ABSTRACT

Asymptotics for mean field games of market competition

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The goal of this thesis is to analyze the limiting behavior of solutions to a system of mean field games developed by Chan and Sircar to model Bertrand and Cournot competition. We first provide a basic introduction to control theory, game theory, and ultimately mean field game theory. With these preliminaries out of the way, we then introduce the model first proposed by Chan and Sircar, namely a coupled system of two nonlinear partial differential equations. This model contains a parameter ϵ that measures the degree of interaction between players; we are interested in the regime ϵ goes to 0. We then prove a collection of theorems which give estimates on the limiting behavior of solutions as ϵ goes to 0 and ultimately obtain recursive growth bounds of polynomial approximations to solutions. Finally, we state some open questions for further research.

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To my parents

CHAPTER ONE

Introductions of Pertinent Concepts

The goal of this chapter is to build the necessary framework required to understand mean field games mathematically and inuitively. It is broken up into three sections: optimal control theory, game theory, and mean field games. Each section will build on concepts explained in the previous sections.

Optimal Control Theory

The theory of optimal control is a product of the natural desire to achieve a goal in the most effective and efficient way possible; we want the most, while using the least. From a thermodynamic perspective, it would seem that the universe is programmed toward optimization, always minimizing energy use in any given process. The mathematical theory of optimal control does not seek to wrestle with the philosophical implications of this, but rather seeks to define optimization problems in a mathematically cogent way, and to eventually find solutions to these often-complicated problems. The basic ideas and intuitions behind pertinent results from optimal control theory will be presented here, but for a more detailed exposition of optimal control theory, see Liberzon (2011).

To begin to define an optimal control problem, we first must define the *control system*, given by

$$\dot{x} = f(t, x(t), \alpha(t)), \quad x(t_0) = x_0 \tag{1}$$

Here, t is *time*; x is called the *state* and is a function of t with values in $\mathbb{R}^n$; and α is the *control input* and is likewise a function of t with its values in some control space $\Omega \subseteq \mathbb{R}^m$. t_0 is the starting or *initial time*, making x_0 the *initial state*. Thus, this equation explains that the way a state changes over time is a function of time, the state itself at a given time, and some control input. The control input is the mathematical representation of how we can influence and change the state of a system. It is not enough to simply be able to define a dynamic system, but also explain how we can alter it toward a certain end. For instance, we could model the state of a rocket as it journies into orbit, however, this does not help us understand how we could change its flight path to potentially minimize fuel costs. Thus, we need a control input within the dynamic system that we are observing and want to optimize.

We have now defined a control system, but we need a mathematical representation of something to optimize. This leads to the cost functional J, given by:

$$J(\alpha) = \int_{t_0}^{t_f} \mathcal{L}(t, x(t), \alpha(t))\, \mathrm{d}t + \mathcal{K}(t_f, x_f). \tag{2}$$

Here we have t_f, the *final time*, and x_f, the *final state* (given by $x_f = x(t_f)$). $\mathcal{L}$ is the *running cost*, also known as the *Lagrangian*. We can sete that for any $t \in [t_0, t_f]$, $\mathcal{L}$ is the associated cost that depends on both the state and the control at time t, so as time passes, our control system accrues a certain cost given by $\mathcal{L}$. $\mathcal{K}$ is the *terminal cost* and is a function that depends on only the final conditions. It is apparent that the cost functional is dependent on t_0, x_0, t_f, and α. However, depicting α as the argument of J not only simplifies the notation, but also reinforces the fact that we wish to minimize J over all $\alpha \in \Omega$. We now have the neccesary framework to mathematically define the control problem: find some control, α^* that minimizes (at least locally) the cost functional $J(\alpha)$. From this understanding of a control problem, we now seek to define the *principle of optimality*.

To understand where the principle of optimality comes from, we must slightly shift our initial understanding of the problem at hand. Up to this point we have discussed the optimization problem as a forward-in-time process. However, it turns out that approaching this problem from a backwards-in-time perspective simplifies the process of checking for optimality. This can be intuitively understood by examining the discrete case, where the trajectory is now given by a path, $\{x_k\}_{k=0}^{T} \subset X$, $T \in \mathbb{N}$, where $k \in \{0, 1, \ldots T\}$ represents a discrete time step. We also have decisions $\alpha_k \in A$, where X and A are finite sets with N and M elements respectively. Going forward, to find an optimal solution we would have to tediously examine and compare every single possible path as k moves step-by-step from 0 to 1, from 1 to 2, eventually up to T. Note that there are $M^T T$ additions required to evaluate a cost, since there are M^T possible paths and T time steps.

Starting at T and moving to $T-1$, we still need to find the path and control within this time step with the least cost (if two or more paths have the same least cost, choose one at random). However, by the construction of this process, if a point x_k for some decision α_k lies on an optimal path, then that path from k onwards must be an optimal path with respect to k acting as the initial condition. This is the principle of optimality for the discrete case, which ensures that there does not exist some other trajectory made up of pieces of non-optimal paths that is more optimal than the path found through this process. Note that there are now NMT additions required to calculate cost, since there are T time steps and M possible decisions for N possible path points to choose going from $k+1$ to k. Thus, computationally, approaching the problem this way reduces the amount of work required to solve the problem.

To derive an infinitesimal version of the principle of optimality, we now reframe the cost functional given by (2) into this backwards-in-time approach:

$$J(t, x, \alpha) = \int_t^{t_1} \mathcal{L}(s, x(s), \alpha(s))\, \mathrm{d}s + \mathcal{K}(t_1, x(t_1)). \tag{3}$$

The cost functional is still dependent on α, but is now also dependent on t and x, where x in the argument of J is fixed. From here, we define the *value function*:

$$u(t, x) \coloneqq \inf_{\alpha|_{[t,t_1]}} J(t, x, \alpha). \tag{4}$$

The existence of a minimizer α^* is not assumed, indicated by the use of an infimum, taken over all controls α restricted to the interval $[t, t_1]$. From this definition we can see that $u(t_1, x) = \mathcal{K}(x)\ \forall x \in \mathbb{R}^n$. Our optimization problem is now concerned with minimizing cost functionals of the form (3). The principle of optimality is as follows: $\forall (t, x) \in [t_0, t_1) \times \mathbb{R}^n$ and every $\Delta t \in (0, t_1 - t]$,

$$u(t,x) = \inf_{\alpha|_{[t,t+\Delta t]}} \left\{ \int_t^{t+\Delta t} \mathcal{L}(s, x(s), \alpha(s))\,\mathrm{d}s + u(t+\Delta t, x(t+\Delta t)) \right\}, \tag{5}$$

with $x(\cdot)$ corresponding to $\alpha|_{[t,t+\Delta t]}$ and $x(t) = x$.

We are now ready to derive the *Hamilton-Jacobi-Bellman* (HJB) *equation*, which is the differential form of the principle of optimality expressed in (5). Using the chain rule and the relation in (1), we can write $u\left(t+\Delta t, x(t+\Delta t)\right)$ as a first order Taylor expansion centered at $\Delta t = 0$,

$$u\left(t+\Delta t, x(t+\Delta t)\right) = u(t,x) + u_t(t,x)\Delta t + D_x u(x,t) \cdot f(t,x,\alpha(t))\Delta t + o(\Delta t) \tag{6}$$

Assuming that $\mathcal{L}$ and α are continuous, we can see that

$$\int_t^{t+\Delta t} \mathcal{L}(s, x(s), \alpha(s))\,\mathrm{d}s = \mathcal{L}(t, x, \alpha(t))\Delta t + o(\Delta t). \tag{7}$$

With the equalities given in (6) and (7), (5) can be written as

$$\begin{aligned} u(t,x) = \inf_{\alpha|_{[t,t+\Delta t]}} \big\{ &\mathcal{L}(t,x,\alpha(t))\Delta t + u(t,x) + u_t(t,x)\Delta t + \\ &D_x u(t,x) \cdot f(t,x,\alpha(t))\Delta t + o(\Delta t) \big\}. \end{aligned} \tag{8}$$

Since $u(t,x)$ and $u_t(t,x)$ are not dependent on α, they can be pulled out of the infimum.

We now divide through by Δt, taking the limit as $\Delta t \to 0$ to get

$$u_t(t,x) + \inf_{\alpha \in \Omega} \left\{ \mathcal{L}(t,x,\alpha(t)) + D_x u(t,x) \cdot f(t,x,\alpha(t)) \right\} = 0. \tag{9}$$

Defining the *Hamiltonian* (a term that comes out of the calculus of variation)

$$\mathcal{H}\left(t,x,-D_x u(t,x)\right) := \sup_{\alpha \in \Omega} \left\{ -D_x u \cdot f(t,x,\alpha) - \mathcal{L}(t,x,\alpha(t)) \right\}, \tag{10}$$

we finally arrive at the HJB equation in its simplest form:

$$u_t(t,x) - \mathcal{H}\left(t,x,-D_x u(t,x)\right) = 0. \tag{11}$$

The HJB equation can also be derived for a stochastic process, where the trajectory is subject to random motion (Brownian motion). The control system is now given by

$$\mathrm{d}x = f(t,x(t),\alpha(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \tag{12}$$

which is the equation for Brownian motion. Here σ is a positive constant and $\mathrm{d}W(t)$ is the Brownian motion. The second order HJB is subsequently given by

$$u_t + \frac{\sigma^2}{2}\Delta u - \mathcal{H}\left(t,x,-D_x u(t,x)\right) = 0, \tag{13}$$

where $\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i \partial x_i}$ is the Laplace operator, indicating a diffusion term. The derivation for (13) is similar to that of (11), with the essential difference that we use the Ito formula from stochastic calculus. See for example Øksendal (2003).

To summarize, starting with a dynamical system, one can frame it into the context of a control system. The goal is then to find a certain control that minimizes a cost functional. The solution is then embedded in the HJB equation, which is a

PDE expressing the principle of optimality. These are the basic ideas of control theory necessary to progress to the idea of a mathematical game.

Game Theory

Games have long been a part of the human tradition as a means of entertainment, their history extending as far into the past as ancient Egypt. However, game theory is a relatively modern mathematical discipline, with much of its formalism coming out of the early and mid twentieth century (Myerson, 2013). The general idea of a game is an activity where players operate under set rules to achieve a certain goal, either individually or cooperatively. Since the goal of any game is to "win," a natural question is how to find the best strategy that either guarantees, or statistically improves chances of winning. More generally, a mathematical *game* is defined as a social situation involving two or more players (Myerson, 2013). This could refer to a game that is simply used as entertainment, but this definiton also allows for applications in economics, where there exist multiple players competing to maximize profit. It follows that *game theory* is the study of mathematical models of conflict and cooperation between rational decision makers (Myerson, 2013). It is important to consider only rational players, because the premise of a game is built on the fact that those playing would not do something to hurt their own chances of winning. With these ideas in mind, we can express a game in terms of the language of control theory developed earlier. The concepts explained here only scratch the surface of the vast and growing field of game theory. For a more detailed exposition on the subject, see Myerson (2013) or Gibbons (1997).

In any game, there exist N players denoted by $i \in \{1, 2, \dots, N\}$. There is then a control system for each player:

$$\dot{x}_i = f_i(t, x_i(t), \alpha_i(t)), \quad x_i(t_0) = x_{i_0}. \tag{14}$$

In this expression, each player's trajectory, $x_i \in \mathbb{R}^n$, changes over time as a function of time, the trajectory itself, and any decision at a given time, $\alpha_i(t) \in \Omega_i \subseteq \mathbb{R}^m$, where Ω_i is the set of permissible decisions for player i. The associated cost functional for a particular player is then given by:

$$J_i(\alpha_1, \ldots, \alpha_N) = \int_{t_0}^{t_f} \mathcal{L}_i(t, x_1(t), \ldots, x_N(t), \alpha_1(t), \ldots, \alpha_N(t))\, \mathrm{d}t + \mathcal{K}_i(t_f, x_{1_f}, \ldots, x_{f_N}), \tag{15}$$

where $x_i(t_f) = x_{i_f}$. This expression can be thought of as an indication of a player's success. This is visibly more involved than the cost functional in (2), since, even if we are examining one particular individual, we must still take into account every other player's trajectory as they make their own decisions. This is a result of the fact that every player is rational; in an attempt to maximize their own success, each player is playing in a way that considers how other players are playing. The cost functional in (15) can be thought of as what players need to minimize in order to win, or at least be the most successful. We now have an associated value function for a particular player:

$$\begin{aligned} u^i(t,x) &= \inf_{\alpha_i \in \Omega_i} \left\{ J(\alpha_1, \ldots, \alpha_N) \right\} = \\ &\inf_{\alpha_i \in \Omega_i} \left\{ \int_{t_0}^{t_f} \mathcal{L}_i(t, x_1(t), \ldots, x_N(t), \alpha_1(t), \ldots, \alpha_N(t))\, \mathrm{d}t + \mathcal{K}_i(t_f, x_{1_f}, \ldots, x_{N_f}) \right\}, \end{aligned} \tag{16}$$

where $x = (x_1, \ldots, x_N)$. Notice that the infimum is taken over all possible decisions that player i can make, which makes sense as player i cannot directly make decisions for the other players; they can only react to the choices other players are making.

