ABSTRACT

Stable Up-Downwind Finite Difference Methods for Solving Heston Stochastic Volatility Equations

Chong Sun, Ph.D.

Advisor: Qin Sheng, Ph.D.

This dissertation explores effective and efficient computational methodologies for solving two-dimensional Heston stochastic volatility option pricing models with multiple financial engineering applications. Dynamically balanced up-downwind finite difference methods taking care of cross financial derivative terms in the partial differential equations involved are implemented and rigorously analyzed. Semi-discretization strategies are utilized over variable data grids for highly vibrant financial market simulations. Moving mesh adaptations are incorporated with experimental validations.

The up-downwind finite difference schemes derived are proven to be numerically stable with first order accuracy in approximations. Discussions on concepts of the stability and convergence are fulfilled. Simulation experiments are carefully designed and carried out to illustrate and validate our conclusions. Multiple convergence and relative error estimates are obtained via computations with reality data. The novel new methods developed are highly satisfactory with distinguished simplicity and straightforwardness in programming realizations for option markets, especially when unsteady stocks' markets are major concerns. The research also reveals promising directions for continuing accomplishments in financial mathematics and computations.

Stable Up-Downwind Finite Difference Methods for Solving Heston Stochastic Volatility Equations

by

Chong Sun, B.A., M.S.

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Lance L. Littlejohn, Ph.D., Chairperson

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Approved by the Dissertation Committee

Qin Sheng, Ph.D., Chairperson

Jameson Graber, Ph.D.

David Kahle, Ph.D.

Lance L. Littlejohn, Ph.D.

Ronald Morgan, Ph.D.

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J. Larry Lyon, Ph.D., Dean

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To my parents

CHAPTER ONE

Introduction

1.1 Black-Scholes-Merton Framework

A financial derivative is a contract between two or more parties whose value is based on an agreed-upon underlying financial asset such as a security or set of assets such as an index. Options are special types of a financial derivative. A call option gives the holder of the option the right to buy an asset by a certain date for a certain price. A put option gives the holder the right to sell an asset by a certain date for a certain price. The date specified in the contract is known as the expiration date or the maturity date. The price specified in the contract is known as the exercise, or the strike price. Options can be traditionally divided into the European type of options or American type of options, with an emerging Asian option type. While an European option can be exercised only at the expiration date, American options can be exercised at any time up to the expiration date. Our investigations in this dissertation are mainly focused upon European put options.

One of the effective mathematical models for pricing aforementioned options was proposed by Fisher Black, Myron Scholes and Robert Merton in 1970s [5, 43]. The theory leads to the Black-Scholes-Merton (BSM) model which plays as a backbone in modern financial modeling and computations [13, 28, 41, 51, 64].

In deriving a BSM formula for the value of an option as a function of asset prices and time to maturity, the following seven assumptions must be stipulated.

(1) We are able to know the short-term riskless interest rate and the rate is constant during the time period we are considering.

(2) The stock prices are assumed to follow geometric Brownian motion, *that is,*

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW(t), \quad t > 0, \tag{1.1}$$

where S is the price of the underlying asset at time t, μ is the expected constant return of the asset, r is the continuous dividend yielded proportional to the asset price, and σ is the constant volatility of the asset returns. The function $W(t)$ is a standard Brownian motion. The volatility in the Black-Scholes-Merton model is considered to be constant.

(3) The stocks pay no dividends to the holders.

(4) No commission fees or any other types of costs are charged in buying or selling the stocks or options.

(5) No restrictions or penalties are posed for short selling.

(6) Any fraction of the price of a security is available for borrowing at the short-term riskless interest rate.

(7) Only European types of options are concerned.

Values of the options should depend only on asset prices and time to maturity of options. To get rid of the uncertainty associated with the Brownian motion in (1.1), a risk-neutral portfolio can be constructed. To this end, we let $P(t)$ denote the value of the portfolio consisting of an option of value $v(S,t)$ and Δ shares of the underlying asset. It follows that

$$P(t) = S(t) - v(S,t)\Delta.$$

Set $\Delta = 1/v_S(S,t)$. The change in the value of the portfolio in a short time interval $[t, t + \Delta t]$, where $\Delta t \ll 1$, is given by

$$\xi = \Delta S(t) - \frac{\Delta u(S,t)}{u_S(S,t)}, \tag{1.2}$$

where $\Delta S(t) = S(t + \Delta t) - S(t)$ and $\Delta u(S,t) = u(S + \Delta S, t + \Delta t) - u(S,t)$.

The value of Δu can be expanded to yield the following equation:

$$\Delta v = v_S \Delta S + \frac{1}{2}\sigma^2 S^2 v_{SS} \Delta t + v_t \Delta t. \tag{1.3}$$

Substituting (1.3) into (1.2), the change in the value of the portfolio is expressed as

$$\begin{aligned}
\xi &= \Delta S(t) - \frac{\Delta u(S,t)}{u_S(S,t)} = \Delta S(t) - \frac{v_S \Delta S + \frac{1}{2}\sigma^2 S^2 v_{SS}\Delta t + v_t \Delta t}{v_S(S,t)} \\
&= -\left(\frac{1}{2}\sigma^2 S^2 v_{SS} + v_t\right)\frac{\Delta t}{v_S}.
\end{aligned} \tag{1.4}$$

Since the return on the portfolio as expressed in (1.4) does not depend on a Brownian motion, thus it is deterministic. Therefore, the return must be equivalent to the riskless security in the period of $[t, t+\Delta t]$. Because the return of riskless securities during this particular time period is $r\Delta t$, we obtain subsequently that

$$-\left(\frac{1}{2}\sigma^2 S^2 v_{SS} + v_t\right)\frac{\Delta t}{v_S} = \left(S - \frac{v}{v_S}\right) r\Delta t. \tag{1.5}$$

After cancelling Δt from both sides of the equation and a rearrangement of variables, we arrive at the following partial differential equation,

$$v_t = -\frac{1}{2}\sigma^2 S^2 v_{SS} - rSv_S + rv, \quad S \in (0,\infty),\ t \in [0,T]. \tag{1.6}$$

In addition, for European put options with a strike price K, we have

$$v(S,T) = \begin{cases} K - S, & S \le K, \\ 0, & S > K. \end{cases} \tag{1.7}$$

It has been shown that solutions to (1.6) together with the terminal value condition (1.7) is unique [5, 31]. To derive the solution, we need the following two basic lemmas.

Lemma 1.1. (Itô's Lemma [28]) *Assume X_t to be an Itô drift-diffusion process that satisfies the stochastic differential equation*

$$dX_t = \mu_t dt + \sigma_t dW_t,$$

where W_t is a Wiener process. If $f(t,x)$ is a twice-differentiable scalar function, then

$$df = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t.$$

Theorem 1.2. [28] *If X is lognormally distributed, and*

$$\ln X \sim \mathbb{N}(\mu, \sigma^2), \tag{1.8}$$

then the expectation

$$\mathbb{E}(\max(K - X, 0)) = \mathbb{E}(X)F(d_1) - KF(d_2),$$

where

$$d_1 = \frac{-\ln[\mathbb{E}(X)/K] + \sigma^2/2}{\sigma}, \;\; d_2 = \frac{-\ln[\mathbb{E}(X)/K] - \sigma^2/2}{\sigma}.$$

Proof. Herewith we give a more straightforward proof which is different from any existing mathematical verification to our best knowledge. For this, we denote $f(X)$ as the probability density function of X. Hence,

$$\mathbb{E}(\max(K - X, 0)) = \int_{-\infty}^{K} (K - X)f(X)dX. \tag{1.9}$$

Since X is lognormally distributed, we have, from the property of lognormally distributed variables,

$$\mathbb{E}(\ln X) = \ln[\mathbb{E}(V)] - \frac{\sigma^2}{2}. \tag{1.10}$$

Let $m = \mathbb{E}(\ln X)$ and

$$Y = \frac{\ln X - m}{\sigma}. \tag{1.11}$$

Thus, Y is normally distributed and subsequently,

$$Y \sim \mathbb{N}(0, 1).$$

Denote $g(Y)$ as the probability density function of Y. Thus,

$$g(Y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Y^2}{2}}.$$

With (1.11), we may replace (1.9) by

$$\mathbb{E}(\max(K - X, 0)) \;\; = \;\; \int_{-\infty}^{(\ln K - m)/\sigma} (K - e^{Y\sigma + m})g(Y)dY$$

$$
\begin{aligned}
&= K\int_{-\infty}^{(\ln K-m)/\sigma} g(Y)dY - \int_{-\infty}^{(\ln K-m)/\sigma} e^{Y\sigma+m}g(Y)dY \\
&= KF\left(\frac{\ln K-m}{\sigma}\right) - \int_{-\infty}^{(\ln K-m)/\sigma} e^{Y\sigma+m}g(Y)dY. \qquad (1.12)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
e^{Y\sigma+m}g(Y) &= \frac{1}{\sqrt{2\pi}}e^{\frac{-Q^2+2Q\sigma+2m}{2}} \\
&= \frac{1}{\sqrt{2\pi}}e^{\frac{-(Q-\sigma)^2+2Q\sigma+2m+\sigma^2}{2}} \\
&= \frac{e^{m+\frac{\sigma^2}{2}}}{\sqrt{2\pi}}e^{\frac{-(Q-\sigma)^2}{2}} \\
&= e^{m+\frac{\sigma^2}{2}}g(Y-\sigma).
\end{aligned}
$$

Based on the above, (1.12) can be conveniently reformulated to

$$
\begin{aligned}
\mathbb{E}(\max(K-X,0)) &= KF\left(\frac{\ln K-m}{\sigma}\right) - e^{m+\frac{\sigma^2}{2}}\int_{-\infty}^{(\ln K-m)/\sigma} g(Y-\sigma)dY \\
&= KF\left(\frac{\ln K-m}{\sigma}\right) - e^{m+\frac{\sigma^2}{2}}F\left(\frac{\ln K-m}{\sigma}-\sigma\right) \\
&= KF\left(\frac{\ln K-m}{\sigma}\right) - \mathbb{E}(V)F\left(\frac{\ln K-m}{\sigma}-\sigma\right).
\end{aligned}
$$

Substituting (1.10) for m into the above equality, we arrive at

$$
\begin{aligned}
\mathbb{E}(\max(K-X,0)) &= KF\left(\frac{-\ln(\mathbb{E}(V)/K)+\sigma^2/2}{\sigma}\right) \\
&\quad - \mathbb{E}(V)F\left(\frac{-\ln(\mathbb{E}(V)/K)-\sigma^2/2}{\sigma}\right)
\end{aligned}
$$

which ensures the theorem. □

1.2 Risk-Neutral Valuations

Consider the Black-Scholes-Merton modeling equation (1.6). The partial differential equation depends purely on the volatility of asset return, asset price, time to maturity and risk-free interest rate. Since no risk preferences are referenced in (1.6), the partial differential equation is independent of investors' risk preferences. Therefore we may assume that all investors are risk-neutral under Black-Scholes-Merton's mathematical formulation.

In a risk-neutral world, returns of all assets are equal to the risk-free interest rate or there will be arbitrage opportunities [5, 43, 51]. To calculate a solution of (1.6), we may adopt a put option on a stock of price S_0, maturing at time T that does not pay dividends before the maturity date. We assume that this put option has a strike price of K. In addition, we may let r denote a constant risk-free interest rate. Hence, the variable price of the put option v in the risk-neutral world is given by

$$\begin{aligned} v(S,t) &= e^{-r(T-t)}\mathbb{E}[\max(K - S_T, 0)] \\ &= e^{-r(T-t)}[S_0 s^{r(T-t)F(d_1) - KF(d_2)}] \\ &= S_0 F(d_1) - K e^{-r(T-t)} F(d_2), \end{aligned}$$

where

$$d_1 = \frac{-\ln[\mathbb{E}(S_T)/K] + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}, \; d_2 = \frac{-\ln[\mathbb{E}(S_T)/K] - \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}.$$

1.3 Limitations of the Black-Scholes-Merton Model

However, there are two issues that cannot be handled well by the standard Black-Scholes-Merton model due to multiple assumptions used.

I. Asymmetry and Excess Kurtosis: The asymmetry and fat tails of the empirical distribution of daily log return of different indices have been observed in the stock market and other asset markets. Further, large movements in asset prices occur more often than a model with normal distribution assumption predicts. It is the main reason for considering asset return with jumps.

II. Stochastic Volatility: Estimated volatilities changes over short period of time. Also, there seems to be a succession of periods with high return variance and with low return variance. This phenomenon is empirically observed in different stock market indexes.

1.4 A Stochastic Volatility Model

Steven Heston introduced a highly effective stochastic volatility model in 1993 [19]. The new model is not based on the assumptions used upon a traditional Black-Scholes-Merton model. Rather, it provides a closed-form solution for the price of a European call option when the spot price for the underlying asset is correlated with its volatility. It improves the accuracy of option-pricing for incorporating the stochastic volatility. [28, 32, 64].

Heston begins by assuming that the spot price of the underlying asset at time t follows from the following stochastic process [19]

$$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{y(t)}dW_1(t), \tag{1.13}$$

where $S(t)$ is the spot price of the underlying asset at time t, μ is the expected return of underlying asset, $y(t)$ is the volatility of the asset at time t and $W_1(t)$ is a Wiener process. In addition, he proposes that the volatility follows the Ornstein-Uhlenbeck process [11, 19, 52],

$$dy(t) = \kappa(\eta - y(t)) + \sigma\sqrt{y(t)}dW_2(t), \tag{1.14}$$

where η is the expected volatility, κ is the rate of reversion of the volatility $y(t)$ to its expected value κ, σ is the volatility of the volatility and $W_2(t)$ is a Weiner process. We assume that the correlation between $W_1(t)$ and $W_2(t)$ is ρ, *that is,*

$$\mathbb{E}[W_1(t)W_2(t)] = \rho.$$

Let $v(S, y, t)$, $t \geq 0$, denote the value of a European put option that is a function of the asset price S, volatility y, and time t. An application of Itô's Lemma and standard non-arbitrage principle with a construction of risk-less portfolio leads to [13, 16, 19, 30],

$$v_t + \frac{1}{2}yS^2v_{SS} + \rho\sigma ySv_{Sy} + \frac{\sigma^2 y}{2}v_{yy} + rSv_S + \kappa(\eta - y)v_y = rv, \ \ S, y > 0. \tag{1.15}$$

Let

$$v(S,y,T) = \max\{K - S, 0\}, \quad S, y \geq 0,$$

be the terminal condition associated with (1.15), where T is the maturity time and K is the strike price. We adopt the following mixed boundary conditions for $S, y > 0$ and $T > t \geq 0$ [12–14],

$$v(0,y,t) = Ke^{-r(T-t)}, \tag{1.16}$$

$$\lim_{S\to\infty} v(S,y,t) = 0, \tag{1.17}$$

$$v_y(S,0,t) = 0, \tag{1.18}$$

$$\lim_{y\to\infty} v_y(S,y,t) = 0. \tag{1.19}$$

CHAPTER TWO

Mathematics Preliminary

2.1 Parabolic Partial Differential Equations

Since (1.15) is in fact a typical parabolic partial differential equation, we dedicate this chapter to the studying of properties of general parabolic partial differential equations.

Definition 2.1. [15,21] Let U be an open bounded subset of $\mathbf{R}^n$ and $U_T = U \times (0, T]$ for some fixed time $T > 0$. We have the equation

$$u_t + Lu = f \quad \text{in} \quad U_T \tag{2.1}$$

where $f : U_T \to \mathbf{R}$ and $g : U \to \mathbf{R}$ is the unknown, $u = u(x, t)$. The letter L denotes for each time t, a linear second-order partial differential operator, having the form

$$L = -\sum_{i,j=1}^{n} a^{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x,t)\frac{\partial}{\partial x_i} + c(x,t),$$

for given coefficients a^{ij}, b^i, and c for $i, j = 1, \ldots, \ldots, n$. We say that the partial differential equation (2.1) is parabolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \quad \leq \quad \theta \mid \xi \mid^2$$

for all $(x, t) \in U_T$, $\xi \in \mathbf{R}^n$.

General second-order parabolic partial differential equations describe, in physical applications, the time-evolution of the density of some quantity u, say a chemical concentration, within the region Ω. The second-order term $\sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i x_j}$ describes the diffusion, the first-order term $\sum_{i=1}^{n} b^i(x,t)u_{x_i}$ describes transport, and the zeroth-order term cu describes creation or depletion.

We start by reviewing the properties of different differential operators. The first and foremost important ones are the eigenvalues and eigenfunctions of operators.

Definition 2.2. [35] The function v, not identically zero, is said to be an eigenfunction of the differential operator L in domain Ω and λ is the corresponding eigenvalue if v vanishes along $\partial\Omega$ and satisfies with Ω the equation $Lv = \lambda v$.

2.2 Finite Difference Methods

It has been known that, of the numerical approximation methods available for solving financial differential equations such as (2.1), those based on finite-differences or finite element settings are used more frequently and desirable than others utilizing Monte Carlo simulation or finite volume methods [8, 9, 31, 44]. Among the first two, finite difference methods are still more friendly to use than finite element methods for solving financial differential equations. Possible reasons include the following. Firstly, finite difference methods are convenient to formulate and implement. Further, in financial fields, domains for modeling differential equations are often rectangle which makes the finite difference approximations a better choice than weak finite element formulations. Usually finite difference methods are used on uniform grids for simplicity. However implementation of finite difference methods on uniform grids can be costly to get desired accuracy. In later situations, nonuniform grids and adaptive methods become more affordable. To further explain our particular finite difference methods, let us introduce the following definitions and notations.

Definition 2.3. Let $\Omega \subset \mathbb{R}^n$ be a connected open set with piecewise smooth boundary. A grid defined on Ω is a set $G \subset \Omega$ such that

$$G = \{(x_{1,i_1}, x_{2,i_2} \cdots, x_{n,i_n}) \mid i_j = 0, 1, 2, \cdots N_j + 1,\ j = 1, 2, \cdots n\},$$

where $x_{j,i_j+1} > x_{j,i_j}$. Further, we require that $B = \{(x_{1,i_1}, x_{2,i_2} \cdots, x_{n,i_n}) \mid \exists\ j = 1, 2, \cdots, n, \text{such that } i_j = 0 \text{ or } i_j = N_j+1\} \subset \partial\Omega$. Each point $(x_{1,i_1}, x_{2,i_2} \cdots, x_{n,i_n}) \in G$ is called an interior grid point. And points in B are called boundary points. Further, we define the i_jth step size in the x_j-direction as

$$h_{x_j,i_j} = x_{j,i_j+1} - x_{j,i_j},\ \ j = 1, 2, \cdots, n,\ i_j = 0, 1, \cdots N_j.$$

If $h_{x_j,i_j} = h_{x_j,i_j+1}$ for $\forall j = 1, 2, \cdots, n,\ i_j = 0, 1, \cdots N_j$, we call G is a uniform grid. Otherwise, we call G a nonuniform grid.

To explain the backbone of a multi-dimensional finite difference methods on a non-uniform mesh, we consider a twice-differentiable function $g : \Omega \to \mathbb{R}$. We let

$$\begin{aligned} g_{i_1,i_2,\cdots i_n} &= g(x_{1,i_1}, x_{2,i_2} \cdots, x_{n,i_n}), \\ h_{x_i} &= \max_{i_j=1,2,\cdots N_j} h_{x_i,i_j}, \\ h &= \max_{i=1,2,\cdots n} h_{x_i}. \end{aligned}$$

We define the following linear operators

$$\Delta_{x_j,-} g_{i_1,i_2,\cdots i_n} = \frac{g_{i_1,i_2,\cdots i_j,\cdots i_n} - g_{i_1,i_2,\cdots i_j-1,\cdots i_n}}{h_{x_j,i_j}}, \tag{2.2}$$

$$\Delta_{x_j,+} g_{i_1,i_2,\cdots i_n} = \frac{g_{i_1,i_2,\cdots i_j+1,\cdots i_n} - g_{i_1,i_2,\cdots i_j,\cdots i_n}}{h_{x_j,i_j}}, \tag{2.3}$$

$$\Delta_{x_j,0} g_{i_1,i_2,\cdots i_n} = \frac{g_{i_1,i_2,\cdots i_j+1/2,\cdots i_n} - g_{i_1,i_2,\cdots i_j-1/2,\cdots i_n}}{h_{x_j,i_j}}, \tag{2.4}$$

$$\begin{aligned} \Delta^2_{x_j,0} g_{i_1,i_2,\cdots i_n} = {} & \frac{2g_{i_1,i_2,\cdots i_j+1,\cdots i_n}}{h_{x_j,i_j}(h_{x_j,i_j-1} + h_{x_j,i_j})} - \frac{2g_{i_1,i_2,\cdots i_j,\cdots i_n}}{h_{x_j,i_j} h_{x_j,i_j-1}} \\ & + \frac{2g_{i_1,i_2,\cdots i_j-1,\cdots i_n}}{h_{x_j,i_j-1} h_{x_j,i_j-1}}. \end{aligned} \tag{2.5}$$

Linear operators defined in (2.2), (2.3) and (2.4) are called backward difference operator, forward difference operator and central difference operator, respectively. These operators are used for the approximation of the first derivatives, *that is,*

$$\begin{aligned} \Delta_{x_j,-} g_{i_1,i_2,\cdots i_n} &= \frac{\partial g}{\partial x_j}(x_{1,i_1}, x_{2,i_2} \cdots, x_{n,i_n}) + \mathcal{O}(h), \\ \Delta_{x_j,+} g_{i_1,i_2,\cdots i_n} &= \frac{\partial g}{\partial x_j}(x_{1,i_1}, x_{2,i_2} \cdots, x_{n,i_n}) + \mathcal{O}(h), \\ \Delta_{x_j,0} g_{i_1,i_2,\cdots i_n} &= \frac{\partial g}{\partial x_j}(x_{1,i_1}, x_{2,i_2} \cdots, x_{n,i_n}) + \mathcal{O}(h^2). \end{aligned}$$

The operator defined in (2.5) is also a central difference operator. However, it is for an approximation of the second derivative of g at the mesh point $(x_{1,i_1}, x_{2,i_2} \cdots, x_{n,i_n})$. It is readily to show that

$$\Delta^2_{x_j,0} g_{i_1,i_2,\cdots i_n} = \frac{\partial^2 g}{\partial x_j^2}(x_{1,i_1}, x_{2,i_2} \cdots, x_{n,i_n}) + \mathcal{O}(h^p),\ p \geq 1.$$

We further observe that $\Delta^2_{x_j,0} g_{i_1,i_2,\cdots i_n} = \Delta_{x_j,0}(\Delta_{x_j,0} g_{i_1,i_2,\cdots i_n})$. So essentially, $\Delta^2_{x_j,0} = \Delta_{x_j,0}\Delta_{x_j,0}$.

To further investigate properties of the aforementioned operators, we need the concepts of consistency and accuracy of a finite difference scheme [35,61,63]. We let $D_{\mathbf{x}}$ denote a finite difference operator evaluated at a point $\mathbf{x} \in \Omega$. Further, we let $I_{\mathbf{x}},\ \mathbf{x} \in G$ be the evaluation operator such that for any function ϕ whose domain contains Ω, we have $I_{\mathbf{x}}\phi = \phi(\mathbf{x})$.

Definition 2.4. [63] Let P a differential operator. Let $P\phi(\mathbf{x}) = f(\mathbf{x}),\ \mathbf{x} \in \Omega$ be a well-posed partial differential equation. We say that the finite difference scheme defined by $D_{\mathbf{x}}\phi = I_{\mathbf{x}} f,\ \mathbf{x} \in G$ is consistent with the partial differential equation if

$$I_{\mathbf{x}} P\phi - D_{\mathbf{x}}\phi \to 0,\ h \to 0,\ \mathbf{x} \in \Omega.$$

The convergence is point-wise at each mesh point in $\mathbf{x} \in \Omega$.

