ABSTRACT

Colliding Branes and Formation of Spacetime Singularities in Superstring Theory

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A systematic study of spacetimes containing two timelike colliding branes is made, in the framework of 10-dimensional string theory. After developing the general formulas to describe such events, we study several classes of exact solutions and spacetime singularities in both the $D+d$-dimensional string theory and its D-dimensional effective theory, obtained by Kaluza-Klein compactification. It is found that spacetime singularities in the low dimensional effective theory may or may not remain after lifted to the $D+d$-dimensional string theory, depending on the specific solutions.

In some cases, solutions of the low dimensional effective theory are free of singularities, but after they are lifted to string theory, the higher dimensional spacetimes become singular. Therefore, simply lifting low dimensional effective theories to high dimensions seemingly does not solve the singularity problem, and additional physical mechanisms are needed. In general however, the spacetime is singular, due to the mutual focus of the two colliding branes. Non-singular cases also exist, but with the price that both of the colliding branes violate all the three energy conditions, weak, dominant, and strong.

Colliding Branes and Formation of Spacetime Singularities in Superstring Theory

by

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A Dissertation

Approved by the Department of Physics

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Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

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Accepted by the Graduate School
May 2009

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ACKNOWLEDGMENTS

It has been an arduous journey. The long hours were never a concern for me, as I had realized at an early age that the process of scientific discovery was not one of instant gratification. It was in the way we teach science that my deepest concerns always lay, as it was not the homework, nor the intricate notions, not even the untapped talent one always hopes is laying dormant, waiting to be unleashed, that I found most disheartening. My deepest concern was that to learn, one must be matched with an appropriate teacher, that would evolve into a mentor who could in turn share his train of thought with me, and whom, of course, I could follow.

It is for this reason, and so many others, I feel profoundly blessed to have had Dr. Anzhong Wang guide me through my graduate studies. Other than being the most hard working human being I have ever met, after my father, he has served as an ideal role model for me, one whom I will strive to live up to for the rest of my professional career as well as personal life. A robust scientist, with impeccable integrity, actively engaged in the open proliferation of ideas that is so important in academia, Dr. Wang has led me and his group through cutting edge research in theoretical physics, the likes of which I never expected to be able to undertake, even as a graduate student. The courses that Dr. Wang teaches are completely transparent, in that he delivers the formalism and applications of these topics in a way which can immediately be applied to ones research with confidence. I feel I could continue my description without ever feeling I have adequately imparted my appreciation. I will thus simply say, that I have thoroughly enjoyed these many years as Dr. Wang's graduate student, and I look forward to building a lifelong friendship with him.

If at all possible, I would extend my gratitude to both my parents, Constantine and Olga (Lola) Tziolas and to my beautiful wife Zoi, simultaneously at this stage.

Zoi Maroudas-Tziolas, my wonderful wife, has shown me there are still good people in the world, simply because she is the best, and statistically speaking there must be some more drifting along the distribution curve somewhere. She is as beautiful as she is kind, as clever as she is personable and as caring as I always wanted the mother of my children to be. She has never taken her eye off of me, not even for an instant, constantly making sure that I am loved, fed, taken care of and even indulged in, at times where I guess I didn't know better. She is certainly, the best thing that ever happened to me. Always at my side, smart, supportive, happy and upbeat, she never denied me explaining a complicated problem to her, and has often been the source of my inspiration. I love her deeply.

My parents always held unwavering belief in me and my abilities and have constantly encouraged me to pursue my dreams, certainly beyond what I once thought I was capable of. My father, Constantine is the most peaceful and relaxing person I have ever met, whose pride in me has made me turn exhaustion into adrenalin infused vigor countless times, during the long hours sprawled over graduate textbooks. My mother's charisma, remarkable moral reasoning and innate processing capabilities, have often made me laugh out loud at her ability to cut completely and utterly to the chase, in what I can only think of as the perfect coupling of clear thought, a mother's instinct and a woman's sixth sense (where perhaps some extra dimensional effects have come into play).

My brother Tasos, with whom I shared what I consider to be an ideal childhood, has always been able to remind me of what is important in life - this very ability makes him a completely unique individual, whom I hope I will be able to spend more time with, on frequent visits to the homeland, in Greece, along with my beloved cousins Katerina and Nandia.

Zoi's parents, Athina and Dionissis Maroudas, carry a profound wisdom and kindness, from which I have benefited greatly ever since we met. Having them as my parents-in-law, has been far beyond my wildest dreams. My step-brother Georgios, with whom we share a love of the outdoors, has also been a remarkable source of support for me. During times when the weight of my workload was getting the best of me, he turned my doubts into aspiration and helped me continue my pursuits. In fact, all of my nieces and nephews, Athina, Denny, Christina, Hristo and Dionissi have always been wonderful in supporting me, by believing that I have an extra brain embedded somewhere in my forehead!

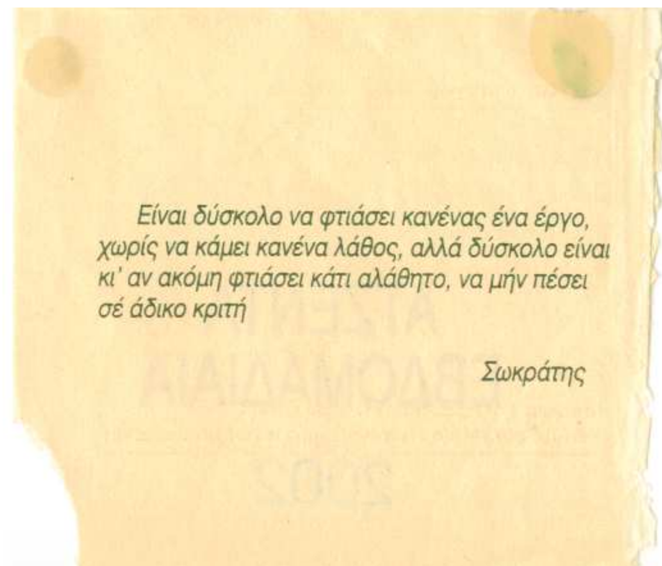
It has been a privilege to study physics under the many talented professors at Baylor University. Dr. Gregory Benesh has done an exceptional job as department chair, always keeping the best interests of the graduate students a priority, for which I know we have all been grateful. Dr. Dwight Russell has been instrumental in helping me develop my teaching style, which I am sure will serve me well in the future. We have shared many enjoyable nights probing the universe with our telescopes, which has been a great motivation for me to continue my academic career in astrophysics. A special thanks goes to Dr. Cleaver for teaching me quantum field theory and brane cosmology, and who I have always been happy to see working late, up to the crack of dawn in his office. I would like to also thank Dr. Qin (Tim) Sheng for teaching me the importance of a careful and rigorous mathematical process, through his courses on advanced numerical methods which I have taken to heart, frequently sending me back to graduate textbooks in mathematical physics to verify my procedures.

I would like to especially thank my closest friend, Dr. Richard Obousy, for the deep intellectual bond that we have shared, ever since our days as undergraduates. Our countless conversations on physics and the ideals of life, have been instrumental at filling in the cracks between knowledge and understanding. It was a true honor to be his best man at his wedding, as his wife Annie Whitney has been a wonderful friend to both me and my wife.

There have been many fantastic people at Baylor, such as Sharma Preet, David Katz, Michael Whittaker, Tim Renner, Martin Frank, Kristin Pechan, James Creel, Michael Devin, Quinyu (Theresa) Ma, Qiang (Bob) Wu and Ben Dundee, all of which I would like to thank for their companionship.

It was at Baylor University, that I met my amazing wife Zoi, which is why I will always think of Baylor as a place from which I took far more than I gave.

DEDICATION



Είναι δύσκολο να φτιάσει κανένας ένα έργο, χωρίς να κάμει κανένα λάθος, αλλά δύσκολο είναι κι' αν ακόμη φτιάσει κάτι αλάθητο, να μήν πέσει σέ άδικο κριτή

Σωκράτης

It is certainly difficult for one to create something,
without making any mistakes. It is also difficult
to imagine, even if one creates something perfectly,
that it will not happen upon unfair criticism.

Socrates

CHAPTER ONE

Introduction to String Theory

1.1 String Theory

The past decade has seen significant advances in cosmology [1]. The discovery of dark energy [2, 3] has decisively established the presence of a non-zero cosmological constant in our universe. Simultaneously, the constant influx of cosmic microwave background (CMB) data [4, 5, 6] has revealed that the early universe is in striking agreement with the basic predictions of inflationary scenarios. The resulting paradigm of a universe undergoing inflation [7, 8, 9] at early times, and dominated by cold dark matter and dark energy at late times, has sometimes been referred to as the ΛCDM (Lambda Cold Dark Matter) model for cosmology.

Superstring theory represents the most promising candidate for a unified theory of the fundamental interactions, including gravity [10, 11]. One of the strongest constraints on any aspiring cosmological theory is that it should be consistent with the standard model of the very early universe [12].

At its inception, string theory was formulated as a theory of hadrons, although quantum chromodynamics (QCD) has proven to be a more consistent description of quark interactions. The theory has evolved however, and is now being applied to cosmological models [13]. Borrowing from its quantum mechanical origins, the characteristic length scale of a string can be implied by the Plank length and energies,

$$l_P = \left(\frac{\hbar G}{c^3}\right)^{3/2} = 1.6 \times 10^{-33}\text{cm}, \tag{1.1}$$

$$m_P = \left(\frac{\hbar c}{G}\right)^{1/2} = 1.2 \times 10^{19}\text{GeV/c}^2, \tag{1.2}$$

As these energy scales are well beyond the reach of foreseeable collider technologies, we anticipate that evidence for the validity of string theory will come from cosmological considerations. With the application of supersymmetry to string theory, where for every boson a corresponding fermion is assumed and vice versa, superstring theory has been infused with many versatile solutions that can be applied to various cosmological schemes.

A definitive prediction of string theory is the existence of a scalar field, φ that couples directly to matter, and is referred to as the *dilaton* [14, 15]. There are two further massless excitations that are common to all string theories. These are the tensor field, $g_{\mu\nu}$, known as the *graviton*, and a rank two *anti-symmetric tensor field*, $B_{\mu\nu}$. Its vacuum expectation value determines the strengths of both the gauge and gravitational couplings.

The inverse string tension α' defines the characteristic string length scale,

$$l_s \equiv \sqrt{\hbar c \alpha'}. \tag{1.3}$$

On the other hand, the effective Planck length in a D-dimensional spacetime, is dependent upon both α' and the value of the dilaton [16],

$$l_{Plank}^{(D)} \equiv e^{\varphi/(D-2)} \sqrt{\hbar c \alpha'}, \tag{1.4}$$

If we invoke Plank units $\hbar = c = 1$, but retain units of length (or equivalently $^1/_{mass}$), the gauge coupling strength is given by [16],

$$\alpha_{gauge} g_s^2 \equiv e^{\varphi} = \left(\frac{l_{Plank}^{(D)}}{l_s} \right)^{D-2}, \tag{1.5}$$

where α_{gauge} is the gauge coupling constant and g_s is the string coupling strength. Thus we enter the weak coupling regime of string theory for $e^{\varphi} \ll 1$, where the dilaton can be treated as a massless particle.

The cosmological consequences of the dilaton field in this regime are profound and its dynamical effects lead to a radical departure from the standard picture of early universe cosmology as seen from the standpoint of 4-dimensional General Relativity.

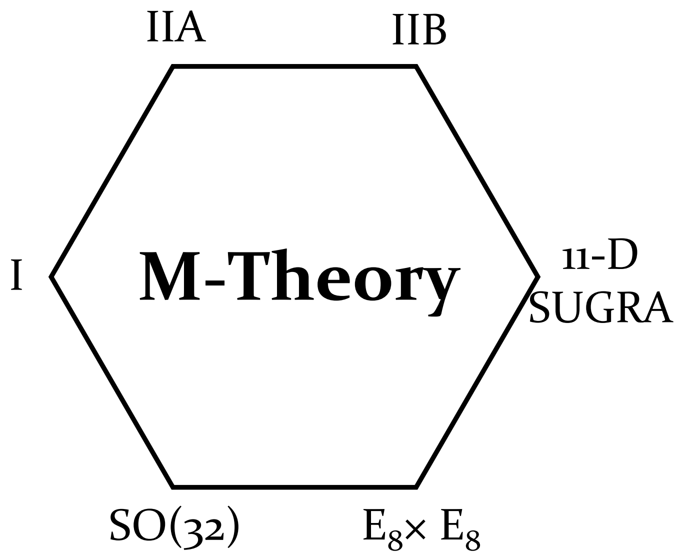


Figure 1.1: A popular way of describing the dualities in M-theory that lead to the five main superstring vacua. 11D Supergravity (SUGRA) can also be shown to be a dual theory, thus connecting M-theory to other gauge theories with effective Yang-Mills solutions, and AdS/CFT correspondence.

1.2 M-Theory

In 1995 string theory experienced a significant revelation [13], in what is now being called the 'second superstring revolution'. Specifically, it was realized that the various types of string theories could be connected via 'dualities', which relate all five superstring theories in ten dimensions to one another. As such, the different theories can be seen as just perturbative expansions of a unique underlying framework, made up of five different, consistent quantum vacua [17, 18]. The equation of motion of this completely unique theory of nature, thus admits many vacua and is a very attractive background from which one may work towards a grand unified theory.

