O\left(\epsilon^{10}\right)\right]}\right\}^2 \\
&= \epsilon^2\left[1+\epsilon^2+\frac{7\epsilon^4}{8}+O\left(\epsilon^6\right)\right]^{1/2}\left[1+O\left(\epsilon^8\right)\right]\left[1-\frac{\epsilon^4}{8}+O\left(\epsilon^6\right)\right]^2 \\
&= \epsilon^2\left\{1+\frac{1}{2}\left[\epsilon^2+\frac{7\epsilon^4}{8}+O\left(\epsilon^6\right)\right]\right. \\
&\quad \left.-\frac{1}{8}\left[\epsilon^2+\frac{7\epsilon^4}{8}+O\left(\epsilon^6\right)\right]^2+O\left(\epsilon^6\right)\right\}\left[1-\frac{\epsilon^4}{4}+O\left(\epsilon^6\right)\right] \\
&= \left[1+\frac{\epsilon^2}{2}+\frac{5\epsilon^4}{16}+O\left(\epsilon^6\right)\right]\left[\epsilon^2-\frac{\epsilon^6}{4}+O\left(\epsilon^8\right)\right] \\
&= \epsilon^2+\frac{\epsilon^4}{2}+\frac{\epsilon^6}{16}+O\left(\epsilon^8\right).
\end{aligned}
$$

Taking

$$
K = K\left(k^2\right),\ \phi = \sin^{-1}\frac{w}{\sqrt{a^2-b^2}},\ \text{and } u = \frac{2K\phi}{\pi}
$$

shows [37]

$$
\begin{aligned}
sn\left(u,k^2\right) &= \frac{2\pi}{kK}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin\left[(2n+1)\phi\right] \\
&= \frac{2\pi}{kK}\left(\frac{q^{1/2}}{1-q}\sin\phi+\frac{q^{3/2}}{1-q^3}\sin 3\phi+\frac{q^{5/2}}{1-q^5}\sin 5\phi+\cdots\right),
\end{aligned}
$$

where the so-called nome q, the same q that appears in the definitions for the elliptic functions θ_j, can be expressed in terms of the parameter k, with $0 < k \ll 1$. Indeed, [44]

$$
q = m + O\left(m^5\right)
$$

where

$$
2m = \frac{1-\sqrt{k'}}{1+\sqrt{k'}}
$$

and $k^2+\left(k'\right)^2 = 1$. A Taylor expansion about $k = 0$ shows

$$
2m = \frac{1-\left(1-k^2\right)^{1/4}}{1+\left(1-k^2\right)^{1/4}} = \frac{1}{8}k^2+\frac{1}{16}k^4+\frac{21}{512}k^6+\frac{31}{1024}k^8+O\left(k^{10}\right).
$$

Thus

$$q=\frac{1}{16}k^2+\frac{1}{32}k^4+\frac{21}{1024}k^6+\frac{31}{2048}k^8+O\left(k^{10}\right).$$

Since [44]

$$\frac{2kK}{\pi}=4q^{1/2}\left[1+q^2+O\left(q^6\right)\right]^2,$$

we have

$$\begin{aligned}
& \frac{2\pi}{kK} \\
= & \frac{4\pi}{2kK} \\
= & \frac{1}{q^{1/2}\left[1+q^2+O\left(q^6\right)\right]^2} \\
= & \left[\frac{1}{16}k^2+\frac{1}{32}k^4+\frac{21}{1024}k^6+\frac{31}{2048}k^8+O\left(k^{10}\right)\right]^{-1/2} \\
& \times\left[1+\frac{1}{256}k^4+\frac{1}{256}k^6+O\left(k^8\right)\right]^{-2} \\
= & \frac{4}{k}\left[1+\frac{1}{2}k^2+\frac{21}{64}k^4+\frac{31}{128}k^6+O\left(k^8\right)\right]^{-1/2}\left[1+\frac{1}{128}k^4+\frac{1}{128}k^6+O\left(k^8\right)\right]^{-1} \\
= & \frac{4}{k}\left[1-\frac{1}{2}k^2-\frac{5}{64}k^4-\frac{5}{128}k^6+O\left(k^8\right)\right]^{1/2}\left[1-\frac{1}{128}k^4-\frac{1}{128}k^6+O\left(k^8\right)\right] \\
= & \frac{4}{k}\left[1-\frac{1}{4}k^2-\frac{9}{128}k^4-\frac{19}{512}k^6+O\left(k^8\right)\right]\left[1-\frac{1}{128}k^4-\frac{1}{128}k^6+O\left(k^8\right)\right] \\
= & \frac{4}{k}\left[1-\frac{1}{4}k^2-\frac{5}{64}k^4-\frac{11}{256}k^6+O\left(k^8\right)\right]=\frac{4}{k}-k-\frac{5}{16}k^3-\frac{11}{64}k^5+O\left(k^7\right).
\end{aligned}$$

We find

$$\begin{aligned}
\frac{q^{1/2}}{1-q} &= \frac{\frac{k}{4}\left[1+\frac{1}{2}k^2+\frac{21}{64}k^4+O\left(k^6\right)\right]^{1/2}}{1-\left[\frac{1}{16}k^2+\frac{1}{32}k^4+O\left(k^6\right)\right]} \\
&= \frac{k}{4}\left[1+\frac{1}{4}k^2+\frac{17}{128}k^4+O\left(k^6\right)\right]\left[1+\frac{1}{16}k^2+\frac{9}{256}k^4+O\left(k^6\right)\right] \\
&= \frac{1}{4}k+\frac{5}{64}k^3+\frac{47}{1024}k^5+O\left(k^7\right).
\end{aligned}$$

Similarly,

$$\frac{q^{3/2}}{1-q^3}=\frac{1}{64}k^3+\frac{3}{256}k^5+O\left(k^7\right)$$

and

$$\frac{q^{5/2}}{1-q^5} = \frac{1}{1024}k^5 + O\left(k^7\right).$$

Using the above expansions and trigonometric identities, we find

$$\begin{aligned}
& sn\left(u,k^2\right) \\
= & \left[\frac{4}{k} - k - \frac{5}{16}k^3 - \frac{11}{64}k^5 + O\left(k^7\right)\right] \\
& \times\left[\left(\frac{1}{4}k + \frac{5}{64}k^3 + \frac{47}{1024}k^5\right)\sin\phi + \left(\frac{1}{64}k^3 + \frac{3}{256}k^5\right)\sin 3\phi \right. \\
& \left. + \frac{1}{1024}k^5 \sin 5\phi + O\left(k^7\right)\right] \\
= & \sin\phi + \frac{\sin\phi}{4}\left(1-\sin^2\phi\right)k^2 + \frac{\sin\phi}{64}\left(9 - 13\sin^2\phi + 4\sin^4\phi\right)k^4 + O\left(k^6\right).
\end{aligned}$$

Since $a, b, \epsilon > 0$, we have

$$\sqrt{a^2-b^2} = \sqrt{a^2 - \left(a\sqrt{1-\epsilon^2}\right)^2} = a\epsilon = \frac{\epsilon\sqrt{ab}\sqrt{a}}{\sqrt{b}} = \frac{\epsilon R}{\sqrt{b}}\sqrt{\frac{b}{\sqrt{1-\epsilon^2}}} = \frac{\epsilon R}{\left(1-\epsilon^2\right)^{1/4}}.$$