Definition 2.5. [63] A scheme $D_{\mathbf{x}}\phi = f(\mathbf{x})$ that is consistent with the differential equation $Pu = f$ is accurate of order $p > 0$ if for any smooth funtion $\phi\ :\ \Omega \to \mathbb{R}$,

$$D_{\mathbf{x}}\phi - I_{\mathbf{x}} P\phi = \mathcal{O}(h^p), \quad h \to 0^+,\ \mathbf{x} \in G.$$

We say that such a scheme is accurate of order p.

It is convenient to check by Taylor expansion that the operators given in (2.2)-(2.5) are of first-order if we are considering nonuniform grids. If instead, uniform grids are employed, then the accuracies of schemes (2.4) and (2.5) increase to second-order while (2.2) and (2.3) stay as first-order schemes.

2.2.1 Exponential Splitting and Padé Approximation

We consider exponential splitting methods for solving a two-dimensional parabolic partial differential equation in this section. The methodology discussed in this section can be extended to multi-dimensional option modeling cases.

To this end, we consider the following linear initial-boundary value problem,

$$u_t = au_{xx} + bu_{yy}, \quad (x, y) \in \Omega, \; 0 < t \leq T, \tag{2.6}$$

$$u = 0, \quad (x, y) \in \partial\Omega, \; 0 < t \leq T, \tag{2.7}$$

$$u = u_0, \quad (x, y) \in \bar{\Omega}, \; t = 0, \tag{2.8}$$

where $a, \; b$ and u_0 are functions of x and y, and Ω is a finite domain in $\mathbb{R}^2$.

A semi-discretized system corresponding to (2.6)-(2.8) can be formulated to

$$\mathbf{u}_t = A\mathbf{u} + B\mathbf{u}, \quad 0 < t \leq T, \tag{2.9}$$

$$\mathbf{u} = \mathbf{u}_0, \quad t = 0, \tag{2.10}$$

where matrix A is derived from the discretization of the au_{xx} term and matrix B is from bu_{yy} in (2.6), together with (2.7) [35, 47–50, 57–59]. The formal solution to the system (2.9) and (2.10) is given by

$$\mathbf{u}(t) = e^{t(A+B)}\mathbf{u}_0. \tag{2.11}$$

It will be costly if we try to evaluate $e^{t(A+B)}$ directly, since sizes of the matrices A and B can be huge and the matrix $A + B$ is relatively dense. However, both A and B will be relatively sparse and easier to handle separately. This motivates a dimensional splitting. There are different splitting formulas. Herewith, we only consider three most well-established ones [40, 47, 50, 54, 55].

The first formula is the first-order exponential splitting method

$$e^{t(A+B)} = e^{tA}e^{tB} + \mathcal{O}(t^2), \quad t \to 0^+,$$

or

$$e^{t(A+B)} = e^{tB}e^{tA} + \mathcal{O}(t^2), \quad t \to 0^+.$$

It can be verified via a Taylor expansion that

$$e^{t(A+B)} - e^{tA}e^{tB} = e^{t(A+B)} - e^{tB}e^{tA} = \mathcal{O}(t^2), \quad t \to 0^+.$$

The Strang splitting method proposed by Gilbert Strang [40, 54, 62] is given by

$$e^{t(A+B)} = e^{\frac{t}{2}B} e^{tA} e^{\frac{t}{2}B} + \mathcal{O}(t^3), \quad t \to 0^+,$$

or

$$e^{t(A+B)} = e^{\frac{t}{2}A} e^{tB} e^{\frac{t}{2}A} + \mathcal{O}(t^3), \quad t \to 0^+.$$

Both above formulas are of second order.

Our third exponential splitting method is the parallel splitting,

$$e^{t(A+B)} = \frac{1}{2}\left(e^{tA}e^{tB} + e^{tB}e^{tA}\right) + \mathcal{O}(t^3), \quad t \to 0^+.$$

The parallel splitting method is also of second order accuracy.

Splitting methods shown above can be extended for multi-dimensional applications. To see this, we consider the following matrix exponential function

$$M = \exp\left\{ t \sum_{j=1}^{m} A_j \right\}, \tag{2.12}$$

which is typical from a semi-discretization of $m-$dimensional evolution partial differential equation. Although the size of each of the matrices $A_j,\ j = 1, 2, \cdots, m$ can be very large, each of the matrices are relatively sparse as compared to $\sum_{j=1}^{m} A_j$. Thus, the approximation of the matrix exponential M via an exponential splitting procedure may simplify the computation and improve the efficiency [40, 54]. To this end, either the first-order splitting, Strang splitting and parallel splitting methods can be used:

$$M = \prod_{j=1}^{m} e^{tA_j} + \mathcal{O}(t^3), \quad t \to 0^+, \tag{2.13}$$

$$M = e^{\frac{t}{2}A_1} e^{\frac{t}{2}A_2} \cdots e^{\frac{t}{2}A_{m-1}} e^{tA_m} e^{\frac{t}{2}A_{m-1}} \cdots e^{\frac{t}{2}A_2} e^{\frac{t}{2}A_1} + \mathcal{O}(t^3), \quad t \to 0^+, \tag{2.14}$$

$$M = \frac{1}{2}\left(\prod_{j=1}^{m} e^{tA_j} + \prod_{j=m}^{1} e^{tA_j} \right) + \mathcal{O}(t^3), \quad t \to 0^+. \tag{2.15}$$

It was proven by Qin Sheng in 1989 that the highest order of accuracy of a stable exponential splitting method without iusing negative or complex steps is two [54]. Numerous researchers have made progress since then in developing higher order splitting schemes, in particularly by introducing complex or negative time steps [6,7]. However here we will concentrate on stable splitting schemes with only positive real-valued time steps for better applicabilities in financial applications. To use iterative methods to get the final solution of (2.11), we can derive the fully discretized system by approximating the matrix exponential with or without exponential splitting. A large literature has been devoted to studying the efficient ways for approximating the matrix exponentials [45]. Here we are mainly concerned with a particular method called Padé approximation.

The basic idea of Padé approximation is to approximate an exponential function with a rational function. Now let $\mathbb{P}_d$ be the set of all polynomials with degree of $d \geq 0$, and $\mathbb{P}_{\alpha/\beta}$ be the set of all rational functions with the form $\dfrac{p_\alpha}{p_\beta}$, where $p_\alpha \in \mathbb{P}_\alpha$ and $p_\beta \in \mathbb{P}_\beta$.

Definition 2.6. Let $p \geq 0$. We say that $\hat{r}(z) \in \mathbb{P}_{\alpha/\beta}$ is an order p α/β Padé approximation to an exponential function $f(z) = e^z$ if

$$f(z) - \hat{r}(z) = \mathcal{O}(z^{p+1}), \quad \|z\| \to 0^+.$$

The most widely used Padé approximations are 1/0, 0/1 and 1/1 Padé approximations. Each of them leads to a fully discretized a linear system from the semi-discretized scheme (2.11).

Denote $N = A + B$ in (2.11). The fully discretized system of (2.11) under an 1/0 Padé approximation is

$$\mathbf{u} = (I + tN)\mathbf{u}_0, \tag{2.16}$$

which is in fact the Euler's method. On the other hand, an application of the 0/1

Padé approximation yields

$$\mathbf{u} = (I - tN)^{-1}\mathbf{u}_0, \tag{2.17}$$

which is the backward Euler's method. Finally an 1/1 Padé approximation gives us

$$\mathbf{u} = \left(I - \frac{t}{2}N\right)^{-1}\left(I + \frac{t}{2}N\right)\mathbf{u}_0, \tag{2.18}$$

which is the standard Crank-Nicolson method [63].

While both Euler's method (2.16) and backward Euler's method (2.17) are of first order accurate, the Crank-Nicolson's method (2.18) is a second-order method.

2.3 Logarithmic Norm

For the sake of stability analysis, we need a very important concept called logarithmic norm.

Definition 2.7. We let $A \in \mathbb{R}^{m\times m}$, and $\|\cdot\|$ be an induced matrix norm. Then the logarithmic norm $\mu : \mathbb{R}^{m\times m} \to \mathbb{R}$ of A is defined as

$$\mu(A) = \lim_{h\to 0^+} \frac{\|I + hA\| - 1}{h}.$$

Lemma 2.8. [35] *The corresponding logarithmic norm to the spectral norm* $\|\cdot\|_2$ *is given by*

$$\mu_2(A) = \frac{1}{2}\left\|A + A^T\right\|_2.$$

The logarithmic norm corresponding to the $\infty-$*norm is given by*

$$\mu_\infty(A) = \sup_{i=1,2,\cdots,n}\left([A]_{ii} + \sum_{j\neq i}|[A]_{ij}|\right),$$

where $[A]_{ij}$ *denotes the element of* A *in the* i*th row and* j*th column.*

2.4 Moving Mesh Methods

Adaptive numerical methods are promising because of their capability to place grid points or computational degrees of freedom at locations where the largest computational errors may occur without the treatments. Moving mesh methods are typical adaptive strategies used for finite difference schemes. They can be very effective as long as two fundamental requirements are satisfied: (1) The region of the domain where refinement is required must be limited to a relatively small fraction of the computation domain. Roughly no more than 1/3 of the domain should be at the finest grid spacing. (2) The numerical order of a scheme should be as close as possible to the numerical order of the computational data and should not exceed it. That is, if one has a solution flow that is essentially shock dominated, roughly of piecewise linear structure, and if the shocks never fill more than 1/3 of the domain, then low order adaptive schemes can offer a large computational savings when compared to other numerical methods of calculations.

In the past two decades, several moving mesh adaptations have been well developed. They can be categorized as follows [24–26].

- h-method. The $h-$method automatically refines or coarsens the mesh grids based on certain error estimate function called monitor function.

- $p-$method. The $p-$method involves the adaptive enrichment of the polynomial order.

- $r-$method, also known as moving mesh method (MMM). It relocates grid points in a mesh having a fixed number of nodes in such a way that the nodes remain concentrated in regions where rapid variation of the solution is observed.

The moving mesh methods require to generate a bijective mapping from a set of mesh grids to another set of mesh grids on the same domain by allocating more grids in the areas with high error estimates [2–4]. The key ingredients of the moving mesh methods include the following two elements [24–26, 56].

- Monitor functions. A monitor function is function used to redistribute the mesh grids depending on the solution arc-length (in 1D), curvature, or certain error estimates.

- Interpolations. If the mesh equations are time-dependent and are solved simultaneously with the given differential equations, then interpolation of dependent variables from the original mesh grids to the new mesh grids is unnecessary. Otherwise, some kind of interpolation is required to pass the solution on the original mesh grids to the newly generated mesh grids.

CHAPTER THREE

Semi-Discretized Scheme for Heston Volatility Model Equations

3.1 Up-downwind Formulation and Semi-Discretization

A closed form solution of the model was obtained by Steven Heston under a set of specific boundary and initial values for assets of the European type [19]. However, to meet a growing demand from American options and other assets, pricing equations often need to be placed together with more realistic initial boundary conditions or even free boundary conditions. Closed forms of solutions are in general unavailable. Thus, numerical approximations of such solutions have become important and necessary. This chapter is concerned with European options. The scheme we plan to develop can also be extended directly to price Asian and American options. However, due to free boundary conditions associated with Asian and American options, the stability and convergence analysis of the scheme becomes rather complicated. Therefore they have become our continuing endeavor.

There have been numerous recent publications on the numerical solution of Heston modeling equations [17, 30, 32–34, 38, 44, 51, 53, 60, 66, 67]. For instance, certain first-order up-downwind algorithms are proposed and studied by several recent investigators [38]. Stability analysis are carried out via standard von Neumann analysis for Cauchy problems or problems with periodic boundary conditions. Although numerical stabilities have been under investigations for even high order schemes on nonuniform grids [13], rigorous analysis are only available in cases where cross-derivative terms have been neglected. The challenge for a stable method continues [38, 47, 56].

But cross-derivatives are essential in a Heston Option Process. Heston modeling formulations also require more realistic Dirichlet, Neumann, or mixed boundary conditions [1, 19]. These have motivated our approach. In this study, we are particularly interested in computations based on a Heston put option model [12,13,30,32,67]. We are primarily interested in the numerical stability over nonuniform data grids. We also wish to effectively reduce the computational costs and raise the algorithmic efficiency for put option computations. These results can be extended for call options in similar ways.

To implement our finite difference methods for solving the Heston stochastic volatility partial differential equation (1.15), we need to carry out some transformation for the original modeling equation.

Set $\tau = T - t$. Equation (1.15) can be rewritten as

$$v_\tau = \frac{yS^2}{2}v_{SS} + \rho\sigma y S v_{Sy} + \frac{\sigma^2 y}{2}v_{yy} + rSv_S + \kappa(\eta - y)v_y - rv, \quad T > \tau > 0.$$

Let $x = \ln\frac{S}{K},\ u = \frac{v}{K}e^{r\tau}$. For $-\infty < x < \infty,\ y > 0,\ T > \tau > 0$, we observe that

$$u_\tau = \frac{y}{2}u_{xx} + \rho\sigma y u_{xy} + \frac{\sigma^2 y}{2}u_{yy} - \left(\frac{y}{2} - r\right)u_x + \kappa(\eta - y)u_y, \tag{3.1}$$

together with constraints [12, 13, 67],

$$\begin{aligned}
u(x,y,0) &= \max\{1 - e^x, 0\}, \quad -\infty < x < \infty,\ y > 0, && (3.2)\\
\lim_{x\to-\infty} u(x,y,\tau) &= 1, \quad y > 0,\ T \ge \tau > 0, && (3.3)\\
\lim_{x\to\infty} u(x,y,\tau) &= 0, \quad y > 0,\ T \ge \tau > 0, && (3.4)\\
u_y(x,0,\tau) &= 0, \quad -\infty < x < \infty,\ T \ge \tau > 0, && (3.5)\\
\lim_{y\to\infty} u_y(x,y,\tau) &= 0, \quad -\infty < x < \infty,\ T \ge \tau > 0. && (3.6)
\end{aligned}$$

We may extend the temporal domain for (3.1)-(3.6) by allowing $T = \infty$. Further, for the sake of computations, we consider a truncated spatial domain $\Omega = \{(x,y) : -X < x < X;\ 0 < y < Y\}$ for sufficiently large X and Y in the rest of our investigations.

Let $-X = x_0 < x_1 < \cdots < x_M < x_{M+1} = X,\ 0 = y_0 < y_1 < \cdots < y_N < y_{N+1} = Y$, for which $x_m - x_{m-1} = h_m,\ y_n - y_{n-1} = k_n,\ 0 < h_m, k_n \ll 1$, $m = 1, 2, \ldots, M+1,\ n = 1, 2, \ldots, N+1$.

Let $z_{m,n} = z_{m,n}(\tau)$ be an approximation of $z(x_m, y_n, \tau)$, $0 \le m \le M+1,\ 0 \le n \le N+1,\ 0 < \tau < T$. Further, let $\Delta_{\ell,+},\ \Delta_{\ell,-}$ and $\Delta_{\ell,0}$ be forward, backward and central difference operators in the ℓ-direction, respectively, where $\ell \in \{x, y\}$ [47]. Similarly, for appropriate indexes, we define

$$\Delta_{x,0}^2 z_{m,n} = \frac{2z_{m+1,n}}{h_{m+1}(h_{m+1}+h_m)} - \frac{2z_{m,n}}{h_{m+1}h_m} + \frac{2z_{m-1,n}}{h_m(h_{m+1}+h_m)}, \tag{3.7}$$

$$\Delta_{y,0}^2 z_{m,n} = \frac{2z_{m,n+1}}{k_{n+1}(k_{n+1}+k_n)} - \frac{2z_{m,n}}{k_{n+1}k_n} + \frac{2z_{m,n-1}}{k_n(k_{n+1}+k_n)}. \tag{3.8}$$

We now approximate the diffusion terms in (3.1) by using the above, and derivatives in (3.5) and (3.6) via the following,

$$u_y(x_m, 0, \tau) \approx \frac{1}{h_y}\Delta_{y,+}u_{m,0}(\tau),\ u_y(x_m, Y, \tau) \approx \frac{1}{h_y}\Delta_{y,-}u_{m,N+1}(\tau),\ 0 < \tau < T.$$

We approximate the advection terms in (3.1) through three different channels depending upon relations between values of η and r.

Case 1: $\eta > 2r$.

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,+}u_{m,n},\ u_y(x_m, y_n, \tau) \approx \Delta_{y,+}u_{m,n}, \qquad 2r \ge y > 0, \tag{3.9}$$

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,-}u_{m,n},\ u_y(x_m, y_n, \tau) \approx \Delta_{y,+}u_{m,n}, \qquad \eta \ge y > 2r, \tag{3.10}$$

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,-}u_{m,n},\ u_y(x_m, y_n, \tau) \approx \Delta_{y,-}u_{m,n}, \qquad Y > y > \eta. \tag{3.11}$$

Case 2: $\eta \le 2r$.

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,+}u_{m,n},\ u_y(x_m, y_n, \tau) \approx \Delta_{y,+}u_{m,n}, \qquad \eta \ge y > 0, \tag{3.12}$$

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,+}u_{m,n},\ u_y(x_m, y_n, \tau) \approx \Delta_{y,-}u_{m,n}, \qquad 2r \ge y > \eta, \tag{3.13}$$

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,-}u_{m,n},\ u_y(x_m, y_n, \tau) \approx \Delta_{y,-}u_{m,n}, \qquad Y > y > 2r. \tag{3.14}$$

Define

$$h_{\min} = \min_{m=1,2\cdots M} h_m,\ h_{\max} = \max_{m=1,2\cdots M} h_m;\ k_{\min} = \min_{n=1,2\cdots N} k_n,\ k_{\max} = \max_{n=1,2\cdots N} k_n.$$

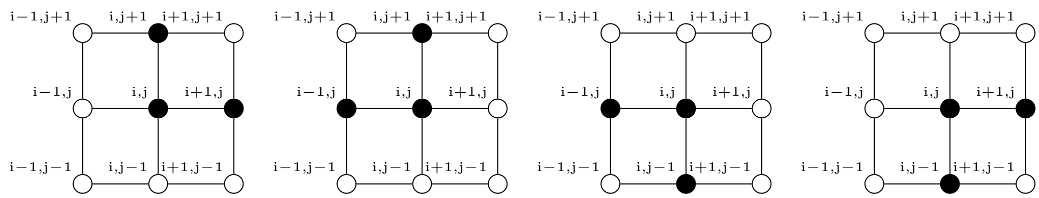

Figure 3.1. Computational stencil for (3.9) - (3.13).

To approximate the cross-derivative in (3.1) dynamically, we have

3.2 Case for $\rho \in [-1, 0]$.

For the smoothness of nonuniform grids [24, 56], we require that

$$-\rho k_{\max} \leq \sigma h_{\min} \leq \sigma h_{\max} \leq -\frac{1}{\rho} k_{\min}. \tag{3.15}$$

We propose that

$$u_{xy}(x_m, y_n, \tau) = \frac{1}{2}(\Delta_{x,+}\Delta_{y,-} + \Delta_{x,-}\Delta_{y,+})u_{m,n}(\tau) + \mathcal{O}(h_{\max} + k_{\max}). \tag{3.16}$$

Substitute all spacial derivative approximations into (3.1) and let w denote the approximate solution to u. We acquire the following linear system,

$$w'(\tau) = Aw(\tau) + f(\tau), \tag{3.17}$$

where $w, f \in \mathbb{R}^{MN}$ and $A \in \mathbb{R}^{MN \times MN}$ is a block tridiagonal matrix in the form of

$$A = \begin{bmatrix} D_1 & Q_1 & \cdots & \cdots & \cdots & 0 \\ P_2 & D_2 & Q_2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & P_{M-2} & D_{M-2} & Q_{M-2} & 0 \\ \cdots & \cdots & \cdots & P_{M-1} & D_{M-1} & Q_{M-1} \\ 0 & \cdots & \cdots & \cdots & P_M & D_M \end{bmatrix},$$

where $P_i, D_j, Q_k \in \mathbb{R}^{N\times N},\ i=2,3,\dots,M;\ j=1,2,\dots,M;\ k=1,2,\dots,M-1.$ Nontrivial entries of the matrices P_m, D_m and Q_m are as follows.

$$
\begin{aligned}
p_{n,n}^{(m)} &= \begin{cases} \frac{y_n}{h_m(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}}, & 0 < y_n \le 2r, \\ \frac{y_n}{h_m(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{y_n-2r}{2h_m}, & 2r < y_n < Y - k_{N+1}, \\ \frac{y_N}{h_m(h_m+h_{m+1})} + \frac{y_N-2r}{2h_m}, & y_n = Y - k_{N+1}; \end{cases} \\
p_{n,n+1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_m k_n}; \\
d_{n,n-1}^{(m)} &= \begin{cases} \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n}, & k_1 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{\kappa(\eta-y_n)}{k_n}, & \eta < y_n \le Y - k_{N+1}; \end{cases} \\
d_{n,n}^{(m)} &= \begin{cases} \alpha_{m,1} + \frac{y_1-2r}{2h_{m+1}} - \frac{\kappa(\eta-y_1)}{k_2}, & y_n = k_1, \\ \beta_{m,n} + \frac{y_n-2r}{2h_{m+1}} - \frac{\kappa(\eta-y_n)}{k_{n+1}}, & k_1 < y_n \le 2r, \\ \beta_{m,n} - \frac{y_n-2r}{2h_m} - \frac{\kappa(\eta-y_n)}{k_{n+1}}, & 2r < y_n \le \eta, \\ \beta_{m,n} - \frac{y_n-2r}{2h_m} + \frac{\kappa(\eta-y_n)}{k_n}, & \eta < y_n < Y - k_{N+1}, \\ \gamma_{m,N} - \frac{y_N-2r}{2h_m} + \frac{\kappa(\eta-y_N)}{k_N}, & y_N = Y - k_{N+1}; \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
d_{n,n+1}^{(m)} &= \begin{cases} \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{\kappa(\eta-y_n)}{k_{n+1}}, & 0 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}}, & \eta < y_n < Y - k_{N+1}; \end{cases} \\
q_{n,n-1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_{m+1}k_n}, \quad y_n > k_1; \\
q_{n,n}^{(m)} &= \begin{cases} \frac{y_1}{h_{m+1}(h_m+h_{m+1})} - \frac{y_1-2r}{2h_{m+1}}, & y_n = k_1, \\ \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{y_n-2r}{2h_{m+1}}, & k_1 < y \le 2r, \\ \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n}, & 2r < y_n \le Y - k_{N+1}, \end{cases}
\end{aligned}
$$

where

$$
\alpha_{m,n} = -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_{n+1}},
$$

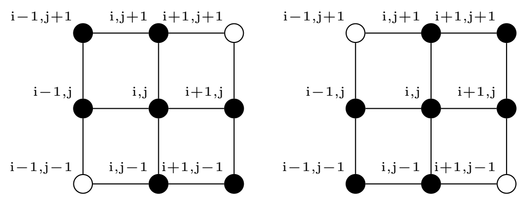

Figure 3.2. Computational stencils of (3.16) **(left)** and (3.19) **(right)**.

$$\begin{aligned}\beta_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{\rho\sigma y_n}{2h_m k_{n+1}},\\ \gamma_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_n}.\end{aligned}$$

It is observed that in the event if $\rho = -1$, we have $h_{\min} = h_{\max} = h,\ k_{\min} = k_{\max} = k,\ k = \sigma h$ due to (3.15). They indicate that uniform spacial grids must be deployed. Thus, (3.17) reduces to

$$w'(\tau) = A_s w(\tau) + f_s(\tau).$$

Nontrivial entries of A_s are readily to obtain based on above discussions.

