ABSTRACT

Uniqueness Implies Uniqueness and Existence for Nonlocal Boundary Value Problems for Third Order Ordinary Differential Equations

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For the third order ordinary differential equation, $y''' = f(x, y, y', y'')$, it is assumed that, for some $m \geq 4$, solutions of nonlocal boundary value problems satisfying

$$y(x_1) = y_1,\ y(x_2) = y_2,$$
$$y(x_m) - \sum_{i=3}^{m-1} y(x_i) = y_3,$$

$a < x_1 < x_2 < \cdots < x_m < b$, and $y_1, y_2, y_3 \in \mathbb{R}$, are unique when they exist. It is proved that, for all $3 \leq k \leq m$, solutions of nonlocal boundary value problems satisfying

$$y(x_1) = y_1,\ y(x_2) = y_2,$$
$$y(x_k) - \sum_{i=3}^{k-1} y(x_i) = y_3,$$

$a < x_1 < x_2 < \cdots < x_k < b$, and $y_1, y_2, y_3 \in \mathbb{R}$, are unique when they exist. It is then shown that solutions do indeed exist.

Uniqueness Implies Uniqueness and Existence for Nonlocal Boundary Value Problems for Third Order Ordinary Differential Equations

by

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CHAPTER ONE

Introduction

In this dissertation, we will be concerned with solutions of nonlocal boundary value problems for the third order ordinary differential equation,

$$y''' = f(x, y, y', y''). \tag{1.1}$$

In particular, we will discuss the uniqueness of solutions of certain boundary value problems for (1.1) implying the uniqueness of solutions of other boundary value problems, and we will see that certain conditions for the uniqueness of solutions to boundary value problems for (1.1) imply that solutions exist. More precisely, we will study questions concerning solutions of (1.1) satisfying

$$y(x_1) = y_1, \ y(x_2) = y_2, \tag{1.2}$$

$$y(x_m) - \sum_{i=3}^{m-1} y(x_i) = y_3, \tag{1.3}$$

where $m \geq 4$ is a positive integer, $a < x_1 < x_2 < \cdots < x_m < b$, and $y_1, y_2, y_3 \in \mathbb{R}$, relative to solutions of (1.1) satisfying boundary conditions (1.2) and

$$y(x_k) - \sum_{i=3}^{k-1} y(x_i) = y_3, \tag{1.4}$$

where $3 \leq k \leq m$, and the boundary condition in (1.4) is interpreted as $y(x_3) = y_3$ in the case $k = 3$.

Boundary value problems for third order ordinary differential equations have been studied both for their use in applications and for purely theoretical interest. Such equations can arise in models for boundary layer theory in fluid mechanics [1, 23, 47, 48, 49, 50], for problems involving convection in a porous medium or a flow adjacent to a standing wall [6], and in models attempting to explain large-scale [9] or one-layer [14] ocean circulation. Theoretical papers for third order problems have dealt with upper and lower solutions [8, 36], multiple solutions and eigenvalue problems [2], periodic solutions [44], monotone boundary conditions [7], limit point/limit circle criteria [4], and so on.

Nonlocal boundary value problems have also received a share of attention in both applied and theoretical settings. For a few examples, see papers by Bai and Fang [3], Feng and Webb [16, 17], Guo, Shan, and Ge [19], Gupta, Ntouyas, and Tsamatos [20], Ma [41, 42, 43], Thompson and Tisdell [46], and Zhang and Wang [51].

The third order, nonlocal boundary value problems in this dissertation are similar in form to a recent paper by Clark and Henderson [10], and expound upon the ideas in a paper by Jackson and Schrader [33]. For other discussions of third order, nonlocal boundary value problems for ordinary differential equations, please see papers by Liu, Zhong, and Jiang [40], Benbouziane, Boucherif, and Bouguima [5], and Du, Lin, and Ge [13].

CHAPTER TWO

Uniqueness Implies Uniqueness

2.1 Brief Overview

For the third order ordinary differential equation,

$$y''' = f(x, y, y', y''),$$

it is assumed that, for some $m \geq 4$, solutions of nonlocal boundary value problems satisfying

$$y(x_1) = y_1,\ y(x_2) = y_2,$$
$$y(x_m) - \sum_{i=3}^{m-1} y(x_i) = y_3,$$

$a < x_1 < x_2 < \cdots < x_m < b$, and $y_1, y_2, y_3 \in \mathbb{R}$, are unique when they exist. It is proved that, for all $3 \leq k \leq m$, solutions of nonlocal boundary value problems satisfying

$$y(x_1) = y_1,\ y(x_2) = y_2,$$
$$y(x_k) - \sum_{i=3}^{k-1} y(x_i) = y_3,$$

$a < x_1 < x_2 < \cdots < x_k < b$, and $y_1, y_2, y_3 \in \mathbb{R}$, are unique when they exist.

This "uniqueness implies uniqueness" result plays an important roll in demonstrating the existence of the solutions in question. For other examples of work on the topic, please see papers by DeBortoli, Henderson, and Pruet [12], Henderson and Jackson [27], Lasota and Opial [39], and Lasota and Luczyński [37, 38].

2.2 Preliminary Result

We will begin by proving a theorem concerning the uniqueness of solutions for similar boundary value problems.

Theorem 2.1 (Uniqueness Implies Uniqueness). *Suppose that the differential equation*

$$y''' = f(x, y, y', y''), \tag{2.1}$$

satisfies the following three conditions:

(A) $f : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous;*

(B) Solutions of initial value problems for (2.1) *are unique and exist on all of* (a, b)*; and,*

(C) Solutions of the boundary value problem for (2.1) *with boundary conditions*

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_5) - y(x_4) - y(x_3) = y_3, \tag{2.2}$$

for any $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, are unique when they exist.*

Then solutions of the boundary value problem for (2.1) *with boundary conditions*

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_4) - y(x_3) = y_3, \tag{2.3}$$

for any $a < x_1 < x_2 < x_3 < x_4 < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, are unique when they exist.*

Proof: The proof is by contradiction. In particular, suppose (A), (B), and (C) hold, and that $u(x)$ and $v(x)$ are distinct solutions of the boundary value problem (2.1), (2.3), for some points $a < x_1 < x_2 < x_3 < x_4 < b$ and $y_1, y_2, y_3 \in \mathbb{R}$. Let $w(x) = u(x) - v(x)$. Since $u(x)$ and $v(x)$ satisfy the boundary conditions in (2.3), we have

$$\begin{aligned} u(x_1) &= v(x_1), \\ u(x_2) &= v(x_2), \\ u(x_4) - u(x_3) &= v(x_4) - v(x_3), \end{aligned}$$

or $w(x_1) = w(x_2) = w(x_4) - w(x_3) = 0$. Under Condition (B), either $w'(x_2) \neq 0$ or $w''(x_2) \neq 0$.

Case 1: $w'(x_2) \neq 0$. Assume, without loss of generality, that $w'(x_2) > 0$. Then there exists an $\alpha > 0$ such that $x_2 + \alpha < x_3$ and $w(x)$ is strictly increasing (and therefore positive since $w(x_2) = 0$) on $(x_2, x_2 + \alpha]$. Now since $w(x_2 + \alpha) \geq w(x)$ for all $x \in [x_2, x_2 + \alpha]$, there exists $x_2 < t_1 < t_2 < t_3 \leq x_2 + \alpha$ such that $w(t_1) + w(t_2) = w(t_3)$, so that we have

$$\begin{aligned} w(x_1) = w(x_2) &= 0, \\ w(t_3) - w(t_2) - w(t_1) &= 0. \end{aligned}$$

That is, we have

$$\begin{aligned} u(x_1) &= v(x_1), \\ u(x_2) &= v(x_2), \\ u(t_3) - u(t_2) - u(t_1) &= v(t_3) - v(t_2) - v(t_1). \end{aligned}$$

Then $u(x) \equiv v(x)$ on (a, b) by Condition (C), which contradicts our assumption that $u(x)$ and $v(x)$ are distinct.

Case 2: $w'(x_2) = 0, w''(x_2) \neq 0$. We may assume, without loss of generality, that $w''(x_2) > 0$. Then there exists a $\beta > 0$ such that $x_2 + \beta < x_3$ and $w''(x) > 0$ on $(x_2, x_2 + \beta]$. Hence $w'(x)$ is strictly increasing (and therefore positive since $w'(x_2) = 0$) on $(x_2, x_2 + \beta]$. Since $w(x_2) = 0$, we have $w(x)$ is positive and increasing on $(x_2, x_2 + \beta]$, and we may repeat the argument from Case 1. The proof is complete. □

2.3 General Case

The previous uniqueness implies uniqueness result may be generalized as follows.

Theorem 2.2 (Uniqueness Implies Uniqueness). *For the differential equation*

$$y''' = f(x, y, y', y''), \tag{2.4}$$

assume that the following conditions hold:

(A) $f : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous;*

(B) Solutions of initial value problems for (2.4) *are unique and exist on all of* (a, b)*; and,*

(C) For some integer $m \geq 4$*, solutions of the boundary value problem for* (2.4) *with boundary conditions*

$$y(x_1) = y_1,\ \ y(x_2) = y_2, \tag{2.5}$$

$$y(x_m) - \sum_{i=3}^{m-1} y(x_i) = y_3, \tag{2.6}$$

$a < x_1 < x_2 < \cdots < x_m < b$ *and* $y_1, y_2, y_3 \in \mathbb{R}$*, are unique, when they exist.*

Then solutions of the differential equation (2.4) *with boundary conditions* (2.5) *and*

$$y(x_k) - \sum_{i=3}^{k-1} y(x_i) = y_3, \tag{2.7}$$

for all positive integers k *with* $3 \leq k \leq m$ *(where the boundary condition in* (2.7) *is interpreted as* $y(x_3) = y_3$ *in the case* $k = 3$*), for any* $a < x_1 < x_2 < \cdots < x_k < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, are unique when they exist.*

Proof: The proof is by induction. We will begin by showing the theorem is true for the case $m = 4,\ k = 3$. We are assuming Conditions (A) and (B) hold, and that solutions for the differential equation (2.4) with boundary conditions (2.5) and $y(x_4) - y(x_3) = y_3$, for $a < x_1 < x_2 < x_3 < x_4 < b$ and $y_1, y_2, y_3 \in \mathbb{R}$, are unique when they exist.

Now, suppose that $u(x)$ and $v(x)$ are distinct solutions of (2.4) with boundary conditions (2.5) and $y(x_3) = y_3$, for $a < x_1 < x_2 < x_3 < b$ and $y_1, y_2, y_3 \in \mathbb{R}$. Let

$w(x) = u(x) - v(x)$. We have

$$\begin{aligned} u(x_1) &= v(x_1), \\ u(x_2) &= v(x_2), \\ u(x_3) &= v(x_3), \end{aligned}$$

so that

$$w(x_1) = w(x_2) = w(x_3) = 0.$$

We have by Condition (B) that either $w'(x_2) \neq 0$ or $w''(x_2) \neq 0$. Let us examine each case.

Case 1: $w'(x_2) \neq 0$. We may assume, without loss of generality, that $w'(x_2) > 0$. Therefore, since $w(x_2) = 0$, $w(x_3) = 0$, and $w(x) = u(x) - v(x)$ is continuous, $w(x)$ has a local maximum on (x_2, x_3), say at $x = \beta$. Then we must have $\alpha \in (x_2, \beta)$ and $\gamma \in (\beta, x_3)$ such that

$$\begin{aligned} w(\alpha) &= w(\gamma), \\ u(\alpha) - v(\alpha) &= u(\gamma) - v(\gamma), \\ u(\alpha) - u(\gamma) &= v(\alpha) - v(\gamma). \end{aligned}$$

We have that $u(x)$ and $v(x)$ satisfy

$$\begin{aligned} u(x_1) &= v(x_1), \\ u(x_2) &= v(x_2), \\ u(\alpha) - u(\gamma) &= v(\alpha) - v(\gamma). \end{aligned}$$

Therefore, $u(x) \equiv v(x)$ on (a, b), by Condition (C). This contradicts our assumption that $u(x)$ and $v(x)$ are distinct.

Case 2: $w'(x_2) = 0$ but $w''(x_2) \neq 0$. Assume, without loss of generality, that $w''(x_2) > 0$. Then there exists a $\delta > 0$ such that $x_2 < x_2 + \delta < x_3$ and $w''(x) > 0$ on $[x_2, x_2 + \delta]$. Thus $w'(x)$ is strictly increasing on $[x_2, x_2 + \delta]$. Since $w'(x_2) = 0$, it is the case that $w'(x)$ is positive on $(x_2, x_2 + \delta]$; and so $w(x)$ is increasing on $[x_2,\ x_2 + \delta]$, and therefore positive on $(x_2, x_2 + \delta]$, since $w(x_2) = 0$. We see that $w(x)$ must in fact be positive on all of $(x_2,\ x_3)$ by Condition (A) and the fact that the next zero of $w(x)$ occurs at x_3, so we may now repeat the argument of Case 1.

In both cases we reach a contradiction to our assumption that $u(x)$ and $v(x)$ are distinct solutions of the selected boundary value problem. We conclude that the theorem holds for $m = 4$ and $k = 3$. To complete the proof, we now show that the theorem holds for an arbitrary positive integer $m > 4$ and the case $k = m - 1$.

