ABSTRACT

Comparison of Smallest Eigenvalues and Extremal Points for Third and Fourth Order Three Point Boundary Value Problems

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The theory of u_0-positive operators with respect to a cone in a Banach space is applied to the linear differential equations $u^{(4)} + \lambda_1 p(x)u = 0$ and $u^{(4)} + \lambda_2 q(x)u = 0$, $0 \leq x \leq 1$, with each satisfying the boundary conditions $u(0) = u'(r) = u''(r) = u'''(1) = 0$, $0 < r < 1$. The existence of smallest positive eigenvalues is established, and a comparison theorem for smallest positive eigenvalues is obtained. These results are then extended to the nth order problem using two different methods. One method involves finding the Green's function for $-u^{(n)} = 0$ satisfying the higher order boundary conditions, and the other involves making a substitution that allows us to work with a variation of the fourth order problem. Extremal points via Krein-Rutman theory are then found. Analogous results are then obtained for the eigenvalue problems $u''' + \lambda_1 p(x)u = 0$ and $u''' + \lambda_2 q(x)u = 0$, with each satisfying $u(0) = u'(r) = u''(1) = 0$, $0 < 1/2 < r < 1$.

Comparison of Smallest Eigenvalues and Extremal Points for Third and Fourth Order Three Point Boundary Value Problems

by

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CHAPTER ONE

Introduction

1.1 Overview

In this dissertation, we will consider two eigenvalue problems. First, we consider the comparison of eigenvalues for the eigenvalue problems

$$u^{(4)} + \lambda_1 p(x)u = 0, \tag{1.1}$$

$$u^{(4)} + \lambda_2 q(x)u = 0, \tag{1.2}$$

satisfying the boundary conditions

$$u(0) = u'(r) = u''(r) = u'''(1) = 0, \tag{1.3}$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0, 1]$.

The focus in the second chapter of the dissertation will be on comparing the smallest eigenvalues for these eigenvalue problems. First, using the theory of u_0-positive operators with respect to a cone in a Banach space, we establish the existence of smallest eigenvalues for (1.1),(1.3), and (1.2),(1.3), and then compare these smallest eigenvalues after assuming a relationship between $p(x)$ and $q(x)$. We then extend these results to the nth order case using two different methods. First, in Chapter 3, we establish the properties of the Green's function for the nth order problem, and by using these properties, we are able to again establish the existence of smallest eigenvalues and then derive the comparison results. In Chapter 4, we use a substitution method so that we can work with fourth order eigenvalue problems that have the same eigenvalues as the nth order problem. Comparison results are then obtained.

We then consider the comparison of eigenvalues for the eigenvalue problems

$$u''' + \lambda_1 p(x)u = 0, \tag{1.4}$$

$$u''' + \lambda_2 q(x)u = 0, \tag{1.5}$$

each of which satisfies the boundary conditions

$$u(0) = u'(r) = u''(1) = 0, \tag{1.6}$$

where $0 < 1/2 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0, 1]$. Results analogous to the ones in Chapters 2, 3, and 4 are found for these eigenvalue problems in Chapters 5, 6, and 7.

The technique for the comparison of these eigenvalues involve the application of sign properties of the Green's function, followed by the application of u_0-positive operators with respect to a cone in a Banach space. These applications are presented in books by Krasnoselskii [23] and by Krein and Rutman [22].

Several authors have before applied these techniques in comparing eigenvalues for different boundary problems than the ones seen here. Previous work has been devoted to boundary value problems for ordinary differential equations involving conjugate, Lidstone, and right focal conditions. For example, Eloe and Henderson have studied smalleset eigenvalue comparisons for a class of two-point boundary value problems [8], and for a class of multipoint boundary value problems [9]. Karna has also studied smallest eigenvalue comparisons for m-point boundary value problems [18] and three-point boundary value problems [19]. In addition, comparison results have been obtained for difference equations [14] and for boundary value problems on time scales [2, 4, 16, 17, 24]. For additional work on this field, see [3, 10, 11, 13, 15, 20, 27, 28].

In the final chapter, we characterize extremal points for both a fourth order problem and the third order problem via Krein-Rutman theory. We show there exists a smallest interval such that there exists at least one nontrivial solution for a fourth order three point problem and a third order three point problem. For the theory used in this chapter, we refer the reader to Amann [1], Deimling [5], Krein and Rutman [22], Schmidt and Smith [26], and Zeidler [29].

There has also been work done on extremal points. Eloe, Hankerson, and Henderson characterized extremal points for a class of multipoint boundary value problem [6] and for a class of two point boundary value problems [7]. Eloe, Henderson, and Thompson characterized extremal points for impulsive Lidstone boundary value problems [12].

1.2 *Preliminary Definitions and Theorems*

Definition 1.1. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and

(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

Definition 1.2. A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^\circ$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

Remark 1.1. Krasnosel'skii [23] showed that every solid cone is reproducing.

Definition 1.3. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, $u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

Definition 1.4. A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $0 \neq u_0 \in \mathcal{P}$ such that for each $0 \neq u \in \mathcal{P}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$.

The following three results are fundamental to our comparison results and are attributed to Krasnosel'skii [23]. The proof of Lemma 1.1 is provided, the proof of

Theorem 1.1 can be found in Krasnosel'skii's book [23], and the proof of Theorem 1.2 is provide by Keener and Travis [21] as an extension of Krasonel'skii's results.

Lemma 1.1. *Let $\mathcal{B}$ be a Banach space over the reals, and let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $M : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $M : \mathcal{P}\backslash\{0\} \to \mathcal{P}^\circ$, then M is u_0-positive with respect to $\mathcal{P}$.*

Proof. Choose any $u_0 \in \mathcal{P}\backslash\{0\}$, and let $u \in \mathcal{P}\backslash\{0\}$. So $Mu \in \Omega \subset \mathcal{P}^\circ$. Choose $k_1 > 0$ sufficiently small and k_2 sufficiently large so that $Mu - k_1 u_0 \in \mathcal{P}^\circ$ and $u_0 - \frac{1}{k_2} Mu \in \mathcal{P}^\circ$. So $k_1 u_0 \leq Mu$ with respect to $\mathcal{P}$ and $Mu \leq k_2 u_0$ with respect to $\mathcal{P}$. Thus $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$ and so M is u_0-positive with respect to P. □

Theorem 1.1. *Let $\mathcal{B}$ be a real Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive, linear operator. Then L has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 1.2. *Let $\mathcal{B}$ be a real Banach space and $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators and assume that at least one of the operators is u_0-positive. If $M \leq N$, $Mu_1 \geq \lambda_1 u_1$ for some $u_1 \in \mathcal{P}$ and some $\lambda_1 > 0$, and $Nu_2 \leq \lambda_2 u_2$ for some $u_2 \in \mathcal{P}$ and some $\lambda_2 > 0$, then $\lambda_1 \leq \lambda_2$. Futhermore, $\lambda_1 = \lambda_2$ implies u_1 is a scalar multiple of u_2.*

CHAPTER TWO

The Fourth Order Problem

In this chapter, we consider the fourth order eigenvalue problems

$$u^{(4)} + \lambda_1 p(x) u = 0, \tag{2.1}$$

$$u^{(4)} + \lambda_2 q(x) u = 0, \tag{2.2}$$

satisfying the boundary conditions

$$u(0) = u'(r) = u''(r) = u'''(1) = 0, \tag{2.3}$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0,1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0,1]$.

We derive comparison results for these fourth order eigenvalue problems by applying the theorems mentioned in the Introduction. To do this, we will define integral operators whose kernel is the Green's function for $-u^{(4)} = 0$ satisfying (2.3).

This Green's function is given by

$$G(x,s) = \begin{cases} \frac{s^3}{6}, & s \le r, s \le x, \\ \frac{(x-r)^3 + r^3}{6}, & s > r, s > x, \\ \frac{(x-s)^3 + s^3}{6}, & s \le r, s > x, \\ \frac{r^3 + (s-x)^3 + (x-r)^3}{6}, & s > r, s \le x. \end{cases}$$

So $u(x)$ solves (2.1),(2.3) if and only if $u(x) = \lambda_1 \int_0^1 G(x,s) p(s) u(s) ds$, and $u(x)$ solves (2.2),(2.3) if and only if $u(x) = \lambda_2 \int_0^1 G(x,s) q(s) u(s) ds$. Note $G(x,s) \ge 0$ on $[0,1] \times [0,1]$, $G(x,s) > 0$ on $(0,1] \times (0,1]$, and $\frac{\partial}{\partial x} G(x,s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 1.1 and 1.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{u \in C^1[0,1] \mid u(0) = 0\}$$

with the norm

$$||u|| = \sup_{0 \le x \le 1} |u'(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(x) \ge 0 \text{ on } [0,1]\}.$$

Notice that for $u \in \mathcal{B}$, $0 \le x \le 1$,

$$\begin{aligned} |u(x)| = |u(x) - u(0)| &= \left| \int_0^x u'(s)ds \right| \\ &\le ||u||x \\ &\le ||u||, \end{aligned}$$

and so $\sup\limits_{0 \le x \le 1} |u(x)| \le ||u||$.

Lemma 2.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u(x) > 0 \text{ on } (0,1] \text{ and } u'(0) > 0\}.$$

Note $\Omega \subset \mathcal{P}$. Choose $u \in \Omega$ and define $B_\epsilon(u) = \{v \in B \mid ||u - v|| < \epsilon\}$ for $\epsilon > 0$. Choose $\epsilon_0 > 0$ such that $u'(0) - \epsilon_0 > 0$. So for $v \in B_{\epsilon_0}(u)$, $\sup\limits_{0 \le x \le 1} |v'(x) - u'(x)| < \epsilon_0$. So $v'(0) > u'(0) - \epsilon_0 > 0$. Also, $|v(x) - u(x)| \le ||v - u|| < \epsilon_0$, and so $v(x) > 0$ on $(0,1]$. So $v \in \Omega$ and hence $B_{\epsilon_0}(u) \subset \Omega \subset \mathcal{P}$ and $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. $\square$

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mu(x) = \int_0^1 G(x,s)p(s)u(s)ds, \ 0 \le x \le 1,$$

and

$$Nu(x) = \int_0^1 G(x,s)q(s)u(s)ds, \ 0 \le x \le 1.$$

Lemma 2.2. *The linear operators M and N are compact.*

Proof. We prove the statement for M only. The proof for N is similiar. We will use the Arzelá-Ascoli theorem to show that M is a compact operator. To do this, we need to show that M is continuous, and for any bounded sequence $\{u_n\}$ in $\mathcal{B}$, the sequence $\{Mu_n\}$ is unformly bounded and equicontinuous.

Let $u, v \in \mathcal{B}$. Since $p(x)$ is a nonnegative continuous function on $[0,1]$, $p(x)$, has a maximum value. Define this maximum value, $\sup_{0\le x\le 1}\{p(x)\} = L$. Since $\frac{\partial}{\partial x}G(x,s)$ is bounded, let $K = \sup_{(x,s)\in[0,1]\times[0,1]}\left\{\frac{\partial}{\partial x}G(x,s)\right\}$. Then, for $\epsilon > 0$, there exists $\delta = \dfrac{\epsilon}{LK} > 0$ such that if $||u-v|| < \delta$, for any $x \in [0,1]$,

$$\begin{aligned}
|Mu'(x) - Mv'(x)| &= \left|\int_0^1 \frac{\partial}{\partial x}G(x,s)p(s)(u(s)-v(s))ds\right| \\
&\le \int_0^1 \frac{\partial}{\partial x}G(x,s)p(s)|(u(s)-v(s))|ds \\
&< LK\delta = \epsilon.
\end{aligned}$$

So, if $||u-v|| < \delta$, then $\sup_{0\le x\le 1}|Mu'(x) - Mv'(x)| < \epsilon$. Thus, for $||u-v|| < \delta$, $||Mu - Mv|| < \epsilon$. Hence M is continuous.

Let $\{u_n\}$ be a bounded sequence in $\mathcal{B}$ and let $||u_n|| \le K_0$ for all n. Since $Mu_n(x) = \int_0^1 G(x,s)p(s)u_n(s)ds$, we have

$$\begin{aligned}
|Mu_n'(x)| &= \left|\int_0^1 \frac{\partial}{\partial x}G(x,s)p(s)u_n(s)ds\right| \\
&\le KK_0L,
\end{aligned}$$

for all n. So $\{Mu_n\}$ is uniformly bounded.

Finally, since $\frac{\partial}{\partial x}G(x,s)$ is continuous for any fixed s, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|x_1 - x_2| < \delta$, $\left|\frac{\partial}{\partial x}G(x_1,s) - \frac{\partial}{\partial x}G(x_2,s)\right| < \frac{\epsilon}{LK_0}$. Then, for any n,

$$\begin{aligned}
|Mu_n'(x_1) - Mu_n'(x_2)| &\le \int_0^1 \left|\frac{\partial}{\partial x}G(x_1,s) - \frac{\partial}{\partial x}G(x_2,s)\right| p(s)u_n(s)ds \\
&< \frac{\epsilon}{LK_0}LK_0 = \epsilon.
\end{aligned}$$

So, if $|x_1 - x_2| < \delta$, then $||Mu_n(x_1) - Mu_n(x_2)|| < \epsilon$ for all n. Therefore, $\{Mu_n\}$ is equicontinuous. Therefore, by the Arzela-Ascoli theorem, M is a compact operator.

$\square$

Lemma 2.3. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(x) \geq 0$. Then since $G(x,s) \geq 0$ on $[0,1] \times [0,1]$ and $p(x) \geq 0$ on $[0,1]$,

$$Mu(x) = \int_0^1 G(x,s)p(s)u(s)ds \geq 0,$$

for $0 \leq x \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0,1]$ such that $u(x) > 0$ and $p(x) > 0$ for all $x \in [\alpha, \beta]$. Then, since $G(x,s) > 0$ on $(0,1] \times (0,1]$,

$$\begin{aligned} Mu(x) &= \int_0^1 G(x,s)p(s)u(s)ds \\ &\geq \int_\alpha^\beta G(x,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < x \leq 1$. Also, since $\frac{\partial}{\partial x}G(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mu)'(0) &= \int_0^1 \frac{\partial}{\partial x}G(0,s)p(s)u(s)ds \\ &\geq \int_\alpha^\beta \frac{\partial}{\partial x}G(0,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

and so $Mu \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 1.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. $\square$

Remark 2.1. Notice that

$$\Lambda u = Mu = \int_0^1 G(x,s)p(s)u(s)ds,$$

if and only if

$$u(x) = \frac{1}{\Lambda} \int_0^1 G(x,s)p(s)u(s)ds,$$

if and only if

$$-u^{(4)}(x) = \frac{1}{\Lambda} p(x)u(x),\ 0 \le x \le 1,$$

with

$$u(0) = u'(r) = u''(r) = u'''(1) = 0.$$

So the eigenvalues of (2.1),(2.3) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (2.2),(2.3) are reciprocals of eigenvalues of N, and conversely.

Theorem 2.1. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 1.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. □

Theorem 2.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \le q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 2.1 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \le q(x)$ on $[0,1]$. So for any $u \in \mathcal{P}$, and $x \in [0,1]$,

$$(Nu - Mu)(x) = \int_0^1 G(x,s)(q(s) - p(s))u(s)ds \ge 0.$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \le N$ with respect to $\mathcal{P}$. Then by Theorem 1.2, $\Lambda_1 \le \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0, 1]$. Then $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = M u_1 + \epsilon u_1 \leq N u_1$, implying $N u_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $N u_2 = \Lambda_2 u_2$, by Theorem 1.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. □

By Remark 2.1, the following theorem is an immediate consequence of Theorems 2.1 and 2.2.