3.3 Case for $\rho \in (0, 1]$.

We need the following restrictions on mesh steps in the case [20, 47]:

$$\rho k_{\max} \le \sigma h_{\min} \le \sigma h_{\max} \le \frac{1}{\rho} k_{\min}. \tag{3.18}$$

Apparently, when $\rho = 1$, a uniform spacial mesh with $h = \sigma k$ again must be used. Otherwise, different from (3.16), we consider a new dynamically balanced formula,

$$u_{xy}(x_m, y_n, \tau) = \frac{1}{2}(\Delta_{x,-}\Delta_{y,-} + \Delta_{x,+}\Delta_{y,+})u_{m,n}(\tau) + \mathcal{O}(h_{\max} + k_{\max}). \tag{3.19}$$

Computational stencils for (3.16) and (3.19) are shown in Figure 2.

In this circumstance, we obtain the following new system,

$$w'(\tau) = \tilde{A}w(\tau) + \tilde{f}(\tau), \tag{3.20}$$

where $w, \tilde{f}(\tau) \in \mathbb{R}^{MN}$ and $\tilde{A} \in \mathbb{R}^{MN \times MN}$ is block tridiagonal, *that is,*

$$\tilde{A} = \begin{bmatrix} \tilde{D}_1 & \tilde{Q}_1 & \cdots & \cdots & \cdots & 0 \\ \tilde{P}_2 & \tilde{D}_2 & \tilde{Q}_2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \tilde{P}_{M-2} & \tilde{D}_{M-2} & \tilde{Q}_{M-2} & 0 \\ \cdots & \cdots & \cdots & \tilde{P}_{M-1} & \tilde{D}_{M-1} & \tilde{Q}_{M-1} \\ 0 & \cdots & \cdots & \cdots & \tilde{P}_M & \tilde{D}_M \end{bmatrix}.$$

Nontrivial entries of $\tilde{P}_m$, $\tilde{D}_m$ and $\tilde{Q}_m$ within their respective ranges of m are given by

$$\tilde{p}^{(m)}_{n,n-1} = \frac{\rho\sigma y_n}{2h_m k_n}, \quad y_n > k_1;$$

$$\tilde{p}^{(m)}_{n,n} = \begin{cases} \frac{y_1}{h_m(h_m+h_{m+1})}, & y_n = k_1, \\ \frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n}, & k_1 < y_n \le 2r, \\ \frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n-2r}{2h_m}, & 2r < y_n \le Y - k_{N+1}; \end{cases}$$

$$\tilde{d}^{(m)}_{n,n-1} = \begin{cases} \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n}, & k_1 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta-y_n)}{k_n}, & \eta < y_n \le Y - k_{N+1}; \end{cases}$$

$$\tilde{d}^{(m)}_{n,n} = \begin{cases} \tilde{\alpha}_{m,1} + \frac{y_1-2r}{2h_{m+1}} - \frac{\kappa(\eta-y_1)}{k_{n+1}}, & y_1 = k_1, \\ \tilde{\beta}_{m,n} + \frac{y_n-2r}{2h_{m+1}} - \frac{\kappa(\eta-y_n)}{k_{n+1}}, & k_1 < y_n \le 2r, \\ \tilde{\beta}_{m,n} - \frac{y_n-2r}{2h_m} - \frac{\kappa(\eta-y_n)}{k_{n+1}}, & 2r < y_n \le \eta, \\ \tilde{\beta}_{m,n} - \frac{y_n-2r}{2h_m} + \frac{\kappa(\eta-y_n)}{k_n}, & \eta < y_n < Y - k_{N+1}, \\ \tilde{\gamma}_{m,N} - \frac{y_N-2r}{2h_m} + \frac{\kappa(\eta-y_N)}{k_N}, & y_N = Y - k_{N+1}; \end{cases}$$

$$\tilde{d}^{(m)}_{n,n+1} = \begin{cases} \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{\kappa(\eta-y_n)}{k_{n+1}}, & 0 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, & \eta < y_n < Y - k_{N+1}; \end{cases}$$

$$\tilde{q}^{(m)}_{n,n} = \begin{cases} \frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{y_n - 2r}{2h_{m+1}}, & 0 < y_n \leq 2r, \\ \frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, & 2r < y_n < Y - k_{N+1}, \\ \frac{y_N}{h_{m+1}(h_m+h_{m+1})}, & y_N = Y - k_{N+1}; \end{cases}$$

$$\tilde{q}^{(m)}_{n,n+1} = \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \quad 0 < y_n < Y - k_{N+1},$$

where

$$\begin{aligned} \tilde{\alpha}_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n}, \\ \tilde{\beta}_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n}, \\ \tilde{\gamma}_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_n}. \end{aligned}$$

Apparently, (3.20) reduces to a uniform scheme with nontrivial matrix $\tilde{A}$ when $\rho = 1$,

$$w'(\tau) = \frac{1}{2h^2}\tilde{A}_s w(\tau) + \tilde{f}(\tau).$$

3.4 von Neumann Necessary Condition for Stability

Lemma 3.1. [35, 37] *The semi-discretized scheme* (3.17) *is stable if*

$$\lim_{h,k\to 0}\left(\max_{\tau\in[0,\tau^*]} \left\|e^{\tau A}\right\|_2\right) \leq c(\tau^*),$$

where $t^* \in (0, T)$.

Lemma 3.1 is called the von Neumann necessary condition for stability. We prove that our scheme possess the Neumann stability property.

Lemma 3.2. [22, 23, 35, 36, 39] *Let* $B \in \mathbb{C}^{d\times d}$. *Then* $\sigma(B) \subset \cup_{i=1}^{d} S_i$, *where*

$$S_i = \left\{ z \in C : |z - b_{i,i}| \leq \sum_{j=1, j\neq i}^{d} |b_{i,j}| \right\}$$

and $\sigma(B)$ *is the set containing all eigenvalues of* B. *Moreover,* $\lambda \in \sigma(B)$ *may lie on* ∂S_{i^0} *for some* $i^0 \in \{1, 2, \ldots, d\}$ *only if* $\lambda \in \partial S_i$ *for all* $i = 1, 2, \ldots, d$. *The* S_i *are known as Geršgorin discs.*

Lemma 3.3. [22] *The matrix exponential,* e^{tA}, *tends to a zero matrix as* $t \to +\infty$ *if and only if all the eigenvalues of* A *have strictly negative real parts.*

Theorem 3.4. *The semi-discretized schemes* (3.17) *and* (3.20) *are linearly stable.*

Proof. We only need to show that each of the MN Geršhgorin discs of A lies on the left side of the complex plane [38, 44]. There are four types of discs to consider:

(1) discs centered at an internal mesh point;

(2) discs centered on one of the Dirichlet boundaries;

(3) discs centered on one of the Neumann boundaries;

(4) discs centered at one of the intersection mesh points of one Dirichlet boundary and the Neumann boundary.

When $\rho \in (0, 1]$ and $2r < \eta$, we have the following.

CASE 1: Consider the situation in which $\eta < y_n \leq Y$. Let $z \in S_i$ be any complex number, where S_i is a Geršhgorin disc centered at an internal point of the spacial grids. Hence,

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|. \qquad (3.21)
\end{aligned}$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.21). As a consequence, (3.21) renders to

$$\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n}$$

$$\leq \left|\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+ \left|\frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_m k_n}\right|$$
$$+ \left|\frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|. \tag{3.22}$$

Recall (3.18) and that $\rho > 0$. We have

$$\frac{2}{\rho\sigma}k_n,\ \frac{2}{\rho\sigma}k_{n+1} \geq h_m + h_{m+1} \text{ and } h_m,\ h_{m+1} \geq \frac{\rho}{\sigma}(k_n + k_{n+1}).$$

The above lead to

$$\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} \geq \frac{\rho\sigma y_n}{2h_m k_n}, \qquad \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} \geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}},$$
$$\frac{y_n}{h_{m+1}(h_m+h_{m+1})} \geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \qquad \frac{y_n}{h_m(h_m+h_{m+1})} \geq \frac{\rho\sigma y_n}{2h_m k_n}.$$

Furthermore, since $y > \eta > 2r$, we conclude that

$$-\frac{\kappa(\eta - y_n)}{k_n} \geq 0 \text{ and } \frac{y_n - 2r}{2h_m} \geq 0.$$

Therefore, the term inside each pair of absolute signs in (3.22) must be positive. We may remove all absolute signs in (3.22), and, subsequently, yields

$$\alpha \leq 0,$$

which is what we expect. Generalizations of the discussion for cases involving $y \leq \eta$ are straightforward. Therefore all eigenvalues contained in S_i must lie on the left half of the complex plane. Now consider the situation in which $2r < y_n \leq \eta$.

$$\left|z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n}\right|$$
$$\leq \left|\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+ \left|\frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_m k_n}\right|$$
$$+ \left|\frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|. \tag{3.23}$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.23). As a consequence, (3.23) renders to

$$\begin{aligned}
&\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\qquad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\qquad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|. \qquad (3.24)
\end{aligned}$$

Recall (3.18) and that $\rho > 0$. We have

$$\frac{2}{\rho\sigma} k_n, \ \frac{2}{\rho\sigma} k_{n+1} \geq h_m + h_{m+1} \text{ and } h_m, \ h_{m+1} \geq \frac{\rho}{\sigma}(k_n + k_{n+1}).$$

The above lead to

$$\begin{aligned}
\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} &\geq \frac{\rho\sigma y_n}{2h_m k_n}, \qquad & \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} &\geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \\
\frac{y_n}{h_{m+1}(h_m + h_{m+1})} &\geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \qquad & \frac{y_n}{h_m(h_m + h_{m+1})} &\geq \frac{\rho\sigma y_n}{2h_m k_n}.
\end{aligned}$$

Furthermore, since $\eta > y > 2r$, we conclude that

$$\frac{\kappa(\eta - y_n)}{k_n} \geq 0 \text{ and } \frac{y_n - 2r}{2h_m} \geq 0.$$

Therefore, the term inside each pair of absolute signs in (3.24) must be positive. We may remove all absolute signs in (3.24), and, subsequently, yields

$$\alpha \leq 0,$$

which is what we expect. Generalizations of the discussion for cases involving $y \leq \eta$ are straightforward. Therefore all eigenvalues contained in S_i must lie on the left half of the complex plane.

Now consider the situation in which $0 < y_n \leq 2r.$. Hence,

$$
\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\qquad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\qquad\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\qquad\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} \right|. \qquad (3.25)
\end{aligned}
$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.25). As a consequence, (3.25) renders to

$$
\begin{aligned}
&\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \\
&\qquad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\qquad\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\qquad\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} \right|. \qquad (3.26)
\end{aligned}
$$

Recall (3.18) and that $\rho > 0$. We have

$$
\frac{2}{\rho\sigma}k_n,\ \frac{2}{\rho\sigma}k_{n+1} \geq h_m + h_{m+1} \text{ and } h_m,\ h_{m+1} \geq \frac{\rho}{\sigma}(k_n + k_{n+1}).
$$

The above lead to

$$
\begin{aligned}
\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} &\geq \frac{\rho\sigma y_n}{2h_m k_n}, & \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} &\geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \\
\frac{y_n}{h_{m+1}(h_m + h_{m+1})} &\geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, & \frac{y_n}{h_m(h_m + h_{m+1})} &\geq \frac{\rho\sigma y_n}{2h_m k_n}.
\end{aligned}
$$

Furthermore, since $\eta > y > 2r$, we conclude that

$$
\frac{\kappa(\eta - y_n)}{k_n} \geq 0 \text{ and } -\frac{y_n - 2r}{2h_m} \geq 0.
$$

Therefore, the term inside each pair of absolute signs in (3.26) must be positive. We may remove all absolute signs in (3.26), and, subsequently, yields

$$\alpha \leq 0,$$

which is what we expect. Generalizations of the discussion for cases involving $y \leq \eta$ are straightforward. Therefore all eigenvalues contained in S_i must lie on the left half of the complex plane.

CASE 2: Now consider the case of $x = x_1$ and $\eta < y < Y$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\qquad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|.
\end{aligned}$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}} \left(\frac{1}{h_m + h_{m+1}} - \frac{1}{h_m} \right) - \frac{y_n - 2r}{2h_m} < 0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Now consider the case of $x = x_1$ and $2r < y \leq \eta$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\qquad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|.
\end{aligned}$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}} \left(\frac{1}{h_m + h_{m+1}} - \frac{1}{h_m} \right) - \frac{y_n - 2r}{2h_m} < 0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Now consider the case of $x = x_1$ and $0 < y \leq 2r$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\qquad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|.
\end{aligned}$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}} \left(\frac{1}{h_m + h_{m+1}} - \frac{1}{h_m} \right) + \frac{y_n - 2r}{2h_m} < 0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Case 3: In the circumstance, when $\eta < y < Y$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\qquad + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_m h_{m+1}} < 0.$$

In the case when $2r < y < \eta$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence we have

$$\begin{aligned}
&\left|z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n}\right| \\
&\qquad \leq \left|\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| \\
&\qquad + \left|\frac{\rho\sigma y_n}{2h_m k_n}\right| + \left|\frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_m h_{m+1}} < 0.$$

In the case when $0 < y < 2r$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left|z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n}\right| \\
&\qquad \leq \left|\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| \\
&\qquad + \left|\frac{\rho\sigma y_n}{2h_m k_n}\right| + \left|\frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m}\right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_m h_{m+1}} < 0.$$

CASE 4: In the circumstance, when $\eta < y < Y$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left|z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1} + k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N - 2r}{2h_m} - \frac{\kappa(\eta - y_N)}{k_N}\right| \\
&\qquad \leq \left|\frac{\sigma^2 y_N}{k_N(k_N + k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{\kappa(\eta - y_N)}{k_N}\right| + \left|\frac{y_N}{h_{m+1}(h_m + h_{m+1})}\right| \\
&\qquad + \left|\frac{\rho\sigma y_N}{2h_m k_N}\right| + \left|\frac{y_N}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N - 2r}{2h_m}\right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_N}{(h_m + h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

When $2r < y < \eta$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left| z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1} + k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N - 2r}{2h_m} + \frac{\kappa(\eta - y_N)}{k_N} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_N}{k_N(k_N + k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{\kappa(\eta - y_N)}{k_N} \right| + \left| \frac{y_N}{h_{m+1}(h_m + h_{m+1})} \right| \\
&\quad + \left| \frac{\rho\sigma y_N}{2h_m k_N} \right| + \left| \frac{y_N}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N - 2r}{2h_m} \right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_N}{(h_m + h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

When $0 < y \leq 2r$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left| z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1} + k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{y_N - 2r}{2h_m} + \frac{\kappa(\eta - y_N)}{k_N} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_N}{k_N(k_N + k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{\kappa(\eta - y_N)}{k_N} \right| + \left| \frac{y_N}{h_{m+1}(h_m + h_{m+1})} \right| \\
&\quad + \left| \frac{\rho\sigma y_N}{2h_m k_N} \right| + \left| \frac{y_N}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{y_N - 2r}{2h_m} \right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_N}{(h_m + h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

When $\rho \in (0, 1]$ and $\eta < 2r$, we have the following.

Case 1: Consider the situation in which $2r < y_n \leq Y$. Let $z \in S_i$ be any complex number, where S_i is a Geršhgorin disc centered at an internal point of the spacial grids. Hence,

$$\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n} \right|$$

$$
\begin{aligned}
&\leq \left|\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| \\
&\quad + \left|\frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_m k_n}\right| \\
&\quad + \left|\frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|. \qquad (3.27)
\end{aligned}
$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.27). As a consequence, (3.27) renders to

$$
\begin{aligned}
&\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n} \\
&\quad \leq \left|\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| \\
&\quad + \left|\frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_m k_n}\right| \\
&\quad + \left|\frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|. \qquad (3.28)
\end{aligned}
$$

Recall (3.18) and that $\rho > 0$. We have

$$
\frac{2}{\rho\sigma}k_n,\ \frac{2}{\rho\sigma}k_{n+1} \geq h_m + h_{m+1} \text{ and } h_m,\ h_{m+1} \geq \frac{\rho}{\sigma}(k_n + k_{n+1}).
$$

The above lead to

$$
\begin{aligned}
\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} &\geq \frac{\rho\sigma y_n}{2h_m k_n}, \qquad & \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} &\geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \\
\frac{y_n}{h_{m+1}(h_m+h_{m+1})} &\geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \qquad & \frac{y_n}{h_m(h_m+h_{m+1})} &\geq \frac{\rho\sigma y_n}{2h_m k_n}.
\end{aligned}
$$

Furthermore, since $y > 2r > \eta$, we conclude that

$$
-\frac{\kappa(\eta - y_n)}{k_n} \geq 0 \text{ and } \frac{y_n - 2r}{2h_m} \geq 0.
$$

Therefore, the term inside each pair of absolute signs in (3.28) must be positive. We may remove all absolute signs in (3.28), and, subsequently, yields

$$
\alpha \leq 0.
$$

Now consider the situation in which $\eta < y_n \le 2r.$. Hence,

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\qquad \le \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\qquad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\qquad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|. \qquad (3.29)
\end{aligned}$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.29). As a consequence, (3.29) renders to

$$\begin{aligned}
&\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \\
&\qquad \le \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\qquad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\qquad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|. \qquad (3.30)
\end{aligned}$$

Recall (3.18) and that $\rho > 0$. We have

$$\frac{2}{\rho\sigma}k_n,\ \frac{2}{\rho\sigma}k_{n+1} \ge h_m + h_{m+1} \text{ and } h_m,\ h_{m+1} \ge \frac{\rho}{\sigma}(k_n + k_{n+1}).$$

The above lead to

$$\begin{aligned}
\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} &\ge \frac{\rho\sigma y_n}{2h_m k_n}, \qquad & \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} &\ge \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \\
\frac{y_n}{h_{m+1}(h_m + h_{m+1})} &\ge \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \qquad & \frac{y_n}{h_m(h_m + h_{m+1})} &\ge \frac{\rho\sigma y_n}{2h_m k_n}.
\end{aligned}$$

Furthermore, since $2r > y > \eta$, we conclude that

$$\frac{\kappa(\eta - y_n)}{k_n} \ge 0 \text{ and } \frac{y_n - 2r}{2h_m} \ge 0.$$

Therefore, the term inside each pair of absolute signs in (3.30) must be positive. We may remove all absolute signs in (3.30), and, subsequently, yields $\alpha \leq 0$, which is what we expect.

Now consider the situation in which $0 < y_n \leq \eta.$. Hence,

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\quad\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} \right|. \qquad (3.31)
\end{aligned}$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.31). As a consequence, (3.31) renders to

$$\begin{aligned}
&\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\quad\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} \right|. \qquad (3.32)
\end{aligned}$$

Recall (3.18) and that $\rho > 0$. We have

$$\frac{2}{\rho\sigma}k_n,\ \frac{2}{\rho\sigma}k_{n+1} \geq h_m + h_{m+1} \text{ and } h_m,\ h_{m+1} \geq \frac{\rho}{\sigma}(k_n + k_{n+1}).$$

The above lead to

$$\begin{aligned}
\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} &\geq \frac{\rho\sigma y_n}{2h_m k_n}, \qquad & \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} &\geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \\
\frac{y_n}{h_{m+1}(h_m + h_{m+1})} &\geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \qquad & \frac{y_n}{h_m(h_m + h_{m+1})} &\geq \frac{\rho\sigma y_n}{2h_m k_n}.
\end{aligned}$$

Furthermore, since $\eta > y > 0$, we conclude that

$$\frac{\kappa(\eta - y_n)}{k_n} \geq 0 \text{ and } \quad -\frac{y_n - 2r}{2h_m} \geq 0.$$

Therefore, the term inside each pair of absolute signs in (3.32) must be positive. We may remove all absolute signs in (3.32), and, subsequently, yields $\alpha \leq 0$, which is what we expect.

Case 2: Now consider the case of $x = x_1$ and $2r < y < Y$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|.
\end{aligned}$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}} \left(\frac{1}{h_m + h_{m+1}} - \frac{1}{h_m} \right) - \frac{y_n - 2r}{2h_m} < 0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Now consider the case of $x = x_1$ and $\eta < y \leq 2r$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|.
\end{aligned}$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}} \left(\frac{1}{h_m + h_{m+1}} - \frac{1}{h_m} \right) - \frac{y_n - 2r}{2h_m} < 0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Now consider the case of $x = x_1$ and $0 < y \leq \eta$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|.
\end{aligned}$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}} \left(\frac{1}{h_m + h_{m+1}} - \frac{1}{h_m} \right) + \frac{y_n - 2r}{2h_m} < 0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

CASE 3: In the circumstance, when $2r < y < Y$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_m h_{m+1}} < 0.$$

In the case when $\eta < y < 2r$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\left|z+\frac{y_n}{h_mh_{m+1}}+\frac{\sigma^2y_n}{k_nk_{n+1}}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}-\frac{\rho\sigma y_n}{2h_mk_n}+\frac{y_n-2r}{2h_m}-\frac{\kappa(\eta-y_n)}{k_n}\right|$$
$$\leq\left|\frac{\sigma^2y_n}{k_n(k_n+k_{n+1})}-\frac{\rho\sigma y_n}{2h_mk_n}-\frac{\kappa(\eta-y_n)}{k_n}\right|+\left|\frac{\sigma^2y_n}{k_{n+1}(k_n+k_{n+1})}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+\left|\frac{\rho\sigma y_n}{2h_mk_n}\right|+\left|\frac{y_n}{h_m(h_m+h_{m+1})}-\frac{\rho\sigma y_n}{2h_mk_n}+\frac{y_n-2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy $\alpha\leq\frac{y_n}{h_m(h_m+h_{m+1})}-\frac{y_n}{h_mh_{m+1}}<0$.

In the case when $0<y<\eta$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z\in S_i$ we have

$$\left|z+\frac{y_n}{h_mh_{m+1}}+\frac{\sigma^2y_n}{k_nk_{n+1}}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}-\frac{\rho\sigma y_n}{2h_mk_n}-\frac{y_n-2r}{2h_m}-\frac{\kappa(\eta-y_n)}{k_n}\right|$$
$$\leq\left|\frac{\sigma^2y_n}{k_n(k_n+k_{n+1})}-\frac{\rho\sigma y_n}{2h_mk_n}-\frac{\kappa(\eta-y_n)}{k_n}\right|+\left|\frac{\sigma^2y_n}{k_{n+1}(k_n+k_{n+1})}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+\left|\frac{\rho\sigma y_n}{2h_mk_n}\right|+\left|\frac{y_n}{h_m(h_m+h_{m+1})}-\frac{\rho\sigma y_n}{2h_mk_n}-\frac{y_n-2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha\leq\frac{y_n}{h_m(h_m+h_{m+1})}-\frac{y_n}{h_mh_{m+1}}<0.$$

Case 4: In the circumstance, when $2r<y<Y$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z\in S_i$ we have

$$\left|z+\frac{y_N}{h_mh_{m+1}}+\frac{\sigma^2y_N}{k_N(k_{N+1}+k_N)}-\frac{\rho\sigma y_N}{2h_mk_N}+\frac{y_N-2r}{2h_m}-\frac{\kappa(\eta-y_N)}{k_N}\right|$$
$$\leq\left|\frac{\sigma^2y_N}{k_N(k_N+k_{N+1})}-\frac{\rho\sigma y_N}{2h_mk_N}-\frac{\kappa(\eta-y_N)}{k_N}\right|+\left|\frac{y_N}{h_{m+1}(h_m+h_{m+1})}\right|$$
$$+\left|\frac{\rho\sigma y_N}{2h_mk_N}\right|+\left|\frac{y_N}{h_m(h_m+h_{m+1})}-\frac{\rho\sigma y_N}{2h_mk_N}+\frac{y_N-2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha\leq\frac{y_N}{(h_m+h_{m+1})^2}-\frac{y_N}{h_mh_{m+1}}<0.$$

When $\eta < y < 2r$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left| z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1}+k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N - 2r}{2h_m} + \frac{\kappa(\eta - y_N)}{k_N} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_N}{k_N(k_N + k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{\kappa(\eta - y_N)}{k_N} \right| + \left| \frac{y_N}{h_{m+1}(h_m + h_{m+1})} \right| \\
&\quad + \left| \frac{\rho\sigma y_N}{2h_m k_N} \right| + \left| \frac{y_N}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N - 2r}{2h_m} \right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_N}{(h_m + h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

When $0 < y \leq \eta$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left| z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1}+k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{y_N - 2r}{2h_m} + \frac{\kappa(\eta - y_N)}{k_N} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_N}{k_N(k_N + k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{\kappa(\eta - y_N)}{k_N} \right| + \left| \frac{y_N}{h_{m+1}(h_m + h_{m+1})} \right| \\
&\quad + \left| \frac{\rho\sigma y_N}{2h_m k_N} \right| + \left| \frac{y_N}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{y_N - 2r}{2h_m} \right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_N}{(h_m + h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

When $\rho \in [-1, 0]$ and $2r < \eta$, we have the following.

