ABSTRACT

Nature through the Lens of Number Theory

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Though we might not recognize it, mathematics is manifested in the world around us. To begin to see an example of this, we first develop the theory that links together the Fibonacci numbers, the Golden Ratio, and continued fractions to ultimately present some well-known (and some less well-known) results on the best approximation of irrational numbers by fractions. A discussion on irrationality, specifically with regard to the Golden Ratio, brings us to the conclusion that the Golden Ratio is among the most irrational numbers of the real line. Armed with this theory, we then examine the mechanics behind plant growth and discover surprising connections between mathematics and flower development. Through an in-depth understanding of the mathematical tools and number-theoretic results initially outlined, we are brought to a deeper appreciation of the Creation that surrounds us as we recognize the order and pattern that exist in the natural world.

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PREFACE

The natural world in which we live is both complex and beautiful; though most people would agree that mathematics is a complex subject, it is unfortunately much less likely to be associated with the word ``beautiful". It seems that a lack of understanding of mathematics (or perhaps merely a lack of interest in it) has made the subject as a whole into one that is perceived as irrelevant to everyday life, but the truth is that this is not the case. The realm of mathematics contains a wealth of knowledge that is not only important in its own right, but is also a vital, ever-present part of nearly all aspects of our life. If this is the case, though, why does math seem to be such an under-appreciated subject among the general population?

The subject of mathematics itself certainly does not help remedy this issue of under-appreciation, for it seems to be one of the hardest subjects to communicate to individuals outside of the field. Higher level mathematics certainly deserves recognition as a rich, well-developed, and imperative area of knowledge, but as research continues, the various branches of mathematics begin to grow just like a tree: new branches form off of old ones, each with a narrower focus, each with its own set of vocabulary, notation, definitions, and theorems, making it difficult even for mathematicians to be conversant in areas in which they do not specialize. How much more daunting is the task of communicating such areas of math to individuals outside of the subject! It was with this task in mind, though, that I chose to present the following exposition of an area of Number Theory centering on continued fractions: by laying out all terminology, explaining all notation, and providing rigorous proof to verify claims that develop into some interesting final results, it is my hope that the material presented here can be understood and appreciated by anyone who has taken a basic course in Calculus.

In case the mathematics on its own does not merit its own appreciation in the eyes of the general reader, the technical explanation in Chapter One will be applied to nature in Chapter Two. For just as we can appreciate the natural beauty that we see around us every day, we should also appreciate the role that mathematics plays in nature and biology once we are introduced to the principles that govern those relevant areas of math. The links between the created world around us and certain mathematical principles can inform our understanding of the order and pattern seen in nature. By delving into the realm of Number Theory, we can see an example of how math speaks volumes in the world of plant development and growth. The hope is that exploring the areas of Fibonacci numbers, the Golden Ratio, continued fractions, best-approximation theory, and irrationality, and how they relate to plant growth will bring us to a deeper appreciation for Creation as we recognize the order and pattern in the natural world.

ACKNOWLEDGMENTS

Thank you, Dr. Hunziker, for your patience, encouragement, and support throughout this process; it was a pleasure getting to work with you on this project. Thank you, Dr. Dugas and Dr. Mathis, for serving on my defense committee. I am very thankful to have had the chance to work with and learn from all of you these past few years, and I am grateful for the time you have invested in my education and personal development.

DEDICATION

To everyone who has ever had to listen to me attempt to adequately explain mathematics.

CHAPTER ONE

From Fibonacci to Irrationality

What do a Medieval mathematician and a sunflower seed have in common? A guess would be "not much"; however, plant life is more closely related to the mathematical interests of Fibonacci than one would think. The astounding symmetry, patterns, and order exhibited by plants are not just sheer happenstance or evolutionary chance. Indeed, the overwhelming presence of order in nature suggests that it is not just a happy coincidence that nature looks and functions as it does. By observing patterns in leaves, seeds, petals, or branches and connecting these patterns to some accessible yet powerful mathematics, the mechanisms and principles that drive plant growth and development can begin to be understood, bringing with them an appreciation for the complexity of a Creation that functions almost flawlessly.

Counting Rabbits and Other Computations

The Medieval Age was by no means a hotbed of academic progress; in keeping with such a theme, the "Dark Ages", as they are thus aptly named, did not feature much mathematical development. In this era of relatively small mathematical advancements, however, one man stands apart as a Medieval pioneer in mathematics: Leonardo Pisano, known to the modern world as Fibonacci. The author of a number of books that feature a wide variety of mathematical topics, Fibonacci sought to introduce Arabic numerals and methods of arithmetic to the Roman world, and his advances are monumental enough to affect today's society: even basic addition and subtraction would be cumbersome when using Roman numerals! Fibonacci's most well-known work is, of course, *Liber Abaci*. Although the book requires well over 500 pages in a current translation (see, for example, [6]), *Liber Abaci*'s crown jewel requires hardly a page of print. The "rabbit problem", as it has come to be known,

	baby rabbit pairs	adult rabbit pairs	total rabbit pairs
beginning	0	1	1
end of month 1	1	1	2
end of month 2	1	2	3
end of month 3	2	3	5
end of month 4	3	5	8
end of month 5	5	8	13
end of month 6	8	13	21
end of month 7	13	21	34
end of month 8	21	34	55
end of month 9	34	55	89
end of month 10	55	89	144
end of month 11	89	144	233
end of month 12	144	233	377

TABLE 1. Rabbit Population Growth

is not even particularly challenging, but this has not stopped it from giving birth to one of the most celebrated and well-known mathematical entities: the Fibonacci sequence.

The Fibonacci sequence arises as one attempts to solve the rabbit problem, the rules of which are summarized below, taken from [6]:

(1) There exist a pair of adult rabbits consisting of one male and one female; rabbits of this species never die.
(2) At the beginning of the first month, these rabbits mate.
(3) At the end of a month, one male and one female rabbit are born to the rabbit parents, becoming a new rabbit pair.
(4) It takes a pair of baby rabbits a month to mature into adult rabbits, at which point they mate and begin producing rabbit pairs, one pair per month.

How many pairs of rabbits exist at the end of a year? The solution is summarized in Table 1. In [6], Fibonacci goes on to explain that the total for a given month can be found by adding the totals of the two previous months together. The author also asserts that one can continue this pattern indefinitely to find the total number of rabbits for any given month. Thus the Fibonacci sequence was born. The sequence

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	...
1	1	2	3	5	8	13	21	34	55	89	144	...

TABLE 2. The Fibonacci Sequence

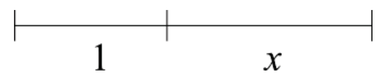


FIGURE 1. In deriving of the Golden Ratio, we discover that $x = \phi$.

is defined by

$$F_1 = 1; F_2 = 1; F_{n+1} = F_n + F_{n-1}, n \geq 2,$$

and the first few terms of the sequence are shown in Table 2. The Fibonacci sequence, though relatively old in the mathematical world, continues to be studied with great vigor today. There are myriad fascinating properties of this particular sequence of numbers. One such property deals with the Golden Ratio, but before the connection between Fibonacci numbers and the Golden Ratio is given, the Golden Ratio itself will be introduced.

The Golden Ratio and the Golden Angle

The Golden Ratio goes by a number of names and a number of symbols; it will be denoted here by ϕ. The numerical value of ϕ is $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803398....$ The Golden Ratio is not simply an arbitrarily chosen value. It is derived from a special set of ratios. Consider a line segment divided into two sections, one of unit length, as labelled in Figure 1. Suppose that the ratio of the longer section to the unit length section is equal to the ratio of the length of the whole segment to the longer section, meaning $x + 1 : x = x : 1$. Then $\frac{x+1}{x} = \frac{x}{1}$ and $x^2 = x + 1$. This means that $x^2 - x - 1 = 0$, so by the Quadratic Formula, $x = \frac{1 \pm \sqrt{5}}{2} = \phi$. Thus the Golden Ratio is aptly named, as it is derived from a set of ratios, and it satisfies the equation $x^2 - x - 1 = 0$. Note that the conjugate $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ also satisfies this equation.

It will be important to note that the Golden Ratio is an *irrational number*, meaning that it cannot be represented as a fraction of form $\frac{p}{q}$ with p and q being integers. The following proof, based on that found in [1], will verify this claim.

Theorem 1. *ϕ is irrational.*

Proof. Suppose to the contrary that $\phi = \frac{p}{q}$ is rational, with $p, q \in \mathbb{Z}$ and $\frac{p}{q}$ in lowest terms (meaning $\gcd(p, q) = 1$). Since ϕ satisfies the relation $\phi^2 - \phi - 1 = 0$, it must be that $(\frac{p}{q})^2 - \frac{p}{q} - 1 = 0$. From this, algebraic manipulation gives

$$\frac{p^2}{q^2} = \frac{p}{q} + 1$$

$$p^2 = pq + q^2 = q(p + q).$$

Since $p + q \in \mathbb{Z}$, this means $q \mid p^2$. Then $\gcd(p, q) = 1$ implies $q = 1$. The relation $(\frac{p}{q})^2 - \frac{p}{q} - 1 = 0$ also gives

$$\frac{p^2}{q^2} = \frac{p}{q} + 1$$

$$1 = \frac{q}{p} - \frac{q^2}{p^2}$$

$$\frac{q^2}{p^2} = \frac{q}{p} - 1$$

$$q^2 = qp - p^2 = p(q - p).$$

Since $q - p \in \mathbb{Z}$, this means $p \mid q^2$. Again, $\gcd(p, q) = 1$ implies $p = 1$. Thus $\phi = \frac{p}{q} = \frac{1}{1} = 1$. However, that means $\phi^2 - \phi - 1 = 1^2 - 1 - 1 = -1 \neq 0$, a contradiction. Therefore $\phi \neq \frac{p}{q}$ is irrational. $\square$

A similar entity to the Golden Ratio is the Golden Angle. When one takes a circle and divides it into two segments, one being of unit length, and when the ratio of the length of the other segment to the unit length segment is equal to the ratio of the circumference to the length of the other segment, the angle θ is referred to as the Golden Angle. This can be seen in Figure 2: if y is the angle we seek and the relationships between the arc segments are those just described, then $2\pi - y : y = 2\pi :$

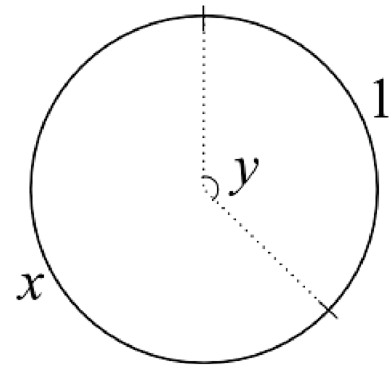


FIGURE 2. In deriving the Golden Angle, we discover that $y = \theta$.

$2\pi - y$ and hence $\frac{2\pi - y}{y} = \frac{2\pi}{2\pi - y}$. This gives the quadratic equation $y^2 + y(-6\pi) + 4\pi^2 = 0$, which has roots $y = \pi(3 \pm \sqrt{5})$; we take the smaller root, namely $y = \pi(3 - \sqrt{5}) = \theta$ to be the Golden Angle. Its value is approximately $\theta \approx 137.50776405...^\circ$.

The relationship between the Fibonacci numbers and the Golden Ratio is a simple one, namely that the limit as n approaches infinity of the ratio of two consecutive Fibonacci numbers is equal to the Golden Ratio:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} = \phi.$$

In order to verify that this is true, an equation known as Binet's Formula is needed. Binet's Formula is a non-recursive formula that can find the value of any number in the Fibonacci sequence, and it is presented as a lemma (with proof taken from [5]) below that will then allow us to verify the truth of this Fibonacci-Golden Ratio relationship.

Lemma 1. *(Binet's Formula) For all $n \in \mathbb{N}$,*

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

Proof. Recall the equation $x^2 - x - 1 = 0$ examined earlier along with its roots, which are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. This means, of course, that $\alpha^2 - \alpha - 1 = 0$ and $\beta^2 - \beta - 1 = 0$, or $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. Multiplying these last two equations by α^n and β^n, respectively, we see that $\alpha^{n+2} = \alpha^{n+1} + \alpha^n$ and $\beta^{n+2} = \beta^{n+1} + \beta^n$. Subtracting these two equalities yields

$$\alpha^{n+2} - \beta^{n+2} = \alpha^{n+1} - \beta^{n+1} + \alpha^n - \beta^n,$$

and we divide by $\alpha - \beta$ to finally obtain

$$\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1} + \alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

For $k \geq 0$, let $H_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}$. Then the previous equation can be rewritten as $H_{n+2} = H_{n+1} + H_n$; notice that this follows the same recursion formula as the Fibonacci sequence, with each term being the sum of the previous two terms. The question that remains is whether the initial values H_1 and H_2 match those of the Fibonacci sequence as well. Since

$$H_1 = \frac{\alpha - \beta}{\alpha - \beta} = 1 = F_1$$

and

$$H_2 = \frac{\alpha^2 - \beta^2}{\alpha - \beta} = \frac{(\alpha - \beta)(\alpha + \beta)}{\alpha - \beta} = \alpha + \beta = \frac{1 + \sqrt{5}}{2} + \frac{1 - \sqrt{5}}{2} = \frac{2}{2} = 1 = F_2,$$

we see that this is, in fact, the case. Thus $F_n = H_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$ for all $n \geq 1$. $\square$

Binet's Formula is useful in its own right: calculating F_{371} can be done much more quickly by employing the formula rather that using the recursion relation that defines the Fibonacci sequence to calculate the 370 terms that come before F_{371} in addition to F_{371} itself. But the real value in Binet's Formula, at least in terms of our investigation, is its usefulness in showing that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$; the proof of this fact, presented below in Theorem 2, is taken from [9].