Theorem 2.3. *Assume the hypotheses of Theorem 2.2. Then there exists smallest positive eigenvalues* λ_1 *and* λ_2 *of* (2.1),(2.3) *and* (2.2),(2.3)*, respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to* λ_1 *and* λ_2 *may be chosen to belong to* $\mathcal{P}^\circ$. *Finally,* $\lambda_1 \geq \lambda_2$, *and* $\lambda_1 = \lambda_2$ *if and only if* $p(x) = q(x)$ *for* $0 \leq x \leq 1$.

CHAPTER THREE

Extending the Fourth Order Problem Using the Green's Function

3.1 Introduction

In this chapter, we will extend the results of the previous fourth order problem to the nth order problems

$$u^{(n)} + \lambda_1 p(x) u = 0,$$

$$u^{(n)} + \lambda_2 q(x) u = 0,$$

satisfying the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-4)}(0) = u^{(n-3)}(r) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0,$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0, 1]$.

To do this, we will find the sign properties of the Green's function for $-u^{(n)} = 0$ satisfying the boundary conditions just stated. We will need the Green's function for the fourth order problem to find these sign properties. We first show the extension of the fourth order problem to the fifth order problem. Then, we extend to the nth order problem.

3.2 The Fifth Order Problem

We now consider the eigenvalue problems

$$u^{(5)} + \lambda_1 p(x) u = 0, \tag{3.1}$$

$$u^{(5)} + \lambda_2 q(x) u = 0, \tag{3.2}$$

satisfying the boundary conditions

$$u(0) = u'(0) = u''(r) = u'''(r) = u^{(4)}(1) = 0, \tag{3.3}$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0,1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0,1]$.

Here we will use methods similar to the methods used in the previous chapter to derive comparison theorems for these fifth order eigenvalue problems. We will do this by finding the Green's function, which we will call $G_5(x,s)$, for $-u^{(5)} = 0$ satisfying (3.3). This Green's function is continuous and $\frac{\partial}{\partial x}G_5(x,s) = G(x,s)$, where $G(x,s)$ is as defined earlier. Therefore, the Green's function is

$$G_5(x,s) = \begin{cases} \frac{4s^3x-s^4}{24}, & s \le r, s \le x, \\ \frac{(x-r)^4+4r^3x-r^4}{24}, & s > r, s > x, \\ \frac{(x-s)^4+4s^3-s^4}{24}, & s \le r, s > x, \\ \frac{4r^3x-(s-x)^4+(x-r)^4}{24}, & s > r, s \le x. \end{cases}$$

Now $u(x)$ solves (3.1),(3.3) if and only if $u(x) = \lambda_1 \int_0^1 G_5(x,s)p(s)u(s)ds$, and $u(x)$ solves (3.2),(3.3) if and only if $u(x) = \lambda_2 \int_0^1 G_5(x,s)q(s)u(s)ds$.

Since $\frac{\partial}{\partial x}G_5(x,s) = G(x,s)$, $\frac{\partial}{\partial x}G_5(x,s) \ge 0$ on $[0,1]\times[0,1]$ and $\frac{\partial}{\partial x}G_5(x,s) > 0$ on $(0,1]\times(0,1]$. Also, since $\frac{\partial^2}{\partial x^2}G_5(x,s) = \frac{\partial}{\partial x}G(x,s)$, then $\frac{\partial^2}{\partial x^2}G_5(x,s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 1.1 and 1.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{u \in C^2[0,1] \mid u(0) = u'(0) = 0\},$$

with the norm

$$||u|| = \sup_{0\le x\le 1} |u''(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u'(x) \ge 0 \text{ on } [0,1]\}.$$

Notice that for $u \in \mathcal{B}$, $0 \le x \le 1$,

$$|u'(x)| = |u'(x) - u'(0)| = \left|\int_0^x u''(s)ds\right|$$

$$\leq ||u||x$$

$$\leq ||u||,$$

and so $\sup_{0\leq x\leq 1} |u'(x)| \leq ||u||$.

Lemma 3.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u'(x) > 0 \text{ on } (0,1] \text{ and } u''(0) > 0\}.$$

Note $\Omega \subset \mathcal{P}$. Choose $u \in \Omega$ and define $B_\epsilon(u) = \{v \in B \mid ||u - v|| < \epsilon\}$ for $\epsilon > 0$. Choose $\epsilon_0 > 0$ such that $u''(0) - \epsilon_0 > 0$. So for $v \in B_{\epsilon_0}(u)$, $\sup_{0\leq x\leq 1} |v''(x) - u''(x)| < \epsilon_0$. So $v''(0) > u''(0) - \epsilon_0 > 0$. Also, $|v'(x) - u'(x)| \leq ||v - u|| < \epsilon_0$, and so $v'(x) > 0$ on $(0,1]$. So $v \in \Omega$ and hence $B_{\epsilon_0}(u) \subset \Omega \subset \mathcal{P}$ and $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. $\square$

Next, we define our linear operators M and N by

$$Mu(x) = \int_0^1 G_5(x,s)p(s)u(s)ds,\ 0 \leq x \leq 1,$$

and

$$Nu(x) = \int_0^1 G_5(x,s)q(s)u(s)ds,\ 0 \leq x \leq 1.$$

Since $G_5(0,s) = \frac{\partial}{\partial x}G_5(x,s)|_{x=0} = 0$, $M, N : \mathcal{B} \to \mathcal{B}$. A standard application of the Arzelá-Ascoli theorem shows that M and N are compact.

Lemma 3.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(x) \geq 0$. Then, since $\frac{\partial}{\partial x}G_5(x,s) = G(x,s) \geq 0$ on $[0,1] \times [0,1]$ and $p(x) \geq 0$ on $[0,1]$,

$$Mu'(x) = \int_0^1 \frac{\partial}{\partial x} G_5(x,s)p(s)u(s)ds \geq 0,$$

for $0 \leq x \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0, 1]$ such that $u(x) > 0$ and $p(x) > 0$ for all $x \in [\alpha, \beta]$. Then, since $\frac{\partial}{\partial x} G_5(x, s) > 0$ on $(0, 1] \times (0, 1]$,

$$\begin{aligned} Mu'(x) &= \int_0^1 \frac{\partial}{\partial x} G_5(x,s)p(s)u(s)ds \\ &\geq \int_\alpha^\beta \frac{\partial}{\partial x} G_5(x,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < x \leq 1$. Also, since $\frac{\partial^2}{\partial x^2} G_5(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mu)''(0) &= \int_0^1 \frac{\partial^2}{\partial x^2} G_5(0,s)p(s)u(s)ds \\ &\geq \int_\alpha^\beta \frac{\partial^2}{\partial x^2} G_5(0,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

and so $Mu \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 1.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. □

Remark 3.1. Notice that

$$\Lambda u = Mu = \int_0^1 G_5(x,s)p(s)u(s)ds,$$

if and only if

$$u(x) = \frac{1}{\Lambda} \int_0^1 G_5(x,s)p(s)u(s)ds,$$

if and only if

$$-u^{(5)}(x) = \frac{1}{\Lambda} p(x)u(x), \ 0 \leq x \leq 1,$$

with

$$u(0) = u'(0) = u''(r) = u'''(r) = u^{(4)}(1) = 0.$$

So the eigenvalues of (3.1),(3.3) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (3.2),(3.3) are reciprocals of eigenvalues of N, and conversely.

Theorem 3.1. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 1.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. $\square$

Theorem 3.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \leq q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 3.1 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \leq q(x)$ on $[0,1]$. So for any $u \in \mathcal{P}$, and $x \in [0,1]$,

$$(Nu - Mu)'(x) = \int_0^1 \frac{\partial}{\partial x} G_5(x,s)(q(s) - p(s))u(s)ds \geq 0.$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 1.2, $\Lambda_1 < \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0,1]$. Then $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N-M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 1.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. $\square$

By Remark 3.1, the following theorem is an immediate consequence of Theorems 3.1 and 3.2.

Theorem 3.3. *Assume the hypotheses of Theorem 3.2. Then there exists smallest positive eigenvalues* λ_1 *and* λ_2 *of* (3.1),(3.3) *and* (3.2),(3.3)*, respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to* λ_1 *and* λ_2 *may be chosen to belong to* $\mathcal{P}^\circ$*. Finally,* $\lambda_1 \geq \lambda_2$*, and* $\lambda_1 = \lambda_2$ *if and only if* $p(x) = q(x)$ *for* $0 \leq x \leq 1$.

3.3 The nth Order Problem

Let $n \in \mathbb{N}$, $n \geq 5$. In this section, we will consider the eigenvalue problems

$$u^{(n)} + \lambda_1 p(x)u = 0, \tag{3.4}$$

$$u^{(n)} + \lambda_2 q(x)u = 0, \tag{3.5}$$

satisfying the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-4)}(0) = u^{(n-3)}(r) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0, \tag{3.6}$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0,1]$, where neither $p(x)$ nor $q(x)$ vanish identically on any compact subinterval of $[0,1]$.

Again, we will use methods similar to the methods used previously to derive comparison theorems for these nth order eigenvalue problems. We will do this by finding the the sign properties of the Green's function, which we will call $G_n(x,s)$, for $-u^{(n)} = 0$ satisfying (3.6). This Green's function, as a function of x, is $C^{(n-4)}[0,1]$, and $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s) = G(x,s)$, where $G(x,s)$ is as defined earlier.

Now $u(x)$ solves (3.4),(3.6) if and only if $u(x) = \lambda_1 \int_0^1 G_n(x,s)p(s)u(s)ds$, and $u(x)$ solves (3.5),(3.6) if and only if $u(x) = \lambda_2 \int_0^1 G_n(x,s)q(s)u(s)ds$.

Since $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s) = G(x,s)$, then $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s) \geq 0$ on $[0,1] \times [0,1]$ and $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s) > 0$ on $(0,1] \times (0,1]$. Also, since $\frac{\partial^{n-3}}{\partial x^{n-3}} G_n(x,s) = \frac{\partial}{\partial x} G(x,s)$, then $\frac{\partial^{n-3}}{\partial x^{n-3}} G_n(x,s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 1.1 and 1.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{u \in C^{(n-3)}[0,1] \mid u(0) = u'(0) = \cdots = u^{(n-4)}(0) = 0\}$$

with the norm

$$||u|| = \sup_{0 \le x \le 1} |u^{(n-3)}(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u^{(n-4)}(x) \ge 0 \text{ on } [0,1]\}.$$

Notice that for $u \in \mathcal{B}$, $0 \le x \le 1$,

$$\begin{aligned} |u^{(n-4)}(x)| = |u^{(n-4)}(x) - u^{(n-4)}(0)| &= \left| \int_0^x u^{(n-3)}(s)ds \right| \\ &\le ||u||x \\ &\le ||u||, \end{aligned}$$

and so $\sup_{0 \le x \le 1} |u^{(n-4)}(x)| \le ||u||$.

Lemma 3.3. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u^{(n-4)}(x) > 0 \text{ on } (0,1] \text{ and } u^{(n-3)}(0) > 0\}.$$

Note $\Omega \subset \mathcal{P}$. Choose $u \in \Omega$ and define $B_\epsilon(u) = \{v \in B \mid ||u - v|| < \epsilon\}$ for $\epsilon > 0$. Choose $\epsilon_0 > 0$ such that $u^{(n-3)}(0) - \epsilon_0 > 0$. So for $v \in B_{\epsilon_0}(u)$, $\sup_{0 \le x \le 1} |v^{(n-3)}(x) - u^{(n-3)}(x)| < \epsilon_0$. So $v^{(n-3)}(0) > u^{(n-3)}(0) - \epsilon_0 > 0$. Also, $|v^{(n-4)}(x) - u^{(n-4)}(x)| \le ||v - u|| < \epsilon_0$, and so $v^{(n-4)}(x) > 0$ on $(0,1]$. So $v \in \Omega$ and hence $B_{\epsilon_0}(u) \subset \Omega \subset \mathcal{P}$, and $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. $\square$

Next, we define our linear operators M and N by

$$Mu(x) = \int_0^1 G_n(x,s)p(s)u(s)ds, \ 0 \le x \le 1,$$

and

$$Nu(x) = \int_0^1 G_n(x,s)q(s)u(s)ds,\ 0 \le x \le 1.$$

Note that since $\frac{\partial^{n-i}}{\partial x^{n-i}} G_n(x,s)|_{x=0} = 0$ for $i = 4, 5, \ldots, n$, then $M, N : \mathcal{B} \to \mathcal{B}$. A standard application of the Arzela-Ascoli theorem shows that M and N are compact.

Lemma 3.4. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(x) \ge 0$. Then, since $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s) = G(x,s) \ge 0$ on $[0,1] \times [0,1]$ and $p(x) \ge 0$ on $[0,1]$,

$$Mu^{(n-4)}(x) = \int_0^1 \frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s)p(s)u(s)ds \ge 0,$$

for $0 \le x \le 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0,1]$ such that $u(x) > 0$ and $p(x) > 0$ for all $x \in [\alpha, \beta]$. Then, since $\frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s) > 0$ on $(0,1] \times (0,1]$,

$$\begin{aligned} Mu^{(n-4)}(x) &= \int_0^1 \frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s)p(s)u(s)ds \\ &\ge \int_\alpha^\beta \frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s)(x,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < x \le 1$. Also, since $\frac{\partial^{n-3}}{\partial x^{n-3}} G_n(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mu)^{(n-3)}(0) &= \int_0^1 \frac{\partial^{n-3}}{\partial x^{n-3}} G_n(0,s)p(s)u(s)ds \\ &\ge \int_\alpha^\beta \frac{\partial^{n-3}}{\partial x^{n-3}} G_n(0,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

and so $Mu \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 1.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. $\square$

Remark 3.2. Notice that

$$\Lambda u = Mu = \int_0^1 G_n(x,s)p(s)u(s)ds,$$

if and only if

$$u(x) = \frac{1}{\Lambda}\int_0^1 G_n(x,s)p(s)u(s)ds,$$

if and only if

$$-u^{(n)}(x) = \frac{1}{\Lambda}p(x)u(x),\ 0 \le x \le 1,$$

with

$$u(0) = u'(0) = \cdots = u^{(n-4)}(0) = u^{(n-3)}(r) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0.$$

So the eigenvalues of (3.4),(3.6) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (3.5),(3.6) are reciprocals of eigenvalues of N, and conversely.

Theorem 3.4. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 1.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. $\square$

Theorem 3.5. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \le q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 3.4 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$ and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \le q(x)$ on $[0,1]$. So for any $u \in \mathcal{P}$, and $x \in [0,1]$,

$$(Nu - Mu)^{(n-4)}(x) = \int_0^1 \frac{\partial^{n-4}}{\partial x^{n-4}} G_n(x,s)(q(s)-p(s))u(s)ds \ge 0.$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 1.2, $\Lambda_1 \leq \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0, 1]$. Then $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 1.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. $\square$

By Remark 3.2, the following theorem is an immediate consequence of Theorems 3.4 and 3.5.

Theorem 3.6. *Assume the hypotheses of Theorem 3.5. Then there exists smallest positive eigenvalues λ_1 and λ_2 of* (3.4),(3.6) *and* (3.5),(3.6)*, respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$ and $\lambda_1 = \lambda_2$ if and only if $p(x) = q(x)$ for $0 \leq x \leq 1$.*

CHAPTER FOUR

Extending the Fourth Order Problem Using Substitution

4.1 Introduction

In this chapter, we will again extend the results of the fourth order problem to the nth order problem

$$u^{(n)} + \lambda_1 p(x) u = 0,$$

$$u^{(n)} + \lambda_2 q(x) u = 0,$$

satisfying the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-4)}(0) = u^{(n-3)}(r) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0,$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0, 1]$.