Case 1: Consider the situation in which $\eta < y_n \leq Y$. Hence,

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} \right| \\
&\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|. \qquad (3.33)
\end{aligned}$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.33). As a consequence, (3.33) renders to

$$\begin{aligned}
\alpha &+ \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n} \\
&\leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|. \qquad (3.34)
\end{aligned}$$

Recall (3.18) and that $\rho > 0$. We have

$$-\frac{2}{\rho\sigma}k_n, \quad -\frac{2}{\rho\sigma}k_{n+1} \geq h_m + h_{m+1} \text{ and } h_m, \ h_{m+1} \geq -\frac{\rho}{\sigma}(k_n + k_{n+1}).$$

The above lead to

$$\begin{aligned}
\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} &\geq -\frac{\rho\sigma y_n}{2h_m k_n}, \qquad & \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} &\geq -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \\
\frac{y_n}{h_{m+1}(h_m + h_{m+1})} &\geq -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \qquad & \frac{y_n}{h_m(h_m + h_{m+1})} &\geq -\frac{\rho\sigma y_n}{2h_m k_n}.
\end{aligned}$$

Furthermore, since $y > \eta > 2r$, we conclude that

$$-\frac{\kappa(\eta - y_n)}{k_n} \geq 0 \text{ and } \frac{y_n - 2r}{2h_m} \geq 0.$$

Therefore, the term inside each pair of absolute signs in (3.34) must be positive. We may remove all absolute signs in (3.34), and, subsequently, yields

$$\alpha \leq 0,$$

which is what we expect. Generalizations of the discussion for cases involving $y \leq \eta$ are straightforward. Therefore all eigenvalues contained in S_i must lie on the left half of the complex plane.

Now consider the situation in which $2r < y_n \leq \eta$. Hence,

$$
\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|. \qquad (3.35)
\end{aligned}
$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.35). As a consequence, (3.35) renders to

$$
\begin{aligned}
&\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|. \qquad (3.36)
\end{aligned}
$$

Recall (3.15) and that $\rho \leq 0$. We have

$$
-\frac{2}{\rho\sigma}k_n, \ -\frac{2}{\rho\sigma}k_{n+1} \geq h_m + h_{m+1} \text{ and } h_m, \ h_{m+1} \geq -\frac{\rho}{\sigma}(k_n + k_{n+1}).
$$

The above lead to

$$
\begin{aligned}
\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} &\geq -\frac{\rho\sigma y_n}{2h_m k_n}, \qquad & \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} &\geq -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \\
\frac{y_n}{h_{m+1}(h_m + h_{m+1})} &\geq -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \qquad & \frac{y_n}{h_m(h_m + h_{m+1})} &\geq -\frac{\rho\sigma y_n}{2h_m k_n}.
\end{aligned}
$$

Furthermore, since $\eta > y > 2r$, we conclude that

$$
\frac{\kappa(\eta - y_n)}{k_n} \geq 0 \text{ and } \frac{y_n - 2r}{2h_m} \geq 0.
$$

Therefore, the term inside each pair of absolute signs in (3.36) must be positive. We may remove all absolute signs in (3.36), and, subsequently, yields

$$\alpha \le 0,$$

which is what we expect. Generalizations of the discussion for cases involving $y \le \eta$ are straightforward. Therefore all eigenvalues contained in S_i must lie on the left half of the complex plane.

Now consider the situation in which $0 < y_n \le 2r.$. Hence,

$$\begin{aligned} &\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\ &\quad \le \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\ &\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_m k_n} \right| \\ &\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} \right|. \end{aligned} \tag{3.37}$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.37). As a consequence, (3.37) renders to

$$\begin{aligned} &\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \\ &\quad \le \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\ &\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| -\frac{\rho\sigma y_n}{2h_m k_n} \right| \\ &\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} \right|. \end{aligned} \tag{3.38}$$

Recall (3.18) and that $\rho > 0$. We have

$$-\frac{2}{\rho\sigma}k_n, \;\; -\frac{2}{\rho\sigma}k_{n+1} \ge h_m + h_{m+1} \text{ and } h_m, \; h_{m+1} \ge -\frac{\rho}{\sigma}(k_n + k_{n+1}).$$

The above lead to

$$\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} \geq -\frac{\rho\sigma y_n}{2h_m k_n}, \qquad \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} \geq -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}},$$
$$\frac{y_n}{h_{m+1}(h_m+h_{m+1})} \geq -\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \qquad \frac{y_n}{h_m(h_m+h_{m+1})} \geq -\frac{\rho\sigma y_n}{2h_m k_n}.$$

Furthermore, since $y > \eta > 2r$, we conclude that

$$\frac{\kappa(\eta - y_n)}{k_n} \geq 0 \text{ and } \ -\frac{y_n - 2r}{2h_m} \geq 0.$$

Therefore, the term inside each pair of absolute signs in (3.38) must be positive. We may remove all absolute signs in (3.38), and, subsequently, yields

$$\alpha \leq 0,$$

which is what we expect. Therefore all eigenvalues contained in S_i must lie on the left half of the complex plane.

Case 2: Now consider the case of $x = x_1$ and $\eta < y < Y$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\qquad + \left| \frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|.
\end{aligned}$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}} \left(\frac{1}{h_m + h_{m+1}} - \frac{1}{h_m} \right) - \frac{y_n - 2r}{2h_m} < 0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Now consider the case of $x = x_1$ and $2r < y \leq \eta$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right|$$
$$\leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|$$
$$+ \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|.$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}} \left(\frac{1}{h_m + h_{m+1}} - \frac{1}{h_m} \right) - \frac{y_n - 2r}{2h_m} < 0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Now consider the case of $x = x_1$ and $0 < y \leq 2r$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right|$$
$$\leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|$$
$$+ \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|.$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}} \left(\frac{1}{h_m + h_{m+1}} - \frac{1}{h_m} \right) + \frac{y_n - 2r}{2h_m} < 0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Case 3: In the circumstance, when $\eta < y < Y$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\left|z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n}\right|$$
$$\leq \left|\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+ \left|\frac{\rho\sigma y_n}{2h_m k_n}\right| + \left|\frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_m h_{m+1}} < 0.$$

In the case when $2r < y < \eta$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\left|z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n}\right|$$
$$\leq \left|\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+ \left|\frac{\rho\sigma y_n}{2h_m k_n}\right| + \left|\frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_m h_{m+1}} < 0.$$

In the case when $0 < y < 2r$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\left|z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n}\right|$$
$$\leq \left|\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+ \left|\frac{\rho\sigma y_n}{2h_m k_n}\right| + \left|\frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_m h_{m+1}} < 0.$$

Case 4: In the circumstance, when $\eta < y < Y$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left| z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1} + k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N - 2r}{2h_m} - \frac{\kappa(\eta - y_N)}{k_N} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_N}{k_N(k_N + k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{\kappa(\eta - y_N)}{k_N} \right| + \left| \frac{y_N}{h_{m+1}(h_m + h_{m+1})} \right| \\
&\quad + \left| \frac{\rho\sigma y_N}{2h_m k_N} \right| + \left| \frac{y_N}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N - 2r}{2h_m} \right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_N}{(h_m + h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

When $2r < y < \eta$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left| z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1} + k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N - 2r}{2h_m} + \frac{\kappa(\eta - y_N)}{k_N} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_N}{k_N(k_N + k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{\kappa(\eta - y_N)}{k_N} \right| + \left| \frac{y_N}{h_{m+1}(h_m + h_{m+1})} \right| \\
&\quad + \left| \frac{\rho\sigma y_N}{2h_m k_N} \right| + \left| \frac{y_N}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N - 2r}{2h_m} \right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_N}{(h_m + h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

When $0 < y \leq 2r$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\left| z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1} + k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{y_N - 2r}{2h_m} + \frac{\kappa(\eta - y_N)}{k_N} \right|$$

$$\le \left|\frac{\sigma^2 y_N}{k_N(k_N+k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{\kappa(\eta - y_N)}{k_N}\right| + \left|\frac{y_N}{h_{m+1}(h_m+h_{m+1})}\right|$$
$$+ \left|\frac{\rho\sigma y_N}{2h_m k_N}\right| + \left|\frac{y_N}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{y_N - 2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \le \frac{y_N}{(h_m+h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

When $\rho \in (0,1]$ and $\eta < 2r$, we have the following. Case 1: Consider the situation in which $2r < y_n \le Y$. Let $z \in S_i$ be any complex number, where S_i is a Geršhgorin disc centered at an internal point of the spacial grids. Hence,

$$\left|z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n}\right|$$
$$\le \left|\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta-y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$

$$+ \left|\frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_m k_n}\right|$$
$$+ \left|\frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|. \tag{3.39}$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.39). As a consequence, (3.39) renders to

$$\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n}$$
$$\le \left|\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta-y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+ \left|\frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| + \left|\frac{\rho\sigma y_n}{2h_m k_n}\right|$$
$$+ \left|\frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|. \tag{3.40}$$

Recall (3.18) and that $\rho > 0$. We have

$$\frac{2}{\rho\sigma}k_n,\ \frac{2}{\rho\sigma}k_{n+1} \geq h_m + h_{m+1} \text{ and } h_m,\ h_{m+1} \geq \frac{\rho}{\sigma}(k_n + k_{n+1}).$$

The above lead to

$$\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} \geq \frac{\rho\sigma y_n}{2h_m k_n}, \qquad \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} \geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}},$$
$$\frac{y_n}{h_{m+1}(h_m + h_{m+1})} \geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \qquad \frac{y_n}{h_m(h_m + h_{m+1})} \geq \frac{\rho\sigma y_n}{2h_m k_n}.$$

Furthermore, since $y > 2r > \eta$, we conclude that

$$-\frac{\kappa(\eta - y_n)}{k_n} \geq 0 \text{ and } \frac{y_n - 2r}{2h_m} \geq 0.$$

Therefore, the term inside each pair of absolute signs in (3.40) must be positive. We may remove all absolute signs in (3.40), and, subsequently, yields

$$\alpha \leq 0,$$

Now consider the situation in which $\eta < y_n \leq 2r.$. Hence,

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} \right|. \qquad (3.41)
\end{aligned}$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.41). As a consequence, (3.41) renders to

$$\begin{aligned}
&\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|
\end{aligned}$$

$$+\left|\frac{y_n}{h_{m+1}(h_m+h_{m+1})}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|+\left|\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|+\left|\frac{\rho\sigma y_n}{2h_mk_n}\right|$$
$$+\left|\frac{y_n}{h_m(h_m+h_{m+1})}-\frac{\rho\sigma y_n}{2h_mk_n}+\frac{y_n-2r}{2h_m}\right|. \tag{3.42}$$

Recall (3.18) and that $\rho > 0$. We have

$$\frac{2}{\rho\sigma}k_n,\ \frac{2}{\rho\sigma}k_{n+1}\geq h_m+h_{m+1} \text{ and } h_m,\ h_{m+1}\geq\frac{\rho}{\sigma}(k_n+k_{n+1}).$$

The above lead to

$$\frac{\sigma^2y_n}{k_n(k_n+k_{n+1})}\geq\frac{\rho\sigma y_n}{2h_mk_n},\qquad \frac{\sigma^2y_n}{k_{n+1}(k_n+k_{n+1})}\geq\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}},$$
$$\frac{y_n}{h_{m+1}(h_m+h_{m+1})}\geq\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}},\qquad \frac{y_n}{h_m(h_m+h_{m+1})}\geq\frac{\rho\sigma y_n}{2h_mk_n}.$$

Furthermore, since $2r > y > \eta$, we conclude that

$$\frac{\kappa(\eta-y_n)}{k_n}\geq 0 \text{ and } \frac{y_n-2r}{2h_m}\geq 0.$$

Therefore, the term inside each pair of absolute signs in (3.42) must be positive. We may remove all absolute signs in (3.42), and, subsequently, yields

$$\alpha\leq 0,$$

which is what we expect.

Now consider the situation in which $0 < y_n \leq \eta$. . Hence,

$$\left|z+\frac{y_n}{h_mh_{m+1}}+\frac{\sigma^2y_n}{k_nk_{n+1}}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}-\frac{\rho\sigma y_n}{2h_mk_n}-\frac{y_n-2r}{2h_m}+\frac{\kappa(\eta-y_n)}{k_n}\right|$$
$$\leq\left|\frac{\sigma^2y_n}{k_n(k_n+k_{n+1})}-\frac{\rho\sigma y_n}{2h_mk_n}+\frac{\kappa(\eta-y_n)}{k_n}\right|+\left|\frac{\sigma^2y_n}{k_{n+1}(k_n+k_{n+1})}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+\left|\frac{y_n}{h_{m+1}(h_m+h_{m+1})}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|+\left|\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|+\left|\frac{\rho\sigma y_n}{2h_mk_n}\right|$$
$$+\left|\frac{y_n}{h_m(h_m+h_{m+1})}-\frac{\rho\sigma y_n}{2h_mk_n}-\frac{y_n-2r}{2h_m}\right|. \tag{3.43}$$

Let α be the real part of z. Since we are concerned only about the upper bound of the real part of the eigenvalues, we may replace z by α via a triangle inequality, and remove absolute value sign on the left hand side of (3.43). As a consequence, (3.43) renders to

$$\begin{aligned}
&\alpha + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n} \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_m k_n} \right| \\
&\quad\quad + \left| \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} \right|. \qquad (3.44)
\end{aligned}$$

Recall (3.18) and that $\rho > 0$. We have

$$\frac{2}{\rho\sigma}k_n,\ \frac{2}{\rho\sigma}k_{n+1} \geq h_m + h_{m+1} \text{ and } h_m,\ h_{m+1} \geq \frac{\rho}{\sigma}(k_n + k_{n+1}).$$

The above lead to

$$\begin{aligned}
\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} &\geq \frac{\rho\sigma y_n}{2h_m k_n}, \qquad \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} \geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \\
\frac{y_n}{h_{m+1}(h_m + h_{m+1})} &\geq \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, \qquad \frac{y_n}{h_m(h_m + h_{m+1})} \geq \frac{\rho\sigma y_n}{2h_m k_n}.
\end{aligned}$$

Furthermore, since $\eta > y > 0$, we conclude that

$$\frac{\kappa(\eta - y_n)}{k_n} \geq 0 \text{ and } \ -\frac{y_n - 2r}{2h_m} \geq 0.$$

Therefore, the term inside each pair of absolute signs in (3.44) must be positive. We may remove all absolute signs in (3.44), and, subsequently, yields $\alpha \leq 0$.

Case 2: Now consider the case of $x = x_1$ and $2r < y < Y$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\begin{aligned}
&\left| z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n} \right| + \left| \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| \\
&\quad\quad + \left| \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right| + \left| \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \right|.
\end{aligned}$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}}\left(\frac{1}{h_m+h_{m+1}}-\frac{1}{h_m}\right)-\frac{y_n-2r}{2h_m}<0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Now consider the case of $x = x_1$ and $\eta < y \leq 2r$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\begin{aligned}
&\left|z+\frac{y_n}{h_m h_{m+1}}+\frac{\sigma^2 y_n}{k_n k_{n+1}}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}-\frac{\rho\sigma y_n}{2h_m k_n}+\frac{y_n-2r}{2h_m}+\frac{\kappa(\eta-y_n)}{k_n}\right| \\
&\qquad\leq\left|\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})}-\frac{\rho\sigma y_n}{2h_m k_n}+\frac{\kappa(\eta-y_n)}{k_n}\right|+\left|\frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| \\
&\qquad\quad+\left|\frac{y_n}{h_{m+1}(h_m+h_{m+1})}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|+\left|\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|.
\end{aligned}$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}}\left(\frac{1}{h_m+h_{m+1}}-\frac{1}{h_m}\right)-\frac{y_n-2r}{2h_m}<0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Now consider the case of $x = x_1$ and $0 < y \leq \eta$. Thus, for any complex number $z \in S_i$, where S_i is a Geršhgorin disc satisfying

$$\begin{aligned}
&\left|z+\frac{y_n}{h_m h_{m+1}}+\frac{\sigma^2 y_n}{k_n k_{n+1}}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}-\frac{\rho\sigma y_n}{2h_m k_n}-\frac{y_n-2r}{2h_m}+\frac{\kappa(\eta-y_n)}{k_n}\right| \\
&\qquad\leq\left|\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})}-\frac{\rho\sigma y_n}{2h_m k_n}+\frac{\kappa(\eta-y_n)}{k_n}\right|+\left|\frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right| \\
&\qquad\quad+\left|\frac{y_n}{h_{m+1}(h_m+h_{m+1})}-\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|+\left|\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|.
\end{aligned}$$

Similar to the previous case, we take α, the real part of z. Thus,

$$\alpha \leq \frac{y_n}{h_{m+1}}\left(\frac{1}{h_m+h_{m+1}}-\frac{1}{h_m}\right)+\frac{y_n-2r}{2h_m}<0.$$

The above apparently implies that such an S_i must lie strictly on the left half of the complex plane, and the origin cannot be on its boundary. This ensures our expectation.

Case 3: In the circumstance, when $2r < y < Y$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\left|z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n}\right|$$
$$\leq \left|\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+ \left|\frac{\rho\sigma y_n}{2h_m k_n}\right| + \left|\frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_m h_{m+1}} < 0.$$

In the case when $\eta < y < 2r$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\left|z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n}\right|$$
$$\leq \left|\frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+ \left|\frac{\rho\sigma y_n}{2h_m k_n}\right| + \left|\frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_m h_{m+1}} < 0.$$

In the case when $0 < y < \eta$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\left|z + \frac{y_n}{h_m h_{m+1}} + \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_n}\right|$$

$$\le \left|\frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta-y_n)}{k_n}\right| + \left|\frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\right|$$
$$+ \left|\frac{\rho\sigma y_n}{2h_m k_n}\right| + \left|\frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{y_n-2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \le \frac{y_n}{h_m(h_m+h_{m+1})} - \frac{y_n}{h_m h_{m+1}} < 0.$$

CASE 4: In the circumstance, when $2r < y < Y$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\left|z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1}+k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N-2r}{2h_m} - \frac{\kappa(\eta-y_N)}{k_N}\right|$$
$$\le \left|\frac{\sigma^2 y_N}{k_N(k_N+k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{\kappa(\eta-y_N)}{k_N}\right| + \left|\frac{y_N}{h_{m+1}(h_m+h_{m+1})}\right|$$
$$+ \left|\frac{\rho\sigma y_N}{2h_m k_N}\right| + \left|\frac{y_N}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N-2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \le \frac{y_N}{(h_m+h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

When $\eta < y < 2r$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\left|z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1}+k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N-2r}{2h_m} + \frac{\kappa(\eta-y_N)}{k_N}\right|$$
$$\le \left|\frac{\sigma^2 y_N}{k_N(k_N+k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{\kappa(\eta-y_N)}{k_N}\right| + \left|\frac{y_N}{h_{m+1}(h_m+h_{m+1})}\right|$$
$$+ \left|\frac{\rho\sigma y_N}{2h_m k_N}\right| + \left|\frac{y_N}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{y_N-2r}{2h_m}\right|.$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \le \frac{y_N}{(h_m+h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

When $0 < y \leq \eta$, Geršhgorin discs, S_i, concerned are centered at boundary points where a Neumann condition is imposed. Hence, for any $z \in S_i$ we have

$$\begin{aligned}
&\left| z + \frac{y_N}{h_m h_{m+1}} + \frac{\sigma^2 y_N}{k_N(k_{N+1}+k_N)} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{y_N - 2r}{2h_m} + \frac{\kappa(\eta - y_N)}{k_N} \right| \\
&\quad \leq \left| \frac{\sigma^2 y_N}{k_N(k_N+k_{N+1})} - \frac{\rho\sigma y_N}{2h_m k_N} + \frac{\kappa(\eta - y_N)}{k_N} \right| + \left| \frac{y_N}{h_{m+1}(h_m+h_{m+1})} \right| \\
&\quad + \left| \frac{\rho\sigma y_N}{2h_m k_N} \right| + \left| \frac{y_N}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_N}{2h_m k_N} - \frac{y_N - 2r}{2h_m} \right|.
\end{aligned}$$

The above indicates that α, the real part of z, must satisfy

$$\alpha \leq \frac{y_N}{(h_m + h_{m+1})^2} - \frac{y_N}{h_m h_{m+1}} < 0.$$

Since the origin cannot lie on the boundary of every Geršhgorin disc [44], combining results from the three cases, we conclude immediately that all eigenvalues of A must be strictly on the left half complex plane. Thus, we must have

$$\lim_{h_{\max}, k_{\max} \to 0^+} \left(\max_{\tau \in [0,\tau^*]} \left\| e^{\tau A} \right\|_2 \right) \leq c(\tau^*),$$

which ensures the numerical stability. $\square$

3.5 Simulation Experiments

Recall (3.1)-(3.6). Similar to existing experiments (see [67] and references therein), we fix $X = 8,\ Y = 1$. We first concentrate on experiments with $\rho = -0.5$ and $T = 0.5$. Next, to test against extreme cases in the option market, we proceed with $\rho = -1$ and $T = 5$. For demonstrating our numerical procedure and its rate of convergence, we first consider uniform spacial grids. Results over nonuniform grids will be presented afterwards. For this, let $h_m = h,\ k_n = k = \sigma h,\ \ m = 1, 2, \ldots, M;\ n = 1, 2, \ldots, N.$

Some key parameters used are shown in Table 3.1. Further, $\Delta\tau$ be our temporal step. We experiment with different values of $\lambda = \Delta\tau / c^2$, where $c = \min\{h, k\}$. To numerically examine this through experiments, we employ a generalized Milne's device [47]. Then, for a selected terminal time T, we denote the numerical solution at point (x_m, y_n, T), $1 \le m \le M; 1 \le n \le N$, as $u_{m,n;h}$ for any particular spatial step $0 < h \ll 1$. Likewise, we let $u_{m,n;h/2}$ and $u_{m,n;h/4}$ be computed solutions obtained by using $h/2$ and $h/4$, respectively. Thus, the point-wise rate of spatial convergence at T can be evaluated via

$$R^h_{m,n} \approx \frac{1}{\ln 2} \ln \frac{\left| u_{m,n;h} - u_{m,n;h/2} \right|}{\left| u_{m,n;h/2} - u_{m,n;h/4} \right|}. \tag{3.45}$$

Table 3.1. Key parameter values for numerical simulations

key parameter	value used
strike price	$K = 100$
volatility of volatility	$\sigma = 1$
risk-free interest rate	$r = 0.05$
mean reversion speed	$\kappa = 2$
long-run mean of volatility	$\eta = 0.1$

Table 3.2. Rates of convergence with $\sigma = 1$, $\rho = -0.5$, $T = 0.5$ and $h = 0.01$

conv. rates	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
$\min_{m,n}(R^h_{m,n})$	0.6193	0.6134	0.6026
$\max_{m,n}(R^h_{m,n})$	1.0024	0.9976	0.9811
$\text{mean}_{m,n}(R^h_{m,n})$	0.9026	0.90438	0.9053

Table 3.3. Rates of convergence with $\sigma = 1$, $\rho = -0.5$, $T = 0.5$ and $h = 0.02$

conv. rates	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
$\min_{m,n}(R^h_{m,n})$	0.6324	0.6221	0.6206
$\max_{m,n}(R^h_{m,n})$	0.9674	1.0007	1.0151
$\text{mean}_{m,n}(R^h_{m,n})$	0.8342	0.8300	0.8296

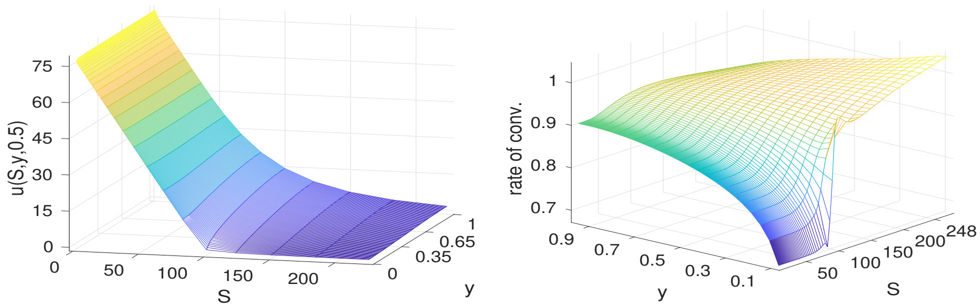

Figure 3.3. Price of an European put option and rate of convergence estimate.