This implies

$$\sin\phi = \frac{w\left(1-\epsilon^2\right)^{1/4}}{\epsilon R}.$$

Note

$$\sqrt{1-\epsilon^2} = 1 - \frac{\epsilon^2}{2} - \frac{\epsilon^4}{8} + O\left(\epsilon^6\right) \Rightarrow \left(1-\epsilon^2\right)^{1/4} = 1 - \frac{\epsilon^2}{4} - \frac{3\epsilon^4}{32} + O\left(\epsilon^6\right)$$

and

$$\sqrt{k} = \epsilon\sqrt{1 + \frac{\epsilon^2}{2} + \frac{\epsilon^4}{16} + O\left(\epsilon^6\right)} = \epsilon + \frac{\epsilon^3}{4} + O\left(\epsilon^7\right).$$

The expansions above and a bit of algebra shows

$$g\left(w\right) = R\sqrt{k}sn\left(u,k^2\right) = w - \frac{w^3}{4R^2}\epsilon^2 + \frac{3R^4w - 4R^2w^3 + 2w^5}{32R^4}\epsilon^4 + O\left(\epsilon^6\right), \quad (5.2.1)$$

which agrees with [21] up to order ϵ^2. Focusing on $g'(w)$, we find

$$\frac{du}{dw} = \frac{2K}{\pi w}\ \tan\phi.$$

Furthermore, [44] gives

$$\frac{d}{du}\, sn(u,k^2) = cn(u,k^2)dn(u,k^2),$$

where the representations

$$cn(u,k^2) = \frac{2\pi}{Kk}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\ \cos\left[(2n+1)\,\phi\right]$$

and

$$dn(u,k^2) = \frac{\pi}{2K} + \frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{q^{n}}{1+q^{2n}}\ \cos(2n\phi)$$

are valid. Note the formula for $cn(u,k^2)$ in [44] has a typo. Then, proceeding in a manner analogous to how (5.2.1) was obtained, we find

$$g'\left(w\right) = 1 - \frac{3w^2}{4R^2}\epsilon^2 + \frac{3R^4 - 12R^2w^2 + 10w^4}{32R^4}\epsilon^4 + O\left(\epsilon^6\right). \qquad (5.2.2)$$

5.3 The Functional Determinant

In this section, we use the results of the previous section to find a perturbative expansion of $F = \ln|f'|$. We then use this expansion of F to give an expansion of $\zeta'_{\mathcal{E}_\epsilon}(0)$. Let

$$z = g\left(w\right) \in \mathcal{D}_R \text{ and } w = f\left(z\right) \in \mathcal{E}_\epsilon.$$

We take (5.2.2) and use the inverse function theorem followed by the geometric series formula, which is valid for $\epsilon \approx 0$, to obtain

$$\begin{aligned} f'\left(z\right) &= \frac{1}{g'\left(f\left(z\right)\right)} = \left\{1 - \left[\frac{3w^2}{4R^2}\epsilon^2 - \frac{3R^4 - 12R^2w^2 + 10w^4}{32R^4}\epsilon^4 + O\left(\epsilon^6\right)\right]\right\}^{-1} \\ &= 1 + \frac{3w^2}{4R^2}\epsilon^2 - \frac{3R^4 - 12R^2w^2 - 8w^4}{32R^4}\epsilon^4 + O\left(\epsilon^6\right). \end{aligned}$$

Note

$$z = w + O\left(\epsilon^2\right)$$

implies

$$w = z + O\left(\epsilon^2\right).$$

We want the above expansion of $f'(z)$ accurate up to $O(\epsilon^6)$, so we must find A and B such that

$$w = z + A\epsilon^2 + B\epsilon^3 + O\left(\epsilon^4\right).$$

We find from (5.2.1) that

$$\begin{aligned}
z &= f(w) = w - \frac{w^3}{4R^2}\epsilon^2 + O\left(\epsilon^4\right) \\
&= z + A\epsilon^2 + B\epsilon^3 - \frac{1}{4R^2}\left(z + A\epsilon^2 + B\epsilon^3\right)^3\epsilon^2 + O\left(\epsilon^4\right) \\
&\Rightarrow \; 0 = A\epsilon^2 + B\epsilon^3 - \frac{z^3}{4R^2}\epsilon^2 + O\left(\epsilon^4\right) \\
&\Rightarrow \; A = \frac{z^3}{4R^2} \text{ and } B = 0.
\end{aligned}$$

So

$$w = z + \frac{z^3}{4R^2}\epsilon^2 + O\left(\epsilon^4\right),$$

which implies

$$\begin{aligned}
f'(z) &= 1 + \frac{3}{4R^2}\left(z + \frac{z^3}{4R^2}\epsilon^2\right)^2\epsilon^2 \\
&\quad - \frac{1}{32R^4}\left[3R^4 - 12R^2\left(z + \frac{z^3}{4R^2}\epsilon^2\right)^2 - 8z^4\right]\epsilon^4 + O\left(\epsilon^6\right) \\
&= 1 + \frac{3z^2}{4R^2}\epsilon^2 + \frac{20z^4 + 12R^2z^2 - 3R^4}{32R^4}\epsilon^4 + O\left(\epsilon^6\right).
\end{aligned}$$

Letting $z = re^{i\theta}$ yields

$$\begin{aligned}
&\left|f'\left(re^{i\theta}\right)\right|^2 \\
&= f'\left(re^{i\theta}\right)\overline{f'\left(re^{i\theta}\right)} \\
&= \left(1 + \frac{3r^2}{4R^2}e^{2i\theta}\epsilon^2 + \frac{20r^4e^{4i\theta} + 12R^2r^2e^{2i\theta} - 3R^4}{32R^4}\epsilon^4\right) \\
&\quad \times \left(1 + \frac{3r^2}{4R^2}e^{-2i\theta}\epsilon^2 + \frac{20r^4e^{-4i\theta} + 12R^2r^2e^{-2i\theta} - 3R^4}{32R^4}\epsilon^4\right) + O\left(\epsilon^6\right) \\
&= 1 + \frac{3r^2}{4R^2}\left(e^{2i\theta} + e^{-2i\theta}\right)\epsilon^2 + \frac{9r^4}{16R^4}\epsilon^4 \\
&\quad + \frac{1}{32R^4}\left(20r^4e^{4i\theta} + 12R^2r^2e^{2i\theta} - 3R^4 + 20r^4e^{-4i\theta} + 12R^2r^2e^{-2i\theta} - 3R^4\right)\epsilon^4
\end{aligned}$$

$$
\begin{aligned}
& +O\left(\epsilon^6\right) \\
= \quad & 1+\frac{3r^2}{2R^2}\cos 2\theta\epsilon^2+\frac{1}{16R^4}\left(20r^4\cos 4\theta+12R^2r^2\cos 2\theta-3R^4+9r^4\right)\epsilon^4 \\
& +O\left(\epsilon^6\right).
\end{aligned}
$$