Instead of using the sign properties of the Green's function for the nth order equation to derive the comparison theorems, we will instead make a substitution and work with a variation of the fourth order problem. This method has its benefits, since we do not need to find the sign properties of the Green's function of the nth order problem, and can instead work with the fourth order problem. We will again start with the fifth order problem and then look at the nth order problem.

4.2 The Fifth Order Problem

We now consider the eigenvalue problems

$$u^{(5)} + \lambda_1 p(x) u = 0, \tag{4.1}$$

$$u^{(5)} + \lambda_2 q(x) u = 0, \tag{4.2}$$

satisfying the boundary conditions

$$u(0) = u'(0) = u''(r) = u'''(r) = u^{(4)}(1) = 0, \tag{4.3}$$

and the eigenvalue problems

$$v^{(4)} + \lambda_1 p(x) \int_0^x v(s)ds = 0, \tag{4.4}$$

$$v^{(4)} + \lambda_2 q(x) \int_0^x v(s)ds = 0, \tag{4.5}$$

satisfying the boundary condtions

$$v(0) = v'(r) = v''(r) = v'''(1) = 0, \tag{4.6}$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0, 1]$.

First we note that if $u(x)$ is a solution to (4.1),(4.3), then $u'(x)$ solves (4.4),(4.6). Also, if $v(x)$ is a solution to (4.4),(4.6), then $\int_0^x v(s)ds$ is a solution to (4.1),(4.3). Similarly, if $u(x)$ is a solution to (4.2),(4.3), then $u'(x)$ solves (4.5),(4.6), and if $v(x)$ is a solution to (4.5),(4.6), then $\int_0^x v(s)ds$ is a solution to (4.2),(4.3).

Now let λ be an eigenvalue of (4.1),(4.3) with the corresponding eigenvector $u(x)$. Then $u'(x)$ is a solution to (4.4),(4.6) with the same eigenvalue λ. Also, if λ is an eigenvalue of (4.4),(4.6) with corresponding eigenvector $v(x)$, then $\int_0^x v(s)ds$ is a solution to (4.1),(4.3) with the corresponding eigenvalue λ. So eigenvalues of (4.1),(4.3) are eigenvalues of (4.4),(4.6), and vice versa. Similarly, eigenvalues of (4.2),(4.3) are eigenvalues of (4.5),(4.6), and vice versa. So any comparison theorems for (4.4),(4.6), and (4.5),(4.6) will apply to (4.1),(4.3), and (4.2),(4.3).

For these reasons, we will derive comparison theorems for eigenvalue problems (4.4),(4.6), and (4.5),(4.6), and then use these theorems to derive the comparison theorems for (4.1),(4.3), and (4.2),(4.3).

Let $G(x, s)$ by the Green's function for $-v^{(4)} = 0$ satisfying (4.6), which was defined earlier. So $v(x)$ solves (4.4),(4.6) if and only if

$$v(x) = \lambda_1 \int_0^1 G(x, s)p(s) \int_0^s v(t)dtds,$$

and $v(x)$ solves (4.5),(4.6) if and only if

$$v(x) = \lambda_2 \int_0^1 G(x,s)q(s) \int_0^s v(t)dtds.$$

Again, note $G(x,s) \geq 0$ on $[0,1]\times[0,1]$, $G(x,s) > 0$ on $(0,1]\times(0,1]$, and $\frac{\partial}{\partial x}G(x,s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 1.1 and 1.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{v \in C^1[0,1] \mid v(0) = 0\}$$

with the norm

$$||v|| = \sup_{0\leq x\leq 1} |v'(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{v \in \mathcal{B} \mid v(x) \geq 0 \text{ on } [0,1]\}.$$

Notice that for $v \in \mathcal{B}$, $0 \leq x \leq 1$,

$$\begin{aligned} |v(x)| = |v(x) - v(0)| &= \left| \int_0^x v'(s)ds \right| \\ &\leq ||v||x \\ &\leq ||v||, \end{aligned}$$

and so $\sup_{0\leq x\leq 1} |v(x)| \leq ||v||$.

Lemma 4.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{v \in \mathcal{B} \mid v(x) > 0 \text{ on } (0,1] \text{ and } v'(0) > 0\}.$$

It was shown in Chapter 2 that $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. □

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mv(x) = \int_0^1 G(x,s)p(s)\int_0^s v(t)dtds,\ 0 \le x \le 1,$$

and

$$Nv(x) = \int_0^1 G(x,s)q(s)\int_0^s v(t)dtds,\ 0 \le x \le 1.$$

A standard application of the Arzelá-Ascoli theorem shows that M and N are compact.

Lemma 4.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $v \in \mathcal{P}$. So $v(x) \ge 0$. Then since $G(x,s) \ge 0$ on $[0,1]\times[0,1]$, $p(x) \ge 0$ on $[0,1]$ and $\int_0^x v(s)ds \ge 0$,

$$Mv(x) = \int_0^1 G(x,s)p(s)\int_0^s v(t)dtds \ge 0,$$

for $0 \le x \le 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $v \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha,\beta] \subset [0,1]$ such that $\int_0^x v(s)ds > 0$ and $p(x) > 0$ for all $x \in [\alpha,\beta]$. Then, since $G(x,s) > 0$ on $(0,1]\times(0,1]$,

$$\begin{aligned} Mv(x) &= \int_0^1 G(x,s)p(s)\int_0^s v(t)dtds \\ &\ge \int_\alpha^\beta G(x,s)p(s)\int_0^s v(t)dtds \\ &> 0, \end{aligned}$$

for $0 < x \le 1$. Also, since $\frac{\partial}{\partial x}G(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mv)'(0) &= \int_0^1 \frac{\partial}{\partial x}G(0,s)p(s)\int_0^s v(t)dtds \\ &\ge \int_\alpha^\beta \frac{\partial}{\partial x}G(0,s)p(s)\int_0^s v(t)dtds \\ &> 0, \end{aligned}$$

and so $Mv \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 1.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. $\square$

Remark 4.1. Notice that

$$\Lambda v = Mv = \int_0^1 G(x,s)p(s)\int_0^s v(t)dtds,$$

if and only if

$$v(x) = \frac{1}{\Lambda}\int_0^1 G(x,s)p(s)\int_0^s v(t)dtds,$$

if and only if

$$-v^{(4)}(x) = \frac{1}{\Lambda}p(x)\int_0^x v(s)ds,\ 0 \le x \le 1,$$

with

$$v(0) = v'(r) = v''(r) = v'''(1) = 0.$$

So the eigenvalues of (4.4),(4.6) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (4.5),(4.6) are reciprocals of eigenvalues of N, and conversely.

Theorem 4.1. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 1.1, M has an essentially unique eigenvector, say $v \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $v \neq 0$, $Mv \in \Omega \subset \mathcal{P}^\circ$ and $v = M\left(\frac{1}{\Lambda}v\right) \in \mathcal{P}^\circ$. $\square$

Theorem 4.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \le q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 4.1 associated with M and N, respectively, with the essentially unique eigenvectors v_1 and $v_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \leq q(x)$ on $[0,1]$. So for any $v \in \mathcal{P}$, and $x \in [0,1]$,

$$(Nv - Mv)(x) = \int_0^1 G(x,s)(q(s) - p(s)) \int_0^s v(t)dtds \geq 0.$$

So $Nv - Mv \in \mathcal{P}$ for all $v \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 1.2, $\Lambda_1 \leq \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0,1]$. Then $(N - M)v_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N - M)v_1 - \epsilon v_1 \in \mathcal{P}$. So $\Lambda_1 v_1 + \epsilon v_1 = Mv_1 + \epsilon v_1 \leq Nv_1$, implying $Nv_1 \geq (\Lambda_1 + \epsilon)v_1$. Since $N \leq N$ and $Nv_2 = \Lambda_2 v_2$, by Theorem 1.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. □

By Remark 4.1, the following theorem is an immediate consequence of Theorems 4.1 and 4.2.

Theorem 4.3. *Assume the hypotheses of Theorem 4.2. Then there exists smallest positive eigenvalues λ_1 and λ_2 of* (4.4),(4.6) *(and hence* (4.1),(4.3)*) and* (4.5),(4.6) *(and hence* (4.2),(4.3)*), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(x) = q(x)$ for $0 \leq x \leq 1$.*

4.3 The nth Order Problem

Let $n \in \mathbb{N}$, $n \geq 5$. In this section, we consider the eigenvalue problems

$$u^{(n)} + \lambda_1 p(x)u = 0, \tag{4.7}$$

$$u^{(n)} + \lambda_2 q(x)u = 0, \tag{4.8}$$

satisfying the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-4)}(0) = u^{(n-3)}(r) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0, \tag{4.9}$$

and the eigenvalue problems

$$v^{(4)} + \lambda_1 p(x) \frac{1}{(n-5)!} \int_0^x (x-s)^{n-5} v(s) ds = 0, \tag{4.10}$$

$$v^{(4)} + \lambda_2 q(x) \frac{1}{(n-5)!} \int_0^x (x-s)^{n-5} v(s) ds = 0, \tag{4.11}$$

satisfying the boundary condtions

$$v(0) = v'(r) = v''(r) = v'''(1) = 0, \tag{4.12}$$

where $0 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0, 1]$.

First we note that if $u(x)$ is a solution to (4.7),(4.9), then $u^{(n-4)}(x)$ solves (4.10),(4.12). Also, if $v(x)$ is a solution to (4.10),(4.12), then $\frac{1}{(n-5)!} \int_0^x (x-s)^{n-5} v(s) ds$ is a solution to (4.7),(4.9). Similarly, if $u(x)$ is a solution to (4.8),(4.9), then $u^{(n-4)}(x)$ solves (4.11),(4.12) and if $v(x)$ is a solution to (4.11),(4.12), then $\frac{1}{(n-5)!} \int_0^x (x - s)^{n-5} v(s) ds$ is a solution to (4.8),(4.9).

Now let λ be an eigenvalue of (4.7),(4.9) with the corresponding eigenvector $u(x)$. Then $u^{(n-4)}(x)$ is a solution to (4.10),(4.12) with the same eigenvalue λ. Also, if λ is an eigenvalue of (4.10),(4.12) with corresponding eigenvector $v(x)$, then $\frac{1}{(n-5)!} \int_0^x (x - s)^{n-5} v(s) ds$ is a solution to (4.7),(4.9) with the corresponding eigenvalue λ. So eigenvalues of (4.7),(4.9) are eigenvalues of (4.10),(4.12), and vice versa. Similarly, eigenvalues of (4.8),(4.9) are eigenvalues of (4.11),(4.12), and vice versa. So any comparison theorems for (4.10),(4.12), and (4.11),(4.12) will apply to (4.7),(4.9), and (4.8),(4.9).

For these reasons, we will derive comparison theorems for eigenvalue problems (4.10),(4.12), and (4.11),(4.12), and then use these theorems to derive the comparison theorems for (4.7),(4.9), and (4.8),(4.9).

Let $G(x,s)$ by the Green's function for $-v^{(4)}=0$ satisfying (4.12), which was defined earlier. So $v(x)$ solves (4.10),(4.12) if and only if

$$v(x)=\lambda_1\int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds,$$

and $v(x)$ solves (4.11),(4.12) if and only if

$$v(x)=\lambda_2\int_0^1 G(x,s)q(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds.$$

Again, note $G(x,s)\geq 0$ on $[0,1]\times[0,1]$, $G(x,s)>0$ on $(0,1]\times(0,1]$, and $\frac{\partial}{\partial x}G(x,s)|_{x=0}>0$ for $0<s<1$.

To apply Theorems 1.1 and 1.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P}\subset\mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B}=\{v\in C^1[0,1]\mid v(0)=0\}$$

with the norm

$$||v||=\sup_{0\leq x\leq 1}|v'(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P}=\{v\in\mathcal{B}\mid v(x)\geq 0 \text{ on } [0,1]\}.$$

Notice that for $v\in\mathcal{B}$, $0\leq x\leq 1$,

$$\begin{aligned}|v(x)|=|v(x)-v(0)|&=\left|\int_0^x v'(s)ds\right|\\&\leq ||v||x\\&\leq ||v||,\end{aligned}$$

and so $\sup_{0\leq x\leq 1}|v(x)|\leq ||v||$.

Lemma 4.3. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega=\{v\in\mathcal{B}\mid v(x)>0 \text{ on } (0,1] \text{ and } v'(0)>0\}.$$

It was shown in Chapter 2 that $\Omega\subset\mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. $\square$

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mv(x) = \int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds,\ 0 \le x \le 1,$$

and

$$Nv(x) = \int_0^1 G(x,s)q(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds,\ 0 \le x \le 1.$$

A standard application of the Arzelá-Ascoli theorem shows that M and N are compact.

Lemma 4.4. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $v \in \mathcal{P}$. So $v(x) \ge 0$. Then since $G(x,s) \ge 0$ on $[0,1]\times[0,1]$, $p(x) \ge 0$ on $[0,1]$ and $\frac{1}{(n-5)!}\int_0^x (x-s)^{n-5}v(s)ds \ge 0$,

$$Mv(x) = \int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \ge 0,$$

for $0 \le x \le 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $v \in \mathcal{P}\backslash\{0\}$. Since $(x-s)^{n-5} > 0$ for $0 \le s < x$, there exists a compact interval $[\alpha,\beta] \subset [0,1]$ such that $\frac{1}{(n-5)!}\int_0^x (x-s)^{n-5}v(s)ds > 0$ and $p(x) > 0$ for all $x \in [\alpha,\beta]$. Then, since $G(x,s) > 0$ on $(0,1]\times(0,1]$,

$$\begin{aligned} Mv(x) &= \int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \\ &\ge \int_\alpha^\beta G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \\ &> 0, \end{aligned}$$

for $0 < x \le 1$. Also, since $\frac{\partial}{\partial x}G(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mv)'(0) &= \int_0^1 \frac{\partial}{\partial x}G(0,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \\ &\ge \int_\alpha^\beta \frac{\partial}{\partial x}G(0,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \\ &> 0, \end{aligned}$$

and so $Mv \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 1.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. $\square$

Remark 4.2. Notice that

$$\Lambda v = Mv = \int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds,$$

if and only if

$$v(x) = \frac{1}{\Lambda}\int_0^1 G(x,s)p(s)\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds,$$

if and only if

$$-v^{(4)}(x) = \frac{1}{\Lambda}p(x)\frac{1}{(n-5)!}\int_0^x (x-s)^{n-5}v(s)ds, \ 0 \le x \le 1,$$

with

$$v(0) = v'(r) = v''(r) = v'''(1) = 0.$$

So the eigenvalues of (4.10),(4.12) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (4.11),(4.12) are reciprocals of eigenvalues of N, and conversely.

Theorem 4.4. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 1.1, M has an essentially unique eigenvector, say $v \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $v \neq 0$, $Mv \in \Omega \subset \mathcal{P}^\circ$ and $v = M\left(\frac{1}{\Lambda}v\right) \in \mathcal{P}^\circ$. $\square$

Theorem 4.5. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \le q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 4.4 associated with M and N, respectively, with the essentially unique eigenvectors v_1 and $v_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \leq q(x)$ on $[0,1]$. So for any $v \in \mathcal{P}$, and $x \in [0,1]$,

$$(Nv - Mv)(x) = \int_0^1 G(x,s)(q(s)-p(s))\frac{1}{(n-5)!}\int_0^s (s-t)^{n-5}v(t)dtds \geq 0.$$

So $Nv - Mv \in \mathcal{P}$ for all $v \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 1.2, $\Lambda_1 \leq \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0,1]$. Then $(N-M)v_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N-M)v_1 - \epsilon v_1 \in \mathcal{P}$. So $\Lambda_1 v_1 + \epsilon v_1 = Mv_1 + \epsilon v_1 \leq Nv_1$, implying $Nv_1 \geq (\Lambda_1 + \epsilon)v_1$. Since $N \leq N$ and $Nv_2 = \Lambda_2 v_2$, by Theorem 1.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. □

By Remark 4.2, the following theorem is an immediate consequence of Theorems 4.4 and 4.5.