Theorem 2.

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1 + \sqrt{5}}{2}.$$

Proof. Once again, we denote the Golden Ratio by $\phi = \frac{1+\sqrt{5}}{2}$ and its conjugate by $\bar{\phi} = \frac{1-\sqrt{5}}{2}$; note that $\phi - \bar{\phi} = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$. With this notation, Binet's Formula takes the form $F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \bar{\phi}^n\right)$. This means

$$\frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}}\left(\phi^{n+1} - \bar{\phi}^{n+1}\right)}{\frac{1}{\sqrt{5}}\left(\phi^n - \bar{\phi}^n\right)} = \frac{\phi^{n+1} + \left(\phi\bar{\phi}^n - \phi\bar{\phi}^n\right) - \bar{\phi}^{n+1}}{\phi^n - \bar{\phi}^n}$$

$$= \frac{\phi\left(\phi^n - \bar{\phi}^n\right)}{\phi^n - \bar{\phi}^n} + \frac{\bar{\phi}^n\left(\phi - \bar{\phi}\right)}{\phi^n - \bar{\phi}^n} = \phi + \frac{\sqrt{5}\bar{\phi}^n}{\phi^n - \bar{\phi}^n}$$

$$= \phi + \sqrt{5}\left(\frac{\phi^n - \bar{\phi}^n}{\bar{\phi}^n}\right)^{-1} = \phi + \sqrt{5}\left(\frac{\phi^n}{\bar{\phi}^n} - 1\right)^{-1}.$$

Now,

$$\frac{\phi}{\bar{\phi}} = \frac{\frac{1+\sqrt{5}}{2}}{\frac{1-\sqrt{5}}{2}} = \left(\frac{\frac{1+\sqrt{5}}{2}}{\frac{1-\sqrt{5}}{2}}\right)\left(\frac{\frac{1+\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2}}\right) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2}{\frac{1-5}{4}} = -\frac{(1+\sqrt{5})^2}{4} = -\frac{6+2\sqrt{5}}{4} = -\frac{3+\sqrt{5}}{2},$$

and $|\frac{\phi}{\bar{\phi}}| > 1$, so we have the following string of limits:

$$\lim_{n\to\infty}\left(\frac{\phi}{\bar{\phi}}\right)^n = \infty;$$

$$\lim_{n\to\infty}\left(\frac{\phi}{\bar{\phi}}\right)^n - 1 = \infty;$$

$$\lim_{n\to\infty}\left(\left(\frac{\phi}{\bar{\phi}}\right)^n - 1\right)^{-1} = 0;$$

$$\lim_{n\to\infty}\sqrt{5}\left(\left(\frac{\phi}{\bar{\phi}}\right)^n - 1\right)^{-1} = 0.$$

Thus

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \lim_{n\to\infty}\left[\phi + \sqrt{5}\left(\frac{\phi^n}{\bar{\phi}^n} - 1\right)^{-1}\right] = \phi = \frac{1+\sqrt{5}}{2}.$$

□

This tidy Theorem 2 establishes a very close connection between Fibonacci numbers and the Golden Ratio, but what exactly does it mean? A useful interpretation is that if you take two consecutive numbers of the Fibonacci sequence and create an improper fraction by placing the larger of the two numbers in the numerator and the smaller in the denominator, you can arrive at a pretty good rational approximation to ϕ, the Golden Ratio. The further along in the Fibonacci sequence you are when you create this fraction, the better your approximation will be. Thus the relationship between the Fibonacci sequence and the Golden Ratio is useful: rational approximations to irrational numbers, especially irrational numbers (like ϕ) that appear in such abundance in certain spheres of the sciences, make computation much easier and quicker at times. In fact, such approximations are so useful and sought-after that many mathematicians devote their research to best-approximation methods in a number of areas, only one of which is irrational numbers. In the case that we're considering with the Golden Ratio and rational approximations using the Fibonacci sequence, we've clearly found a set of fraction approximations to ϕ. But are these the "best" rational approximations we can get for the irrational ϕ?

An Introduction to Continued Fractions

To answer this question, we need a bit more mathematical machinery, namely an intriguing topic in Number Theory called *continued fractions*. Continued fractions look like this:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots}}}}}$$

but are usually denoted by $[a_0; a_1, a_2, a_3, ...]$. Each a_i is a real number (usually positive, although a_0 need not be). When $a_i \in \mathbb{Z}$ for all a_i's in the continued fraction, the continued fraction is called *simple*. There are also both finite and infinite continued fractions. A finite continued fraction terminates (for example, $a = [a_0; a_1, a_2, ..., a_n]$) while an infinite continued fraction does not (for example, $b = [b_0; b_1, b_2, b_3, ...]$ with an infinite number of terms). Simply put, rational numbers have finite continued fractions and irrational numbers have infinite continued fractions, which is actually very intuitive. If we have a finite continued fraction

$$x = [a_0; a_1, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}},$$

then we can use basic algebraic manipulation to work this continued fraction into something that looks like a regular fraction, namely, $\frac{p}{q}$ for some $p, q \in \mathbb{Z}$. But if we have an infinite continued fraction

$$x = [a_0; a_1, a_2, ...] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},$$

there is no ending term a_n, so there is no starting point at which we can begin manipulating the continued fraction into a more recognizable form. Although this is not proof of the fact that infinite continued fractions are irrational numbers, it does intuitively illustrate that it should be impossible to write an infinite continued fraction as a rational number $\frac{p}{q}$.

We are interested in one irrational number in particular; what is the (infinite) continued fraction expansion for ϕ? There is more than one way to find a continued fraction expansion given any real number, and the most widely-used method involves successive uses of the greatest integer function. However, our knowledge of the Golden Ratio runs relatively deep at this point, so there is a more straight-forward approach that can be used to answer this question. Recall the identity $\phi^2 - \phi - 1 = 0$, or $\phi^2 = \phi + 1$. Dividing by ϕ yields $\phi = \frac{\phi+1}{\phi} = 1 + \frac{1}{\phi}$. Having established this relationship, we can replace the ϕ in the denominator on the right-hand side by $1 + \frac{1}{\phi}$. Repeating this process gives

$$\phi = 1 + \frac{1}{\phi} = 1 + \cfrac{1}{1 + \frac{1}{\phi}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}.$$

Thus we can write the continued fraction expansion for the Golden Ratio immediately as $\phi = [1; 1, 1, 1, ...]$, which is infinite, as expected.

The next notion needed to arrive at our destination of rational approximations to ϕ is that of *convergents*. The "kth convergent" of a continued fraction $a = [a_0; a_1, a_2, ...]$ is found by simply truncating the continued fraction after the term a_k. This is denoted by $C_k = [a_0; a_1, a_2, ..., a_k]$. Each convergent is clearly a finite continued fraction, so each convergent of any continued fraction (finite or infinite) will be a rational number, say $C_k = \frac{p_k}{q_k}$. The value of the convergent C_k can be found by untangling the continued fraction using basic algebraic manipulation or by employing some well-known formulas for calculating such values. As we will eventually hope to calculate a number of convergents for ϕ, a formula would certainly be helpful. Letting $C_k = \frac{p_k}{q_k}$, the kth convergent can be calculated using the recursive formula

$$C_k = \frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}, k \geq 2,$$

for the continued fraction $[a_0; a_1, a_2, ...]$; it is easily seen that $C_0 = a_0$ and $C_1 = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$. We include the simple proof of this formula, taken from [5], since it will be employed as we continue our investigation.

Theorem 3. *The kth convergent of the simple continued fraction $[a_0; a_1, a_2, ...]$ is $C_k = \frac{p_k}{q_k}$ (as outlined above) for $k \geq 0$.*

Proof. We proceed by induction on k, so first consider a set of base cases for $k = 0, k = 1$, and $k = 2$:

$$C_0 = a_0 = \frac{p_0}{q_0};$$

$$C_1 = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{p_1}{q_1};$$

$$C_2 = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = a_0 + \frac{1}{\frac{a_1 a_2 + 1}{a_2}} = a_0 + \frac{a_2}{a_1 a_2 + 1} = \frac{a_0(a_1 a_2 + 1) + a_2}{a_1 a_2 + 1}$$

$$= \frac{a_2(a_0 a_1 + 1) + a_0}{a_1 a_2 + 1} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} = \frac{p_2}{q_2}.$$

Now, assume $C_n = \frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$ for $2 \leq n \leq k$ and consider the convergent $C_{n+1} = [a_0; a_1, a_2, ..., a_n, a_{n+1}]$, which can be rewritten as $C_{n+1} = [a_0; a_1, a_2, ..., a_n, a_{n+1}] = [a_0; a_1, a_2, ..., a_n + \frac{1}{a_{n+1}}]$. As this finite continued fraction has $n+1$ terms, the inductive hypothesis implies

$$C_{n+1} = \frac{(a_n + \frac{1}{a_{n+1}})p_{n-1} + p_{n-2}}{(a_n + \frac{1}{a_{n+1}})q_{n-1} + q_{n-2}} = \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}} = \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}},$$

the desired result. Thus the formula holds. □

For a finite continued fraction $x = [a_0; a_1, ..., a_n]$, it is easy to see that the nth convergent C_n is actually equal to the number given by $x = [a_0; a_1, ..., a_n]$; hence $x = C_n = [a_0; a_1, ..., a_n]$, meaning that the last convergent of a finite continued fraction is actually equal to the original value x. For an infinite continued fraction $\alpha = [a_0; a_1, a_2, ...]$, define $a'_k = [a_k, a_{k+1}, ...]$. Comparing this to our notation for convergents, we see that the convergent C_k gives the value of the truncated infinite continued fraction up to the term a_k, while a'_k ignores all terms that come before a_k, giving only the tail of the infinite continued fraction from a_k onward. With this new notation, we can rewrite $\alpha = [a_0; a_1, a_2, ..., a_n, a'_{n+1}]$, noting that a'_{n+1} need not be an integer or even a rational number, but will in fact be itself an infinite continued fraction if α is irrational. This is, of course, an abuse of notation, since we defined continued fractions be composed of integers only, but we will see that the resulting theory will function quite well if we can overlook this notational abuse. Since this new representation of α's continued fraction now looks like a finite continued fraction, it is safe to say that the "last convergent" of α is equal to the actual value of α. This is a slight variation of our original notion of a convergent, since this final convergent will be defined in terms of a'_{n+1}, which is not an integer, as it has been in the past. Because of this, we will use a special notation to refer to this last convergent, namely

C^*_{n+1}. Hence we have, writing $\alpha = [a_0; a_1, a_2, ..., a_n, a'_{n+1}]$, that

$$\alpha = C^*_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \frac{a'_{n+1}p_n + p_{n-1}}{a'_{n+1}q_n + q_{n-1}}.$$

Using the formula from Theorem 3, we can now compute as many convergents of $\phi = [1; 1, 1, 1, ...]$ as we desire:

$$C_0 = a_0 = 1;$$

$$C_1 = a_0 + \frac{1}{a_1} = 1 + \frac{1}{1} = 2;$$

$$C_2 = \frac{1(2) + 1}{1(1) + 1} = \frac{3}{2};$$

$$C_3 = \frac{1(3) + 2}{1(2) + 1} = \frac{5}{3};$$

$$C_4 = \frac{1(5) + 3}{1(3) + 2} = \frac{8}{5}.$$

Before we get too carried away with this process, note should be taken: since $a_i = 1$ for all $i \geq 0$, the convergent formula becomes

$$C_k = \frac{p_k}{q_k} = \frac{1(p_{k-1}) + p_{k-2}}{1(q_{k-1}) + q_{k-2}} = \frac{p_{k-1} + p_{k-1}}{q_{k-1} + q_{k-2}},$$

when we are considering the continued fraction of ϕ. Now, given the previous formulas for the convergents $C_0 = \frac{p_0}{q_0}$ and $C_1 = \frac{p_1}{q_1}$ as well as the continued fraction $\phi = [1; 1, 1, ...]$, it is clear that $p_0 = 1$, $p_1 = 2$, $q_0 = 1$, and $q_1 = 1$ in this case. Breaking the convergent formula apart to examine the numerators and denominators separately, we see that $p_k = p_{k-1} + p_{k-2}$ and $q_k = q_{k-1} + q_{k-2}$, which are recursion formulas of the same form as that used for the Fibonacci sequence. Noting that $p_0 = F_2$, $p_1 = F_3$, $q_0 = F_1$, and $q_1 = F_2$, we can quickly conclude that $p_k = F_{k+1} + F_k$ and $q_k = F_k + F_{k-1}$, from which we see that

$$C_k = \frac{p_k}{q_k} = \frac{F_{k+1} + F_k}{F_k + F_{k-1}} = \frac{F_{k+2}}{F_{k+1}}.$$

Then

$$\lim_{k\to\infty}\frac{F_{k+2}}{F_{k+1}} = \lim_{k\to\infty} C_k = \lim_{k\to\infty}[1;1,1,...,1] = [1;1,1,1,...] = \phi,$$

further verification that $\lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n} = \phi$.