For later use, we use the geometric series formula to calculate

$$
\begin{aligned}
& \left|f'\left(re^{i\theta}\right)\right|^{-2} \\
& =\Bigg\{1-\Bigg[-\frac{3r^2}{2R^2}\cos 2\theta\epsilon^2-\frac{1}{16R^4}(20r^4\cos 4\theta+ \\
& 12R^2r^2\cos 2\theta-3R^4+9r^4)\epsilon^4+O\left(\epsilon^6\right)\Bigg]\Bigg\}^{-1} \qquad (5.3.1)\\
& =1-\frac{3r^2}{2R^2}\cos 2\theta\epsilon^2 \\
& \quad+\frac{1}{16R^4}\left(3R^4+9r^4-2r^4\cos 4\theta-12R^2r^2\cos 2\theta\right)\epsilon^4+O\left(\epsilon^6\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
& \ln\left|f'\left(re^{i\theta}\right)\right|^2 \\
= \quad & \ln\Bigg[1+\frac{3r^2}{2R^2}\cos 2\theta\epsilon^2+\frac{1}{16R^4}\left(20r^4\cos 4\theta+12R^2r^2\cos 2\theta-3R^4+9r^4\right)\epsilon^4 \\
& +O\left(\epsilon^6\right)\Bigg] \\
= \quad & \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\Bigg[\frac{3r^2}{2R^2}\cos 2\theta\epsilon^2 \\
& +\frac{1}{16R^4}\left(20r^4\cos 4\theta+12R^2r^2\cos 2\theta-3R^4+9r^4\right)\epsilon^4+O\left(\epsilon^6\right)\Bigg]^n \\
= \quad & \frac{3r^2}{2R^2}\cos 2\theta\epsilon^2+\frac{1}{16R^4}\left(11r^4\cos 4\theta+12R^2r^2\cos 2\theta-3R^4\right)\epsilon^4+O\left(\epsilon^6\right),
\end{aligned}
$$

which is valid for $\epsilon\approx 0$. Now,

$$
\begin{aligned}
F(r,\theta) \quad = \quad & \frac{1}{2}\ln\left|f'\left(re^{i\theta}\right)\right|^2=\frac{3r^2}{4R^2}\cos 2\theta\epsilon^2 \\
& +\frac{1}{32R^4}\left(11r^4\cos 4\theta+12R^2r^2\cos 2\theta-3R^4\right)\epsilon^4+O\left(\epsilon^6\right)
\end{aligned}
$$

implies

$$
F_r(r,\theta)=\frac{3r}{2R^2}\cos 2\theta\epsilon^2+\frac{1}{16R^4}\left(22r^3\cos 4\theta+12R^2r\cos 2\theta\right)\epsilon^4+O\left(\epsilon^6\right).
$$

So

$$F_r F(r,\theta) = \frac{9r^3}{8R^4}\cos^2 2\theta\epsilon^4 + O\left(\epsilon^6\right).$$

Since

$$\int_0^{2\pi} \cos^2 2\theta d\theta = \pi \text{ and } \int_0^{2\pi} d\theta \cos n\theta = 0 \text{ for } n \in \mathbb{N},$$

we find

$$\begin{aligned} & \int_0^{2\pi} R d\theta \left(F\left[2R^{-1} + F_r\right] + 3F_r\right)(R,\theta) \\ = & \left(\frac{2(-6)}{32} + \frac{9}{8}\right)\pi\epsilon^4 + O\left(\epsilon^6\right) = \frac{3\pi}{4}\epsilon^4 + O\left(\epsilon^6\right). \end{aligned}$$

By (2.4.2), we have

$$\begin{aligned} \zeta'_{E_\epsilon}(0) &= \zeta'_{D_R}(1;0) + \frac{1}{12\pi}\int_0^{2\pi} R d\theta \left(F\left[2R^{-1} + F_r\right] + 3F_r\right)(R,\theta) \\ &= \zeta'_{D_R}(0) + \frac{1}{16}\epsilon^4 + O\left(\epsilon^6\right). \end{aligned} \tag{5.3.2}$$

From [5], we know

$$\zeta'_{\mathcal{D}_R}(0) = \zeta'_{\mathcal{D}}(0) + \frac{1}{3}\ln R$$

where

$$\zeta'_{\mathcal{D}}(0) = \frac{5}{12} + 2\zeta'(-1) + \frac{1}{2}\ln\pi + \frac{1}{6}\ln 2$$

and ζ is the Riemann zeta function.

5.4 Eigenpairs on the Disk

Before applying the perturbative expansion formulas of Chapter Four, we study the eigenpairs of

$$-\Delta_{\mathcal{D}_R}\varphi = \lambda\varphi \tag{5.4.1}$$

with $\varphi|_{\partial\mathcal{D}_R} = 0$. Let

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(m+k)!}\left(\frac{x}{2}\right)^{m+2k}$$

be the mth Bessel function of the first kind. Separating variables and solving the resulting two one-dimensional eigenvalue problems shows the eigenpairs of $-\Delta_{\mathcal{D}_R}$ are [3]

$$\left(\left(E_{|m|,n}\right)^2, u_{m,n}\right)_{m\in\mathbb{Z},n\in\mathbb{N}}$$

where $E_{|m|,n}R$ is the n th positive zero of $J_{|m|}$ and

$$u_{m,n} = J_{|m|}\left(E_{|m|,n}r\right)e^{im\theta}C_{m,n}$$

with $C_{m,n}$ a normalization constant. From [42],

$$E_{|m|,n}r \in \mathbb{R},$$

$$\overline{J_m(z)} = J_{\overline{m}}(\overline{z}),$$

and

$$\int_0^R rdrJ_{|m|}^2\left(E_{|m|,n}r\right) = \frac{R^2}{2}J_{|m+1|}^2\left(E_{|m|,n}R\right).$$

Using this, we find

$$\begin{aligned}
\langle u_{m,n}|u_{k.l}\rangle &= C_{m,n}C_{k,l}\int_0^{2\pi}d\theta e^{(k-m)i\theta}\int_0^R rdrJ_{|m|,n}\left(E_{|m|,n}r\right)J_{|k|.l}\left(E_{|k|,l}r\right) \\
&= \begin{cases} C_{m,n}^2R^2\pi J_{|m+1|}^2\left(E_{|m|,n}R\right) & (k,l)=(m,n) \\ 0 & (k,l)\neq(m,n) \end{cases}.
\end{aligned}$$

By Bourget's Theorem [42], $J_{|m+1|}\left(E_{|m|,n}R\right)\neq 0$. Therefore

$$C_{m,n} = \frac{1}{R\sqrt{\pi}\left|J_{|m+1|}\left(E_{|m|,n}R\right)\right|}.$$

The angular part of the eigenfunctions ensures they are mutually orthogonal. Our choice of $C_{m,n}$ ensures normality. So, even though $E_{m,n} = E_{-m,n}$, the set of eigenfunctions $u_{m,n}$is orthonormal. We conclude that although the eigenvalues on the disk have degeneracy two, the orthonormal eigenfunctions as defined above are such that (4.2.2) and (4.2.3)apply. In the next section, we will apply the perturbative expansions of Chapter Four to study $\zeta_{\mathcal{E}_\alpha}(s)$.