Table 3.4. Rates of convergence with $\sigma = 1$, $\rho = -0.5$, $T = 0.5$ and $h = 0.04$

conv. rates	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
$\min_{m,n}(R^h_{m,n})$	0.5824	0.5971	0.6179
$\max_{m,n}(R^h_{m,n})$	0.9941	0.9437	0.9586
$\text{mean}_{m,n}(R^h_{m,n})$	0.7952	0.8015	0.8142

Let $h = 0.01$ and $\sigma = 1$. For simplicity of notations, we use the same letter v for the approximate solution to (1.15). We show the solution v for $\rho = -0.5$ and $\rho = -1$ in Figures 3.3 and 3.4 , respectively. It can be observed that the European put option price is a decreasing function of the stock price S. This coincides well with the financial theory that a put option price should have a negative correlation with the underline stock price [1, 28].

Let us plot the computed rate of convergence surfaces for cases when $\rho = -0.5$ and $\rho = -1$ in Figure 3.3 and Figure 3.4, respectively. In addition, a summary of point-wise convergence rates for the former case on different spacial grids is given in Table 3.2. Minor disturbances can be observed in regions where the solution changes fast, in particularly in extreme situations with $\rho = -1$ as being demonstrated in Figure 3.4. Further, the mean convergence rate for the two cases are given in the caption of Figure 3.3 and Figure 3.4. In the extreme case when $\rho = -1$ and $T = 5$,

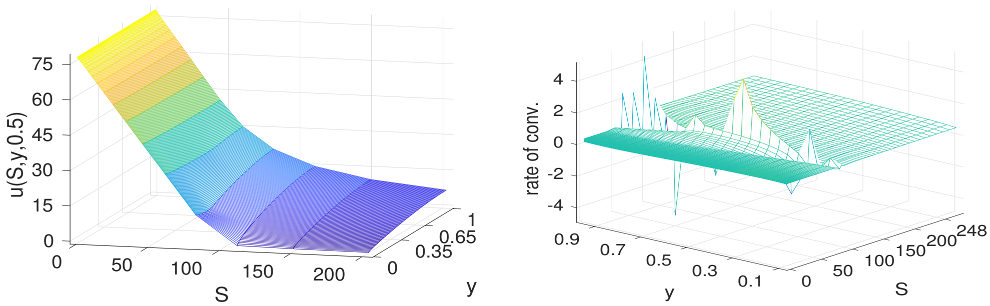

Figure 3.4. Price of an European put option and rate of convergence estimate.

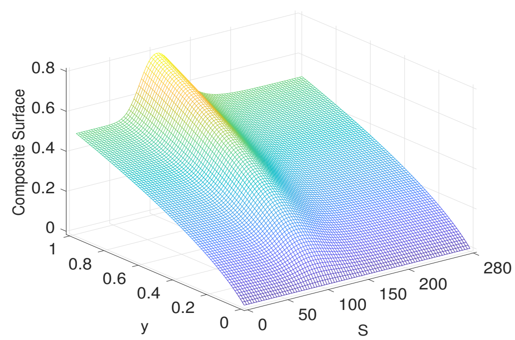

Figure 3.5. A composite surface plot of $z_1(S)z_2(y)$

we observed a smaller mean convergence rate. This is within our expectation due to the more iterations to get the solution and the decrease in the well-posedness of the original PDE. These results are consistent with those from well-established high-order schemes [12, 13, 32, 38, 67]. A Matlab platform is used.

Now, consider simulations over nonuniform spacial grids. We are particularly interested in the following nonlinear grid distribution governing functions [44, 47]

$$z_1(S) = \sqrt{\frac{1}{2.56} + \frac{25(S/K)^{10}}{2.56[1+(S/K)^5]^4}}, \quad S_{\min} \le S \le S_{\max}, \tag{3.46}$$

$$z_2(y) = \frac{10\sqrt{0.5y}}{7}, \quad y_{\min} \le y \le y_{\max}. \tag{3.47}$$

In our simulation experiments, selections of monitoring functions z_1, z_2, are based initially on the numerical solution v acquired on uniform spacial meshes. They are chosen to reflect trends of solution curvatures [13, 47].

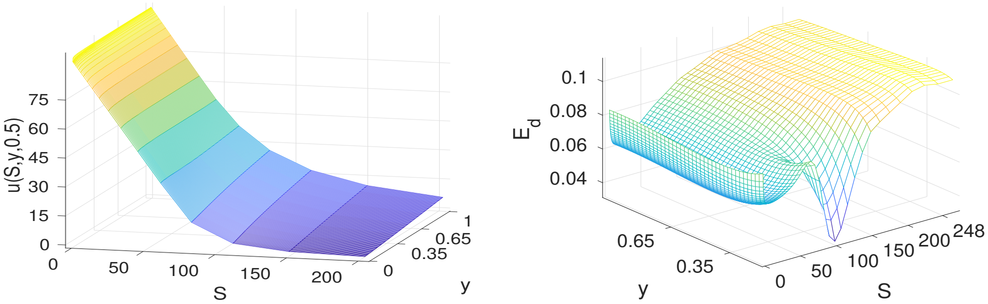

Figure 3.6. Solution on nonuniform grids and the relative difference .

Our nonuniform grids are generated via an arc-length equal-distribution principal for functions z_1, z_2 in S- and y-directions, respectively. The principal is commonly utilized in adaptive computations and serves as an initial exploration for more sophisticated adaptations [30, 47].

A composite surface plot of the mesh distribution function $z_1 z_2$ is given in Figure 3.5. It characterizes the 2-dimensional profile of our grids distribution. The numerical solution acquired over such nonuniform grids, with $\rho = -0.5$ at $T = 0.5$ is given in Figure 3.6 .

Let $\Omega_{N,M}$ be a reference spacial mesh which can also be either our uniform mesh or nonuniform mesh. We may map solutions v_{unif} and v_{nonunif}, numerical solutions obtained on the uniform mesh and nonuniform mesh, respectively, to $\Omega_{N,M}$.

We plot the following point-wise relative error,

$$E_d(S, y, t) = \frac{|v_{\text{unif}}(S, y, t) - v_{\text{nonunif}}(S, y, t)|}{|v_{\text{unif}}(S, y, t)|}, \quad (S, y, t) \in \Omega_{N,M}, \; 0 < t \leq T. \tag{3.48}$$

in Figure 3.6. The mean relative difference E_{mean} is given in the caption of Figure 3.6. We can see that the solutions on the uniform and nonuniform grids agrees with each other due to the small E_{mean}.

3.6 Summary

A numerically stable and dynamically balanced up-downwind semi-discretized finite difference method is constructed and analyzed in this chapter based on arbitrary option data grids which can be determined through a proper moving mesh principal. The algorithm acquired is extremely convenient to use in realities. It is reliable and effective for computing Heston stochastic volatility option pricing model solutions with cross-derivative terms in market realities. Rigorous mathematical proofs are given to ensure the stability and convergence.

Simulation experiments further confirm our theoretical expectations on both uniform and arbitrary spacial data grids given.

Our next endeavors include improving the computational efficiency through exponential splitting methods, particularly variable step ADI or LOD approximations [10, 32, 47, 51, 56]. Compact schemes for raising the accuracy have also been introduced in our study with initial successes in handling cross-derivatives dynamically and well balances for pricing American and some Asian options [1, 16, 30, 41, 58, 67]. Initial investigations have been very promising.

CHAPTER FOUR

Exponentially Split Up-Downwind Method for Stochastic Heston Volatilities.

A nonuniform spacial mesh will be introduced in this chapter. Based on it, a semi-discretized scheme will be introduced for approximating (3.1)-(3.6). Another dynamically balanced up-downwind difference approximations will be implemented. Numerical stability analysis will be conducted rigorously in Section 3. Computational experiments will be carried out in Section 4. Orders of convergence will also be computationally evaluated. Last but not least, conclusions and further remarks will be given in Section 5.

4.1 Dynamically Balanced Up-Downwind Semi-Discretization

Consider an arbitrary mesh $\Omega_{h,k} = \{(x_m, y_n)\}_{n,m=1}^{M,N}$ over a optional domain Ω. We further denote $x_0 = -X,\ x_{M+1} = X;\ y_0 = 0,\ y_{N+1} = Y$ and let $0 < x_m - x_{m-1} = h_m \ll 1,\ m = 1, 2, \ldots, M,\ 0 < y_n - y_{n-1} = k_n \ll 1,\ n = 1, 2, \ldots, N.$

Let $z_{m,n}(\tau)$ be an approximation of the function value $z(x_m, y_n, \tau)$, $(x_m, y_n) \in \Omega_{h,k},\ 0 < \tau < T$. If $\Delta_{\ell,+},\ \Delta_{\ell,-}$ and $\Delta^2_{\ell,0}$ are finite difference operators defined as follows in the ℓ-direction, where $\ell \in \{x, y\}$, then

$$
\begin{aligned}
\Delta_{x,+} z_{m,n} &= \frac{z_{m+1,n} - z_{m,n}}{h_{m+1}} + \mathcal{O}(h),\ \Delta_{x,-} z_{m,n} = \frac{z_{m,n} - z_{m-1,n}}{h_m} + \mathcal{O}(h),\\
\Delta_{y,+} z_{m,n} &= \frac{z_{m,n+1} - z_{m,n}}{k_{n+1}} + \mathcal{O}(k),\ \Delta_{y,-} z_{m,n} = \frac{z_{m,n} - z_{m,n-1}}{k_n} + \mathcal{O}(k),\\
\Delta^2_{x,0} z_{m,n} &= \frac{2z_{m+1,n}}{h_{m+1}(h_{m+1} + h_m)} - \frac{2z_{m,n}}{h_{m+1}h_m} + \frac{2z_{m-1,n}}{h_m(h_{m+1} + h_m)} + \mathcal{O}(h),\\
\Delta^2_{y,0} z_{m,n} &= \frac{2z_{m,n+1}}{k_{n+1}(k_{n+1} + k_n)} - \frac{2z_{m,n}}{k_{n+1}k_n} + \frac{2z_{m,n-1}}{k_n(k_{n+1} + k_n)} + \mathcal{O}(k),
\end{aligned}
$$

where $h = \max\{h_m, h_{m+1}\},\ k = \max\{k_n, k_{n+1}\}$ for appropriate indexes [35, 47, 56].

We now approximate the diffusion terms in (3.1) by using the above operators, and derivatives in (3.5) and (3.6) via the following for $0 < \tau < T$.

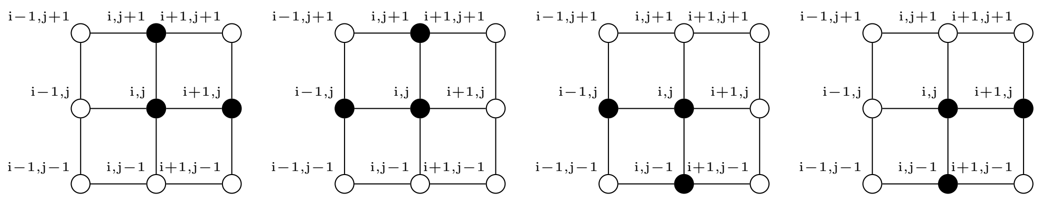

Figure 4.1. Computational stencils for (3.9) - (3.13).

At the same time, we approximate advection terms in (3.1) dynamically through three different channels depending upon relations between η and r.

Case 1: $\eta > 2r$.

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,+}u_{m,n},\ \ u_y(x_m, y_n, \tau) \approx \Delta_{y,+}u_{m,n}, \qquad 2r \geq y > 0, \tag{4.1}$$

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,-}u_{m,n},\ \ u_y(x_m, y_n, \tau) \approx \Delta_{y,+}u_{m,n}, \qquad \eta \geq y > r, \tag{4.2}$$

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,-}u_{m,n},\ \ u_y(x_m, y_n, \tau) \approx \Delta_{y,-}u_{m,n}, \qquad Y > y > \eta. \tag{4.3}$$

Case 2: $\eta \leq 2r$.

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,+}u_{m,n},\ \ u_y(x_m, y_n, \tau) \approx \Delta_{y,+}u_{m,n}, \qquad \eta \geq y > 0, \tag{4.4}$$

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,+}u_{m,n},\ \ u_y(x_m, y_n, \tau) \approx \Delta_{y,-}u_{m,n}, \qquad 2r \geq y > \eta, \tag{4.5}$$

$$u_x(x_m, y_n, \tau) \approx \Delta_{x,-}u_{m,n},\ \ u_y(x_m, y_n, \tau) \approx \Delta_{y,-}u_{m,n}, \qquad Y > y > 2r. \tag{4.6}$$

Computational stencils for Case 1 and Case 2 are shown in Figure 4.1. Define

$$h_{\min} = \min_{m=1,2\cdots M} h_m,\ \ h_{\max} = \max_{m=1,2\cdots M} h_m;\ \ k_{\min} = \min_{n=1,2\cdots N} k_n,\ \ k_{\max} = \max_{n=1,2\cdots N} k_n.$$

We now approximate the cross-derivative in (3.1) dynamically. In particularly, we have

4.1.1 Case for $\rho \in [-1, 0)$.

First, for the smoothness of nonuniform grids [25, 47], we require that

$$-\rho k_{\max} \leq \sigma h_{\min} \leq \sigma h_{\max} \leq -\frac{1}{\rho} k_{\min}. \tag{4.7}$$

We propose the following:

$$u_{xy}(x_m, y_n, \tau) = \frac{1}{2}(\Delta_{x,+}\Delta_{y,-} + \Delta_{x,-}\Delta_{y,+})u_{m,n}(\tau) + \mathcal{O}(h_{\max} + k_{\max}). \tag{4.8}$$

Substitute all spacial derivative approximations into (3.1) and let w denote the semi-discretized approximationn to u. We acquire immediately the following linear ordinary differential equation system with a proper initial value,

$$\begin{aligned} w'(\tau) &= Aw(\tau) + f(\tau), \quad \tau > 0, \\ w(0) &= w_0, \end{aligned} \tag{4.9}$$

where $w, f \in \mathbb{R}^{MN}$ and $A \in \mathbb{R}^{MN \times MN}$ is block tridiagonal in the form of

$$A = \begin{bmatrix} D_1 & Q_1 & \cdots & \cdots & \cdots & 0 \\ P_2 & D_2 & Q_2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & P_{M-2} & D_{M-2} & Q_{M-2} & 0 \\ \cdots & \cdots & \cdots & P_{M-1} & D_{M-1} & Q_{M-1} \\ 0 & \cdots & \cdots & \cdots & P_M & D_M \end{bmatrix},$$

where $P_i, D_j, Q_k \in \mathbb{R}^{N \times N}$, $i = 2, 3, \ldots, M;\ j = 1, 2, \ldots, M;\ k = 1, 2, \ldots, M-1$. Nontrivial entries of the matrices P_m, D_m and Q_m for their respective ranges of m are thee following.

$$\begin{aligned} p_{n,n}^{(m)} &= \begin{cases} \frac{y_n}{h_m(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}}, & 0 < y_n \le 2r, \\ \frac{y_n}{h_m(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{y_n - 2r}{2h_m}, & 2r < y_n < Y - k_{N+1}, \\ \frac{y_N}{h_m(h_m+h_{m+1})} + \frac{y_N - 2r}{2h_m}, & y_n = Y - k_{N+1}; \end{cases} \\ p_{n,n+1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_m k_n}; \\ d_{n,n-1}^{(m)} &= \begin{cases} \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1} k_n}, & k_1 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1} k_n} - \frac{\kappa(\eta - y_n)}{k_n}, & \eta < y_n \le Y - k_{N+1}; \end{cases} \end{aligned}$$

$$
d_{n,n}^{(m)} = \begin{cases} \alpha_{m,1} + \frac{y_1-2r}{2h_{m+1}} - \frac{\kappa(\eta-y_1)}{k_2}, & y_n = k_1, \\ \beta_{m,n} + \frac{y_n-2r}{2h_{m+1}} - \frac{\kappa(\eta-y_n)}{k_{n+1}}, & k_1 < y_n \le 2r, \\ \beta_{m,n} - \frac{y_n-2r}{2h_m} - \frac{\kappa(\eta-y_n)}{k_{n+1}}, & 2r < y_n \le \eta, \\ \beta_{m,n} - \frac{y_n-2r}{2h_m} + \frac{\kappa(\eta-y_n)}{k_n}, & \eta < y_n < Y - k_{N+1}, \\ \gamma_{m,N} - \frac{y_N-2r}{2h_m} + \frac{\kappa(\eta-y_N)}{k_N}, & y_N = Y - k_{N+1}; \end{cases}
$$

$$
d_{n,n+1}^{(m)} = \begin{cases} \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{\kappa(\eta-y_n)}{k_{n+1}}, & 0 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}}, & \eta < y_n < Y - k_{N+1}; \end{cases}
$$

$$
q_{n,n-1}^{(m)} = -\frac{\rho\sigma y_n}{2h_{m+1}k_n}, \quad y_n > k_1;
$$

$$
q_{n,n}^{(m)} = \begin{cases} \frac{y_1}{h_{m+1}(h_m+h_{m+1})} - \frac{y_1-2r}{2h_{m+1}}, & y_n = k_1, \\ \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{y_n-2r}{2h_{m+1}}, & k_1 < y \le 2r, \\ \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n}, & 2r < y_n \le Y - k_{N+1}, \end{cases}
$$

where

$$
\begin{aligned}
\alpha_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_{n+1}}, \\
\beta_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{\rho\sigma y_n}{2h_m k_{n+1}}, \\
\gamma_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_n}.
\end{aligned}
$$

It is observed that in the event if $\rho = -1$, we have the following due to (4.7):

$$
h_{\min} = h_{\max} = h,\ k_{\min} = k_{\max} = k,\ k = \sigma h,
$$

which indicate that uniform spacial grids must be employed. Thus, (4.9) reduces to

$$
w'(\tau) = A_s w(\tau) + f_s(\tau).
$$

Nontrivial entries of A_s are readily to obtain based on above discussions.

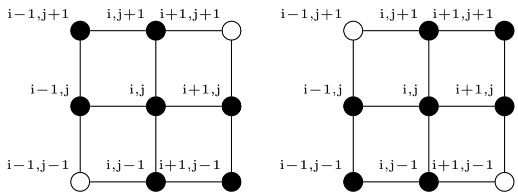

Figure 4.2. Computational stencils of (4.8) (left) and (4.11) (right).

4.1.2 Case for $\rho \in [0, 1]$.

We need the following restrictions on mesh steps in the case [47]:

$$\rho k_{\max} \leq \sigma h_{\min} \leq \sigma h_{\max} \leq \frac{1}{\rho} k_{\min}. \tag{4.10}$$

Apparently, when $\rho = 1$, the above implies that a uniform spacial mesh with $h = \sigma k$ must be used.

Different from (4.8), we consider a new dynamically balanced cross-derivative approximation,

$$u_{xy}(x_m, y_n, \tau) = \frac{1}{2}(\Delta_{x,-}\Delta_{y,-} + \Delta_{x,+}\Delta_{y,+})u_{m,n}(\tau) + \mathcal{O}(h_{\max} + k_{\max}). \tag{4.11}$$

Computational stencils for (4.8) and (4.11) are shown in Figure 4.2.

In this circumstance, we obtain the following new system,

$$w'(\tau) = \tilde{A}w(\tau) + \tilde{f}(\tau), \tag{4.12}$$

where $w, \tilde{f}(\tau) \in \mathbb{R}^{MN}$ and $\tilde{A} \in \mathbb{R}^{MN \times MN}$ is block tridiagonal, *that is,*

$$\tilde{A} = \begin{bmatrix} \tilde{D}_1 & \tilde{Q}_1 & \cdots & \cdots & \cdots & 0 \\ \tilde{P}_2 & \tilde{D}_2 & \tilde{Q}_2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \tilde{P}_{M-2} & \tilde{D}_{M-2} & \tilde{Q}_{M-2} & 0 \\ \cdots & \cdots & \cdots & \tilde{P}_{M-1} & \tilde{D}_{M-1} & \tilde{Q}_{M-1} \\ 0 & \cdots & \cdots & \cdots & \tilde{P}_M & \tilde{D}_M \end{bmatrix},$$

where

$$\tilde{p}_{n,n-1}^{(m)} = \frac{\rho\sigma y_n}{2h_m k_n}, \quad y_n > k_1;$$

$$
\begin{aligned}
\tilde{p}^{(m)}_{n,n} &= \begin{cases} \frac{y_1}{h_m(h_m+h_{m+1})}, \quad y_n = k_1, \\ \frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n}, \quad k_1 < y_n \le 2r, \\ \frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n - 2r}{2h_m}, \quad 2r < y_n \le Y - k_{N+1}; \end{cases} \\
\tilde{r}^{(m)}_{n,n-1} &= \begin{cases} \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n}, \quad k_1 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_n}, \quad \eta < y_n \le Y - k_{N+1}; \end{cases} \\
\tilde{r}^{(m)}_{n,n} &= \begin{cases} \tilde{\alpha}_{m,1} + \frac{y_1 - 2r}{2h_{m+1}} - \frac{\kappa(\eta - y_1)}{k_{n+1}}, \quad y_1 = k_1, \\ \tilde{\beta}_{m,n} + \frac{y_n - 2r}{2h_{m+1}} - \frac{\kappa(\eta - y_n)}{k_{n+1}}, \quad k_1 < y_n \le 2r, \\ \tilde{\beta}_{m,n} - \frac{y_n - 2r}{2h_m} - \frac{\kappa(\eta - y_n)}{k_{n+1}}, \quad 2r < y_n \le \eta, \\ \tilde{\beta}_{m,n} - \frac{y_n - 2r}{2h_m} + \frac{\kappa(\eta - y_n)}{k_n}, \quad \eta < y_n < Y - k_{N+1}, \\ \tilde{\gamma}_{m,N} - \frac{y_N - 2r}{2h_m} + \frac{\kappa(\eta - y_N)}{k_N}, \quad y_N = Y - k_{N+1}; \end{cases} \\
\tilde{r}^{(m)}_{n,n+1} &= \begin{cases} \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} + \frac{\kappa(\eta - y_n)}{k_{n+1}}, \quad 0 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}}, \quad \eta < y_n < Y - k_{N+1}; \end{cases} \\
\tilde{q}^{(m)}_{n,n} &= \begin{cases} \frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} - \frac{y_n - 2r}{2h_{m+1}}, \quad 0 < y_n \le 2r, \\ \frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}}, \quad 2r < y_n < Y - k_{N+1}, \\ \frac{y_N}{h_{m+1}(h_m+h_{m+1})}, \quad y_N = Y - k_{N+1}; \end{cases} \\
\tilde{q}^{(m)}_{n,n+1} &= \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}}, \quad 0 < y_n < Y - k_{N+1},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\alpha}_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n}, \\
\tilde{\beta}_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n}, \\
\tilde{\gamma}_{m,n} &= -\frac{y_n}{h_m h_{m+1}} - \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_n}.
\end{aligned}
$$

The semi-discretized method (4.12) reduces to a uniform scheme when $\rho = 1$, *that is,*

$$w'(\tau) = \frac{1}{2h^2}\tilde{A}_s w(\tau) + \tilde{f}(\tau).$$

Nontrivial elements of $\tilde{A}$ and $\tilde{A}_s$ can be derived conveniently from simplifications of the above formulas.