Assume Conditions (A), (B), and (C) hold for a positive integer $m > 4$. Let $u(x)$ and $v(x)$ be distinct solutions of the differential equation (2.4) with boundary conditions (2.5) and

$$y(x_k) - \sum_{i=3}^{k-1} y(x_i) = y_3, \tag{2.8}$$

for $k = m - 1$, any $a < x_1 < x_2 < \cdots < x_k < b$, and any $y_1, y_2, y_3 \in \mathbb{R}$. Let $w(x) = u(x) - v(x)$. We have

$$\begin{gathered} w(x_1) = w(x_2) = 0, \\ w(x_k) - \sum_{i=3}^{k-1} w(x_i) = 0. \end{gathered}$$

By Condition (B), we know either $w'(x_2) \neq 0$ or $w''(x_2) \neq 0$. We will examine each case.

Case 1: $w'(x_2) \neq 0$. Assume, without loss of generality, that $w'(x_2) > 0$. Then there exists $\alpha > 0$ such that $x_2 < x_2 + \alpha < x_3$ and $w(x)$ is strictly increasing on $(x_2, x_2 + \alpha)$. Observe that $w(x)$ is also positive on $(x_2, x_2 + \alpha)$ since $w(x_2) = 0$. Therefore we may choose $x_2 < t_1 < t_2 < \cdots < t_{k-1} \leq x_2 + \alpha$ such that

$$w(t_{k-1}) - \sum_{i=1}^{k-2} w(t_i) = 0.$$

That is, we have

$$\begin{aligned} u(x_1) &= v(x_1), \\ u(x_2) &= v(x_2), \\ u(t_{k-1}) - \sum_{i=1}^{k-2} u(t_i) &= v(t_{k-1}) - \sum_{i=1}^{k-2} v(t_i). \end{aligned}$$

Therefore, $u(x) \equiv v(x)$ on (a, b), by Condition (C). This contradicts our assumption that $u(x)$ and $v(x)$ are distinct.

Case 2: $w'(x_2) = 0$ but $w''(x_2) \neq 0$. Assume, without loss of generality, that $w''(x_2)$ is positive. Then there exists a $\delta > 0$ such that $x_2 < x_2 + \delta < x_3$ and $w'(x)$ is strictly increasing on $[x_2, x_2 + \delta]$. Since $w'(x_2) = 0$, it is the case that $w'(x)$ is positive on $(x_2, x_2 + \delta]$, and so $w(x)$ is increasing on $[x_2, x_2 + \delta]$, hence positive on $[x_2,\ x_2 + \delta]$ since $w(x_2) = 0$. We may now choose the appropriate values of t_i, $i = 1, 2, \ldots, k - 1$, from $(x_2, x_2 + \delta]$ to repeat the argument of Case 1.

We conclude that the result holds for the case $k = m - 1$. This completes the proof of the theorem. $\square$

CHAPTER THREE

Uniqueness Implies Existence

3.1 Brief Overview

For the third order ordinary differential equation,

$$y''' = f(x, y, y', y''),$$

it is assumed that, for some $m \geq 4$, solutions of nonlocal boundary value problems satisfying

$$y(x_1) = y_1, \; y(x_2) = y_2,$$
$$y(x_m) - \sum_{i=3}^{m-1} y(x_i) = y_3,$$

for any $a < x_1 < x_2 < \cdots < x_m < b$ and $y_1, y_2, y_3 \in \mathbb{R}$, are unique when they exist. It is proved that, for all $3 \leq k \leq m$, solutions of nonlocal boundary value problems satisfying

$$y(x_1) = y_1, \; y(x_2) = y_2,$$
$$y(x_k) - \sum_{i=3}^{k-1} y(x_i) = y_3,$$

for any $a < x_1 < x_2 < \cdots < x_k < b$ and $y_1, y_2, y_3 \in \mathbb{R}$, do in fact exist.

3.1.1 History

The existence of solutions to differential equations has long been a topic of interest for applied mathematicians. In particular, many papers have been written that demonstrate the existence of solutions when one of the underlying assumptions is that, if there is a solution, it must be unique. For a few examples, please see papers by Davis and Henderson [11], Henderson [25, 26], Henderson and McGwier [29], Jackson [32], Klaasen [34, 35], and Lasota and Opial [39].

In this dissertation, the "shooting method" of obtaining solutions is used. That is, a fixed set of data is introduced, along with a family of solutions that all satisfy the set at some points. One then proceeds to show that at least one of the solutions hits a specified target point, whereby one accomplishes proving that all desirable points get hit by some solution. For examples of other work that employs the shooting method, the reader is invited to see papers by Jackson and Schrader [33] and Henderson, Karna, and Tisdell [28].

3.1.2 Prerequisite Results

In this section, we state a number of results that are fundamental to our uniqueness implies existence theorems. In particular, we include the statement of a continuous dependence of solutions upon initial conditions result, a crucial pre-compactness condition, a uniqueness implies existence theorem for conjugate boundary value problems, and a theorem on invariance of domain. Since each of these is in the existing literature, we state them here without proof.

Theorem 3.1 (Continuous Dependence of Solutions Upon Initial Conditions). *Let $g(t,x)$ be continuous on an open set $D \subseteq \mathbb{R} \times \mathbb{R}^n$, and assume that initial value problems for $x' = g(t,x)$ on D have unique solutions. Given any $(t_0, x_0) \in D$, let $x(t; t_0, x_0)$ denote the solution of*

$$x' = g(t,x)$$

$$x(t_0) = x_0$$

with maximal interval $(\alpha(t_0,x_0), \omega(t_0,x_0))$. Then for every $\varepsilon > 0$ and every compact $[a,b] \subseteq (\alpha(t_0,x_0), \omega(t_0,x_0))$, there exists a $\delta > 0$ such that $(t_1,x_1) \in D$, $|t_0 - t_1| < \delta$, and $\|x_1 - x_0\| < \delta$ imply that $[a,b] \subseteq (\alpha(t_1,x_1), \omega(t_1,x_1))$, the maximal interval of existence of the solution $x(t; t_1, x_1)$ of

$$x' = g(t,x)$$

$$x(t_1) = x_1,$$

and $\|x(t; t_1, x_1) - x(t; t_0, x_0)\| < \varepsilon$ *on* $[a, b]$.

The norm in the Theorem is the usual Euclidean norm on $\mathbb{R}^n$.

The next two theorems are due to Jackson and Schrader [33].

Theorem 3.2. *Assume the differential equation*

$$y''' = f(x, y, y', y'') \tag{3.1}$$

satisfies the following three conditions:

(A) $f : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous;*

(B) Solutions of initial value problems for (3.1) *are unique and exist on all of* (a, b)*; and,*

(C) Solutions of the boundary value problem for (3.1) *with boundary conditions*

$$y(x_1) = y_1, \ y(x_2) = y_2, \ y(x_3) = y_3, \tag{3.2}$$

for any $a < x_1 < x_2 < x_3 < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, are unique when they exist.*

Let $[c, d]$ *be a closed subinterval of* (a, b) *and let* $\{y_n(x)\}$ *be a sequence of solutions of* (3.1) *such that* $|y_n(x)| \le M$ *on* $[c, d]$ *for some* $M > 0$ *and all* $n \ge 1$. *Then* $\{y_n(x)\}$ *contains a subsequence* $\{y_{n_j}(x)\}$ *such that* $\{y_{n_j}^{(i)}(x)\}$ *converges uniformly on each compact subinterval of* (a, b) *for* $i = 0, 1, 2$.

Theorem 3.3 (Uniqueness Implies Existence). *Assume that* (3.1) *satisfies (A), (B), and (C) from the previous theorem. Then, given any* $a < x_1 < x_2 < x_3 < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, the boundary value problem* (3.1), (3.2) *has a solution.*

The last theorem in this section is due to L. E. J. Brouwer [30, 45].

Theorem 3.4 (Brouwer Invariance of Domain Theorem). *If* $\phi : G \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and one-to-one, and if* G *is an open set, then* $\phi(G)$ *is an open set and* ϕ *is a homeomorphism.*

3.2 Preliminary Result

Our first uniqueness implies existence result deals with the boundary value problem

$$y''' = f(x, y, y', y''), \tag{3.3}$$

$$y(x_1) = y_1,\ y(x_2) = y_2, \tag{3.4}$$

$$y(x_4) - y(x_3) = y_3. \tag{3.5}$$

We will assume the following conditions throughout this section:

(A) $f : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ is continuous;

(B) Solutions of initial value problems for (3.3) are unique and exist on all of (a, b); and,

(C) For any $a < x_1 < x_2 < x_3 < x_4 < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, solutions of the boundary value problem (3.3)–(3.5) are unique when they exist.

We begin by proving a continuous dependence theorem that will be needed in the proof of our uniqueness implies existence theorem.

Theorem 3.5 (Continuous Dependence). *Assume Conditions (A), (B), and (C) are satisfied. Let* $z(x)$ *be an arbitrary but fixed solution of* (3.3)*. Then given any* $a < x_1 < x_2 < x_3 < x_4 < b$*, any* c *and* d *with* $a < c < x_1$ *and* $x_4 < d < b$*, and any* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that* $|x_i - t_i| < \delta$*,* $i = 1, 2, 3, 4$*, and* $\max\{|z(x_1) - y_1|,\ |z(x_2) - y_2|,\ |z(x_4) - z(x_3) - y_3|\} < \delta$ *imply that* (3.3) *has a solution* $y(x)$ *satisfying*

$$y(t_1) = y_1,\ y(t_2) = y_2,\ y(t_4) - y(t_3) = y_3,$$

and $\left|z^{(i)}(x) - y^{(i)}(x)\right| < \varepsilon$ *on* $[c, d]$*, for* $i = 0, 1, 2$*.*

Proof: We will make use of the Brouwer Invariance of Domain Theorem in our proof. Let us define our open set G and our function ϕ.

Fix $t_0 \in (a,b)$. Define $G \subseteq \mathbb{R}^7$ by

$$G = \{(t_1, t_2, t_3, t_4, c_1, c_2, c_3) : a < t_1 < t_2 < t_3 < t_4 < b;\ c_1, c_2, c_3 \in \mathbb{R}\}.$$

G is open in $\mathbb{R}^7$. Now define $\phi : G \subseteq \mathbb{R}^7 \to \mathbb{R}^7$ by

$$\phi\big((t_1, t_2, t_3, t_4, c_1, c_2, c_3)\big) = \big(t_1, t_2, t_3, t_4, u(t_1), u(t_2), u(t_4) - u(t_3)\big)$$

where $u(x)$ is a solution of (3.3) satisfying

$$u(t_0) = c_1,\ u'(t_0) = c_2,\ u''(t_0) = c_3.$$

The function ϕ is continuous since solutions of (3.3) depend continuously upon initial conditions, by Theorem 3.1.

We claim ϕ is one-to-one. To see this, suppose

$$\phi\big((s_1, s_2, s_3, s_4, h_1, h_2, h_3)\big) = \phi\big((t_1, t_2, t_3, t_4, c_1, c_2, c_3)\big).$$

Clearly $s_i = t_i$, $i = 1, 2, 3, 4$. We have

$$w(t_0) = h_1,\ w'(t_0) = h_2,\ w''(t_0) = h_3,$$

and

$$u(t_0) = c_1,\ u'(t_0) = c_2,\ u''(t_0) = c_3,$$

for solutions $w(x)$ and $u(x)$ of (3.3). Then

$$u(t_1) = w(s_1) = w(t_1),\ u(t_2) = w(s_2) = w(t_2),$$

and

$$u(t_4) - u(t_3) = w(s_4) - w(s_3) = w(t_4) - w(t_3).$$

Then by condition (C) we have $w \equiv u$ on (a,b), which implies that $h_i = c_i$, $i = 1, 2, 3$, and ϕ is one-to-one. By the Brouwer Invariance of Domain Theorem, $\phi(G)$ is open and ϕ^{-1} is continuous on $\phi(G)$.

We will use the continuity of ϕ^{-1} to establish the result of our theorem. Let $z(x)$ be a solution of (3.3). Choose $a < x_1 < x_2 < x_3 < x_4 < b$, as well as c and d such that $a < c < x_1$, $x_4 < d < b$, and choose $\varepsilon > 0$. By continuity with respect to initial conditions, there exists $\eta > 0$ such that, for our fixed solution $z(x)$, $\left|z^{(i-1)}(t_0) - c_i\right| < \eta$, for $i = 1,2,3$, implies $\left|u^{(i-1)}(x) - z^{(i-1)}(x)\right| < \varepsilon$ on $[c,d]$, $i = 1,2,3$, where $u(x)$ is the solution of (3.3) with $u^{(i-1)}(t_0) = c_i$, $i = 1,2,3$. Now $\big(x_1, x_2, x_3, x_4, z(x_1), z(x_2), z(x_4) - z(x_3)\big) \in \phi(G)$, $\phi(G)$ is open, and $\phi^{-1} : \phi(G) \to G$ is continuous. Hence, there exists $\delta > 0$ such that $|x_i - t_i| < \delta$, for $i = 1,2,3,4$, and $\max\{|z(x_1) - y_1|,\ |z(x_2) - y_2|,\ |z(x_4) - z(x_3) - y_3|\} < \delta$ imply that $(t_1, t_2, t_3, t_4, y_1, y_2, y_3) \in \phi(G)$ and $\phi^{-1}\big((t_1, t_2, t_3, t_4, y_1, y_2, y_3)\big)$ is in the open ball of radius η centered at

$$\phi^{-1}\Big(\big(x_1, x_2, x_3, x_4, z(x_1), z(x_2), z(x_4) - z(x_3)\big)\Big) = \big(x_1, x_2, x_3, x_4, z(t_0), z'(t_0), z''(t_0)\big).$$

Say that

$$\phi^{-1}\big((t_1, t_2, t_3, t_4, y_1, y_2, y_3)\big) = (t_1, t_2, t_3, t_4, d_1, d_2, d_3).$$

If $u(x)$ is the solution of (3.3) satisfying

$$u(t_0) = d_1,\ u'(t_0) = d_2,\ u''(t_0) = d_3,$$

then $\left|u^{(i-1)}(x) - z^{(i-1)}(x)\right| < \varepsilon$ on $[c,d]$, $i = 1,2,3$. Moreover,

$$\begin{aligned}
(t_1, t_2, t_3, t_4, y_1, y_2, y_3) &= \phi\big((t_1, t_2, t_3, t_4, d_1, d_2, d_3)\big) \\
&= \phi\Big(\big(t_1, t_2, t_3, t_4, u(t_0), u'(t_0), u''(t_0)\big)\Big) \\
&= \big(t_1, t_2, t_3, t_4, u(t_1), u(t_2), u(t_4) - u(t_3)\big),
\end{aligned}$$

so that

$$u(t_1) = y_1,\ u(t_2) = y_2,\ u(t_4) - u(t_3) = y_3.$$

The proof of Theorem 3.5 is complete. $\square$

We now have the necessary machinery to prove a uniqueness implies existence theorem for the boundary value problem (3.3)–(3.5).