By building up the theory behind continued fractions and examining how that theory applies to the Golden Ratio, we've arrived again at the conclusion that ratios of consecutive Fibonacci numbers make decent rational approximations to ϕ. But again, are we any closer to finding out if these fraction approximations are best-possible? The answer is yes, though we might not yet see how, and one final push in the area of convergents and continued fractions will shine the light on this result. The Best-Approximation Theorem (Theorem 4 to come) for irrational numbers will seal the deal, but before we conclude the current section, a number of lemmas will first be introduced in order to make the flow of logic easier to understand in the section to come. The proof of these three lemmas are taken from [5].

Lemma 2. *Let $C_n = \frac{p_n}{q_n}$ be the nth convergent of the (simple) continued fraction given by $\alpha = [a_0; a_1, a_2, a_3, ...]$. Then for $n \geq 1$, we have*

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}.$$

Proof. We proceed by induction on n. For the base case in which $n = 1$, we see that

$$p_1 q_0 - q_1 p_0 = (a_0 a_1 + 1)(1) - (a_1)(a_0) = 1 = (-1)^0 = (-1)^{1-1},$$

using the formulas for p_0, p_1, q_0, and q_1 previously outlined. Now, assume that $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$ and recall that $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$. Then consider:

$$p_{n+1} q_n - q_{n+1} p_n = (a_{n+1} p_n + p_{n-1}) q_n - (a_{n+1} q_n + q_{n-1}) p_n$$

$$= a_{n+1}(p_n q_n - p_n p_n) + p_{n-1} q_n - q_{n-1} p_n$$

$$= -(p_n q_{n-1} - q_n p_{n-1}) = -(-1)^{n-1} = (-1)^n,$$

using the inductive hypothesis to arrive at the desired conclusion. Thus the formula holds for all $n \geq 1$. $\square$

Lemma 3. *Let $C_n = \frac{p_n}{q_n}$ be the nth convergent of the (simple) continued fraction given by $\alpha = [a_0; a_1, a_2, a_3, ...]$. Then $\gcd(p_n, q_n) = 1$ for all $n \geq 1$.*

Proof. Let $d = \gcd(p_n, q_n)$. Since $p_n \geq 0$ and $q_n \geq 0$ by definition for $n \geq 1$, $d \geq 0$ also. By Lemma 2, $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$. Now, $d \mid p_n$ and $d \mid q_n$, so $d \mid p_n q_{n-1} - q_n p_{n-1}$ and thus $d \mid (-1)^{n-1}$. Potential divisors of $(-1)^{n-1}$ are only -1 and 1, but since $d \geq 0$, it must be that $d = 1$. Therefore $\gcd(p_n, q_n) = 1$ for all $n \geq 1$. $\square$

Lemma 4. *Let $C_n = \frac{p_n}{q_n}$ be the nth convergent of the (simple) continued fraction given by $\alpha = [a_0; a_1, a_2, ...]$. Then $\alpha \geq \frac{p_n}{q_n}$ if n is even, and $\alpha \leq \frac{p_n}{q_n}$ if n is odd.*

Proof. From a previous discussion, we can write

$$\alpha = [a_0; a_1, a_2, ...] = \alpha = [a_0; a_1, ..., a_n, a'_{n+1}],$$

so that

$$\alpha = C^*_{n+1} = \frac{a'_{n+1} p_n + p_{n-1}}{a'_{n+1} q_n + q_{n-1}}.$$

Then, employing Lemma 2, we have

$$x - \frac{p_n}{q_n} = \frac{a'_{n+1} p_n + p_{n-1}}{a'_{n+1} q_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{a'_{n+1} p_n q_n + p_{n-1} q_n}{q_n(a'_{n+1} q_n + q_{n-1})} - \frac{a'_{n+1} q_n p_n + q_{n-1} p_n}{q_n(a'_{n+1} q_n + q_{n-1})}$$

$$= \frac{a'_{n+1} p_n q_n + p_{n-1} q_n - a'_{n+1} q_n p_n - q_{n-1} p_n}{q_n(a'_{n+1} q_n + q_{n-1})}$$

$$= \frac{p_{n-1} q_n - q_{n-1} p_n}{q_n(a'_{n+1} q_n + q_{n-1})} = \frac{(-1)^n}{q_n(a'_{n+1} q_n + q_{n-1})}.$$

Now, the denominator of this last fraction contains only positive values and thus is itself positive. Thus the sign of $\frac{(-1)^n}{q_n(a'_{n+1} q_n + q_{n-1})}$ will be negative if n is odd and positive if n is even, giving

$$\begin{cases} x - \frac{p_n}{q_n} \leq 0, & n \text{ odd}, \\ x - \frac{p_n}{q_n} \geq 0, & n \text{ even}. \end{cases}$$

Therefore $x \leq \frac{p_n}{q_n}$ when n is odd and $x \geq \frac{p_n}{q_n}$ when n is even. □

Some Best-Approximation Results

The search for the set of rational numbers that constitute the best fractional approximations to the Golden Ratio is not yet over, but the Best-Approximation Theorem is now within sight. The first lemma provides only a preliminary bound for the distance between α and the nth convergent of α, but Lemma 6 gets very close to the heart of the matter and in fact makes the proof of the Best-Approximation Theorem quite simple; the proof of this lemma, along with the proof of Lemma 5 and of Theorem 4, is taken from [5].

Lemma 5. *Let $\alpha = [a_0; a_1, a_2, ...]$ be irrational with convergents $C_n = \frac{p_n}{q_n}$ for all $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$,*

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

Proof. From a previous discussion, we can write

$$\alpha = [a_0; a_1, a_2, ...] = [a_0; a_1, ..., a_n, a'_{n+1}],$$

so that

$$\alpha = C^*_{n+1} = \frac{a'_{n+1}p_n + p_{n-1}}{a'_{n+1}q_n + q_{n-1}}.$$

Then, employing Lemma 2,

$$\alpha - \frac{p_n}{q_n} = \frac{a'_{n+1}p_n + p_{n-1}}{a'_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{a'_{n+1}p_nq_n + p_{n-1}q_n}{q_n(a'_{n+1}q_n + q_{n-1})} - \frac{a'_{n+1}q_np_n + q_{n-1}p_n}{q_n(a'_{n+1}q_n + q_{n-1})}$$

$$= \frac{a'_{n+1}p_nq_n + p_{n-1}q_n - a'_{n+1}q_np_n - q_{n-1}p_n}{q_n(a'_{n+1}q_n + q_{n-1})}$$

$$= \frac{p_{n-1}q_n - q_{n-1}p_n}{q_n(a'_{n+1}q_n + q_{n-1})} = \frac{(-1)^n}{q_n(a'_{n+1}q_n + q_{n-1})},$$

so

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(a'_{n+1}q_n + q_{n-1})},$$

noting that this last denominator is necessarily positive. Now, since

$$a'_k = [a_k; a_{k+1}, a_{k+2}, ...] = a_k + \frac{1}{a_{k+1} + \frac{1}{\ddots}},$$

it is clear that $a_k < a'_k$ for all $k \in \mathbb{N}$. Hence we have $a_{n+1} < a'_{n+1}$, giving $a_{n+1}q_n + q_{n-1} < a'_{n+1}q_n + q_{n-1}$ and $q_n(a_{n+1}q_n + q_{n-1}) < q_n(a'_{n+1}q_n + q_{n-1})$, and finally

$$\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{q_n(a'_{n+1}q_n + q_{n-1})} < \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} = \frac{1}{q_n q_{n+1}} \leq \frac{1}{q_n^2},$$

with this last inequality being simply the result of the fact that $q_n \leq q_{n+1}$ for all $n \in \mathbb{N}$, a clear implication of the recursive formula for convergents seen in Theorem 3. □

Lemma 6. *Let $C_n = \frac{p_n}{q_n}$ be the nth convergent of the infinite continued fraction $x = [a_0; a_1, a_2, a_3, ...]$, and let $a, b \in Z$ with $1 \leq b < q_{n+1}$. Then $|q_n x - p_n| \leq |bx - a|$.*

Proof. Begin by considering the following system of equations:

$$\begin{cases} p_n\alpha + p_{n+1}\beta = a, \\ q_n\alpha + q_{n+1}\beta = b. \end{cases}$$

Written in matrix form, this system corresponds to:

$$\begin{bmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

The determinant of the matrix A is $p_n q_{n+1} - q_n p_{n+1} = -(p_{n+1}q_n - q_{n+1}p_n) = -(-1)^n = (-1)^{n+1}$ by the identity in Lemma 2; since this determinant is nonzero, there exists a solution (α, β) to this system. A relatively tedious amount of row-reduction and algebraic manipulation (in which Lemma 2 is again useful) allow one to arrive at the solution

$$\begin{cases} \alpha = (-1)^{n+1}(aq_{n+1} - bp_{n+1}), \\ \beta = (-1)^{n+1}(bp_n - aq_n). \end{cases}$$

Now, suppose $\alpha = 0$; then clearly $(-1)^{n+1}(aq_{n+1} - bp_{n+1}) = 0$, implying that $aq_{n+1} - bp_{n+1} = 0$ and $aq_{n+1} = bp_{n+1}$. From this, it can be seen that $q_{n+1} \mid bp_{n+1}$. But $\gcd(p_{n+1}, q_{n+1}) = 1$ by Lemma 3, so the conclusion is that $q_{n+1} \mid b$. However, if $q_{n+1} \mid b$, then it is obvious that $q_{n+1} \leq b$, which is a contradiction to $1 \leq b < q_{n+1}$. Thus we must have $\alpha \neq 0$, which provides the condition $|\alpha| > 0$; since $\alpha \in \mathbb{Z}$, this really gives us that $|\alpha| \geq 1$. In the case where $\beta = 0$, we read off from the matrix equation above that describes the system under consideration that $\alpha p_n = a$ and $\beta q_n = b$. Then

$$|bx - a| = |\alpha q_n x - \alpha p_n| = |\alpha||q_n x - p_n| \geq |q_n x - p_n|,$$

which is the desired result. This concludes the case where $\beta = 0$, so we assume that $\beta \neq 0$ for the remainder of the proof. Note that, since $\beta \in \mathbb{Z}$, the fact that $|\beta| \geq 1$ is implicit in this assumption.

Now, suppose $\beta < 0$. The second equation in our system is $q_n\alpha + q_{n+1}\beta = b$, or $q_n\alpha = b - q_{n+1}\beta$. By definition, $b > 0$ and $q_{n+1} > 0$; this combined with $\beta < 0$ implies that $q_n\alpha \geq 0$, and since $q_n \geq 0$ also by definition, we conclude that $\alpha > 0$ in this case. Supposing instead that $\beta > 0$ and recalling that we've chosen b such that $b < q_{n+1}$, we have $b < \beta q_{n+1}$ (since $|\beta| \geq 1$ for all β implies $\beta \geq 1$ in this case) and hence $0 < b - \beta q_{n+1}$. The right-hand side of the equation considered in the previous case, namely $q_n\alpha = b - q_{n+1}\beta$, is thus negative in value, so $q_n\alpha$ is negative as well. Again, since $q_n \geq 0$, this means $\alpha < 0$ in this case. Thus $\beta < 0$ implies $\alpha > 0$, and $\beta > 0$ implies $\alpha < 0$. This means β and α always have opposite signs. What is more, $\frac{p_n}{q_n} - x$

and $\frac{p_{n+1}}{q_{n+1}} - x$ also have opposite signs by Lemma 4, implying the same about $q_n x - p_n$ and $q_{n+1}x - p_{n+1}$. This, along with the fact that α and β have opposite signs, shows that $\alpha(q_n x - p_n)$ and $\beta(q_{n+1}x - p_{n+1})$ must have the same sign. This allows us to see, finally, that

$$|bx - a| = |(\alpha q_n + \beta q_{n+1})x - (\alpha p_n + \beta p_{n+1})| = |\alpha(q_n x - p_n) + \beta(q_{n+1}x - p_{n+1})|$$

$$= |\alpha(q_n x - p_n)| + |\beta(q_{n+1}x - p_{n+1})| = |\alpha||(q_n x - p_n)| + |\beta||(q_{n+1}x - p_{n+1})|$$

$$> |\alpha||q_n x - p_n| \geq |q_n x - p_n|,$$

noting that for any $x, y \in \mathbb{R}$ having the same sign, we know that $|x + y| = |x| + |y|$. Therefore $|q_n x - p_n| \leq |bx - a|$, and the final result is obtained. $\square$

Finally, on to the coveted best-approximation theorem:

Theorem 4. *Let x be irrational with (infinite) continued fraction expansion $x = [a_0; a_1, a_2, a_3, ...]$, and let $C_n = \frac{p_n}{q_n}$ be the nth convergent of x's continued fraction expansion. If $1 \leq b \leq q_n$ and $a, b \in \mathbb{Z}$, then $\frac{a}{b} \in \mathbb{Q}$ satisfies*

$$\left| x - \frac{p_n}{q_n} \right| \leq \left| x - \frac{a}{b} \right|.$$

Proof. Suppose to the contrary that $\left| x - \frac{p_n}{q_n} \right| > \left| x - \frac{a}{b} \right|$. Then

$$|q_n x - p_n| = |q_n| \left| x - \frac{p_n}{q_n} \right| \geq |b| \left| x - \frac{p_n}{q_n} \right| > |b| \left| x - \frac{a}{b} \right| = |bx - a|,$$

which is clearly a contradiction to Lemma 6. Therefore $\left| x - \frac{p_n}{q_n} \right| \leq \left| x - \frac{a}{b} \right|$. $\square$

The claim is that this theorem proves that ratios of consecutive Fibonacci numbers are the best rational approximations to the irrational ϕ, but this implication may not be obvious, so we follow the conclusion of the theorem logically to arrive at this desired result. The term on the left-hand side of the inequality in the statement of the previous theorem is $|\alpha - \frac{p_n}{q_n}|$, which is the difference between the nth convergent and the irrational number α viewed in absolute value. Another way to look at this is as an error term: this absolute value ultimately tells us how close the convergent C_n is to the actual value of α, or how good of an approximation the fraction C_n is to α. The full inequality in the statement of the theorem tells us that this error term is bounded above by $|\alpha - \frac{a}{b}|$, where b is any integer between 1 and q_n (the denominator of C_n) and a is any integer. This right-hand-side term $|\alpha - \frac{a}{b}|$ is another error term, this time communicating how close the fraction $\frac{a}{b}$ is to the actual value of α. Thus the full inequality $|\alpha - \frac{p_n}{q_n}| \leq |\alpha - \frac{a}{b}|$ tells us that convergents make the best rational approximations to any given irrational α in that any fraction whose numerator is unrestricted and whose denominator is positive and bounded above by q_n is farther away from α than $\frac{p_n}{q_n}$ is to α.