This underlying theory, called 'M-Theory', can be shown to live in 11 spacetime dimensions [19, 20, 21], the low-energy limit of which is ironically 11-dimensional supergravity [22, 23]. All five superstring theories can be thought of as originating from M-Theory as depicted schematically in Fig. 1.1. The main aspects of the dual theories are listed in Table 1.1 for reference purposes.

Table 1.1: *A summary of the main features of the 5 main string theories. M-Theory showed how these can be related through dualities, which provides string theory a great flexibility, especially when dealing with infinities that arise around spacetime singularities, such as black holes or the big bang.*

Type	Dimensions	Characteristics
Bosonic	26	Bosons only, thus only forces are present. No matter, both open and closed strings, contains tachyons, which have imaginary mass and travel faster than the speed of light.
I	10	Supersymmetry, both open and closed strings, no tachyon, group symmetry SO(32).
IIA	10	Supersymmetry, closed strings only, no tachyon, massless fermions are non-chiral (they spin both ways).
IIB	10	Supersymmetry, closed strings only, no tachyon, massless fermions are chiral (they spin one way).
SO(32)	10	Supersymmetry, closed strings only, no tachyon, heterotic (right and left moving strings differ), group symmetry SO(32).
$E_8 \times E_8$	10	Supersymmetry, closed strings only, no tachyon, heterotic (right and left moving strings differ), group symmetry $E_8 \times E_8$.

In addition to the fundamental strings dualities, M-theory goes further and admits a variety of stable domain wall solutions, called 'p-branes' [24, 25], where p is the number of *spatial extensions* of the objects. Especially important in this regard are the 'Dirichlet p-branes' [26, 27, 11], or 'D-branes' for short, which are p-dimensional soliton-like hyperplanes in spacetime whose quantum dynamics are governed by the theory of open strings with ends constrained to move on them, as depicted in Fig. 1.2.

Today, M-theory is far from being well understood and remains a central theme in theoretical physics, be it in high energy particle physics or the cosmology of the early universe. Witten playfully suggested M should stand for 'Magic', 'Mystery' or 'Membrane', while others have come to call it the 'Mother' theory, addressing its undeniable potential for contributing to a unified theory of nature.

We will now turn our attention to the elements of superstring theory that are relevant to this dissertation, starting with a review of the basic elements of Type II strings and continuing on to D-branes and their interactions.

1.3 Type II Superstrings

The effective bosonic action of the Type IIA superstring is N = 2, D = 10, non-chiral supergravity and is given by [12],

$$\begin{aligned} S_{IIA} &= \frac{1}{16\pi\alpha'}\Big\{\int d^{10}x\sqrt{|g_{10}|}\Big[e^{-\Phi}\Big(R_{10}+(\nabla\Phi)^2-\frac{1}{12}H_3^2\Big) \\ &\quad -\frac{1}{4}F_2^2-\frac{1}{48}(F_4')^2\Big]+\frac{1}{2}\int B_2\wedge F_4\wedge F_4\Big\} \end{aligned} \tag{1.6}$$

where R_{10} is the Ricci scalar curvature of the spacetime with metric g_{MN}, $g_{10} = det(g_{MN})$, N is the number of supersymmetry generators, and D is of course, the number of dimensions. Strings sweep out geodesic surfaces with respect to the metric g_{MN}. The dilaton field, Φ, determines the value of the string coupling parameter, $g_s^2 = e^{\Phi}$.

It is interesting to note that the dilaton-graviton sector of this action may be interpreted as a ten-dimensional Brans-Dicke theory [28], where the coupling between the dilaton and graviton is specified by the Brans-Dicke parameter $\omega = -1$.

The antisymmetric tensor field strengths are defined by,

$$\begin{aligned} H_3 &= dB_2, \qquad & F_4 = dA_3, \\ F_2 &= dA_1, \qquad & F_4' = F_4 + A_1\wedge H_3, \end{aligned} \tag{1.7}$$

where d is the exterior derivative and $(F_p,\ H_p)$ denote antisymmetric p-form potentials. The last term in Eq.(1.6) is a 'Chern-Simons' term and is a necessary consequence of supersymmetry. For the backgrounds we will be concerned with however, this term can be neglected and we do not consider it further. This is an appropriate choice for all but a few models as, in general, the equations resulting from Eq. (1.6) are very difficult to solve.

In fact, we will be working exclusively in the *Neveu-Schwarz – Neveu-Schwarz* (NS-NS) sector of the action which only contains the graviton, the antisymmetric 2-form potential and the dilaton field. The *Ramond-Ramond* (RR) sector on the other hand, contains the antisymmetric p-form potentials, where p is odd. Their difference is that the NS-NS sector couples directly to the dilaton, but the RR fields do not [11, 18].

The corresponding action in the conformally related Einstein frame describes the on-brane physical manifestations of the appropriate string theory, its derivation being a fundamental step in extracting the relevant phenomenology [29, 30]. This is achieved through the toroidal compactification of the NS-NS action [31], which reduces Type II string theory to the conformally covariant effective Einstein frame, as outlined in Chapter 2 and shown in detail in Appendix B.

The NS-NS sector for the Type IIB theory has the same form as that of the Type IIA action. Their main difference is in the type of D-branes that can reside in the different Type II theories:

Type IIA Dp - Branes : These branes exist for all even values of p,

$$p = 0 \quad 2 \quad 4 \quad 6 \quad 8 \tag{1.8}$$

The case p = 0 is a D-particle, while p = 8 describes a domain wall in ten dimensional spacetime (in light of the solitonic description of D-branes [19]). The D0-brane and D6-brane are electromagnetic duals of each other, as are the D2-brane and D4-brane.

Type IIB Dp - Branes : Here we find branes for all odd values of p,

$$p = -1 \quad 1 \quad 3 \quad 5 \quad 7 \quad 9 \tag{1.9}$$

The case p = -1 describes an object which is localized in time and corresponds to a ‘D-instanton’, while p = 1 is a ‘D-string’. The D-instanton and D7-brane are electromagnetic duals of one another, as are the D1-brane and the D5-brane. The D9-branes are spacetime filling branes.

Of particular importance is the p = 3 case, which yields the self-dual D3-brane, with worldvolume that of the observable 3+1-dimensions we experience. As such Type IIB superstring theory has been a necessary ingredient in string gas cosmological models, and is the flavor that we will be most concerned with in this work.

1.4 D-Branes

Dp-branes describe a p+1 dimensional hypersurface in spacetime onto which open strings can attach, as seen in Fig. 1.2. Such objects arise when we choose Dirichlet rather than Neumann boundary conditions for the open strings. More precisely, the Dp-brane is specified by choosing Neumann boundary conditions in the directions along the hypersurface,

$$\partial_\sigma x^\mu \Big|_{\sigma=0,\pi} = 0 \qquad (1.10)$$

and Dirichlet boundary conditions in the transverse directions,

$$\delta_\sigma x^\nu \Big|_{\sigma=0,\pi} = 0 \qquad (1.11)$$

where $\mu \in [0, p]$ and $\nu \in [p+1, 9]$.

Although simple in description, brane dynamics are not very well understood. Their study over recent years however, has had a remarkable impact in high-energy physics, contributing to the microscopic explanation of black hole entropy and the emission of Hawking radiation [32, 33] and probes of short-distances in spacetime [34, 35, 36, 37, 38], where quantum gravitational fluctuations become important and classical general relativity breaks down.

String theories generally have perturbative elementary closed string states, because their amplitudes are functions of the string coupling g_s. Explicit realizations of the dualities inherent in D-brane models however, have shown these states can be mapped to a D-brane state in a dual theory, by an S-duality transformation. In

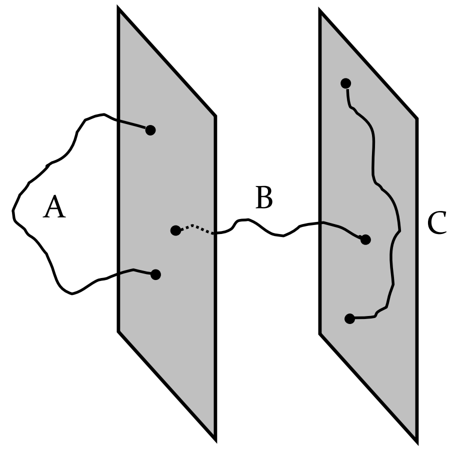


Figure 1.2: A pair of D-branes (shaded regions) with open strings (wavy lines) attached (with Dirichlet boundary conditions). The string ends are free to move along the hyperplanes. The corresponding open string coordinates satisfy Neumann boundary conditions in the directions along the D-branes and Dirichlet boundary conditions in the directions transverse to the D-branes.

this new theory, the string state depends on $1/g_s$ and is therefore non-perturbative. This infuses the theory with the remarkable ability to probe energetically divergent regions of spacetime, such as black holes, as one can in some cases move between $g_s \rightarrow 1/g_s$ states where the spacetime is better understood.

Another shining example of the applicability of string theory, is the gauge theory/gravity or AdS/CFT correspondence [39, 40]. Consider how D-branes carry gauge fields, while on the other hand they admit a dual description as solutions of the classical equations of motion of string theory and supergravity. Demanding now that these two descriptions are equivalent implies, for some special cases, that string theory is equivalent to a gauge field theory. This is an explicit realization of the old ideas that Yang-Mills theory may be represented as some sort of string theory, and prove to be very useful in certain studies where solutions are obtained via numerical methods, by drastically reducing the necessary calculations.

D-branes can be studied in static manifolds, such as the popular Randall-Sundrum models [41, 42]. In brane world scenarios however, D-branes are usually considered to be dynamic, in that they can move together, collide and recoil in a higher dimensional bulk, thus forming a 'brane gas'. In these scenarios, our universe is modeled as a D-brane [43] interacting in a higher dimensional bulk, which leads to a potential explanation as to why gravity couples so weakly to matter. In effect, gravitons permeate throughout the whole spacetime and are thus diluted, providing us with a potential resolution of the hierarchy problem.

1.5 Brane Gas Cosmology

The goal of superstring cosmology is to examine the dynamical evolution D-branes and re-examine cosmological questions in the light of our new understanding of string theory [44]. In fact, substantial theoretical progress in string theory has brought forth a diverse new generation of cosmological models, some of which are subject to direct observational tests.

One key advance is the emergence of methods of moduli stabilization [45, 46]. Compactification of string theory from a total dimension D down to four dimensions introduces many gravitationally-coupled scalar fields - moduli - in the effective four-dimensional theory [47, 48, 49]. Divergent or light moduli are extremely problematic in cosmology, as in any realistic model they must be shown to be metastable, or adhere to a finely tuned balance of forces. Nevertheless, a few light scalars could prove to be a valuable resource, since they could address the issue of dark energy, dark matter, or provide a theoretical motivation for inflation [50].

Brane Gas Cosmology (BGC) is an approach to string cosmology that attempts to take advantage of the new tools that supersymmetry, M-Theory and D-branes have provided. As mentioned in section 1.3, generically the equations resulting from Eq.(1.6) are very difficult to solve. However, by invoking some approximations which draw from cosmological observations, the equations can be made tractable. By doing so we can define the objectives and requirements that Brane Gas Cosmology should work towards.

Homogeneous Fields: The background fields (i.e. metric, flux, and dilaton) are assumed to be homogeneous and therefore at most functions of time. The generalization to inhomogeneous fields can then follow from the effective Einstein frame solutions and the extensive studies of Bianchi type metric perturbations [50, 51].

Adiabatic approximation: The background fields are assumed to be evolving slowly enough that higher derivative corrections, i.e. α' corrections, can be ignored. This means that locally, string sources will not be influenced by the expansion and their evolution can be characterized by their scaling behavior.

Weak Coupling: We work in the region of weak coupling (i.e. $g_s \ll 1$), and we will choose initial conditions for the dilaton that preserve this condition. Thus, higher orders corrections in g_s, can be neglected.

Toroidal Spatial Dimensions: We assume that spatial dimensions are toroidal and therefore admit non-trivial one cycles. In the past this assumption was believed to be crucial, however it was later shown that this condition may be relaxed in some cases, allowing for more phenomenologically motivated backgrounds [52].

From the point of view of cosmology, all of these approximations are familiar. However, both the adiabatic and weak coupling approximation are very restrictive from the string theory perspective. The string corrections we are choosing to ignore may be very important for early universe cosmology, especially near cosmological singularities [53, 54, 55]. The motivation here is to take a modest approach by slowly turning on stringy effects, as one extrapolates the known cosmological equations backward in time to better understand the departures from standard big-bang cosmology.

We adopt the same approach in this work, by first examining the general properties of a spacetime where the effects from branes are initially taken to be negligible, before exploring the dynamics of brane collisions.

CHAPTER TWO

The Collision of Branes in String Theory

Branes in string/M-Theory are fundamental constituents [11, 56, 57], and of particular relevance to cosmology [58, 59, 12, 60, 61, 62]. These substances can move freely in bulk, collide, recoil, reconnect, and thereby form a brane gas in the early universe [63, 50], or create an ekpyrotic/cyclic universe [64]. Understanding these processes is fundamental to both string/M-Theory and their applications to cosmology.