To undestand what a solution to a game might be, we must discuss the concept of *Nash equilibrium*. Nash equilibrium refers to a state in which no one player can unilaterally improve their situation (Gibbons, 1997). That is to say, a Nash equilibrium strategy is a collection of controls $\left\{\alpha_1^*(t), \ldots, \alpha_N^*(t)\right\}$ such that

$$J_i(\alpha_1^*, \ldots, \alpha_N^*) = \min_{\alpha_i \in \Omega_i} J_i\left(\alpha_i, (\alpha_j^*)_{j \neq i}\right), \quad \forall i \in \{1, \ldots, N\}, \tag{17}$$

with $\left(\alpha_i, (\alpha_j^*)_{j \neq i}\right) = \left(\alpha_1^*, \ldots, \alpha_{i-1}^*, \alpha_i, \alpha_{i+1}^*, \ldots, \alpha_N^*\right)$. The control $\alpha^* = \left(\alpha_1^*, \ldots, \alpha_N^*\right)$ is called a *Nash point*. It is important to note that a Nash equilibrium does not necessarily exist. Similarly, this definition neither implies uniqueness of a Nash equlibrium, nor that this state is what is best for everyone. Nash equilibrium, if it exists at all, is simply a state in which no one can singlehandedly improve their situation by making any of their permissible decisions. That is, it could be the case that $J_i(\alpha_1^*, \ldots, \alpha_N^*) \neq u^i(t, x_i)$ for some, or for all $i \in \{1, \ldots, N\}$. Nash equilibrium is a solution that takes into account every player. Since games are competitive, not everyone can be a winner, however, we can look for the state in which no one can do any better given their decisions able to be made. So, we can still have players who are worse off than others, but there exists at least a stability associated with this particular state; none of the players can gain more than they already have. This is not a typical notion of "winning," but it is the best way in which everyone can "win" to some degree. Perhaps more pessimistically, there is solace in the fact that a player is not losing as badly as they could be.

Recall that the HJB equation is a differential equation that describes the principle of optimality, which loosely says that travelling backwards-in-time along infintesimal paths that are optimal ensures that the entire path travelled is optimal upon arriving at t_0. To define a HJB equation for each player, we must define the Hamiltonians. We say the set of Lagrangians, $\{\mathcal{L}_i\}_{i=1}^N$, has a Nash point α^* at x if

$$\begin{aligned}
&- \nabla u_i \cdot f_i\left(x, \alpha^*\right) - \mathcal{L}_i\left(x, \alpha^*\right) \\
&\quad = \min_{\alpha_i \in \Omega_i} \left\{ -\nabla u_i \cdot f_i\left(x, \alpha_i, (\alpha_j^*)_{j \neq i}\right) - \mathcal{L}_i\left(x, \alpha_i, (\alpha_j^*)_{j \neq i}\right) \right\}, \forall i \in \{1, \ldots, N\}.
\end{aligned} \tag{18}$$

Note that $\nabla u_i = \left(\frac{\partial u_i}{\partial x_1}, \dots, \frac{\partial u_i}{\partial x_N} \right)$ is the spatial gradient of u_i. If such a point exists, the Hamiltonian, $\mathcal{H}_i\left(x, -\nabla u_1, \dots, -\nabla u_N\right)$, is defined as the system (18). We can now express the HJB equation for each player:

$$\partial_t u_i(t,x) + \frac{\sigma^2}{2} \Delta u^i - \mathcal{H}_i\left(x, -\nabla u_1, \dots, -\nabla u_N\right) = 0 \tag{19}$$

Here, $\Delta u_i = \sum_{j=1}^{N} \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \sum_{j=1}^{N} \sum_{k=1}^{n} \frac{\partial^2 u_i}{\partial x_{j_k} \partial x_{j_k}}$. It is important to mention that all of the HJB equations are coupled together through nonlinearities in ∇u_i, found in the Hamiltonians, $\mathcal{H}_i$. Since the equations in system (19) are all coupled together through the spatial gradients of each player's value function, it is apparent that as each player's trajectory changes, not only is their own value function affected, but also the value functions of the other players. This reflects the idea that we are dealing with rational players who will make logical decisions based off of the trajectories and successes of the others in the game.

While it may be the case that each player's trajectory affects the other players' value functions, if we take the number of players N to infinity, the impact of any one player on anyone else, and thus the entire system, becomes negligible. This is analogous to many processes in physics where the microscopic view of any one particle does little to inform about the macroscopic view of the entire system. For instance, electrical current is made up of an astronomically large number of electrons moving across a wire, yet any particular electron, because it is constantly colliding with atoms in the wire, in sum is moving glacially slow across the length of the wire. It turns out that this general physical concept of the microscipic trivially informing the macroscopic applies to games, in particular in economic systems with large numbers of players.

As one can see, the mathematical description of a game is much more complex than that of a single control system. Subsequently, solutions are that much harder to come by. Even in the case of a single player, it can be difficult to solve the

HJB equation with an explicit solution rare. Solving such a PDE often requires numerical methods and approximations to obtain a value function, but even then, the computation can quickly get out of hand. However, despite the hopelessness of finding a general solution to any and every game, there are constraints we can put on the games and the players that can abate some of the complexites of the problem.

Mean Field Games

In our attempt to simplify the problem at hand, it is important to consider constraints that are reasonable; these constraints should still allow the game to resemble real-world systems. One specific type of game that has received much attention in the study of game theory is a *mean field game*. A mean field game is a type of game with a very large number of rational players, whose individual impact is negligible on the entire system. In addition, these players are identical in their goals and pursuits and anonymous to one another, so their information is limited to what they are able to obseve. This ensures that no player has an unfair advantage over another, for example, some sort of outside information on other players' strategies. The goal of this section is to develop a mathematical framework for describing mean field games. For a more rigorous treatment of mean field games and strategies on analyzing them, see Lasry and Lions (2007).

We are dealing with a large number of players, but the players are identical, so there is no need to index the control system of each player as done in (14). Instead, more simply, a player's dynamics is given by (1). However, the cost functional does not remain the same as in (2). Instead, a player's cost functional is also dependent on the probability density of a trajectory, $x(t)$, or a mean field. This is due to the fact that we are dealing with an infinite number of players who negligibly impact the system individually. We let $m(x,t) = m\left(x(t)\right)$ represent the probability density for

x at time t. Our cost functional J is now

$$J(\alpha, m) = \int_{t_0}^{t_f} \mathcal{L}\big(t, x(t), \alpha(t), m(x,t)\big)\, \mathrm{d}t + \mathcal{K}(t_0, x_0). \tag{20}$$

It follows that the value function is given by

$$u(t,x) = \inf_{\alpha|_{[t,T]}} \left\{ \int_t^T \mathcal{L}\big(s, x(s), \alpha(s), m(x,s)\big)\, \mathrm{d}s + u\big(T, x(T)\big) \right\}. \tag{21}$$

Notice that since the Lagrangian is dependent on m, the Hamiltonian is also dependent on m. This means that our HJB equation looks like this:

$$u_t + \frac{\sigma^2}{2}\Delta u - \mathcal{H}(t, x, -D_x u, m) = 0. \tag{22}$$

Since we are examining the limiting case as N goes to infinity, we see that a player's value function is no longer dependent on the dynamics of another player as in (16). Rather, a player's value function is now dependent on the average density of the other players' trajectories. While there is no explicit expression for m, m can be implicitly described by a PDE known as the Fokker-Planck equation.

Since m is a probability density for $x(t)$, the probability that the trajectory x at time t is in a given set Ω is given by

$$\mathbb{P}\big[x(t) \in \Omega\big] = \int_\Omega m(x,t)\, \mathrm{d}x. \tag{23}$$

Using a conservation of mass argument, we can view (23) as an amount of mass in the set Ω. Hence, the change in the amount of mass in Ω is equal to the negative flux of mass along $\partial\Omega$, the boundary of Ω:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega m(x,t)\, \mathrm{d}x = -\int_{\partial\Omega} m(x,t) v(x,t) \cdot \mathbf{n}\, \mathrm{d}S, \tag{24}$$

where $v : \mathbb{R}^n_x \times \mathbb{R}_t \to \mathbb{R}^n$, given by $v(t,x) = \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$, is a vector field that acts on the mass. Thus, by the divergence theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m(x,t)\,\mathrm{d}x = -\int_\Omega \nabla_x \cdot (m(x,t)v(x,t))\,\mathrm{d}x, \tag{25}$$

which implies

$$\partial_t m = -\nabla_x \cdot (mv). \tag{26}$$

This is the most basic form of the Fokker-Planck equation.

Alternatively, we can take some smooth test function, φ, and the expected value, $\mathbb{E}\left[\varphi(t,x)\right] = \int_{\mathbb{R}^n} \varphi(t,x)m(t,x)\,\mathrm{d}x$, and take the derivative in time on each side. Recalling that $\frac{\mathrm{d}x}{\mathrm{d}t} = v$, we see on one hand,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\varphi(t,x)\right] &= \mathbb{E}\left[\partial_t\varphi(t,x) + \nabla_x\varphi(t,x)\frac{\mathrm{d}x}{\mathrm{d}t}\right] \\ &= \int_{\mathbb{R}^n}\partial_t\varphi(t,x)m(t,x)\,\mathrm{d}x + \int_{\mathbb{R}^n}\nabla_x\varphi(t,x)v(t,x)m(t,x)\,\mathrm{d}x,\end{aligned} \tag{27}$$

and on the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\varphi(t,x)m(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^n}\partial_t\varphi(t,x)m(t,x)\,\mathrm{d}x + \int_{\mathbb{R}^n}\varphi(t,x)\partial_t m(t,x)\,\mathrm{d}x. \tag{28}$$

Setting the results of (27) and (28) equal to each other yields

$$\int_{\mathbb{R}^n}\varphi(t,x)\partial_t m(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^n}\nabla_x\varphi(t,x)v(t,x)m(t,x)\,\mathrm{d}x. \tag{29}$$

Integrating by parts, we get

$$\int_{\mathbb{R}^n}\varphi(t,x)\partial_t m(t,x)\,\mathrm{d}x = -\int_{\mathbb{R}^n}\varphi(t,x)\nabla_x\cdot\big(v(t,x)m(t,x)\big)\,\mathrm{d}x. \tag{30}$$

But φ is arbitrary, so we obtain (26), the same result described by following the physical intuition. We can also consider a stochastic case and obtain by Ito's formula

$$\partial_t m - \frac{\sigma^2}{2}\Delta m + \nabla_x \cdot (mv) = 0 \tag{31}$$

Recall from the discussion of games that Nash equilibrium is a particular state in which no one player can unilaterally improve their situation. Since we are looking at the case as N goes to infinity, a Nash equilibrium strategy is a control α^* and the distribution m^* (obtained from plugging in α^* as a control into the control system) such that

$$J(\alpha^*, m^*) = \min_{\alpha \in \Omega} J(\alpha, m^*) \tag{32}$$

This seems like it might be impossible to find a solution, since m^* is defined by α^*, yet to find α^*, we seem to need to know m^*. However, the differential equations for u and m are actually coupled, so in general there is a dependence of u (and subsequently α^*) on m and vice versa.

To see how the equations are coupled, we can write v in terms of $\mathcal{H}$, because at Nash equilibrium each player will be operating under optimal conditions, that is $v = v^*$. By the definition of v and the Hamiltonian given in (10), if $p = -D_x u$, then $v^* = D_p\mathcal{H}$, so that we have

$$\partial_t m - \frac{\sigma^2}{2}\Delta m + \nabla \cdot \big(m D_p \mathcal{H}(t, x, -D_x u, m)\big) = 0, \tag{33}$$

which is now a Kolmogorov equation. We now have the system of coupled, second-order mean field equations:

$$\begin{aligned} &\partial_t u + \frac{\sigma^2}{2}\Delta u - \mathcal{H}(t, x, -D_x u, m) = 0 \\ &\partial_t m - \frac{\sigma^2}{2}\Delta m + \nabla \cdot \big(m D_p \mathcal{H}(t, x, -D_x u, m)\big) = 0. \end{aligned} \tag{34}$$

On one hand, the HJB equation is coupled with the Kolmogorov equation through m itself, whereas the Kolmogorov equation is coupled to the HJB equation through the Hamiltonian, and hence the spacial gradient of u. We can also see that the HJB equation is a backwards in time equation, while the Kolmogorov equation is defined forwards in time. Thus, while the constraints of a mean field game simplify the general construction of a game, there are other complexities within their mathematical representation which are not a consequence of the number of unique players present.