In the case in which we are interested, $\alpha = \phi$ and $C_n = \frac{p_n}{q_n} = \frac{F_{n+2}}{F_{n+1}}$, so rewriting Theorem 4 in these terms gives

$$\left|\phi - \frac{F_{n+2}}{F_{n+1}}\right| \leq \left|\phi - \frac{a}{b}\right|$$

for $1 \leq b \leq F_{n+1}$. This means that, when a bound is placed on the size of the denominator, ratios of consecutive Fibonacci numbers provide the best rational approximations to the Golden Ratio, exactly what we set out for in the first place!

A Most Irrational Discussion

We now know that ratios of consecutive Fibonacci numbers provide the best rational approximations to the irrational number ϕ, and generally, we know that convergents of any irrational give the best rational approximations to it. But what is most interesting is that ϕ is the "most irrational" of the irrational numbers, and it is with this concept that we conclude this section. By "most irrational", we mean that the rational approximations of ϕ are farther away from ϕ than rational approximations of all other irrational numbers are from any other irrational number. Since we approximate irrational numbers with convergents and the approximations improve the farther along we are in the sequence of convergents, we can rephrase the notion of a "most irrational" number by saying that the sequence of convergents of ϕ converges to ϕ slower than any other sequence of convergents converges to its respective irrational number. In order to solidify our claim that ϕ is the most irrational number, we will present a series of theorems to come; but before we get there, we must introduce and develop a new definition that will make the flow of these last theorems a bit smoother.

For $\alpha, \beta \in \mathbb{R}$, α is said to be *equivalent* to β if there exist integers $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$ such that $\alpha = \frac{a\beta+b}{c\beta+d}$; it turns out that this equivalence condition is, in fact, an *equivalence relation.* The importance of this fact will be explained following its proof, which is taken from [7].

Lemma 7. *The definition of equivalent numbers gives an equivalence relation.*

Proof. There are three conditions to check in order to verify an equivalence relation:

(1) *For any $\alpha \in \mathbb{R}$, α is equivalent to α.*

Let $\alpha \in \mathbb{R}$. Then $\alpha = \frac{1(\alpha)+0}{0(\alpha)+1}$ with $1(1) - 0(0) = 1$, so α is equivalent to α.

(2) *If α is equivalent to β, then β is equivalent to α.*

Let α be equivalent to β. Then there exist $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = \pm 1$ and $\alpha = \frac{a\beta+b}{c\beta+d}$. We then have the following:

$$\alpha = \frac{a\beta + b}{c\beta + d}$$

$$\alpha(c\beta + d) = c\alpha\beta + d\alpha = a\beta + b$$

$$c\alpha\beta - a\beta = -d\alpha + b$$

$$\beta(c\alpha - a) = -d\alpha + b$$

$$\beta = \frac{-d\alpha + b}{c\alpha - a},$$

with $(-d)(-a) - bc = ad - bc = \pm 1$. Thus β is equivalent to α.

(3) *If α is equivalent to β and β is equivalent to γ, then α is equivalent to γ.*

Let α be equivalent to β and β be equivalent to γ. Then there exist

$a, a_0, b, b_0, c, c_0, d, d_0 \in \mathbb{Z}$ such that $\alpha = \frac{a\beta+b}{c\beta+d}$ and $\beta = \frac{a_0\gamma+b_0}{c_0\gamma+d_0}$ with $ad-bc = \pm 1$ and $a_0d_0 + b_0c_0 = \pm 1$. Combining these equalities gives

$$\alpha = \frac{a\beta+b}{c\beta+d} = \frac{a\left(\frac{a_0\gamma+b_0}{c_0\gamma+d_0}\right)+b}{c\left(\frac{a_0\gamma+b_0}{c_0\gamma+d_0}\right)+d} = \frac{\frac{aa_0\gamma+ab_0+b(c_0\gamma+d_0)}{c_0\gamma+d_0}}{\frac{ca_0\gamma+cb_0+d(c_0\gamma+d_0)}{c_0\gamma+d_0}}$$

$$= \frac{aa_0\gamma+ab_0+bc_0\gamma+bd_0}{ca_0\gamma+cb_0+dc_0\gamma+dd_0} = \frac{(aa_0+bc_0)\gamma+(ab_0+bd_0)}{(ca_0+dc_0)\gamma+(cb_0+dd_0)}.$$

Clearly, $(aa_0+bc_0), (ab_0+bd_0), (ca_0+dc_0), (cb_0+dd_0) \in \mathbb{Z}$. Also,

$$(aa_0+bc_0)(cb_0+dd_0) - (ab_0+bd_0)(ca_0+dc_0)$$

$$= aa_0cb_0 + aa_0dd_0 + bc_0cb_0 + bc_0dd_0 - (ab_0ca_0 + ab_0dc_0 + bd_0ca_0 + bd_0dc_0)$$

$$= aa_0dd_0 + bc_0cb_0 - ab_0dc_0 - bd_0ca_0$$

$$= a_0d_0(ad-bc) + c_0b_0(bc-ad)$$

$$= a_0d_0(ad-bc) - b_0c_0(ad-bc)$$

$$= (a_0d_0 - b_0c_0)(ad-bc) = (\pm 1)(\pm 1) = \pm 1.$$

Thus α is equivalent to γ.

Since these three properties hold, we conclude that the above definition of equivalent numbers gives an equivalence relation on the real numbers. ☐

What does it mean that this is an equivalence relation on $\mathbb{R}$? An equivalence relation produces *equivalence classes*; the equivalence class of a number $\alpha \in \mathbb{R}$ is simply the set of all numbers that are equivalent to α (i.e. the set of all $\beta \in \mathbb{R}$ such that there exist $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$ and $\alpha = \frac{a\beta+b}{c\beta+d}$). An equivalence relation actually forms a *partition* of $\mathbb{R}$ through these equivalence classes, meaning two things:

(1) Every $\alpha \in \mathbb{R}$ falls in one equivalence class; this means that the union of all equivalence classes is equal to $\mathbb{R}$.

(2) Every $\alpha \in \mathbb{R}$ is a member of only one equivalence class; this means that equivalence classes are disjoint.

This discussion on equivalent numbers, equivalence relations, and equivalence classes seems like only an excuse to use the words "equivalent" and "equivalence" more than usual, but these notions of equivalent numbers will be useful as we continue to develop the idea of a "most irrational" number. The following lemma (the proof of which comes from [7]) gives a characterization of equivalent numbers that will aid in the proof of Theorem 7; this characterization will prove itself to be significantly more enlightening than the original definition of equivalent numbers, and it will connect back to the previous discussion on continued fractions.

Lemma 8. *Two irrational numbers $\alpha, \beta \in \mathbb{R}$ are equivalent if and only if $\alpha = [a_0; a_1, a_2, ..., a_k, b_0, b_1, b_2, ...]$ and $\beta = [c_0; c_1, c_2, ..., c_n, b_0, b_1, b_2, ...]$ for some $k, n \in \mathbb{N}$.*

Proof. First, let $\alpha = [a_0; a_1, a_2, ..., a_k, b_0, b_1, b_2, ...]$ and $\beta = [c_0; c_1, c_2, ..., c_n, b_0, b_1, b_2, ...]$ for some $k, n \in N$. Let $\gamma = [][b_0; b_1, b_2, ...]$. Then $\alpha = [a_0; a_1, ..., a_k, \gamma] = \frac{p_k\gamma + p_{k-1}}{q_k\gamma + q_{k-1}}$ with $p_k q_{k-1} - q_k p_{k-1} = \pm 1$ by Lemma 2, so α is equivalent to γ. Similarly, $\beta = [b_0; b_1, ..., b_n, \gamma] = \frac{r_n\gamma + r_{n-1}}{s_n\gamma + s_{n-1}}$ with $r_n s_{n-1} - s_n r_{n-1} = \pm 1$, again by Lemma 2, so β is equivalent to γ. By Theorem 7, then, α is equivalent to β.

Now suppose that α and β are equivalent. Then there exist $a, b, c, d, \in \mathbb{Z}$ such that $\beta = \frac{a\alpha + b}{c\alpha + d}$ with $ad - bc = \pm 1$. Without loss of generality, we may suppose that $c\alpha + d > 0$, for if this were not the case, we could replace a, b, c, and d with $-a, -b, -c$, and $-d$ to achieve a positive denominator. Now, let $\alpha = [a_0; a_1, a_2, ...]$ be the continued fraction of α. Then we can write

$$\alpha = [a_0; a_1, a_2, ...] = [a_0; a_1, a_2, ..., a_{k-1}, a'_k] = \frac{p_{k-1}a'_k + p_{k-2}}{q_{k-1}a'_k + q_{k-2}},$$

where $\{\frac{p_n}{q_n}\}_{n=1}^{\infty}$ denotes the sequence of convergents of α. But then we have

$$\beta = \frac{a\left(\frac{p_{k-1}a'_k+p_{k-2}}{q_{k-1}a'_k+q_{k-2}}\right)+b}{c\left(\frac{p_{k-1}a'_k+p_{k-2}}{q_{k-1}a'_k+q_{k-2}}\right)+d}$$

$$= \frac{\frac{ap_{k-1}a'_k+ap_{k-2}+bq_{k-1}a'_k+bq_{k-2}}{q_{k-1}a'_k+q_{k-2}}}{\frac{cp_{k-1}a'_k+cp_{k-2}+dq_{k-1}a'_k+dq_{k-2}}{q_{k-1}a'_k+q_{k-2}}}$$

$$= \frac{ap_{k-1}a'_k + ap_{k-2} + bq_{k-1}a'_k + bq_{k-2}}{cp_{k-1}a'_k + cp_{k-2} + dq_{k-1}a'_k + dq_{k-2}}$$

$$= \frac{(ap_{k-1} + bq_{k-1})a'_k + (ap_{k-2} + bq_{k-2})}{(cp_{k-1} + dq_{k-1})a'_k + (cp_{k-2} + dq_{k-2})}$$

$$= \frac{Pa'_k + R}{Qa'_k + S},$$

where $P = ap_{k-1}+bq_{k-1}$, $Q = cp_{k-1}+dq_{k-1}$, $R = ap_{k-2}+bq_{k-2}$, and $S = cp_{k-2}+dq_{k-2}$. We see that

$$PQ - RS = (ap_{k-1} + bq_{k-1})(cp_{k-1} + dq_{k-1}) - (ap_{k-2} + bq_{k-2})(cp_{k-2} + dq_{k-2})$$

$$= acp_{k-1}p_{k-2} + bdq_{k-1}q_{k-2} + adp_{k-1}q_{k-2} + bcp_{k-2}q_{k-1}$$

$$-acp_{k-1}p_{k-2} - bdq_{k-1}q_{k-2} - adp_{k-2}q_{k-1} - bcp_{k-1}q_{k-2}$$

$$= ad(p_{k-1}q_{k-2}) - bc(p_{k-1}q_{k-2} - p_{k-2}q_{k-1}) = (ad - bc)(p_{k-1}q_{k-2}) = (\pm 1)(\pm 1) = \pm 1,$$

using Lemma 2. Now, by Lemma 5, we know $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$, so $\frac{-1}{q_n^2} < \alpha - \frac{p_n}{q_n} < \frac{1}{q_n^2}$ and hence $\frac{-1}{q_n} < q_n\alpha - p_n < \frac{1}{q_n}$. This means that $q_n\alpha - p_n = \frac{\delta_n}{q_n}$ for any $n \in \mathbb{N}$ and some $\delta_n \in \mathbb{R}$ such that $|\delta_n| < 1$, so we can write $p_{k-1} = \alpha q_{k-1} + \frac{\delta_1}{q_{n-1}}$ and $p_{n-2} = \alpha q_{n-2} + \frac{\delta_2}{q_{n-2}}$, where $|\delta_1| < 1$ and $|\delta_2| < 1$. Then

$$Q = cp_{k-1} + dq_{k-1} = c\alpha q_{k-1} + \frac{c\delta_1}{q_{k-1}} + dq_{k-1} = (c\alpha + d)q_{k-1} + \frac{c\delta_1}{q_{k-1}}$$

and

$$S = cp_{k-2} + dq_{k-2} = c\alpha q_{k-2} = \frac{c\delta_2}{q_{k-2}} + dq_{k-2} = (c\alpha + d)q_{k-2} + \frac{c\delta_2}{q_{k-2}}.$$