Theorem 4.6. *Assume the hypotheses of Theorem 4.5. Then there exists smallest positive eigenvalues λ_1 and λ_2 of* (4.10),(4.12) *(and hence* (4.7),(4.9)*) and* (4.11),(4.12) *(and hence* (4.8),(4.9)*), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(x) = q(x)$ for $0 \leq x \leq 1$.*

CHAPTER FIVE

The Third Order Problem

In this chapter, we consider the third order eigenvalue problems

$$u''' + \lambda_1 p(x)u = 0, \tag{5.1}$$

$$u''' + \lambda_2 q(x)u = 0, \tag{5.2}$$

each of which satisfies the boundary conditions

$$u(0) = u'(r) = u''(1) = 0, \tag{5.3}$$

where $0 < 1/2 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0,1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0,1]$.

We derive comparison results for these third order eigenvalue problems by applying the theorems mentioned in the introduction. To do this, we will define integral operators, each of whose kernel is the Green's function for $-u^{(3)} = 0$ satisfying (5.3).

This Green's function is given by

$$H(x,s) = \begin{cases} \frac{s^2}{2}, & s \le r, s \le x, \\ \frac{x(2r-x)}{2}, & s \ge r, s \ge x, \\ \frac{x(2s-x)}{2}, & s \le r, s \ge x, \\ \frac{x(2r-x)+(x-s)^2}{2}, & s \ge r, s \le x. \end{cases}$$

So $u(x)$ solves (5.1),(5.3) if and only if $u(x) = \lambda_1 \int_0^1 H(x,s)p(s)u(s)ds$, and $u(x)$ solves (5.2),(5.3) if and only if $u(x) = \lambda_2 \int_0^1 H(x,s)q(s)u(s)ds$. Also, note $H(x,s) \ge 0$ on $[0,1]\times[0,1]$, $H(x,s) > 0$ on $(0,1]\times(0,1]$, and $\frac{\partial}{\partial x}H(x,s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 1.1 and 1.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{u \in C^1[0,1] \mid u(0) = 0\}$$

with the norm

$$||u|| = \sup_{0\le x\le 1} |u'(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(x) \ge 0 \text{ on } [0,1]\}.$$

Notice that for $u \in \mathcal{B}$, $0 \le x \le 1$,

$$\begin{aligned} |u(x)| = |u(x) - u(0)| &= \left| \int_0^x u'(s)ds \right| \\ &\le ||u||x \\ &\le ||u||, \end{aligned}$$

and so $\sup_{0\le x\le 1} |u(x)| \le ||u||$.

Lemma 5.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u(x) > 0 \text{ on } (0,1] \text{ and } u'(0) > 0\}.$$

Note $\Omega \subset \mathcal{P}$. Choose $u \in \Omega$ and define $B_\epsilon(u) = \{v \in B \mid ||u - v|| < \epsilon\}$ for $\epsilon > 0$. Choose $\epsilon_0 > 0$ such that $u'(0) - \epsilon_0 > 0$. So for $v \in B_{\epsilon_0}(u)$, $\sup_{0\le x\le 1} |v'(x) - u'(x)| < \epsilon_0$. So $v'(0) > u'(0) - \epsilon_0 > 0$. Also, $|v(x) - u(x)| \le ||v - u|| < \epsilon_0$, and so $v(x) > 0$ on $(0,1]$. So $v \in \Omega$ and hence $B_{\epsilon_0}(u) \subset \Omega \subset \mathcal{P}$ and $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. $\square$

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mu(x) = \int_0^1 H(x,s)p(s)u(s)ds, \ 0 \le x \le 1,$$

and

$$Nu(x) = \int_0^1 H(x,s)q(s)u(s)ds, \ 0 \le x \le 1.$$

A standard application of the Arzelá-Ascoli theorem shows that M and N are compact.

Lemma 5.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(x) \geq 0$. Then since $H(x,s) \geq 0$ on $[0,1] \times [0,1]$ and $p(x) \geq 0$ on $[0,1]$,

$$Mu(x) = \int_0^1 H(x,s)p(s)u(s)ds \geq 0,$$

for $0 \leq x \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0,1]$ such that $u(x) > 0$ and $p(x) > 0$ for all $x \in [\alpha, \beta]$. Then, since $H(x,s) > 0$ on $(0,1] \times (0,1]$,

$$\begin{aligned} Mu(x) &= \int_0^1 H(x,s)p(s)u(s)ds \\ &\geq \int_\alpha^\beta H(x,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < x \leq 1$. Also, since $\frac{\partial}{\partial x} H(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mu)'(0) &= \int_0^1 \frac{\partial}{\partial x} H(0,s)p(s)u(s)ds \\ &\geq \int_\alpha^\beta \frac{\partial}{\partial x} H(0,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

and so $Mu \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 1.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. $\square$

Remark 5.1. Notice that

$$\Lambda u = Mu = \int_0^1 H(x,s)p(s)u(s)ds,$$

if and only if

$$u(x) = \frac{1}{\Lambda} \int_0^1 H(x,s)p(s)u(s)ds,$$

if and only if

$$-u^{(3)}(x) = \frac{1}{\Lambda}p(x)u(x),\ 0 \le x \le 1,$$

with

$$u(0) = u'(r) = u''(1) = 0.$$

So the eigenvalues of (5.1),(5.3) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (5.2),(5.3) are reciprocals of eigenvalues of N, and conversely.

Theorem 5.1. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 1.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. $\square$

Theorem 5.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \le q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 5.1 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \le q(x)$ on $[0,1]$. So for any $u \in \mathcal{P}$, and $x \in [0,1]$,

$$(Nu - Mu)(x) = \int_0^1 H(x,s)(q(s) - p(s))u(s)ds \ge 0.$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \le N$ with respect to $\mathcal{P}$. Then by Theorem 1.2, $\Lambda_1 \le \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0,1]$. Then $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \le Nu_1$, implying

$Nu_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 1.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. $\square$

By Remark 5.1, the following theorem is an immediate consequence of Theorems 5.1 and 5.2.

Theorem 5.3. *Assume the hypotheses of Theorem 5.2. Then there exists smallest positive eigenvalues* λ_1 *and* λ_2 *of* (5.1),(5.3) *and* (5.2),(5.3), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to* λ_1 *and* λ_2 *may be chosen to belong to* $\mathcal{P}^\circ$. *Finally,* $\lambda_1 \geq \lambda_2$, *and* $\lambda_1 = \lambda_2$ *if and only if* $p(x) = q(x)$ *for* $0 \leq x \leq 1$.

CHAPTER SIX

Extending the Fourth Order Problem Using the Green's Function

6.1 Introduction

In this chapter, we will extend the results of the previous third order problem to the nth order problem

$$u^{(n)} + \lambda_1 p(x) u = 0,$$

$$u^{(n)} + \lambda_2 q(x) u = 0,$$

satisfying the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-1)}(r) = u^{(n-1)}(1) = 0,$$

where $0 < 1/2 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0, 1]$.

To do this, we will find the sign properties of the Green's function for $-u^{(n)} = 0$ satisfying the boundary conditions just stated. We will need the Green's function for the third order problem to find these sign properties. We first show the extension of the third order problem to the fourth order problem. Then, we extend to the nth order problem.

6.2 The Fourth Order Problem

We now consider the eigenvalue problems

$$u^{(4)} + \lambda_1 p(x) u = 0, \tag{6.1}$$

$$u^{(4)} + \lambda_2 q(x) u = 0, \tag{6.2}$$

satisfying the boundary conditions

$$u(0) = u'(0) = u''(r) = u'''(1) = 0, \tag{6.3}$$

where $0 < 1/2 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0,1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0,1]$.

Here we will use methods similar to the methods used in the previous chapter to derive comparison theorems for these fourth order eigenvalue problems. We will do this by finding the Green's function, which we will call $H_4(x,s)$, for $-u^{(4)} = 0$ satisfying (6.3). This Green's function is continuous and $\frac{\partial}{\partial x} H_4(x,s) = H(x,s)$, where $H(x,s)$ is as defined earlier. Therefore, the Green's function is

$$H_4(x,s) = \begin{cases} \frac{3s^2x - s^3}{6}, & s \le r, s \le x, \\ \frac{3rx^2 - x^3}{6}, & s \ge r, s \ge x, \\ \frac{3sx^2 - x^3}{6}, & s \le r, s \ge x, \\ \frac{3rx^2 - 2x^3 + (x-s)^3}{6}, & s \ge r, s \le x. \end{cases}$$

Now $u(x)$ solves (6.1),(6.3) if and only if $u(x) = \lambda_1 \int_0^1 H_4(x,s)p(s)u(s)ds$, and $u(x)$ solves (6.2),(6.3) if and only if $u(x) = \lambda_2 \int_0^1 H_4(x,s)q(s)u(s)ds$.

Note that since $\frac{\partial}{\partial x} H_4(x,s) = H(x,s)$, $\frac{\partial}{\partial x} H_4(x,s) \ge 0$ on $[0,1] \times [0,1]$ and $\frac{\partial}{\partial x} H_4(x,s) > 0$ on $(0,1] \times (0,1]$. Also, since $\frac{\partial^2}{\partial x^2} H_4(x,s) = \frac{\partial}{\partial x} H(x,s)$, $\frac{\partial^2}{\partial x^2} H_4(x,s)|_{x=0}$ > 0 for $0 < s < 1$.

To apply Theorems 1.1 and 1.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{u \in C^2[0,1] \mid u(0) = u'(0) = 0\},$$

with the norm

$$||u|| = \sup_{0 \le x \le 1} |u''(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u'(x) \ge 0 \text{ on } [0,1]\}.$$

Notice that for $u \in \mathcal{B}$, $0 \leq x \leq 1$,

$$\begin{aligned} |u'(x)| = |u'(x) - u'(0)| &= \left| \int_0^x u''(s)ds \right| \\ &\leq ||u||x \\ &\leq ||u||, \end{aligned}$$

and so $\sup\limits_{0\leq x\leq 1} |u'(x)| \leq ||u||$.

Lemma 6.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u'(x) > 0 \text{ on } (0,1] \text{ and } u''(0) > 0\}.$$

Note $\Omega \subset \mathcal{P}$. Choose $u \in \Omega$ and define $B_\epsilon(u) = \{v \in B \mid ||u - v|| < \epsilon\}$ for $\epsilon > 0$. Choose $\epsilon_0 > 0$ such that $u''(0) - \epsilon_0 > 0$. So for $v \in B_{\epsilon_0}(u)$, $\sup\limits_{0\leq x\leq 1} |v''(x) - u''(x)| < \epsilon_0$. So $v''(0) > u''(0) - \epsilon_0 > 0$. Also, $|v'(x) - u'(x)| \leq ||v - u|| < \epsilon_0$, and so $v'(x) > 0$ on $(0, 1]$. So $v \in \Omega$ and hence $B_{\epsilon_0}(u) \subset \Omega \subset \mathcal{P}$ and $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. □

Next, we define our linear operators M and N by

$$Mu(x) = \int_0^1 H_4(x, s)p(s)u(s)ds, \ 0 \leq x \leq 1,$$

and

$$Nu(x) = \int_0^1 H_4(x, s)q(s)u(s)ds, \ 0 \leq x \leq 1.$$

Since $H_4(0, s) = \frac{\partial}{\partial x} H_4(x, s)|_{x=0} = 0$, $M, N : \mathcal{B} \to \mathcal{B}$. A standard application of the Arzelá-Ascoli theorem shows that M and N are compact.

Lemma 6.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(x) \geq 0$. Then, since $\frac{\partial}{\partial x} H_4(x,s) = H(x,s) \geq 0$ on $[0,1] \times [0,1]$ and $p(x) \geq 0$ on $[0,1]$,

$$Mu'(x) = \int_0^1 \frac{\partial}{\partial x} H_4(x,s)p(s)u(s)ds \geq 0,$$

for $0 \leq x \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0,1]$ such that $u(x) > 0$ and $p(x) > 0$ for all $x \in [\alpha, \beta]$. Then, since $\frac{\partial}{\partial x} H_4(x,s) > 0$ on $(0,1] \times (0,1]$,

$$\begin{aligned} Mu'(x) &= \int_0^1 \frac{\partial}{\partial x} H_4(x,s)p(s)u(s)ds \\ &\geq \int_\alpha^\beta \frac{\partial}{\partial x} H_4(x,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < x \leq 1$. Also, since $\frac{\partial^2}{\partial x^2} H_4(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mu)''(0) &= \int_0^1 \frac{\partial^2}{\partial x^2} H_4(0,s)p(s)u(s)ds \\ &\geq \int_\alpha^\beta \frac{\partial^2}{\partial x^2} H_4(0,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

and so $Mu \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 1.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. $\square$

Remark 6.1. Notice that

$$\Lambda u = Mu = \int_0^1 H_4(x,s)p(s)u(s)ds,$$

if and only if

$$u(x) = \frac{1}{\Lambda} \int_0^1 H_4(x,s)p(s)u(s)ds,$$

if and only if

$$-u^{(4)}(x) = \frac{1}{\Lambda} p(x)u(x), \ 0 \leq x \leq 1,$$

with

$$u(0) = u'(0) = u''(r) = u'''(1) = 0.$$

So the eigenvalues of (6.1),(6.3) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (6.2),(6.3) are reciprocals of eigenvalues of N, and conversely.

Theorem 6.1. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 1.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. $\square$

Theorem 6.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \leq q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 6.1 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \leq q(x)$ on $[0,1]$. So for any $u \in \mathcal{P}$, and $x \in [0,1]$,

$$(Nu - Mu)'(x) = \int_0^1 \frac{\partial}{\partial x} H_4(x,s)(q(s) - p(s))u(s)ds \geq 0.$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 1.2, $\Lambda_1 \leq \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0,1]$. Then $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 1.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. $\square$

By Remark 6.1, the following theorem is an immediate consequence of Theorems 6.1 and 6.2.

Theorem 6.3. *Assume the hypotheses of Theorem 6.2. Then there exists smallest positive eigenvalues* λ_1 *and* λ_2 *of* (6.1),(6.3) *and* (6.2),(6.3) *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to* λ_1 *and* λ_2 *may be chosen to belong to* $\mathcal{P}^\circ$. *Finally,* $\lambda_1 \geq \lambda_2$ *and* $\lambda_1 = \lambda_2$ *if and only if* $p(x) = q(x)$ *for* $0 \leq x \leq 1$.

6.3 The nth Order Problem

Let $n \in \mathbb{N}$, $n \geq 4$. In this section, we will consider the eigenvalue problems

$$u^{(n)} + \lambda_1 p(x) u = 0, \tag{6.4}$$

$$u^{(n)} + \lambda_2 q(x) u = 0, \tag{6.5}$$

satisfying the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-3)}(0) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0, \tag{6.6}$$

where $0 < 1/2 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0,1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0,1]$.

Again, we will use methods similar to the methods used previously to derive comparison theorems for these nth order eigenvalue problems. We will do this by finding the the sign properties of the Green's function, which we will call $H_n(x,s)$, for $-u^{(n)} = 0$ satisfying (6.6). This Green's function is $C^{(n-3)}[0,1]$ and $\frac{\partial^{(n-4)}}{\partial x^{(n-3)}} H_n(x,s) = H(x,s)$, where $H(x,s)$ is as defined earlier.