Because the equations in (34) are coupled and non-linear, the task to solve them is difficult, but many techniques for their analysis have been developed over the past couple of decades. Mean field games have recieved much attention in the topic of game theory, and much has been discovered about their solutions. For a more in-depth treatment of the analysis of these equations, see Cardaliaguet (2010).

CHAPTER TWO

Bertrand and Cournot Mean Field Games

The purpose of this chapter is to explain the motivation behind the system of mean field game equations developed by Patrick Chan and Ronnie Sircar in their paper *Bertrand and Cournot Mean Field Games* (Chan and Sircar, 2015). These equations, while constructed via idealized situations and assumptions, have applications to real-world economics as demonstrated by Chan and Sircar in their paper *Fracking, Renewables, and Mean Field Games*, which analyzes the rapid decline of oil prices from June 2014 to January 2015 (Chan and Sircar, 2017). It is imperative that these equations are understood, as the equations, and more specifically their solutions, are the focus of the analysis given in Chapter 3. For other applications of mean field games to idealized situations see Guéant et al. (2011).

A major application of mean field games is in economics. We can look at an oligopoly of agents, each trying to compete with each other to maximize their profits. The basic assumptions of a mean field game hold with the agents being anonymous, uncooperative, and rational. In addition, they are selling identical products, so no one agent has an unfair advantage over another. There are two basic models of competition, Bertrand and Cournot competition. In *Bertrand competition* the control available to agents is price. Agents directly control the price at which to sell their product, which by the law of supply and demand affects the quantity of product produced. In *Cournot competition*, the control given to the agents is the quantity of product. Agents are free to control how much of their product is being produced, which similarly by the law of supply and demand affects the price at which the product is sold. For a finite number of players, Bertrand competition leads to a Nash equilibrium in which the agents make no profit. This is a consequence of the fact that to be competitive, an agent must sell their product at or less than the price of any other agent. In Cournot competition, a profit can still be made as agents are not required to directly lower prices to remain competitive. This may seem like Cournot competition is superior, however, as the number of agents tends toward infinity, Bertrand and Cournot competions are actually equivalent; the optimal price in Bertrand competition implies the optimal quantity in Cournot competition (Chan and Sircar, 2015).

In *Bertrand and Cournot Mean Field Games*, Chan and Sircar develop a mean field game for Bertrand/Cournot competition with exhaustible resources. Since resources are exhaustible, if an agent uses all of their resources, then they are no longer a part of the competition. In this setting $m(x,t)$ is the density of agents with a positive amount of resources at time $t > 0$. At $t = 0$, we let $M(x)$ denote the density of players with a positive initial amount of resources. Since $M(x)$ is a probability measure, $\int_{(0,\infty)} M(x)\,\mathrm{d}x = 1$. As some agents will start with more resources than others, this initial condition allows us to distinguish which agents have more or fewer initial reserves. As time progresses, agents will naturally exhaust their resources; $\eta(t)$ is the fraction of active firms for a given time t with

$$\eta(t) = \int_{(0,\infty)} m(t,x)\,\mathrm{d}x. \tag{35}$$

The first time $\eta = 0$ is the exhaustion time or the final time T.

The price set by the agents is given by $p(t,x)$ and the quantity $q(t,x)$ can be expressed as a linear demand function of p:

$$q(t,x) = a(\eta(t)) - p(t,x) + c(\eta(t))\bar{p}(t), \tag{36}$$

where $\bar{p}(t) = \dfrac{1}{\eta(t)} \displaystyle\int_{(0,\infty)} p^*(t,x)m(t,x)\,\mathrm{d}x$ is the average price, and p^* is the optimal price. Also, $a(\eta) = \dfrac{1}{1+\epsilon\eta}$ and $c(\eta) = 1 - a(\eta)$, where $\epsilon \geq 0$ is a parameter that measures the degree of interaction between players. A small value of ϵ indicates a low amount of competition between agents. If $\epsilon = 0$, then there is no competition, and each agent is a monopolist. Conversely, as epsilon goes to infinity, the system tends toward perfect competition. It is also the case that the amount of demand can fluctuate randomly, therefore some Brownian motion is introduced into the remaining resources:

$$\mathrm{d}X_t = -q(t,x)\,\mathrm{d}t + \sigma \mathbb{1}_{(X_t>0)}\,\mathrm{d}W_t, \tag{37}$$

where $\mathbb{1}_{(X_t>0)}$ is the characteristic function for the set of all resources greater than 0. We also have that $\sigma \geq 0$ and $x = X_t$.

Before introducing the value function, we also consider that agents are trying to maximize lifetime profit, discounted at a rate $r > 0$. The value function is then

$$u(t,x) = \sup_p \mathbb{E}\left[\int_t^T e^{-r(s-t)} p(s,x)q(s,x)\,\mathrm{d}s\right]. \tag{38}$$

The corresponding Hamiltonian is

$$\mathcal{H} = \sup_p \{pq - \partial_x u \cdot q\} = \sup_p \left\{\big(a(\eta(t)) - p(t,x) + c(\eta(t))\bar{p}(t)\big)\left(p(t,x) - \partial_x u(t,x)\right)\right\}. \tag{39}$$

Thus, the HJB equation is

$$\partial_t u + \frac{\sigma^2}{2}\partial_{xx}^2 u - ru + \sup_p \left\{\big(a(\eta(t)) - p(t,x) + c(\eta(t))\bar{p}(t)\big)\left(p(t,x) - \partial_x u(t,x)\right)\right\} = 0. \tag{40}$$

We can solve for the optimal price p^* by differentiating the expression being maximized in (39) with respect to p, setting the derivative equal to 0 (first order condition) and solving for p to get

$$p^*(t,x) = \frac{1}{2}\big(a(\eta(t)) + \partial_x u(t,x) + c(\eta(t))\bar{p}(t)\big). \tag{41}$$

We can now solve for q^* by substituting (41) into (36) to obtain

$$q^*(t,x) = \frac{1}{2}\left(a(\eta(t)) - \partial_x u(t,x) + c(\eta(t))\bar{p}(t)\right). \tag{42}$$

It follows that

$$\mathcal{H} = q^*\left(p^* - \partial_x u\right) = \left(q^*\right)^2. \tag{43}$$

Our HJB equation is now

$$\partial_t u + \frac{\sigma^2}{2}\partial_{xx}^2 u - ru + \frac{1}{4}\Big(a(\eta(t)) - \partial_x u(t,x) + c(\eta(t))\bar{p}(t)\Big)^2 = 0. \tag{44}$$

Note that we have left Dirichlet boundary conditions, since when an agent runs out of resources x, there is no longer a way to generate profit, so $u(t,0) = 0$. As T is the time at which all agents have exhausted their resources, we can see that $u(T,x) = 0$.

Notice that the differential equation for u is dependent on m through $\bar{p}$. From Chapter 1 we can see that m satisfies the equation

$$\partial_t m - \frac{\sigma^2}{2}\partial_{xx}^2 m + \partial_x\left[D_\rho\mathcal{H}\cdot m\right] = 0, \tag{45}$$

with $\rho = \partial_x u$. However,

$$D_\rho\mathcal{H} = \frac{\mathrm{d}}{\mathrm{d}\rho}\left[\left(q^*\right)^2\right] = 2q^*\frac{\mathrm{d}}{\mathrm{d}\rho}[q^*] = 2q^*\left(-\frac{1}{2}\right) = -q^*, \tag{46}$$

so that the equation for m is now

$$\partial_t m - \frac{\sigma^2}{2}\partial_{xx}^2 m + \partial_x\left[-\frac{1}{2}\left(a(\eta(t)) - \partial_x u(t,x) + c(\eta(t))\bar{p}(t)\right)m\right] = 0. \tag{47}$$

Recall that m has the initial condition $m(0,x) = M(x)$. We also have left Dirichlet boundary conditions for m, since for small values of x, the local dynamics are dominated by the Brownian motion, but Chan and Sircar assert that Brownian motion is not significant enough to save an agent from exhausting their resources (Chan and Sircar, 2015). Thus, as a player's resources approach 0, they soon run out of resources altogether and drop out of the competition, hence $m(t,0) = 0$. We have now fully defined the system of mean field equations:

$$\begin{cases} (i) \quad \partial_t u + \dfrac{\sigma^2}{2}\partial^2_{xx}u - ru + \dfrac{1}{4}\Big(a(\eta(t)) - \partial_x u(t,x) + c(\eta(t))\bar{p}(t)\Big)^2 = 0 \\ (ii) \quad \partial_t m - \dfrac{\sigma^2}{2}\partial^2_{xx}m + \partial_x\left[-\dfrac{1}{2}\left(a(\eta(t)) - \partial_x u(t,x) + c(\eta(t))\bar{p}(t)\right)m\right] = 0 \\ (iii) \quad m(t,0) = 0, m(0,x) = M(x), u(t,0) = 0, u(T,x) = 0 \end{cases} \tag{48}$$

While a solution to a mean field game requires solving this system of only two coupled differential equations (as opposed to solving the more complex system of N coupled differential equations each with different Hamiltonians that describes a general game of N players) an explicit solution does not exist. However, assuming a solution exists for any value of ϵ, we can approximate numerical solutions of (48) using nth order Taylor series expansions. Recall that $a(\eta)$ and $c(\eta)$ are functions of the parameter ϵ, so solutions to (48) will depend on ϵ. Thus, we will formally differentiate u and m with respect to ϵ to produce a Taylor expansion centered at $\epsilon = 0$:

$$\begin{aligned} u(t,x) &= u_0(t,x) + \epsilon u_1(t,x) + \frac{\epsilon^2}{2}u_2(t,x) + \cdots + \frac{\epsilon^n}{n!}u_n(t,x) + o(\epsilon^n) \\ m(t,x) &= m_0(t,x) + \epsilon m_1(t,x) + \frac{\epsilon^2}{2}m_2(t,x) + \cdots + \frac{\epsilon^n}{n!}m_n(t,x) + o(\epsilon^n) \end{aligned} \tag{49}$$

Here, $u_i(t,x) = \dfrac{\partial^i}{\partial \epsilon^i}u(t,x) \mid_{\epsilon=0}$ and $m_i(t,x) = \dfrac{\partial^i}{\partial \epsilon^i}m(t,x) \mid_{\epsilon=0}$. Notice that when $\epsilon = 0$, $c(\eta(t)) = 0$, so that the equations in (48) decouple. Thus, the functions u_n and m_n in the asymptotic expansion can be solved more or less explicitly, because the equations they solve are likewise decoupled.

With this approximation set up, it is of interest to know if these series converge, and if so, what are their radii of convergence? These questions concerning the asymptotics of solutions to (48) are central to the analysis given in Chapter 3 and while we do not prove analyticity, we obtain error estimates on the numerical solutions and provide a useful linear approximation to solutions of the system.