Recently, Maeda and his collaborators numerically studied the collision of two branes in a five-dimensional bulk, and found that the formation of a spacelike singularity after the collision is generic [65, 66, 67, 68]. This is a very important result, as it implies a low-energy description of colliding branes breaks down at some point, and without a complete theory of quantum gravity, predictability is effectively lost. Similar conclusions were obtained from the studies of two colliding orbifold branes [69, 70]. Lately however, it has been argued that, from the point of view of the higher dimensional spacetime, these singularities are very mild and can be easily regularised [71].

In this work, we will construct various flavors of brane worlds, as we explore the predictions that General Relativity and String Theory offer after careful analysis of the resulting phenomenologies. Specifically, we are in search of evidence that may be used to explain the current state of the observed universe, be it the cosmic microwave background radiation, the currently observed anomalous expansion of our universe or the nature of dark energy and dark matter.

In practice, finding analytical solutions to brane world scenarios is very difficult. The differential equations that describe the solutions we seek, usually do not take standard forms and numerical methods requiring sophisticated computational algorithms become necessary. In this work however, we aspire to find explicit analytical results, using spacetime models, which are constructed using the fundamental properties of spacetime and matter, as we understand them today. In such, we arrive at a fully transparent and accountable body of work, that can be used to construct more complicated models, as we explore the implications of higher dimensional spacetimes populated by any number of interacting branes.

2.1 The Model

In the brane worlds we will be considering, we are ultimately interested in the properties of two colliding timelike branes. In this section, we provide a recipe for how such models can be constructed, and a road map to our plan of approach. Specific cases are considered in the chapters that follow, in which we will discuss the solutions in detail. However, at this stage we wish to provide a clear and succinct outline, as an introduction to our general approach to exploring brane world scenarios analytically.

We begin visualizing our model by forming the metric that will describe our spacetime. We start with a $(D+d)$- dimensional metric,

$$\begin{aligned} d\hat{s}^2_{D+d} &= \hat{g}_{AB}dx^A dx^B = \\ &= \gamma_{\mu\nu}(x^\lambda)dx^\mu dx^\nu + \hat{\Phi}^2(x^\lambda)\hat{\gamma}_{ab}(z^c)dz^a dz^b, \end{aligned} \tag{2.1}$$

where the capital Latin letters span the whole dimensionality of the spacetime, $A, B, C = 0, 1, ..., D+d-1$. We divide the metric into a D-dimensional *external* spacetime and a d-dimensional *internal* spacetime. Greek letters represent the external coordinates, $\mu, \nu, \lambda = 0, 1, 2, ..., D-1$ and lower case Latin letters represent internal coordinates, $a, b, c = D, D+1, ..., D+d-1$. Thus, $\gamma_{\mu\nu}(x^\lambda)$ and $\hat{\Phi}^2(x^\lambda)$ depend only on the external coordinates x^λ of the spacetime M_D, and $\hat{\gamma}_{ab}(z^c)$ only on the internal coordinates z^c of the space M_d.

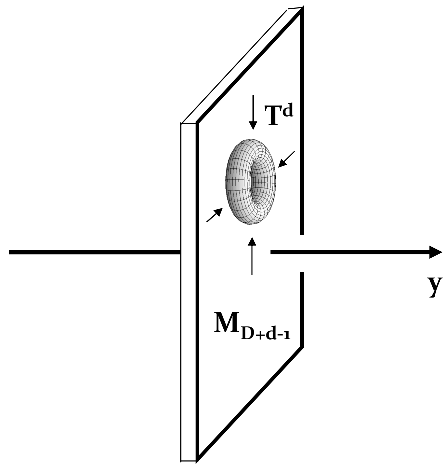


Figure 2.1: A schematic of a d-dimensional torus occupying the internal spacetime in our model. The compactification allows us to integrate around the corresponding internal coordinates z^c coordinates on M_d. The $(D+d-1)$-dimensional spacetime is presented here with the extra dimension y for clarity.

We will assume that the matter fields are all independent of z^c, then the internal space M_d must be Ricci flat,

$$R[\hat{\gamma}] = 0. \tag{2.2}$$

For the purpose of the current work, it is sufficient to assume that the internal space, M_d is a $d-$dimensional torus, $T^d = S^1 \times S^1 \times ... \times S^1$, as in Fig.2.1.

As shown explicitly in Appendix B, we find that the Ricci scalar for the spacetime M_{D+d} after compactification, is,

$$\begin{aligned} \hat{R}_{D+d}[\hat{g}] &= R_D[\gamma] + \frac{d(d-1)}{\hat{\Phi}^2}\,\gamma^{\mu\nu}(\nabla_\mu\hat{\Phi})(\nabla_\nu\hat{\Phi}) \\ &\quad -\frac{2}{\hat{\Phi}^d}\,\gamma^{\mu\nu}(\nabla_\mu\nabla_\nu\hat{\Phi}^d). \end{aligned} \tag{2.3}$$

2.1.1 The NS-NS Sector of the Action in String Theory

Let us consider the toroidal compactification of the NS-NS sector of the string action in Type II sting theory. In this case, our (D+d)-dimensional manifold $\hat{M}_{D+d} = M_D \times M_d$, corresponds to the string theory with $D+d = 10$. Then, the action takes the form [1, 12, 50],

$$S_{D+d} = -\frac{1}{2\kappa^2_{D+d}} \int d^{D+d}x \sqrt{|\hat{g}_{D+d}|} e^{-\hat{\Gamma}} \Big\{ \hat{R}_{D+d}[\hat{g}] + \hat{g}^{AB}(\hat{\nabla}_A \hat{\Gamma})(\hat{\nabla}_B \hat{\Gamma}) - \frac{1}{12}\hat{H}^2 \Big\}. \tag{2.4}$$

If we now ignore the dilaton $\hat{\Gamma}$ and the form fields $\hat{H}$,

$$\hat{\Gamma} = \hat{H} = 0, \tag{2.5}$$

we find that by using Eq. (2.3) the integration of the action (2.4) over the internal space yields, (Appendix B)

$$S^{(S)}_D = -\frac{1}{2\kappa^2_D} \int d^D x \sqrt{|\gamma|}\, \hat{\Phi}^d \Big\{ R_D[\gamma] + \frac{d(d-1)}{\hat{\Phi}^2}\, \gamma^{\mu\nu} (\nabla_\mu \hat{\Phi})(\nabla_\nu \hat{\Phi}) \Big\}, \tag{2.6}$$

where

$$\kappa^2_D \equiv \frac{\kappa^2_{D+d}}{V_s}, \tag{2.7}$$

and V_s is the volume of the internal space, defined as

$$V_s \equiv \int \sqrt{\hat{\gamma}} d^d z. \tag{2.8}$$

For a string scale compactification, we have $V_s = \left(2\pi\sqrt{\alpha'}\right)^d$, where $(2\pi\alpha')$ is the inverse string tension.

After the conformal transformation,

$$g_{\mu\nu} = \hat{\Phi}^{\frac{2d}{D-2}} \gamma_{\mu\nu}, \tag{2.9}$$

the D-dimensional effective action of Eq.(2.6) can be cast in the minimally coupled form,

$$S^{(E)}_D = -\frac{1}{2\kappa^2_D} \int d^D x \sqrt{|g_D|}\, \big\{ R_D[g] - \kappa^2_D (\nabla\phi)^2 \big\}, \tag{2.10}$$

where

$$\phi \equiv \pm \left[\frac{(D+d-2)d}{\kappa^2_D (D-2)} \right]^{1/2} \ln \hat{\Phi}. \tag{2.11}$$

The action of Eq.(2.6) is usually referred to as *the string frame*, and the one of Eq.(2.10) as *the Einstein frame*. It should be noted that solutions related by this conformal transformation can have completely different physical and geometrical properties in the two frames.

In particular, in one frame a solution can be singular, while in the other it can be totally free from any kind of singularities. A simple example is the flat FRW universe which can be written as,

$$\gamma_{ab} = a^2(\tau)\eta_{ab} \tag{2.12}$$

But the spacetime described by γ_{ab} usually has a big bang singularity, while the one described by η_{ab} is Minkowski, and does not have any kind of spacetime singularities.

The dimensionally reduced actions (2.6) and (2.10) may be viewed as the prototype actions for string cosmology, because they contain many of the key features common to more general actions. Cosmological solutions to these actions have been extensively discussed in the literature, both in the homogeneous and inhomogeneous contexts (see [12] for a review).

2.1.2 The Action of a Brane

To study the collision of two branes, we add the following brane actions to $S_D^{(E)}$ of Eq.(2.10),

$$S_{D-1,m}^{(E,I)} = \int_{M_{D-1}^{(I)}} \sqrt{\left|g_{D-1}^{(I)}\right|} \left(\mathcal{L}_{D-1}^{(m,I)}(\psi) - V_{D-1}^{(I)}(\phi)\right) d^{D-1}\xi_{(I)}, \tag{2.13}$$

where $V_{D-1}^{(I)}(\phi)$ denotes the potential of the scalar field ϕ on the I-th brane ($I = 1,\ 2$) and $\xi_{(I)}^{\mu}$'s are the intrinsic coordinates of the I-th brane, where $\mu,\ \nu,\ \lambda = 0, 1, 2, ..., D-2$. $\mathcal{L}_{D-1}^{(m,I)}(\psi)$ is the Lagrangian density of matter fields located on the I-th brane, denoted collectively by ψ. It should be noted that the above action does not include kinetic terms of the scalar field on the branes.

This setup is quite similar to the Horava-Witten heterotic M-Theory on S^1/Z_2 [17, 29, 72, 73], in which the two potentials $V_4^{(1)}(\phi)$ and $V_4^{(2)}(\phi)$ have opposite signs. It is also similar to the modulus stabilization mechanism of Goldberger and Wise [74], which have recently been applied to orbifold branes in string theory [30].

The two branes are localized on the surfaces,

$$\Phi_I\left(x^a\right) = 0, \tag{2.14}$$

or equivalently

$$x^a = x^a\left(\xi^\mu_{(I)}\right), \tag{2.15}$$

and $g^{(I)}_{D-1}$ denotes the determinant of the reduced metric $g^{(I)}_{\mu\nu}$ of the I-th brane, defined as

$$g^{(I)}_{\mu\nu} \equiv g_{ab} e^{(I)a}_{(\mu)} e^{(I)b}_{(\nu)}\Big|_{M^{(I)}_{D-1}}, \tag{2.16}$$

where

$$e^{(I)\,a}_{(\mu)} \equiv \left.\frac{\partial x^a}{\partial \xi^\mu_{(I)}}\right|_{M^{(I)}_{D-1}}. \tag{2.17}$$

A notable difference between our model and those applied to M-theory and Randall-Sundrum models, is that we will not assume Z_2 reflection symmetry between our branes.

2.1.3 The Total Action

Then, the total action is given by,

$$S^{(E)}_{total} = S^{(E)}_D + \sum_{I=1}^{2} S^{(E,I)}_{D-1,m}. \tag{2.18}$$

Variation of the total action (2.18) with respect to g_{ab} yields the D-dimensional gravitational field equations,

$$\begin{aligned} R_{ab} - \frac{1}{2} R g_{ab} &= \kappa^2_D \Bigg\{ T^\phi_{ab} + \sum_{I=1}^{2} \left(T^{(m,I)}_{\mu\nu} + g^{(I)}_{\mu\nu} V^{(I)}_{D-1}(\phi) \right) \\ &\quad \times e^{(I,\mu)}_a e^{(I,\nu)}_b \sqrt{\left|\frac{g^{(I)}_{D-1}}{g_D}\right|}\, \delta\left(\Phi_I\right) \Bigg\}, \end{aligned} \tag{2.19}$$

where

$$
\begin{aligned}
T^{\phi}_{ab} &= \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \left(\nabla \phi\right)^2 , \\
T^{(m,I)}_{\mu\nu} &= 2 \frac{\delta \mathcal{L}^{(m,I)}_{D-1}}{\delta g^{(I)\,\mu\nu}} - g^{(I)}_{\mu\nu} \mathcal{L}^{(m,I)}_{D-1} ,
\end{aligned}
\tag{2.20}
$$

and $\nabla_a \left(\nabla^{(I)}_{\mu}\right)$ denotes the covariant derivative with respect to $g_{ab} \left(g^{(I)}_{\mu\nu}\right)$.

Variation of the total action with respect to ϕ, on the other hand, yields the Klein-Gordon field equations,

$$
\Box \phi = - \sum_{I=1}^{2} \frac{\partial V^{(I)}_{D-1}(\phi)}{\partial \phi} \sqrt{\left| \frac{g^{(I)}_{D-1}}{g_D} \right|} \, \delta \left(\Phi_I\right),
\tag{2.21}
$$

where $\Box \equiv g^{ab} \nabla_a \nabla_b$. We also have

$$
\nabla^{(I)}_{\nu} T^{(m,I)\,\mu\nu} = 0.
\tag{2.22}
$$

As an example, we have shown explicitly in Appendix C, how the appropriate equations of motion follow from the variation of the action, for the simplest case of the Einstein-Hilbert action. We will next examine the mechanism that we employ to affect the collision of the two timelike branes in our model.