Now $u(x)$ solves (6.4),(6.6) if and only if $u(x) = \lambda_1 \int_0^1 H_n(x,s) p(s) u(s) ds$, and $u(x)$ solves (6.5),(6.6) if and only if $u(x) = \lambda_2 \int_0^1 H_n(x,s) q(s) u(s) ds$.

Note that since $\frac{\partial^{n-3}}{\partial x^{n-3}}H_n(x,s) = H(x,s)$, $\frac{\partial^{n-3}}{\partial x^{n-3}}H_n(x,s) \geq 0$ on $[0,1]\times[0,1]$ and $\frac{\partial^{n-3}}{\partial x^{n-3}}H_n(x,s) > 0$ on $(0,1]\times(0,1]$. Also, since $\frac{\partial^{n-2}}{\partial x^{n-2}}H_n(x,s) = \frac{\partial}{\partial x}H(x,s)$, $\frac{\partial^{n-2}}{\partial x^{n-2}}H_n(x,s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 1.1 and 1.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{u \in C^{(n-2)}[0,1] \mid u(0) = u'(0) = \cdots = u^{(n-3)}(0) = 0\},$$

with the norm

$$||u|| = \sup_{0\leq x\leq 1} |u^{(n-2)}(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u^{(n-3)}(x) \geq 0 \text{ on } [0,1]\}.$$

Notice that for $u \in \mathcal{B}$, $0 \leq x \leq 1$,

$$\begin{aligned} |u^{(n-3)}(x)| = |u^{(n-3)}(x) - u^{(n-3)}(0)| &= \left|\int_0^x u^{(n-2)}(s)ds\right| \\ &\leq ||u||x \\ &\leq ||u||, \end{aligned}$$

and so $\sup_{0\leq x\leq 1} |u^{(n-3)}(x)| \leq ||u||$.

Lemma 6.3. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u^{(n-3)}(x) > 0 \text{ on } (0,1] \text{ and } u^{(n-2)}(0) > 0\}.$$

Note $\Omega \subset \mathcal{P}$. Choose $u \in \Omega$ and define $B_\epsilon(u) = \{v \in B \mid ||u-v|| < \epsilon\}$ for $\epsilon > 0$. Choose $\epsilon_0 > 0$ such that $u^{(n-2)}(0) - \epsilon_0 > 0$. So for $v \in B_{\epsilon_0}(u)$, $\sup_{0\leq x\leq 1} |v^{(n-2)}(x) - u^{(n-2)}(x)| < \epsilon_0$. So $v^{(n-2)}(0) > u^{(n-2)}(0) - \epsilon_0 > 0$. Also, $|v^{(n-3)}(x) - u^{(n-3)}(x)| \leq ||v-u|| < \epsilon_0$, and so $v^{(n-3)}(x) > 0$ on $(0,1]$. So $v \in \Omega$ and hence $B_{\epsilon_0}(u) \subset \Omega \subset \mathcal{P}$ and $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. $\square$

Next, we define our linear operators M and N by

$$Mu(x) = \int_0^1 H_n(x,s)p(s)u(s)ds,\ 0 \le x \le 1,$$

and

$$Nu(x) = \int_0^1 H_n(x,s)q(s)u(s)ds,\ 0 \le x \le 1.$$

Note that since $\frac{\partial^{n-i}}{\partial x^{n-i}} H_n(x,s)|_{x=0} = 0$ for $i = 3, 4, \ldots, n$, $M, N : \mathcal{B} \to \mathcal{B}$. A standard application of the Arzelá-Ascoli theorem shows that M and N are compact.

Lemma 6.4. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(x) \ge 0$. Then, since $\frac{\partial^{n-3}}{\partial x^{n-3}} H_n(x,s) = H(x,s) \ge 0$ on $[0,1] \times [0,1]$ and $p(x) \ge 0$ on $[0,1]$,

$$Mu^{(n-3)}(x) = \int_0^1 \frac{\partial^{n-3}}{\partial x^{n-3}} H_n(x,s)p(s)u(s)ds \ge 0,$$

for $0 \le x \le 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha, \beta] \subset [0,1]$ such that $u(x) > 0$ and $p(x) > 0$ for all $x \in [\alpha, \beta]$. Then, since $\frac{\partial^{n-3}}{\partial x^{n-3}} H_n(x,s) > 0$ on $(0,1] \times (0,1]$,

$$\begin{aligned} Mu^{(n-3)}(x) &= \int_0^1 \frac{\partial^{n-3}}{\partial x^{n-3}} H_n(x,s)p(s)u(s)ds \\ &\ge \int_\alpha^\beta \frac{\partial^{n-3}}{\partial x^{n-3}} H_n(x,s)(x,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < x \le 1$. Also, since $\frac{\partial^{n-2}}{\partial x^{n-2}} H_n(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mu)^{(n-2)}(0) &= \int_0^1 \frac{\partial^{n-2}}{\partial x^{n-2}} H_n(0,s)p(s)u(s)ds \\ &\ge \int_\alpha^\beta \frac{\partial^{n-2}}{\partial x^{n-2}} H_n(0,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

and so $Mu \in \Omega \subset \mathcal{P}^{\circ}$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^{\circ}$. Therefore by Lemma 1.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. $\square$

Remark 6.2. Notice that

$$\Lambda u = Mu = \int_0^1 H_n(x,s)p(s)u(s)ds,$$

if and only if

$$u(x) = \frac{1}{\Lambda}\int_0^1 H_n(x,s)p(s)u(s)ds,$$

if and only if

$$-u^{(n)}(x) = \frac{1}{\Lambda}p(x)u(x),\ 0 \leq x \leq 1,$$

with

$$u(0) = u'(0) = \cdots = u^{(n-3)}(0) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0.$$

So the eigenvalues of (6.4),(6.6) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (6.5),(6.6) are reciprocals of eigenvalues of N, and conversely.

Theorem 6.4. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^{\circ}$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 1.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^{\circ}$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^{\circ}$. $\square$

Theorem 6.5. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \leq q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 6.4 associated with M and N respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^{\circ}$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \le q(x)$ on $[0,1]$. So for any $u \in \mathcal{P}$, and $x \in [0,1]$,

$$(Nu - Mu)^{(n-3)}(x) = \int_0^1 \frac{\partial^{n-3}}{\partial x^{n-3}} H_n(x,s)(q(s) - p(s))u(s)ds \ge 0.$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \le N$ with respect to $\mathcal{P}$. Then by Theorem 1.2, $\Lambda_1 \le \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \ne q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0,1]$. Then $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \le Nu_1$, implying $Nu_1 \ge (\Lambda_1 + \epsilon)u_1$. Since $N \le N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 1.2, $\Lambda_1 + \epsilon \le \Lambda_2$, or $\Lambda_1 < \Lambda_2$. $\square$

By Remark 6.2, the following theorem is an immediate consequence of Theorems 6.4 and 6.5.

Theorem 6.6. *Assume the hypotheses of Theorem 6.5. Then there exists smallest positive eigenvalues λ_1 and λ_2 of* (6.4),(6.6) *and* (6.5),(6.6)*, respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \ge \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(x) = q(x)$ for $0 \le x \le 1$.*

CHAPTER SEVEN

Extending the Third Order Problem Using Substitution

7.1 Introduction

In this chapter, we will again extend the results of the third order problem to the nth order problems

$$u^{(n)} + \lambda_1 p(x)u = 0,$$

$$u^{(n)} + \lambda_2 q(x)u = 0,$$

satisfying the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-3)}(0) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0,$$

where $0 < 1/2 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0,1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0,1]$.

Instead of using the sign properties of the Green's function for the nth order equation to derive the comparison theorems, we will instead make a substitution and work with a variation of the third order problem. This method has its benefits, since we do not need to find the sign properties of the Green's function for the nth order problem, and can instead work with the third order problem. We will again start with the fourth order problem and then look at the nth order problem.

7.2 The Fourth Order Problem

We now consider the eigenvalue problems

$$u^{(4)} + \lambda_1 p(x)u = 0, \tag{7.1}$$

$$u^{(4)} + \lambda_2 q(x)u = 0, \tag{7.2}$$

satisfying the boundary conditions

$$u(0) = u'(0) = u''(r) = u'''(1) = 0, \tag{7.3}$$

and the eigenvalue problems

$$v^{(3)} + \lambda_1 p(x) \int_0^x v(s)ds = 0, \tag{7.4}$$

$$v^{(3)} + \lambda_2 q(x) \int_0^x v(s)ds = 0, \tag{7.5}$$

satisfying the boundary condtions

$$v(0) = v'(r) = v''(1) = 0, \tag{7.6}$$

where $0 < 1/2 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanish identically on any compact subinterval of $[0, 1]$.

First we note that if $u(x)$ is a solution to (7.1),(7.3), then $u'(x)$ solves (7.4),(7.6). Also, if $v(x)$ is a solution to (7.4),(7.6), then $\int_0^x v(s)ds$ is a solution to (7.1),(7.3). Similarly, if $u(x)$ is a solution to (7.2),(7.3), then $u'(x)$ solves (7.5),(7.6), and if $v(x)$ is a solution to (7.5),(7.6), then $\int_0^x v(s)ds$ is a solution to (7.2),(7.3).

Now let λ be an eigenvalue of (7.1),(7.3) with the corresponding eigenvector $u(x)$. Then $u'(x)$ is a solution to (7.4),(7.6) with the same eigenvalue λ. Also, if λ is an eigenvalue of (7.4),(7.6) with corresponding eigenvector $v(x)$, then $\int_0^x v(s)ds$ is a solution to (7.1),(7.3) with the corresponding eigenvalue λ. So eigenvalues of (7.1),(7.3) are eigenvalues of (7.4),(7.6), and vice versa. Similarly, eigenvalues of (7.2),(7.3) are eigenvalues of (7.5),(7.6), and vice versa. So any comparison theorems for (7.4),(7.6), and (7.5),(7.6) will apply to (7.1),(7.3), and (7.2),(7.3).

For these reasons, we will derive comparison theorems for eigenvalue problems (7.4),(7.6), and (7.5),(7.6), and then use these theorems to derive the comparison theorems for (7.1),(7.3), and (7.2),(7.3).

Let $H(x,s)$ by the Green's function for $-v^{(3)} = 0$ satisfying (7.6), which was defined earlier. So $v(x)$ solves (7.4),(7.6) if and only if $v(x) = \lambda_1 \int_0^1 H(x,s)p(s) \int_0^s v(t)dtds$ and $v(x)$ solves (7.5),(7.6) if and only if $v(x) = \lambda_2 \int_0^1 H(x,s)q(s) \int_0^s v(t)dtds$. Also, note $H(x,s) \geq 0$ on $[0,1] \times [0,1]$, $H(x,s) > 0$ on $(0,1] \times (0,1]$, and $\frac{\partial}{\partial x} H(x,s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 1.1 and 1.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{v \in C^1[0,1] \mid v(0) = 0\},$$

with the norm

$$||v|| = \sup_{0 \leq x \leq 1} |v'(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{v \in \mathcal{B} \mid v(x) \geq 0 \text{ on } [0,1]\}.$$

Notice that for $v \in \mathcal{B}$, $0 \leq x \leq 1$,

$$\begin{aligned} |v(x)| = |v(x) - v(0)| &= \left| \int_0^x v'(s)ds \right| \\ &\leq ||v||x \\ &\leq ||v||, \end{aligned}$$

and so $\sup_{0 \leq x \leq 1} |v(x)| \leq ||v||$.

Lemma 7.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{v \in \mathcal{B} \mid v(x) > 0 \text{ on } (0,1] \text{ and } v'(0) > 0\}.$$

It was shown in Chapter 5 that $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. □

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mv(x) = \int_0^1 H(x,s)p(s)\int_0^s v(t)dtds, \ 0 \le x \le 1,$$

and

$$Nv(x) = \int_0^1 H(x,s)q(s)\int_0^s v(t)dtds, \ 0 \le x \le 1.$$

A standard application of the Arzelá-Ascoli theorem shows that M and N are compact.

Lemma 7.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $v \in \mathcal{P}$. So $v(x) \ge 0$. Then since $H(x,s) \ge 0$ on $[0,1]\times[0,1]$, $p(x) \ge 0$ on $[0,1]$ and $\int_0^x v(s)ds \ge 0$,

$$Mv(x) = \int_0^1 H(x,s)p(s)\int_0^s v(t)dtds \ge 0,$$

for $0 \le x \le 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $v \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha,\beta] \subset [0,1]$ such that $\int_0^x v(s)ds > 0$ and $p(x) > 0$ for all $x \in [\alpha,\beta]$. Then, since $H(x,s) > 0$ on $(0,1]\times(0,1]$,

$$\begin{aligned} Mv(x) &= \int_0^1 H(x,s)p(s)\int_0^s v(t)dtds \\ &\ge \int_\alpha^\beta H(x,s)p(s)\int_0^s v(t)dtds \\ &> 0, \end{aligned}$$

for $0 < x \le 1$. Also, since $\frac{\partial}{\partial x}H(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned} (Mv)'(0) &= \int_0^1 \frac{\partial}{\partial x}H(0,s)p(s)\int_0^s v(t)dtds \\ &\ge \int_\alpha^\beta \frac{\partial}{\partial x}H(0,s)p(s)\int_0^s v(t)dtds \\ &> 0, \end{aligned}$$

and so $Mv \in \Omega \subset \mathcal{P}^\circ$. So $M : \mathcal{P} \backslash \{0\} \to \Omega \subset \mathcal{P}^\circ$. Therefore by Lemma 1.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. $\square$

Remark 7.1. Notice that

$$\Lambda v = Mv = \int_0^1 H(x,s)p(s) \int_0^s v(t)dtds,$$

if and only if

$$v(x) = \frac{1}{\Lambda} \int_0^1 H(x,s)p(s) \int_0^s v(t)dtds,$$

if and only if

$$-v^{(3)}(x) = \frac{1}{\Lambda} p(x) \int_0^x v(s)ds, \ 0 \leq x \leq 1,$$

with

$$v(0) = v'(r) = v''(1) = 0.$$

So the eigenvalues of (7.4),(7.6) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (7.5),(7.6) are reciprocals of eigenvalues of N, and conversely.

Theorem 7.1. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 1.1, M has an essentially unique eigenvector, say $v \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $v \neq 0$, $Mv \in \Omega \subset \mathcal{P}^\circ$ and $v = M\left(\frac{1}{\Lambda}v\right) \in \mathcal{P}^\circ$. $\square$

Theorem 7.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \leq q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 7.1 associated with M and N, respectively, with the essentially unique eigenvectors v_1 and $v_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \le q(x)$ on $[0,1]$. So for any $v \in \mathcal{P}$, and $x \in [0,1]$,

$$(Nv - Mv)(x) = \int_0^1 H(x,s)(q(s)-p(s)) \int_0^s v(t)dtds \ge 0.$$

So $Nv - Mv \in \mathcal{P}$ for all $v \in \mathcal{P}$, or $M \le N$ with respect to $\mathcal{P}$. Then by Theorem 1.2, $\Lambda_1 \le \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \ne q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0,1]$. Then $(N-M)v_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N-M)v_1 - \epsilon v_1 \in \mathcal{P}$. So $\Lambda_1 v_1 + \epsilon v_1 = Mv_1 + \epsilon v_1 \le Nv_1$, implying $Nv_1 \ge (\Lambda_1 + \epsilon)v_1$. Since $N \le N$ and $Nv_2 = \Lambda_2 v_2$, by Theorem 1.2, $\Lambda_1 + \epsilon \le \Lambda_2$, or $\Lambda_1 < \Lambda_2$. □

By Remark 7.1, the following theorem is an immediate consequence of Theorems 7.1 and 7.2.