We see that

$$\lim_{k\to\infty}(Q-S)=\lim_{k\to\infty}\left((c\alpha+d)q_{k-1}+\frac{c\delta_1}{q_{k-1}}-\left((c\alpha+d)q_{k-2}+\frac{c\delta_2}{q_{k-2}}\right)\right)$$
$$=\lim_{k\to\infty}((c\alpha+d)(q_{k-1}-q_{k-2}))+\lim_{k\to\infty}\left(\frac{c\delta_1}{q_{k-1}}-\frac{c\delta_2}{q_{k-2}}\right)$$
$$=(c\alpha+d)\lim_{k\to\infty}(q_{k-1}-q_{k-2})+0,$$

since the general formula for q_n, given by $q_n=a_nq_{n-1}+q_{n-2}$, implies $q_n\to\infty$ as $n\to\infty$. This general formula for q_n also tells us that $q_n>q_{n-1}$ for all $n\in\mathbb{N}$, implying that $q_{k-1}-q_{k-2}>0$ for all $k\in\mathbb{N}$. This, along with the assumption that $c\alpha+d>0$, implies that

$$\lim_{k\to\infty}(Q-S)=(c\alpha+d)\lim_{k\to\infty}(q_{k-1}-q_{k-2})>0.$$

This means that there exists a $k_1\in\mathbb{N}$ such that $k\geq k_1$ implies $Q-S>0$, or $Q>S$. We also see that

$$\lim_{k\to\infty}S=\lim_{k\to\infty}\left((c\alpha+d)q_{k-2}+\frac{c\delta_2}{q_{k-2}}\right)=\lim_{k\to\infty}(c\alpha+d)q_{k-2}+\lim_{k\to\infty}\frac{c\delta_2}{q_{k-2}}$$
$$=(c\alpha+d)\lim_{k\to\infty}q_{k-2}+0>0,$$

for similar reasons to those above, so there exists $k_2\in\mathbb{N}$ such that $k\geq k_2$ implies $S>0$. Taking $k_0=\max\{k_1,k_2\}$, we have that $k\geq k_0$ implies $Q>S>0$. We now know that $\beta=\frac{Pa'_k+R}{Qa'_k+S}$ with $PS-QR=\pm1$ and $Q>S>0$ for all $k\geq k_0$. Since $\frac{P}{Q}\in\mathbb{Q}$, we can write this fraction as a finite continued fraction:

$$\frac{P}{Q}=[c_0;c_1,...,c_n]=c_0+\cfrac{1}{c_1+\cfrac{1}{c_2+\ddots\frac{1}{c_n}}}=c_0+\cfrac{1}{c_1+\cfrac{1}{c_2+\ddots\frac{1}{c_n-1+1}}}$$
$$=c_0+\cfrac{1}{c_1+\cfrac{1}{c_2+\ddots\frac{1}{(c_n-1)+\frac{1}{1}}}}=[c_0;c_1,c_2,...,c_n-1,1].$$

This means we can write the finite continued fraction of $\frac{P}{Q}$ in two ways, one way with an even number of terms and one way with an odd number of terms. We will then choose to write $\frac{P}{Q} = [c_0; c_1, ..., c_n]$ in such a way that n satisfies $PS - QR = \pm 1 = (-1)^{n-1}$. Now, we will denote the set of convergents of $\frac{P}{Q}$ by $\{\frac{r_0}{s_0}, \frac{r_1}{s_1}, ..., \frac{r_n}{s_n}\}$. Then, since $\frac{P}{Q}$ has a finite continued fraction, we know that $\frac{P}{Q} = \frac{r_n}{s_n}$. Cross-multiplication gives $Ps_n = Qr_n$. By Lemma 3, we know that $\gcd(r_n, s_n) = 1$; since $PS - QR = \pm 1$, it is clear that $\gcd(P, Q) = 1$. From $Ps_n = Qr_n$, then, we can conlcude that $s_n \mid Q$, $r_n \mid P$, $P \mid r_n$, and $Q \mid s_n$; this implies that $r_n = P$ and $s_n = Q$. Now, employing Lemma 2 again, we can write

$$PS - QR = r_n S - s_n R = (-1)^{n-1} = r_n s_{n-1} - r_{n-1} s_n,$$

giving $r_n S - s_n R = r_n s_{n-1} - r_{n-1} s_n$, so $r_n S - r_n s_{n-1} = s_n R - s_n r_{n-1}$ and $r_n(S - s_{n-1}) = s_n(R - r_{n-1})$. Again, since $\gcd(r_n, s_n) = 1$, this implies that $r_n \mid (R - r_{n-1})$ and $s_n \mid (R - s_{n-1})$. Now, we have that $s_n = Q > S > 0$ and $s_n > s_{n-1}$. First, suppose that $S - s_{n-1} \geq 0$. Then $0 \leq |S - s_{n-1}| = S - s_{n-1} < s_n - s_{n-1} < s_n$. Suppose now that $S - s_{n-1} < 0$. Then $s_{n-1} - S > 0$ and $0 < |S - sn - 1| = |s_{n-1} - S| = s_{n-1} - S < s_n - S < s_n$. In both cases, then, we see that $|S - s_{n-1}| < s_n$. But $s_n \mid (S - s_{n-1})$, so it must be the case that $S - s_{n-1} = 0$, or $S = s_{n-1}$. Then $r_n S - s_n R = r_n s_{n-1} - r_{n-1} s_n$ gives $r_n s_{n-1} - s_n R = r_n s_{n-1} - r_{n-1} s_n$; hence $s_n R = r_{n-1} s_n$ and $R = \frac{r_{n-1} s_n}{s_n} = r_{n-1}$. This means that $\frac{P}{Q} = \frac{r_n}{s_n}$ and $\frac{R}{S} = \frac{r_{n-1}}{s_{n-1}}$ are the nth and $(n-1)$st convergents of any continued fraction that begins with $[c_0; c_1, ..., c_n]$. But we also have that $\beta = \frac{Pa'_k + R}{Qa'_k + S} = \frac{r_n a'_k + r_{n-1}}{s_n a'_k + s_{n-1}}$; this is precisely the formula for the $(n+1)$st convergent of the continued fraction $[c_0; c_1, ..., c_n, a'_k]$. Since there are $n+1$ terms in this continued fraction, we know that the $(n+1)$st convergent C_{n+1^*} is actually the value of the continued fraction itself; this means that $\beta = C_{n+1^*} \frac{r_n a'_k + r_{n-1}}{s_n a'_k + s_{n-1}} = \frac{r_{n+1}}{s_{n+1}} = [c_0; c_1, ..., c_n, a'_k]$, so we conclude that $\beta = [c_0; c_1, ..., c_n, a'_k] = [c_0; c_1, ..., c_n, a_k, a_{k+1}, a_{k+2}, ...]$, as desired. $\square$

The above lemma, though quite lengthy, is actually very important to our discussion because it gives a direct link between equivalent numbers and continued fractions. Specifically, it tells us that two numbers are equivalent precisely when they share the same tail of their continued fraction; the common tail may begin at different points for each number without affecting their equivalence. To illustrate this new look at equivalent numbers, here is an example. Recalling that the Golden Ratio is $\phi = [1; 1, 1, 1, 1, ...]$, the number given by $[2; 3, 3, 1, 1, 2, 1, 1, 1, ...]$ is equivalent to ϕ because they eventually have a common tail. Similarly, $[2; 3, 3, 1, 1, 2, 1, 4, 1, 2, 1, 2, 1, 2, ...]$ is equivalent to $[5; 9, 7, 2, 1, 4, 1, 2, 1, 2, 1, 2, ...]$. But $[1; 1, 1, 1, 1, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...]$ is not equivalent to ϕ because its tail will never become entirely an endless string of 1's. We are now well-prepared to see a series of theorems that speak to the irrationality of the Golden Ratio. The proofs of Theorems 5, 6, and 7 are taken from [7].

Theorem 5. *For any irrational α, there are infinitely many rational numbers $\frac{p}{q}$ such that $|\frac{p}{q} - \alpha| < \frac{1}{q^2\sqrt{5}}$.*

Proof. To show this, we will show that for every three consecutive convergents of $\alpha = [a_0; a_1, a_2, ...]$, one of those three convergents will satisfy $|\frac{p_n}{q_n} - \alpha| < \frac{1}{q_n^2\sqrt{5}}$, where $\frac{p_n}{q_n} = C_n$ is the nth convergent of α. To begin, we see that

$$\left|\frac{p_n}{q_n} - \alpha\right| = \left|\frac{p_n}{q_n} - \frac{a'_{n+1}p_n + p_{n-1}}{a'_{n+1}q_n + q_{n-1}}\right|$$

$$= \left|\frac{p_n a'_{n+1} q_n + p_n q_{n-1} - a'_{n+1} p_n q_n - p_{n-1} q_n}{a'_{n+1} q_n^2 + q_{n-1} q_n}\right| = \left|\frac{p_n q_{n-1} - p_{n-1} q_n}{a'_{n+1} q_n^2 + q_{n-1} q_n}\right|$$

$$= \left|\frac{(-1)^{k-1}}{a'_{n+1} q_n^2 + q_{n-1} q_n}\right| = \frac{1}{a'_{n+1} q_n^2 + q_{n-1} q_n} = \frac{1}{q_n^2\left(a'_{n+1} + \frac{q_{n-1}}{q_n}\right)},$$

by appealing to a previous notational discussion and Lemma 2. If we can show that $a'_i + \frac{q_{i-2}}{q_{i-1}} > \sqrt{5}$ for at least one of the values $i = n - 1$, $i = n$, and $i = n + 1$, this will complete the proof. To show this, suppose to the contrary that $a'_i + \frac{q_{i-2}}{q_{i-1}} \leq \sqrt{5}$ for $i = n - 1$, $i = n$ and $i = n + 1$. The following then hold for $i = n - 1$, $i = n$, and

$i = n + 1$:

$$a'_i = [a_i, a_{i+1}, a_{i+2}, ...] = a_i + \frac{1}{a'_{i+1}}, \tag{1}$$

by definition; defining $b_n = \frac{q_{n-2}}{q_{n-1}}$ for all $n \in \mathbb{N}$,

$$\frac{1}{b_{i+1}} = \frac{q_i}{q_{i-1}} = \frac{q_i - q_{i-2} + q_{i-2}}{q_{i-1}} = \frac{q_i - q_{i-2}}{q_{i-1}} + \frac{q_{i-2}}{q_{i-1}} = a_i + b_i, \tag{2}$$

since $q_i = a_i q_{i-1} + q_{i-2}$; and, combining these,

$$\frac{1}{a'_{i+1}} + \frac{1}{b_{i+1}} = a'_i - a_i + a_i + b_i = a'_i + b_i \leq \sqrt{5}, \tag{3}$$

since we assume $a'_i + b_i = a'_i + \frac{q_{i-2}}{q_{i-1}} \leq \sqrt{5}$ for $i = n - 1$, $i = n$, and $i = n + 1$. Then we have

$$1 = a'_i \frac{1}{a'_i} \leq (\sqrt{5} - b_i)\left(\sqrt{5} - \frac{1}{b_i}\right),$$

for $i = n$ and $i = n + 1$, employing equation (3). This gives the following string of inequalities, again for $i = n$ and $i = n + 1$:

$$1 \leq (\sqrt{5} - b_i)\left(\sqrt{5} - \frac{1}{b_i}\right) = 6 - \sqrt{5}\left(b_i + \frac{1}{b_i}\right)$$

$$\sqrt{5}\left(b_i + \frac{1}{b_i}\right) \leq 5$$

$$b_i + \frac{1}{b_i} \leq \frac{5}{\sqrt{5}} = \sqrt{5}.$$

Since $b_i = \frac{q_{i-2}}{q_{i-1}} \in \mathbb{Q}$ and $\sqrt{5} \notin \mathbb{Q}$, we actually have strict inequality in the last equation, giving the following:

$$b_i + \frac{1}{b_i} - \sqrt{5} < 0$$

$$b_i^2 + 1 - b_i\sqrt{5} < 0$$

$$b_i^2 - b_i\sqrt{5} + \frac{5}{4} < \frac{1}{4}$$

$$\left(\frac{\sqrt{5}}{2} - b_i\right) < \frac{1}{4}$$

$$\frac{\sqrt{5}}{2} - b_i < \frac{1}{2}$$

$$\frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{1}{2}(\sqrt{5}-1) < b_i,$$

once again for $i = n$ and $i = n+1$. Using this last inequality and also equation 2, we finally arrive at

$$a_n = \frac{1}{b_{n+1}} - b_n < \frac{1}{\frac{1}{2}(\sqrt{5}-1)} - \frac{1}{2}(\sqrt{5}-1)$$

$$= \frac{2}{(\sqrt{5}-1)}\frac{(\sqrt{5}+1)}{\sqrt{5}+1} - \frac{1}{2}(\sqrt{5}-1) = \frac{1}{2}(\sqrt{5}+1) - \frac{1}{2}(\sqrt{5}-1) = 1,$$

a contradiction, since we take $a_n \geq 1$ for all $n \in \mathbb{N}$. Hence $a_i' + \frac{q_{i-2}}{q_{i-1}} > \sqrt{5}$ for at least one of the values $i = n-1$, $i = n$, and $i = n+1$, meaning

$$\frac{1}{a_i' + \frac{q_{i-2}}{q_{i-1}}} < \frac{1}{\sqrt{5}}$$

holds for at least one out of every three consecutive convergents of α and hence holds for infinitely many convergents $\frac{p_n}{q_n}$. Thus

$$\left|\frac{p_n}{q_n} - \alpha\right| = \frac{1}{q_n^2\left(a_{n+1}' + \frac{q_{n-1}}{q_n}\right)} < \frac{1}{q_n^2\sqrt{5}}$$

for infinitely many $n \in \mathbb{N}$. □

Theorem 6. *In Theorem 5, the $\sqrt{5}$ is best possible; if it is replaced by any number larger than $\sqrt{5}$ the theorem does not hold for all irrational numbers.*