4.2 Exponential Splitting

Let us first introduce the theory of exponential splitting by considering the following exploration formulations.

Let $\mathcal{D}$ be a two-dimensional spacial domain and consider the following partial differential equation:

$$\frac{\partial u}{\partial t} = \mathcal{F}u + \mathcal{G}u, \ (x, y) \in \mathcal{D}, \ t > t_0, \tag{4.13}$$

where $\mathcal{F}, \ \mathcal{G}$ are linear spacial differential operators. Assume that an appropriate semidiscretization of (4.13), together with proper boundary conditions, yields the following system:

$$v' = Av + Bv + f, \quad t > t_0, \tag{4.14}$$

where $A, \ B \in \mathbb{C}^{n\times n}$, $AB \neq BA$ in general, $v, f \in \mathbb{C}^n$ and v approximates u on $\mathcal{D}$. Let $v(t_0) = v_0$ be an initial vector given. Then for arbitrary $\tau > 0$, the exact solution of (4.14) can be provided by the variation-of-constant formula,

$$v(t+\tau) = e^{\tau(A+B)}v(t) + \int_t^{t+\tau} e^{(\xi-t)(A+B)} f d\xi, \ t > t_0. \tag{4.15}$$

The matrix exponential in (4.15) can be approximated by

$$e^{\tau(A+B)} \ = \ e^{\tau A}e^{\tau B} + \mathcal{O}(\tau^2), \quad \tau \to 0^+. \tag{4.16}$$

Now, an application of the [0/1] Padé approximation to (4.16), dropping all truncation errors, yields the following fully discretized implicit scheme

$$w(t+\tau) \ = \ (I - \tau A)^{-1}(I - \tau B)^{-1}w(t).$$

We further apply the exponential splitting and Padé approximation introduced above to our semidiscretized schemes (4.9) and (4.12) for derivations of our fully discretized schemes respectively in the following two consective subsections.

4.2.1 Cases for $\rho \in [-1, 0]$.

The former solution to (4.9) is

$$w(\tau_{n+1}) = e^{\Delta\tau A} w(\tau_n) + \int_{\tau_n}^{\tau_{n+1}} e^{(t-\tau_n)A} f(t)dt, \quad n = 0, 1, \ldots, \tag{4.17}$$

Applying the first-order exponential splitting (4.16) to the matrix exponential term in (4.17), we acquire that

$$w(\tau_{n+1}) = e^{\Delta\tau A_1} e^{\Delta\tau A_2} w(\tau_n) + \int_{\tau_n}^{\tau_{n+1}} e^{(t-\tau_n)A} f(t)dt, \quad n = 0, 1, \ldots, \tag{4.18}$$

where $A_1 + A_2 = A$. Details of A_1 and A_2 are shown below.

$$A_1 = \begin{bmatrix} G_1 & K_1 & \cdots & \cdots & \cdots & 0 \\ H_2 & G_2 & K_2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & H_{M-2} & G_{M-2} & K_{M-2} & 0 \\ \cdots & \cdots & \cdots & H_{M-1} & G_{M-1} & K_{M-1} \\ 0 & \cdots & \cdots & \cdots & H_M & G_M \end{bmatrix},$$

$$A_2 = \begin{bmatrix} T_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ L_2 & T_2 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & L_{M-2} & T_{M-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & L_{M-1} & T_{M-1} & \cdots \\ 0 & \cdots & \cdots & \cdots & L_M & T_M \end{bmatrix}.$$

Here matrices $H_i, G_j, K_k, R_i, T_j \in \mathbb{R}^{N\times N},\ i = 2, 3, \ldots, M;\ j = 1, 2, \ldots, M;\ k = 1, 2, \ldots, M-1$. Nontrivial entries of matrices H_m, G_m, K_m, R_m and T_m within their respective ranges of m are as follows.

$$h_{n,n}^{(m)} = \begin{cases} \frac{y_n}{h_m(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}}, & 0 < y_n \le 2r, \\ \frac{y_n}{h_m(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{y_n-2r}{2h_m}, & 2r < y_n < Y - k_{N+1}, \\ \frac{y_N}{h_m(h_m+h_{m+1})} + \frac{y_N-2r}{2h_m}, & y_n = Y - k_{N+1}; \end{cases}$$

$$g_{n,n}^{(m)} = \begin{cases} -\frac{y_n}{h_m h_{m+1}} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{y_n-2r}{2h_{m+1}}, & k_1 \le y_n \le 2r, \\ -\frac{y_n}{h_m h_{m+1}} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} - \frac{y_n-2r}{2h_m}, & 2r < y_n \le Y - k_{N+1}, \\ -\frac{y_N}{h_m h_{m+1}} - \frac{y_N-2r}{2h_m}, & y_N = Y - k_{N+1}; \end{cases}$$

$$k_{n,n-1}^{(m)} = -\frac{\rho\sigma y_n}{2h_{m+1}k_n}, \quad y_n > k_1;$$

$$k_{n,n}^{(m)} = \begin{cases} \frac{y_1}{h_{m+1}(h_m+h_{m+1})} - \frac{y_1-2r}{2h_{m+1}}, & y_n = k_1, \\ \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{y_n-2r}{2h_{m+1}}, & k_1 < y_n \le 2r, \\ \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n}, & 2r < y_n \le Y - k_{N+1}, \end{cases}$$

$$l_{n,n+1}^{(m)} = -\frac{\rho\sigma y_n}{2h_m k_n};$$

$$t_{n,n-1}^{(m)} = \begin{cases} \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n}, & k_1 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{\kappa(\eta-y_n)}{k_n}, & \eta < y_n \le Y - k_{N+1}; \end{cases}$$

$$t_{n,n}^{(m)} = \begin{cases} -\frac{\sigma^2 y_1}{k_2(k_2+k_1)} - \frac{\kappa(\eta-y_1)}{k_2}, & y_n = k_1, \\ -\frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{\kappa(\eta-y_n)}{k_{n+1}}, & k_1 < y_n \le \eta, \\ -\frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_n} + \frac{\kappa(\eta-y_n)}{k_n}, & \eta < y_n < Y - k_{N+1}, \\ -\frac{\sigma^2 y_N}{k_N(k_N+k_{N+1})} - \frac{\rho\sigma y_N}{2h_{m+1}k_N} + \frac{\kappa(\eta-y_N)}{k_N}, & y_N = Y - k_{N+1}; \end{cases}$$

$$t_{n,n+1}^{(m)} = \begin{cases} \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{\kappa(\eta-y_n)}{k_{n+1}}, & 0 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}}, & \eta < y_n < Y - k_{N+1}; \end{cases}$$

4.2.2 Case for $\rho \in (0, 1]$.

By the same token, the former solution to (4.12) is

$$w(\tau_{n+1}) = e^{\Delta\tau\tilde{A}} w(\tau_n) + \int_{\tau_n}^{\tau_{n+1}} e^{(t-\tau_n)\tilde{A}} \tilde{f}(t)dt, \quad n = 0, 1, \ldots, \tag{4.19}$$

Applying the first-order splitting scheme (4.16) to the matrix exponential term in (4.19), we get

$$w(\tau_{n+1}) = e^{\Delta\tau\tilde{A}_1} e^{\Delta\tau\tilde{A}_2} w(\tau_n) + \int_{\tau_n}^{\tau_{n+1}} e^{(t-\tau_n)\tilde{A}} f(t)dt, \quad n = 0, 1, \ldots, \tag{4.20}$$

where $\tilde{A}_1 + \tilde{A}_2 = \tilde{A}$. $\tilde{A}_1$ and $\tilde{A}_2$ are described as follows.

$$\tilde{A}_1 \quad = \quad \begin{bmatrix} \tilde{G}_1 & \tilde{K}_1 & \cdots & \cdots & \cdots & 0 \\ \tilde{H}_2 & \tilde{G}_2 & \tilde{K}_2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \tilde{H}_{M-2} & \tilde{G}_{M-2} & \tilde{K}_{M-2} & 0 \\ \cdots & \cdots & \cdots & \tilde{H}_{M-1} & \tilde{G}_{M-1} & \tilde{K}_{M-1} \\ 0 & \cdots & \cdots & \cdots & \tilde{H}_M & \tilde{G}_M \end{bmatrix},$$

and

$$\tilde{A}_2 \quad = \quad \begin{bmatrix} \tilde{T}_1 & \tilde{L}_2 & \cdots & \cdots & \cdots & 0 \\ \cdots & \tilde{T}_2 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \tilde{T}_{M-2} & \tilde{L}_{M-2} & 0 \\ \cdots & \cdots & \cdots & \cdots & \tilde{T}_{M-1} & \tilde{L}_{M-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & \tilde{T}_M \end{bmatrix}.$$

Here matrices $\tilde{H}_i, \tilde{G}_j, \tilde{K}_k, \tilde{R}_i, \tilde{T}_j \in \mathbb{R}^{N\times N}$, $i = 2, 3, \ldots, M;\ j = 1, 2, \ldots, M;\ k = 1, 2, \ldots, M-1$.

Nontrivial entries of the matrices $\tilde{H}_m, \tilde{G}_m, \tilde{K}_m, \tilde{R}_m$ and $\tilde{T}_m$ for their respective ranges of m are as follows.

$$\tilde{h}_{n,n-1} \quad = \quad \frac{\rho\sigma y_n}{2h_m k_n};$$

$$\tilde{h}_{n,n}^{(m)} = \begin{cases} \frac{y_1}{h_m(h_m+h_{m+1})}, & y_n = k_1, \\ \frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n}, & k_1 < y_n \le 2r, \\ \frac{y_n}{h_m(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_n} + \frac{y_n-2r}{2h_m}, & 2r < y_n \le Y - k_{N+1}; \end{cases}$$

$$\tilde{g}_{n,n}^{(m)} = \begin{cases} -\frac{y_n}{h_m h_{m+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{y_n-2r}{2h_{m+1}}, & k_1 \le y_n \le 2r, \\ -\frac{y_n}{h_m h_{m+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{y_n-2r}{2h_m}, & 2r < y_n \le Y - k_{N+1}, \\ -\frac{y_N}{h_m h_{m+1}} - \frac{y_N-2r}{2h_m}, & y_N = Y - k_{N+1}; \end{cases}$$

$$\tilde{k}_{n,n}^{(m)} = \begin{cases} \frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} - \frac{y_n-2r}{2h_{m+1}}, & 0 < y_n \le 2r, \\ \frac{y_n}{h_{m+1}(h_m+h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, & 2r < y_n < Y - k_{N+1}, \\ \frac{y_N}{h_{m+1}(h_m+h_{m+1})}, & y_N = Y - k_{N+1}. \end{cases}$$

$$\tilde{t}_{n,n-1}^{(m)} = \begin{cases} \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n}, & k_1 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta-y_n)}{k_n}, & \eta < y_n \le Y - k_{N+1}; \end{cases}$$

$$\tilde{t}_{n,n}^{(m)} = \begin{cases} -\frac{\sigma^2 y_1}{k_2(k_2+k_1)} + \frac{\rho\sigma y_n}{2h_m k_1} - \frac{\kappa(\eta-y_1)}{k_2}, & y_n = k_1, \\ -\frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta-y_n)}{k_{n+1}}, & k_1 < y_n \le \eta, \\ -\frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} + \frac{\kappa(\eta-y_n)}{k_n}, & \eta < y_n < Y - k_{N+1}, \\ -\frac{\sigma^2 y_N}{k_N(k_N+k_{N+1})} + \frac{\rho\sigma y_N}{2h_m k_N} + \frac{\kappa(\eta-y_N)}{k_N}, & y_N = Y - k_{N+1}; \end{cases}$$

$$\tilde{t}_{n,n+1}^{(m)} = \begin{cases} \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{\kappa(\eta-y_n)}{k_{n+1}}, & 0 < y_n \le \eta, \\ \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}, & \eta < y_n < Y - k_{N+1}; \end{cases}$$

$$\tilde{l}_{n,n+1}^{(m)} = \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}.$$

4.3 Lax-Richtmyer Sufficient Condition for Stability

To obtain a fully discretized scheme, we use [0/1] Padé approximation to each of the matrix exponentials in (4.17) and (4.19). We arrive at the following equations, respectively:

$$w_{n+1} = (I-\tau A_1)^{-1}(I-\tau A_2)^{-1}w_n; \tag{4.21}$$

$$w_{n+1} = (I-\tau \tilde{A}_1)^{-1}(I-\tau \tilde{A}_2)^{-1}w_n. \tag{4.22}$$

In this section we are concerned with a sufficient condition for stability provided by Lax and Richtmyer [37].

Theorem 4.1. [37] *Let* $\|\cdot\|_\star$ *be a well defined matrix norm. The schemes* (4.21) *and* (4.22) *are linearly stable if*

$$\begin{aligned} \|(I-\tau A_1)^{-1}(I-\tau A_2)^{-1}\|_\star &\leq 1, \\ \|(I-\tau \tilde{A}_1)^{-1}(I-\tau \tilde{A}_2)^{-1}\|_\star &\leq 1. \end{aligned}$$

Before we can prove our scheme is linearly stable, we need the following lemmas.

Lemma 4.2. *Diagonal elements of matrices* $A_1, A_2, \tilde{A}_1$ *and* $\tilde{A}_2$ *are negative.*

Proof. For the four matrices, we prove the case when $k_1 < y_n \leq 2r$. Other cases follow immediately from the proof.

Note that the diagonal entries of A_1 satisfy the following inequalities.

$$\begin{aligned} -\frac{y_n}{h_m h_{m+1}} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{y_n - 2r}{2h_{m+1}} &\leq -\frac{y_n}{h_m h_{m+1}} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\ &= \frac{y_n}{h_m h_{m+1}}\left(-1 + \frac{\sigma h_{m+1}}{2(-k_{n+1}/\rho)}\right) \\ &\leq \frac{y_n}{h_m h_{m+1}}\left(-1 + \frac{1}{2}\right) \\ &= -\frac{y_n}{2h_m h_{m+1}} \\ &< 0. \end{aligned}$$

Further, the diagonal entries of A_2 satisfy

$$-\frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{\kappa(\eta - y_n)}{k_{n+1}} \leq -\frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1}k_n}$$

$$
\begin{aligned}
&= \frac{\sigma^2 y_n}{k_n k_{n+1}}\left(-1+\frac{-\rho k_{n+1}}{2h_{m+1}/\sigma}\right)\\
&\le \frac{\sigma^2 y_n}{k_n k_{n+1}}\left(-1+\frac{1}{2}\right)\\
&= -\frac{\sigma^2 y_n}{2k_n k_{n+1}}\\
&< 0.
\end{aligned}
$$

The diagonal entries of $\tilde{A}_1$ satisfy

$$
\begin{aligned}
-\frac{y_n}{h_m h_{m+1}}+\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}+\frac{y_n-2r}{2h_{m+1}} &\le -\frac{y_n}{h_m h_{m+1}}+\frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}}\\
&\le \frac{y_n}{h_m h_{m+1}}\left(-1+\frac{\rho h_m}{2(k_{n+1}/\sigma)}\right)\\
&\le \frac{y_n}{h_m h_{m+1}}\left(-1+\frac{\sigma h_m}{2(k_{n+1}/\rho)}\right)\\
&\le \frac{y_n}{h_m h_{m+1}}\left(-1+\frac{1}{2}\right)\\
&= -\frac{y_n}{2h_m h_{m+1}}\\
&< 0.
\end{aligned}
$$

The diagonal entries of $\tilde{A}_2$ satisfy

$$
\begin{aligned}
-\frac{\sigma^2 y_n}{k_n k_{n+1}}+\frac{\rho\sigma y_n}{2h_m k_n}-\frac{\kappa(\eta-y_n)}{k_{n+1}} &\le -\frac{\sigma^2 y_n}{k_n k_{n+1}}+\frac{\rho\sigma y_n}{2h_m k_n}\\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}}\left(-1+\frac{\rho k_{n+1}}{2(h_m/\sigma)}\right)\\
&\le \frac{\sigma^2 y_n}{k_n k_{n+1}}\left(-1+\frac{1}{2}\right)\\
&= -\frac{\sigma^2 y_n}{2k_n k_{n+1}}\\
&< 0.
\end{aligned}
$$

□

Lemma 4.3. *The off-diagonal elements of* $A_1, A_2, \tilde{A}_1$ *and* $\tilde{A}_2$ *are nonnegative.*

Proof. We provide proofs for A_1 and A_2 for $0 < y < 2r$. These proofs can be generalized to other cases. We need the condition (4.7) in each of the following discussions.

$$
\begin{aligned}
h_{n,n}^{(m)} &= \frac{y_n}{h_m(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= \frac{y_n\sigma}{h_m}\left[\frac{1}{\sigma(h_m+h_{m+1})} - \frac{1}{2(-k_{n+1}/\rho)}\right] \\
&\geq \frac{y_n\sigma}{h_m}\left(\frac{1}{2\sigma h_{max}} - \frac{1}{2\sigma h_{max}}\right) \\
&= 0. \\
k_{n,n-1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_{m+1}k_n} \;>\; 0. \\
k_{n,n}^{(m)} &= \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{y_n-2r}{2h_{m+1}} \\
&> \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&= \frac{\sigma y_n}{h_{m+1}}\left[\frac{1}{\sigma(h_m+h_{m+1})} - \frac{1}{2(-k_n/\rho)}\right] \\
&\geq \frac{y_n\sigma}{h_{m+1}}\left(\frac{1}{2\sigma h_{max}} - \frac{1}{2\sigma h_{max}}\right) \\
&= 0. \\
l_{n,n+1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_m k_{n+1}} \;>\; 0. \\
t_{n,n-1}^{(m)} &= \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&= -\frac{\rho\sigma^2 y_n}{k_n}\left[\frac{1}{-\rho(k_n+k_{n+1})} - \frac{1}{2\sigma h_{m+1}}\right] \\
&\geq -\frac{\rho\sigma^2 y_n}{k_n}\left(\frac{1}{2\sigma h_{\min}} - \frac{1}{2\sigma h_{\min}}\right) \\
&= 0. \\
t_{n,n+1}^{(m)} &= \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{\kappa(\eta-y_n)}{k_{n+1}} \\
&> \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= -\frac{\rho\sigma^2 y_n}{k_{n+1}}\left[\frac{1}{-\rho(k_n+k_{n+1})} - \frac{1}{2\sigma h_m}\right] \\
&\geq -\frac{\rho\sigma^2 y_n}{k_{n+1}}\left(\frac{1}{2\sigma h_{\min}} - \frac{1}{2\sigma h_{\min}}\right) \\
&= 0
\end{aligned}
$$

when $k_1 < y_n \leq 2r$. Proofs of other cases follow from above discussions.

On the other hand, diagonal entries of A_1 satisfy

$$\begin{aligned}
-\frac{y_n}{h_m h_{m+1}} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{y_n - 2r}{2h_{m+1}} &\leq -\frac{y_n}{h_m h_{m+1}} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= \frac{y_n}{h_m h_{m+1}} \left(-1 + \frac{\sigma h_{m+1}}{2(-k_{n+1}/\rho)}\right) \\
&\leq \frac{y_n}{h_m h_{m+1}} \left(-1 + \frac{1}{2}\right) \\
&= -\frac{y_n}{2h_m h_{m+1}} \\
&< 0.
\end{aligned}$$

The diagonal entries of A_2 satisfy

$$\begin{aligned}
-\frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1} k_n} - \frac{\kappa(\eta - y_n)}{k_{n+1}} &\leq -\frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1} k_n} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} \left(-1 + \frac{-\rho k_{n+1}}{2h_{m+1}/\sigma}\right) \\
&\leq \frac{\sigma^2 y_n}{k_n k_{n+1}} \left(-1 + \frac{1}{2}\right) \\
&= -\frac{\sigma^2 y_n}{2k_n k_{n+1}} \\
&< 0.
\end{aligned}$$

The diagonal entries of $\tilde{A}_1$ satisfy

$$\begin{aligned}
-\frac{y_n}{h_m h_{m+1}} + \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} + \frac{y_n - 2r}{2h_{m+1}} &\leq -\frac{y_n}{h_m h_{m+1}} + \frac{\rho\sigma y_n}{2h_{m+1} k_{n+1}} \\
&\leq \frac{y_n}{h_m h_{m+1}} \left(-1 + \frac{\rho h_m}{2(k_{n+1}/\sigma)}\right) \\
&\leq \frac{y_n}{h_m h_{m+1}} \left(-1 + \frac{\sigma h_m}{2(k_{n+1}/\rho)}\right) \\
&\leq \frac{y_n}{h_m h_{m+1}} \left(-1 + \frac{1}{2}\right) \\
&= -\frac{y_n}{2h_m h_{m+1}} \\
&< 0.
\end{aligned}$$

The diagonal entries of $\tilde{A}_2$ satisfy

$$-\frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_{n+1}} \leq -\frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n}$$

$$
\begin{aligned}
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} \left(-1 + \frac{\rho k_{n+1}}{2(h_m/\sigma)} \right) \\
&\leq \frac{\sigma^2 y_n}{k_n k_{n+1}} \left(-1 + \frac{1}{2} \right) \\
&= -\frac{\sigma^2 y_n}{2 k_n k_{n+1}} \\
&< 0.
\end{aligned}
$$

□

Lemma 4.4. *The off-diagonal elements of $A_1, A_2, \tilde{A}_1$ and $\tilde{A}_2$ are nonnegative.*

Proof. We provide proof for A_1 and A_2, for $0 < y < 2r$. The proof can be generalized to other cases. We need the condition (4.7) in each of the following piece of the proof.

$$
\begin{aligned}
h_{n,n}^{(m)} &= \frac{y_n}{h_m(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= \frac{y_n \sigma}{h_m} \left[\frac{1}{\sigma(h_m + h_{m+1})} - \frac{1}{2(-k_{n+1}/\rho)} \right] \\
&\geq \frac{y_n \sigma}{h_m} \left(\frac{1}{2\sigma h_{max}} - \frac{1}{2\sigma h_{max}} \right) \\
&= 0. \\
k_{n,n-1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_{m+1} k_n} \; > \; 0. \\
k_{n,n}^{(m)} &= \frac{y_n}{h_{m+1}(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1} k_n} - \frac{y_n - 2r}{2h_{m+1}} \\
&> \frac{y_n}{h_{m+1}(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1} k_n} \\
&= \frac{\sigma y_n}{h_{m+1}} \left[\frac{1}{\sigma(h_m + h_{m+1})} - \frac{1}{2(-k_n/\rho)} \right] \\
&\geq \frac{y_n \sigma}{h_{m+1}} \left(\frac{1}{2\sigma h_{max}} - \frac{1}{2\sigma h_{max}} \right) \\
&= 0. \\
l_{n,n+1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_m k_{n+1}} \; > \; 0. \\
t_{n,n-1}^{(m)} &= \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1} k_n} \\
&= -\frac{\rho\sigma^2 y_n}{k_n} \left[\frac{1}{-\rho(k_n + k_{n+1})} - \frac{1}{2\sigma h_{m+1}} \right]
\end{aligned}
$$

$$
\begin{aligned}
&\geq -\frac{\rho\sigma^2 y_n}{k_n}\left(\frac{1}{2\sigma h_{\min}} - \frac{1}{2\sigma h_{\min}}\right) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
t_{n,n+1}^{(m)} &= \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{\kappa(\eta - y_n)}{k_{n+1}} \\
&> \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= -\frac{\rho\sigma^2 y_n}{k_{n+1}}\left[\frac{1}{-\rho(k_n + k_{n+1})} - \frac{1}{2\sigma h_m}\right] \\
&\geq -\frac{\rho\sigma^2 y_n}{k_{n+1}}\left(\frac{1}{2\sigma h_{\min}} - \frac{1}{2\sigma h_{\min}}\right) \\
&= 0
\end{aligned}
$$

for $k_1 < y_n \leq 2r$. Other cases follow immediately from the above discussion.