2.2 Brane Collision Mechanics

In this dissertation, we shall consider two colliding branes, located on the two timelike surfaces, $\Phi_1 = 0$ and $\Phi_2 = 0$, where

$$
\begin{aligned}
\Phi_1(t,y) &= t - ay, \\
\Phi_2(t,y) &= t + by,
\end{aligned}
\tag{2.23}
$$

where $a^2 > 1$, and $b^2 > 1$, as shown in Appendix D. As we will see in the following sections, our choices of a and b result in different dynamics and can be used to catalog the phenomenology of the model. This is not unexpected, as these parameters can be thought of as the 'velocity' with which the branes collide.

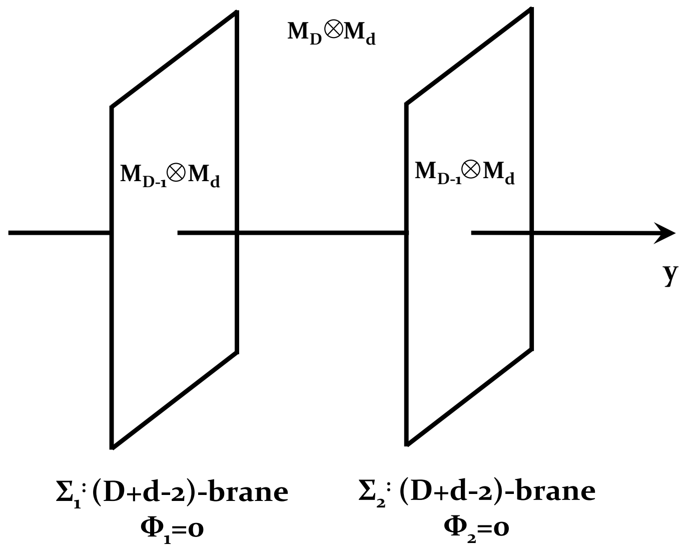


Figure 2.2: A graphical representation of a (D+d)-dimensional spacetime. A (D+d-1)-dimensional surface represents the (D+d-2)-brane, which can be said to represent the observable universe in some cases. In String theory, many such branes are allowed to move independently, collide, recoil and exchange energy and finally form a *brane gas*. The *bulk* is (D+d)-dimensional and can be taken to be a vacuum, or contain any number of scalar or vector fields, all of which have different brane interactions. In this work, the extra dimension will be taken to be y.

To motivate the dynamics of colliding branes, we promote Φ_1 and Φ_2 to Heaviside functions,

$$\begin{aligned} \Phi_1 &\Rightarrow \bar{\Phi}_1 = \Phi_1 H(\Phi_1), \\ \Phi_2 &\Rightarrow \bar{\Phi}_2 = \Phi_2 H(\Phi_2), \end{aligned} \tag{2.24}$$

where we define the Heaviside function as,

$$H(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases} \tag{2.25}$$

By doing this we are essentially *turning on* the brane interaction after the two branes *collide*, at $\Phi_1 = \Phi_2 = 0$, (where we have mapped $\bar{\Phi}_{1,2}$ back to $\Phi_{1,2}$ for simplicity). Before the collision, the equations that describe the physics on the branes are frozen and energy is exchanged between them only after the collision.

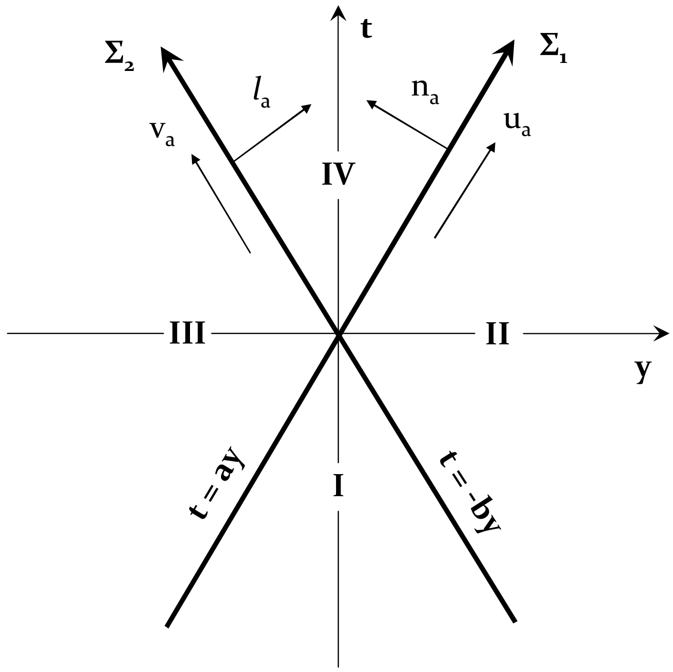


Figure 2.3: The two branes Φ_1 and Φ_2, on the (t, y)-plane for $a > 1$, $b > 1$ follow the trajectories described by Eq.(2.28). The four regions, $I - IV$ are defined by Eq.(2.26). Note that the D+d-2 dimensions not shown in this figure are projected onto the branes that travel along the hypersurfaces Σ_1 and Σ_2.

Note also, that Eq.(2.25) is undefined at $x = 0$. When approaching a spacetime singularity we must turn to Quantum Gravity (QG) [36, 37, 38], which is not well understood yet. However when we regard the spaces from the standpoint of String Theory, we will be able to examine the general characteristics of the cosmological singularity.

The two colliding branes divide the whole spacetime into four regions, $I - IV$, which are defined, respectively, as

$$\begin{aligned}
\text{Region I} &\equiv \{x^a : \Phi_1 < 0,\ \Phi_2 < 0\}, \\
\text{Region II} &\equiv \{x^a : \Phi_1 > 0,\ \Phi_2 < 0\}, \\
\text{Region III} &\equiv \{x^a : \Phi_1 < 0,\ \Phi_2 > 0\}, \\
\text{Region IV} &\equiv \{x^a : \Phi_1 > 0,\ \Phi_2 > 0\}, \qquad (2.26)
\end{aligned}$$

as shown schematically in Fig. 2.3. In each of these regions, we define

$$F^A \equiv F(t,y)\Big|_{Region\ A}, \tag{2.27}$$

where $A = I,\ II,\ III,\ IV$. We also define the hypersurfaces Σ_1 and Σ_2, as the trajectories of the branes,

$$\begin{aligned} \Sigma_1 &= \{x^a : \Phi_1 = 0\}, \\ \Sigma_2 &= \{x^a : \Phi_2 = 0\}. \end{aligned} \tag{2.28}$$

We next define an orthonormal coordinate system on the brane, that will be used to define the direction of energy flow on the branes. It can be shown that these vectors have the following properties,

$$\begin{aligned} n_a n^a &= l_a l^a = -1, && \textit{spacelike vectors}, \\ u_a u^a &= v_a v^a = +1, && \textit{timelike vectors}, \\ n_a u^a &= l_a v^a = \ 0. && \textit{normal to eachother}. \end{aligned} \tag{2.29}$$

We define the normal vectors to each of these two surfaces, n_a and l_a, as,

$$\begin{aligned} n_a &= \frac{\partial \Phi_1}{\partial x^a} = N(\delta^t_a - a\delta^y_a), \\ l_a &= \frac{\partial \Phi_2}{\partial x^a} = L(\delta^t_a + b\delta^y_a), \end{aligned} \tag{2.30}$$

where N and L are normalization terms, that depend on the metric. We also introduce the two timelike vectors u_a and v_a via the relations,

$$\begin{aligned} n_a u^a = 0 \Rightarrow u_a &= N(a\delta^t_a - \delta^y_a), \\ l_a v^a = 0 \Rightarrow v_a &= L(b\delta^t_a + \delta^y_a). \end{aligned} \tag{2.31}$$

Having determined the geometry of our model, we proceed with analyzing the dynamics by determining the equations of motion of the matter fields on each brane.

As we will see in the following sections, we will be able to separate our results into distinct solutions on the regions of Eq.(2.26) and the hypersurfaces of Eq.(2.28). Using distribution theory [75], we will express the Einstein field equations, as,

$$\begin{aligned} G_{ab} &= G_{ab}^{(\Phi_1)}\delta(\Phi_1) + G_{ab}^{(\Phi_2)}\delta(\Phi_2) + G_{ab}^{(IV)}H(u)H(v) + G_{ab}^{(III)}H(u)[1-H(v)] \\ &\quad + G_{ab}^{(II)}H(v)[1-H(u)] + G_{ab}^{(I)}[1-H(u)][1-H(v)], \end{aligned} \tag{2.32}$$

where each region can be examined independently.

2.3 Energy Conditions

Appendix A contains a list of energy-momentum tensors for various types of matter fields, that are frequently used in cosmological models. In the actual universe however, where matter fields are not discretely present or isolated, the total energy momentum tensor of a spacetime, will be the sum of all contributions,

$$T_{ab}^{total} = \sum_{i=1}^{n} T_{ab}^{i} \tag{2.33}$$

Not any given T_{ab} is considered physical, and certain conditions must be imposed, the so-called 'energy conditions' [76]. In the following, we review these conditions, as they apply to a perfect fluid with energy momentum tensor,

$$T_{ab} = (\rho + p)u_a u_b - p g_{ab} \tag{2.34}$$

where ρ is the energy density of the fluid, and p is its pressure.

2.3.1 The Weak Energy Condition

The weak energy condition is equivalent to saying that the energy density, as measured by any observer, is non-negative, and can be expressed as,

$$\begin{aligned} \rho &\geq 0 \\ \rho + p &\geq 0 \end{aligned} \tag{2.35}$$

This condition holds automatically for massless scalar fields.

2.3.2 The Dominant Energy Condition

The dominant energy condition is expressed as,

$$\rho \geq 0$$
$$\rho \pm p \geq 0 \tag{2.36}$$

This statement is considered to be slightly stronger than the weak condition.

Comparing Eq.(2.36) with Eq.(2.35), we can also state that, the dominant energy condition is the weak energy condition, with the additional requirement that the pressure should not exceed the energy density and by extension, that matter cannot travel faster than light. This condition holds for all known forms of matter.

2.3.3 The Strong Energy Condition

The strong energy condition can be interpreted as stating that gravity is attractive. It is expressed as,

$$\rho + p \geq 0$$
$$\rho + 3p \geq 0 \tag{2.37}$$

This condition holds for electromagnetic fields, as well as massless scalar fields. It is notable that this condition holds independently from the weak and dominant conditions.

As we impose these conditions to our models, we must keep an open mind, as higher dimensional solutions often result in branes that contain their own brew of physics. Even if this is the case, it is a very interesting test to perform, even as a baseline comparison of other branes to our own universe.

2.4 The Nature of Spacetime Singularities

Spacetime singularities are divided into two major classes, coordinate singularities and spacetime singularities. The former is due to a bad choice of coordinates and can be removed, while the latter are real spacetime singularities, and cannot be removed by coordinate transformations. Real spacetime singularities can be further divided into two sub-classes: the scalar singularities and the non-scalar singularities.

We will now examine the nature of spacetime singularities and their relationship to the coordinates which are used to describe the manifold we are working in. In certain cases, the equations of general relativity are easier to solve with an appropriate choice of coordinates.

For instance in the Schwarzschild solution, which describes the gravitational field produced by a point-like particle, where coordinates with spherical symmetry (r, θ, ϕ), are appropriate,

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.38}$$

However the coordinate system itself is degenerate, as the coordinates do not cover the axis $\theta = (0, \pi)$, because the line element becomes degenerate there and the metric ceases to be of rank 4 ($\sin(0) = \sin(\pi) = 0$). This degeneracy is easily removable however, with the introduction of Cartesian coordinates (x, y, z). Such singularities are said to be *removable* [77].

To identify the *curvature* singularities, one must construct scalar invariants that are independent of the coordinate system used by which the singular nature of the spacetime can be explored objectively. One may also choose to examine the dynamical nature of the spacetime, to better understand the effects on an observer traveling along a timelike congruence. In the following, we describe how these will be used in our current study.

2.4.1 Scalar Singularities

Even when one starts with well defined energy conditions, it is very difficult to predict the evolution of such systems. Our current understanding of the behavior of matter under extreme conditions of density and pressure is very limited. This is exactly why studies of black holes and the circumstances surrounding the big bang are subject to heated debates in the field [1, 78].

In addition to these qualitative restrictions on the form of physical matter, we may also turn to geometrical considerations in our search for singularities, by examining any number of tensors that encode geometrical information about the universe, in search of areas of high curvature. These divergences are suspected to coincide with curvature singularities, providing us with an extra tool for exploring the nature of the spacetime that exists before and after our branes collide.

Using invariant geometrical methods, we can construct fourteen scalar quantities out of the Riemann tensor, [79, 80], shown explicitly in Appendix E. These scalars are by definition invariant under coordinate transformations, and can thus be used to identify singularities by identifying where they become divergent. The Kretschmann scalar, is the most representative and frequently used invariant curvature scalar,

$$I = R_{abcd}R^{abcd} \tag{2.39}$$

2.4.2 Non-Scalar Singularities

When these scalars vanish, or are non-singular one can not be certain that the space is free from singularities [81]. In such cases, we may attempt to identify the singularities by examining the tidal forces and resulting distortions. In particular, tidal forces experienced by an observer may become infinitely large, under certain conditions [82].