Proof. We show that if $\sqrt{5}$ is replaced by any larger number, say $A > \sqrt{5}$, then Theorem 5 does not hold specifically for $\alpha = \phi$, meaning there do not exist infinitely many fractions that approximate ϕ with an error term less than $\frac{1}{q^2A}$. Suppose to the contrary that there are infinitely many $\frac{p}{q} \in \mathbb{Q}$ such that $|\frac{p}{q} - \phi| < \frac{1}{q^2A}$. Then there are infinitely many $\frac{p}{q} \in \mathbb{Q}$ such that $-\frac{1}{q^2A} < \frac{p}{q} - \phi < \frac{1}{q^2A}$, meaning that there are infinitely many such rationals such that $\phi = \frac{p}{q} + \frac{\delta}{q^2}$ where $\delta \in \mathbb{R}$ is fixed with $|\delta| < \frac{1}{A} < \frac{1}{\sqrt{5}}$. This gives the following string of equivalences:

$$\phi = \frac{p}{q} + \frac{\delta}{q^2}$$

$$q\left(\frac{1+\sqrt{5}}{2}\right) - p = \frac{\delta}{q}$$

$$\frac{q}{2} - p = \frac{\delta}{q} - \frac{\sqrt{5}q}{2}$$

$$\left(\frac{q}{2} - p\right)^2 = \left(\frac{\delta}{q} - \frac{\sqrt{5}q}{2}\right)^2$$

$$\frac{q^2}{4} + p^2 - pq = \frac{\delta^2}{q^2} + \frac{5q^2}{4} - \delta\sqrt{5}$$

$$q^2 + 4p^2 - 4pq = \frac{4\delta^2}{q^2} + 5q^2 - 4\delta\sqrt{5}$$

$$-4q^2 + 4p^2 - 4pq = \frac{4\delta^2}{q^2} - 4\delta\sqrt{5}$$

$$-q^2 + p^2 - pq = \frac{\delta^2}{q^2} - \delta\sqrt{5}.$$

Let's take a closer look at this last equation. For all $q \in \mathbb{Z}$, we have $-q^2+p^2-pq \in \mathbb{Z}$, so $\frac{\delta^2}{q^2} - \delta\sqrt{5} \in \mathbb{Z}$ for all $q \in \mathbb{Z}$. But we also have

$$\lim_{|q|\to\infty}\left(\frac{\delta^2}{q^2} - \delta\sqrt{5}\right) = \lim_{|q|\to\infty}\left(\frac{\delta^2}{q^2}\right) - \delta\sqrt{5} = -\delta\sqrt{5},$$

so since $|\delta| < \frac{1}{\sqrt{5}}$, we know $\frac{-1}{\sqrt{5}} < \delta < \frac{1}{\sqrt{5}}$ and

$$1 = \frac{\sqrt{5}}{\sqrt{5}} > -\delta\sqrt{5} = \lim_{|q|\to\infty}\left(\frac{\delta^2}{q^2} - \delta\sqrt{5}\right) > \frac{-\sqrt{5}}{\sqrt{5}} = -1.$$

This means that if we pick $q_0 \in \mathbb{Z}$ with $|q_0|$ large enough, we can ensure that $-1 < \frac{\delta^2}{q^2} - \delta\sqrt{5} < 1$ for all $q \in \mathbb{Z}$ with $|q| \geq |q_0|$, giving $-1 < -q^2 + p^2 - pq < 1$ for all such $q \in \mathbb{Z}$. But since $-q^2+p^2-pq \in \mathbb{Z}$, this implies that $-q^2+p^2-pq = p^2-pq-q^2 = 0$ for all such $q \in \mathbb{Z}$. Viewing $p^2 - pq - q^2$ as a quadratic equation in the variable p, the

Quadratic Formula gives

$$p = \frac{q \pm \sqrt{q^2 - 4(-q^2)}}{2} = \frac{q \pm \sqrt{5q^2}}{2} = \frac{q}{2}(1 \pm \sqrt{5}).$$

But we know that $p \in \mathbb{Z}$, so since $\frac{q}{2}(1 \pm \sqrt{5})$ is irrational for all non-zero $q \in \mathbb{Z}$, it must be that $q = 0$. This contradicts $\frac{p}{q} \in \mathbb{Q}$. Hence for all $q \in \mathbb{Z}$ with $|q| \geq |q_0|$, the equation $-q^2 + p^2 - pq = \frac{\delta^2}{q^2} - \delta\sqrt{5}$ is false. Thus there are only finitely many $q \in \mathbb{Z}$ such that $-q^2 + p^2 - pq = \frac{\delta^2}{q^2} - \delta\sqrt{5}$ holds (specifically, all $q \in \mathbb{Z}$ with $-q_0 < q < q_0$). For a fixed $q \in (-q_0, q_0)$, there are at most two options for $p \in \mathbb{Z}$ such that $-q^2 + p^2 - pq = \frac{\delta^2}{q^2} - \delta\sqrt{5}$ holds. This gives only finitely many $\frac{p}{q} \in \mathbb{Q}$ such that $-q^2 + p^2 - pq = \frac{\delta^2}{q^2} - \delta\sqrt{5}$, a contradiction. Thus there must be finitely many $\frac{p}{q} \in \mathbb{Q}$ such that $|\frac{p}{q} - \phi| < \frac{1}{q^2 A}$ for $A > \sqrt{5}$. $\square$

Theorem 7. *For any irrational α that is not equivalent to ϕ, there are infinitely many rational numbers $\frac{p}{q}$ such that $|\frac{p}{q} - \alpha| < \frac{1}{q^2 2\sqrt{2}}$.*

Proof. As in the proof of Theorem 5, we show that for any three consecutive convergents $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$, and $\frac{p_{n+1}}{q_{n+1}}$ of α, at least one must satisfy $|\frac{p_i}{q_i} - \alpha| < \frac{1}{q_i^2 2\sqrt{2}}$. To do this, we again note that

$$\left|\frac{p_n}{q_n} - \alpha\right| = \frac{1}{q_n^2\left(a'_{n+1} + \frac{q_{n-1}}{q_n}\right)},$$

so it is enough to show that $a'_{i+1} + \frac{q_{i-1}}{q_i} > 2\sqrt{2}$ for at least one of $i = n-1$, $i = n$, and $i = n+1$. Suppose to the contrary that $a'_{i+1} + \frac{q_{i-1}}{q_i} > 2\sqrt{2}$ for $i = n-1$, $i = n$, and $i = n+1$, and once again we will define $b_n = \frac{q_{n-2}}{q_{n-1}}$ for all $n \geq 2$. As in the proof of Theorem 5, equations (1) and (2) hold for $i = n-1$, $i = n$, and $i = n+1$, as does the following modified version of equation (3):

$$\frac{1}{a'_{i+1}} + \frac{1}{b_{i+1}} = a'_i - a_i + a_i + b_i = a'_i + b_i \leq 2\sqrt{2}. \tag{4}$$

Then, for $i = n$ and $i = n+1$, we have the following:

$$1 = a_i' \frac{1}{a_i'} \leq (2\sqrt{2} - b_i)\left(2\sqrt{2} - \frac{1}{b_i}\right) = 9 - 2\sqrt{2}\left(b_i + \frac{1}{b_i}\right)$$

$$b_i + \frac{1}{b_i} \leq \frac{8}{2\sqrt{2}} = 2\sqrt{2};$$

since $b_i = \frac{q_{i-2}}{q_{i-1}} \in \mathbb{Q}$ but $2\sqrt{2} \notin \mathbb{Q}$, we have strict inequality in this last equation. Then

$$b_i^2 + 1 - 2\sqrt{2} < 0$$

$$(\sqrt{2} - b_i)^2 < 1$$

$$\sqrt{2} - b_i < 1$$

for $i = n$ and $i = n+1$. This gives

$$a_n = \frac{1}{b_{n+1}} - b_n < \frac{1}{\sqrt{2}-1} - \sqrt{2} + 1 = \frac{1}{\sqrt{2}-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) - \sqrt{2} + 1 = \frac{\sqrt{2}+1}{1} - \sqrt{2} + 1 = 2,$$

which holds for any $n \in \mathbb{N}$. But since α is not equivalent to ϕ, there are infinitely many $k \in \mathbb{N}$ such that $a_k \geq 2$, which gives a contradiction in the above equation for all such $k \in \mathbb{N}$. Then for each such $k \in \mathbb{N}$, at least one of $i = k-1$, $i = k$, and $i = k+1$ must be such that $a_{i+1}' + \frac{q_{i-1}}{q_i} > 2\sqrt{2}$. This means that

$$\left|\frac{p_i}{q_i} - \alpha\right| = \frac{1}{q_i^2\left(a_{i+1}' + \frac{q_{i-1}}{q_i}\right)} < \frac{1}{q_i^2 2\sqrt{2}}$$

holds for at least one of $i = k-1$, $i = k$, and $i = k+1$, where $k \in \mathbb{N}$ is one of the infinitely many integers such that $a_k \geq 2$. This gives infinitely many rational approximations $\frac{p}{q}$ to any α (not equivalent to ϕ) such that

$$\left|\frac{p}{q} - \alpha\right| < \frac{1}{q^2 2\sqrt{2}}.$$

□

Now let's revisit these theorems to determine what they actually tell us. In Theorem 5, we learn that, for any irrational α, there are infinitely many fractions $\frac{p}{q}$ (for example, all convergents of α) that are within a distance of $\frac{1}{q^2\sqrt{5}}$ from α. However, in the proof of Theorem 6, we discover that the Golden Ratio has only finitely many fractions $\frac{p}{q}$ that are within a distance of $\frac{1}{q^2 A}$ from α for any $A > \sqrt{5}$. What is more, in Theorem 7, we learn that for any irrational α whose continued fraction does not end in an infinite string of 1's, there are infinitely many fractions $\frac{p}{q}$ (once again, all convergents of α will work) that are within a distance of $\frac{1}{q^2 2\sqrt{2}}$ from α. So while the Golden Ratio ϕ is as well-approximable by fractions as other irrationals are when we have an error bound of $\frac{1}{q^2\sqrt{5}}$, ϕ fails to be well-approximable by fractions when the error bound is decreased to $\frac{1}{q^2 2\sqrt{2}}$. Since all irrationals that are not in the equivalence class of ϕ (using the equivalence relation that follows from the definition of equivalent numbers) are as well-approximable with the bound of $\frac{1}{q^2 2\sqrt{2}}$ as they are with the bound of $\frac{1}{q^2\sqrt{5}}$, these facts tell us that the equivalence class of ϕ contains exactly those irrational numbers that have the worst rational approximations, in the sense that they do not have infinitely many rational approximations that fall within $\frac{1}{q^2 2\sqrt{2}}$ of α while all other irrationals do. This does not mean that we cannot get arbitrarily close to ϕ using fractions, for we already know that $\lim\limits_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$, meaning that there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|\frac{F_{n+1}}{F_n} - \phi| < \varepsilon$ for any arbitrarily small $\varepsilon > 0$, even for $\varepsilon < \frac{1}{q^2\sqrt{5}}$. The difference is that the $\varepsilon > 0$ that we chose here was fixed and independent of the chosen $\frac{p}{q}$ (in this case, the chosen $\frac{F_{n+1}}{F_n}$), whereas the error bounds discussed in the above theorems depend directly on the denominator of the rational approximation $\frac{p}{q}$ to ϕ, meaning the bounds get smaller as the denominator increases. While this seems like a very nuanced and vague distinction, it is important in that it amounts to the Golden Ratio (and those irrationals in its equivalence class) being least like a fraction as a number can be; ϕ is a representative of the class of the most irrational numbers on the number line.

CHAPTER TWO

How Do Plants Grow, and How Do They Relate to Fibonacci?

At this point, we have mastered much in the realm of Fibonacci numbers and continued fractions, but exactly how does this relate at all to plants? Moreover, why should we even believe that there is a potential relationship between this area of number theory and plant structure? Answering this second question is as easy as taking a closer look at the inhabitants of the closest flowerbed. Examining the head of a sunflower, one can immediately see much order and pattern, but most noticeable are two different sets of spirals that seem to emanate from the center of the flower head; these spirals are called *parastichies*. One family of parastichies on the flower head turns in the clockwise direction while the other family turns in the counter-clockwise direction. These two sets of spirals intertwine together perfectly to create circular symmetry while filling the flower head. If one was to pick any such flower head and count the number of parastichies in the clockwise direction and the number of parastichies in the counter clockwise direction, the odds are quite good that both of these numbers will actually be Fibonacci numbers. What is more, they are often consecutive numbers in the Fibonacci sequence. Turning next to the nearest tree and examining leaf arrangement around a twig, if one estimates the fraction of a full turn between two consecutive leaves, the estimate will usually be a ratio of two Fibonacci numbers (for example, 2/5 of a full turn, 1/3 of a full turn, 8/13 of a full turn, etc.). These numbers are often referred to as *phyllotactic ratios.* If you buy a pineapple at the grocery store and take a good look at the pattern that wraps around its outside, you can see three sets of parallel spirals moving in different directions. Again, the number of spirals in each of the three families should be a Fibonacci number. While these observations may seem coincidental, further investigation will show that spiral patterns and Fibonacci numbers are unbelievably common in nature: they can be seen in pinecones, artichokes, asparagus, broccoli, celery, trees, thorns, flowers, and many other places. With these observations, it should not surprise us that Fibonacci is quite closely related to more than just hypothetical rabbit populations. Fibonacci's sequence is everywhere in nature!