CHAPTER THREE

Proof of Error Estimates

The system of PDE that this chapter is concerned with comes from the ideas developed by Chan and Sircar (Chan and Sircar, 2015), discussed in Chapter 2 and is given by

$$\begin{cases} (i) & u_t + \dfrac{\sigma^2}{2} u_{xx} - ru + F(\epsilon)^2 = 0, & 0 \leq x \leq L,\ \ 0 \leq t \leq T \\ (ii) & m_t - \dfrac{\sigma^2}{2} m_{xx} - \big[F(\epsilon)m\big]_x = 0, & 0 \leq x \leq L,\ \ 0 \leq t \leq T \\ (iii) & m(x,0) = M(x),\ \ u(x,T) = u_T(x), & 0 \leq x \leq L \\ (iv) & u(0,t) = m(0,t) = 0,\ \ u_x(t,L) = 0, & 0 \leq t \leq T \\ (v) & \dfrac{\sigma^2}{2} m_x(t,L) + F(\epsilon)m(t,L) = 0, & 0 \leq t \leq T \end{cases} \tag{50}$$

where $M(x)$ and $u_T(x)$ are known, smooth functions and F is given by

$$F(\epsilon) = \frac{1}{2}\left(\frac{2}{2+\epsilon} + \frac{\epsilon}{2+\epsilon} \int_0^L u_x m \,\mathrm{d}x - u_x \right), \tag{51}$$

for some parameter $\epsilon \geq 0$. We will use the notation $u^\epsilon = u(x,t;\epsilon)$, $m^\epsilon = m(x,t;\epsilon)$ to indicate respective solutions to (i) and (ii) in (50) when the parameter ϵ is nonzero. Similarly, $u^0 = u(x,t;0)$ and $m^0 = m(x,t;0)$ will indicate respective solutions to (i) and (ii) in (50) when $\epsilon = 0$, the case in which the two equations are decoupled.
In order to state our results, let us introduce the following notation:

- For $0 < \alpha \leq 1$, $C^\alpha([0,L])$ is the space of all α-Hölder continuous functions on $[0,L]$; its norm given by

$$|\phi|_\alpha = \|\phi\|_\infty + [\phi]_\alpha,\ \ [\phi]_\alpha := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x-y|^\alpha}.$$

- For $0 < \alpha \le 1$, n a positive integer, $C^{n+\alpha}([0,L])$ is the space of all α-Hölder continuous functions ϕ on $[0,L]$ such that the jth derivative $\phi^{(j)}$ for $1 \le j \le n$ is also α-Hölder continuous; its norm is given by

$$|\phi|_{n+\alpha} = \sum_{i=0}^{n} \left\| \phi^{(i)} \right\|_\infty + [\phi^{(n)}]_\alpha.$$

- $C^{\alpha,\alpha/2}([0,L]\times[0,T])$ is the space of all functions $\phi = \phi(x,t)$ on $[0,L]\times[0,T]$ such that ϕ is C^α in the x-variable, $C^{\alpha/2}$ in the t-variable; its norm is given by

$$|\phi|_{\alpha,\alpha/2} = \|\phi\|_\infty + [\phi]_{x,\alpha} + [\phi]_{t,\alpha/2}$$

where

$$[\phi]_{x,\alpha} := \sup_{x \neq y, t} \frac{|\phi(x,t) - \phi(y,t)|}{|x-y|^\alpha}, \; [\phi]_{t,\alpha/2} := \sup_{x, t \neq s} \frac{|\phi(x,t) - \phi(x,s)|}{|t-s|^{\alpha/2}}.$$

- $C^{2+\alpha,1+\alpha/2}([0,L]\times[0,T])$ is the space of all functions $\phi = \phi(x,t)$ on $[0,L]\times[0,T]$ such that ϕ is $C^{2+\alpha}$ in the x-variable, $C^{1+\alpha/2}$ in the t-variable; its norm is given by

$$|\phi|_{2+\alpha,1+\alpha/2} = \|\phi\|_\infty + \|\phi_t\|_\infty + \|\phi_x\|_\infty + \|\phi_{xx}\|_\infty + [\phi_{xx}]_{x,\alpha} + [\phi_t]_{t,\alpha/2}.$$

The following theorem is a result from Graber and Bensoussan (2018).

Theorem 1. *Let $u_T, M \in C^{2+\gamma}([0,L])$, for some $0 < \gamma \le 1$. There exists a unique pair of solutions (u^ϵ, m^ϵ) to (50) and a constants C and $\alpha \in (0,1]$ such that $|u^\epsilon|_{2+\alpha,1+\alpha/2}, |m^\epsilon|_{2+\alpha,1+\alpha/2} \le C$, for any $\epsilon \in [0,\epsilon_0]$ and C depending only on ϵ_0 and the data.*

The following three lemmas will establish useful bounds for solutions to the types of PDE we will encounter in latter theorems of this chapter.

Lemma 2. *Let φ be a smooth function such that $\varphi(0) = 0$. Suppose u is the solution to the following PDE*

$$u_t + \frac{\sigma^2}{2} u_{xx} + a(x,t)u + b(x,t)\varphi(u_x) = c(x,t), \tag{52}$$

with boundary conditions satisfying $\big|u(0,t)\big|, \big|u(L,t)\big|, \big|u_T(x)\big| \leq C_1$. For $\min_{(x,t)}\{a(x,t)\} = a_0$ and $\big|c(x,t)\big| \leq C_2$, $|u| \leq e^{-a_0 T}\left(C_1 e^{-a_0 T} + C_2 T\right)$.

Proof. Let $a_0 = \min_{(x,t)}\{a(x,t)\}$. We can add and subtract $a_0 u$ on the left hand side of (52) to get

$$u_t + \frac{\sigma^2}{2} u_{xx} + a_0 u + (a - a_0)u + b(x,t)\varphi(u_x) = c(x,t) \tag{53}$$

Let $\tilde{u} = u e^{a_0(t-T)}$. Multiplying through (52) by $e^{a_0(t-T)}$, we obtain

$$\tilde{u}_t + \frac{\sigma^2}{2}\tilde{u}_{xx} + (a - a_0)\tilde{u} + e^{a_0(t-T)} b(x,t)\varphi(\tilde{u}_x e^{-a_0(t-T)}) = c(x,t)e^{a_0(t-T)} \tag{54}$$

Since $\big|c(x,t)\big| \leq C_2$,

$$\left|\tilde{u}_t + \frac{\sigma^2}{2}\tilde{u}_{xx} + (a - a_0)\tilde{u} + e^{a_0(t-T)} b(x,t)\varphi(\tilde{u}_x e^{-a_0(t-T)})\right| \leq C_2 e^{-a_0 T}. \tag{55}$$

Case 1: Let $\hat{u} = \tilde{u} - C_2 e^{-a_0 T} t$. It follows that $\hat{u}$ satisfies the PDE

$$\hat{u}_t + \frac{\sigma^2}{2}\hat{u}_{xx} + (a - a_0)\hat{u} + e^{a_0(t-T)} b(x,t)\varphi(\hat{u}_x e^{-a_0(t-T)}) \leq 0. \tag{56}$$

Using the maximum principle on $\hat{u}$ (see Evans (2010), Chapter 7.1, Theorem 9) and the estimates on the boundary conditions for u, we have that

$$\max \hat{u} \le \max \left\{ u(0,t)e^{a_0(t-T)} - C_2 e^{-a_0 T} t, \right. \\ \left. u(L,t)e^{a_0(t-T)} - C_2 e^{-a_0 T} t, u_T(x) - C_2 e^{-a_0 T} T \right\}, \quad (57)$$

so that $\hat{u} \le C_1 e^{-a_0 T}$, which implies that

$$u \le e^{-a_0 T} \left(C_1 e^{-a_0 T} + C_2 T \right).$$

Case 2: Let $\hat{u} = \tilde{u} + C_2 e^{-a_0 T} t$. It follows that $\hat{u}$ satisfies the PDE

$$\hat{u}_t + \frac{\sigma^2}{2} \hat{u}_{xx} + (a - a_0)\hat{u} + e^{a_0(t-T)} b(x,t) \varphi(\hat{u}_x e^{-a_0(t-T)}) \ge 0. \quad (58)$$

Using the minimum principle on $\hat{u}$ and the estimates on the boundary conditions for u, we have that

$$\min \tilde{u} \ge \min \left\{ u(0,t)e^{a_0(t-T)} + C_2 e^{-a_0 T} t, \right. \\ \left. u(L,t)e^{a_0(t-T)} t + C_2 e^{-a_0 T} t, u_T(x) + C_2 e^{-a_0 T} T \right\}, \quad (59)$$

so that $\hat{u} \ge -C_1 e^{-a_0 T}$, which implies that

$$u \ge -e^{-a_0 T} \left(C_1 e^{-a_0 T} + C_2 T \right).$$

We now have

$$|u| \le e^{-a_0 T} \left(C_1 e^{-a_0 T} + C_2 T \right)$$

as desired. $\square$

Lemma 3. *Suppose u is the solution to the following PDE*

$$u_t + \frac{\sigma^2}{2} u_{xx} + a(x,t)u + b(x,t)u_x = c(x,t) \tag{60}$$

with boundary conditions satisfying $u(0,t) = 0$, $u_x(L,t) = 0$, and $\left|u_T'(x)\right| \leq C_1$, $a, b, c \in L^\infty$, then

$$\left|u_x(0,t)\right| \leq kM,$$

where $k = \max\left\{\left\|u(x,t)\right\|_\infty, 1\right\}$,

$$M = \max\left\{Ce^L, k^{-1}C_1 \exp\left[k^{-1}\left\|u_T(x)\right\|_\infty + L\right]\right\},$$

and C is given by

$$C = e\left(\frac{1}{4}\left\|b(x,t)\right\|_\infty^2 + \left\|a(x,t)\right\|_\infty + \left\|c(x,t)\right\|_\infty\right).$$

Proof. Let $u = \varphi(w)$, then $u_t = \varphi'(w)w_t$, $u_x = \varphi'(w)w_x$, and $u_{xx} = \varphi''(w)w_x^2 + \varphi'(w)w_{xx}$ so that (60) becomes

$$\varphi'(w)w_t + \varphi''(w)w_x^2 + \varphi'(w)w_{xx} + a(x,t)\varphi(w) + b(x,t)\varphi'(w)u_x = c(x,t). \tag{61}$$

Dividing through by $\varphi'(w)$ and rearranging terms, we obtain

$$w_t + w_{xx} = -\frac{\varphi''(w)}{\varphi'(w)}w_x^2 - b(x,t)w_x - a(x,t)\frac{\varphi(w)}{\varphi'(w)} + \frac{c(x,t)}{\varphi'(w)}. \tag{62}$$

Let $\varphi(w) = k\log(1+w)$ for $k \geq 1$, so that $\varphi'(w) = \frac{k}{1+w}$ and $\varphi''(w) = \frac{-k}{(1+w)^2}$. Notice that w has the same boundary conditions as u. We now have

$$w_t + w_{xx} = \frac{w_x^2}{1+w} - b(x,t)w_x - \frac{1}{k}a(x,t)u(x,t)(1+w) - \frac{1}{k}c(x,t)(1+w). \tag{63}$$

By Young's inequality, $\left|b(x,t)w_x\right| \leq \frac{w_x^2}{1+w} + \frac{|b|^2}{4(1+w)}$, so that

$$\begin{aligned} w_t + w_{xx} &\geq -\left(\frac{1}{4}\left|b(x,t)\right|^2 + \frac{1}{k}\left|a(x,t)\right|\left|u(x,t)\right| + \frac{1}{k}\left|c(x,t)\right|\right)(1+w) \\ &\geq -\left(\frac{1}{4}\left\|b(x,t)\right\|_\infty^2 + \frac{1}{k}\left\|a(x,t)\right\|_\infty\left\|u(x,t)\right\|_\infty + \frac{1}{k}\left\|c(x,t)\right\|_\infty\right) e^{\frac{1}{k}\left\|u(x,t)\right\|_\infty}. \end{aligned} \tag{64}$$

Choosing $k = \max\left\{\left\|u(x,t)\right\|_\infty, 1\right\}$, we now have

$$w_t + w_{xx} \geq -e\left(\frac{1}{4}\left\|b(x,t)\right\|_\infty^2 + \left\|a(x,t)\right\|_\infty + \left\|c(x,t)\right\|_\infty\right). \tag{65}$$

Hence,

$$w_t + w_{xx} \geq -C. \tag{66}$$

Now, let $\tilde{w} = w + Me^{-x}$ so that our PDE for $\tilde{w}$ is now

$$\tilde{w}_t + \tilde{w}_{xx} \geq -C + Me^{-x}. \tag{67}$$

We need to choose M such that $Me^{-L} \geq C$ and $\tilde{w}_x(x,T) \leq 0$, so choose

$$M = \max\left\{Ce^L, k^{-1}C_1 \exp\left\{k^{-1}\left\|u_T(x)\right\|_\infty + L\right\}\right\}.$$

By the maximum principle,

$$\max \tilde{w} = \max\left\{\tilde{w}(0,t), \tilde{w}(x,T)\right\}, \tag{68}$$

but as $\tilde{w}_x(x,T) \leq 0$ and $\tilde{w}(0,t) = \tilde{w}(0,T) = M$, we have $\tilde{w}(x,T) \leq M$ and it must be the case that $\max \tilde{w} = \tilde{w}(0,t) = M$. And as $\tilde{w}(x,t) \leq \tilde{w}(0,t)$, we have that $\tilde{w}_x(0,t) \leq 0$. Finally, as $w = e^{k^{-1}u} - 1$, it follows that

$$u_x(0,t) \leq kM.$$

For the other direction, the proof that

$$u_x(0,t) \geq kM$$

is similar, so is omitted. $\square$

Lemma 4. *Let m be the solution to the following PDE*

$$m_t - \frac{\sigma^2}{2} m_{xx} - (bm)_x - w_x = 0, \tag{69}$$

with $m(0,t) = w(0,t) = 0$, $\frac{\sigma^2}{2} m_x(L,t) + bm(L,t) + w(L,t) = 0$, $|b| \leq C_1$, *and* $|w| \leq C_2$, *then* $|m| \leq C$ *and* $C = \frac{64L}{\sigma^6}\left(\|b^\epsilon\|_\infty^2 + TL + 2\right)^2$.