Theorem 3.6 (Uniqueness Implies Existence). *Assume Conditions (A), (B), and (C) hold. Then for any* $a < x_1 < x_2 < x_3 < x_4 < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, the boundary value problem* (3.3)–(3.5) *has a solution.*

Proof: Assume that (3.3) satisfies Conditions (A), (B), and (C). Choose $a < x_1 < x_2 < x_3 < x_4 < b$ and $y_1, y_2, y_3 \in \mathbb{R}$. Now, from Theorem 2.2, we also have the following condition satisfied:

(D) For any $a < x_1 < x_2 < x_3 < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, solutions of the boundary value problem

$$y''' = f(x, y, y', y''),$$

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3$$

are unique when they exist.

Therefore, Theorem 3.3 implies we have a unique solution, say $z(x)$, to the three point boundary value problem

$$y''' = f(x, y, y', y''),$$

$$y(x_2) = y_2,\ y(x_3) = 0,\ y(x_4) = y_3.$$

Now, define the set S by

$$S = \big\{u(x_1) : u \text{ is a solution of (3.3) with } u(x_2) = y_2,\ u(x_4) - u(x_3) = y_3\big\}.$$

Certainly $z(x_1) \in S$, so S is nonempty. We will show that S is both open and closed, and since $\mathbb{R}$ is connected, we will have shown that $S = \mathbb{R}$. Therefore we will have $y(x_1) = y_1 \in S$, and our proof will be complete.

S is open: Let $p_0 \in S$. Then $p_0 = u(x_1)$ for some solution $u(x)$ of (3.3) with $u(x_2) = y_2$ and $u(x_4) - u(x_3) = y_3$. By Theorem 3.5, there exists a $\delta > 0$ sufficiently small such that if $|p - p_0| < \delta$, then there is a solution $u_p(x)$ of (3.3) with

$$\begin{aligned} u_p(x_1) &= p, \\ u_p(x_2) &= u(x_2) = y_2, \\ u_p(x_4) - u_p(x_3) &= u(x_4) - u(x_3) = y_3. \end{aligned}$$

We have $p \in S$, whence $(p_0 - \delta, p_0 + \delta) \subseteq S$, so S is open.

S is closed: Assume S is not closed. Then there exists an r_0 which is a limit point of S, but r_0 is not in S. Then there exists an infinite sequence of distinct points $\{r_t\}_{t=1}^{\infty} \subset S$ such that $r_t \to r_0$. Without loss of generality we may assume $\{r_t\}_{t=1}^{\infty}$ is strictly monotone, say strictly monotone increasing. Since $\{r_t\}_{t=1}^{\infty} \subset S$, we have a sequence of functions $\{y_t(x)\}_{t=1}^{\infty}$ such that each $y_t(x)$ is a solution of (3.3) satisfying

$$\begin{aligned} y_t(x_1) &= r_t, \\ y_t(x_2) &= z(x_2) = y_2, \\ y_t(x_4) - y_t(x_3) &= z(x_4) - z(x_3) = y_3. \end{aligned}$$

By our uniqueness condition and our assumption that $\{r_t\}_{t=1}^{\infty}$ is strictly monotone increasing, it must be the case that $y_{t+1}(x) > y_t(x)$ on (a, x_2). Choose $\tau \in (a, x_1)$. By Theorem 3.3, there exists a solution $w(x)$ of (3.3) satisfying

$$w(\tau) = 0,\ w(x_1) = r_0,\ w(x_2) = y_2.$$

It follows from Theorem 3.2 that $\{y_t(x)\}_{t=1}^{\infty}$ cannot be uniformly bounded on any compact subinterval of (a, b). To see this, suppose there exist $[c, d] \subset (a, b)$ and $M > 0$ such that $\{|y_t(x)|\}_{t=1}^{\infty} < M$ on $[c, d]$. Then $\{y_t(x)\}_{t=1}^{\infty}$ contains a subsequence $\{y_{t_j}(x)\}_{j=1}^{\infty}$ such that $\{y_{t_j}(x)\}_{j=1}^{\infty}$ converges uniformly on $[x_1, x_4]$, say $y_{t_j}(x)_{j=1}^{\infty}$ converges uniformly to $y(x)$ on $[x_1, x_4]$. But $y_{t_j}(x_1) \to r_0$, $y_{t_j}(x_2) = y_2$, and $y_{t_j}(x_4) - y_{t_j}(x_3) = y_3$, so $y(x)$ would be a solution of (3.3) satisfying

$$y(x_1) = r_0,\ y(x_2) = y_2,\ y(x_4) - y(x_3) = y_3,$$

which contradicts our assumption that r_0 is not in S. Hence, there exists a positive integer T_1 such that $y_t(\tau) > w(\tau) = 0$ for all $t \geq T_1$. Likewise, there exists $\theta \in (x_1, x_2)$ and a positive integer T_2 such that $y_t(\theta) > w(\theta)$ for all $t \geq T_2$. Therefore, for some $T \geq \max\{T_1, T_2\}$, we have

$$y_T(\tau) > w(\tau),\ y_T(x_1) < w(x_1) = r_0,\ y_T(\theta) > w(\theta).$$

It follows that $w(x) - y_T(x)$ must have a zero on (τ, x_1), say $w(\alpha) = y_T(\alpha)$ for $\tau < \alpha < x_1$. Similarly, $w(x) - y_T(x)$ must have a zero on (x_1, θ), say $w(\beta) = y_T(\beta)$ for $x_1 < \beta < \theta$. We have then that

$$w(\alpha) = y_T(\alpha),\ w(\beta) = y_T(\beta),\ w(x_2) = y_T(x_2) = y_2.$$

Since both $w(x)$ and $y_T(x)$ are assumed to be solutions of (3.3), Condition (D) implies that $w(x) \equiv y_T(x)$ on (a, b). But then

$$r_0 = w(x_1) = y_T(x_1) = r_T < r_0,$$

which is a contradiction. We must conclude that S contains all of its limit points, and is therefore closed. □

3.3 General Case

It is possible to generalize the results of the previous section. We begin by proving a continuous dependence theorem that is analogous to Theorem 3.5.

Theorem 3.7 (Continuous Dependence). *Assume that the differential equation*

$$y'''(x) = f(x, y, y', y'') \tag{3.6}$$

satisfies the following three conditions:

(A) $f : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous;*

(B) Solutions of initial value problems for (3.6) *are unique and exist on all of* (a, b)*; and,*

(C) For some positive integer $m \geq 4$, any $a < x_1 < x_2 < \cdots < x_m < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, solutions of the boundary value problem for (3.6) with boundary conditions

$$y(x_1) = y_1,\ y(x_2) = y_2, \tag{3.7}$$

$$y(x_m) - \sum_{i=3}^{m-1} y(x_i) = y_3, \tag{3.8}$$

are unique when they exist.

Choose a positive integer k such that $3 \leq k \leq m$. Let $z(x)$ be an arbitrary but fixed solution of (3.6). Then for any $a < x_1 < x_2 < \cdots < x_k < b$, any c and d with $a < c < x_1$ and $x_k < d < b$, and any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x_i - t_i| < \delta$, $i = 1, 2, \ldots, k$, and $\max\left\{|z(x_1) - y_1|, |z(x_2) - y_2|, \left|z(x_k) - \sum_{i=3}^{k-1} z(x_i) - y_3\right|\right\} < \delta$ imply that (3.6) has a solution $y(x)$ with

$$y(t_1) = y_1,\ y(t_2) = y_2,\ y(t_k) - \sum_{i=3}^{k-1} y(t_i) = y_3,$$

and $\left|z^{(i)}(x) - y^{(i)}(x)\right| < \varepsilon$ on $[c, d]$, $i = 0, 1, 2$.

Proof: Assume (3.6) satisfies Conditions (A), (B), and (C) for some positive integer $m \geq 4$. Choose k such that $3 \leq k \leq m$. Fix $t_0 \in (a, b)$. Define the set $G \subset \mathbb{R}^{k+3}$ by

$$G = \{(t_1, t_2, \ldots, t_k, c_1, c_2, c_3 : a < t_1 < t_2 < \cdots < t_k < b;\ c_1, c_2, c_3 \in \mathbb{R}\}.$$

G is open in $\mathbb{R}^{k+3}$. Now define $\phi : G \subset \mathbb{R}^{k+3} \to \mathbb{R}^{k+3}$ by

$$\phi\big((t_1, t_2, \ldots, t_k, c_1, c_2, c_3)\big) = \left(t_1, t_2, \ldots, t_k, u(t_1), u(t_2), u(t_k) - \sum_{i=3}^{k-1} u(t_i)\right),$$

where $u(x)$ is the solution of (3.6) satisfying

$$u(t_0) = c_1,\ u'(t_0) = c_2,\ u''(t_0) = c_3.$$

The function ϕ is continuous since solutions of (3.6) depend continuously upon initial conditions, by Theorem 3.1. We claim that ϕ is one-to-one. To see this, suppose that

$$\phi\big((s_1, s_2, \ldots, s_k, h_1, h_2, h_3)\big) = \phi\big((t_1, t_2, \ldots, t_k, c_1, c_2, c_3)\big).$$

It is clear from the definition of ϕ that $s_i = t_i$, $i = 1, 2, \ldots, k$. We have

$$w(t_0) = h_1,\ w'(t_0) = h_2,\ w''(t_0) = h_3,$$

and

$$u(t_0) = c_1,\ u'(t_0) = c_2,\ u''(t_0) = c_3,$$

for solutions $w(x)$ and $u(x)$ of(3.6). Then

$$u(t_1) = w(s_1) = w(t_1),\ u(t_2) = w(s_2) = w(t_2),$$

and

$$u(t_k) - \sum_{i=3}^{k-1} u(t_i) = w(s_k) - \sum_{i=3}^{k-1} w(s_i) = w(t_k) - \sum_{i=3}^{k-1} w(t_i).$$

Then by Theorem 2.2 and Condition (C), we have $w \equiv u$ on (a, b), which implies that $h_i = c_i$, for $i = 1, 2, 3$. Thus ϕ is one-to-one. By the Brouwer Invariance of Domain Theorem, $\phi(G)$ is open and ϕ^{-1} is continuous on $\phi(G)$.

Now, let $z(x)$ be a solution of (3.6). Choose $a < x_1 < x_2 < \cdots < x_k < b$, any c and d with $a < c < x_1$ and $x_k < d < b$, and any $\varepsilon > 0$. By continuous dependence upon initial conditions, there exists an $\eta > 0$ such that, for our fixed solution $z(x)$, $\left|z^{(i-1)}(t_0) - c_i\right| < \eta$, for $i = 1, 2, 3$, implies $\left|u^{(i-1)}(x) - z^{(i-1)}(x)\right| < \varepsilon$ on $[c, d]$, $i = 1, 2, 3$, where $u(x)$ is the solution of (3.6) with $u^{(i-1)}(t_0) = c_i$, $i = 1, 2, 3$. We have

$$\left(x_1, x_2, \ldots, x_k, z(x_1), z(x_2), z(x_k) - \sum_{i=3}^{k-1} z(x_i)\right) \in \phi(G),$$

$\phi(G)$ is open, and $\phi^{-1} : \phi(G) \to G$ is continuous. Therefore, there exists a $\delta > 0$ such that $|x_i - t_i| < \delta$, for $i = 1, 2, \ldots, k$, and

$$\max\left\{|z(x_1) - y_1|,\ |z(x_2) - y_2|,\ \left|z(x_k) - \sum_{i=3}^{k-1} z(x_i) - y_3\right|\right\} < \delta$$

imply that $(t_1, t_2, \ldots, t_k, y_1, y_2, y_3) \in \phi(G)$ and $\phi^{-1}\big((t_1, t_2, \ldots, t_k, y_1, y_2, y_3)\big)$ is in the open ball of radius η centered at

$$\phi^{-1}\left(x_1, x_2, \ldots, x_k, z(x_1), z(x_2), z(x_k) - \sum_{i=3}^{k-1} z(x_i)\right)$$

$$= (x_1, x_2, \ldots, x_k, z(t_0), z'(t_0), z''(t_0)) .$$

Suppose that

$$\phi^{-1}\big((t_1, t_2, \ldots, t_k, y_1, y_2, y_3)\big) = (t_1, t_2, \ldots, t_k, d_1, d_2, d_3).$$

If $u(x)$ is the solution of (3.6) satisfying

$$u(t_0) = d_1,\ u'(t_0) = d_2,\ u''(t_0) = d_3,$$

then $\left|u^{(i-1)}(x) - z^{(i-1)}(x)\right| < \varepsilon$ on $[c, d]$, $i = 1, 2, 3$. Moreover,

$$\begin{aligned}(t_1, t_2, \ldots, t_k, y_1, y_2, y_3) &= \phi\left((t_1, t_2, \ldots, t_k, d_1, d_2, d_3)\right)\\ &= \phi\left((t_1, t_2, \ldots, t_k, u(t_0), u'(t_0), u''(t_0))\right)\\ &= \left(t_1, t_2, \ldots, t_k, u(t_1), u(t_2), u(t_k) - \sum_{i=3}^{k-1} u(t_i)\right),\end{aligned}$$

so that $u(x)$ is the solution of (3.6) satisfying

$$u(t_1) = y_1,\ u(t_2) = y_2,\ u(t_k) - \sum_{i=3}^{k-1} u(t_i) = y_3.$$

□

Everything is in place to prove our last theorem of this chapter, a uniqueness implies existence result. This theorem is a generalization of Theorem 3.6.