Further, diagonal entries of A_1 satisfy

$$
\begin{aligned}
-\frac{y_n}{h_m h_{m+1}} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{y_n - 2r}{2h_{m+1}} &\leq -\frac{y_n}{h_m h_{m+1}} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= \frac{y_n}{h_m h_{m+1}}\left(-1 + \frac{\sigma h_{m+1}}{2(-k_{n+1}/\rho)}\right) \\
&\leq \frac{y_n}{h_m h_{m+1}}\left(-1 + \frac{1}{2}\right) \\
&= -\frac{y_n}{2h_m h_{m+1}} \\
&< 0.
\end{aligned}
$$

Diagonal entries of A_2 satisfy

$$
\begin{aligned}
-\frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1} k_n} - \frac{\kappa(\eta - y_n)}{k_{n+1}} &\leq -\frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\rho\sigma y_n}{2h_{m+1} k_n} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}}\left(-1 + \frac{-\rho k_{n+1}}{2h_{m+1}/\sigma}\right) \\
&\leq \frac{\sigma^2 y_n}{k_n k_{n+1}}\left(-1 + \frac{1}{2}\right) \\
&= -\frac{\sigma^2 y_n}{2k_n k_{n+1}} \\
&< 0.
\end{aligned}
$$

Diagonal entries of $\tilde{A}_1$ satisfy

$$\begin{aligned}
-\frac{y_n}{h_m h_{m+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} + \frac{y_n - 2r}{2h_{m+1}} &\leq -\frac{y_n}{h_m h_{m+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_{n+1}} \\
&\leq \frac{y_n}{h_m h_{m+1}}\left(-1 + \frac{\rho h_m}{2(k_{n+1}/\sigma)}\right) \\
&\leq \frac{y_n}{h_m h_{m+1}}\left(-1 + \frac{\sigma h_m}{2(k_{n+1}/\rho)}\right) \\
&\leq \frac{y_n}{h_m h_{m+1}}\left(-1 + \frac{1}{2}\right) \\
&= -\frac{y_n}{2h_m h_{m+1}} \\
&< 0.
\end{aligned}$$

Diagonal entries of $\tilde{A}_2$ satisfy

$$\begin{aligned}
-\frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} - \frac{\kappa(\eta - y_n)}{k_{n+1}} &\leq -\frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_n} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}}\left(-1 + \frac{\rho k_{n+1}}{2(h_m/\sigma)}\right) \\
&\leq \frac{\sigma^2 y_n}{k_n k_{n+1}}\left(-1 + \frac{1}{2}\right) \\
&= -\frac{\sigma^2 y_n}{2k_n k_{n+1}} \\
&< 0.
\end{aligned}$$

The above completes our proof. $\square$

Lemma 4.5. *All off-diagonal elements of $A_1, A_2, \tilde{A}_1$ and $\tilde{A}_2$ are nonnegative.*

Proof. We provide proof for A_1 and A_2, for $0 < y < 2r$. The proof can be generalized to other cases. We use the condition (4.7) in each segment of the proof.

$$\begin{aligned}
h_{n,n}^{(m)} &= \frac{y_n}{h_m(h_m + h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= \frac{y_n\sigma}{h_m}\left[\frac{1}{\sigma(h_m + h_{m+1})} - \frac{1}{2(-k_{n+1}/\rho)}\right] \\
&\geq \frac{y_n\sigma}{h_m}\left(\frac{1}{2\sigma h_{max}} - \frac{1}{2\sigma h_{max}}\right) \\
&= 0.
\end{aligned}$$

$$k^{(m)}_{n,n-1} = -\frac{\rho\sigma y_n}{2h_{m+1}k_n} > 0.$$

$$\begin{aligned}
k^{(m)}_{n,n} &= \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{y_n-2r}{2h_{m+1}} \\
&> \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&= \frac{\sigma y_n}{h_{m+1}}\left[\frac{1}{\sigma(h_m+h_{m+1})} - \frac{1}{2(-k_n/\rho)}\right] \\
&\geq \frac{y_n\sigma}{h_{m+1}}\left(\frac{1}{2\sigma h_{max}} - \frac{1}{2\sigma h_{max}}\right) \\
&= 0.
\end{aligned}$$

$$l^{(m)}_{n,n+1} = -\frac{\rho\sigma y_n}{2h_m k_{n+1}} > 0.$$

$$\begin{aligned}
t^{(m)}_{n,n-1} &= \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&= -\frac{\rho\sigma^2 y_n}{k_n}\left[\frac{1}{-\rho(k_n+k_{n+1})} - \frac{1}{2\sigma h_{m+1}}\right] \\
&\geq -\frac{\rho\sigma^2 y_n}{k_n}\left(\frac{1}{2\sigma h_{\min}} - \frac{1}{2\sigma h_{\min}}\right) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
t^{(m)}_{n,n+1} &= \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{\kappa(\eta-y_n)}{k_{n+1}} \\
&> \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= -\frac{\rho\sigma^2 y_n}{k_{n+1}}\left[\frac{1}{-\rho(k_n+k_{n+1})} - \frac{1}{2\sigma h_m}\right] \\
&\geq -\frac{\rho\sigma^2 y_n}{k_{n+1}}\left(\frac{1}{2\sigma h_{\min}} - \frac{1}{2\sigma h_{\min}}\right) \\
&= 0
\end{aligned}$$

for $2r < y < \eta$. The discussion can be generalized to show other cases. Let us again use condition (4.7) in each part of the following proof.

$$h^{(m)}_{n,n} = \frac{y_n}{h_m(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}}$$

$$
\begin{aligned}
&= \frac{y_n\sigma}{h_m}\left[\frac{1}{\sigma(h_m+h_{m+1})} - \frac{1}{2(-k_{n+1}/\rho)}\right] \\
&\geq \frac{y_n\sigma}{h_m}\left(\frac{1}{2\sigma h_{max}} - \frac{1}{2\sigma h_{max}}\right) \\
&= 0. \\
k_{n,n-1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_{m+1}k_n} \;>\; 0. \\
k_{n,n}^{(m)} &= \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{y_n-2r}{2h_{m+1}} \\
&> \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&= \frac{\sigma y_n}{h_{m+1}}\left[\frac{1}{\sigma(h_m+h_{m+1})} - \frac{1}{2(-k_n/\rho)}\right] \\
&\geq \frac{y_n\sigma}{h_{m+1}}\left(\frac{1}{2\sigma h_{max}} - \frac{1}{2\sigma h_{max}}\right) \;=\; 0. \\
l_{n,n+1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_m k_{n+1}} \;>\; 0. \\
t_{n,n-1}^{(m)} &= \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&= -\frac{\rho\sigma^2 y_n}{k_n}\left[\frac{1}{-\rho(k_n+k_{n+1})} - \frac{1}{2\sigma h_{m+1}}\right] \\
&\geq -\frac{\rho\sigma^2 y_n}{k_n}\left(\frac{1}{2\sigma h_{\min}} - \frac{1}{2\sigma h_{\min}}\right) \\
&= 0. \\
t_{n,n+1}^{(m)} &= \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} - \frac{\kappa(\eta-y_n)}{k_{n+1}} \\
&> \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= -\frac{\rho\sigma^2 y_n}{k_{n+1}}\left[\frac{1}{-\rho(k_n+k_{n+1})} - \frac{1}{2\sigma h_m}\right] \\
&\geq -\frac{\rho\sigma^2 y_n}{k_{n+1}}\left(\frac{1}{2\sigma h_{\min}} - \frac{1}{2\sigma h_{\min}}\right) \\
&= 0
\end{aligned}
$$

for $\eta < y < Y$. The discussion can be again generalized to other cases. We use the condition (4.7) in each of the following investigations.

$$
\begin{aligned}
h_{n,n}^{(m)} &= \frac{y_n}{h_m(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= \frac{y_n\sigma}{h_m}\left[\frac{1}{\sigma(h_m+h_{m+1})} - \frac{1}{2(-k_{n+1}/\rho)}\right] \\
&\geq \frac{y_n\sigma}{h_m}\left(\frac{1}{2\sigma h_{max}} - \frac{1}{2\sigma h_{max}}\right) \\
&= 0. \\
k_{n,n-1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_{m+1}k_n} \;>\; 0. \\
k_{n,n}^{(m)} &= \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} + \frac{y_n-2r}{2h_{m+1}} \\
&> \frac{y_n}{h_{m+1}(h_m+h_{m+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&= \frac{\sigma y_n}{h_{m+1}}\left[\frac{1}{\sigma(h_m+h_{m+1})} - \frac{1}{2(-k_n/\rho)}\right] \\
&\geq \frac{y_n\sigma}{h_{m+1}}\left(\frac{1}{2\sigma h_{max}} - \frac{1}{2\sigma h_{max}}\right) \\
&= 0. \\
l_{n,n+1}^{(m)} &= -\frac{\rho\sigma y_n}{2h_m k_{n+1}} \;>\; 0. \\
t_{n,n-1}^{(m)} &= \frac{\sigma^2 y_n}{k_n(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&= -\frac{\rho\sigma^2 y_n}{k_n}\left[\frac{1}{-\rho(k_n+k_{n+1})} - \frac{1}{2\sigma h_{m+1}}\right] \\
&\geq -\frac{\rho\sigma^2 y_n}{k_n}\left(\frac{1}{2\sigma h_{\min}} - \frac{1}{2\sigma h_{\min}}\right) \\
&= 0. \\
t_{n,n+1}^{(m)} &= \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} - \frac{\kappa(\eta-y_n)}{k_{n+1}} \\
&> \frac{\sigma^2 y_n}{k_{n+1}(k_n+k_{n+1})} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= -\frac{\rho\sigma^2 y_n}{k_{n+1}}\left[\frac{1}{-\rho(k_n+k_{n+1})} - \frac{1}{2\sigma h_m}\right] \\
&\geq -\frac{\rho\sigma^2 y_n}{k_{n+1}}\left(\frac{1}{2\sigma h_{\min}} - \frac{1}{2\sigma h_{\min}}\right) \\
&= 0.
\end{aligned}
$$

These inequalities ensure our proof. □

Lemma 4.6. *$A_1, A_2, \tilde{A}_1$ and $\tilde{A}_2$ defined in Chapter 3 are diagonally dominant.*

Proof. We consider the case when $k_1 < y_n \leq 2r$ for matrices A_1 and A_2. We let $[A_l]_{ij}$ to represent the entry of A_l in the ith row and jth column for $l = 1, 2$. Lemma 4.1 and Lemma 4.2 results are used for each of the following derivations.

$$\begin{aligned}
|[A_1]_{ii}| - \sum_{j \neq i} \left|[A_1]_{ij}\right| &= \left|g_{n,n}^{(m)}\right| - \left|h_{n,n}^{(m)}\right| - \left|k_{n,n}^{(m)}\right| - \left|k_{n,n-1}^{(m)}\right| \\
&= -g_{n,n}^{(m)} - h_{n,n}^{(m)} - k_{n,n}^{(m)} - k_{n,n-1}^{(m)} \\
&= \frac{y_n}{h_m h_{m+1}} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} - \frac{y_n - 2r}{2h_{m+1}} \\
&\quad - \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&\quad - \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1} k_n} + \frac{y_n - 2r}{2h_{m+1}} + \frac{\rho\sigma y_n}{2h_{m+1} k_n} \\
&= \frac{y_n}{h_m h_{m+1}} - \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_{m+1}(h_m + h_{m+1})} \\
&= \frac{y_n}{h_m h_{m+1}} - \frac{y_n}{h_m h_{m+1}} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
|[A_2]_{ii}| - \sum_{j \neq i} \left|[A_2]_{ij}\right| &= \left|t_{n,n}^{(m)}\right| - \left|t_{n,n-1}^{(m)}\right| - \left|t_{n,n+1}^{(m)}\right| - \left|l_{n,n+1}^{(m)}\right| \\
&= -t_{n,n}^{(m)} - t_{n,n-1}^{(m)} - t_{n,n+1}^{(m)} - l_{n,n+1}^{(m)} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1} k_n} + \frac{\kappa(\eta - y_n)}{k_{n+1}} \\
&\quad - \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1} k_n} \\
&\quad - \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} - \frac{\kappa(\eta - y_n)}{k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\sigma^2 y_n}{k_n k_{n+1}} \\
&= 0.
\end{aligned}$$

When $2r < y \leq \eta$,

$$|[A_1]_{ii}| - \sum_{j \neq i} \left|[A_1]_{ij}\right| = \left|g_{n,n}^{(m)}\right| - \left|h_{n,n}^{(m)}\right| - \left|k_{n,n}^{(m)}\right| - \left|k_{n,n-1}^{(m)}\right|$$

$$
\begin{aligned}
&= -g_{n,n}^{(m)} - h_{n,n}^{(m)} - k_{n,n}^{(m)} - k_{n,n-1}^{(m)} \\
&= \frac{y_n}{h_m h_{m+1}} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} - \frac{y_n - 2r}{2h_{m+1}} \\
&\quad - \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&\quad - \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_n} + \frac{y_n - 2r}{2h_{m+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&= \frac{y_n}{h_m h_{m+1}} - \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_{m+1}(h_m + h_{m+1})} \\
&= \frac{y_n}{h_m h_{m+1}} - \frac{y_n}{h_m h_{m+1}} \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
|[A_2]_{ii}| - \sum_{j \neq i} \left|[A_2]_{ij}\right| &= \left|t_{n,n}^{(m)}\right| - \left|t_{n,n-1}^{(m)}\right| - \left|t_{n,n+1}^{(m)}\right| - \left|l_{n,n+1}^{(m)}\right| \\
&= -t_{n,n}^{(m)} - t_{n,n-1}^{(m)} - t_{n,n+1}^{(m)} - l_{n,n+1}^{(m)} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{\kappa(\eta - y_n)}{k_{n+1}} \\
&\quad - \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&\quad - \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{\kappa(\eta - y_n)}{k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\sigma^2 y_n}{k_n k_{n+1}} \\
&= 0.
\end{aligned}
$$

When $2r < y \leq \eta$,

$$
\begin{aligned}
|[A_1]_{ii}| - \sum_{j \neq i} \left|[A_1]_{ij}\right| &= \left|g_{n,n}^{(m)}\right| - \left|h_{n,n}^{(m)}\right| - \left|k_{n,n}^{(m)}\right| - \left|k_{n,n-1}^{(m)}\right| \\
&= -g_{n,n}^{(m)} - h_{n,n}^{(m)} - k_{n,n}^{(m)} - k_{n,n-1}^{(m)} \\
&= \frac{y_n}{h_m h_{m+1}} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{y_n - 2r}{2h_{m+1}} \\
&\quad - \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&\quad - \frac{y_n}{h_{m+1}(h_m + h_{m+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{y_n - 2r}{2h_{m+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&= \frac{y_n}{h_m h_{m+1}} - \frac{y_n}{h_m(h_m + h_{m+1})} - \frac{y_n}{h_{m+1}(h_m + h_{m+1})}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{y_n}{h_m h_{m+1}} - \frac{y_n}{h_m h_{m+1}} \\
&= 0. \\
|[A_2]_{ii}| - \sum_{j\neq i} \left|[A_2]_{ij}\right| &= \left|t^{(m)}_{n,n}\right| - \left|t^{(m)}_{n,n-1}\right| - \left|t^{(m)}_{n,n+1}\right| - \left|l^{(m)}_{n,n+1}\right| \\
&= -t^{(m)}_{n,n} - t^{(m)}_{n,n-1} - t^{(m)}_{n,n+1} - l^{(m)}_{n,n+1} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} + \frac{\rho\sigma y_n}{2h_{m+1}k_n} - \frac{\kappa(\eta - y_n)}{k_{n+1}} \\
&\quad - \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_{m+1}k_n} \\
&\quad - \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} - \frac{\rho\sigma y_n}{2h_m k_{n+1}} + \frac{\kappa(\eta - y_n)}{k_{n+1}} + \frac{\rho\sigma y_n}{2h_m k_{n+1}} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\sigma^2 y_n}{k_n(k_n + k_{n+1})} - \frac{\sigma^2 y_n}{k_{n+1}(k_n + k_{n+1})} \\
&= \frac{\sigma^2 y_n}{k_n k_{n+1}} - \frac{\sigma^2 y_n}{k_n k_{n+1}} \\
&= 0.
\end{aligned}
$$

The above completes our proof. □

We let $\mu_\infty(A)$ denote the logarithmic norm of $A \in \mathbb{R}^{m\times m}$ associated with the $\infty-$norm $\|\cdot\|_\infty$ [22, 23, 54].

Theorem 4.7. [29]

$$
\mu_\infty(A) \;=\; \sup_i \left([A]_{ii} + \sum_{j\neq i}^{m} |[A]_{ij}| \right), \qquad i = 1, 2, \ldots, m.
$$

Theorem 4.8. [29] *Let* $A \in \mathbb{C}^{m\times m}$ *and* $\tau > 0,\ \omega \in \mathbb{R}$. *We have*

$$
\mu(A) \leq \omega \Leftrightarrow \|(I - \tau A)^{-1}\| \leq \frac{1}{1 - \tau\omega} \quad \textit{whenever} \quad 1 - \tau\omega > 0.
$$

Lemma 4.9. *We have*

$$
\max\{\mu_\infty(A_1), \mu_\infty(A_2), \mu_\infty(\tilde{A}_1), \mu_\infty(\tilde{A}_2)\} \leq 0.
$$

Proof. The result is a straightforward application of the Lemmas 4.2 and 4.3. □

Theorem 4.10. *The fully discretized schemes* (4.21), (4.22) *are linearly stable.*

Proof. By Theorem 4.7 and Lemma 4.8, we have

$$\begin{aligned}
\|(I-\tau A_1)^{-1}(I-\tau A_2)^{-1}\|_\infty &\leq \|(I-\tau A_1)^{-1}\|_\infty\|(I-\tau A_2)^{-1}\|_\infty \\
&\leq \frac{1}{1-0}\times\frac{1}{1-0} \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
\|(I-\tau \tilde{A}_1)^{-1}(I-\tau \tilde{A}_2)^{-1}\|_\infty &\leq \|(I-\tau \tilde{A}_1)^{-1}\|_\infty\|(I-\tau \tilde{A}_2)^{-1}\|_\infty \\
&\leq \frac{1}{1-0}\times\frac{1}{1-0} \\
&= 1.
\end{aligned}$$

□

Theorem 4.11. [29] *Consider a linear system $w'(t) = Aw(t)$. If A satisfies*

(1) $[A]_{ij} \geq 0$ *for* $i \neq j$ *and* $[A]_{ii} \geq -\alpha$ *for all* i*, with* $\alpha > 0$;

(2) A *has no eigenvalues on the positive real axis, then it implies positivity for backward Euler for any step size* $\tau > 0$.

Theorem 4.12. [37,63] *A consistent finite difference scheme for solving a linear partial differential equation for which the initial value problem is well-posed is convergent if and only if it is linearly stable.*

Theorem 4.13. *The fully discretized schemes* (4.21), (4.22) *are convergent.*

Proof. The result is true based on Theorem 4.4 and Theorem 4.11. □

4.4 Simulation Experiments

Recall (3.1)-(3.6). Based on similar arguments in [64,67], in our option computational experiments, we fix $X = 8,\ Y = 1$. We first concentrate on experiments with $\rho = -0.5$ and $T = 0.5$. Next, to test against extreme cases in nowadays option markets, we proceed with $\rho = -1$ and $T = 5$. For demonstrating our numerical procedure and its rate of convergence, we first consider uniform spacial grids. Results

Table 4.1. Key parameter values for numerical simulations

.

key parameter	value used
strike price	$K = 100$
volatility of volatility	$\sigma = 1$
risk-free interest rate	$r = 0.05$
mean reversion speed	$\kappa = 2$
long-run mean of volatility	$\eta = 0.1$

Table 4.2. Rates of convergence with $\sigma = 1$, $\rho = -0.5$, $T = 0.5$ and $h = 0.01$

conv. rates	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
$\min_{m,n}(R^h_{m,n})$	0.8394	0.8406	0.8841
$\max_{m,n}(R^h_{m,n})$	1.2622	1.2410	1.2837
$\text{mean}_{m,n}(R^h_{m,n})$	0.9980	0.9984	0.9969

over nonuniform grids will be presented afterwards. For the considerations, we let $h_m = h,\ k_n = k = \sigma h,\ \ m = 1, 2, \ldots, M;\ n = 1, 2, \ldots, N$. Some key parameters adopted are shown in Table 4.1. Further, $\Delta\tau$ be our temporal step. We experiment with different values of $\lambda = \Delta\tau/c^2$, where $c = \min\{h, k\}$. To numerically examine the numerical error and rate of convergence, we employ a generalized Milne's device [47]. Then, for a selected terminal time T, we denote the numerical solution at point $(x_m, y_n, T),\ 1 \le m \le M; 1 \le n \le N$, as $u_{m,n;h}$ for any particular spatial step $0 < h \ll 1$. Likewise, we let $u_{m,n;h/2}$ and $u_{m,n;h/4}$ be computed solutions obtained by using $h/2$ and $h/4$, respectively. Thus, the point-wise rate of spatial convergence at T can then be effectively evaluated via

$$R^h_{m,n} \approx \frac{1}{\ln 2} \ln \frac{\left|u_{m,n;h} - u_{m,n;h/2}\right|}{\left|u_{m,n;h/2} - u_{m,n;h/4}\right|}. \tag{4.23}$$

Let $h = 0.01$ and $\sigma = 1$. For simplicity of notations, we use the same letter v for the approximate solution to (1.15). We show the solution v for $\rho = -0.5$ and $\rho = -1$ in Figures 4.3 and 4.6 , respectively. It can be observed that the European put option price is a decreasing function of the stock price S. This coincides well with the financial theory that a put option price should have a negative correlation

Table 4.3. Rates of convergence with $\sigma = 1$, $\rho = -0.5$, $T = 0.5$ and $h = 0.02$

conv. rates	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
$\min_{m,n}(R^h_{m,n})$	0.6324	0.6221	0.6206
$\max_{m,n}(R^h_{m,n})$	0.9674	1.0007	1.0151
$\mathrm{mean}_{m,n}(R^h_{m,n})$	0.8342	0.8300	0.8296

Table 4.4. Rates of convergence with $\sigma = 1$, $\rho = -0.5$, $T = 0.5$ and $h = 0.04$

conv. rates	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
$\min_{m,n}(R^h_{m,n})$	0.5824	0.5971	0.6179
$\max_{m,n}(R^h_{m,n})$	0.9941	0.9437	0.9586
$\mathrm{mean}_{m,n}(R^h_{m,n})$	0.7952	0.8015	0.8142

with the underline stock price [1, 28].