5.5 Another Perturbation

In order for a term-by-term comparison of two perturbative expansions to be valid, the coefficients of the expansions must not vary with respect to the perturbation parameter. We now explore a perturbation that will allow us to carry out term-by-term comparisons. To this end, we let

$$b = 1 \text{ and } a = 1 + \alpha.$$

In this case, $\mathcal{D}_{\sqrt{a}}$ is the resulting disk and we will write $\mathcal{E}_\alpha$ to refer to the corresponding ellipse. Let $\mathcal{D} = \mathcal{D}_1$ and define $h : \mathcal{D} \to \mathcal{E}_\alpha$ by $h(z) = f(\sqrt{a}z)$ where $f : \mathcal{D}_{\sqrt{a}} \to \mathcal{E}_\alpha$. Then

$$|h'(z)|^{-2} = a^{-1} \left| f'\left(\sqrt{a}z\right)\right|^{-2}.$$

Since $R^2 = 1 + \alpha$ and

$$\epsilon^2 = 2\alpha - 3\alpha^2 + O\left(\alpha^3\right), \tag{5.5.1}$$

we have

$$\frac{\epsilon^2}{R^2} = 2\alpha - 5\alpha^2 + O\left(\alpha^3\right).$$

From (5.3.1), we have

$$\begin{aligned}
& \left|h'\left(re^{i\theta}\right)\right|^{-2} \\
= \; & \left(1 - \alpha + \alpha^2\right)[1 - \left(3r^2\cos 2\theta\right)\alpha \\
& + \frac{1}{2}\left(\frac{9}{4}r^4 + \frac{3}{4} + \frac{9}{2}r^2\cos 2\theta - \frac{1}{2}r^4\cos 4\theta\right)\alpha^2] + O\left(\alpha^3\right) \\
= \; & 1 - \left(1 + 3r^2\cos 2\theta\right)\alpha - \frac{1}{4}\left(2r^4\cos 4\theta - 30r^2\cos 2\theta - 9r^4 - 7\right)\alpha^2 + O\left(\alpha^3\right).
\end{aligned}$$

Then

$$H_1 = \left(1 + 3r^2\cos 2\theta\right)\Delta_{\mathcal{D}}$$

and

$$H_2 = \frac{1}{4}\left(2r^4\cos 4\theta - 30r^2\cos 2\theta - 9r^4 - 7\right)\Delta_{\mathcal{D}}.$$

By (4.2.2) we have

$$E^1_{|m|,n} = \left\langle u^0_n \middle| \left(1 + 3r^2 \cos 2\theta\right) \Delta_D u^0_n \right\rangle = -\left(E^0_{|m|,n}\right)^2 .$$

Note

$$\zeta_{\mathcal{D}}(s) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}} \left(E^0_{|m|,n}\right)^{-2s}$$

since the square of each $E^0_{|m|,n}$ is a solution of (5.4.1) with $R = 1$. From (4.3.2) and (4.3.3),

$$\zeta^1_{\mathcal{E}_\alpha}(s) = -s \sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}} \left(E^0_{|m|,n}\right)^{-2s} \frac{E^1_{|m|,n}}{\left(E^0_{|m|,n}\right)^2} = s\zeta_D(s),$$

and

$$\zeta^2_{\mathcal{E}_\alpha}(s) = \frac{1}{2}\left(s^2 + s\right)\zeta_D(s) - s\Lambda(s)$$

where

$$\Lambda(s) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}} \left(E^0_{|m|,n}\right)^{-2s-2} E^2_{|m|,n}.$$

Here $E^2_{|m|,n}$ is as in (4.2.3), and depends on H_2. By (4.3.1),

$$\zeta_{\mathcal{E}_\alpha}(s) = \zeta_{\mathcal{D}}(s) + s\zeta_{\mathcal{D}}(s)\alpha + \left[\frac{1}{2}\left(s^2 + s\right)\zeta_D(s) - s\Lambda(s)\right]\alpha^2 + O\left(\alpha^3\right), \qquad (5.5.2)$$

which agrees with [23] up to order α and is valid for more general s.

Differentiating (5.5.2), we find

$$\begin{aligned} \zeta'_{\mathcal{E}_\alpha}(s) &= \zeta'_{\mathcal{D}}(s) + \left[\zeta_{\mathcal{D}}(s) + s\zeta'_{\mathcal{D}}(s)\right]\alpha \\ &\quad + \left[\frac{1}{2}(2s+1)\zeta_{\mathcal{D}}(s) + \frac{1}{2}\left(s^2 + s\right)\zeta'_{\mathcal{D}}(s) - \Lambda(s) - s\Lambda'(s)\right]\alpha^2 + O\left(\alpha^3\right). \end{aligned}$$

Since [15, 16]

$$\zeta_{\mathcal{D}}(0) = \frac{1}{6},$$

evaluating at $s = 0$ gives

$$\zeta'_{\mathcal{E}_\alpha}(0) = \zeta'_{\mathcal{D}}(0) + \frac{1}{6}\alpha + \left[\frac{1}{12} - \Lambda(0) - s\Lambda'(s)\,|_{s=0}\right]\alpha^2 + O\left(\alpha^3\right). \qquad (5.5.3)$$

Note

$$\frac{1}{6}\ln R^2 = \frac{1}{6}\alpha - \frac{1}{12}\alpha^2 + O\left(\alpha^3\right)$$

and

$$\epsilon^4 = 4\alpha^2 + O\left(\alpha^3\right). \tag{5.5.4}$$

Then, from (5.3.2) we have

$$\zeta'_{\mathcal{E}_\alpha}(0) = \zeta'_{\mathcal{D}}(0) + \frac{1}{6}\alpha + \frac{1}{6}\alpha^2 + O\left(\epsilon^6\right).$$

Comparing with (5.5.3) gives

$$\Lambda(0) + s\Lambda'(s)\,|_{s=0} = -\frac{1}{12}.$$

In Chapter Six we show

$$s\Lambda'(s)\,|_{s=0} = 0.$$

Therefore, the analytic continuation of $\Lambda(s)$ to $s = 0$ is such that

$$\Lambda(0) = -\frac{1}{12} = \zeta(-1).$$

We now investigate

$$E^2_{|m|,n} = \left\langle u^0_m | H_2 | u^0_m \right\rangle + \sum_{\substack{(j,k)\in\mathbb{Z}\times\mathbb{N} \\ (|j|,k)\neq(|m|,n)}} \frac{\left|\langle u^0_k | H_1 | u^0_m\rangle\right|^2}{E^0_m - E^0_k}.$$

We find

$$\left\langle u^0_{m,n} | H_2 | u^0_{m,n} \right\rangle = \left(E^0_{|m|,n}\right)^2 \left(\frac{7}{4} + \frac{9\pi}{2} I\right),$$

where

$$I = \int_0^1 r^5 J^2_{|m|}\left(E^0_{|m|,n} r\right) dr.$$

In Appendix B, we use Schafheitlin's reduction formula [42] to rewrite I in terms of m, $\lambda^0_{|m|,n}$, and $J^2_{|m+1|}\left(E^0_{|m|,n}\right)$. Next,

$$\sum_{\substack{(j,k)\in\mathbb{Z}\times\mathbb{N} \\ (|j|,k)\neq(|m|,n)}} \left|\left\langle u^0_{m,n} | H_1 | u^0_{j,k} \right\rangle\right|^2 = \sum_{\substack{(j,k)\in\mathbb{Z}\times\mathbb{N} \\ (|j|,k)\neq(|m|,n)}} \left(E^0_{|m|,n}\right)^4 \left|1 + 3\left\langle u^0_{m,n} | 3r^2 \cos 2\theta | u^0_{j,k} \right\rangle\right|^2$$