Proof. Begin by multiplying each term in (69) by some $\Phi'(m)$ with $\Phi''(m) \geq 0$ and $\Phi(0) = 0$, then integrate in space:

$$\begin{aligned}
&\int_0^L \Phi'(m) m_t \,\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^L \Phi(m)\,\mathrm{d}y \\
&\frac{\sigma^2}{2}\int_0^L \Phi'(m) m_{xx}\,\mathrm{d}y = \frac{\sigma^2}{2}\Phi'(m) m_x \Big|_0^L - \frac{\sigma^2}{2}\int_0^L \Phi''(m) m_x^2 \,\mathrm{d}y \\
&\int_0^L \Phi'(m)\left(b^\epsilon m\right)_x \,\mathrm{d}y = \Phi'(m)\left(b^\epsilon m\right)\Big|_0^L - \int_0^L \Phi''(m) m_x \left(b^\epsilon m\right)\,\mathrm{d}y \\
&\int_0^L \Phi'(m) w_x \,\mathrm{d}y = \Phi'(m) w \Big|_0^L - \int_0^L \Phi''(m) m_x w \,\mathrm{d}y.
\end{aligned} \tag{70}$$

Note that upon adding the boundary terms, we are left with 0 becasue of the boundary conditions on the PDE for m. Our equation is now

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \Phi(m)\,\mathrm{d}y + \frac{\sigma^2}{2}\int_0^L \Phi''(m) m_x^2 \,\mathrm{d}y = -\int_0^L \Phi''(m) m_x \left(b^\epsilon m\right)\,\mathrm{d}y - \int_0^L \Phi''(m) m_x w \,\mathrm{d}y. \tag{71}$$

By Young's inequality,

$$m_x\left(b^\epsilon m\right) \leq \frac{\sigma^2}{8} m_x^2 + \frac{2}{\sigma^2}\left(b^\epsilon m\right)^2 \text{ and } m_x w \leq \frac{\sigma^2}{8} m_x^2 + \frac{2}{\sigma^2} w^2,$$

and our expression is now

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \Phi(m)\,\mathrm{d}y + \frac{\sigma^2}{2}\int_0^L \Phi''(m)m_x^2\,\mathrm{d}y \\ \leq \int_0^L \Phi''(m)\left(\frac{\sigma^2}{8}m_x^2 + \frac{2}{\sigma^2}\left(b^\epsilon m\right)^2\right)\,\mathrm{d}y + \int_0^L \Phi''(m)\left(\frac{\sigma^2}{8}m_x^2 + \frac{2}{\sigma^2}w^2\right)\,\mathrm{d}y. \quad (72)$$

After combining like terms,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \Phi(m)\,\mathrm{d}y + \frac{\sigma^2}{4}\int_0^L \Phi''(m)m_x^2\,\mathrm{d}y \leq \frac{2}{\sigma^2}\int_0^L \Phi''(m)\left(b^\epsilon m\right)^2\,\mathrm{d}y + \frac{2}{\sigma^2}\int_0^L \Phi''(m)w^2\,\mathrm{d}y. \quad (73)$$

Integrating in time we get

$$\int_0^L \Phi(m(y,t))\,\mathrm{d}y - \int_0^L \Phi(m(y,0))\,\mathrm{d}y + \frac{\sigma^2}{4}\int_0^t\int_0^L \Phi''(m)m_x^2\,\mathrm{d}y\,\mathrm{d}s \\ \leq \frac{2}{\sigma^2}\int_0^t\int_0^L \Phi''(m)\left(b^\epsilon m\right)^2\,\mathrm{d}y\,\mathrm{d}s + \frac{2}{\sigma^2}\int_0^t\int_0^L \Phi''(m)w^2\,\mathrm{d}y\,\mathrm{d}s, \quad (74)$$

and rearranging terms we have

$$\int_0^L \Phi(m(y,t))\,\mathrm{d}y + \frac{\sigma^2}{4}\int_0^t\int_0^L \Phi''(m)m_x^2\,\mathrm{d}y\,\mathrm{d}s \\ \leq \int_0^L \Phi(m(y,0))\,\mathrm{d}y + \frac{2}{\sigma^2}\|b^\epsilon\|_\infty^2\int_0^t\int_0^L \Phi''(m)m^2\,\mathrm{d}y\,\mathrm{d}s + \frac{2}{\sigma^2}\int_0^t\int_0^L \Phi''(m)w^2\,\mathrm{d}y\,\mathrm{d}s. \quad (75)$$

Let $\Phi(m) = m^p$ for $p \geq 2$. It follows that $\Phi''(m) = p(p-1)m^{p-2}$, and our equation is now

$$\int_0^L m^p\,\mathrm{d}y + \frac{\sigma^2 p(p-1)}{4}\int_0^t\int_0^L m^{p-2}m_x^2\,\mathrm{d}y\,\mathrm{d}s \leq \int_0^L m^p(y,0)\,\mathrm{d}y \\ + \frac{2p(p-1)}{\sigma^2}\|b^\epsilon\|_\infty^2\int_0^t\int_0^L m^p\,\mathrm{d}y\,\mathrm{d}s + \frac{2p(p-1)}{\sigma^2}\int_0^t\int_0^L m^{p-2}w^2\,\mathrm{d}y\,\mathrm{d}s. \quad (76)$$

By Young's inequality,

$$m^{p-2}w \leq \frac{p-2}{p}m^p + \frac{2}{p}w^p \leq m^p + w^p,$$

so that

$$\begin{aligned}\int_0^L m^p\,\mathrm{d}y + \frac{\sigma^2 p(p-1)}{4}\int_0^t\int_0^L m^{p-2}m_x^2\,\mathrm{d}y\,\mathrm{d}s \leq \int_0^L m^p(y,0)\,\mathrm{d}y \\ + \frac{2p(p-1)}{\sigma^2}\|b^\epsilon\|_\infty^2\int_0^t\int_0^L m^p\,\mathrm{d}y\,\mathrm{d}s + \frac{2p(p-1)}{\sigma^2}\int_0^t\int_0^L (m^p + w^p)\,\mathrm{d}y\,\mathrm{d}s,\end{aligned} \tag{77}$$

and after combining like terms,

$$\begin{aligned}\int_0^L m^p\,\mathrm{d}y + \frac{\sigma^2 p(p-1)}{4}\int_0^t\int_0^L m^{p-2}m_x^2\,\mathrm{d}y\,\mathrm{d}s \leq \int_0^L m^p(y,0)\,\mathrm{d}y \\ + \frac{2p(p-1)}{\sigma^2}\left(\|b^\epsilon\|_\infty^2 + 1\right)\int_0^t\int_0^L m^p\,\mathrm{d}y\,\mathrm{d}s + \frac{2p(p-1)}{\sigma^2}\int_0^t\int_0^L w^p\,\mathrm{d}y\,\mathrm{d}s.\end{aligned} \tag{78}$$

For $q > p$,

$$\begin{aligned}\int_0^T\int_0^L m^q\,\mathrm{d}x\,\mathrm{d}t &\leq \int_0^T\int_0^L \sup_{y\in[0,L]} m^p(y,t)m^{q-p}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \int_0^T \sup_{y\in[0,L]} m^p(y,t)\int_0^L m^{q-p}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \int_0^T \sup_{y\in[0,L]} m^p(y,t)\,\mathrm{d}t\,\sup_{s\in[0,T]}\int_0^L m^{q-p}(x,s)\,\mathrm{d}x\end{aligned} \tag{79}$$

Choose $q = 2p$, then (79) yields

$$\int_0^T\int_0^L m^{2p}\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T \sup_{y\in[0,L]} m^p(y,t)\,\mathrm{d}t\,\sup_{s\in[0,T]}\int_0^L m^p(x,s)\,\mathrm{d}x. \tag{80}$$

Note that

$$m^{p-2}m_x^2 = \left(m^{\frac{p}{2}-1}m_x\right)^2 = \left(\frac{2}{p}\left(\frac{\partial}{\partial x}m^{\frac{p}{2}}\right)\right)^2 = \frac{4}{p^2}\left(m^{\frac{p}{2}}\right)_x^2, \tag{81}$$

so that

$$
\begin{aligned}
m^{\frac{p}{2}}(x,t) &= m^{\frac{p}{2}}(0,t) + \int_0^x \frac{\partial}{\partial y}\left(m^{\frac{p}{2}}(y,t)\right)\,\mathrm{d}y \\
&\leq \int_0^L \left|\left(m^{\frac{p}{2}}(y,t)\right)_x\right|\,\mathrm{d}y \\
&\leq \left(\int_0^L \left|\left(m^{\frac{p}{2}}(y,t)\right)_x\right|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_0^L \mathrm{d}y\right)^{\frac{1}{2}}.
\end{aligned} \tag{82}
$$

Squaring both sides, we have

$$
\begin{aligned}
m^p(x,t) &\leq L\int_0^L \left(m^{\frac{p}{2}}(y,t)\right)_x^2\,\mathrm{d}y \\
\sup_{y\in[0,L]} m^p(y,t) &\leq L\int_0^L \left(m^{\frac{p}{2}}(y,t)\right)_x^2\,\mathrm{d}y.
\end{aligned} \tag{83}
$$

Integrating in time we see that

$$
\begin{aligned}
\int_0^T \sup_{y\in[0,L]} m^p(y,t)\,\mathrm{d}t &\leq L\int_0^T\int_0^L \left(m^{\frac{p}{2}}(y,t)\right)_x^2\,\mathrm{d}y\,\mathrm{d}t \\
&= L\frac{p^2}{4}\int_0^T\int_0^L m^{p-2}(y,t)m_x^2(y,t)\,\mathrm{d}y\,\mathrm{d}t \\
\frac{\sigma^2(p-1)}{Lp}\int_0^T \sup_{y\in[0,L]} m^p(y,t)\,\mathrm{d}t &\leq \frac{\sigma^2 p(p-1)}{4}\int_0^T\int_0^L m^{p-2}(y,t)m_x^2(y,t)\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned} \tag{84}
$$

It follows that

$$
\begin{aligned}
\frac{\sigma^2(p-1)}{Lp}\int_0^T \sup_{y\in[0,L]} m^p(y,t)\,\mathrm{d}t &\leq \int_0^L m^p(y,0)\,\mathrm{d}y \\
&+ \frac{2p(p-1)}{\sigma^2}\left(\|b^\epsilon\|_\infty^2+1\right)\int_0^t\int_0^L m^p\,\mathrm{d}y\,\mathrm{d}s + \frac{2p(p-1)}{\sigma^2}\int_0^t\int_0^L w^p\,\mathrm{d}y\,\mathrm{d}s,
\end{aligned} \tag{85}
$$

and

$$
\begin{aligned}
\sup_{s\in[0,T]}\int_0^L m^p(x,s)\,\mathrm{d}x &\leq \int_0^L m^p(y,0)\,\mathrm{d}y \\
&+ \frac{2p(p-1)}{\sigma^2}\left(\|b^\epsilon\|_\infty^2+1\right)\int_0^t\int_0^L m^p\,\mathrm{d}y\,\mathrm{d}s + \frac{2p(p-1)}{\sigma^2}\int_0^t\int_0^L w^p\,\mathrm{d}y\,\mathrm{d}s,
\end{aligned} \tag{86}
$$

so that

$$\int_0^T\int_0^L m^{2p}\,\mathrm{d}x\,\mathrm{d}t \le \frac{Lp}{\sigma^2(p-1)}\Bigg(\int_0^L m^p(y,0)\,\mathrm{d}y$$
$$+\frac{2p(p-1)}{\sigma^2}\left(\|b^\epsilon\|_\infty^2+1\right)\int_0^t\int_0^L m^p\,\mathrm{d}y\,\mathrm{d}s+\frac{2p(p-1)}{\sigma^2}\int_0^t\int_0^L w^p\,\mathrm{d}y\,\mathrm{d}s\Bigg)^2. \quad (87)$$

As $\frac{p}{2(p-1)} \le 1$,

$$\int_0^T\int_0^L m^{2p}\,\mathrm{d}x\,\mathrm{d}t \le \frac{L}{\sigma^2}\Bigg(\int_0^L m^p(y,0)\,\mathrm{d}y$$
$$+\frac{2p(p-1)}{\sigma^2}\left(\|b^\epsilon\|_\infty^2+1\right)\int_0^T\int_0^L m^p\,\mathrm{d}y\,\mathrm{d}s+\frac{2p(p-1)}{\sigma^2}\int_0^T\int_0^L w^p\,\mathrm{d}y\,\mathrm{d}s\Bigg)^2. \quad (88)$$

Let $M_p = \max\left\{\left\|m(0,t)\right\|_p, \left\|m(x,t)\right\|_p, \|w\|_\infty, 1\right\}$. Then

$$\begin{aligned}
M_{2p}^{2p} &\le \frac{L}{\sigma^2}\left(M_p^p+\frac{2p(p-1)}{\sigma^2}\left(\|b^\epsilon\|_\infty^2+1\right)M_p^p+\frac{2p(p-1)}{\sigma^2}TLM_p^p\right)^2\\
&\le \frac{L}{\sigma^2}\left(\frac{2p^2}{\sigma^2}M_p^p+\frac{2p^2}{\sigma^2}\left(\|b^\epsilon\|_\infty^2+1\right)M_p^p+\frac{2p^2}{\sigma^2}TLM_p^p\right)^2\\
&\le \frac{4Lp^4}{\sigma^6}\left(\|b^\epsilon\|_\infty^2+TL+2\right)^2 M_p^{2p}\\
M_{2p} &\le \left(L^{\frac{1}{2}}p^2C'\right)^{\frac{1}{p}} M_p
\end{aligned} \quad (89)$$

Where $C' = \frac{2}{\sigma^3}\left(\|b^\epsilon\|_\infty^2+TL+2\right)$.