Theorem 3.8 (Uniqueness Implies Existence). *Assume that the differential equation* (3.6) *satisfies Conditions (A), (B), and (C) of Theorem 3.7. Then for any integer* $3 \le k \le m$*, any* $a < x_1 < x_2 < \cdots < x_k < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, the boundary value problem for* (3.6) *with boundary conditions* (3.7) *and*

$$y(x_k) - \sum_{i=3}^{k-1} y(x_i) = y_3 \tag{3.9}$$

has a solution. The boundary condition in (3.9) *is interpreted as* $y(x_3) = y_3$ *in the case* $k = 3$.

Proof: The proof is by induction on m. Note that Theorem 3.6 gives us our result for the case $m = 4$. Let $m \geq 4$ be given. For inductive purposes, let k such that $4 \leq k \leq m$ be given, and assume that, for all $4 \leq h \leq k-1$, any $a < x_1 < x_2 < \cdots < x_h < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, there exists a solution for (3.6) satisfying (3.7) and

$$y(x_h) - \sum_{i=3}^{h-1} y(x_i) = y_3. \tag{3.10}$$

Let $y_1, y_2, y_3 \in \mathbb{R}$ be chosen. Then there is a unique solution, say $z(x)$, to the boundary value problem for (3.6) such that

$$\begin{aligned} z(x_2) &= y_2, \\ z(x_3) &= 0, \\ z(x_k) - \sum_{i=4}^{k-1} z(x_i) &= y_3. \end{aligned}$$

We have that $z(x)$ satisfies the boundary conditions

$$y(x_2) = y_2, \tag{3.11}$$

$$y(x_k) - \sum_{i=3}^{k-1} y(x_i) = y_3. \tag{3.12}$$

Define the set S by

$$S = \{u(x_1) : u(x) \text{ is a solution of (3.6) satisfying (3.11)} - \text{(3.12)}\}.$$

We will show that S contains y_1.

Certainly $z(x_1) \in S$, so S is nonempty. Since $\mathbb{R}$ is a connected set, the proof of the Theorem will be complete when we show that S is both open and closed, whereby $S = \mathbb{R}$.

S is open: Suppose $p_0 \in S$. Then $p_0 = u(x_1)$ for some solution $u(x)$ of (3.6) with $u(x_2) = y_2$ and $u(x_k) - \sum_{i=3}^{k-1} u(x_i) = y_3$. By Theorem 3.7, there exists a $\delta > 0$ sufficiently small such that, if $|p - p_0| < \delta$, then there is a solution $u_p(x)$ of (3.6)

satisfying

$$\begin{aligned}
u_p(x_1) &= p, \\
u_p(x_2) &= u(x_2) = y_2, \\
u_p(x_k) - \sum_{i=3}^{k-1} u_p(x_i) &= u(x_k) - \sum_{i=3}^{k-1} u(x_i) = y_3.
\end{aligned}$$

That is, $p \in S$, whence $(p_0 - \delta, p_0 + \delta) \subset S$, so S is open.

S is closed: Assume to the contrary that S is not closed. Then S does not contain some of its limit points, so there exists an r_0 that is a limit point of S, but r_0 is not in S. Therefore, there is an infinite sequence of distinct points $\{r_t\}_{t=1}^{\infty} \subset S$ such that $r_t \to r_0$. Without loss of generality, assume $\{r_t\}_{t=1}^{\infty}$ is strictly monotone, say strictly monotone increasing. Since $\{r_t\}_{t=1}^{\infty} \subset S$, we have a sequence of functions $\{y_t(x)\}_{t=1}^{\infty}$ such that each $y_t(x)$ is a solution of (3.6) satisfying

$$\begin{aligned}
y_t(x_1) &= r_t, \\
y_t(x_2) &= z(x_2) = y_2, \\
y_t(x_k) - \sum_{i=3}^{k-1} y_t(x_i) &= z(x_k) - \sum_{i=3}^{k-1} z(x_i) = y_3.
\end{aligned}$$

Choose $\tau \in (a, x_1)$. By Theorem 2.2 and Theorem 3.3, there exists a solution $w(x)$ of (3.3) satisfying

$$w(\tau) = 0,\ w(x_1) = r_0,\ w(x_2) = y_2.$$

By our uniqueness condition and our assumption that $\{r_t\}_{t=1}^{\infty}$ is strictly monotone increasing, it must be the case that $y_{t+1}(x) > y_t(x)$ on (a, x_2) for all $t \in \mathbb{N}$. It follows from Theorem 3.2 and our assumption that r_0 is not in S that $\{y_t(x)\}_{t=1}^{\infty}$ cannot be uniformly bounded on any compact subinterval of (a, b). (See the argument in the proof of Theorem 3.6.) Hence, there exists a positive integer T_1 such that $y_t(\tau) > w(\tau) = 0$ for all $t \geq T_1$. Likewise, there exists $\theta \in (x_1, x_2)$ and a positive integer T_2 such that $y_t(\theta) > w(\theta)$ for all $t \geq T_2$. Therefore, for some $T \geq \max\{T_1, T_2\}$, we have

$$y_T(\tau) > w(\tau),\ \ y_T(x_1) < w(x_1) = r_0,\ \ y_T(\theta) > w(\theta).$$

It follows that $w(x) - y_T(x)$ must have a zero on (τ, x_1), say $w(\alpha) = y_T(\alpha)$, $\tau < \alpha < x_1$. Similarly, $w(x) - y_T(x)$ must have a zero on (x_1, θ), say $w(\beta) = y_T(\beta)$, $x_1 < \beta < \theta$. We have then that

$$\begin{aligned} w(\alpha) &= y_T(\alpha), \\ w(\beta) &= y_T(\beta), \\ w(x_2) &= y_T(x_2) = y_2. \end{aligned}$$

Since both $w(x)$ and $y_T(x)$ are assumed to be solutions of (3.6), Theorem 2.2 and Condition (C) imply that $w \equiv y_T$ on (a, b). Then we have

$$r_0 = w(x_1) = y_T(x_1) = r_T < r_0,$$

a contradiction. We must conclude that S contains all of its limit points and is therefore closed. $\square$

CHAPTER FOUR

Advanced Results

In the final chapter of this dissertation, we will build on our earlier results by the addition of a left side nonlocal boundary condition similar in form to the right side nonlocal boundary condition of our preceding work. As before, we will proceed with a uniqueness implies uniqueness theorem, a continuous dependence theorem, and finally, a uniqueness implies existence theorem.

4.1 Uniqueness Implies Uniqueness

Theorem 4.1 (Uniqueness Implies Uniqueness). *Suppose the following three conditions hold for the differential equation*

$$y''' = f(x, y, y', y''). \tag{4.1}$$

(A) $f : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous;*

(B) Solutions of initial value problems for (4.1) *are unique and exist on all of* (a, b)*; and,*

(C) For some $m, n \in \mathbb{N}$ *with* $n > 1$*, any* $a < x_1 < x_2 < \cdots < x_{m+n} < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, solutions of the boundary value problem for* (4.1) *with boundary conditions*

$$y(x_1) - \sum_{i=2}^{n-1} y(x_i) = y_1, \tag{4.2}$$

$$y(x_n) = y_2, \tag{4.3}$$

$$y(x_{m+n}) - \sum_{j=n+1}^{m+n-1} y(x_j) = y_3, \tag{4.4}$$

are unique when they exist. We take the boundary condition (4.2) *to mean* $y(x_1) = y_1$ *in the case that* $n = 2$*, and the boundary condition* (4.4) *is taken to be* $y(x_3) = y_3$ *in the case that* $m = 1$*.*

Then for any integers p and q such that $1 \le p \le m$, $1 < q \le n$, solutions for the boundary value problem for (4.1) *with boundary conditions*

$$\begin{aligned} y(x_1) - \sum_{i=2}^{q-1} y(x_i) &= y_1, & (4.5)\\ y(x_q) &= y_2, & (4.6)\\ y(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y(x_j) &= y_3, & (4.7) \end{aligned}$$

where the boundary condition (4.5) *is taken to mean $y(x_1) = y_1$ in the case that $q = 2$, and the boundary condition* (4.7) *is taken to be $y(x_3) = y_3$ in the case that $p = 1$, are unique when they exist.*

Proof: There are three nontrivial cases: $p = m$ and $1 < q < n$, $1 \le p < m$ and $q = n$, and $1 \le p < m$ and $1 < q < n$. We will prove each case by induction.

Proof of Case 1: $p = m$ and $1 < q < n$. Assume Conditions (A) and (B) hold, and that Condition (C) holds for some positive integer m and for $n = 3$. Set $p = m$, $q = 2$. We are assuming the uniqueness of solutions of the boundary value problem for (4.1) with boundary conditions

$$\begin{aligned} y(x_1) - y(x_2) &= y_1,\\ y(x_3) &= y_2,\\ y(x_{p+3}) - \sum_{j=4}^{p+2} y(x_j) &= y_3, \end{aligned}$$

for any $a < x_1 < x_2 < \cdots < x_{p+3} < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$. Suppose that, for some $a < x_1 < x_2 < \cdots < x_{p+2} < b$ and some $y_1, y_2, y_3 \in \mathbb{R}$, there are distinct solutions $u(x)$ and $v(x)$ of the boundary value problem for (4.1) with boundary conditions

$$\begin{aligned} y(x_1) &= y_1,\\ y(x_2) &= y_2,\\ y(x_{p+2}) - \sum_{j=3}^{p+1} y(x_j) &= y_3. \end{aligned}$$

Set $w(x) = u(x) - v(x)$. Since $u(x)$ and $v(x)$ both solve the boundary value problem, we have that

$$w(x_1) = w(x_2) = w(x_{p+2}) - \sum_{j=3}^{p+1} w(x_j) = 0.$$

By Condition (B), we know that either $w'(x_1) \neq 0$ or $w''(x_1) \neq 0$. Suppose $w'(x_1) \neq 0$. Without loss of generality, assume $w'(x_1) > 0$. Therefore, since $w(x_1) = w(x_2) = 0$, it must be the case that $w(x)$ is positive on (x_1, x_2) and has a local maximum on (x_1, x_2). Suppose the local maximum occurs at $x = \alpha \in (x_1, x_2)$. Now, there must exist $\beta \in (x_1, \alpha)$ and $\gamma \in (\alpha, x_2)$ such that $w(\beta) = w(\gamma)$. This gives us $a < \beta < \gamma < x_2 < \cdots < x_{m+2}$ with

$$w(\beta) - w(\gamma) = w(x_2) = w(x_{p+2}) - \sum_{j=3}^{p+1} w(x_j) = 0.$$

In other words, we have

$$u(\beta) - u(\gamma) = v(\beta) - v(\gamma),$$

$$u(x_2) = v(x_2),$$

$$u(x_{p+2}) - \sum_{j=3}^{p+1} u(x_j) = v(x_{p+2}) - \sum_{j=3}^{p+1} v(x_j).$$

Our uniqueness condition gives us that $u(x) \equiv v(x)$ on (a, b), but this contradicts our assumption that $u(x)$ and $v(x)$ are distinct.

If we assume $w'(x_1) = 0$, then it must be true that $w''(x_1) \neq 0$. Assume, without loss of generality, that $w''(x_1) > 0$. Thus $w'(x)$ is increasing on a right-neighborhood of x_1, and in fact $w'(x)$ is positive on such a neighborhood since $w'(x_1) = 0$. Therefore, since $w(x_1) = w(x_2) = 0$ and by our uniqueness assumption, we have $w(x) > 0$ on (x_1, x_2), and $w(x)$ obtains a local maximum on (x_1, x_2). We may repeat our argument for the assumption $w'(x_1) \neq 0$. We conclude the result holds for $p = m$, $n = 3$, $q = 2$.

Now, assume that Conditions (A) and (B) hold, and that Condition (C) holds for m and some positive integer $n > 3$. That is, we are assuming uniqueness for solutions of (4.1) satisfying boundary conditions

$$y(x_1) - \sum_{i=2}^{n-1} y(x_i) = y_1, \tag{4.8}$$

$$y(x_n) = y_2, \tag{4.9}$$

$$y(x_{m+n}) - \sum_{j=n+1}^{m+n-1} y(x_j) = y_3, \tag{4.10}$$

for any $a < x_1 < x_2 < \cdots < x_{m+n} < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$. Set $p = m$, $q = n - 1$. Suppose that, for some $a < x_1 < x_2 < \cdots < x_{p+q}$ and $y_1, y_2, y_3 \in \mathbb{R}$, we have distinct solutions $u(x)$ and $v(x)$ of the boundary value problem for (4.1) satisfying boundary conditions

$$y(x_1) - \sum_{i=2}^{q-1} y(x_i) = y_1, \tag{4.11}$$

$$y(x_q) = y_2, \tag{4.12}$$

$$y(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y(x_j) = y_3. \tag{4.13}$$

Set $w(x) = u(x) - v(x)$. We have $u(x_1) - \sum_{i=2}^{q-1} u(x_i) = v(x_1) - \sum_{i=2}^{q-1} v(x_i)$, so that $w(x_1) - \sum_{i=2}^{q-1} w(x_i) = 0$. Likewise we have $u(x_q) = v(x_q)$, so that $w(x_q) = 0$, and $u(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u(x_j) = v(x_{p+q}) - \sum_{j=q+1}^{p+q-1} v(x_j)$, so that $w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) = 0$.