Since

$$\int_0^{2\pi} d\theta \cos 2\theta e^{i(j-m)\theta} = \begin{cases} \pi & j = m \pm 2 \\ 0 & \text{otherwise} \end{cases},$$

we have

$$\begin{aligned} & \sum_{\substack{(j,k)\in\mathbb{Z}\times\mathbb{N} \\ (|j|,k)\neq(|m|,n)}} \left\langle u^0_{m,n} | r^2 \cos 2\theta | u^0_{j,k} \right\rangle \\ = & \sum_{\substack{(j,k)\in\mathbb{Z}\times\mathbb{N} \\ j=m\pm 2}} \pi C_{m,n} C_{j,k} \int_0^1 r^3 J_{|m|}\left(\lambda^0_{|m|,n} r\right) J_{|j|}\left(\lambda^0_{|j|,k} r\right) dr. \end{aligned}$$

So

$$\begin{aligned} E^2_{|m|,n} &= \left(E^0_{|m|,n}\right)^2 \left(\frac{7}{4} + \frac{9\pi}{2} I\right) \\ &+ \sum_{\substack{(j,k)\in\mathbb{Z}\times\mathbb{N} \\ j=m\pm 2}} \left(E^0_{|m|,n}\right)^4 \frac{\left|1 + 3\pi C_{m,n} C_{j,k} \int_0^1 r^3 J_{|m|}\left(E^0_{|m|,n} r\right) J_{|j|}\left(E^0_{|j|,k} r\right)\right|^2}{\left(E^0_{|m|,n}\right)^2 - \left(E^0_{|j|,k}\right)^2}. \end{aligned}$$

CHAPTER SIX

Heat Kernel Coefficient Calculations

6.1 The Heat Kernel

Let $\mathcal{M}$ be a d-dimensional smooth Riemannian manifold with smooth boundary and P a strongly elliptic second order differential operator with local boundary conditions on $\mathcal{M}$. Let ζ_P be the spectral zeta function associated with P. The heat kernel

$$K(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}$$

is related to ζ_P by

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} K(t).$$

For t small, we may write

$$K(t) \sim \sum_{n=0,1/2,1,3/2,\cdots}^{\infty} a_n t^{n-d/2}$$

where the coefficients a_n depend on both P and the boundary conditions under consideration. If P is minus the Laplacian then

$$a_{d/2-s} = \Gamma(s) \operatorname{Res}\zeta_P(s) \tag{6.1.1}$$

for $s = d/2, (d-1)/2, \cdots, 1/2, -\frac{2n+1}{2}$, $n \in \mathbb{N}_0$ and

$$a_{d/2+s} = \frac{(-1)^s}{s!} \zeta_P(-s) \tag{6.1.2}$$

for $s \in \mathbb{N}_0$ [20].

The heat kernel coefficients a_n are of geometric significance. To see this, suppose $\Omega \subset \mathbb{R}^2$ is a simply connected open set with smooth boundary, $\partial\Omega$. In two dimensions the heat kernel coefficients can be expressed in terms of the area of Ω, the length of $\partial\Omega$, and the curvature, κ, of $\partial\Omega$. Adapting the results of [39] to the coefficient and index conventions used in [20], we have

$$\begin{aligned}
a_0 &= \frac{1}{4\pi}(\text{area of } \Omega) \\
a_{1/2} &= -\frac{1}{8\sqrt{\pi}}(\text{length of } \partial\Omega) \\
a_1 &= \frac{1}{12\pi}\int_{\partial\Omega} \kappa d\sigma \\
a_{3/2} &= \frac{1}{256\sqrt{\pi}}\int_{\partial\Omega} \kappa^2 d\sigma \\
a_2 &= \frac{1}{315\pi}\int_{\partial\Omega} \kappa^3 d\sigma \\
a_{5/2} &= \frac{37}{2^{15}\sqrt{\pi}}\int_{\partial\Omega} \kappa^4 d\sigma - \frac{1}{2^{12}\sqrt{\pi}}\int_{\partial\Omega}\left(\frac{d\kappa}{d\sigma}\right)^2 d\sigma \\
a_3 &= \frac{68}{4545\pi}\int_{\partial\Omega} \kappa^5 d\sigma - \frac{4}{3465\pi}\int_{\partial\Omega}\kappa\left(\frac{d\kappa}{ds}\right)^2 d\sigma
\end{aligned} \tag{6.1.3}$$

for minus the Laplacian with the Dirichlet boundary condition. It is worth mentioning that a κ' term shows up in $a_{3/2}$ and a_2 but integrates to zero. Although heat kernel coefficients are worth studying in their own right, as in [15, 16], we will be concerned with using (6.1.1) and (6.1.2) to compare perturbative zeta function expansions with perturbative heat kernel coefficient expansions.

6.2 Heat Kernel Coefficient Calculations

Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$, defined by

$$\gamma(t) = \langle a\cos t, b\sin t\rangle,$$

be a parameterization of $\partial\mathcal{E}_\epsilon$. Then

$$\kappa = \frac{|\det(\gamma', \gamma'')|}{|\gamma'|^3} = \frac{b}{a^2\left(1-\epsilon^2\cos^2 t\right)^{3/2}}$$

and

$$d\sigma = \frac{d\sigma}{dt}dt = a\sqrt{1-\epsilon^2\cos^2 t}\,dt$$

imply

$$(\kappa')^2 d\sigma = \left(\frac{d\kappa}{dt}\frac{dt}{d\sigma}\right)^2 \frac{d\sigma}{dt}dt$$

$$
\begin{aligned}
&= \left(\frac{d\kappa}{dt}\right)^2 \frac{dt}{d\sigma} dt \\
&= \frac{9b^2\epsilon^4 \cos^2 t \sin^2 t}{a^5 \left(1-\epsilon^2 \cos^2 t\right)^{11/2}} dt.
\end{aligned}
$$

Let

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

be the beta function [44]. From [17], the hypergeometric function is given for $\Re w > \Re v > 0$ by

$${}_2F_1(u,v;w;z) = \frac{1}{\beta(v,w-v)} \int_0^1 t^{v-1}(1-t)^{w-v-1}(1-tz)^{-u}\,dt.$$

Taking $t = \cos^2\theta$ yields

$$\int_0^{\pi/2} \frac{\cos^{2v-1}\theta \sin^{2(w-v)-1}\theta}{\left(1-z\cos^2\theta\right)^u} d\theta = \frac{\Gamma(v)\Gamma(w-v)}{2\Gamma(w)}\ {}_2F_1(u,v;w;z).$$

For

$$\text{length of } \partial\Omega = \int_{\partial\Omega} d\sigma = 4a \int_0^{\pi/2} \sqrt{1-\epsilon^2\cos^2 t}\,dt,$$

we have

$$\text{length of } \partial\Omega = 2a\pi\ {}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;\epsilon^2\right).$$

Similarly,

$$\int_{\partial\Omega} \kappa d\sigma = \frac{4b}{a}\int_0^{\pi/2} \frac{dt}{1-\epsilon^2\cos^2 t} = \frac{2\pi b}{a}\ {}_2F_1\left(1,\frac{1}{2};1;\epsilon^2\right).$$