Tidal Forces. To see how the tidal forces arise, we consider the timelike geodesics defined by the projection of our metric onto a free falling frame. We define λ, the proper time of the timelike geodesics and,

$$e^{\mu}_{(0)} = \frac{dx^{\mu}}{d\lambda}, \tag{2.40}$$

the unit vectors that define the free falling frame, with properties,

$$\begin{aligned} e^{\mu}_{(\alpha)} e^{\nu}_{(\beta)} &= \eta_{\alpha\beta}, \\ e^{\mu}_{(\alpha);\nu} e^{\nu}_{(0)} &= 0, \end{aligned} \tag{2.41}$$

where $\eta_{\alpha\beta} = \text{diag.}\{-1, 1, ..., 1\}$. Projecting the Ricci tensor onto the above frame, we find,

$$R_{(\alpha)(\beta)} \equiv R_{\mu\nu} e^{\mu}_{(\alpha)} e^{\nu}_{(\beta)}, \tag{2.42}$$

where now non-scalar singularities can rise, whenever any of the components of $R_{(\alpha)(\beta)}$ is singular.

Distortion. The distortion, which is proportional to the double integral of $R_{(\alpha)(\beta)}$ with respect to proper time λ, is given by,

$$D_{(a)(b)} = \int d\lambda \int R_{(\alpha)(\beta)} d\lambda. \tag{2.43}$$

In combination with the information from the tidal forces that arise, any non-scalar singularities can now be identified. Although this method is mathematically more time consuming, in combination with the study of curvature scalars, it is a robust tool for exploring the singularities in the space.

If any component $R_{(\alpha)(\beta)}$ is singular, but none of $D_{(a)(b)}$ are singular, we call the singularity *weak*, and if any component of $R_{(\alpha)(\beta)}$ is singular, and $D_{(a)(b)}$ is also singular, we call such singularities *strong*.

The remainder of this dissertation has as follows: In Chapter 3, we will first explore these spacetimes in both the string and Einstein frames. We will specifically compare the singularities which arise within each frame, and the dimensionally reduced effective theories, to see if the singularities indeed persist. In certain cases, it is believed [83], that moving to a higher dimensional theory would, in effect, *dilute* some singularities and allow them to be examined. We will systematically analyze each case and present our findings.

In Chapter 4 we will explore a simple brane collision scenario in 5D Einstein spacetime. There, we will explore the singularities that arise, in conjunction with the energy conditions on each brane. This is the first analytical verification of numerical results on brane collisions studied by Maeda et, al [65, 66, 67, 68]. In addition, it will provide us with the general behavior of the branes as they interact outside of the realm of string theory.

In Chapter 5 brane collisions will be explored in the NS-NS sector of string theory, and results will be contrasted and compared to those of Chapter 3 and 4. In Chapter 6 we will provide our conclusions and present our ideas for the future.

CHAPTER THREE

Spacetime Singularities in String Theory and Effective Low Dimensional Theories

In this chapter we will systematically study the singularities present in the NS-NS sector of 11-dimensional Type II string theory, with the characteristics of the space adopted in our model. For the sake of simplicity, we shall consider specific cases in which the effects of the branes are negligible. We will thus derive the effective D-dimensional einstein equations, as they emerge from the dimensional reduction to the spacetime characteristic of our model. In a sense, we are paving the road to our brane collision study, by first establishing the underlying characteristics of this spacetime.

Specifically, spacetime singularities are studied here in both the D+d dimensional string theory and its D-dimensional effective theory, obtained by Kaluza-Klein compactification [84]. It is found that spacetime singularities in the low dimensional effective theory may or may not remain after lifted to the D+d-dimensional string theory, depending on particular solutions. It is also found that in some cases solutions of the low dimensional effective theory are not singular, but after they are lifted to string theory, the higher dimensional spacetimes become singular. Therefore, simply lifting low dimensional effective theories to high dimensions seemingly does not solve the singularity problem, and additional physical mechanisms must be employed.

Now we proceed to the formal development of these arguments and conclusions. In Section 3.1, we study two classes of exact solutions, by paying particular attention to their local and global singular behavior. In Section 3.2, we first lift these solutions to the $D + d$-dimensional spacetime of string theory, and then study their singular behavior. Section 3.3 contains our conclusions and discussing remarks.

3.1 Solutions in D-dimensional Spacetimes in the Einstein Frame

In this section, we shall first construct analytical solutions and then study their local and global properties in the Einstein frame.

The variation of the action in the einstein frame, Eq.(2.10), with respect to $g_{\mu\nu}$ and ϕ yields the D-dimensional Einstein-scalar field equations,

$$R_{\mu\nu} = \kappa_D^2 \phi_{,\mu}\phi_{,\nu}, \tag{3.1}$$

$$\nabla_\lambda \nabla^\lambda \phi = 0, \tag{3.2}$$

where $(\ )_{,\mu} \equiv \partial(\ )/\partial x^\mu$.

In this chapter we consider the D-dimensional spacetimes described by the metric

$$ds^2_{D,E} = 2e^{2\sigma(u,v)} du dv - e^{2h(u,v)} d\Sigma^2_{D-2}, \tag{3.3}$$

where

$$d\Sigma^2_{D-2} \equiv \sum_{i=2}^{D-1} \left(dx^i\right)^2. \tag{3.4}$$

It should be noted that metric (3.3) is invariant under the coordinate transformation,

$$u = f(\bar{u}), \quad v = g(\bar{v}), \tag{3.5}$$

where $f(\bar{u})$ and $g(\bar{v})$ are arbitrary functions of their indicated arguments. Then, the Einstein-scalar equations (3.1) and (3.2) yield,

$$h_{,uv} + (D-2) h_{,u} h_{,v} = 0, \tag{3.6}$$

$$2\phi_{,uv} + (D-2)\left(h_{,u}\phi_{,v} + h_{,v}\phi_{,u}\right) = 0, \tag{3.7}$$

$$h_{,uu} + {h_{,u}}^2 - 2h_{,u}\sigma_{,u} = -\frac{\kappa_D^2}{D-2}{\phi_{,u}}^2, \tag{3.8}$$

$$h_{,vv} + {h_{,v}}^2 - 2h_{,v}\sigma_{,v} = -\frac{\kappa_D^2}{D-2}{\phi_{,v}}^2, \tag{3.9}$$

$$\left(h_{,uv} + h_{,u}h_{,v}\right)(D-2) + 2\sigma_{,uv} = -\kappa_D^2 \phi_{,u}\phi_{,v}. \tag{3.10}$$

From Eq.(3.6) we find that the general solution of $h(u,v)$ is given by

$$h(u,v) = \frac{1}{D-2}\ln\left(F(u) + G(v)\right), \tag{3.11}$$

where $F(u)$ and $G(v)$ are arbitrary functions. To study the solutions further, it is convenient to distinguish the three cases:

(1) $F'(u) \neq 0,\ G'(v) = 0,$

(2) $F'(u) = 0,\ G'(v) \neq 0,$ and

(3) $F'(u)G'(v) \neq 0,$

where a prime denotes the ordinary differentiation. The second case can be obtained from the first one by exchanging the u and v coordinates. Thus, without loss of generality, we need consider only the (1): Class I and (3): Class II, solutions.

3.1.1 *Class I: $F'(u) \neq 0,\ G'(v) = 0$*

In this case, from Eq.(3.9) we find that $\phi = \phi(u)$. Hence, Eq.(3.10) yields

$$\sigma(u, v) = a(u) + b(v), \tag{3.12}$$

where $a(u)$ and $b(v)$ are other arbitrary functions. Using the gauge freedom of Eq.(3.5), without loss of generality we can always set $a(u) = b(v) = 0$, so that this class of solutions are given by

$$\begin{aligned} \sigma(u, v) &= 0, \\ h(u, v) &= \ln \alpha(u), \\ \phi(u, v) &= \pm\sqrt{\frac{D-2}{\kappa_D^2}} \int^u \left(-\frac{\alpha''(u')}{\alpha(u')}\right)^{1/2} du' + \phi_0, \end{aligned} \tag{3.13}$$

where $\alpha(u) \equiv F(u)^{1/(D-2)}$, and ϕ_0 is an integration constant.

It should be noted that for the above solutions all the scalars built from the Riemann curvature tensor are zero, therefore, in the present case scalar curvature singularities are always absent [81]. However, non-scalar curvature singularities might also exist. In particular, tidal forces experienced by an observer may become infinitely large under certain conditions [82].

To see how this can happen, let us consider the timelike geodesics in the (u, v)-plane, which in the present case are simply given by

$$\dot{u} = \gamma_0, \qquad \dot{v} = \frac{1}{2\gamma_0}, \qquad \dot{x}^i = 0, \tag{3.14}$$

where $i = 2, ..., D-1$, γ_0 is an integration constant, and an overdot denotes the ordinary derivative with respect to the proper time, λ, of the timelike geodesics. Defining $e^{\mu}_{(0)} = dx^{\mu}/d\lambda$, we find that the unit vectors, given by

$$\begin{aligned} e^{\mu}_{(0)} &= \gamma_0 \delta^{\mu}_{u} + \frac{1}{2\gamma_0}\delta^{\mu}_{v}, \\ e^{\mu}_{(1)} &= \gamma_0 \delta^{\mu}_{u} - \frac{1}{2\gamma_0}\delta^{\mu}_{v}, \\ e^{\mu}_{(i)} &= \frac{1}{\alpha(u)}\delta^{\mu}_{i}, \end{aligned} \tag{3.15}$$

form a freely falling frame,

$$e^{\mu}_{(\alpha)} e^{\nu}_{(\beta)} g_{\mu\nu} = \eta_{(\alpha)(\beta)}, \qquad e^{\mu}_{(\alpha);\nu}\, e^{\nu}_{(0)} = 0, \tag{3.16}$$

where $\eta_{\alpha\beta} = \text{diag.}\,\{-1,\ 1, ...,\ 1\}$. Projecting the Ricci tensor onto the above frame, we find that

$$\begin{aligned} R_{(\alpha)(\beta)} &\equiv R_{\mu\nu} e^{\mu}_{(\alpha)} e^{\nu}_{(\beta)} \\ &= -\gamma_0^2 (D-2) \left(\frac{\alpha''(u)}{\alpha(u)}\right) [\delta^u_\alpha \delta^u_\beta - \left(\delta^u_\alpha \delta^v_\beta + \delta^v_\alpha \delta^u_\beta\right) + \delta^v_\alpha \delta^v_\beta]. \end{aligned} \tag{3.17}$$

Clearly, the tidal forces remain finite over the whole spacetime, as long as α''/α is finite. A particular case is where

$$\frac{\alpha''(u)}{\alpha(u)} = -\omega^2, \tag{3.18}$$

where ω is a real constant. In this case the solution is given by

$$\begin{aligned} \alpha(u) &= \alpha_0 \sin\left(\omega u + \Delta\right), \\ \phi(u) &= \pm\sqrt{\frac{D-2}{\kappa_D}}\,\omega u + \phi_0, \end{aligned} \tag{3.19}$$

with Δ and ϕ_0 being the integration constants. A representative of the singular case is given by

$$\frac{\alpha''(u)}{\alpha(u)} = -\frac{\omega^2}{(u-u_0)^\gamma}, \tag{3.20}$$

for which we have

$$\phi(u,v) = \phi_0 + \left(\frac{\omega^2(D-2)}{\kappa_D^2}\right)^{1/2} \times \begin{cases} \frac{2}{2-\gamma}(u-u_0)^{1-\gamma/2}, & \gamma \neq 2 \\ \ln(u-u_0), & \gamma = 2 \end{cases} \tag{3.21}$$

where u_0 is an arbitrary constant, and without loss of generality, we can always set $u_0 = 0$. The constants ω and γ have to satisfy the conditions $\alpha(u) > 0$ and $\alpha''(u)/\alpha(u) < 0$, so that the metric has the correct signs and the scalar field is real. When $\gamma = 0$ it reduces to the solution given by Eq.(3.19). Without loss of generality, in the following we study these solutions together.

From Eq.(3.14) we find that $u \sim \gamma_0 \lambda$, where the proper time λ was chosen such that $u = 0$ corresponds to $\lambda = 0$. Then, the distortion, which is proportional to the double integral of $R_{(\alpha)(\beta)}$ with respect to proper time λ, is given by

$$\int d\lambda \int R_{(a)(b)} d\lambda \sim \begin{cases} \lambda(\ln\lambda - 1), & \gamma = 1 \\ \ln\lambda, & \gamma = 2 \\ \lambda^{2-\gamma}, & \gamma \neq 1, 2 \end{cases} \tag{3.22}$$

as $\lambda \to 0$. Thus, as long as $\gamma < 2$, the distortion exerted on the observer is finite.

The interesting case is where $0 < \gamma < 2$, for which the tidal forces become unbound, while the distortion remains finite, as $u \to 0$. In this sense, the singularity at $u = 0$ is usually said to be weak, and the spacetime beyond this surface may be extensible [34]. When $\gamma \geq 2$ however, both the tidal forces and the distortion become unbound, and the singularity now is said to be strong, for which the spacetime beyond $u = 0$ is usually believed to not be extensible. When $\gamma \leq 0$, both the tidal forces and the distortion remain finite at $u = 0$, and the spacetime is free of spacetime singularities.