Now, let $a_k = M_{2^k}$. It follows that

$$\begin{aligned}
a_{k+1} &\le \left(L^{\frac{1}{2}}2^{2k}C'\right)^{\frac{1}{2^k}} M_{2^k}\\
&\le \left(L^{\frac{1}{2}}2^{2k}C'\right)^{\frac{1}{2^k}}\left(L^{\frac{1}{2}}2^{2(k-1)}C'\right)^{\frac{1}{2^{k-1}}} M_{2^{k-1}}\\
&\vdots\\
&\le \prod_{n=0}^{k}\left(L^{\frac{1}{2}}2^{2n}C'\right)^{\frac{1}{2^n}}.
\end{aligned} \quad (90)$$

We can see that

$$\begin{aligned}\log\left(\prod_{n=0}^{\infty}\left(L^{\frac{1}{2}}2^{2n}C'\right)^{\frac{1}{2^n}}\right) &= \sum_{n=0}^{\infty}\frac{1}{2^n}\left(\frac{1}{2}\log L+\log\left(2^{2n}\right)+\log C'\right)\\ &= \sum_{n=0}^{\infty}\frac{\log\left(2^{2n}\right)}{2^n}+\sum_{n=0}^{\infty}\frac{\frac{1}{2}\log L+\log C'}{2^n}\\ &= \sum_{n=0}^{\infty}\frac{\log\left(2^{2n}\right)}{2^n}+\log L+2\log C'.\end{aligned} \tag{91}$$

Note that

$$\lim_{n\to\infty}\left|\frac{\log\left(2^{2(n+1)}\right)}{2^{n+1}}\cdot\frac{2^n}{\log\left(2^{2n}\right)}\right|=\frac{1}{2}<1, \tag{92}$$

so that by the ratio test, $\sum_{n=0}^{\infty}\frac{\log\left(2^{2n}\right)}{2^n}$ converges absolutely. Furthermore, for $|x|<1$, $\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$, so that

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty}x^n\right]=\sum_{n=0}^{\infty}nx^n=x\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{1-x}\right]=\frac{x}{(1-x)^2}. \tag{93}$$

With $x=\frac{1}{2}$, it follows that $\sum_{n=0}^{\infty}\frac{n}{2^n}=2$, showing that

$$\sum_{n=0}^{\infty}\frac{\log\left(2^{2n}\right)}{2^n}=\log\left(2^2\right)\sum_{n=0}^{\infty}\frac{n}{2^n}=\log 16. \tag{94}$$

We now have

$$\log\left(\prod_{n=0}^{\infty}\left(L^{\frac{1}{2}}2^{2n}C'\right)^{\frac{1}{2^n}}\right)\le\log 16+\log L+2\log C', \tag{95}$$

and

$$\prod_{n=0}^{\infty}\left(L^{\frac{1}{2}}2^{2n}C'\right)^{\frac{1}{2^n}}\le 16LC'^2, \tag{96}$$

which, in particular, means

$$\|m\|_\infty \le M_\infty \le 16LC'^2. \tag{97}$$

This shows that $|m| \le C$, with $C = 16LC'^2$. $\square$

The following result comes from Ladyžhenskaja et al. (1968), Theorem IV.9.1.

Theorem 5. *Let $\phi_T = \phi_T(x) \in C^2([0,L])$ and $f = f(x,t) \in L^\infty([0,L]\times[0,T])$. Then there exists a unique (weak) solution ϕ to the following boundary value problem:*

$$\begin{aligned} \phi_t + \frac{\sigma^2}{2}\phi_{xx} &= f(x,t), \\ \phi(0,t) = 0,\ \phi_x(L,t) = 0,&\ \phi(x,T) = \phi_T(x). \end{aligned} \tag{98}$$

Moreover, for any $0 < \alpha < 1$ there exists a constant κ_α depending on σ, T, L such that

$$|\phi|_{\alpha,\alpha/2}, |\phi_x|_{\alpha,\alpha/2} \le \kappa \left(|\phi_T|_2 + \|f\|_\infty\right). \tag{99}$$

Corollary 6. *There exists a constant such that $\big|u(x,t;\epsilon) - u(x,t;0)\big| \le C\epsilon$.*

Proof. Define $v = \frac{1}{\epsilon}\left(u^\epsilon - u^0\right)$. We now plug u^ϵ and u^0 respectively into (i) of (50) and subtract the two equations to get

$$v_t + \frac{\sigma^2}{2}v_{xx} - rv + \frac{1}{\epsilon}\left(F(\epsilon)^2 - F(0)^2\right) = 0. \tag{100}$$

Note that

$$F(\epsilon)^2 - F(0)^2 = (F(\epsilon) - F(0))^2 + 2F(0)(F(\epsilon) - F(0)), \tag{101}$$

so that we have

$$v_t + \frac{\sigma^2}{2} v_{xx} - rv + \frac{\epsilon}{4}\left(\frac{-1}{2+\epsilon} + \frac{1}{2+\epsilon}\int_0^L u_x^\epsilon m^\epsilon \,\mathrm{d}x - v_x\right)^2 + \frac{1}{2}\left(1-u_x^0\right)\left(\frac{-1}{2+\epsilon} + \frac{1}{2+\epsilon}\int_0^L u_x^\epsilon m^\epsilon \,\mathrm{d}x - v_x\right) = 0. \quad (102)$$

With

$$b(x,t) = -\frac{\epsilon}{2}\left(\frac{-1}{2+\epsilon} + \frac{1}{2+\epsilon}\int_0^L u_x^\epsilon m^\epsilon \,\mathrm{d}x\right) - \frac{1}{2}\left(1-u_x^0\right)$$

and

$$c(x,t) = -\frac{\epsilon}{4}\left(\frac{-1}{2+\epsilon} + \frac{1}{2+\epsilon}\int_0^L u_x^\epsilon m^\epsilon \,\mathrm{d}x\right)^2 - \frac{1}{2}\left(1-u_x^0\right)\left(\frac{-1}{2+\epsilon} + \frac{1}{2+\epsilon}\int_0^L u_x^\epsilon m^\epsilon \,\mathrm{d}x\right),$$

we have

$$v_t + \frac{\sigma^2}{2} v_{xx} - rv + b(x,t) v_x + \epsilon v_x^2 = c(x,t). \quad (103)$$

As b and c are bounded, it follows from Theorem 1 and Lemma 2 that $|v| \le C$. □

Corollary 7. *There exists a constant such that* $\left|u_x(x,t;\epsilon) - u_x(x,t;0)\right| \le C\epsilon$.

Proof. Let $v = \frac{1}{\epsilon}\left(u^\epsilon - u^0\right)$ and let $w = v_x$. Differentiate through (100) to obtain

$$\begin{aligned} w_t + \frac{\sigma^2}{2} w_{xx} - rw - \frac{2}{\epsilon}\left(F(\epsilon) u_{xx}^\epsilon - F(0) u_{xx}^0\right) = 0 \\ w_t + \frac{\sigma^2}{2} w_{xx} - rw - 2\left(\frac{1}{\epsilon} u_{xx}^\epsilon \left(F(\epsilon) - F(0)\right) - F(0) w_x\right) = 0. \end{aligned} \quad (104)$$

After expanding $F(\epsilon)$ and $F(0)$ and combining like terms, we obtain

$$
\begin{aligned}
&w_t + \frac{\sigma^2}{2} w_{xx} + (u^\epsilon_{xx} - r)w + \left(1 - u^0_x\right) w_x - u^\epsilon_{xx}\left(\frac{-1}{2+\epsilon} + \frac{1}{2+\epsilon}\int_0^L u^\epsilon_x m^\epsilon \,\mathrm{d}x\right) = 0 \\
&w_t + \frac{\sigma^2}{2} w_{xx} - a(x,t)w + b(x,t)w_x = c(x,t).
\end{aligned}
\tag{105}
$$

It follows from Theorem 1, Lemma 2, Lemma 3 and Theorem 5 that $|w| \le C$. □

Corollary 8. *For $v = \frac{1}{\epsilon}\left(u^\epsilon - u^0\right)$, there exists a constant such that $|v|_{\alpha,\alpha/2} \le C$.*

Proof. Let $v = \frac{1}{\epsilon}\left(u^\epsilon - u^0\right)$, then by Corollary 6 and rearranging (103),

$$
\begin{aligned}
v_t + \frac{\sigma^2}{2} &= rv - b(x,t)v_x - \epsilon v_x^2 + c(x,t) \\
&= f(x,t).
\end{aligned}
\tag{106}
$$

Notice that v satisfies the following boundary conditions:

$$
\begin{aligned}
&v(0,t) = \tfrac{1}{\epsilon}\left(u(0,t;\epsilon) - u(0,t;0)\right) = 0, \text{ as } u(0,t) = 0 \\
&v_x(L,t) = \tfrac{1}{\epsilon}\left(u_x(L,t;\epsilon) - u_x(L,t;0)\right) = 0, \text{ as } u_x(L,t) = 0 \\
&v(x,T) = v_T(x) = \tfrac{1}{\epsilon}\left(u_T(x;\epsilon) - u_T(x;0)\right) \in C^2([0,L]), \text{ as } u_T(x) \in C^{2+\alpha}.
\end{aligned}
\tag{107}
$$

We also have that b and c are bounded. As $|v| \le C$ by Corollary 6, and $|v_x| \le C'$ by Corollary 7, it follows that $f(x,t) \in L^\infty([0,L] \times [0,T])$. Thus, by Theorem 5, the result follows. □

Corollary 9. *There exists a constant such that $\left|m(x,t;\epsilon) - m(x,t;0)\right| \le C\epsilon$.*

Proof. Define $\rho = \frac{1}{\epsilon}\left(m^\epsilon - m^0\right)$. Substituting m^ϵ and m^0 into (69), subtracting the two PDE and dividing by ϵ, we get

$$
\rho_t - \rho_{xx} - \frac{1}{\epsilon}\left(F(\epsilon)m^\epsilon - F(0)m^0\right)_x = 0. \tag{108}
$$

Note that

$$F(\epsilon)m^\epsilon - F(0)m^0 = m^\epsilon \big(F(\epsilon) - F(0)\big) + F(0)\left(m^\epsilon - m^0\right), \tag{109}$$

so that we have

$$\rho_t - \rho_{xx} - \big(F(0)\rho\big)_x - \frac{1}{\epsilon}\left[m^\epsilon \big(F(\epsilon) - F(0)\big)\right]_x = 0. \tag{110}$$

With $w = \frac{1}{\epsilon}\big(F(\epsilon) - F(0)\big)\, m^\epsilon$ and $b = F(0)$, it is clear from Theorem 1 that $\|w\|_\infty < \infty$ and $\|b\|_\infty < \infty$, so that by Lemma 4, $|\rho| \leq C$. □

Error Terms: Definitions and Useful Formulas

To make future theorems and expressions more compact, we introduce the following notation. Let $E_n^\epsilon(f) = f(\epsilon) - \sum_{i=0}^{n} \frac{\epsilon^i f^{(i)}(0)}{i!}$, the error of the nth order Maclaurin expansion in ϵ. Also, let $\hat{E}_n^\epsilon(f) = \frac{1}{\epsilon^{n+1}} E_n^\epsilon(f)$, the normalized error of f. The following two lemmas provide useful expressions for the error of products of functions.