Now, from [17] we know

$${}_2F_1(u,v;w;z) = (1-z)^{w-u-v}\ {}_2F_1(w-u,w-v;w;z)$$

and

$${}_2F_1(u,v;w;z) = \sum_{k=0}^{\infty} \frac{(u)_k (v)_k}{(w)_k} \frac{z^k}{k!} \tag{6.2.1}$$

where

$$(u)_k = \begin{cases} 1 & k=0 \\ \prod_{j=0}^{k-1}(u+j) & k>0 \end{cases}. \tag{6.2.2}$$

Note ${}_2F_1\left(0, v; w; z\right) = 1$. Then, since $\epsilon^2 = 1 - \left(b/a\right)^2$, we have

$$ {}_2F_1\left(1, \frac{1}{2}; 1; \epsilon^2\right) = \left(1 - \epsilon^2\right)^{-1/2} = \frac{a}{b}. $$

It follows that

$$ \int_{\partial\Omega} \kappa d\sigma = 2\pi. $$

We further calculate

$$ \int_{\partial\Omega} \kappa^2 d\sigma = \frac{4b^2}{a^3} \int_0^{\pi/2} \frac{dt}{\left(1 - \epsilon^2 \cos^2 t\right)^{5/2}} = \frac{2\pi b^2}{a^3}\ {}_2F_1\left(\frac{5}{2}, \frac{1}{2}; 1; \epsilon^2\right). $$

Similarly,

$$ \int_{\partial\Omega} \kappa^3 d\sigma = \frac{4b^3}{a^5} \int_0^{\pi/2} \frac{dt}{\left(1 - \epsilon^2 \cos^2 t\right)^4} = \frac{2\pi b^3}{a^5}\ {}_2F_1\left(4, \frac{1}{2}; 1; \epsilon^2\right). $$

Next,

$$ \begin{aligned} \int_{\partial\Omega} \left(\kappa'\right)^2 d\sigma &= \frac{9b^2\epsilon^4}{a^5} \int_0^{2\pi} \frac{\cos^2 t \sin^2 t}{\left(1 - \epsilon^2 \cos^2 t\right)^{11/2}} dt \\ &= \frac{18b^2\epsilon^4}{a^5} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(4\right)}\ {}_2F_1\left(\frac{11}{2}, \frac{3}{2}; 4; \epsilon^2\right) \\ &= \frac{9\pi b^2\epsilon^4}{8a^5}\ {}_2F_1\left(\frac{11}{2}, \frac{3}{2}; 4; \epsilon^2\right) \end{aligned} $$

and

$$ \int_{\partial\Omega} \kappa \left(\kappa'\right)^2 d\sigma = \frac{9\pi b^3\epsilon^4}{8a^7}\ {}_2F_1\left(7, \frac{3}{2}; 4; \epsilon^2\right). $$

Finally,

$$ \text{area of } \Omega = ab\pi. $$

Therefore,

$$ \begin{aligned} a_0 &= \frac{1}{4}ab \\ a_{1/2} &= -\frac{\sqrt{\pi}}{4}a\ {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; \epsilon^2\right) \\ a_1 &= \frac{1}{6} \end{aligned} $$

$$
\begin{aligned}
a_{3/2} &= -\frac{\sqrt{\pi}}{128}\frac{b^2}{a^3}\ {}_2F_1\left(\frac{5}{2},\frac{1}{2};1;\epsilon^2\right) \\
a_2 &= \frac{2b^3}{315a^5}\ {}_2F_1\left(4,\frac{1}{2};1;\epsilon^2\right) \\
a_{5/2} &= \frac{37}{2^{14}}\frac{\sqrt{\pi}b^4}{a^7}\ {}_2F_1\left(\frac{11}{2},\frac{1}{2};1;\epsilon^2\right) - \frac{1}{2^{12}}\frac{9\sqrt{\pi}b^2\epsilon^4}{8a^5}\ {}_2F_1\left(\frac{11}{2},\frac{3}{2};4;\epsilon^2\right) \\
a_3 &= \frac{68}{4545}\frac{2b^4}{a^7}\ {}_2F_1\left(7,\frac{1}{2};1;\epsilon^2\right) - \frac{1}{3465}\frac{9b^3\epsilon^4}{2a^7}\ {}_2F_1\left(7,\frac{3}{2};4;\epsilon^2\right).
\end{aligned} \tag{6.2.3}
$$

These results do not require $\epsilon \approx 0$.

6.3 Comparisons of Expansions

Throughout this section, we will compare (5.5.2) to (6.2.3) using either (6.1.1) or (6.1.2), depending on the value of s, with $d = 2$. For $s = 1$ we use (6.1.1) to obtain

$$\frac{1}{4}(1+\alpha) = a_0 = \mathrm{Res}\zeta_{\mathcal{E}_\alpha}(1).$$

Then from (5.5.2) we have

$$
\begin{aligned}
\zeta_{\mathcal{E}_\alpha}(1+\delta) &= \zeta_{\mathcal{D}}(1+\delta) + (1+\delta)\zeta_{\mathcal{D}}(1+\delta)\alpha \\
&\quad + \left[\frac{1}{2}\left(\delta^2+3\delta+1\right)\zeta_{\mathcal{D}}(1+\delta) - (1+\delta)\Lambda(1+\delta)\right]\alpha^2 + O\left(\alpha^3\right).
\end{aligned}
$$

Letting $\delta \to 0$ gives

$$\mathrm{Res}\zeta_{\mathcal{E}_\alpha}(1) = \mathrm{Res}\zeta_{\mathcal{D}}(1) + \mathrm{Res}\zeta_{\mathcal{D}}(1)\alpha + \left[\mathrm{Res}\zeta_{\mathcal{D}}(1) - \mathrm{Res}\Lambda(1)\right]\alpha^2 + O\left(\alpha^3\right).$$

From (6.1.3), we know

$$\mathrm{Res}\zeta_{\mathcal{D}}(1) = \frac{1}{4}.$$

Therefore

$$\mathrm{Res}\Lambda(1) = \frac{1}{4}.$$

Taking $s = 0$ in (6.1.2) gives

$$\frac{1}{6} = a_1 = \zeta_{\mathcal{E}_\alpha}(0) = \zeta_{\mathcal{D}}(0) - s\Lambda(s)\left.\right|_{s=0}\alpha^2 + O\left(\alpha^3\right).$$

Therefore

$$s\Lambda(s)\,|_{s=0}=0.$$

This implies

$$\Lambda(s)=c_0+c_1 s+O\left(s^2\right)$$

in a neighborhood of $s=0$ for some constants c_0 and c_1. Then

$$\Lambda'(s)=c_1+O(s)$$

implies

$$s\Lambda'(s)\,|_{s=0}=0.$$

This result was used in Chapter Five to conclude $\Lambda(0)=-1/12$.