In order to determine the precise relationship between the mathematics of Chapter One and plant structure, we will now develop a foundational knowledge of plant growth and develop this understanding in some specific areas, namely leaf arrangement on a stem and seed distribution on the head of a flower. Though this is not an exhaustive list of how all parts of all plants grow, it is sufficient to eventually see how the prior discussion on Fibonacci numbers, continued fractions, and the Golden Ratio can teach us about the order and pattern that exist in the seemingly-random world of nature. The discussion contained in this chapter is taken from that in Chapter 10 of [1], except where otherwise indicated.

An Introduction to Plant Growth

As with most things, a very good place to start an exposition on plant growth is at the very beginning: namely, the beginning of a plant's life. When a plant stem first begins to grow from a seed, new plant cells are created at the top of the stem, adding height and leaving behind older cells as the plant gets taller. A region called the *apex* is found at the very tip of the stem, and this is where most new cell production occurs. As cells from the apex multiply, they form groups called *primordia*; these groups of cells are what will eventually become leaves, petals, seeds, and the like. These primordia are formed successively from the apex, and what is most interesting is that they form a spiralling pattern, creating what is called the *generative spiral*. As the stem's height increases, the primordia are left behind at various intervals and begin to develop into whichever of the various plant parts they are destined to become. To illustrate the importance of the generative spiral and its relationship to the mathematics we have covered, we will now take a literal look at the apex and its immediate primordia.

Let's examine a hypothetical newborn plant that is currently nothing more than a stem and an apex; we suppose that no primordia exist currently. Cells in the apex begin to multiply and group themselves together to form the first primordium. As cell production continues from the apex, the height of the stem increases, and this first primordium is left behind (though only slightly) while cell production shifts to contribute to and form the next primordium. Once this second primordium is completed, production shifts again to a new third primordium, and the previous two primordia are left to eventually develop as they will. The main question to examine, then, is this: as primordia are produced and left behind, how are they arranged in relation to the apex and in relation to one another? This arrangement (especially with respect to the primordia that develop into leaves around a stem) and the study of the mechanisms that govern it are referred to as *phyllotaxis*.

The question regarding primordia distribution will take some time to examine thoroughly, though it will not necessarily be difficult to answer given our wide mathematical knowledge. To begin the investigation, we must establish a basic understanding of primordia arrangement. As new primordia develop, they must be placed where there is space; a plant would obviously not choose to leave behind primordia from the apex in a straight line, one behind the other, because there is simply more space for each new primordium if it is placed elsewhere. What is more, this straight-line approach would produce a lopsided plant: it would result in a flower whose leaves, petals, and seeds lined up one on top of the other in a straight line along one side of the stem, which is certainly far from the symmetric design we see around us! If a plant then seeks to place the newest primordium in such a way that it has space to grow and develop, it will then space each primordium at some distance away from its immediate neighbors. The most logical plan of attack for the young plant, then, is to choose a constant angle (say, α) that will be the angle between each pair of consecutive primordia because this will ensure that each primordium is given the space it needs to grow. Of course, the resulting pattern will be a spiral (the generative spiral); it turns out that the most common constant angle between consecutive primordia is the Golden Angle. Though we currently have no argument to back up this claim, we will develop a systematic approach that will eventually arrive at this conclusion. Thus, armed only with the knowledge that there is a constant (yet unknown) angle α between two consecutive primordia and that this distribution of primordia results in a spiral (the generative spiral), we will break down the problem of primordia arrangement around the apex into two parts, investigating various examples of plant growth to illustrate the process and shed light on the question. First, in order to determine how successive primordia are arranged in relation to each other, we will discuss how leaves arrange themselves around a stem. Looking then at seed distribution on the head of a flower, we will see an example of how primordia are arranged in relation to the apex.

Leaf Arrangement

Leaves are a very important part of plant anatomy: they take in sunlight to be used in photosynthesis that will ultimately provide the plant with the energy it needs to grow and survive. This explains the thin, flat shape of leaves; their shape maximizes surface area, allowing sunlight to reach the largest leaf area possible and hence giving the plant as much energy at one time as possible. We don't need much mathematics to see why this leaf shape makes sense, but math will come in handy as we explore why leaves arrange themselves as they do around a stem. All leaves on a particular plant serve the same purpose, meaning that their combined efforts add together to all contribute to the plant's energy supply. Just like each individual leaf is shaped so as to maximize the amount of sunlight that can reach it, all of the leaves on a flower appear to arrange themselves around the stem so as to (once again) maximize

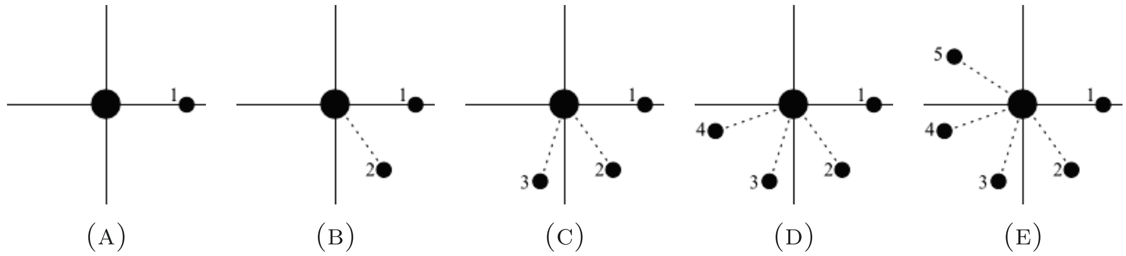
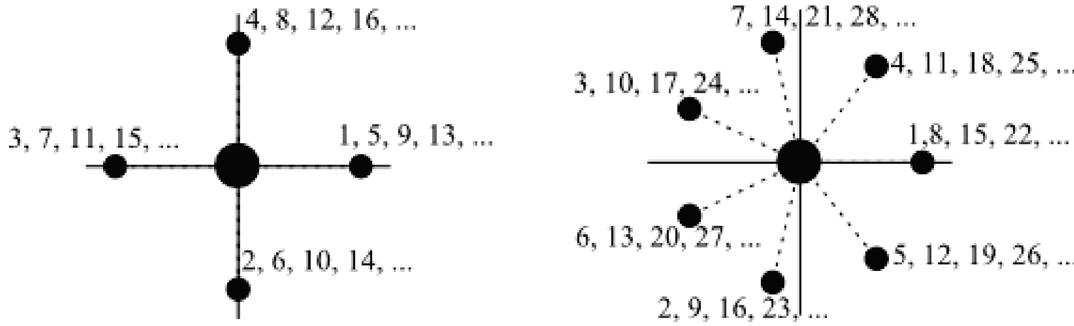


Figure 3. Marking a hypothetical leaf arrangement around a stem (represented by the larger center circle); the bottommost leaf is represented by dot 1, the next lowest by dot 2, etc. Dotted lines represent the rays emanating from the center on which the leaves lie.

the total amount of leaf surface area that can be exposed to sunlight. If this is the case, then it would not make sense, for example, for two leaves to stack one on top of the other because this would prevent the bottom leaf from being able to access sunlight. This brings up a question: how can leaves arrange themselves so that they don't block each other from sunlight?

To answer this question, we must take a literal bird's eye view of plant growth. Armed with the knowledge that primordia appear in a spiralling pattern on a young plant and that some promordia will develop into leaves, we can conclude that leaves appear around a stem in a spiral pattern as well, and at this point we can be fairly confident that the angle between consecutive leaves will be the Golden Angle; this will often be the case, but we would like to arrive at this conclusion by logically developing an argument as to why such an arrangement would make sense. Now, as a stem gets taller, it "leaves behind" leaves, starting at the bottom and working to the top of the stem and distributing the leaves at various intervals around the stem itself. Viewing this process from above, we can imagine a circle with the plant stem at the center. We mark the ray emanating from the stem in the direction towards which the first leaf points as the positive x-axis. Drawing another ray from the center in the direction towards which the second leaf points will give us an angle between these two rays. Drawing such a ray for each successive leaf will give the same angle between each pair of two successive rays. Such a sample progression can be seen in



(A) (B)

FIGURE 4. If we use an angle between two consecutive leaves that is a rational multiple of 360°, we see here that leaves will begin to stack one on top of the other as the cycle of leaf placement repeats. There is an angle of $90° = \frac{1}{4}(360)°$ between consecutive leaves in Diagram (A) and an angle of $\frac{2}{7}(360)°$ between consecutive leaves in Diagram (B). Again, the numbers label the leaves in succession, beginning at the bottom of the stem and working towards the top.

Figure 3; the leaves are labelled successively, from the bottom of the stem upwards. We can use diagrams such as these to help us in seeing the answer to the question of ideal leaf arrangement. Figure 4 shows two such diagrams. Now, the angle between each successive new leaf represents a part of the whole 360° circle. For example, the angle between consecutive leaves in Figure 4(A) is 90°, which is one fourth of the full circle, or $\frac{1}{4}(360)°$. Our question can then be rephrased: what is the angle between successive leaves that results in the best possible leaf arrangement around the stem, or what is the factor by which we must multiply 360° by to get this desired angle? Intuition and the examples in Figure 4(A) and 4(B) show that this factor cannot be a rational number, for if this were the case, we see that leaves would eventually begin stacking on top of each other as the distribution pattern began to repeat, and this would block sunlight from all leaves except the handful that were lucky enough to land on top of the stack. This means that the desired angle between successive leaves cannot be a fraction multiple of 360°, and hence it must be an *irrational* multiple of 360°.

There are only a handful of irrational numbers with which most people are familiar: the number π is perhaps the most well-known, followed by Euler's e, perhaps, $\sqrt{2}$, and, of course, our friend ϕ. This might lead one to think that this irrational-multiple conclusion is a good deal; while we know there are more fractions than we can name, there appear to be only a few irrational numbers, so we must simply test out each $(\pi, e, \sqrt{2}, \phi, \text{etc.})$ to see which gives an ideal leaf arrangement! The truth is, however, that there are many, many more irrational numbers than there are fractions; there are infinitely, infinitely many. How, then, are we supposed to figure out which irrational multiple of 360° gives us the best leaf arrangement? The answer lies, surprisingly, in continued fractions and the Golden Ratio.

Theorem 4, the Best-Approximation Theorem, has shown us that all irrational numbers can be approximated by fractions, and that the fractions that produce the best approximations are in fact the convergents of the irrational number's continued fraction expansion. But Theorem 7 has also shown us that ϕ is in the equivalence class of the least-rational of all irrational numbers, meaning that its convergents give a worse approximation of it than any other convergents approximate any other irrational number. In other words, while all irrational numbers are decidedly not fractions, ϕ is, in a sense, the absolute farthest from being a fraction of all of the irrational numbers. By this logic, it would certainly make sense that the angle that we seek between leaves is the full 360° multiplied by a factor of ϕ. But this is a number we know! We see that $\phi(360) = (1.61803398...)(360) = 582.4922328...$, which amounts to a full 360° rotation around the circle plus an extra 222.4922328...° (in the counter-clockwise direction, as convention dictates); this is, of course, equivalent to simply an angle of 222.4922328...°. But 222.4922328...° in the counter-clockwise direction is the same thing as 137.5077672...° in the clockwise direction, which is precisely θ, the Golden Angle, as predicted.

Let's summarize this conclusion one more time. If we are spacing out leaves around a stem, we have seen quite clearly that if the angle between two successive leaves is a fraction multiple of the full circle, then leaves will very quickly begin to stack on top of each other as the pattern around the stem continues. This will block sunlight from many leaves. Then it must be that the angle between two consecutive leaves must be an irrational sector of the full circle, because an irrational angle will ensure that leaves never line up exactly one on top of the other. However, we have seen that irrational numbers can be approximated with fractions even though their values are not fractions themselves; the better the rational approximations are for a given irrational α, the closer leaves will come to stacking one on top of the other, as if we were using a rational angle between leaves. Therefore, the irrational number that will allow a plant to expose the largest amount of leaf area to sunlight will be the number that has the worst rational approximations. We have seen that this number is precisely the Golden Ratio ϕ, whose angular counter-part is the Golden Angle θ. Thus it would seem that a plant should space out its leaves at an angle of $\theta = 137.5077672...^\circ$ apart so as to create the most effective use of leaf surface area in harvesting sunlight.