Theorem 7.3. *Assume the hypotheses of Theorem 7.2. Then there exists smallest positive eigenvalues λ_1 and λ_2 of* (7.4),(7.6) *(and hence* (7.1),(7.3)*) and* (7.5),(7.6) *(and hence* (7.2),(7.3)*), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \ge \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(x) = q(x)$ for $0 \le x \le 1$.*

7.3 The nth Order Problem

Let $n \in \mathbb{N}$, $n \ge 4$. In this section, we consider the eigenvalue problems

$$u^{(n)} + \lambda_1 p(x)u = 0, \tag{7.7}$$

$$u^{(n)} + \lambda_2 q(x)u = 0, \tag{7.8}$$

satisfying the boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-3)}(0) = u^{(n-2)}(r) = u^{(n-1)}(1) = 0, \tag{7.9}$$

and the eigenvalue problems

$$v^{(3)} + \lambda_1 p(x) \frac{1}{(n-4)!} \int_0^x (x-s)^{n-4} v(s) ds = 0, \tag{7.10}$$

$$v^{(3)} + \lambda_2 q(x) \frac{1}{(n-4)!} \int_0^x (x-s)^{n-4} v(s) ds = 0, \tag{7.11}$$

satisfying the boundary condtions

$$v(0) = v'(r) = v''(1) = 0, \tag{7.12}$$

where $0 < 1/2 < r < 1$, and $p(x)$ and $q(x)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(x)$ nor $q(x)$ vanishes identically on any compact subinterval of $[0, 1]$.

First we note that if $u(x)$ is a solution to (7.7),(7.9), then $u^{(n-3)}(x)$ solves (7.10),(7.12). Also, if $v(x)$ is a solution to (7.10),(7.12), then $\frac{1}{(n-4)!} \int_0^x (x-s)^{n-4} v(s) ds$ is a solution to (7.7),(7.9). Similarly, if $u(x)$ is a solution to (7.8),(7.9), then $u^{(n-3)}(x)$ solves (7.11),(7.12), and if $v(x)$ is a solution to (7.11),(7.12), then $\frac{1}{(n-4)!} \int_0^x (x-s)^{n-4} v(s) ds$ is a solution to (7.8),(7.9).

Now let λ be an eigenvalue of (7.7),(7.9) with the corresponding eigenvector $u(x)$. Then $u^{(n-3)}(x)$ is a solution to (7.10),(7.12) with the same eigenvalue λ. Also, if λ is an eigenvalue of (7.10),(7.12) with corresponding eigenvector $v(x)$, then $\frac{1}{(n-4)!} \int_0^x (x-s)^{n-4} v(s) ds$ is a solution to (7.7),(7.9) with the corresponding eigenvalue λ. So eigenvalues of (7.7),(7.9) are eigenvalues of (7.10),(7.12), and vice versa. Similarly, eigenvalues of (7.8),(7.9) are eigenvalues of (7.11),(7.12), and vice versa. So any comparison theorems for (7.10),(7.12), and (7.11),(7.12) will apply to (7.7),(7.9), and (7.8),(7.9).

For these reasons, we will derive comparison theorems for eigenvalue problems (7.10),(7.12), and (7.11),(7.12), and then use these theorems to derive the comparison theorems for (7.7),(7.9), and (7.8),(7.9).

Let $H(x,s)$ by the Green's function for $-v^{(3)} = 0$ satisfying (7.12), which was defined earlier. So $v(x)$ solves (7.10),(7.12) if and only if

$$v(x) = \lambda_1 \int_0^1 H(x,s)p(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds,$$

and $v(x)$ solves (7.11),(7.12) if and only if

$$v(x) = \lambda_2 \int_0^1 H(x,s)q(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds.$$

Also, note $H(x,s) \geq 0$ on $[0,1]\times[0,1]$, $H(x,s) > 0$ on $(0,1]\times(0,1]$, and $\frac{\partial}{\partial x}H(x,s)|_{x=0} > 0$ for $0 < s < 1$.

To apply Theorems 1.1 and 1.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{v \in C^1[0,1] \mid v(0) = 0\},$$

with the norm

$$||v|| = \sup_{0\leq x\leq 1} |v'(x)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{v \in \mathcal{B} \mid v(x) \geq 0 \text{ on } [0,1]\}.$$

Notice that for $v \in \mathcal{B}$, $0 \leq x \leq 1$,

$$\begin{aligned} |v(x)| = |v(x) - v(0)| &= \left|\int_0^x v'(s)ds\right| \\ &\leq ||v||x \\ &\leq ||v||, \end{aligned}$$

and so $\sup_{0\leq x\leq 1} |v(x)| \leq ||v||$.

Lemma 7.3. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{v \in \mathcal{B} \mid v(x) > 0 \text{ on } (0,1] \text{ and } v'(0) > 0\}.$$

It was shown in Chapter 5 that $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. $\square$

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mv(x) = \int_0^1 H(x,s)p(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds, \; 0 \leq x \leq 1,$$

and

$$Nv(x) = \int_0^1 H(x,s)q(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds, \; 0 \leq x \leq 1.$$

A standard application of the Arzelá-Ascoli theorem shows that M and N are compact.

Lemma 7.4. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $v \in \mathcal{P}$. So $v(x) \geq 0$. Then since $H(x,s) \geq 0$ on $[0,1] \times [0,1]$, $p(x) \geq 0$ on $[0,1]$ and $\frac{1}{(n-4)!}\int_0^x (x-s)^{n-4}v(s)ds \geq 0$,

$$Mv(x) = \int_0^1 H(x,s)p(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds \geq 0,$$

for $0 \leq x \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $v \in \mathcal{P}\backslash\{0\}$. Since $(x-s)^{n-4} > 0$ for $0 \leq s < x$, there exists a compact interval $[\alpha, \beta] \subset [0,1]$ such that $\frac{1}{(n-4)!}\int_0^x (x-s)^{n-4}v(s)ds > 0$ and $p(x) > 0$ for all $x \in [\alpha, \beta]$. Then, since $H(x,s) > 0$ on $(0,1] \times (0,1]$,

$$\begin{aligned}
Mv(x) &= \int_0^1 H(x,s)p(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds \\
&\geq \int_\alpha^\beta H(x,s)p(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds \\
&> 0,
\end{aligned}$$

for $0 < x \leq 1$. Also, since $\frac{\partial}{\partial x}H(x,s)|_{x=0} > 0$ for $0 < s < 1$,

$$\begin{aligned}
(Mv)'(0) &= \int_0^1 \frac{\partial}{\partial x}H(0,s)p(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds \\
&\geq \int_\alpha^\beta \frac{\partial}{\partial x}H(0,s)p(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds \\
&> 0,
\end{aligned}$$

and so $Mv \in \Omega \subset \mathcal{P}^{\circ}$. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^{\circ}$. Therefore by Lemma 1.1, M is u_0-positive with respect to $\mathcal{P}$. A similar argument for N completes the proof. $\square$

Remark 7.2. Notice that

$$\Lambda v = Mv = \int_0^1 H(x,s)p(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds,$$

if and only if

$$v(x) = \frac{1}{\Lambda}\int_0^1 H(x,s)p(s)\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds,$$

if and only if

$$-v^{(3)}(x) = \frac{1}{\Lambda}p(x)\frac{1}{(n-4)!}\int_0^x (x-s)^{n-4}v(s)ds, \ 0 \le x \le 1,$$

with

$$v(0) = v'(r) = v''(1) = 0.$$

So the eigenvalues of (7.10),(7.12) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (7.11),(7.12) are reciprocals of eigenvalues of N, and conversely.

Theorem 7.4. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^{\circ}$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 1.1, M has an essentially unique eigenvector, say $v \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $v \neq 0$, $Mv \in \Omega \subset \mathcal{P}^{\circ}$ and $v = M\left(\frac{1}{\Lambda}v\right) \in \mathcal{P}^{\circ}$. $\square$

Theorem 7.5. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(x) \le q(x)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 7.4 associated with M and N, respectively, with the essentially unique eigenvectors v_1 and $v_2 \in \mathcal{P}^{\circ}$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(x) = q(x)$ on $[0,1]$.*

Proof. Let $p(x) \leq q(x)$ on $[0,1]$. So for any $v \in \mathcal{P}$, and $x \in [0,1]$,

$$(Nv - Mv)(x) = \int_0^1 H(x,s)(q(s) - p(s))\frac{1}{(n-4)!}\int_0^s (s-t)^{n-4}v(t)dtds \geq 0.$$

So $Nv - Mv \in \mathcal{P}$ for all $v \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 1.2, $\Lambda_1 \leq \Lambda_2$.

If $p(x) = q(x)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(x) \neq q(x)$. So $p(x) < q(x)$ on some subinterval $[\alpha, \beta] \subset [0,1]$. Then $(N - M)v_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N - M)v_1 - \epsilon v_1 \in \mathcal{P}$. So $\Lambda_1 v_1 + \epsilon v_1 = Mv_1 + \epsilon v_1 \leq Nv_1$, implying $Nv_1 \geq (\Lambda_1 + \epsilon)v_1$. Since $N \leq N$ and $Nv_2 = \Lambda_2 v_2$, by Theorem 1.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. $\square$

By Remark 7.2, the following theorem is an immediate consequence of Theorems 7.4 and 7.5.

Theorem 7.6. *Assume the hypotheses of Theorem 7.5. Then there exists smallest positive eigenvalues λ_1 and λ_2 of* (7.10),(7.12) *(and hence* (7.7),(7.9)*) and* (7.11),(7.12) *(and hence* (7.8),(7.9)*), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(x) = q(x)$ for $0 \leq x \leq 1$.*

CHAPTER EIGHT

Extremal Points

8.1 Introduction

In this chapter, we will consider the fourth order boundary value problem,

$$u^{(4)} + p(x)u = 0, \tag{8.1}$$

for $0 \leq x \leq \beta$ satisfying the boundary conditions

$$u(0) = u'(r) = u''(r) = u'''(b) = 0, \tag{8.2}$$

where $0 < r < b \leq \beta$, and $p(x)$ is a nonnegative continuous function on $[0, b]$ which does not vanish identically on any compact subinterval of $[0, b]$. We will also consider the third order boundary value problem,

$$u^{(3)} + p(x)u = 0, \tag{8.3}$$

for $0 \leq x \leq \beta$ satisfying the boundary conditions

$$u(0) = u'(r) = u''(b) = 0, \tag{8.4}$$

where $0 < 1/2 < r < b \leq \beta$, and $p(x)$ is a nonnegative continuous function on $[0, b]$ which does not vanish identically on any compact subintervale of $[0, b]$.

For the fourth order problem, we establish the existence of a largest interval, $[0, b)$, such that on any subinterval $[0, c]$ of $[0, b)$, there exists only the trivial solution of (8.1) satisfying $u(0) = u'(r) = u''(r) = u'''(c) = 0$. For the third order problem, we establish the existence of a largest interval, $[0, b)$, such that on any subinterval $[0, c]$ of $[0, b)$, there exists only the trivial solution of (8.3) satisfying $u(0) = u'(r) = u''(c) = 0$. We accomplish this by characterizing the first extremal point through the existence of a nontrivial solution that lies in a cone, by establishing the spectral radius of a compact operator. Because the spectral radius of the compact operators dealt with in the previous chapters is precisely the largest positive eigenvalues of those operators, this inclusion of this material is natural.

8.2 Definitions and Theorems

Definition 8.1. We say b_0 is the first extremal point of the boundary value problem (8.1),(8.2) (or (8.3),(8.4)), if $b_0 = \inf\{b > 0 \mid$ (8.1),(8.2) (or (8.3),(8.4)) has a nontrivial solution$\}$.

Definition 8.2. A bounded linear operator $N : \mathcal{B} \to \mathcal{B}$ is said to be positive with respect to the cone $\mathcal{P}$ if $N : \mathcal{P} \to \mathcal{P}$.

Throughout this chapter, we will denote the spectral radius of the bounded linear operator N by $r(N)$.

The following four theorems are fundamental to our following results. The first result can be found in [25], and the last three theorems and proofs can be found in [1] or [23]. In each of the following theorems, assume that $\mathcal{P}$ is a reproducing cone, and that $N, N_1, N_2 : \mathcal{B} \to \mathcal{B}$ are compact, linear, and positive with respect to $\mathcal{P}$.

Theorem 8.1. *Let N_b, $\rho \le \beta \le \sigma$ be a family of compact, linear operators on a Banach space such that the mapping $b \mapsto N_b$ is continuous in the uniform operator topology. Then the mapping $b \mapsto r(N_b)$ is continuous.*

Theorem 8.2. *Assume $r(N) > 0$. Then $r(N)$ is an eigenvalue of N, and there is a corresponding eigenvector in $\mathcal{P}$. If, in addition, N is u_0-positive, then $r(N)$ is a simple eigenvalue of N, and the corresponding eigenvector is essentially unique and belongs to $\mathcal{P}^\circ$.*

Theorem 8.3. *If $N_1 \le N_2$ with respect to $\mathcal{P}$, then $r(N_1) \le r(N_2)$.*

Theorem 8.4. *Suppose there exists $\gamma > 0$, $u \in \mathcal{B}$, $-u \notin \mathcal{P}$ such that $\gamma u \leq Nu$ with respect to $\mathcal{P}$. Then N has an eigenvector in $\mathcal{P}$ which corresponds to an eigenvalue, $\gamma \leq \lambda$.*

8.3 The Fourth Order Problem

Consider the fourth order boundary value problem,

$$u^{(4)} + p(x)u = 0, \tag{8.1}$$

for $0 \leq x \leq \beta$ satisfying the boundary conditions

$$u(0) = u'(r) = u''(r) = u'''(b) = 0, \tag{8.2}$$

where $0 < r < b \leq \beta$, and $p(x)$ is a nonnegative continuous function on $[0, b]$ which does not vanish identically on any compact subinterval of $[0, b]$.

We will be defining compact integral operators whose kernels are the Green's function for $-u^{(4)} = 0$ satisfying (8.2). Because $G(x, s)$, which was defined in Chapter 2, has the property that $\frac{\partial^3}{\partial x^3} G(x, s) = 0$ for all $(x, s) \in [r, 1] \times [0, 1]$, $G(x, s)$ satisfies (8.2), and so $G(x, s)$ is the Green's function for $-u^{(4)} = 0$ satisfying (8.2).

To apply Theorems 8.1-8.4, we need to define a family of Banach spaces $\mathcal{B}$ and cones $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ to be

$$\mathcal{B} = \{u \in C^1[0, \beta] \mid u(0) = 0\},$$

with the norm

$$||u|| = \sup_{0 \leq x \leq \beta} |u'(x)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(x) \geq 0 \text{ on } [0, \beta]\}.$$

From earlier, we know that $\mathcal{P}^\circ \neq \emptyset$. In fact,

$$\{u \in \mathcal{B} \mid u(x) > 0 \text{ on } (0, \beta] \text{ and } u'(0) > 0\} \subset \mathcal{P}^\circ.$$

For all $b \in [0, \beta]$, define the Banach space $\mathcal{B}_b$ to be

$$\mathcal{B}_b = \{u \in C^1[0, b] \mid u(0) = 0\},$$

with the norm

$$||u|| = \sup_{0 \le x \le b} |u'(x)|.$$

Define the cone $\mathcal{P}_b \subset \mathcal{B}_b$ to be

$$\mathcal{P}_b = \{u \in \mathcal{B}_b \mid u(x) \ge 0 \text{ on } [0, b]\}.$$

Again, note that for all $b \in [0, \beta]$, $\mathcal{P}_b^\circ \ne \emptyset$. In fact,

$$\{u \in \mathcal{B} \mid u(x) > 0 \text{ on } (0, b] \text{ and } u'(0) > 0\} \subset \mathcal{P}_b^\circ.$$

Now for each $b \in [0, \beta]$, define the linear operator

$$N_b u(x) = \begin{cases} \int_0^b G(x, s)p(s)u(s)ds, & 0 \le x \le b, \\ \int_0^b G(b, s)p(s)u(s)ds + (x - b) \int_0^b \frac{\partial}{\partial x} G(b, s)p(s)u(s)ds, & b \le x \le \beta. \end{cases}$$

Notice that by the way N_b is defined, $N_b u(x) \in C^1[0, \beta]$ for $u(x) \in C^1[0, \beta]$, and $N_b u(0) = 0$. So $N_b : \mathcal{B} \to \mathcal{B}$. Also note that when N_b is restricted to B_b, $N_b : \mathcal{B}_b \to \mathcal{B}_b$ by

$$N_b u(x) = \int_0^b G(x, s)p(s)u(s)ds.$$

So $u(x)$ is a solution to (8.1),(8.2) if and only if $u(x) = N_b u(x) = \int_0^b G(x, s)p(s)u(s)ds$ for $x \in [0, b]$.