As a final comparison of coefficients, we use (6.1.1) with $s=1/2$, which yields

$$-\frac{\sqrt{\pi}}{4}a\ {}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;\epsilon^2\right)=a_{1/2}=\mathrm{Res}\zeta_{\mathcal{E}_\alpha}\left(\frac{1}{2}\right).$$

From (5.5.1) and (5.5.4), we have

$$\epsilon^2=2\alpha-3\alpha^2+O\left(\alpha^3\right)\text{ and }\epsilon^4=4\alpha^2+O\left(\alpha^3\right).$$

Then (6.2.1) and (6.2.2) give

$$\begin{aligned}a\ {}_2F_1\left(\frac{1}{2},-\frac{1}{2};1;\epsilon^2\right) &= a\left[1-\frac{1}{4}\epsilon^2-\frac{3}{64}\epsilon^4+O\left(\epsilon^6\right)\right]\\ &= (1+\alpha)\left[1-\frac{1}{2}\alpha+\frac{9}{16}\alpha^2+O\left(\alpha^3\right)\right]\\ &= 1+\frac{1}{2}\alpha+\frac{1}{16}\alpha^2+O\left(\alpha^3\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\zeta_{\mathcal{E}_\alpha}\left(\frac{1}{2}+\delta\right)\\ =\ &\zeta_{\mathcal{D}}\left(\frac{1}{2}+\delta\right)+\left(\frac{1}{2}+\delta\right)\zeta_{\mathcal{D}}\left(\frac{1}{2}+\delta\right)\alpha\\ &+\left[\frac{1}{2}\left[\left(\frac{1}{2}+\delta\right)^2+\frac{1}{2}+\delta\right]\zeta_{\mathcal{D}}\left(\frac{1}{2}+\delta\right)-\left(\frac{1}{2}+\delta\right)\Lambda\left(\frac{1}{2}+\delta\right)\right]\alpha^2+O\left(\alpha^3\right).\end{aligned}$$

From (6.1.3) we know

$$\mathrm{Res}\zeta_{\mathcal{D}}\left(\frac{1}{2}\right)=-\frac{1}{4}.$$

Then, letting $\delta\to 0$ gives

$$\mathrm{Res}\zeta_{\mathcal{E}_\alpha}\left(\frac{1}{2}\right)=-\frac{1}{4}-\frac{1}{8}\alpha-\left[\frac{1}{2}\mathrm{Res}\Lambda\left(\frac{1}{2}\right)+\frac{3}{32}\right]\alpha^2+O\left(\alpha^3\right).$$

Since $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, from (6.1.1) and (6.2.3) we have

$$\begin{aligned}
& -\frac{\sqrt{\pi}}{4}\left[1+\frac{1}{2}\alpha+\frac{1}{16}\alpha^2+O\left(\alpha^3\right)\right] \\
= & -\frac{\sqrt{\pi}}{4}a\ {}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;\epsilon^2\right) \\
= & \ \Gamma\left(\frac{1}{2}\right)\mathrm{Res}\zeta_{E_\alpha}\left(\frac{1}{2}\right) \\
= & -\frac{\sqrt{\pi}}{4}\left[1+\frac{1}{2}\alpha-4\left[\mathrm{Res}\Lambda\left(\frac{1}{2}\right)-\frac{3}{32}\right]\alpha^2+O\left(\alpha^3\right)\right].
\end{aligned}$$

We conclude

$$\mathrm{Res}\Lambda\left(\frac{1}{2}\right)=-\frac{5}{32}.$$

The above comparisons give strong evidence for the correctness of (5.5.2) up to order α. At order α^2 we are given information about the meromorphic structure of Λ.

APPENDICES

APPENDIX A

Evaluation of Integrals

We first prove

$$\int_0^{2\pi} d\theta F\left(1,\theta\right) = -\pi \ln 4,$$

where

$$F\left(r,\theta\right) = \frac{1}{2}\ln\frac{1-2r^2\cos 2\theta + r^4}{4r^4}.$$

Since

$$F\left(1,\theta\right) = -\frac{1}{2}\ln\frac{2}{1-\cos 2\theta} = \frac{1}{2}\ln\sin^2\theta = \ln\left|\sin\theta\right|$$

for $\theta \neq k\pi$, $k \in \mathbb{Z}$, we have

$$\int_0^{2\pi} d\theta F\left(1,\theta\right) = \int_0^{\pi} d\theta \ln\sin\theta + \int_{\pi}^{2\pi} d\theta \ln\left(-\sin\theta\right).$$

Then

$$\int_{\pi}^{2\pi} d\theta \ln\left(-\sin\theta\right) = \int_{\pi}^{2\pi} d\theta \ln\sin\left(\theta-\pi\right) = \int_0^{\pi} d\varphi \ln\sin\varphi,$$

where $\varphi = \theta - \pi$, implies

$$\int_0^{2\pi} d\theta F\left(1,\theta\right) = 2\int_0^{\pi} d\theta \ln\sin\theta.$$

Note

$$\int_0^{\pi} d\theta \ln\sin\theta = \left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi}\right) d\theta \ln\sin\theta = 2\int_0^{\pi/2} d\theta \ln\sin\theta$$

and

$$\int_0^{\pi/2} d\theta \ln\sin\theta = \int_0^{\pi/2} d\theta \ln\cos\theta.$$

Thus

$$\begin{aligned}\int_0^{\pi/2} d\theta \ln\sin\theta &= \frac{1}{2}\int_0^{\pi/2} d\theta\left(\ln\sin\theta + \ln\cos\theta\right) = \frac{1}{2}\int_0^{\pi/2} d\theta \ln\sin\theta\cos\theta \\ &= \frac{1}{2}\int_0^{\pi/2} d\theta \ln\frac{\sin 2\theta}{2} = \frac{1}{2}\int_0^{\pi/2} d\theta\left(\ln\sin 2\theta - \ln 2\right)\end{aligned}$$

$$= \quad -\frac{\pi}{4}\ln 2+\frac{1}{2}\int_0^{\pi/2} d\theta \ln\sin 2\theta = -\frac{\pi}{4}\ln 2+\frac{1}{4}\int_0^{\pi} d\varphi \ln\sin\varphi,$$

where $\varphi = 2\theta$. Since

$$\int_0^{\pi} d\theta \ln\sin\theta = 2\int_0^{\pi/2} d\theta \ln\sin\theta = -\frac{\pi}{2}\ln 2+\frac{1}{2}\int_0^{\pi} d\varphi \ln\sin\varphi$$

implies

$$\int_0^{\pi} d\theta \ln\sin\theta = -\pi\ln 2,$$

which can also be evaluated using a contour integral [1], we have

$$\int_0^{2\pi} d\theta F\left(1,\theta\right) = -\pi\ln 4.$$

Second, we show

$$\int_0^{2\pi} d\theta F_r\left(r,\theta\right) = 0$$

for $r > 1$, where

$$F_r\left(r,\theta\right) = \frac{2r^2\cos 2\theta - 2}{r - 2r^3\cos 2\theta + r^5}.$$

Periodicity and $\varphi = 2\theta$ imply

$$\int_0^{2\pi} d\theta F_r\left(r,\theta\right) = 2\int_0^{\pi} d\theta F_r\left(r,\theta\right) = \int_0^{2\pi} d\varphi F_r\left(r,\varphi/2\right).$$

Let $z = e^{i\varphi}$. Then

$$dz = izd\varphi \text{ and } \cos\varphi = \frac{1}{2}\left(z+\frac{1}{z}\right).$$