Claim: It cannot be the case that $w(x_i) = 0$ *for every* i *among* $\{2,3,\ldots,q\text{-}1\}$. To establish the claim, assume to the contrary that $w(x_i) = 0$ for all $i \in \{2, 3, \ldots, q-1\}$. Then $w(x_1) = 0$, since $w(x_1) - \sum_{i=2}^{q-1} w(x_i) = 0$. Now, since $w(x_1) = w(x_2) = 0$ and $w(x) = u(x) - v(x)$ is continuous, we may choose $\alpha, \beta \in (x_1, x_2)$ with $\alpha < \beta$ and $w(\alpha) = w(\beta)$, or equivalently, $w(\alpha) - w(\beta) = w(x_1) = 0$. Then we have

$$\begin{aligned} 0 &= w(x_1) - \sum_{i=2}^{q-1} w(x_i) \\ &= w(\alpha) - w(\beta) - \sum_{i=2}^{q-1} w(x_i) \\ &= w(\alpha) - \left(w(\beta) + \sum_{i=2}^{q-1} w(x_i) \right), \end{aligned}$$

so that $u(x)$ and $v(x)$ satisfy

$$\begin{aligned} u(\alpha)-\left(u(\beta)+\sum_{i=2}^{q-1}u(x_i)\right) &= v(\alpha)-\left(v(\beta)+\sum_{i=2}^{q-1}v(x_i)\right), \\ u(x_q) &= v(x_q), \\ u(x_{p+q})-\sum_{j=q+1}^{p+q-1}u(x_j) &= v(x_{p+q})-\sum_{j=q+1}^{p+q-1}v(x_j). \end{aligned}$$

But then, Condition (C) implies that $u(x) \equiv v(x)$ on (a,b), a contradiction of our assumption that $u(x)$ and $v(x)$ are distinct. The claim is established.

We have that $w(x_i) \neq 0$, for some $i \in \{2, \ldots, q-1\}$. Without loss of generality, assume that $w(x_{q-1}) \neq 0$. Then since $w(x_q) = 0$, and $w(x) = u(x) - v(x)$ is a continuous function, we may choose $\alpha, \beta \in (x_{q-1}, x_q)$ such that $\alpha < \beta$ and $w(\alpha) + w(\beta) = w(x_{q-1})$. Therefore, we have

$$w(x_1)-\sum_{i=2}^{q-1}w(x_i) = w(x_1)-\left(\sum_{i=2}^{q-2}w(x_i)+w(\alpha)+w(\beta)\right) = 0,$$

whence $u(x)$ and $v(x)$ satisfy

$$\begin{aligned} u(x_1)-\left(\sum_{i=2}^{q-2}u(x_i)+u(\alpha)+u(\beta)\right) &= v(x_1)-\left(\sum_{i=2}^{q-2}v(x_i)+v(\alpha)+v(\beta)\right), \\ u(x_q) &= v(x_q), \\ u(x_{p+q})-\sum_{j=q+1}^{p+q-1}u(x_j) &= v(x_{p+q})-\sum_{j=q+1}^{p+q-1}v(x_j). \end{aligned}$$

Thus $u(x) \equiv v(x)$, by Condition (C). This contradicts our assumption that $u(x)$ and $v(x)$ are distinct. The proof of Case 1 is complete.

Proof of Case 2: $1 \leq p < m$ and $q = n$. We begin by proving the case $m = 2$, $p = 1$, and $q = n \geq 2$ is some positive integer. That is, we are assuming Conditions (A) and (B), and that, for any $a < x_1 < x_2 < \cdots < x_{n+2} < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, solutions of the boundary value problem for (4.1) satisfying boundary conditions

$$y(x_1)-\sum_{i=2}^{n-1}y(x_i) = y_1, \tag{4.14}$$

$$y(x_n) = y_2, \tag{4.15}$$

$$y(x_{n+2})-y(x_{n+1}) = y_3, \tag{4.16}$$

are unique, when they exist.

Suppose that, for some $a < x_1 < x_2 < \cdots < x_{n+1} < b$ and some $y_1, y_2, y_3 \in \mathbb{R}$, we have distinct solutions $u(x)$ and $v(x)$ of the boundary value problem for (4.1) satisfying boundary conditions

$$\begin{aligned} y(x_1) - \sum_{i=2}^{n-1} y(x_i) &= y_1, \\ y(x_n) &= y_2, \\ y(x_{n+1}) &= y_3. \end{aligned}$$

Set $w(x) = u(x) - v(x)$. We have $u(x_1) - \sum_{i=2}^{n-1} u(x_i) = v(x_1) - \sum_{i=2}^{n-1} v(x_i)$, or $w(x_1) - \sum_{i=2}^{n-1} w(x_i) = 0$, and $u(x_n) = v(x_n)$, so that $w(x_n) = 0$, and finally, $u(x_{n+1}) = v(x_{n+1})$, or $w(x_{n+1}) = 0$. It is the case that $w(x)$ cannot have a zero on (x_n, x_{n+1}), else we would get an immediate contradiction to our assumption that $u(x)$ and $v(x)$ are distinct, by our uniqueness condition. Therefore, since $w(x_n) = 0$, $w(x_{n+1}) = 0$, and $w(x) = u(x) - v(x)$ is a continuous function, it must be the case that $w(x)$ has a local extremum on (x_n, x_{n+1}). Then we may choose $\alpha, \beta \in (x_n, x_{n+1})$ such that $w(\alpha) = w(\beta)$. We have

$$\begin{aligned} w(\alpha) &= w(\beta), \\ u(\alpha) - v(\alpha) &= u(\beta) - v(\beta), \\ u(\alpha) - u(\beta) &= v(\alpha) - v(\beta). \end{aligned}$$

Therefore, $u(x)$ and $v(x)$ are solutions of (4.1) satisfying

$$\begin{aligned} u(x_1) - \sum_{i=2}^{n-1} u(x_i) &= v(x_1) - \sum_{i=2}^{n-1} v(x_i), \\ u(x_n) &= v(x_n), \\ u(\alpha) - u(\beta) &= v(\alpha) - v(\beta). \end{aligned}$$

Then our uniqueness condition implies $u(x) \equiv v(x)$ on (a, b), which contradicts our assumption that $u(x)$ and $v(x)$ are distinct. Case 2 holds if $m = 2$, $p = 1$, and $q = n \geq 2$ is some positive integer.

It remains to prove the case $p = m-1$ and $q = n$. Suppose that Conditions (A), (B), and (C) hold, and let $p = m-1$ and $q = n$. Additionally, for some $a < x_1 < x_2 < \cdots < x_{p+q} < b$ and some $y_1, y_2, y_3 \in \mathbb{R}$, assume that $u(x)$ and $v(x)$ are distinct solutions of the boundary value problem for (4.1) satisfying

$$y(x_1) - \sum_{i=2}^{q-1} y(x_i) = y_1, \tag{4.17}$$

$$y(x_q) = y_2, \tag{4.18}$$

$$y(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y(x_j) = y_3. \tag{4.19}$$

Set $w(x) = u(x) - v(x)$. We have $u(x_1) - \sum_{i=2}^{q-1} u(x_i) = v(x_1) - \sum_{i=2}^{q-1} v(x_i)$, or $w(x_1) - \sum_{i=2}^{q-1} w(x_i) = 0$. Likewise, we have $u(x_q) = v(x_q)$, or $w(x_q) = 0$, and finally, $u(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u(x_j) = v(x_{p+q}) - \sum_{j=q+1}^{p+q-1} v(x_j)$, or $w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) = 0$.

Claim: It cannot be the case that $w(x_j) = 0$ for each $j \in \{q+1, \ldots, p+q-1\}$. To establish the claim, assume to the contrary that $w(x_j) = 0$ for every $j \in \{q+1, \ldots, p+q-1\}$. Then $w(x_{p+q}) = 0$, since $w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) = 0$. Now, since $w(x_{p+q}) = w(x_{p+q-1} = 0$, and $w(x) = u(x) - v(x)$ is continuous, there exist $\alpha, \beta \in (x_{p+q-1}, x_{p+q})$ such that $\alpha < \beta$ and $w(\alpha) = w(\beta)$, or equivalently, $w(\beta) - w(\alpha) = w(x_{p+q}) = 0$. We have

$$\begin{aligned} 0 &= w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) \\ &= w(\beta) - w(\alpha) - \sum_{j=q+1}^{p+q-1} w(x_j) \\ &= w(\beta) - \left(\sum_{j=q+1}^{p+q-1} w(x_j) + w(\alpha) \right), \end{aligned}$$

so that, for $a < x_1 < x_2 < \cdots < x_{p+q-1} < \alpha < \beta < b$, $u(x)$ and $v(x)$ satisfy

$$\begin{aligned} u(x_1) - \sum_{i=2}^{q-1} u(x_i) &= v(x_1) - \sum_{i=2}^{q-1} v(x_i), \\ u(x_q) &= v(x_q), \\ u(\beta) - \left(\sum_{j=q+1}^{p+q-1} u(x_j) + u(\alpha) \right) &= v(\beta) - \left(\sum_{j=q+1}^{p+q-1} v(x_j) + v(\alpha) \right). \end{aligned}$$

Condition (C) implies that $u(x) \equiv v(x)$ on (a, b), but this contradicts our assumption that $u(x)$ and $v(x)$ are distinct. The claim is true.

By our claim, we have that $w(x_j) \neq 0$ for some $i \in \{q+1, \ldots, p+q-1\}$. Without loss of generality, assume $w(x_{q+1}) \neq 0$. Then, since $w(x_q) = 0$, and $w(x) = u(x) - v(x)$ is continuous, we may choose $\alpha, \beta \in (x_q, x_{q+1})$ with $\alpha < \beta$ such that $w(x_{q+1}) = w(\alpha) + w(\beta)$. Then we have

$$w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) = w(x_{p+q}) - \left(w(\alpha) + w(\beta) + \sum_{j=q+2}^{p+q-1} w(x_j) \right) = 0,$$

whence $u(x)$ and $v(x)$ satisfy

$$\begin{aligned} u(x_1) - \sum_{i=2}^{q-1} u(x_i) &= v(x_1) - \sum_{i=2}^{q-1} v(x_i), \\ u(x_q) &= v(x_q), \\ u(x_{p+q}) - \left(u(\alpha) + u(\beta) + \sum_{j=q+2}^{p+q-1} u(x_j) \right) &= v(x_{p+q}) - \left(v(\alpha) + v(\beta) + \sum_{j=q+2}^{p+q-1} v(x_j) \right), \end{aligned}$$

for $a < x_1 < x_2 < \cdots < x_q < \alpha < \beta < x_{q+2} < \cdots < x_{p+q} < b$. We see that $u(x) \equiv v(x)$ by Condition (C), contradicting our assumption that $u(x) \neq v(x)$. This completes the proof of Case 2.

Proof of Case 3: $1 \leq p < m$ *and* $1 < q < n$. Assume that Conditions (A) and (B) hold, and that Condition (C) holds for $m = 2$ and $n = 3$. That is, for any $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, we are assuming uniqueness of solutions of the boundary value problem for (4.1) that satisfy the boundary conditions

$$\begin{aligned} y(x_1) - y(x_2) &= y_1, \\ y(x_3) &= y_2, \\ y(x_5) - y(x_4) &= y_3. \end{aligned}$$

Now, for some $a < x_1 < x_2 < x_3 < b$ and some $y_1, y_2, y_3 \in \mathbb{R}$, suppose that $u(x)$ and $v(x)$ are distinct solutions of the boundary value problem for (4.1) that both satisfy the boundary conditions

$$y(x_1) = y_1, \;\; y(x_2) = y_2, \;\; y(x_3) = y_3.$$

Set $w(x) = u(x) - v(x)$. Then we have $u(x_i) = v(x_i)$, for $i = 1, 2, 3$, so that $w(x_i) = 0$, for $i = 1, 2, 3$. Therefore, since $w(x_i) = 0$, for $i = 1, 2, 3$, and since $w(x) = u(x) - v(x)$ is continuous, we may choose $\alpha, \beta \in (x_1, x_2)$ such that $\alpha < \beta$ and $w(\alpha) = w(\beta)$, and we may choose $\eta, \theta \in (x_2, x_3)$ such that $\eta < \theta$ and $w(\eta) = w(\theta)$. Thus we have

$$\begin{aligned} w(\alpha) &= w(\beta), \\ u(\alpha) - v(\alpha) &= u(\beta) - v(\beta), \\ u(\alpha) - u(\beta) &= v(\alpha) - v(\beta), \end{aligned}$$

and

$$\begin{aligned} w(\eta) &= w(\theta), \\ u(\eta) - v(\eta) &= u(\theta) - v(\theta), \\ u(\eta) - u(\theta) &= v(\eta) - v(\theta), \\ u(\theta) - u(\eta) &= v(\theta) - v(\eta). \end{aligned}$$

Therefore, $u(x)$ and $v(x)$ are solutions of the boundary value problem for (4.1) that satisfy

$$u(\alpha) - u(\beta) = v(\alpha) - v(\beta),$$

$$u(x_2) = v(x_2),$$

$$u(\theta) - u(\eta) = v(\theta) - v(\eta),$$

with $a < \alpha < \beta < x_2 < \eta < \theta < b$. Our uniqueness condition says that $u(x) \equiv v(x)$ on (a, b), which is a contradiction of our assumption that $u(x)$ and $v(x)$ are distinct. We conclude that Case 3 holds for $m = 2$ and $n = 3$.