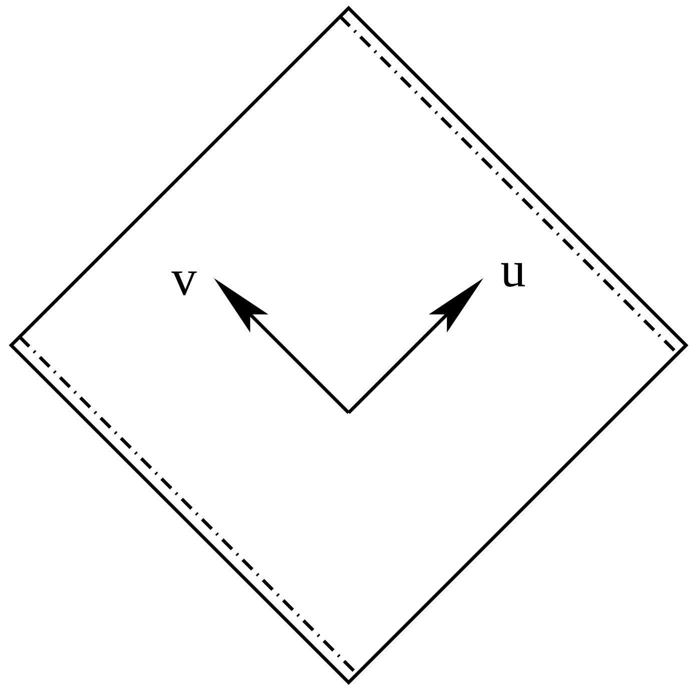


Figure 3.1: The Penrose diagram for the Class Ia, $\gamma = 0$ case of the solution (3.19) in the Einstein frame. The double solid lines denote spacetime singularities, where the distortion exerting on a freely falling observer becomes unbound, although the tidal forces still remain finite there. In the corresponding $(D+d)$-dimensional spacetime the singularity at $u = \infty$ ($u = -\infty$) disappears for $\epsilon_a = +1$ ($\epsilon_a = -1$).

It should be noted that in the case $\gamma = 0$ we have $\phi(u) \to \pm\infty$ as $u \to \pm\infty$. Then, from Eq.(3.22) we can see that the distortion exerted on a free-falling observer becomes unbound, whereas the tidal forces still remain finite (constant). Thus, the spacetime at $u = \pm\infty$ is singular for $\gamma = 0$. The corresponding Penrose diagram in this case is given by Fig. 3.1.

When $\gamma = 2$, Eq.(3.20) has the solution

$$\alpha(u) = \alpha_0 u^{\delta}, \tag{3.23}$$

where $\omega^2 = \delta(1-\delta)$ with $0 < \delta < 1$. Note that the distortion now also becomes unbound at $u = \infty$, although the tidal forces vanish there. Thus, in terms of distortion, the spacetime is singular at $u = \infty$. The corresponding Penrose diagram in this case is given by Fig. 3.2.

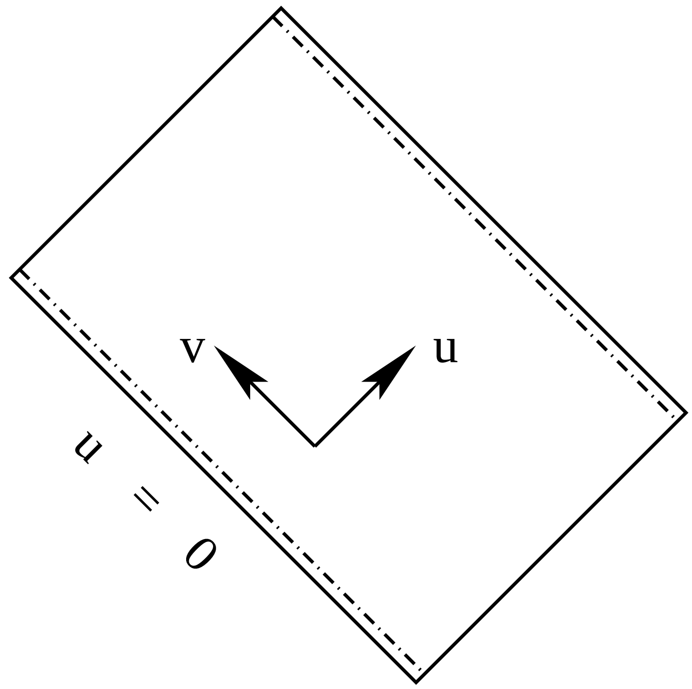


Figure 3.2: The Penrose diagram for the Class Ib, $\gamma = 2$ case of the solution (3.23). The double solid line denotes a spacetime singularities, where both the tidal forces and the distortion exerting on a freely falling observer become unbound. When lifting to $(D+d)$ dimensions, the corresponding Penrose diagram remains the same. In particular, the hypersurfaces $u = 0,\ \infty$ are still singular.

3.1.2 Class II: $F'(u)G'(v) \neq 0$

In this case to solve Eqs.(3.8)-(3.7), it is found convenient first to introduce two new coordinates $\bar{u}$ and $\bar{v}$ via the relations $\bar{u} \equiv F(u)$ and $\bar{v} \equiv G(v)$, using the gauge freedom (3.5). In terms of these new coordinates, the metric (3.3) takes the form,

$$ds^2_{D,E} = 2e^{2\Sigma(\bar{u},\bar{v})}d\bar{u}d\bar{v} - e^{2H(\bar{u},\bar{v})}d\Sigma^2_{D-2}, \tag{3.24}$$

where

$$\begin{aligned} H(\bar{u},\bar{v}) &\equiv h(u,v) = \frac{1}{D-2}\ln\left(\bar{u}+\bar{v}\right), \\ \Sigma(\bar{u},\bar{v}) &\equiv \sigma(u,v) - \frac{1}{2}\ln\left[F'(u)G'(v)\right]. \end{aligned} \tag{3.25}$$

Then, it can be shown that Eqs.(3.7)-(3.8) reduce to

$$
\begin{aligned}
M_{,t} &= \frac{1}{2} t\left({\phi_{,t}}^2 + {\phi_{,y}}^2\right), &(3.26)\\
M_{,y} &= t\phi_{,t}\phi_{,y}, &(3.27)\\
M_{,tt} &- M_{,yy} = -\frac{1}{2}\left({\phi_{,t}}^2 - {\phi_{,y}}^2\right), &(3.28)\\
\phi_{,tt} &+ \frac{1}{t}\phi_{,t} - \phi_{,yy} = 0, &(3.29)
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma &\equiv -\frac{D-3}{2(D-2)}\ln(t) + \kappa_D^2 M,\\
t &\equiv \bar{u} + \bar{v}, \qquad y \equiv \bar{u} - \bar{v}. &(3.30)
\end{aligned}
$$

Eq.(3.28) is the integrability condition of Eqs.(3.26) and (3.27). Thus, once a solution for ϕ is found from Eq.(3.29), then M can be found from Eqs.(3.26) and (3.27) by quadratures.

In the following, we consider three classes of such solutions, to be referred to, respectively, as, Class IIa, IIb, and IIc solutions.

Class IIa Solutions. This class of solutions is given by

$$
\begin{aligned}
M &= \frac{1}{2}\frac{\chi^2}{\kappa_D^2}\ln(t) + M_0,\\
\phi &= \frac{\chi}{\kappa_D}\ln(t) + \phi_0, &(3.31)
\end{aligned}
$$

where $\chi \equiv c\kappa_D$ and c, ϕ_0 and M_0 are integration constants. The solution has a big bang singularity at $t = 0$, as can be seen from the expression,

$$
\begin{aligned}
R_D[g] &\equiv \kappa_D^2 g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\\
&= \frac{A_0}{t^{\left(\chi^2 + \frac{D-1}{D-2}\right)}}, &(3.32)
\end{aligned}
$$

where $A_0 = 2\chi^2/e^{2M_0}$. The corresponding Penrose diagram is given by Fig. 3.3.

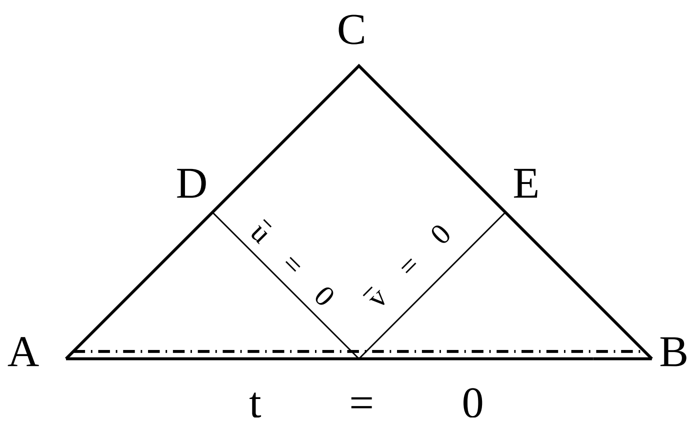


Figure 3.3: The Penrose diagram for the Class IIa solutions given by Eq.(3.31) in D-dimensional spacetime. The horizontal line $t = 0$ represents a big bang singularity.

Class IIb Solutions. lbch:3.2.classIIb This class of solutions is given by

$$\begin{aligned} M &= \frac{1}{2}\frac{\chi^2}{\kappa_D^2}\ln\left(\frac{t^4}{(y^2-t^2)\left(y+\sqrt{y^2-t^2}\right)^2}\right) + M_0, \\ \phi &= \frac{\chi}{\kappa_D}\ln\left(\frac{t^2}{y+\sqrt{y^2-t^2}}\right) + \phi_0, \end{aligned} \tag{3.33}$$

for which we find that

$$\begin{aligned} R_D[g] &\equiv \kappa_D^2 g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \\ &= \frac{2A_0\left(4y^2-3t^2+2\right)}{t^{\left(4\chi^2+\frac{D-1}{D-2}\right)}\left[\sqrt{y^2-t^2}\,\left(y+\sqrt{y^2-t^2}\right)\right]^{1-2\chi^2}} \end{aligned} \tag{3.34}$$

Clearly, the spacetime is singular at $t = 0$. When $\chi^2 < 1/2$, it is also singular on the null hypersurfaces $y^2 = t^2$, and the corresponding Penrose diagram is given by Fig. 3.4.

When $\frac{1}{2} \leq \chi^2 < 1$, the spacetime is not singular at $y = \pm t$, but the metric coefficient Σ is. To extend the metric beyond these surfaces, we can use the gauge freedom (3.5) to introduce two new coordinates $\tilde{u}$ and $\tilde{v}$ via the relations,

$$\bar{u} = \tilde{u}^{2n}, \quad \bar{v} = -\left(-\tilde{v}\right)^{2n}, \tag{3.35}$$

where

$$n \equiv \frac{1}{2(1-\chi^2)}. \tag{3.36}$$

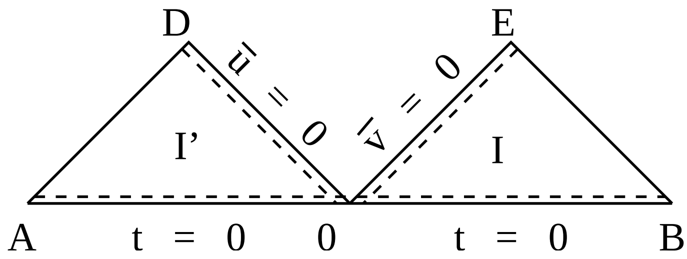


Figure 3.4: The Penrose diagram for the Class IIb solutions given by Eq.(3.33) in the D-dimensional spacetime. The horizontal line $t = 0$ represents a big bang singularity. The spacetime along the line $0D$ and $0E$ are singular for $\chi^2 < 1/2$, and are not singular but represent spacetime null infinities for $\chi^2 \geq 1$. The two regions I and I' are physically disconnected in both cases. When lifted to the (D+d)-dimensional spacetime, with $D = 5 = d$ the solutions are given by Eq.(3.65), and the corresponding spacetime along the line $0D$ and $0E$ become singular for $\chi^2 < 3/4$, although it is still not singular but represent spacetime null infinities for $\chi^2 \geq 1$.

It can be shown that in terms of $\tilde{u}$ and $\tilde{v}$ the coordinate singularity at $y = \pm t$ disappears, and the solutions are valid over the whole half plane $t > 0$, by simply taking $-\infty < \tilde{u},\ \tilde{v} < +\infty$. Then, the corresponding Penrose diagram of the extended solutions is that of Fig. 3.3.

When $\chi^2 \geq 1$, the hypersurfaces $y = \pm t$ represent spacetime null infinities, and the solutions are already geodesically maximal. Indeed, it is found that the null geodesics $\bar{u} = constant$ have the integral,

$$\eta = \begin{cases} \eta_0 \left(-\bar{v}\right)^{1-\chi^2}, & \chi^2 > 1 \\ \eta_0 \ln(-\bar{v}), & \chi^2 = 1 \end{cases} \tag{3.37}$$

near the hypersurface $y = t$ ($\bar{v} = 0$), where η_0 is an integration constant, and η denotes the affine parameter along the null geodesics. Thus, as $\bar{v} \to 0^-$, we always have $|\eta| \to \infty$.

Similar, it can be shown that the hypersurface $y = -t$ also represents a null infinity of the spacetime. Therefore, the corresponding Penrose diagram is given by Fig. 3.4, but the hypersurfaces $y = \pm t$ now are not singular, although the two regions I and I' are still disconnected.