Seed Distribution

Seed distribution is the second verse of the same song as leaf distribution. We can apply the logic of leaf distribution to see precisely why the Golden Angle leads to a closest-packing design of seeds on a flower head, meaning a pattern that allows no overlap of seeds or gaps between seeds. But while in the case of leaves we needed to only determine the optimal angle between consecutive leaves, the case for seeds will add an additional element to the question, for we must now determine the optimal angle between consecutive seeds as well as the optimal distance of each seed from the center of the flower head. Figure 5 provides a diagram to illustrate how seed distribution can be represented graphically. Now, similarly to the case of leaves, placing consecutive seeds at a fractional multiple of 360° apart will result in a wagon-wheel arrangement full of gaps (see Figure 6), so we know we need an angle that is an

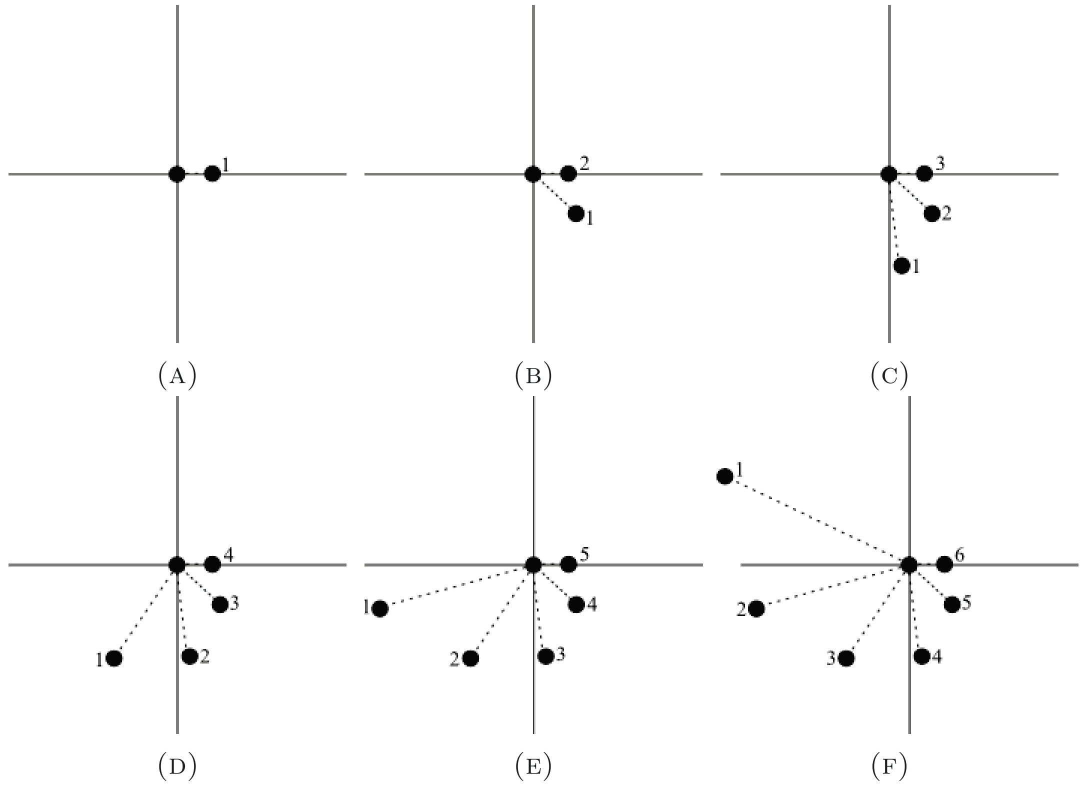
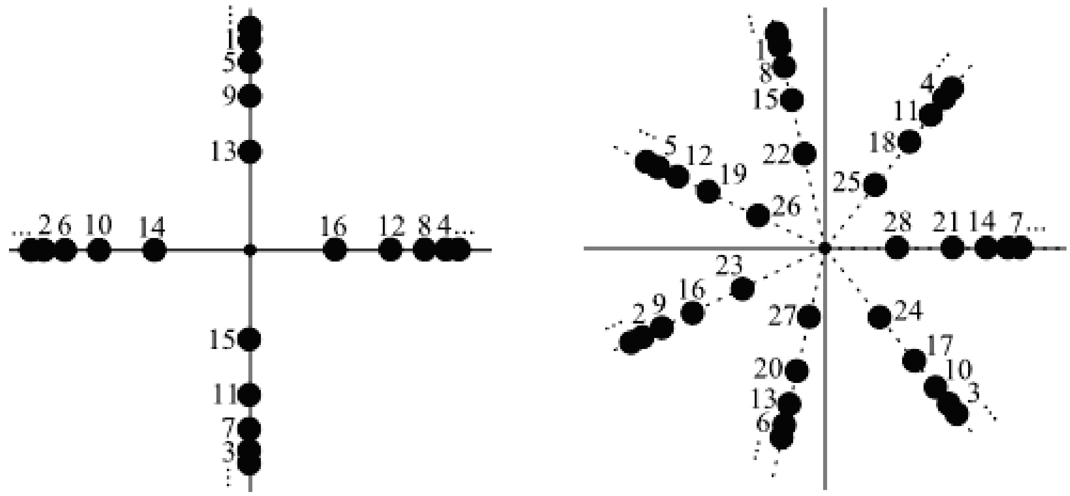


Figure 5. Marking a hypothetical seed arrangement around an apex (represented by the dot at the origin); dot 1 represents the first primordium created from the apex, dot 2 represents the next primordium created by the apex, etc. As more primordia are formed, those already in existence are forced farther away from the center and begin to form a spiralling pattern. Here, the x- and y-axes serve simply as a point of reference; we have arranged the diagrams so that the most recent primordium appears along the positive x-axis.

irrational multiple of 360°. The Golden Angle, of course, is as poorly approximable by fractions as they come, so as the number of seeds increases, the seed arrangement will be as far from resembling the wagon-wheel patterns seen in Figure 6 as possible; no other irrational multiple of 360° will be as far from this pattern as the Golden Angle is, which is a consequence of the Best-Approximation Theorem. The farthest thing from a wagon-wheel pattern is, of course, a closest-packing in which all seeds are evenly spaced apart. Hence the Golden Angle does provide a closest-packing of seeds in that it allows seeds to be spaced out optimally from each other. It is most interesting that even slight deviations from the Golden Angle lead to drastic changes



(A) (B)

FIGURE 6. Figure (A) shows an angle of $90^\circ = \frac{1}{4}(360)^\circ$ between consecutive seeds, while Figure (B) shows an angle of $\frac{2}{7}(360)^\circ$ between consecutive seeds. Again, seeds are labelled in succession, beginning with the earliest-developed seeds, which are forced outward as new seeds are produced from the center. The resulting wagon-wheel pattern will result for any angle between consecutive seeds that is a rational multiple of 360°.

in the pattern seen on a flower head; for examples of this, see [1]. In addition, the two sets of spiralling families of parastichies hinted at in the opening discussion of this chapter are the most clearly observed when the Golden Angle is employed; in [8], Naylor presents flower heads constructed using angles of $\sqrt{2}(360)^\circ$ and $\pi(360)^\circ$. The parastichies in these flower heads, though still visible, become much more difficult to trace, particularly as the number of seeds grows. It appears, then, that employing the Golden Ratio gives a flower head the optimal arrangement of seeds, and that arrangements resulting from other angles actually do not come close to producing such an ideal closest-packing as the Golden Angle does. However, this sheds no light yet on how each seed should be spaced apart from the center of the flower head.

To determine the optimal distance between the center of the flower head and each individual seed, we present a model outlined by Micheal Naylor in [8] and referenced as well in [1]. Suppose that there are n seeds on the head of a flower, each having an area of one square unit. If it is assumed that the seeds are arranged so as to produce a closest-packing, then the area of the flower head will be equal to the sum of the areas of each of the individual seeds. Since there are n seeds each having an area of one square unit, the area of the circular flower head will be approximately $A = n$. The general formula for the area of a circle, however, is $A = \pi r^2$, where r is the radius of the circle. Hence we can write $A = \pi r^2 = n$, where r is the radius of the flower head, giving $r^2 = \frac{n}{\pi}$ and $r = \sqrt{\frac{n}{\pi}} = \frac{1}{\sqrt{\pi}}\sqrt{n}$. This means that the radius of the flower head is equal to a constant (namely, a constant quite close to $\frac{1}{\sqrt{\pi}}$) multiplied by the square root of the total number of seeds on the flower head; we say then that the radius is proportional to the square root of the number n of seeds. What does this tell us about the distance between individual seeds and the center? Assuming that the seed farthest from the center lies on the perimeter of the circle constituting the head of the flower, the distance between this seed and the center will actually be equal to the radius of the flower head, which we know should be proportional to the square root of the total number of seeds. In his model, Naylor takes the constant of proportionality between the radius and the number of seeds to be 1, so that a flower head of n seeds will have a radius of length $\sqrt{n}$. But dismissing the seed farthest away from the center gives a circular patch of $n-1$ seeds, the radius of which must be $\sqrt{n-1}$, according to Naylor's model, giving us the distance between the center and this $(n-1)$st seed. Dismissing the two seeds farthest from the center on our flower head of n seeds gives a circular patch of $n-2$ seeds; its radius (and hence the distance between the $(n-2)$nd seed and the center) will be $\sqrt{n-2}$. In fact, for any $k \leq n$, the distance between the center of the flower head and the kth seed from the center will be $\sqrt{k}$, the radius of the circular flower head containing k seeds.

This then gives us a complete understanding of how seed arrangement on a flower head should optimally work. If we have placed k seeds on the flower head, the $(k+1)$st seed from the center should be placed at a distance of $\sqrt{k+1}$ from the center and an angle of $\phi(360)^\circ$ from the kth seed from the center. This uniquely defines the location of each individual seed, giving us a precise picture of what a closest-packing of seeds on a flower head should look like. This generalizes quite easily to the arrangement of primordia around the apex, because to define the location of each primordium, we must know its distance from the apex and the angle that it makes with the previous primordium. For the precise reasons discussed in the investigation of seed arrangement, each primordium should be at a distance proportional to the square root of the total number of existing primordia away from the apex, and the angle between this primordium and its immediate predecessor should be the Golden Angle. Since seeds, leaves, petals, thorns, and all other plant parts develop from primordia, this gives us a complete model for plant growth: an efficiently-growing plant will space consecutive petals, consecutive seeds, consecutive thorns, etc. apart by the Golden Angle and will space them out from the center of the stem at a distance proportional to the square root of the total number of primordia, when necessary.

The presence of the Fibonacci numbers observed in the opening discussion of this chapter stems from the relationship between Fibonacci numbers and the Golden Ratio as well as the important role that the Golden Ratio plays in plant development. Each individual occurrence of Fibonacci numbers in nature, from the phyllotactic ratios observed in leaf arrangement on a tree to the number of parastichies in a particular spiral family and beyond, has its own story to tell that adds even more intricacies and details to the relationship between plants and the Golden Ratio. While there is not time here to investigate each such occurrence, [8] and [4] are good resources with which one could begin. This discussion on primordia and the Golden Angle simply serves to illustrate the fact that Fibonacci numbers, the Golden Ratio, and continued fractions appear to be intrinsically related to fundamental plant growth and development; the pattern and beauty that we see in flower seeds and petals, leaves on a tree, stalks of a celery plant, and florets of broccoli are all a result of basic primordia development and its relationship to mathematics.

What is most interesting about this optimized model of plant growth is that it is not the only one, nor is it necessarily the most widely accepted. There are many approaches to modelling plant growth, and many of them help us to better understand plant development, but the truth is that, like most models, each is an imperfect representation of true plant growth. These models and theories have been developed, improved, and revised for quite a long time (for a history of phyllotactic theory, see [2]), but they still cease to present a perfect representation of how plants actually grow in practice. It certainly is amazing that even after such intensive and prolonged study, we are unable to fully understand and accurately reproduce something as simple as plant structure and development. Though mathematics provides us with an overabundance of knowledge and insight in many areas of life, even it cannot unlock all of the secrets and dispel all of the mystery that the world around us holds.

Concluding Remarks

We have now seen a few examples of how the Golden Ratio and continued fractions can explain how a logically-functioning plant should arrange its leaves and its seeds in order to make the most efficient use of its space and resources, but this seems to be a hypothetical understanding of plant growth. While it is easy enough to say "if I were a plant, I would choose to space my leaves precisely the Golden Angle apart from each other, because that makes mathematical sense", it is much more difficult to say that all plants *do* actually employ the Golden Angle, and even if we could say that all plants do this, it would be yet another great leap to say that all plants do this *because it makes the most mathematical sense*. The point here is that it is impossible to determine whether or not all plants space their seeds out according to the Golden Angle; there are simply too many plants to check, and what is more, there are plants that do not follow such a rule. That being said, there has been work conducted to determine whether or not this relationship between math and plant growth is strong enough to imply plausible connection. One such in-depth study conducted on pine cones can be seen in [4]. This investigation sought to determine whether spiralling patterns of parastichies in pine cones consistently follow Fibonacci-like patterns. By examining thousands of pine cones from a wide range of conifer trees, investigators saw that for the majority of tree species considered, only between 1% and 2% of cones examined deviated from the usual pattern of two spirals moving in opposite directions around the cone, one consisting of 8 spiral families and the other of 5 families. In a handful of cases, 100% of cones examined conformed to this pattern. Though this is not definitive evidence that the mathematics of Fibonacci numbers, the Golden Ratio, and continued fractions governs plant growth, it does imply a correlation of some form.

If nothing else, the study of math and plant growth at least makes us stop and think about the relationship between these two entities. There is strong evidence for much order and design in plant development, and the potential for such a connection between math and nature suggests a beauty and logic hidden in the world in which we live. Though we are able to do nothing more than conjecture that the Golden Ratio, Fibonacci numbers, and continued fractions govern the development and structure of plants, this apparent connection does suggest that there is some force of order driving plant growth. This order demands the existence of a Creator. Whether He purposefully employed mathematics in this creation or whether this mathematical and biological connection is simply the coincidental result of an entirely unrelated divine blueprint, God chose pattern, logic, purpose, and order to create the captivating and beautiful world in which we live, and through this He has built bridges between the natural world and mathematics that help each discipline deepen one's understanding of the other and of the world in general.

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