Suppose that Conditions (A), (B), and (C) hold for positive integers $m > 2$ and $n > 3$. Set $p = m - 1$ and $q = n - 1$. Assume that, for some $a < x_1 < x_2 < \cdots < x_{p+q} < b$ and some $y_1, y_2, y_3 \in \mathbb{R}$, there exist distinct solutions $u(x)$ and $v(x)$ of the boundary value problem for (4.1) that satisfy the boundary conditions of (4.5)-(4.7).

Set $w(x) = u(x) - v(x)$. We have then that

$$u(x_1) - \sum_{i=2}^{q-1} u(x_i) = v(x_1) - \sum_{i=2}^{q-1} v(x_i),$$
$$u(x_q) = v(x_q),$$
$$u(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u(x_j) = v(x_{p+q}) - \sum_{j=q+1}^{p+q-1} v(x_j),$$

or equivalently,

$$\begin{aligned} w(x_1) - \sum_{i=2}^{q-1} w(x_i) &= 0, \\ w(x_q) &= 0, \\ w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) &= 0. \end{aligned}$$

From arguments in the proofs of Case 1 and Case 2, we know that our assumption that $u(x)$ and $v(x)$ are distinct implies that it cannot be the case that $w(x_i) = 0$ for all $i \in \{1, 2, \ldots, q-1\}$, nor can $w(x_j) = 0$ for all $j \in \{q+1, \ldots, p+q\}$. Assume, without loss of generality, that $w(x_{q-1}) \neq 0$ and $w(x_{q+1}) \neq 0$. Thus, since $w(x_{q-1}) \neq 0$ and $w(x_{q+1}) \neq 0$, $w(x_q) = 0$, and $w(x) = u(x) - v(x)$ is a continuous function, we may choose $\alpha, \beta \in (x_{q-1}, x_q)$ such that $\alpha < \beta$ and $w(\alpha) + w(\beta) = w(x_{q-1})$, and we may choose $\eta, \theta \in (x_q, x_{q+1})$ such that $\eta < \theta$ and $w(\eta) + w(\theta) = w(x_{q+1})$. Hence we have

$$w(x_1) - \sum_{i=2}^{q-1} w(x_i) = w(x_1) - \left(\sum_{i=2}^{q-2} w(x_i) + w(\alpha) + w(\beta) \right) = 0,$$

and

$$w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) = w(x_{p+q}) - \left(w(\eta) + w(\theta) + \sum_{j=q+2}^{p+q-1} w(x_j) \right) = 0,$$

whence $u(x)$ and $v(x)$ satisfy

$$\begin{aligned} u(x_1) - \left(\sum_{i=2}^{q-2} u(x_i) + u(\alpha) + u(\beta) \right) &= v(x_1) - \left(\sum_{i=2}^{q-2} v(x_i) + v(\alpha) + v(\beta) \right), \\ u(x_q) &= v(x_q), \\ u(x_{p+q}) - \left(u(\eta) + u(\theta) + \sum_{j=q+2}^{p+q-1} u(x_j) \right) &= v(x_{p+q}) - \left(v(\eta) + v(\theta) + \sum_{j=q+2}^{p+q-1} v(x_j) \right), \end{aligned}$$

for $a < x_1 < x_2 < \cdots < x_{q-2} < \alpha < \beta < x_q < \eta < \theta < x_{q+2} < \cdots < x_{p+q} < b$. Then Condition (C) implies that $u(x) \equiv v(x)$ on (a, b), which contradicts our assumption that $u(x)$ and $v(x)$ are distinct. This completes the proof of Case 3, and thus the proof of Theorem 4.1 is complete. □

4.2 Continuous Dependence

Theorem 4.2 (Continuous Dependence). *For the differential equation*

$$y''' = f(x, y, y', y''), \tag{4.20}$$

suppose that the following three conditions hold.

(A) $f : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous;*

(B) Solutions of initial value problems for (4.20) *are unique and exist on all of* (a, b)*; and,*

(C) For some $m, n \in \mathbb{N}$ *with* $n > 1$*, any* $a < x_1 < x_2 < \cdots < x_{m+n} < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, solutions of the boundary value problem for* (4.20) *with boundary conditions*

$$\begin{aligned} y(x_1) - \sum_{i=2}^{n-1} y(x_i) &= y_1, & (4.21)\\ y(x_n) &= y_2, & (4.22)\\ y(x_{m+n}) - \sum_{j=n+1}^{m+n-1} y(x_j) &= y_3, & (4.23) \end{aligned}$$

are unique when they exist. We take the boundary condition (4.21) *to mean* $y(x_1) = y_1$ *in the case that* $n = 2$*, and the boundary condition* (4.23) *is taken to be* $y(x_3) = y_3$ *in the case that* $m = 1$*.*

Choose positive integers p *and* q *such that* $1 \le p \le m$ *and* $1 < q \le n$*. Let* $z(x)$ *be an arbitrary but fixed solution of* (4.20)*. Then for any* $a < x_1 < x_2 < \cdots < x_{p+q} < b$*,* *any* c *and* d *with* $a < c < x_1$ *and* $x_{p+q} < d < b$, *and given* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $|x_i - t_i| < \delta$, *for* $i = 1, 2, \ldots, p+q$, *and*

$$max\left\{\left|z(x_1) - \sum_{i=2}^{q-1} z(x_i) - y_1\right|, |z(x_q) - y_2|, \left|z(x_{p+q}) - \sum_{j=q+1}^{p+q-1} z(x_j) - y_3\right|\right\} < \delta$$

imply that (4.20) *has a solution* $y(x)$ *with*

$$\begin{aligned} y(t_1) - \sum_{i=2}^{q-1} y(x_i) &= y_1, \\ y(t_q) &= y_2, \\ y(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y(x_j) &= y_3, \end{aligned}$$

and $\left|z^{(k)}(x) - y^{(k)}(x)\right| < \varepsilon$, *for* $k = 0, 1, 2$, *on* $[c, d]$.

Proof: Assume that Equation (4.20) satisfies Conditions (A), (B), and (C) for a positive integer m and a positive integer $n > 1$. Choose integers p and q such that $1 \leq p \leq m$ and $1 < q \leq n$. Let $\varepsilon > 0$ be given. Fix $t_0 \in (a, b)$. We will use the Brouwer Invariance of Domain Theorem in our proof.

Define the set $G \subset \mathbb{R}^{p+q+3}$ by

$$G = \{(t_1, t_2, \ldots, t_{p+q}, c_1, c_2, c_3) : a < t_1 < t_2 < \cdots < t_{p+q} < b;\ c_1, c_2, c_3 \in \mathbb{R}\}.$$

G is open in $\mathbb{R}^{p+q+3}$. Now define $\phi : G \subset \mathbb{R}^{p+q+3} \to \mathbb{R}^{p+q+3}$ by

$$\phi\big((t_1, t_2, \ldots, t_{p+q}, c_1, c_2, c_3)\big) =$$
$$\left(t_1, t_2, \ldots, t_{p+q}, u(t_1) - \sum_{i=2}^{q-1} u(t_i), u(t_q), u(t_{p+q}) - \sum_{j=q+1}^{p+q-1} u(t_j)\right),$$

where $u(x)$ is the solution of (4.20) satisfying

$$u(t_0) = c_1,\ u'(t_0) = c_2,\ u''(t_0) = c_3.$$

The function ϕ is continuous since solutions of (4.20) depend continuously upon initial conditions, by Theorem 3.1. We claim that ϕ is one-to-one. To see this, suppose that

$$\phi\big((s_1, s_2, \ldots, s_{p+q}, h_1, h_2, h_3)\big) = \phi\big((t_1, t_2, \ldots, t_{p+q}, c_1, c_2, c_3)\big).$$

The definition of ϕ gives us that $s_i = t_i$, for $i = 1, 2, \ldots, p+q$. We have

$$w(t_0) = h_1,\ w'(t_0) = h_2,\ w''(t_0) = h_3,$$

and

$$u(t_0) = c_1,\ u'(t_0) = c_2,\ u''(t_0) = c_3,$$

for solutions $w(x)$ and $u(x)$ of (4.20). Then we have

$$u(t_1) - \sum_{i=2}^{q-1} u(t_i) = w(s_1) - \sum_{i=2}^{q-1} w(s_i) = w(t_1) - \sum_{i=2}^{q-1} w(t_i),$$
$$u(t_q) = w(s_q) = w(t_q),$$
$$u(t_{p+q}) - \sum_{j=q+1}^{p+q-1} u(t_j) = w(s_{p+q}) - \sum_{j=q+1}^{p+q-1} w(s_j) = w(t_{p+q}) - \sum_{j=q+1}^{p+q-1} w(t_j).$$

Then by Theorem 4.1 and Condition (C), we have $w(x) \equiv u(x)$ on (a, b), which implies that $h_i = c_i$, for $i = 1, 2, 3$. Thus ϕ is one-to-one. By the Brouwer Invariance of Domain Theorem, $\phi(G)$ is open and ϕ^{-1} is continuous on $\phi(G)$.

Now, let $z(x)$ be a solution of (4.20). Choose $a < x_1 < x_2 < \cdots < x_{p+q} < b$, any c and d with $a < c < x_1$ and $x_{p+q} < d < b$, and choose $\varepsilon > 0$. By Theorem 3.1, there exists an $\eta > 0$ such that, for our fixed solution $z(x)$, $\left|z^{(i-1)}(t_0) - c_i\right| < \eta$, for $i = 1, 2, 3$, implies $\left|u^{(i-1)}(x) - z^{(i-1)}(x)\right| < \varepsilon$ on $[c, d]$, for $i = 1, 2, 3$, where $u(x)$ is the solution of (4.20) with $u^{(i-1)}(t_0) = c_i$, for $i = 1, 2, 3$. We have

$$\left(x_1, x_2, \ldots, x_{p+q}, z(x_1) - \sum_{i=2}^{q-1} z(x_i), z(x_q), z(x_{p+q}) - \sum_{j=q+1}^{p+q-1} z(x_j)\right) \in \phi(G),$$

$\phi(G)$ is open, and $\phi^{-1} : \phi(G) \to G$ is continuous. Therefore, there exists a $\delta > 0$ such that $|x_i - t_i| < \delta$, for $i = 1, 2, \ldots, p+q$, and

$$\max\left\{\left|z(x_1) - \sum_{i=2}^{q-1} z(x_i) - y_1\right|, |z(x_q) - y_2|, \left|z(x_{p+q}) - \sum_{j=q+1}^{p+q-1} z(x_j) - y_3\right|\right\} < \delta$$

imply that $(t_1, t_2, \ldots, t_{p+q}, y_1, y_2, y_3) \in \phi(G)$, and that $\phi^{-1}\big((t_1, t_2, \ldots, t_{p+q}, y_1, y_2, y_3)\big)$ is in the open ball of radius η centered at

$$\phi^{-1}\left(x_1, x_2, \ldots, x_{p+q}, z(x_1) - \sum_{i=2}^{q-1} z(x_i), z(x_q), z(x_{p+q}) - \sum_{j=q+1}^{p+q-1} z(x_j)\right)$$

$$= \big(x_1, x_2, \ldots, x_{p+q}, z(t_0), z'(t_0), z''(t_0)\big).$$

Suppose that

$$\phi^{-1}\big((t_1, t_2, \ldots, t_{p+q}, y_1, y_2, y_3)\big) = (t_1, t_2, \ldots, t_{p+q}, d_1, d_2, d_3).$$

If $u(x)$ is the solution of Equation (4.20) satisfying

$$u(t_0) = d_1,\ u'(t_0) = d_2,\ u''(t_0) = d_3,$$

then $\left|u^{(i-1)}(x) - z^{(i-1)}(x)\right| < \varepsilon$ on $[c, d]$, for $i = 1, 2, 3$. Moreover,

$$\begin{aligned}
(t_1, t_2, \ldots, t_{p+q}, y_1, y_2, y_3) &= \phi\big((t_1, t_2, \ldots, t_{p+q}, d_1, d_2, d_3)\big) \\
&= \phi\big(t_1, t_2, \ldots, t_{p+q}, u(t_0), u'(t_0), u''(t_0))\big)
\end{aligned}$$

$$= \left(t_1, t_2, \ldots, t_{p+q}, u(t_1) - \sum_{i=2}^{q-1} u(t_i), u(t_q), u(t_{p+q}) - \sum_{j=q+1}^{p+q-1} u(t_j)\right),$$

so that $u(x)$ is the solution of Equation (4.20) satisfying

$$\begin{aligned}
u(t_1) - \sum_{i=2}^{q-1} u(t_i) &= y_1, \\
u(t_q) &= y_2, \\
u(t_{p+q}) - \sum_{j=q+1}^{p+q-1} u(t_j) &= y_3.
\end{aligned}$$

□

4.3 Uniqueness Implies Existence

We end this chapter, and this dissertation, with several uniqueness implies existence results. First, two corollaries follow immediately from Theorem 4.2 by applying Theorems 3.3 and 3.8, respectively.

Corollary 4.1 (Uniqueness Implies Existence). *Suppose that Conditions (A), (B), and (C) from Theorem 4.2 hold. Then for any $a < x_1 < x_2 < x_3 < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, the boundary value problem*

$$y''' = f(x, y, y', y''),$$

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) = y_3,$$

has a unique solution.