Hence

$$\begin{aligned}\int_0^{2\pi} d\varphi F_r\left(r,\varphi/2\right) &= -\oint_{|z|=1}\frac{dz}{iz}\frac{2-r^2\left(z+\frac{1}{z}\right)}{r-r^3\left(z+\frac{1}{z}\right)+r^5} \\ &= -\frac{1}{ir}\oint_{|z|=1}\frac{dz}{z}\frac{r^2z^2-2z+r^2}{r^2z^2-\left(r^4+1\right)z+r^2}.\end{aligned}$$

Let

$$R\left(z\right) = \frac{P\left(z\right)}{Q\left(z\right)} = \frac{r^2z^2-2z+r^2}{z\left[r^2z^2-\left(r^4+1\right)z+r^2\right]}.$$

Then, by the Residue Theorem [1],

$$\oint_{|z|=1} R(z)\, dz = 2\pi i \sum \operatorname{Res}(a; R)$$

where the sum is taken over the finite number of isolated singular points a of $R(z)$ such that $|a| < 1$. If $Q(z)$ has a simple root at $z = a$ and $P(a) \neq 0$ then [18]

$$\operatorname{Res}(a; R) = \frac{P(a)}{Q'(a)}.$$

Note Q has only real simple roots since $r > 1$ implies

$$\left(r^4 + 1\right)^2 - 4r^4 = r^8 + 2r^4 + 1 - 4r^4 = r^8 - 2r^4 + 1 = \left(r^4 - 1\right)^2 > 0.$$

Now, $Q(z) = 0$ implies $z = 0$ or

$$z = \frac{r^4 + 1 \pm (r^4 - 1)}{2r^2}.$$

Since

$$\frac{r^4 + 1 + (r^4 - 1)}{2r^2} = \rho^2 > 1$$

and

$$\frac{r^4 + 1 - (r^4 - 1)}{2r^2} = \frac{1}{r^2} < 1,$$

we have

$$\oint_{|z|=1} R(z)\, dz = 2\pi i \left[\operatorname{Res}(0; R) + \operatorname{Res}\left(\frac{1}{r^2}; R\right)\right].$$

We find

$$P(0) = r^2 > 0 \text{ and } P\left(\frac{1}{r^2}\right) = r^2 - \frac{1}{r^2} > 0.$$

Also,

$$Q'(z) = 3r^2 z^2 - 2\left(r^4 + 1\right) z + r^2$$

implies

$$Q'(0) = r^2 \text{ and } Q'\left(\frac{1}{r^2}\right) = \frac{1}{r^2} - r^2.$$

Thus

$$\operatorname{Res}(0;R)+\operatorname{Res}\left(\frac{1}{r^2};R\right)=\frac{P(0)}{Q'(0)}+\frac{P\left(\frac{1}{r^2}\right)}{Q'\left(\frac{1}{r^2}\right)}=1-1=0.$$

This implies

$$\int_0^{2\pi} d\theta F_r(r,\theta)=0$$

for $r>1$. Finally,

$$\begin{aligned}0 &= \int_1^r ds\int_0^{2\pi} d\theta F_s(s,\theta)=\int_0^{2\pi} d\theta\int_1^r dsF_s(s,\theta)\\ &= \int_0^{2\pi} d\theta\left[F(r,\theta)-F(1,\theta)\right]=\int_0^{2\pi} d\theta F(r,\theta)+\pi\ln 4\end{aligned}$$

implies

$$\int_0^{2\pi} d\theta F(r,\theta)=-\pi\ln 4$$

for $r>1$. We conclude

$$\int_0^{2\pi} d\theta F(r,\theta)=-\pi\ln 4$$

for $r\geq 1$.

APPENDIX B

Schafheitlin's Reduction

Suppose $\left(E_{|m|,n}\right)^2$ is an eigenvalue of (5.4.1) with $R=1$. Then [42]

$$J_{|m|}\left(E_{|m|,n}\right)=0 \text{ and } \left[J'_{|m|}\left(E_{|m|,n}\right)\right]^2=J^2_{|m+1|}\left(E_{|m|,n}\right).$$

We will rewrite

$$I=\int_0^1 r^5 dr J^2_{|m|}\left(E_{|m|,n}r\right).$$

Using Schafheitlin's reduction formula [42],

$$\begin{aligned}
&(\beta+2)\int^w x^{\beta+2}dx J_m^2(x)\\
=\ &(\beta+1)\left[m^2-\frac{1}{4}(\beta+1)^2\right]\int^w x^\beta dx J_m^2(x)\\
&+\frac{w^{\beta+1}}{2}\{\left[wJ'_m(w)-\frac{1}{2}(\beta+1)J_m(w)\right]^2\\
&+\left[w^2-m^2+\frac{1}{4}(\beta+1)^2\right]J_m^2(w)\},
\end{aligned}$$

with $\beta=3$, $x=E_{|m|,n}r$ and $w=E_{|m|,n}$ we find

$$\begin{aligned}
I &= \left(E_{|m|,n}\right)^{-6}\int_0^{E_{|m|,n}} x^5 dx J^2_{|m|}(x)\\
&= \frac{4}{5}\left(q^2-4\right)\left(E_{|m|,n}\right)^{-6}\int_0^{E_{|m|,n}} x^3 dx J^2_{|m|}(x)\\
&\quad+\frac{\left(E_{|m|,n}\right)^{-2}}{10}\{\left[E_{|m|,n}J'_{|m|}\left(E_{|m|,n}\right)-2J_{|m|}\left(E_{|m|,n}\right)\right]^2\\
&\quad+\left[\left(E_{|m|,n}\right)^2-m^2+4\right]J^2_{|m|}\left(E_{|m|,n}\right)\}\\
&= \frac{4}{5}\left(m^2-4\right)\left(E_{|m|,n}\right)^{-6}\int_0^{E_{|m|,n}} x^3 dx J^2_{|m|}(x)+\frac{1}{10}J^2_{|m+1|}\left(E_{|m|,n}\right).
\end{aligned}$$

Reducing again with $\beta=1$ we find

$$\int_0^{E_{|m|,n}} x^3 dx J^2_{|m|}(x)=\frac{2}{3}\left(m^2-1\right)\int_0^{E_{|m|,n}} x dx J^2_{|m|}(x)+\frac{\left(E_{|m|,n}\right)^4}{6}J^2_{|m+1|}\left(E_{|m|,n}\right).$$

Since [42]

$$\int_0^1 rdrJ_{|m|}^2\left(E_{|m|,n}r\right) = \frac{1}{2}J_{|m+1|}^2\left(E_{|m|,n}\right),$$

taking $x = E_{|m|,n}r$ yields

$$\int_0^{E_{|m|,n}} xdxJ_{|m|}^2(x) = \frac{1}{2}\left(E_{|m|,n}\right)^2 J_{|m+1|}^2\left(E_{|m|,n}\right).$$

Then

$$\begin{aligned} I &= [\frac{8}{30}\left(E_{|m|,n}\right)^{-4}\left(m^2-4\right)\left(m^2-1\right) \\ &\quad +\frac{2}{15}\left(E_{|m|,n}\right)^{-2}\left(m^2-4\right)+\frac{1}{10}]J_{|m+1|}^2\left(E_{|m|,n}\right). \end{aligned}$$

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