Lemma 10.

$$E_n^\epsilon(fg) = gE_n^\epsilon(f) + \sum_{i=0}^{n} \frac{\epsilon^i f^{(i)}(0)}{i!} E_{n-i}^\epsilon(g) \tag{111}$$

Proof. Note that

$$\frac{\mathrm{d}^i}{\mathrm{d}\epsilon^i}(fg)\,|_{\epsilon=0} = \sum_{j=0}^{i} \binom{i}{j} f^{(i-j)}(0) g^{(j)}(0), \tag{112}$$

so that by the definition of E_n^ϵ,

$$\begin{aligned} E_n^\epsilon(fg) &= fg - \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{i}{j}\frac{\epsilon^i f^{(i-j)}(0)g^{(j)}(0)}{i!} \\ &= fg - \sum_{i=0}^{n}\sum_{j=0}^{i}\frac{\epsilon^i f^{(i-j)}(0)g^{(j)}(0)}{(i-j)j!} \\ &= fg - \sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{\epsilon^{i+j} f^{(i)}(0)g^{(j)}(0)}{i!j!} \\ &= fg - \sum_{i=0}^{n}\frac{\epsilon^i f^{(i)}(0)}{i!}\sum_{j=0}^{n-i}\frac{\epsilon^j g^{(j)}(0)}{j!}. \end{aligned} \tag{113}$$

We then add and subtract $g\sum_{i=0}^{n}\frac{\epsilon^i f^{(i)}(0)}{i!}$ and combine like terms to get the result:

$$\begin{aligned} &= g\left(f - \sum_{i=0}^{n}\frac{\epsilon^i f^{(i)}(0)}{i!}\right) + \sum_{i=0}^{n}\frac{\epsilon^i f^{(i)}(0)}{i!}\left(g - \sum_{j=0}^{n-i}\frac{\epsilon^j g^{(j)}(0)}{j!}\right) \\ &= gE_n^\epsilon(f) + \sum_{i=0}^{n}\frac{\epsilon^i f^{(i)}(0)}{i!}E_{n-i}^\epsilon(g). \end{aligned} \tag{114}$$

□

Lemma 11. *For all $n \in \mathbb{N}$ and $k = \left\lfloor \frac{n}{2} \right\rfloor$,*

$$E_n^\epsilon\left(f^2\right) = \left[E_k^\epsilon(f)\right]^2 + 2\sum_{i=0}^{k}\frac{\epsilon^i f^{(i)}(0)}{i!}E_{n-i}^\epsilon(f). \tag{115}$$

Proof. Let $k = \left\lfloor \frac{n}{2} \right\rfloor$. We have that

$$\begin{aligned} f^2 &= \left(E_k^\epsilon(f) + \sum_{i=0}^{k}\frac{\epsilon^i f^{(i)}(0)}{i!}\right)^2 \\ &= \left[E_k^\epsilon(f)\right]^2 + 2E_k^\epsilon(f)\sum_{j=0}^{k}\frac{\epsilon^j f^{(j)}(0)}{j!} + \left(\sum_{i=0}^{k}\frac{\epsilon^i f^{(i)}(0)}{i!}\right)^2. \end{aligned} \tag{116}$$

Note that

$$\left(\sum_{i=0}^{k}\frac{\epsilon^i f^{(i)}(0)}{i!}\right)^2=\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{i}{j}\frac{\epsilon^i f^{(i-j)}(0)f^{(j)}(0)}{i!}-2\sum_{i=k+1}^{n}\sum_{j=0}^{n-i}\frac{\epsilon^{i+j}f^{(i)}(0)f^{(j)}(0)}{i!j!}. \tag{117}$$

It follows that

$$\begin{aligned}E_n^\epsilon(f^2)&=f^2-\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{i}{j}\frac{\epsilon^i f^{(i-j)}(0)f^{(j)}(0)}{i!}\\&=\left[E_k^\epsilon(f)\right]^2+2\sum_{j=0}^{k}\frac{\epsilon^j f^{(j)}(0)}{j!}E_k^\epsilon(f)-2\sum_{i=k+1}^{n}\sum_{j=0}^{n-i}\frac{\epsilon^{i+j}f^{(i)}(0)f^{(j)}(0)}{i!j!}\\&=\left[E_k^\epsilon(f)\right]^2+2\sum_{j=0}^{k}\frac{\epsilon^j f^{(j)}(0)}{j!}E_k^\epsilon(f)-2\sum_{j=0}^{n-k-1}\sum_{i=k+1}^{n-j}\frac{\epsilon^{i+j}f^{(i)}(0)f^{(j)}(0)}{i!j!}.\end{aligned} \tag{118}$$

If n is even, $n=2k$, so that $n-k-1=k-1$, hence,

$$\begin{aligned}E_n^\epsilon(f^2)&=\left[E_k^\epsilon(f)\right]^2+2\sum_{j=0}^{k}\frac{\epsilon^j f^{(j)}(0)}{j!}E_k^\epsilon(f)-2\sum_{j=0}^{k-1}\frac{\epsilon^j f^{(j)}(0)}{j!}\sum_{i=k+1}^{n-j}\frac{\epsilon^i f^{(i)}(0)}{i!}\\&=\left[E_k^\epsilon(f)\right]^2+2\frac{\epsilon^k f^{(k)}(0)}{k!}E_k^\epsilon(f)+2\sum_{j=0}^{k-1}\frac{\epsilon^j f^{(j)}(0)}{j!}\left(E_k^\epsilon(f)-\sum_{k+1}^{n-j}\frac{\epsilon^i f^{(i)}(0)}{i!}\right)\\&=\left[E_k^\epsilon(f)\right]^2+2\frac{\epsilon^k f^{(k)}(0)}{k!}E_k^\epsilon(f)+2\sum_{j=0}^{k-1}\frac{\epsilon^j f^{(j)}(0)}{j!}E_{n-j}^\epsilon(f)\\&=\left[E_k^\epsilon(f)\right]^2+2\sum_{j=0}^{k}\frac{\epsilon^j f^{(j)}(0)}{j!}E_{n-j}^\epsilon(f).\end{aligned} \tag{119}$$

If n is odd, $n = 2k+1$, so that $n-k-1 = k$, hence,

$$\begin{aligned} E_n^\epsilon(f^2) &= \left[E_k^\epsilon(f)\right]^2 + 2\sum_{j=0}^{k} \frac{\epsilon^j f^{(j)}(0)}{j!} E_k^\epsilon(f) - 2\sum_{j=0}^{k} \frac{\epsilon^j f^{(j)}(0)}{j!} \sum_{i=k+1}^{n-j} \frac{\epsilon^i f^{(i)}(0)}{i!} \\ &= \left[E_k^\epsilon(f)\right]^2 + 2\sum_{j=0}^{k} \frac{\epsilon^j f^{(j)}(0)}{j!} \left(E_k^\epsilon(f) - \sum_{i=k+1}^{n-j} \frac{\epsilon^i f^{(i)}(0)}{i!} \right) \\ &= \left[E_k^\epsilon(f)\right]^2 + 2\sum_{j=0}^{k} \frac{\epsilon^j f^{(j)}(0)}{j!} E_{n-j}^\epsilon(f). \end{aligned} \tag{120}$$

Thus, for any $n \in \mathbb{N}$ with $k = \left\lfloor \frac{n}{2} \right\rfloor$, we see that

$$E_n^\epsilon(f^2) = \left[E_k^\epsilon(f)\right]^2 + 2\sum_{j=0}^{k} \frac{\epsilon^j f^{(j)}(0)}{j!} E_{n-j}^\epsilon(f). \tag{121}$$

$\square$

Main Result

Recall our derivation for u_n and m_n at the end of Chapter two. Our main result suggests that this formal identification can be made rigorous. In what follows we will identify the symbol $E_n^\epsilon(u)$ with $u^\epsilon - \sum_{i=0}^n \frac{\epsilon^i}{i!} u_i$ and likewise $E_n^\epsilon(m) = m^\epsilon - \sum_{i=0}^n \frac{\epsilon^i}{i!} m_i$. Formally, $E_n^\epsilon(u)$ and $E_n^\epsilon(m)$ have the same meaning as in the previous section; however, it is only by proving that u and m are n times differentiable in ϵ that we can rigorously make this identification. We also define $E_n^\epsilon(u_x)$ to mean $\partial_x E_n^\epsilon(u) = \partial_x u^\epsilon - \sum_{i=0}^n \frac{\epsilon^i}{i!} \partial_x u_i$. Once again we define $\hat{E}_n^\epsilon(u) = \epsilon^{-n-1} E_n^\epsilon(u)$, $\hat{E}_n^\epsilon(u_x) = \frac{1}{\epsilon^{n+1}} E_n^\epsilon(u_x)$, and $\hat{E}_n^\epsilon(m) = \frac{1}{\epsilon^{n+1}} E_n^\epsilon(m)$. Finally, we define

$$C_{u,n} = \sup_{\epsilon \in [0,1]} \left| \hat{E}_n^\epsilon(u) \right|_{2+\alpha, 1+\alpha/2}, \; C_{m,n} = \sup_{\epsilon \in [0,1]} \left| \hat{E}_n^\epsilon(m) \right|_{2+\alpha, 1+\alpha/2}.$$

We may now state our main result.

Theorem 12. *The constants $C_{u,n}$ and $C_{m,n}$ satisfy the following recursive relation:*

$$\max\{C_{u,n}, C_{m,n}\} \leq M(n+2)^3 \max\{C_{u,n-1}, C_{m,n-1}\}^4, \tag{122}$$

where M is a constant depending only on the data. Thus we have the following growth bounds:

$$\max\{C_{u,n}, C_{m,n}\} \leq M^{\frac{7^n-1}{6}} \max\{C_{u,0}, C_{m,0}\}^{7^n}. \tag{123}$$

As a result, we obtain these error estimates:

$$\left| u^\epsilon - \sum_{i=0}^{n} \frac{\epsilon^i}{i!} u_i \right|_{2+\alpha, 1+\alpha/2} \leq M^{\frac{7^n-1}{6}} \max\{C_{u,0}, C_{m,0}\}^{7^n} \epsilon^{n+1}, \tag{124}$$

and

$$\left| m^\epsilon - \sum_{i=0}^{n} \frac{\epsilon^i}{i!} m_i \right|_{2+\alpha, 1+\alpha/2} \leq M^{\frac{7^n-1}{6}} \max\{C_{u,0}, C_{m,0}\}^{7^n} \epsilon^{n+1}. \tag{125}$$

Remark 1. *The error estimates are very crude and are far from giving convergence of the power series. However, they nevertheless provide a rigorous sense in which the solution (u^ϵ, m^ϵ) can be approximated polynomially by explicitly known functions.*

It is easy to see that (122) implies (123) via induction:

Proof of Growth Bounds from Recursion. Let $C_n = \max\{C_{u,n}, C_{m,n}\}$, and let $\tilde{C}_n = \max\{C_n, n+3\}$. Then for $n = 1$, $C_1 \leq M(3)^3 C_0^4$, so that

$$\tilde{C}_1 \leq M\tilde{C}_0^7, \tag{126}$$

and (123) holds for $n = 1$. Now, suppose $\tilde{C}_{n-1} \leq M^{\frac{7^{n-1}-1}{6}} \tilde{C}_0^{7^{n-1}}$. Then as $C_n \leq M(n+2)^3 C_{n-1}$,

$$\tilde{C}_n \leq M\tilde{C}_{n-1}^7 \leq M\left(M^{\frac{7^{n-1}-1}{6}}\tilde{C}_0^{7^{n-1}}\right)^7 \leq M^{\frac{7^n-1}{6}}\tilde{C}_0^{7^n}, \tag{127}$$

which is the desired inequality. □

In the proof of the main theorem, we will use a classical result in the theory of parabolic PDE. The following result can be deduced from the proof of Theorems IV.5.2 and IV.5.3 in Ladyžhenskaja et al. (1968).