Corollary 4.2 (Uniqueness Implies Existence). *Suppose that Conditions (A), (B), and (C) from Theorem 4.2 hold. Then for any positive integer k with $3 \le k \le m+2$, any $a < x_1 < x_2 < \cdots < x_k < b$, and any $y_1, y_2, y_3 \in \mathbb{R}$, the boundary value problem for the differential equation*

$$y''' = f(x, y, y', y''),$$

with boundary conditions

$$\begin{aligned} y(x_1) &= y_1, \\ y(x_2) &= y_2, \\ y(x_k) - \sum_{i=3}^{k-1} y(x_i) &= y_3, \end{aligned}$$

has a unique solution. The third boundary condition is taken to mean $y(x_3) = y_3$ in the case $k = 3$.

Theorem 4.3 (Uniqueness Implies Existence). *For the differential equation*

$$y''' = f(x, y, y', y''), \tag{4.24}$$

suppose that the following three conditions hold.

(A) $f : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous;*

(B) Solutions of initial value problems for (4.24) are unique and exist on all of (a, b); and,

(C) For any $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, solutions of the boundary value problem for (4.24) with boundary conditions

$$y(x_1) - y(x_2) = y_1, \tag{4.25}$$

$$y(x_3) = y_2, \tag{4.26}$$

$$y(x_5) - y(x_4) = y_3, \tag{4.27}$$

are unique, when they exist.

Then given any $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, the boundary value problem* (4.24)-(4.27) *has a solution, which is unique by Condition (C).*

Proof: Let $a < x_1 < x_2 < x_3 < x_4 < x_5 < b$ and $y_1, y_2, y_3 \in \mathbb{R}$ be given. By Corollary 4.2, there exists a unique solution of (4.24), say $z(x)$, satisfying

$$\begin{aligned} z(x_1) &= y_1, \\ z(x_2) &= 0, \\ z(x_5) - z(x_4) &= y_3. \end{aligned}$$

Then $z(x)$ is a solution of the boundary value problem for (4.24) that satisfies the boundary conditions

$$y(x_1) - y(x_2) = y_1, \tag{4.28}$$

$$y(x_5) - y(x_4) = y_3. \tag{4.29}$$

Define the set S by

$$S = \{u(x_3) : u(x) \text{ is a solution of (4.24) satisfying (4.28)} - \text{(4.29)}\}.$$

Our proof will be complete when we demonstrate that $y_2 \in S$.

We have $z(x_3) \in S$, so that S is nonempty. We will show that S is both open and closed, whereby $S = \mathbb{R}$ by the connectedness of $\mathbb{R}$.

S is open: Let $p_0 \in S$. Then $p_0 = u(x_3)$ for some solution $u(x)$ of (4.24) with $u(x_1) - u(x_2) = y_1$ and $u(x_5) - u(x_4) = y_3$. By Theorem 4.2 there exists a $\delta > 0$ sufficiently small such that, if $|p - p_0| < \delta$, then there is a solution $u_p(x)$ of (4.24) with

$$\begin{aligned} u_p(x_3) &= p, \\ u_p(x_1) - u_p(x_2) &= u(x_1) - u(x_2) = y_1, \\ u_p(x_5) - u_p(x_4) &= u(x_5) - u(x_4) = y_3. \end{aligned}$$

That is, $p \in S$, so $(p_0 - \delta, p_0 + \delta) \subset S$, whence S is open. The proof of the Theorem will be complete when we show that S is also closed.

S is closed: Assume S is not closed. Then S does not contain at least one of its limit points. Let r_0 be a limit point of S that is not contained in S. There exists an infinite sequence of distinct points $\{r_t\}_{t=1}^{\infty} \subset S$ such that $r_t \to r_0$. Without loss of generality, we may assume that $\{r_t\}_{t=1}^{\infty}$ is strictly monotone, say strictly monotone increasing. Now, $\{r_t\}_{t=1}^{\infty} \subset S$ implies that there exists a sequence $\{y_t(x)\}_{t=1}^{\infty}$ such that each $y_t(x)$ is a solution of (4.24) with

$$\begin{aligned} y_t(x_3) &= r_t, \\ y_t(x_1) - y_t(x_2) &= z(x_1) - z(x_2) = y_1, \\ y_t(x_5) - y_t(x_4) &= z(x_5) - z(x_4) = y_3. \end{aligned}$$

Choose $\tau \in (x_2, x_3)$. By Corollary 4.2, there exists a unique solution $w(x)$ of the differential equation (4.24) satisfying

$$w(\tau) = 0,\ w(x_3) = r_0,\ w(x_5) - w(x_4) = z(x_5) - z(x_4) = y_3.$$

By Theorem 4.1, we have that the conditions of Theorem 3.2 are met. Also, observe that $y_{t+1}(x_1) > y_t(x_1)$ on (x_2, x_3) for all $t \in \mathbb{N}$, else the $y_t(x)$, and therefore the r_t, would not be distinct. It follows from Theorem 3.2 and our assumption that r_0 is not in S that $\{y_t(x)\}_{t=1}^{\infty}$ cannot be uniformly bounded on any compact subinterval of (a, b). (See the argument in the proof of Theorem 3.6.) Hence, there exists a positive integer T_1 such that $y_t(\tau) > w(\tau) = 0$ for all $t \geq T_1$. Likewise, there exists $\theta \in (x_3, x_4)$ and a positive integer T_2 such that $y_t(\theta) > w(\theta)$ for all $t \geq T_2$. Therefore, for some $T \geq \max\{T_1, T_2\}$, we have

$$y_T(\tau) > w(\tau),\ y_T(x_3) < w(x_3) = r_0,\ y_T(\theta) > w(\theta).$$

It must be the case that $w(x) - y_T(x)$ has a zero on (τ, x_3), say $w(\alpha) = y_T(\alpha)$ for some $\alpha \in (\tau, x_3)$. Similarly, $w(x) - y_T(x)$ must have a zero on (x_3, θ), say $w(\beta) = y_T(\beta)$

for some $\beta \in (x_3, \theta)$. We have then that $w(x)$ and $y_T(x)$ are solutions of (4.24) that satisfy

$$\begin{aligned} w(\alpha) &= y_T(\alpha), \\ w(\beta) &= y_T(\beta), \\ w(x_5) - w(x_4) &= y_T(x_5) - y_T(x_4) = y_3. \end{aligned}$$

Theorem 4.1 implies that $w(x) \equiv y_T(x)$ on (a, b). But that would imply

$$r_0 = w(x_3) = y_T(x_3) = r_T < r_0,$$

a contradiction. We conclude that S contains all of its limit points, whereby S is closed. $\square$

Theorem 4.4 (Uniqueness Implies Existence). *Assume that Conditions (A), (B), and (C) from Theorem 4.3 hold. Then for any* $a < x_1 < x_2 < x_3 < x_4 < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$, *the boundary value problem for*

$$y''' = f(x, y, y', y''), \tag{4.30}$$

satisfying boundary conditions

$$\begin{aligned} y(x_1) - y(x_2) &= y_1, & (4.31) \\ y(x_3) &= y_2, & (4.32) \\ y(x_4) &= y_3, & (4.33) \end{aligned}$$

has a unique solution.

Proof: Let $a < x_1 < x_2 < x_3 < x_4 < b$ and $y_1, y_2, y_3 \in \mathbb{R}$ be given. By Corollary 4.1, there exists a unique solution of (4.30), say $z(x)$, such that

$$z(x_1) = y_1, \; z(x_2) = 0, \; z(x_4) = y_3.$$

That is, $z(x)$ is a solution of the differential equation (4.30) that satisfies the boundary conditions

$$y(x_1) - y(x_2) = y_1, \tag{4.34}$$

$$y(x_4) = y_3. \tag{4.35}$$

Define the set S by

$$S = \{u(x_3) : u(x) \text{ is a solution of } (4.30) \text{ satisfying } (4.34) - (4.35)\}.$$

To complete the proof, we will show that $y_2 \in S$.

We have $z(x_3) \in S$, so that S is nonempty. We will show that S is both open and closed, whereby $S = \mathbb{R}$ by the connectedness of $\mathbb{R}$.

S is open: Let $p_0 \in S$. Then $p_0 = u(x_3)$ for some solution $u(x)$ of (4.30) with $u(x_1) - u(x_2) = y_1$ and $u(x_4) = y_3$. By Theorem 3.5 there exists a $\delta > 0$ sufficiently small such that, if $|p - p_0| < \delta$, then there is a solution $u_p(x)$ of (4.30) with

$$\begin{aligned} u_p(x_3) &= p, \\ u_p(x_1) - u_p(x_2) &= u(x_1) - u(x_2) = y_1, \\ u_p(x_4) &= u(x_4) = y_3. \end{aligned}$$

That is, $p \in S$, so $(p_0 - \delta, p_0 + \delta) \subset S$, whence S is open. The proof of the Theorem will be complete when we show that S is also closed.

S is closed: Assume S is not closed. Then S does not contain at least one of its limit points. Let r_0 be a limit point of S that is not contained in S. There exists an infinite sequence of distinct points $\{r_t\}_{t=1}^{\infty} \subset S$ such that $r_t \to r_0$. Without loss of generality, we may assume that $\{r_t\}_{t=1}^{\infty}$ is strictly monotone, say strictly monotone increasing. Now, $\{r_t\}_{t=1}^{\infty} \subset S$ implies that there exists a sequence $\{y_t(x)\}_{t=1}^{\infty}$ such

that each $y_t(x)$ is a solution of (4.30) with

$$\begin{aligned} y_t(x_3) &= r_t, \\ y_t(x_1) - y_t(x_2) &= z(x_1) - z(x_2) = y_1, \\ y_t(x_4) &= z(x_4) = y_3. \end{aligned}$$

Choose $\tau \in (x_2, x_3)$. By Corollary 4.1, there exists a unique solution $w(x)$ of the differential equation (4.30) satisfying

$$w(\tau) = 0,\ w(x_3) = r_0,\ w(x_4) = y_t(x_4) = y_3.$$

By Theorem 4.1, we have that the conditions of Theorem 3.2 are met. Also, observe that $y_{t+1}(x_1) > y_t(x_1)$ on (x_2, x_4) for all $t \in \mathbb{N}$, else the $y_t(x)$, and therefore the r_t, would not be distinct. It follows from Theorem 3.2 and our assumption that r_0 is not in S that $\{y_t(x)\}_{t=1}^{\infty}$ cannot be uniformly bounded on any compact subinterval of (a, b). (See the argument in the proof of Theorem 3.6.) Hence, there exists a positive integer T_1 such that $y_t(\tau) > w(\tau) = 0$ for all $t \geq T_1$. Likewise, there exists $\theta \in (x_3, x_4)$ and a positive integer T_2 such that $y_t(\theta) > w(\theta)$ for all $t \geq T_2$. Therefore, for some $T \geq \max\{T_1, T_2\}$, we have

$$y_T(\tau) > w(\tau),\ y_T(x_3) < w(x_3) = r_0,\ y_T(\theta) > w(\theta).$$

It must be the case that $w(x) - y_T(x)$ has a zero on (τ, x_3), say $w(\alpha) = y_T(\alpha)$ for some $\alpha \in (\tau, x_3)$. Similarly, $w(x) - y_T(x)$ must have a zero on (x_3, θ), say $w(\beta) = y_T(\beta)$ for some $\beta \in (x_3, \theta)$. We have then that $w(x)$ and $y_T(x)$ are solutions of (4.30) that satisfy

$$w(\alpha) = y_T(\alpha),\ w(\beta) = y_T(\beta),\ w(x_4) = y_T(x_4) = y_3.$$

Theorem 4.1 implies that $w(x) \equiv y_T(x)$ on (a, b). But that would imply

$$r_0 = w(x_3) = y_T(x_3) = r_T < r_0,$$

a contradiction. We conclude that S contains all of its limit points, whereby S is closed. $\square$

Theorem 4.5 (Uniqueness Implies Existence). *For the differential equation*

$$y''' = f(x, y, y', y''), \tag{4.36}$$

suppose that the following three conditions hold.

(A) $f : (a, b) \times \mathbb{R}^3 \to \mathbb{R}$ *is continuous;*

(B) Solutions of initial value problems for (4.36) *are unique and exist on all of* (a, b)*; and,*

(C) For some $m, n \in \mathbb{N}$ *with* $n > 1$*, any* $a < x_1 < x_2 < \cdots < x_{m+n} < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, solutions of the boundary value problem for* (4.36) *with boundary conditions*

$$y(x_1) - \sum_{i=2}^{n-1} y(x_i) = y_1, \tag{4.37}$$

$$y(x_n) = y_2, \tag{4.38}$$

$$y(x_{m+n}) - \sum_{j=n+1}^{m+n-1} y(x_j) = y_3, \tag{4.39}$$

are unique when they exist. We take the boundary condition (4.37) *to mean* $y(x_1) = y_1$ *in the case that* $n = 2$*, and the boundary condition* (4.39) *is taken to be* $y(x_{n+1}) = y_3$ *in the case that* $m = 1$*.*

Then for any positive integers p *and* q *with* $1 \le p \le m$ *and* $1 < q \le n$*, any* $a < x_1 < x_2 < \cdots < x_{p+q} < b$ *and any* $y_1, y_2, y_3 \in \mathbb{R}$*, the boundary value problem for* (4.36) *with boundary conditions*

$$y(x_1) - \sum_{i=2}^{q-1} y(x_i) = y_1, \tag{4.40}$$

$$y(x_q) = y_2, \tag{4.41}$$

$$y(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y(x_j) = y_3, \tag{4.42}$$

has a (unique) solution. As in the supposition, we take the boundary condition (4.40) *to mean* $y(x_1) = y_1$ *in the case that* $q = 2$*, and the boundary condition* (4.42) *is taken to be* $y(x_{q+1}) = y_3$ *in the case that* $p = 1$*.*

Proof: Assume that Conditions (A), (B), and (C) hold. The proof is by induction. We have shown that the Theorem holds for the case $m = 2$ and $n = 3$. Let $m \geq 2$ and $n \geq 3$ be given. For inductive purposes, assume that, for all $2 \leq h \leq m-1$ and $3 \leq k \leq n-1$, any $a < x_1 < x_2 < \cdots < x_{h+k} < b$ and any $y_1, y_2, y_3 \in \mathbb{R}$, there exists a solution for (4.36) satisfying

$$\begin{aligned} y(x_1) - \sum_{i=2}^{k-1} y(x_i) &= y_1, \\ y(x_k) &= y_2, \\ y(x_{h+k}) - \sum_{j=k+1}^{h+k-1} y(x_j) &= y_3. \end{aligned}$$

To complete the proof, we will show that the Theorem holds for each of three nontrivial cases.