Lemma 8.1. *For all $b \in [0, \beta]$, the linear operator N_b is positive with respect to $\mathcal{P}$ and $\mathcal{P}_b$. Also, $N_b : \mathcal{P} \backslash \{0\} \to \mathcal{P}_b^\circ$.*

Proof. Since for $u \in \mathcal{P}$, $G(x, s) \ge 0$, $\frac{\partial}{\partial x} G(b, s) \ge 0$, and $p(s)u(s) \ge 0$, $N_b u(x) \ge 0$ for $0 \le x \le \beta$. So $N_b : \mathcal{P} \to \mathcal{P}$. Similarly, $N_b : \mathcal{P}_b \to \mathcal{P}_b$.

Now set $\Omega_b = \{u \in \mathcal{B} \mid u(x) > 0 \text{ on } (0, \beta] \text{ and } u'(0) > 0\}$. Let $u \in \mathcal{P}_b \backslash \{0\}$. So there exists a compact interval $[c, d] \subset [0, b]$ such that $p(x)u(x) > 0$ for all $x \in [c, d]$. Since $G(x, s) > 0$ for $0 < x \leq b$,

$$\begin{aligned} N_b u(x) &= \int_0^b G(x,s)p(s)u(s)ds \\ &\geq \int_c^d G(x,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < x \leq b$.

Since $\frac{\partial}{\partial x} G(x, s)|_{x=0} > 0$,

$$\begin{aligned} N_b u'(0) &= \int_0^b \frac{\partial}{\partial x} G(0,s)p(s)u(s)ds \\ &\geq \int_c^d \frac{\partial}{\partial x} G(0,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

and so $N_b u \in \Omega_b$. So $N_b : \mathcal{P}\backslash\{0\} \to \Omega_b \subset \mathcal{P}_b^\circ$. $\square$

Lemma 8.2. *The map $b \mapsto N_b$ is continuous in the uniform topology.*

Proof. First note that $\sup\limits_{0 \leq x \leq \beta} |u(x)| \leq \beta ||u||$. Let $f : (0, \beta] \to \{N_b\}$, $b \in [0, \beta]$, such that $f(b) = N_b$. Let $0 < b_1 < b_2 \leq \beta$. Let $\epsilon > 0$. Then

$$\begin{aligned} ||f(b_2) - f(b_1)|| &= ||N_{b_2} - N_{b_1}|| \\ &= \sup_{||u||=1} ||N_{b_2}u - N_{b_1}u|| \\ &= \sup_{||u||=1} \{ \sup_{x \in [0,\beta]} |(N_{b_2}u)'(x) - (N_{b_1}u)'(x)| \}. \end{aligned}$$

Since $\frac{\partial}{\partial x} G(x, s)$ and $p(x)$ are continuous functions for $0 \leq x \leq \beta$, they are bounded above for $0 \leq x \leq \beta$. Choose K and P such that $|\frac{\partial}{\partial x} G(x, s)| \leq K$ and $|p(x)| \leq P$ for $0 \leq x \leq \beta$.

Suppose $x \le b_1$. Then for $|b_2 - b_1| < \delta < \dfrac{\epsilon}{KP\beta}$,

$$
\begin{aligned}
|(N_{b_2}u)'(x) - (N_{b_1}u)'(x)| &= \left| \int_0^{b_2} \frac{\partial}{\partial x} G(x,s)p(s)u(s)ds - \int_0^{b_1} \frac{\partial}{\partial x} G(x,s)p(s)u(s)ds \right| \\
&= \left| \int_{b_1}^{b_2} \frac{\partial}{\partial x} G(x,s)p(s)u(s)ds \right| \\
&\le \int_{b_1}^{b_2} \left| \frac{\partial}{\partial x} G(x,s) \right| |p(s)||u(s)|ds \\
&\le \int_{b_1}^{b_2} KP\beta ds \\
&= KP\beta |b_2 - b_1| \\
&< KP\beta \frac{\epsilon}{KP\beta} = \epsilon
\end{aligned}
$$

Now suppose $b_1 < x \le b_2$. Since $G(x,s) \in C^1[0,\beta]$ in the first variable, for $|b_2 - b_1| < \delta < \dfrac{\epsilon}{2KP\beta}$, $\left|\frac{\partial}{\partial x} G(x,s) - \frac{\partial}{\partial x} G(b_1,s)\right| < \dfrac{\epsilon}{2P\beta^2}$. So

$$
\begin{aligned}
|(N_{b_2}u)'(x) - (N_{b_1}u)'(x)| &= \left| \int_0^{b_2} \frac{\partial}{\partial x} G(x,s)p(s)u(s)ds - \int_0^{b_1} \frac{\partial}{\partial x} G(b_1,s)p(s)u(s)ds \right| \\
&\le \int_{b_1}^{b_2} \left| \frac{\partial}{\partial x} G(x,s) \right| |p(s)||u(s)|ds \\
&\quad + \int_0^{b_1} \left| \frac{\partial}{\partial x} G(x,s) - \frac{\partial}{\partial x} G(b_1,s) \right| |p(s)||u(s)|ds \\
&< \int_{b_1}^{b_2} KP\beta ds + \int_0^{b_1} \frac{\epsilon}{2P\beta^2} P\beta ds \\
&= KP\beta |b_2 - b_1| + \frac{\epsilon}{2P\beta^2} P b_1 \beta \\
&< KP\beta \frac{\epsilon}{2KP\beta} + \frac{\epsilon}{2P\beta^2} P\beta^2 = \epsilon.
\end{aligned}
$$

Now suppose $\beta > x > b_2$. Again, since $G(x,s) \in C^1[0,\beta]$ in the first variable, for $|b_2 - b_1| < \delta < \dfrac{\epsilon}{2KP\beta}$, $\left|\frac{\partial}{\partial x} G(b_2,s) - \frac{\partial}{\partial x} G(b_1,s)\right| < \dfrac{\epsilon}{2P\beta^2}$. So

$$
\begin{aligned}
|(N_{b_2}u)'(x) - (N_{b_1}u)'(x)| &= \left| \int_0^{b_2} \frac{\partial}{\partial x} G(b_2,s)p(s)u(s)ds - \int_0^{b_1} \frac{\partial}{\partial x} G(b_1,s)p(s)u(s)ds \right| \\
&\le \int_{b_1}^{b_2} \left| \frac{\partial}{\partial x} G(b_2,s) \right| |p(s)||u(s)|ds \\
&\quad + \int_0^{b_1} \left| \frac{\partial}{\partial x} G(b_2,s) - \frac{\partial}{\partial x} G(b_1,s) \right| |p(s)||u(s)|ds
\end{aligned}
$$

$$
\begin{aligned}
&< \int_{b_1}^{b_2} KP\beta ds + \int_0^{b_1} \frac{\epsilon}{2P\beta^2} P\beta ds \\
&= KP\beta|b_2 - b_1| + \frac{\epsilon}{2P\beta^2} Pb_1\beta \\
&< KP\beta \frac{\epsilon}{2KP\beta} + \frac{\epsilon}{2P\beta^2} P\beta^2 = \epsilon.
\end{aligned}
$$

So we have that $\sup_{||u||=1} \{ \sup_{x\in[0,\beta]} |(N_{b_2}u)'(x) - (N_{b_1}u)'(x)|\} < \epsilon$ for $|b_2 - b_1| < \delta$. So $||f(b_2) - f(b_1)|| < \epsilon$ for $|b_2 - b_1| < \delta$, and so f is continuous. $\square$

Theorem 8.5. *For $0 < b \le \beta$, $r(N_b)$ is strictly increasing as a function of b.*

Proof. In [22], it is shown that there is a $\lambda > 0$ and $u \in \mathcal{P}_b\backslash\{0\}$ such that $N_b u(x) = \lambda u(x)$. Extend u to $[b, \beta]$ by

$$
u(x) = \frac{1}{\lambda}\left(\int_0^b G(b,s)p(s)u(s)ds + (x-b)\int_0^b \frac{\partial}{\partial x}G(b,s)p(s)u(s)ds\right).
$$

Then for $x \in [0, \beta]$, $N_b u(x) = \lambda u(x)$. Thus for $0 < b \le \beta$, $r(N_b) \ge \lambda > 0$.

Now let $0 < b_1 < b_2 \le \beta$. Since $r(N_{b_1}) > 0$, by Theorem 8.2, there exists a $u_0 \in \mathcal{P}_{b_1}\backslash\{0\}$ such that $N_{b_1}u_0 = r(N_{b_1})u_0$. Let $u_1 = N_{b_1}u_0$ and $u_2 = N_{b_2}u_0$. Then for $x \in (0, b_1]$,

$$
(u_2 - u_1)(x) = \int_{b_1}^{b_2} G(x,s)u(s)p(s)ds > 0.
$$

Also,

$$
(u_2 - u_1)'(0) = \int_{b_1}^{b_2} \frac{\partial}{\partial x}G(0,s)u(s)p(s)ds > 0.
$$

Thus the restriction of $u_2 - u_1$ to $[0, b_1]$ belongs to Ω_{b_1}. So there exists a $\delta > 0$ such that $u_2 - u_1 \ge \delta u_0$ with respect to $\mathcal{P}_{b_1}$. Since $u_2 \in \mathcal{P}$, it follows that $u_2 - u_1 \ge \delta u_0$ with respect to $\mathcal{P}$. Thus

$$
\begin{aligned}
u_2 &\ge u_1 + \delta u_0 \\
&= r(N_{b_1})u + \delta u_0 \\
&= (r(N_{b_1}) + \delta)u_0
\end{aligned}
$$

with respect to $\mathcal{P}$. Thus $N_{b_2}u_0 \geq (r(N_{b_1})+\delta)$ with respect to $\mathcal{P}$, and so by Theorem 8.4, $r(N_{b_2}) \geq r(N_{b_1}) + \delta$. So $r(N_{b_2}) > r(N_{b_1})$ and $r(N_b)$ is strictly increasing. $\square$

Now, we state and prove the main result.

Theorem 8.6. *The following are equivalent:*

(i) b_0 *is the first extremal point of the boundary value problem corresponding to* (8.1),(8.2) *for* $0 \leq x \leq \beta$*;*

(ii) there exists a nontrivial solution u *of the boundary value problem* (8.1),(8.2) *for* $0 \leq x \leq b_0$ *such that* $u \in \mathcal{P}_{b_0}\backslash\{0\}$*;*

(iii) $r(N_{b_0}) = 1$.

Proof. First, we show $(iii) \Rightarrow (ii)$; since $r(N_{b_0}) = 1 > 0$, by Theorem 8.3, $r(N_{b_0})$ is an eigenvalue of N_{b_0}, and so there exists a $u \in \mathcal{P}_{b_0}\backslash\{0\}$ such that $N_{b_0}u(x) = r(N_{b_0})u(x) = u(x)$ for $x \in [0, b_0]$. So (ii) holds.

Next, we prove $(ii) \Rightarrow (i)$. Let $u \in \mathcal{P}_{b_0}\backslash\{0\}$ satisfy (8.1),(8.2), for $0 \leq x \leq b_0$. For $x > b_0$, extend $u(x) = \int_0^{b_0} G(b_0, s)p(s)u(s)ds + (x - b_0)\int_0^{b_0} \frac{\partial}{\partial x}G(b_0, s)p(s)u(s)ds$. So $N_{b_0}u(x) = u(x)$ for $0 \leq x \leq \beta$. So $r(N_{b_0}) \geq 1$.

If $r(N_{b_0}) = 1$, then by Theorem 8.5, for $0 < b < b_0$, $r(N_b) < r(N_{b_0}) = 1$, and so b_0 is the first extremal point of (8.1),(8.2).

Assume $r(N_{b_0}) > 1$. Let $v \in \mathcal{P}_{b_0}\backslash\{0\}$ such that $N_{b_0}v = r(N_{b_0})v$. Now v restricted to $[0, b_0]$ belongs to Ω_{b_0}, and so there exists a $\delta > 0$ such that $u \geq \delta v$ with respect to $\mathcal{P}_{b_0}$. Extend $v(x) = \int_0^{b_0} G(b_0, s)p(s)v(s)ds + (x - b_0)\int_0^{b_0} \frac{\partial}{\partial x}G(b_0, s)p(s)v(s)ds$ for $x \geq b_0$. Then $u \geq \delta v$ with respect to $\mathcal{P}$. Assume δ is maximal such that $u \geq \delta v$. Then

$$u = N_{b_0}u \geq N_{b_0}(\delta v) = \delta N_{b_0}v = \delta r(N_{b_0})v.$$

Since $r(N_{b_0}) > 1$, $\delta r(N_{b_0}) > \delta$. But $u \geq \delta r(N_{b_0})v$, a contradiction to the fact that δ

is the maximal value satisfying $y \geq \delta v$. So $r(N_{b_0}) = 1$, and so b_0 is the first extremal point of (8.1),(8.2).

Last, we show $(i) \Rightarrow (iii)$. If b_0 is the first extremal point of the boundary value problem (8.1),(8.2), there exists a $u \in \mathcal{P}_{b_0} \backslash \{0\}$ such that $r(N_{b_0})u = N_{b_0}u$, and so $r(N_{b_0}) \geq 1$. Assume $r(N_{b_0}) > 1$. By Lemma 8.2, $\lim\limits_{b \to 0} r(N_b) = 0$, and so by the Intermediate Value Theorem, there exists an $\alpha \in (0, b_0)$ such that $r(N_\alpha) = 1$. So there exists a nontrivial solution of (8.1),(2.2) on $[0, \alpha]$, which contradicts the fact that b_0 is the first extremal point. So $r(N_{b_0}) = 1$. $\square$

8.4 The Third Order Problem

In this section, we will consider the third order boundary value problem,

$$u^{(3)} + p(x)u = 0, \tag{8.3}$$

for $0 \leq x \leq \beta$ satisfying the boundary conditions

$$u(0) = u'(r) = u''(b) = 0, \tag{8.4}$$

where $0 < 1/2 < r$, $r < b \leq \beta$, and $p(x)$ is a nonnegative continuous function on $[0, b]$ which does not vanish identically on any compact subinterval of $[0, b]$.

We will be defining compact integral operators whose kernels are the Green's function for $-u^{(3)} = 0$ satisfying (8.4). Because $H(x, s)$, which was defined in Chapter 5, has the property that $\frac{\partial^2}{\partial x^2} H(x, s) = 0$ for all $(x, s) \in [r, 1] \times [0, 1]$, $H(x, s)$ satisfies (8.4), and so $H(x, s)$ is the Green's function for $-u^{(3)} = 0$ satisfying (8.4).