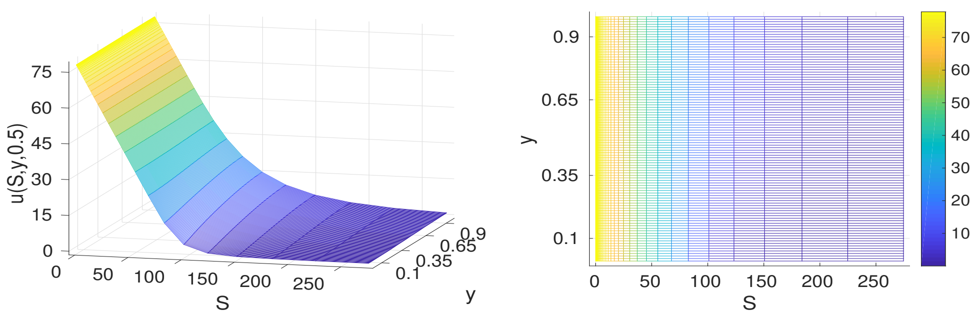

Figure 4.3. Price of an European put option surface and contour plots.

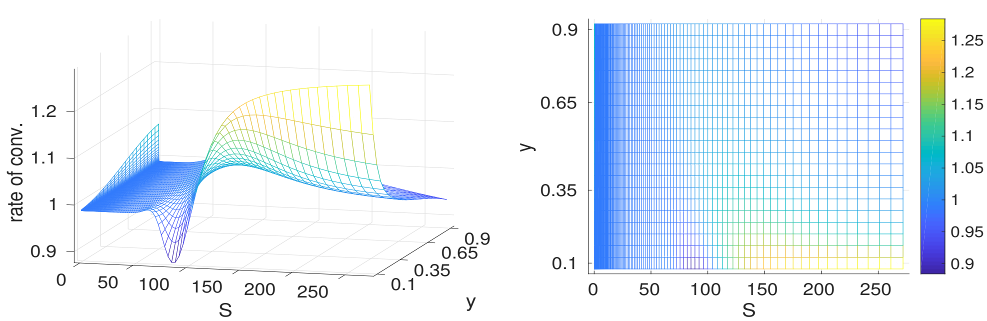

Figure 4.4. Pointwise rate of convergence and rate of convergence estimate contour.

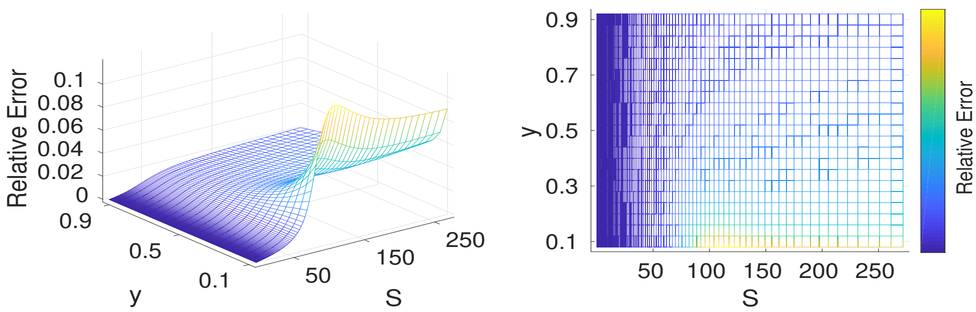

Figure 4.5. Pointwise relative error and rate of convergence contour.

Let us plot the computed rate of convergence surfaces for cases when $\rho = -0.5$ and $\rho = -1$ in Figure 4.4 and Figure 4.5, respectively. In addition, a summary of point-wise convergence rates for the former case on different spacial grids is given in Table 4.3. Minor disturbances can be observed in regions where the solution changes fast, in particularly in extreme situations with $\rho = -1$ as being demonstrated in Figure 4.6.

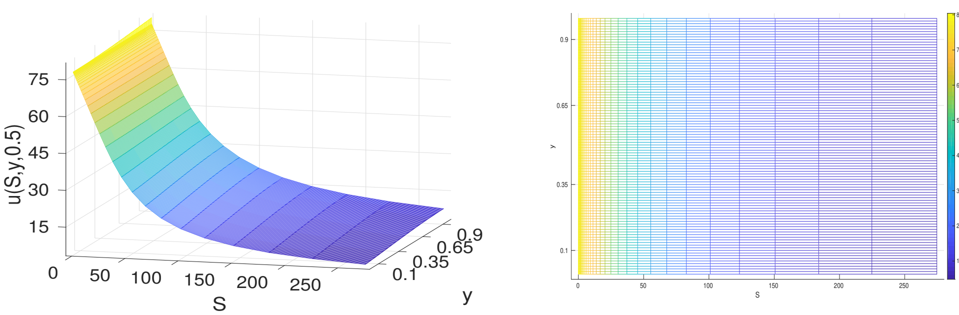

Figure 4.6. Price of an European put option surface and contour plots.

Further, the mean convergence rate for the two cases are given in the caption of Figure 4.4 and Figure 4.5. In the extreme case when $\rho = -1$ and $T = 5$, we observed a smaller mean convergence rate. This is within our expectation due to the more iterations to get the solution and the decrease in the well-posedness of the original PDE. These results are consistent with those from well-established high-order schemes [8, 12, 13, 32, 38, 67]. Again, an effective Matlab platform is used.

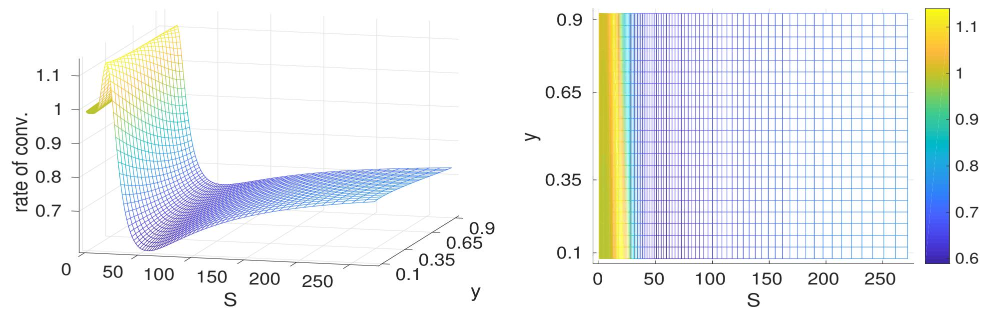

Figure 4.7. Pointwise rate of convergence estimate surface and contour plots.

Now, consider simulations over nonuniform spacial grids. We are particularly interested in the following nonlinear grid distribution governing functions [44, 47].

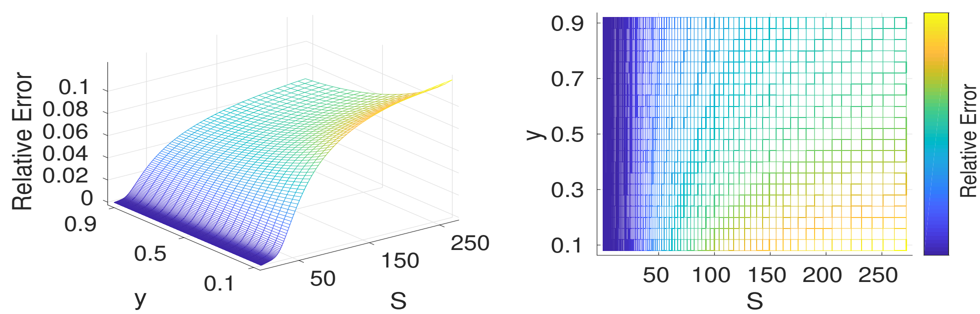

Figure 4.8. Pointwise relative error and rate of convergence estimate.

In our simulation experiments, selections of monitoring functions z_1, z_2, are based initially on the numerical solution v acquired on uniform spacial meshes. They are chosen to reflect trends of solution curvatures [13, 47].

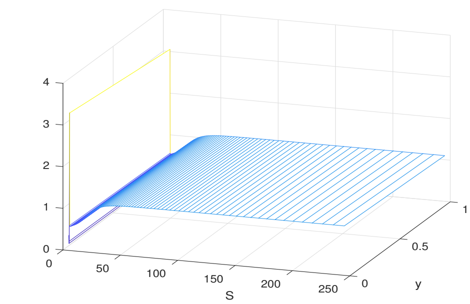

Figure 4.9. A composite surface plot of $z_1(S)z_2(y)$

Our nonuniform grids are generated via an arc-length equal-distribution principal for inverse of mean values of convergence rates for S- and y-directions, respectively. The principal is commonly utilized in adaptive computations and serves as an initial exploration for more sophisticated adaptations [30, 41, 47].

A composite surface plot of the mesh distribution functions in the $S-$ and y $-$directions is given in Figure 4.9. It characterizes the 2-dimensional profile of our grids distribution. The numerical solution acquired over such nonuniform grids, with $\rho = -0.5$ at $T = 0.5$ is given in Figure 4.10 .

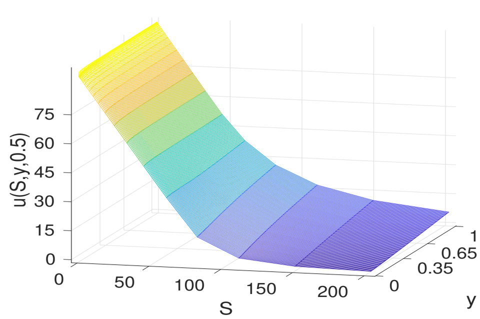

Figure 4.10. Price of an European put option on nonuniform grids

Let $\Omega_{N,M}$ be a reference spacial mesh which can also be either our uniform mesh or nonuniform mesh. We may map solutions v_{unif} and v_{nonunif}, numerical solutions obtained on the uniform mesh and nonuniform mesh, respectively, to $\Omega_{N,M}$. We plot the following point-wise relative error,

$$E_d(S,y,t) = \frac{|v_{\text{unif}}(S,y,t) - v_{\text{nonunif}}(S,y,t)|}{|v_{\text{unif}}(S,y,t)|}, \quad (S,y,t) \in \Omega_{N,M},\ 0 < t \leq T. \tag{4.24}$$

in Figure 3.6. The mean relative difference E_{mean} is given in the caption of Figure 3.6. We can see that the solutions on the uniform and nonuniform grids agrees with each other due to the small E_{mean}.

4.5 Summary

A numerically stable and dynamically balanced up-downwind semi-discretized finite difference method is constructed and analyzed in this chapter based on arbitrary option data grids, which are more preferable in nowadays trading markets. The algorithm acquired is readily to use in financial realities. It is reliable and effective for computing Heston stochastic volatility option pricing model solutions with cross-derivative terms in market realities. Rigorous mathematical proofs are given to ensure the stability and convergence. Simulation experiments further confirm our theoretical expectations on both uniform and arbitrary spacial data given. Our continuing endeavors include further improvements of the computational efficiency through higher order exponential splitting strategies, in particularly with adaptive ADI or LOD formulations [10,32,47,51,56]. Highly accurate EEG monitoring mechanism will also be installed [3].

Compact schemes for raising the accuracy have also been introduced in our study with initial successes in handling cross-derivatives dynamically and well balances for pricing American and some Asian options [1, 16, 30, 67]. Our initial investigations have been very promising.

CHAPTER FIVE

Conclusion

5.1 Contributions

In this dissertation, we developed a novel up-downwind finite difference method for solving two-dimensional Heston stochastic volatility model with a cross-derivative term. The mathematical model has been vital to option trading markets, especially those operating primarily in European call/put systems. We strictly proved that our new schemes are stable in both Neumann and linear senses.

In Chapter One, we derived and solved the basic Black-Shcoles-Merton model and discussed various issues related to the Black-Scholes-Merton model. Dealing with drawbacks of the model, we introduced stochastic volatility to the model and arrived at a two-dimensional partial differential equation with a cross-derivative term. Its associated initial-boundary conditions were also derived from the standard financial theory.

Mathematical preliminaries were discussed in Chapter Two. The mathematical theories needed to understand our methods and analysis include finite difference approximations, exponential splitting strategies, Padé approximants, adaptive grid designs and von Neumann, linear stability theories.

In Chapter Three, we transformed our stochastic volatility model equation to a new form which is suitable for computations. Original infinity domains involved were truncated to finite domains that are large enough for financial market simulations. Our new finite difference methods are built. We creatively introduced dynamically balanced up-downwind scheme by using seven-point stencils shown in Figure 3.2. We discretized the diffusion terms with standard central difference approximations. Discretizations of the cross-derivative term and advection terms consist of a combination of forward differences and backward differences depending on flow directions indicated by the coefficients of those terms. The new numerical method is proved to be first order accurate in space. The following two attributes made our schemes stand out as compared to all existing computational methods.

- We have provided rigorous mathematical proofs of the stability in both von Neumann and linear senses for our scheme when applied to the Heston type stochastic volatility model with a cross-derivative term. In fact, no higher order scheme has been found to be numerical stable for solving our option model equations.

- Our finite difference scheme offers accurate and smooth solution profiles even in cases when the size of the coefficient of the cross-derivative term is extremely large, or when the time to maturity is long. Known higher order schemes are not able to handle such extreme situations [11, 13, 28].

Mathematical proofs of the von Neumann stability of our first scheme are provided in this Chapter Three. Further, an [1/1] Padé approximant was used to build a fully discretized scheme. Numerical experiments were carried out over a MATLAB platform. We visulized the corresponding convergence surface and showed that the average point-wise convergence rate in space is approximately one which well agrees with the theoretical expectations. The finite difference scheme was further then improved over adaptive nonuniform grids. The monitor function is designed to follow changes of the solution surface. Simulation results obtained on the adaptive grids are similar to those we acquired earlier on uniform meshes.

In Chapter Four, our attention was switched from von Neumann stability analysis to linear stability analysis. The ∞-norm was used throughout this chapter. We proved that the ∞-logarithmic norm was negative for the augmentation matrix in the semi-discretized system. Thus the Padé [0/1] and [1/1] approximants yield linearly stable fully discretized schemes. Since the schemes were linearly stable, we were able to state that the schemes were also convergent due to the Lax equivalence theorem. Again, the Heston stochastic volatility model equation with a cross derivative was targeted. A theoretical justification for employing our schemes for solving the stochastic volatility model was provided. In addition, to improve further the efficiency and effectiveness of our scheme, we utilized first-order exponential splitting and Strang splitting in our schemes. Theoretical justifications were again provided for our fully discretized schemes with splitting under the ∞-logarithmic norm. We performed numerical experiments with the aforementioned exponential splitting formulas. The numerical results were similar to those without the splitting. However, with the help of exponential splitting, significant reductions of the running time was observed. Therefore, we achieved greater accuracy with smaller step sizes for the same amount of time. This fact made our schemes much more competitive as compared to existing higher order schemes. Simulated point-wise convergence surface again confirmed that our novel finite difference schemes equipped with exponential splitting are first order accuracy in space.

5.2 Continue Explorations

We plan to focus our investigations on several new topics in deep neural network learning procedures for pricing option prices based on the Heston stochastic volatility model.

The deep neural network is a neural network consists of multiple hidden layers of neurons. Such a network we shall explore first will be based on the Heston model in the sense that parameters from the model will be used as new input features.

The outputs expected will be single numbers that represent prices of the options we are interested in. So, essentially, this is a regression type of problems [9,42].

Immediate difficulties, however, are the following.

- To obtain appropriate data. Since we are using multiple layers of neurons and the feature dimension is relatively large, a large amount of data is needed to avoid the overfitting problem.

- To determine the number of hidden layers and the number of neurons in each layer. It is the same problem as applications of deep neural networks in other areas. We can use grid search technique to let the learning algorithm to choose the parameters automatically. However, the shear possible combinations of layer numbers and neuron numbers per layer will be an intimidating work. Further, the grid search technique is likely to increase the need for even much larger amount of data.

- To choose proper activation functions. Since we do not know yet what will be the best number of layers. If the number of layers are large, then the exploding gradient or dying neuron problem will occur. So the common ReLU activation function will not be appropriate any more. We may need to switch to the computation-demanding ELU activation functions etc.This will likely increase the need for even better computation algorithms, software and hardware.

Our future research will primarily target the last two points of the aforementioned difficulties.

APPENDICES

APPENDIX A

Future Work

In this appendix, we are going to highlight our side projects about predicting stock prices with moving directions through k-mean clustering and deep neural network. These are clearly associated with the next generation of machine learning and artificial intelligence [18, 65]. The research topics have attracted a tremendous amount of recent attentions from researchers in numerical partial differential equations (for instance, see [18, 50] and references therein). Our objectives within the study are to forecast moving directions of stock prices in the next month through unsupervised clustering and neural networks. The reason we are forecasting such a monthly return as opposed to daily return is because very-short-term data tend to be extremely noisy especially in nowadays financial markets [9, 28, 42, 64].

A.1 Methodology

We consider monthly data from Jan. 1990 to Feb. 2017 for both clustering and neural networks training [27]. We first cluster the 326-month data into four groups based on four macroeconomic factors, *that is,* market return, Fed rate, unemployment rate and inflation rate. Then we build a neural network for each group based on three technical factors and one essential company factor: two-month price moving average, three-month price moving average, current month stock return and size of the company. Details of the designs, ideas behind them and results obtained will be briefly addressed in following four subsections.

A.1.1 Experiments with Clustering

Our steps for clustering in the projects are as follows:

(1) Data preprocessing: we normalize the data by find the z-score of each of the four types of macro data-market return, Fed rate, unemployment rate and

inflation rate, respectively.

(2) We employ the K-means++ clustering algorithm to cluster the 326 months into four groups based on the z-scores of four types of data in each month [27]. The data are collected from the website of Bureau of Labor Statistics, an official site of Federal Reserve and Yahoo Finance.

The reason for clustering and choosing four clusters is because:

(1) According to empirical research on business cycles, business cycle consists of four stages: expansion, peak, contraction and trough. Each stage exhibits different recognizable patterns in macro factors.

(2) Stock returns frequently exhibit different patterns during different stages.

A.1.2 Results of Clustering

We have implemented our K-means++ algorithm via Matlab, and all four different clusters are accomplished.

Next, we examine aforementioned four groups closely, and compare them with the business cycle figures found, we can draw following classifications:

(1) Group 1: it is represented by red diamond contains 133 points. It matches roughly the recession period;

(2) Group 2: it is represented by blue asterisk contains 43 points. This group matches the trough of the economy;

(3) Group 3: it is represented by magenta triangle contains 95 points. It matches the expansion period;

(4) Group 4: it is the black circle which matches the peak period [28, 42, 64]. It contains 55 points.

A.1.3 Neural Network Training

We have built one neural network for each group, respectively, via Matlab. The neural networks have four layers and four nodes in the first hidden layer, and three nodes in the second hidden layer [9, 18, 27, 42]. We also use weighted constraint for second and fourth clusters due to small sample size available for these two groups. The augmented error we consider can be expressed as

$$E_{\text{aug}}(\mathbf{w}) = E_{\text{in}}(\mathbf{w}) + \frac{\lambda}{N}\sum_{l,i,j}(w_{ij}^{(l)})^2. \tag{A.1}$$

Now we may proceed forward the training of each neural network with the ideas behind the setting.

The training process of neural networks for each stock consists of:

(1) Data preprocessing in which we normalize the data by calculating z-score of input data for neural networks for two-month price moving average, three-month price moving average, current month stock return and size of the company;

(2) Index of the data by each month they belong to;

(3) Separation of the data points into four groups according to the time index of each point, based on the clustering results obtained earlier in Section 2.1.

(4) Construction of one neural network for each group with forward and backward propagation and regularization. Here we utilize the concept of stochastic gradient descent [41]. We set regularization parameter $\lambda = 0.01$ in the augmented error (A.1) for 2nd and 4th groups. The descent rate is set to be $\eta = 0.1$. Our stopping criteria used is to combine upper bounds for iterations and controls of the error size in (A.1). We further adopt an upper bound for number of iterations to be $n = 10,000$. We count one round of forward and backward propagation for each data point as one round.

Reasons why we prefer the above type of designs of neural networks in our projects include the following.

(1) Stock market data can be highly noisy and nonlinear. Neural networks are suitable for handling such turbulent data [27].

(2) Stock returns in different stages of business cycles often exhibit different patterns. Thus, to build one neural network for each group of clusters can be an optimal decision.

(3) Reasons why we choose two-hidden-layer networks is because that we have experimented with one-hidden-layer neural network, but it takes a while to converge and they tend to have very large in-sample errors [18].

(4) The way of node settings in each hidden layer is due to empirical results.

(5) The need of regularization is that, without regularization, we may obtain computational results that are seriously overfitting. Besides, Groups 2 and 4 have relatively smaller sizes of data as compared with the other two groups.

(6) Our particular descent rate and early stopping settings are due to different experiments with different combinations of the two terms. The use of $\eta = 0.1$ and $n = 10,000$ tend to offer us with lower validation errors within a relatively smaller amount of time comparing to other combinations.

A.1.4 Results of Neural Networks

We have built neural networks for over 30 stocks chosen from different economic sectors. In this way, we may predict whether our model works better for one sector than the other. The observations and analyses for these results can be highlighted to the following: follows.

A.1.5 Some Basic Analysis

(1) Our clustering is not robust. *That is,* different initializations of first center will result in quite different sizes of groups. When we finally chose a group, we must make multiple and different initializations. We chose these clusters if they have minimal errors and the size of smallest cluster is acceptable.

(2) There is still overfitting in the neural networks despite of the effort that we use early stopping and weight restraints. The gap is large even in Group 1 which has the largest data sizes.

(3) The problem of overfitting is especially severe for the Group 2 which only contains 43 points while Group 4 contains 55 data points. As we have seen, in all the four stocks, Groups 2 and 4 have the high out-sample classification errors.

(4) Our weight constrains do help cure some extents from overfitting problems as compared to those without weight constraints.

(5) One thing we wish to clarify is that our base rates are all close to 50% for each group in projects. Our out-sample is relative low comparing to those base rates. So, the predicting power of our computational model is stronger than any purely random guesses.

(6) But nevertheless, aforementioned base rates pose another concern. As we can see our base rates are really close to 50% for all four groups for all four stocks. So, does it mean that the stock movement is purely random even inside each cluster? Is our small out-sample error only because of the small sample effect? Further in-depth researches are need to be fulfilled for answering these questions. Otherwise all types of our neural network models may become less meaningful when stock movements become purely random [18, 65].

A.2 Conclusions and Continue Endeavors

Brief conclusions:

(1) Stock market return is highly noisy and it is hard to design a perfect neural network model.

(2) Even within the same type of models, different initialization and choice of different factors may yield different predictions.

(3) It is safer to put money in index funds than active funds. Those active funds are speculating by their different natures and performances. The track record exhibited in neural networks may be random.

Anticipated continuing research work:

(1) To add more fundamental data factors to our neural network input sets.

(2) To have PCA and regularizations for avoiding overfitting difficulties.

(3) To consider cross-validations for fighting problems with small numbers of financial data points.

(4) To introduce convolutional neural networks.

(5) To carry out more detailed mathematical and numerical analyses.

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