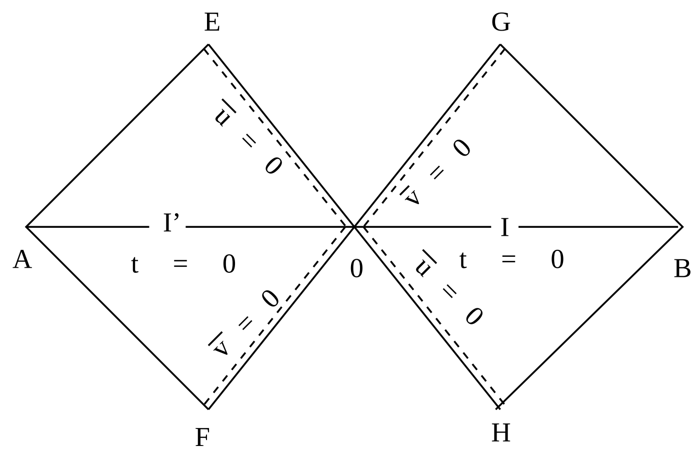


Figure 3.5: The Penrose diagram for the ClassIIc solutions given by Eq.(3.38) in the D-dimensional spacetime. On the null hypersurfaces $\bar{u} = 0$ and $\bar{v} = 0$ the spacetime is singular for $\chi^2 < 1/2$, and not singular for $\chi^2 \geq 1$. In the latter case, these surfaces represent the spacetime null infinities.

Class IIc Solutions. This class of solutions is given by

$$
\begin{aligned}
M &= \frac{1}{2}\frac{\chi^2}{\kappa_D^2} \ln\left(\frac{\left(y + \sqrt{y^2 - t^2}\right)^2}{y^2 - t^2}\right) + M_0, \\
\phi &= \frac{\chi}{\kappa_D} \ln\left(y + \sqrt{y^2 - t^2}\right) + \phi_0,
\end{aligned} \tag{3.38}
$$

for which we have

$$
\begin{aligned}
R_D[g] &\equiv \kappa_D^2 g^{\alpha\beta} \phi_{,\alpha}\phi_{,\beta} \\
&= -\frac{2A_0\, t^{\frac{D-3}{D-2}}\, (y^2 - t^2)^{\chi^2 - 1/2}}{\left(y + \sqrt{y^2 - t^2}\right)^{2\chi^2 + 1}}.
\end{aligned} \tag{3.39}
$$

Clearly, the spacetime is no longer singular at $t = 0$, and the solution needs to be extended across $t = 0$ to the region $t < 0$. Eqs.(3.26)-(3.29) show that if $(M(t, y),\ \phi(t, y))$ is a solution, so is $(M(-t, y),\ \phi(-t, y))$. From this observation, we can see that the solutions in the region $t < 0$ can be obtained from the ones in the region $t > 0$ by simply replacing t by $-t$.

On the other hand, Eq.(3.39) show that the solutions may also be singular on the hypersurfaces $y = \pm t$, depending on the choice of χ. The singular behavior of

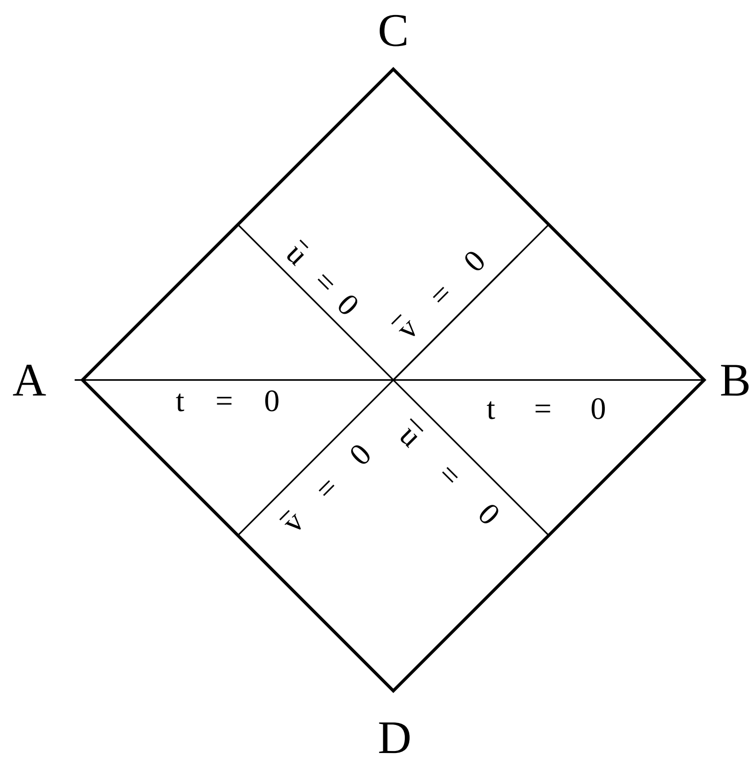


Figure 3.6: The Penrose diagram for the Class IIc solutions given by Eq.(3.38) for $\frac{1}{2} \leq \chi^2 < 1$ in the D-dimensional spacetime.

these solutions at these surfaces is similar to the ones given in the last subsection. In particular, when $\chi^2 < 1/2$, it is singular and the corresponding Penrose diagram is given by Fig. 3.5.

When $\frac{1}{2} \leq \chi^2 < 1$, the spacetime is not singular at $y = \pm t$, although the metric coefficients are. The extension can also be done by introducing the new coordinates $\tilde{u}$ and $\tilde{v}$ defined by Eq.(3.35). Then, the corresponding Penrose diagram of the extended solutions is that of Fig. 3.6.

When $\chi^2 \geq 1$, the hypersurfaces $y = \pm t$ already represent the null infinities of the spacetime, and the corresponding Penrose diagram is that of Fig. 3.5, but now with the hypersurfaces $\bar{u} = 0$ and $\bar{v} = 0$ being non-singular.The two regions, I and I', are still not connected.

3.2 Solutions in $(D+d)$-dimensional Spacetimes in the String Frame

From Eqs.(2.9) and (2.11) we can easily solve the transformation equations, so that from the Einstein solutions we can find the equivalent String frame solutions,

$$
\begin{aligned}
g_{\mu\nu} &= \exp\left\{\epsilon_a \left(\frac{4\kappa_D^2 d}{(D-2)(D+d-2)}\right)^{1/2} \phi\right\} \gamma_{\mu\nu}, \\
\hat{\Phi} &= \exp\left\{-\epsilon_a \left(\frac{\kappa_D^2 (D-2)}{(D+d-2)d}\right)^{1/2} \phi\right\},
\end{aligned}
\tag{3.40}
$$

where $\epsilon_a = \pm 1$. We now repeat our study in the string frame and compare results.

3.2.1 Class I: $F'(u) \neq 0,\ G'(v) = 0$

In this case, the modulus $\phi(u,v)$ is given by Eq.(3.21). In the $(D+d)$-dimensional spacetime, the metric can be written in the form

$$
d\hat{s}^2_{D+d} = 2d\hat{u}d\hat{v} - e^{2\hat{h}(\hat{u})} d^2\Sigma_{D-2} - \Phi^2(\hat{u})\gamma_{ab}(z)dz^a dz^b, \tag{3.41}
$$

where

$$
\begin{aligned}
d\hat{u} &\equiv e^{2\hat{\sigma}} du, \\
\hat{\sigma} &\equiv \sigma - \frac{d}{D-2}\ln\Phi, \\
\hat{h} &\equiv h - \frac{d}{D-2}\ln\Phi.
\end{aligned}
\tag{3.42}
$$

Following what we did in the Einstein frame, we can construct a free-falling frame in the $(D+d)$-dimensions, given by

$$
\begin{aligned}
e^A_{(0)} &= \hat{\gamma}_0 \delta^A_{\hat{u}} + \frac{1}{2\hat{\gamma}_0}\delta^A_v, \\
e^A_{(1)} &= \hat{\gamma}_0 \delta^A_{\hat{u}} - \frac{1}{2\hat{\gamma}_0}\delta^A_v, \\
e^A_{(i)} &= e^{-\hat{h}}\delta^A_i, \\
e^A_{(b)} &= \Phi^{-1}\delta^A_b,
\end{aligned}
\tag{3.43}
$$

which satisfy the relations,

$$
e^A_{(C)} e^B_{(D)} \hat{g}_{AB} = \eta_{CD}, \qquad e^A_{(C);B} e^B_{(0)} = 0, \tag{3.44}
$$

where $\hat{\gamma}_0$ is another integration constant. Then, it can be shown that the Riemann tensor in this case has only two independent components, given by

$$\begin{aligned}
\hat{R}^{(i)}{}_{(0)(j)(0)} &= -\hat{\gamma}_0^2\left(\hat{h}_{,\hat{u}\hat{u}} + \hat{h}^2_{,\hat{u}}\right)\delta^i_j, \\
\hat{R}^{(a)}{}_{(0)(b)(0)} &= -\hat{\gamma}_0^2\left(\frac{\Phi_{,\hat{u}\hat{u}}}{\Phi}\right)\delta^a_b.
\end{aligned} \tag{3.45}$$

To study the solutions further, it is found convenient to consider the two cases $\gamma \neq 2$ and $\gamma = 2$ separately.

Class Ia: $\gamma \neq 2$. In this case, we have

$$\begin{aligned}
d\hat{u} &= e^{au^{(1-\gamma/2)}}du, \\
\hat{h} &= \frac{1}{2}au^{1-\gamma/2} + \alpha(u), \\
\Phi &= e^{bu^{1-\gamma/2}},
\end{aligned} \tag{3.46}$$

where

$$\begin{aligned}
a &\equiv \frac{2\epsilon_a}{2-\gamma}\left(\frac{4\omega^2 d}{D+d-2}\right)^{1/2}, \\
b &\equiv -\frac{2\epsilon_a}{2-\gamma}\left(\frac{\omega^2(D-2)^2}{(D+d-2)d}\right)^{1/2}.
\end{aligned} \tag{3.47}$$

Then, the non-vanishing frame components of the Riemann tensor are given by

$$\begin{aligned}
\hat{R}^{(i)}{}_{(0)(j)(0)} &= \hat{\gamma}_0^2 e^{-2au^{1-\gamma/2}}\left\{\frac{a\gamma(2-\gamma)}{8u^{\gamma/2+1}} + \frac{a^2(2-\gamma)^2 + 16\omega^2}{16u^\gamma}\right\}\delta^i_j, \\
\hat{R}^{(a)}{}_{(0)(b)(0)} &= \hat{\gamma}_0^2\frac{b(2-\gamma)}{4}e^{-2au^{1-\gamma/2}}\left\{\frac{\gamma}{u^{\gamma/2+1}} - \frac{(b-a)(2-\gamma)}{u^\gamma}\right\}\delta^a_b.
\end{aligned} \tag{3.48}$$

Therefore, for the choice $\epsilon_a = +1$ we have

$$\hat{R}^{(A)}{}_{(0)(B)(0)} = \begin{cases} 0, & \gamma > 2 \\ \infty, & 0 < \gamma < 2 \\ \text{constant}, & \gamma = 0 \\ \infty, & -2 < \gamma < 0 \\ \text{finite}, & \gamma \leq -2, \end{cases} \tag{3.49}$$

as $u \to 0$, but now with $A, B = i, a$. Thus, the tidal forces experienced by a free-falling observer remain finite in the string frame at $u = 0$ for all the cases, except for the ones where $0 < \gamma < 2$ or $-2 < \gamma < 0$. As a result, the spacetime is singular at $u = 0$ for these latter solutions. However, the singularity is weak, because the distortion exerted on the observer is still finite,

$$\begin{aligned}\int d\lambda \int \hat{R}^{(A)}{}_{(0)(B)(0)} d\lambda &\sim A_1 \lambda^{2-\gamma} + A_2 \lambda^{1-\gamma/2} \\ &\sim \text{finite}, \end{aligned} \tag{3.50}$$

as $\lambda \to 0$ (or $u \to 0$) for $0 < \gamma < 2$ and $-2 < \gamma < 0$, where A_1 and A_2 are finite constants.

Therefore, for the choice $\epsilon_a = +1$ the strong singularities of the solutions with $\gamma > 2$ at $u = 0$ in the Einstein frame now disappear in the string frame and the corresponding spacetime becomes regular there. The singularities of the solutions with $0 < \gamma < 2$ are weak in both of the two frames. The solutions with $-2 < \gamma < 0$ is free from singularities in the Einstein frame, while they become singular in the string frame, although they remain weak in nature. The solutions with $\gamma = 0$ and $\gamma \leq -2$ are free from singularity at $u = 0$ in both of the two frames.

Note that for $\gamma = 0$ Eq.(3.48) shows that

$$\hat{R}^{(A)}{}_{(0)(B)(0)} \to 0, \tag{3.51}$$

as $u \to \infty$. Thus, in this case the spacetime singularity at $u = \infty$ that appears in the Einstein frame now disappears in the $(D+d)$-dimensional string frame, although the null infinity $u = -\infty$ still remains singular [cf. Fig. 3.1].

When $\epsilon_a = -1$, from Eq.(3.48) we find that

$$\hat{R}^{(A)}{}_{(0)(B)(0)} = \begin{cases} \infty, & \gamma > 0, \\ \text{constant}, & \gamma = 0, \\ \infty, & -2 < \gamma < 0, \\ \text{finite}, & \gamma \leq -2, \end{cases} \tag{3.52}$$

as $u \to 0$. It can be shown that in this case the nature of the singularities of the solutions remains the same in both of the two frames for $\gamma \geq 0$ and $\gamma \leq -2$, that is, in both frames it is strong for $\gamma > 2$, weak for $0 < \gamma < 2$, and free of singularities for $\gamma = 0$ and $\gamma \leq -2$. For $-2 < \gamma < 0$, the solutions are free of singularities in the Einstein frame, but singular in the string frame with the nature of the singularities being still weak.