(1) $p = m - 1$ and $q = n$.

(2) $p = m$ and $q = n - 1$.

(3) $p = m$ and $q = n$.

Proof of Case 1: $p = m - 1$ and $q = n$. Let $a < x_1 < x_2 < \cdots < x_{p+q} < b$ and $y_1, y_2, y_3 \in \mathbb{R}$ be given. By assumption, there exists a unique solution, say $z(x)$, to the boundary value problem for (4.36) satisfying

$$\begin{aligned} z(x_1) - \sum_{i=2}^{q-2} z(x_i) &= y_1, \\ z(x_{q-1}) &= 0, \\ z(x_{p+q}) - \sum_{j=q+1}^{p+q-1} z(x_j) &= y_3. \end{aligned}$$

Then $z(x)$ solves the differential equation (4.36) and satisfies the boundary conditions

$$y(x_1) - \sum_{i=2}^{q-1} y(x_i) = y_1, \tag{4.43}$$

$$y(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y(x_j) = y_3. \tag{4.44}$$

Define the set S by

$$S = \{u(x_q) : u(x) \text{ is a solution of } (4.36) \text{ satisfying } (4.43) - (4.44)\}.$$

We will show that $y_2 \in S$. We have that $z(x_q) \in S$, so that S is nonempty. We will show that S is both open and closed, whereby $S = \mathbb{R}$, since $\mathbb{R}$ is a connected set.

S is open: Let $p_0 \in S$. Then $p_0 = u(x_q)$ for some solution $u(x)$ of (4.36) with

$$\begin{aligned} u(x_1) - \sum_{i=2}^{q-1} u(x_i) &= y_1, \\ u(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u(x_j) &= y_3. \end{aligned}$$

By Theorem 4.2, there is a $\delta > 0$ sufficiently small such that, if $|p - p_0| < \delta$, then there is a solution $u_p(x)$ of (4.36) with

$$\begin{aligned} u_p(x_q) &= p, \\ u_p(x_1) - \sum_{i=2}^{q-1} u_p(x_i) &= u(x_1) - \sum_{i=2}^{q-1} u(x_i) = y_1, \\ u_p(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u_p(x_j) &= u(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u(x_j) = y_3. \end{aligned}$$

Therefore $p \in S$, whence $(p_0 - \delta, p_0 + \delta) \subset S$, so S is open. The proof of Case 1 will be complete when we show that S is also closed.

S is closed: Assume to the contrary that S is not closed. Then S does not contain some of its limit points. Choose r_0 such that r_0 is a limit point of S that is not contained in S. Then there exists an infinite sequence of distinct points $\{r_t\}_{t=1}^{\infty} \subset S$ such that $r_t \to r_0$. Without loss of generality, we may assume that $\{r_t\}_{t=1}^{\infty}$ is strictly monotone, say strictly monotone increasing. Now, $\{r_t\}_{t=1}^{\infty} \subset S$ implies that there exists a sequence $\{y_t(x)\}_{t=1}^{\infty}$ such that, for each $t \in \mathbb{N}$, $y_t(x)$ is a solution of (4.36)

that satisfies

$$\begin{aligned} y_t(x_1) - \sum_{i=2}^{q-1} y_t(x_i) &= z(x_1) - \sum_{i=2}^{q-1} z(x_i), \\ y_t(x_q) &= r_t, \\ y_t(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y_t(x_j) &= z(x_{p+q}) - \sum_{j=q+1}^{p+q-1} z(x_j) = y_3. \end{aligned}$$

Choose $\tau \in (x_{q-1}, x_q)$. By Corollary 4.2, there exists a unique solution $w(x)$ of (4.36) satisfying

$$\begin{aligned} w(\tau) &= 0, \\ w(x_q) &= r_0, \\ w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) &= z(x_{p+q}) - \sum_{j=q+1}^{p+q-1} z(x_j) = y_3. \end{aligned}$$

By Theorem 4.1 we have that the conditions of Theorem 3.2 are met. Also, observe that $y_{t+1}(x_1) > y_t(x_1)$ on (x_{q-1}, x_{q+1}) for all $t \in \mathbb{N}$, else the $y_t(x)$, and therefore the r_t, would not be distinct. It follows from Theorem 3.2 and our assumption that r_0 is not in S that $\{y_t(x)\}_{t=1}^{\infty}$ cannot be uniformly bounded on any compact subinterval of (a, b). (See the argument in the proof of Theorem 3.6.) Hence, there exists a positive integer T_1 such that $y_t(\tau) > w(\tau) = 0$ for all $t \geq T_1$. Likewise, there exists $\theta \in (x_q, x_{q+1})$ and a positive integer T_2 such that $y_t(\theta) > w(\theta)$ for all $t \geq T_2$. Therefore, for some $T \geq \max\{T_1, T_2\}$, we have

$$y_T(\tau) > w(\tau),\ y_T(x_q) < w(x_q) = r_0,\ y_T(\theta) > w(\theta).$$

It must be the case that $w(x) - y_T(x)$ has a zero on (τ, x_q), say $w(\alpha) = y_T(\alpha)$ for some $\alpha \in (\tau, x_q)$. Similarly, $w(x) - y_T(x)$ must have a zero on (x_q, θ), say $w(\beta) = y_T(\beta)$ for some $\beta \in (x_q, \theta)$. We have then that $w(x)$ and $y_T(x)$ are solutions of (4.36) that satisfy

$$\begin{aligned} w(\alpha) &= y_T(\alpha), \\ w(\beta) &= y_T(\beta), \\ w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) &= y_T(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y_T(x_j) = y_3. \end{aligned}$$

Theorem 4.1 gives us that $w(x) \equiv y_T(x)$ on (a, b). But that would imply

$$r_0 = w(x_q) = y_T(x_q) = r_T < r_0,$$

a contradiction. We conclude that S contains all of its limit points, whereby S is closed.

Proof of Case 2: $p = m$ and $q = n - 1$. Let $a < x_1 < x_2 < \cdots < x_{p+q} < b$ and $y_1, y_2, y_3 \in \mathbb{R}$ be given. By assumption, there exists a unique solution, say $z(x)$, to the boundary value problem for (4.36) satisfying

$$\begin{aligned} z(x_1) - \sum_{i=2}^{q-1} z(x_i) &= y_1, \\ z(x_{q+1}) &= 0, \\ z(x_{p+q}) - \sum_{j=q+2}^{p+q-1} z(x_j) &= y_3. \end{aligned}$$

Then $z(x)$ solves the differential equation (4.36) and satisfies the boundary conditions

$$y(x_1) - \sum_{i=2}^{q-1} y(x_i) = y_1, \tag{4.45}$$

$$y(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y(x_j) = y_3. \tag{4.46}$$

Define the set S by

$$S = \{u(x_q) : u(x) \text{ is a solution of (4.36) satisfying (4.45)} - \text{(4.46)}\}.$$

We will show that $y_2 \in S$. We have that $z(x_q) \in S$, so that S is nonempty. We will show that S is both open and closed, whereby $S = \mathbb{R}$, since $\mathbb{R}$ is a connected set.

S is open: Let $p_0 \in S$. Then $p_0 = u(x_q)$ for some solution $u(x)$ of (4.36) with

$$\begin{aligned} u(x_1) - \sum_{i=2}^{q-1} u(x_i) &= y_1, \\ u(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u(x_j) &= y_3. \end{aligned}$$

By Theorem 4.2, there is a $\delta > 0$ sufficiently small such that, if $|p - p_0| < \delta$, then there is a solution $u_p(x)$ of (4.36) with

$$\begin{aligned} u_p(x_q) &= p, \\ u_p(x_1) - \sum_{i=2}^{q-1} u_p(x_i) &= u(x_1) - \sum_{i=2}^{q-1} u(x_i) = y_1, \\ u_p(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u_p(x_j) &= u(x_{p+q}) - \sum_{j=q+1}^{p+q-1} u(x_j) = y_3. \end{aligned}$$

Therefore $p \in S$, whence $(p_0 - \delta, p_0 + \delta) \subset S$, so S is open. The proof of Case 2 will be complete when we show that S is also closed.

S is closed: Assume to the contrary that S is not closed. Then S does not contain some of its limit points. Choose r_0 such that r_0 is a limit point of S that is not contained in S. Then there exists an infinite sequence of distinct points $\{r_t\}_{t=1}^{\infty} \subset S$ such that $r_t \to r_0$. Without loss of generality, we may assume that $\{r_t\}_{t=1}^{\infty}$ is strictly monotone, say strictly monotone increasing. Now, $\{r_t\}_{t=1}^{\infty} \subset S$ implies that there exists a sequence $\{y_t(x)\}_{t=1}^{\infty}$ such that, for each $t \in \mathbb{N}$, $y_t(x)$ is a solution of (4.36) that satisfies

$$\begin{aligned} y_t(x_1) - \sum_{i=2}^{q-1} y_t(x_i) &= z(x_1) - \sum_{i=2}^{q-1} z(x_i), \\ y_t(x_q) &= r_t, \\ y_t(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y_t(x_j) &= z(x_{p+q}) - \sum_{j=q+1}^{p+q-1} z(x_j) = y_3. \end{aligned}$$

Choose $\tau \in (x_{q-1}, x_q)$. By Corollary 4.2, there exists a unique solution $w(x)$ of (4.36) satisfying

$$\begin{aligned} w(\tau) &= 0, \\ w(x_q) &= r_0, \\ w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) &= z(x_{p+q}) - \sum_{j=q+1}^{p+q-1} z(x_j) = y_3. \end{aligned}$$

By Theorem 4.1 we have that the conditions of Theorem 3.2 are met. Also, observe that $y_{t+1}(x_1) > y_t(x_1)$ on (x_{q-1}, x_{q+1}) for all $t \in \mathbb{N}$, else the $y_t(x)$, and therefore the r_t, would not be distinct. It follows from Theorem 3.2 and our assumption that r_0 is not in S that $\{y_t(x)\}_{t=1}^{\infty}$ cannot be uniformly bounded on any compact subinterval of (a, b). (See the argument in the proof of Theorem 3.6.) Hence, there exists a positive integer T_1 such that $y_t(\tau) > w(\tau) = 0$ for all $t \geq T_1$. Likewise, there exists $\theta \in (x_q, x_{q+1})$ and a positive integer T_2 such that $y_t(\theta) > w(\theta)$ for all $t \geq T_2$. Therefore, for some $T \geq \max\{T_1, T_2\}$, we have

$$y_T(\tau) > w(\tau),\ y_T(x_q) < w(x_q) = r_0,\ y_T(\theta) > w(\theta).$$

It must be the case that $w(x) - y_T(x)$ has a zero on (τ, x_q), say $w(\alpha) = y_T(\alpha)$ for some $\alpha \in (\tau, x_q)$. Similarly, $w(x) - y_T(x)$ must have a zero on (x_q, θ), say $w(\beta) = y_T(\beta)$ for some $\beta \in (x_q, \theta)$. We have then that $w(x)$ and $y_T(x)$ are solutions of (4.36) that satisfy

$$\begin{aligned} w(\alpha) &= y_T(\alpha), \\ w(\beta) &= y_T(\beta), \\ w(x_{p+q}) - \sum_{j=q+1}^{p+q-1} w(x_j) &= y_T(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y_T(x_j) = y_3. \end{aligned}$$

Theorem 4.1 gives us that $w(x) \equiv y_T(x)$ on (a, b). But that would imply

$$r_0 = w(x_q) = y_T(x_q) = r_T < r_0,$$

a contradiction. We conclude that S contains all of its limit points, whereby S is closed.

Proof of Case 3: $p = m$ and $q = n$. Let $a < x_1 < x_2 < \cdots < x_{p+q} < b$ and $y_1, y_2, y_3 \in \mathbb{R}$ be given. With the proof of Case 1 we established the existence of a unique solution of (4.36), say $z(x)$, satisfying

$$\begin{aligned} z(x_1) - \sum_{i=2}^{q-2} z(x_i) &= y_1, \\ z(x_{q-1}) &= 0, \\ z(x_{p+q}) - \sum_{j=q+1}^{p+q-1} z(x_j) &= y_3. \end{aligned}$$

The $z(x)$ solves the differential equation (4.36) and satisfies the boundary conditions

$$y(x_1) - \sum_{i=2}^{q-1} y(x_i) = y_1, \tag{4.47}$$

$$y(x_{p+q}) - \sum_{j=q+1}^{p+q-1} y(x_j) = y_3. \tag{4.48}$$

Define the set S by

$$S = \{u(x_q) : u(x) \text{ is a solution of (4.36) satisfying (4.47)} - \text{(4.48)}\}.$$

We may apply the argument from the proof of Case 1 to establish that $S = \mathbb{R}$, whence $y_2 \in S$. This completes the proof of the Theorem. $\square$

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