To apply Theorems 8.1-8.4, we need to define a family of Banach spaces $\mathcal{B}$ and cones $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ to be

$$\mathcal{B} = \{u \in C^1[0, \beta] \mid u(0) = 0\},$$

with the norm

$$||u|| = \sup_{0 \leq x \leq \beta} |u'(x)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(x) \geq 0 \text{ on } [0, \beta]\}.$$

From earlier, we know that $\mathcal{P}^\circ \neq \emptyset$. In fact,

$$\{u \in \mathcal{B} \mid u(x) > 0 \text{ on } (0, \beta] \text{ and } u'(0) > 0\} \subset \mathcal{P}^\circ.$$

For all $b \in [0, \beta]$, define the Banach space $\mathcal{B}_b$ to be

$$\mathcal{B}_b = \{u \in C^1[0, b] \mid u(0) = 0\},$$

with the norm

$$||u|| = \sup_{0 \leq x \leq b} |u'(x)|.$$

Define the cone $\mathcal{P}_b \subset \mathcal{B}_b$ to be

$$\mathcal{P}_b = \{u \in \mathcal{B}_b \mid u(x) \geq 0 \text{ on } [0, b]\}.$$

Again, note that for all $b \in [0, \beta]$, $\mathcal{P}_b^\circ \neq \emptyset$. In fact,

$$\{u \in \mathcal{B} \mid u(x) > 0 \text{ on } (0, b] \text{ and } u'(0) > 0\} \subset \mathcal{P}_b^\circ.$$

Now for each $b \in [0, \beta]$, define the linear operator

$$N_b u(x) = \begin{cases} \int_0^b H(x, s)p(s)u(s)ds, & 0 \leq x \leq b, \\ \int_0^b H(b, s)p(s)u(s)ds + (x - b)\int_0^b \frac{\partial}{\partial x} H(b, s)p(s)u(s)ds, & b \leq x \leq \beta. \end{cases}$$

Notice that by the way N_b is defined, $N_b u(x) \in C^1[0, \beta]$ for $u(x) \in C^1[0, \beta]$, and $N_b u(0) = 0$. So $N_b : \mathcal{B} \to \mathcal{B}$. Also note that when N_b is restricted to B_b, $N_b : \mathcal{B}_b \to \mathcal{B}_b$ by

$$N_b u(x) = \int_0^b H(x, s)p(s)u(s)ds.$$

So $u(x)$ is a solution to (8.3),(8.4) if and only if $u(x) = N_b u(x) = \int_0^b H(x, s)p(s)u(s)ds$ for $x \in [0, b]$.

Lemma 8.3. *For all $b \in [0, \beta]$, the linear operator N_b is positive with respect to $\mathcal{P}$ and $\mathcal{P}_b$. Also, $N_b : \mathcal{P} \backslash \{0\} \to \mathcal{P}_b^{\circ}$.*

Proof. Since for $u \in \mathcal{P}$, $H(x,s) \geq 0$, $\frac{\partial}{\partial x} H(b,s) \geq 0$, and $p(s)u(s) \geq 0$, $N_b u(x) \geq 0$ for $0 \leq x \leq \beta$. So $N_b : \mathcal{P} \to \mathcal{P}$. Similarly, $N_b : \mathcal{P}_b \to \mathcal{P}_b$.

Now set $\Omega_b = \{u \in \mathcal{B} \mid u(x) > 0 \text{ on } (0, \beta] \text{ and } y'(0) > 0\}$. Let $u \in \mathcal{P}_b \backslash \{0\}$. So there exists a compact interval $[c, d] \subset [0, b]$ such that $p(x)u(x) > 0$ for all $x \in [c, d]$. Since $H(x,s) > 0$ for $0 < x \leq b$,

$$\begin{aligned} N_b u(x) &= \int_0^b H(x,s)p(s)u(s)ds \\ &\geq \int_c^d H(x,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < x \leq b$.

Since $\frac{\partial}{\partial x} H(x,s)|_{x=0} > 0$,

$$\begin{aligned} N_b u'(0) &= \int_0^b \frac{\partial}{\partial x} H(0,s)p(s)u(s)ds \\ &\geq \int_c^d \frac{\partial}{\partial x} H(0,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

and so $N_b u \in \Omega_b$. So $N_b : \mathcal{P} \backslash \{0\} \to \Omega_b \subset \mathcal{P}_b^{\circ}$. □

Lemma 8.4. *The map $b \mapsto N_b$ is continuous in the uniform topology.*

Proof. First note that $\sup\limits_{0 \leq x \leq 1} |u(x)| \leq ||u||$. Let $f : (0, \beta] \to \{N_b\}$, $b \in [0, \beta]$, such that $f(b) = N_b$. Let $0 < b_1 < b_2 \leq \beta$. Let $\epsilon > 0$. Then

$$\begin{aligned} ||f(b_2) - f(b_1)|| &= ||N_{b_2} - N_{b_1}|| \\ &= \sup_{||u||=1} ||N_{b_2}u - N_{b_1}u|| \\ &= \sup_{||u||=1} \{ \sup_{x \in [0,\beta]} |(N_{b_2}u)'(x) - (N_{b_1}u)'(x)| \}. \end{aligned}$$

Since $\frac{\partial}{\partial x}H(x,s)$ and $p(x)$ are continuous functions for $0 \leq x \leq \beta$, they are bounded above for $0 \leq x \leq \beta$. Choose K and P such that $|\frac{\partial}{\partial x}H(x,s)| \leq K$ and $|p(x)| \leq P$ for $0 \leq x \leq \beta$.

Suppose $x \leq b_1$. Then for $|b_2 - b_1| < \delta < \dfrac{\epsilon}{KP\beta}$,

$$\begin{aligned}
|(N_{b_2}u)'(x) - (N_{b_1}u)'(x)| &= \left|\int_0^{b_2} \frac{\partial}{\partial x}H(x,s)p(s)u(s)ds - \int_0^{b_1} \frac{\partial}{\partial x}H(x,s)p(s)u(s)ds\right| \\
&= \left|\int_{b_1}^{b_2} \frac{\partial}{\partial x}H(x,s)p(s)u(s)ds\right| \\
&\leq \int_{b_1}^{b_2} \left|\frac{\partial}{\partial x}H(x,s)\right| |p(s)||u(s)|ds \\
&\leq \int_{b_1}^{b_2} KP\beta ds \\
&= KP\beta|b_2 - b_1| \\
&< KP\beta\frac{\epsilon}{KP\beta} = \epsilon
\end{aligned}$$

Now suppose $b_1 < x \leq b_2$. Since $H(x,s) \in C^1[0,\beta]$ in the first variable, for $|b_2 - b_1| < \delta < \dfrac{\epsilon}{2KP\beta}$, $\left|\frac{\partial}{\partial x}H(x,s) - \frac{\partial}{\partial x}H(b_1,s)\right| < \dfrac{\epsilon}{2P\beta^2}$. So

$$\begin{aligned}
|(N_{b_2}u)'(x) - (N_{b_1}u)'(x)| &= \left|\int_0^{b_2} \frac{\partial}{\partial x}H(x,s)p(s)u(s)ds - \int_0^{b_1} \frac{\partial}{\partial x}H(b_1,s)p(s)u(s)ds\right| \\
&\leq \int_{b_1}^{b_2} \left|\frac{\partial}{\partial x}H(x,s)\right| |p(s)||u(s)|ds \\
&\quad + \int_0^{b_1} \left|\frac{\partial}{\partial x}H(x,s) - \frac{\partial}{\partial x}H(b_1,s)\right| |p(s)||u(s)|ds \\
&< \int_{b_1}^{b_2} KP\beta ds + \int_0^{b_1} \frac{\epsilon}{2P\beta^2}P\beta ds \\
&= KP\beta|b_2 - b_1| + \frac{\epsilon}{2P\beta^2}Pb_1\beta \\
&< KP\beta\frac{\epsilon}{2KP\beta} + \frac{\epsilon}{2P\beta^2}P\beta^2 = \epsilon.
\end{aligned}$$

Now suppose $\beta > x > b_2$. Again, since $H(x,s) \in C^1[0,\beta]$ in the first variable, for $|b_2 - b_1| < \delta < \dfrac{\epsilon}{2KP\beta}$, $\left|\frac{\partial}{\partial x}H(b_2,s) - \frac{\partial}{\partial x}H(b_1,s)\right| < \dfrac{\epsilon}{2P\beta^2}$. So

$$|(N_{b_2}u)'(x) - (N_{b_1}u)'(x)| = \left|\int_0^{b_2} \frac{\partial}{\partial x}H(b_2,s)p(s)u(s)ds - \int_0^{b_1} \frac{\partial}{\partial x}H(b_1,s)p(s)u(s)ds\right|$$

$$\leq \int_{b_1}^{b_2} \left| \frac{\partial}{\partial x} H(b_2, s) \right| |p(s)||u(s)|ds$$
$$+ \int_0^{b_1} \left| \frac{\partial}{\partial x} H(b_2, s) - \frac{\partial}{\partial x} H(b_1, s) \right| |p(s)||u(s)|ds$$
$$< \int_{b_1}^{b_2} KP\beta ds + \int_0^{b_1} \frac{\epsilon}{2P\beta^2} P\beta ds$$
$$= KP\beta |b_2 - b_1| + \frac{\epsilon}{2P\beta^2} P b_1 \beta$$
$$< KP\beta \frac{\epsilon}{2KP\beta} + \frac{\epsilon}{2P\beta^2} P\beta^2 = \epsilon.$$

So we have that $\sup_{||u||=1} \{ \sup_{x \in [0,\beta]} |(N_{b_2} u)'(x) - (N_{b_1} u)'(x)| \} < \epsilon$ for $|b_2 - b_1| < \delta$. So $||f(b_2) - f(b_1)|| < \epsilon$ for $|b_2 - b_1| < \delta$, and so f is continuous. $\square$

Theorem 8.7. *For $0 < b \leq \beta$, $r(N_b)$ is strictly increasing as a function of b.*

Proof. In [22], it is shown that there exists $\lambda > 0$ and $u \in \mathcal{P}_b \backslash \{0\}$ such that $N_b u(x) = \lambda u(x)$. Extend u to $[b, \beta]$ by

$$u(x) = \frac{1}{\lambda} \left(\int_0^b H(b,s) p(s) u(s) ds + (x - b) \int_0^b \frac{\partial}{\partial x} H(b, s) p(s) u(s) ds \right).$$

Then for $x \in [0, \beta]$, $N_b u(x) = \lambda u(x)$. Thus for $0 < b \leq \beta$, $r(N_b) \geq \lambda > 0$.

Now let $0 < b_1 < b_2 \leq \beta$. Since $r(N_{b_1}) > 0$, by Theorem 8.2, there exists a $u_0 \in \mathcal{P}_{b_1} \backslash \{0\}$ such that $N_{b_1} u_0 = r(N_{b_1}) u_0$. Let $u_1 = N_{b_1} u_0$ and $u_2 = N_{b_2} u_0$. Then for $x \in (0, b_1]$,

$$(u_2 - u_1)(x) = \int_{b_1}^{b_2} H(x, s) u(s) p(s) ds > 0.$$

Also,

$$(u_2 - u_1)'(0) = \int_{b_1}^{b_2} \frac{\partial}{\partial x} H(0, s) u(s) p(s) ds > 0.$$

Thus the restriction of $u_2 - u_1$ to $[0, b_1]$ belongs to Ω_{b_1}. So there exists a $\delta > 0$ such that $u_2 - u_1 \geq \delta u_0$ with respect to $\mathcal{P}_{b_1}$. Since $u_2 \in \mathcal{P}$, it follows that $u_2 - u_1 \geq \delta u_0$ with respect to $\mathcal{P}$. Thus

$$u_2 \geq u_1 + \delta u_0$$

$$= r(N_{b_1})u + \delta u_0$$

$$= (r(N_{b_1}) + \delta)u_0$$

with respect to $\mathcal{P}$. Thus $N_{b_2}u_0 \geq (r(N_{b_1})+\delta)$ with respect to $\mathcal{P}$, and so by Theorem 8.4, $r(N_{b_2}) \geq r(N_{b_1}) + \delta$. So $r(N_{b_2}) > r(N_{b_1})$ and $r(N_b)$ is strictly increasing. $\square$

Now, we state and prove the main result.

Theorem 8.8. *The following are equivalent:*

(i) b_0 *is the first extremal point of the boundary value problem corresponding to* (8.3),(8.4) *for* $0 \leq x \leq \beta$*;*

(ii) there exists a nontrivial solution u *of the boundary value problem* (8.3),(8.4) *for* $0 \leq x \leq b_0$ *such that* $u \in \mathcal{P}_{b_0} \backslash \{0\}$*;*

(iii) $r(N_{b_0}) = 1$*.*

Proof. First, we show $(iii) \Rightarrow (ii)$; since $r(N_{b_0}) = 1 > 0$, by Theorem 8.3, $r(N_{b_0})$ is an eigenvalue of N_{b_0}, and so there exists a $u \in \mathcal{P}_{b_0} \backslash \{0\}$ such that $N_{b_0}u(x) = r(N_{b_0})u(x) = u(x)$ for $x \in [0, b_0]$. So (ii) holds.

Next, we prove $(ii) \Rightarrow (i)$. Let $u \in \mathcal{P}_{b_0} \backslash \{0\}$ satisfy (8.3),(8.4), for $0 \leq x \leq b_0$. For $x > b_0$, extend $u(x) = \int_0^{b_0} H(b_0, s)p(s)u(s)ds + (x - b_0)\int_0^{b_0} \frac{\partial}{\partial x}H(b_0, s)p(s)u(s)ds$. So $N_{b_0}u(x) = u(x)$ for $0 \leq x \leq \beta$. So $r(N_{b_0}) \geq 1$.

If $r(N_{b_0}) = 1$, then by Theorem 8.7, for $0 < b < b_0$, $r(N_b) < r(N_{b_0}) = 1$, and so b_0 is the first extremal point of (8.3),(8.4).

Assume $r(N_{b_0}) > 1$. Let $v \in \mathcal{P}_{b_0} \backslash \{0\}$ such that $N_{b_0}v = r(N_{b_0})v$. Now v restricted to $[0, b_0]$ belongs to Ω_{b_0}, and so there exists a $\delta > 0$ such that $u \geq \delta v$ with respect to $\mathcal{P}_{b_0}$. Extend $v(x) = \int_0^{b_0} H(b_0, s)p(s)v(s)ds + (x - b_0)\int_0^{b_0} \frac{\partial}{\partial x}H(b_0, s)p(s)v(s)ds$ for $x \geq b_0$. Then $u \geq \delta v$ with respect to $\mathcal{P}$. Assume δ is maximal such that $u \geq \delta v$. Then

$$u = N_{b_0}u \geq N_{b_0}(\delta v) = \delta N_{b_0}v = \delta r(N_{b_0})v.$$

Since $r(N_{b_0}) > 1$, $\delta r(N_{b_0}) > \delta$. But $u \geq \delta r(N_{b_0})v$, a contradiction to the fact that δ is the maximal value satisfying $y \geq \delta v$. So $r(N_{b_0}) = 1$, and so b_0 is the first extremal point of (8.1),(8.2).

Last, we show $(i) \Rightarrow (iii)$. If b_0 is the first extremal point of the boundary value problem (8.3),(8.4), there exists a $u \in \mathcal{P}_{b_0} \backslash \{0\}$ such that $r(N_{b_0})u = N_{b_0}u$, and so $r(N_{b_0}) \geq 1$. Assume $r(N_{b_0}) > 1$. By Lemma 8.4, $\lim_{b \to 0} r(N_b) = 0$, and so by the Intermediate Value Theorem, there exists an $\alpha \in (0, b_0)$ such that $r(N_\alpha) = 1$. So there exists a nontrivial solution of (8.1),(2.2) on $[0, \alpha]$, which contradicts the fact that b_0 is the first extremal point. So $r(N_{b_0}) = 1$. $\square$

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