Theorem 13. *Let $\phi_T = \phi_T(x) \in C^{2+\alpha}([0,L])$, $a = a(x,t) \in C^{\alpha,\alpha/2}([0,L]\times[0,T])$, $b = b(x,t) \in C^{\alpha,\alpha/2}([0,L]\times[0,T])$, and $c = c(x,t) \in C^{\alpha,\alpha/2}([0,L]\times[0,T])$. Then there exists a unique solution $\phi \in C^{2+\alpha,1+\alpha/2}([0,L]\times[0,T])$ to the following boundary value problem:*

$$\begin{aligned} &\phi_t + \frac{\sigma^2}{2}\phi_{xx} + a(x,t)\phi + b(x,t)\phi_x = c(x,t), \\ &\phi(0,t) = 0,\ \ \phi_x(L,t) = 0,\ \ \phi(x,T) = \phi_T(x). \end{aligned} \tag{128}$$

Moreover, there exists a constant κ depending on $\sigma, r, T, |a|_{\alpha,\alpha/2}$, and $|b|_{\alpha,\alpha/2}$ such that

$$|\phi|_{2+\alpha,1+\alpha/2} \leq \kappa(|\phi_T|_{2+\alpha} + |c|_{\alpha,\alpha/2}). \tag{129}$$

We only need to apply Theorem 13 with $\phi_T = 0$, since $u_n(x,T) = 0 = m_n(x,0)$ for all $n = 1,2,3,4,\ldots$.

Lemma 14. *The function $2\hat{E}_n^\epsilon\left(F(\epsilon)\right) + \hat{E}_n^\epsilon(u_x^\epsilon)$ depends only on t. Moreover,*

$$\left|2\hat{E}_n^\epsilon(F(\epsilon)) + \hat{E}_n^\epsilon(u_x^\epsilon)\right|_{\alpha/2} \leq K\left(\frac{1}{2^n} + \sum_{i=0}^{n-1} C_{m,i} + \sum_{i=0}^{n-1} C_{u,i}\sum_{j=0}^{n-i-1}(C_{m,j} + C_{m,j-1})\right). \tag{130}$$

whenever K is a constant such that

$$K \geq \max\left\{\frac{1 + L|u_x^\epsilon|_{\alpha,\alpha/2}|m^\epsilon|_{\alpha,\alpha/2}}{2}, L|u_x^\epsilon|_{\alpha,\alpha/2}, L\right\}. \tag{131}$$

(Here $C_{m,-1} := 0$.) As a corollary,

$$2\left|\hat{E}_n^\epsilon(F(\epsilon))\right|_{\alpha,\alpha/2} \le C_{u,n} + K\left(\frac{1}{2^n} + \sum_{i=0}^{n-1} C_{m,i} + \sum_{i=0}^{n-1} C_{u,i} \sum_{j=0}^{n-i-1} (C_{m,j} + C_{m,j-1})\right). \tag{132}$$

Proof. Note that $\hat{E}_n^\epsilon\left(\frac{2}{2+\epsilon}\right) = \frac{(-1)^{n+1}}{2^n(2+\epsilon)}$ and $\hat{E}_n^\epsilon\left(\frac{\epsilon}{2+\epsilon}\right) = -\hat{E}_n^\epsilon\left(\frac{2}{2+\epsilon}\right)$, so by the linearity of $E_n^\epsilon(\cdot)$ and a successive application of (111),

$$\begin{aligned}\hat{E}_n^\epsilon(F(\epsilon)) = \frac{1}{2}\Bigg[&\frac{(-1)^{n+1}}{2^n(2+\epsilon)} - \frac{(-1)^{n+1}}{2^n(2+\epsilon)}\int_0^L u_x^\epsilon m^\epsilon \,\mathrm{d}x \\ &+ (-1)^n \int_0^L u_x^\epsilon \sum_{i=0}^{n-1}\left(\frac{-1}{2}\right)^{n-(i+1)} \hat{E}_i^\epsilon(m^\epsilon)\,\mathrm{d}x \\ &- \int_0^L \sum_{i=0}^{n-1}\left(\frac{-1}{2}\right)^{n-(i+1)} \hat{E}_i^\epsilon(u_x^\epsilon) \sum_{j=0}^{n-i-1} \frac{(-1)^{n-j+1} m_j}{j!}\,\mathrm{d}x - \hat{E}_n^\epsilon(u_x^\epsilon)\Bigg].\end{aligned} \tag{133}$$

Observe that

$$\frac{m_j}{j!} = \epsilon^{-j}(E_j^\epsilon(m) - E_{j-1}^\epsilon(m)) = \epsilon\hat{E}_j^\epsilon(m) - \hat{E}_{j-1}^\epsilon(m) \;\Rightarrow\; \left|\frac{m_j}{j!}\right|_{\alpha,\alpha/2} \le C_{m,j} + C_{m,j-1}. \tag{134}$$

Thus, (130) follows from (133) and the triangle inequality. $\square$

To simplify our recursive formulas, define $C_i := \max\{1, C_{u,j}, C_{m,j} : 1 \le j \le i\}$.

Corollary 15. *Assume K satisfies (131). Then (130) becomes*

$$\left|2\hat{E}_n^\epsilon(F(\epsilon)) + \hat{E}_n^\epsilon(u_x^\epsilon)\right|_{\alpha/2} \le K(n+1)C_{n-1}^2 \tag{135}$$

and (132) becomes

$$\left|2\hat{E}_n^\epsilon(F(\epsilon))\right|_{\alpha,\alpha/2} \le C_n + K(n+1)C_{n-1}^2. \tag{136}$$

Lemma 16. *Assume $K \geq 1$ satisfies* (131). *Let $c(x,t) = \hat{E}_n^\epsilon\left(F(\epsilon)^2\right) + F(0)\hat{E}_n^\epsilon(u_x^\epsilon)$. Then*

$$|c|_{\alpha,\alpha/2} \leq \tilde{K}(n+2)^3 C_{n-1}^4 \tag{137}$$

as long as

$$\tilde{K} \geq \max\{K^2, K\big|F(0)\big|_{\alpha,\alpha/2}\}. \tag{138}$$

Proof. By Lemma 115, with $k = \lfloor \frac{n}{2} \rfloor$,

$$\begin{aligned}\hat{E}_n^\epsilon\left(F(\epsilon)^2\right) &= \frac{1}{\epsilon^{n+1}}\left[E_k^\epsilon\left(F(\epsilon)\right)\right]^2 + 2\sum_{j=0}^{k}\frac{F^{(j)}(0)}{j!}\hat{E}_{n-j}^\epsilon\left(F(\epsilon)\right) \\ &= \frac{1}{\epsilon^{n+1}}\left[E_k^\epsilon\left(F(\epsilon)\right)\right]^2 + 2\sum_{j=1}^{k}\frac{F^{(j)}(0)}{j!}\hat{E}_{n-j}^\epsilon\left(F(\epsilon)\right) + 2F(0)\hat{E}_n^\epsilon\left(F(\epsilon)\right),\end{aligned} \tag{139}$$

so that

$$c(x,t) = \epsilon^{2k+1-n}\left[\hat{E}_k^\epsilon\left(F(\epsilon)\right)\right]^2 + 2\sum_{j=1}^{k}\frac{F^{(j)}(0)}{j!}\hat{E}_{n-j}^\epsilon\left(F(\epsilon)\right) + F(0)(2\hat{E}_n^\epsilon\left(F(\epsilon)\right) + \hat{E}_n^\epsilon(u_x^\epsilon)). \tag{140}$$

Now we use Corollary 15 to get

$$\begin{aligned}|c|_{\alpha,\alpha/2} &\leq \frac{1}{4}\left[C_k + K(n+1)C_{k-1}^2\right]^2 \\ &+ 2\sum_{j=1}^{k}(C_j + K(n+1)C_{j-1}^2)(C_{n-j} + K(n+1)C_{n-j-1}^2) + \big|F(0)\big|_{\alpha,\alpha/2}K(n+1)C_{n-1}^2 \\ &\leq \frac{1}{4}K^2(n+2)^2C_k^4 + 2kK^2(n+2)^2C_{n-1}^4 + \big|F(0)\big|_{\alpha,\alpha/2}K(n+1)C_{n-1}^2.\end{aligned} \tag{141}$$

The claim follows. ∎

We now prove our main result.

Proof of Theorem 12. The case that $n = 0$ is the content of Corollary 6 and Corollary 9. Now, we prove the recursive relation. Let $\psi^n = \hat{E}_n^\epsilon(u)$. By the linearity of $E_n^\epsilon(\cdot)$, we have that

$$\hat{E}_n^\epsilon\left(u_t+\frac{\sigma^2}{2}u_{xx}-ru+F(\epsilon)^2\right)=\psi_t^n+\frac{\sigma^2}{2}\psi_{xx}^n-r\psi^n+\hat{E}_n^\epsilon\left(F(\epsilon)^2\right)=0. \tag{142}$$

By Lemma 16 and Theorem 13, we obtain

$$|\psi^n|_{2+\alpha,1+\alpha/2}\leq\kappa\tilde{K}(n+2)^3C_{n-1}^4, \tag{143}$$

where $\kappa,\tilde{K}$ depend only on the data. Now let $\varphi^n=\hat{E}_n^\epsilon(m)$. By the linearity of $E_n^\epsilon(\cdot)$, we have that

$$\hat{E}_n^\epsilon\left(m_t-\frac{\sigma^2}{2}m_{xx}-\big[F(\epsilon)m\big]_x\right)=\varphi_t^n-\frac{\sigma^2}{2}\varphi_{xx}^n-\left[\hat{E}_n^\epsilon\left(F(\epsilon)m\right)\right]_x=0. \tag{144}$$

By Lemma 111, we have that

$$\begin{aligned}\hat{E}_n^\epsilon\left(F(\epsilon)m\right)&=m^\epsilon\hat{E}_n^\epsilon\left(F(\epsilon)\right)+\frac{1}{\epsilon^{n+1}}\sum_{i=0}^{n}\frac{\epsilon^iF^{(i)}(0)}{i!}E_{n-i}(m)\\&=m^\epsilon\hat{E}_n^\epsilon\left(F(\epsilon)\right)+\sum_{i=1}^{n-1}\frac{F^{(n-i)}(0)}{(n-i)!}\hat{E}_i(m)+F(0)\varphi^n.\end{aligned} \tag{145}$$

Now, let $b=F(0)$ and $w=m^\epsilon\hat{E}_n^\epsilon\left(F(\epsilon)\right)+\sum_{i=0}^{n-1}\frac{F^{(n-i)}(0)}{(n-i)!}\hat{E}_i(m)$. Our PDE is now

$$\varphi_t^n-\frac{\sigma^2}{2}\varphi_{xx}^n-(b\varphi^n)_x-w_x=0. \tag{146}$$

Again, applying Lemma 16 and Theorem 13, we obtain

$$|\varphi^n|_{2+\alpha,1+\alpha/2}\leq\tilde{\kappa}\tilde{K}(n+2)^3C_{n-1}^4, \tag{147}$$

the constant $\tilde{\kappa}$ being possibly different from κ.

Combining (147) and (143), we obtain the recursive relation

$$C_n \leq M(n+2)^3 C_{n-1}^4, \tag{148}$$

which, in particular, implies (122). $\square$

CHAPTER FOUR

Conclusions and Open Questions

It is apparent that the growth bounds are very crude and do not demonstrate analyticity. Therefore, it is still of interest to know whether or not we can obtain tighter estimates that give analyticity, or even some type of Gevrey class regularity if analyticity is too stringent a goal. As the growth bounds tend to grow to astronomically large numbers even after just the first few iterations, it follows that the first order estimates are of the most use. The first order estimates prove rigorously that $\frac{\partial}{\partial \epsilon} u \mid_{\epsilon=0}$ and $\frac{\partial}{\partial \epsilon} m \mid_{\epsilon=0}$ are equal to u_1 and m_1, respectively. That is, the notation acually represents derivatives in ϵ evaluated at 0 for u and m as we would expect.

There are still some open questions regarding this system of mean field equations. Recall in Chapter 2 that we defined T to be the exhaustion time, the first instance at which all players have dropped out of the competition. It can be treated as a variable and Graber and Bensoussan proved existence and uniqueness of solutions for an arbitrary final time (Graber and Bensoussan, 2018). However, the exit time is not something known a priori and must be determined from the system endogenously. Without coupling of the equations, the exhaustion time can be computed, but when we do have a coupled system, is there a way to determine how long before players run out of resources? And is it even the case that the expected value is finite? Furthermore, Chan and Sircar present the system of mean field games as an infinite time horizon problem, so is the problem still well-posed?

The calculations done in Chapter 3 to obtain error estimates consider the second order case of the mean field game. However, for the first order case, when $\sigma = 0$, is the problem still well posed? Graber and Mouzouni partially addressed this case, but for Neumann boundary conditions (Graber and Mouzouni, 2018). Furthermore, if the problem is indeed well-posed, are there also error estimates for solutions as shown in Chapter 3 for the second order case?

While much of the theory of mean field games has been developed over the past couple of decades, it is clear that there is still much to learn about solutions to these types of equations. This system of PDE, while a simplification of a general system of N coupled differential equations, is still very deep in its implications and has its own unique complexities leaving much to explore about its solutions.

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