Similarly, one can show that for $\gamma = 0$ the spacetime singularity at $u = -\infty$ appearing in the Einstein frame now disappears in the $(D+d)$-dimensional string frame, although the spacetime is still singular at the null infinity $u = +\infty$ [cf. Fig. 3.1].

Class Ib: $\gamma = 2$. When $\gamma = 2$, the corresponding solutions in the Einstein frame are given by Eqs.(3.22) and (3.23). The solutions have a strong singularity at $u = 0$. In the string frame, the corresponding solutions are given by Eq.(3.41) but with

$$\begin{aligned} \hat{h}(\hat{u}) &= \frac{a+2\delta}{2(1+a)} \ln|\hat{u}|, \\ \Phi(\hat{u}) &= [(1+a)\hat{u}]^{\frac{b}{1+a}}, \end{aligned} \tag{3.53}$$

where

$$\hat{u} \equiv \frac{1}{1+a} u^{1+a} = \begin{cases} 0, & a > -1, \\ -\infty, & a < -1, \end{cases} \tag{3.54}$$

as $u \to 0$. Thus, when $a > -1$ the half plane $u \geq 0$ is mapped to the half plane $\hat{u} \geq 0$, and the hypersurface $u = 0$ ($u = \infty$) is mapped to the one $\hat{u} = 0$ ($\hat{u} = \infty$). when $a < -1$ the half plane $u \geq 0$ is mapped to the one $\hat{u} \leq 0$, and the hypersurface $u = 0$ ($u = \infty$) corresponds to the one $\hat{u} = -\infty$ ($\hat{u} = 0$).

It can be shown that now we have

$$
\begin{aligned}
\hat{R}^{(i)}{}_{(0)(j)(0)} &= -\hat{\gamma}_0^2 \frac{(a+2\delta)(2\delta - a - 2)}{4(1+a)^2 \hat{u}^2} \delta^i_j, \\
\hat{R}^{(a)}{}_{(0)(b)(0)} &= -\hat{\gamma}_0^2 \frac{b(b-a-1)}{(1+a)^2 \hat{u}^2} \delta^a_b. \qquad (3.55)
\end{aligned}
$$

Clearly, the spacetime is singular at $\hat{u} = 0$, and the nature of the singularity is strong, because

$$
\int d\lambda \int \hat{R}^{(A)}{}_{(0)(B)(0)} d\lambda \sim \ln \lambda \to -\infty, \qquad (3.56)
$$

as $\lambda \to 0$ (or $\hat{u} \to 0$). Note that the distortion also becomes unbound as $|\hat{u}| \to \infty$ ($|\lambda| \to \infty$), although the tidal forces vanish there.

It should be noted that the above analysis is valid only for $a \neq -1$. When $a = -1$, we find that

$$
\begin{aligned}
\omega^2 &= \frac{D+d-2}{4d}, \\
b &= \frac{D-2}{2d}. \qquad (3.57)
\end{aligned}
$$

The corresponding solutions are given by

$$
\begin{aligned}
\hat{h}(\hat{u}) &= \left(\delta - \frac{1}{2}\right) \hat{u}, \\
\Phi(\hat{u}) &= e^{\frac{D-2}{2d} \hat{u}}, \qquad (3.58)
\end{aligned}
$$

where $u \equiv e^{\hat{u}}$. Then, we find that

$$
\begin{aligned}
\hat{R}^{(i)}{}_{(0)(j)(0)} &= -\hat{\gamma}_0^2 \left(\delta - \frac{1}{2}\right)^2 \delta^i_j, \\
\hat{R}^{(a)}{}_{(0)(b)(0)} &= -\hat{\gamma}_0^2 \left(\frac{D-2}{2d}\right)^2 \delta^a_b, \qquad (3.59)
\end{aligned}
$$

which are finite (constants). However, at the null infinities $\hat{u} = \pm\infty$, which correspond, respectively, to $u = 0$ and $u = \infty$, the distortions are still unbound. As a result, the $(D+d)$-dimensional spacetimes remain singular on these surfaces.

Therefore, when $\gamma = 2$ the corresponding Penrose diagram of the $(D+d)$-dimensional spacetimes is that of Fig. 3.1, where the two hypersurfaces $u = 0$ and $u = \infty$ remain singular.

3.2.2 Class II: $F'(u)G'(v) \neq 0$

Three classes of solutions were studied in the last section. To make them more manageable, in this subsection we shall restrict ourselves only to $D = d = 5$. We generalize the metric (3.41) and using $t = u + v$, $y = u - v$ we arrive at the form:

$$d\hat{s}_{10}^2 = \frac{1}{2}e^{2A(t,y)}(dt^2 - dy^2) - e^{2B(t,y)}d\Sigma_3^2 - e^{2C(t,y)}d\Sigma_{5,z}^2, \tag{3.60}$$

where $d\Sigma_{5,z}^2 \equiv \gamma_{ab}\,(z^c)\,dz^a dz^b$, $(a, b = 1, 2, ..., 5)$, and

$$\begin{aligned} A &= \sigma - \frac{5}{3}\beta\phi = -\frac{1}{3}\ln(t) + \kappa_5^2 M - \frac{5}{3}\beta\phi, \\ B &= h - \frac{5}{3}\beta\phi = \frac{1}{3}\ln(t) - \frac{5}{3}\beta\phi, \\ C &= \beta\phi, \qquad \beta = \epsilon\sqrt{\frac{3\kappa_5^2}{40}}, \qquad \epsilon_a = \pm 1. \end{aligned} \tag{3.61}$$

Class IIa Solutions. In this case, substituting the solution (3.31) into Eq.(3.61) and setting $M_0 = \phi_0 = 0$ without loss of generality, we obtain

$$\begin{aligned} A &= \frac{1}{2}\left[\left(\chi - \epsilon_a\sqrt{\frac{5}{24}}\right)^2 - \frac{7}{8}\right]\ln(t), \\ B &= \left(\frac{1}{3} - \epsilon_a\sqrt{\frac{5}{24}}\chi\right)\ln(t), \\ C &= \epsilon_a\sqrt{\frac{3}{40}}\chi\ln(t). \end{aligned} \tag{3.62}$$

The corresponding Kretschmann scalar is given by,

$$I_{10} \equiv R_{ABCD}R^{ABCD} = \frac{\tilde{I}_{10}}{t^{\alpha_0}}, \tag{3.63}$$

where

$$\tilde{I}_{10} = \tfrac{1}{45}\left[\, 9\chi^4(40\chi^2 + 143) \; + 80(5\chi^2 + 2) - \epsilon_a 52\sqrt{30}\chi^3(3\chi^2 + 4)\, \right]$$
$$\alpha_0 = 2\Big(\chi - \epsilon_a\sqrt{\tfrac{5}{24}}\Big)^2 + \tfrac{9}{4} > 0 \qquad . \quad (3.64)$$

Clearly, the spacetime is always singular at $t = 0$ for any given χ, similar to that in the 5-dimensional case. Therefore, in the present case, the spacetime singularity remains even after lifted from the effective 5-dimensional spacetime to the 10 dimensional bulk.

Class IIb Solutions. In this case, the combination of Eqs.(3.33) and (3.61) yields

$$\begin{aligned}
A &= \left[2\Big(\chi - \epsilon_a\sqrt{\frac{5}{96}}\Big)^2 - \frac{7}{16}\right]\ln(t) - \frac{1}{2}\chi^2\ln\left(y^2 - t^2\right) \\
&\quad -\left[\Big(\chi - \epsilon_a\sqrt{\frac{5}{96}}\Big)^2 - \frac{5}{96}\right]\ln\left(y + \sqrt{y^2 - t^2}\right), \\
B &= \Big(\frac{1}{3} - \epsilon_a\sqrt{\frac{5}{6}}\chi\Big)\ln(t) - \epsilon_a\sqrt{\frac{5}{24}}\chi\ln\left(y + \sqrt{y^2 - t^2}\right), \\
C &= \epsilon_a\sqrt{\frac{3}{10}}\chi\ln(t) - \epsilon_a\sqrt{\frac{3}{40}}\chi\ln\left(y + \sqrt{y^2 - t^2}\right),
\end{aligned} \qquad (3.65)$$

for which we find that

$$I_{10} = \frac{\tilde{I}_{10}}{t^{\alpha_0}(y^2 - t^2)^{\alpha_1}(y + \sqrt{y^2 - t^2})^{\alpha_2}}, \qquad (3.66)$$

where α_0 is given by Eq.(3.64),

$$\begin{aligned}
\alpha_1 &\equiv 2\Big(\frac{3}{4} - \chi^2\Big), \\
\alpha_2 &\equiv \frac{101}{24} - \Big(2\chi - \epsilon_a\sqrt{\frac{5}{24}}\Big)^2,
\end{aligned} \qquad (3.67)$$

and $\tilde{I}_{10} = \tilde{I}_{10}(t, y)$, which is non-zero for $t = 0$ and $y^2 = t^2$, but its expression is too complicated to give it here explicitly.

From the above expression, it can be seen that the spacetime is always singular at $t = 0$, but the strength of the singularity for $y > 0$ and $y < 0$ is different, because when $t = 0$ we have $y + \sqrt{y^2 - t^2} = 0$ for $y \le 0$ and $y + \sqrt{y^2 - t^2} \neq 0$ for $y > 0$. In particular, when $t = 0$ and $y > 0$, we find that

$$I_{10}\Big|_{t=0,\ y>0} \simeq \frac{\tilde{I}_{10}}{t^{\alpha_0}}, \tag{3.68}$$

where

$$\begin{aligned}\tilde{I}_{10}\Big|_{t=0} &= \frac{256}{45} y^8 \big[\, 9\chi^4(160\chi^2 + 143) + 10(10\chi^2 + 1) \\ &\qquad -\epsilon_a 104\sqrt{30}\chi^3(3\chi^2 + 1)\big].\end{aligned} \tag{3.69}$$

On the other hand, when $t = 0$ and $y < 0$, we find that

$$I_{10}\Big|_{t=0,\ y<0} \simeq \frac{y^{\alpha_2 - 2\alpha_1}\tilde{I}_{10}}{t^{32/3}}, \tag{3.70}$$

where $\tilde{I}_{10}$ is still given by Eq.(3.69).

Eqs.(3.66)- (3.67) also show that the spacetime is singular when $y^2 - t^2 = 0$ for $\chi^2 < 3/4$,

$$I_{10}\Big|_{t^2=y^2} \propto \frac{\tilde{I}_{10}}{(y^2 - t^2)^{\alpha_1}}, \tag{3.71}$$

but now with

$$\tilde{I}_{10}\Big|_{t^2=y^2} = \pm\frac{8}{15}t^7\chi^2\Big[20(6\chi^4 - \chi^2 - 1) - 13\epsilon_a\sqrt{30}\chi(2\chi^4 - 1)\Big]. \tag{3.72}$$

The corresponding Penrose diagram for $\chi^2 < 3/4$ is given by Fig.3.4. It is remarkable to note that the solutions with $1/2 \le \chi^2 < 3/4$ is not singular in the 5-dimensional effective theory, as shown explicitly in the Class IIb solutions in the Einstein frame.

Since the solution in the 10-dimensional bulk is not singular across the hypersurfaces $y^2 = t^2$ for $\chi^2 \ge 3/4$, one must extend the solutions beyond these surfaces. The extension is quite similar to the 5-dimensional case for the ones with $\chi^2 \ge 1/2$. In particular, for $3/4 \le \chi^2 < 1$, setting

$$\bar{u} = (y + t)^{2n}, \qquad \bar{v} = (y - t)^{2n}, \tag{3.73}$$

where n is given by Eq.(3.36), one can show that the coordinate singularity at $y^2 = t^2$ disappears in terms of $\bar{u}$ and $\bar{v}$. Then, the Penrose diagram for the extended solutions is given exactly by Fig. 3.3.

When $\chi^2 \geq 1$, the hypersurfaces $y^2 = t^2$ already represent the null infinities, and the corresponding Penrose diagram is given by Fig. 3.4, but now the hypersurfaces $y^2 = t^2$, represent, respectively, by the lines $0D$ and $0E$, are non-singular. The two regions I and I' are physically disconnected.

Class IIc Solutions. In this case, from Eq.(3.38) we find that

$$\begin{aligned} A &= -\frac{1}{3}\ln(t) - \frac{1}{2}\chi^2 \ln(y^2 - t^2) \\ &\quad + \left[\left(\chi - \epsilon_a\sqrt{\frac{5}{96}}\right)^2 \frac{5}{96}\right] \ln\left(y + \sqrt{y^2 - t^2}\right), \\ B &= \frac{1}{3}\ln(t) - \epsilon_a\sqrt{\frac{5}{24}}\chi \ln\left(y + \sqrt{y^2 - t^2}\right), \\ C &= \epsilon_a\sqrt{\frac{3}{40}}\chi \ln\left(y + \sqrt{y^2 - t^2}\right), \end{aligned} \tag{3.74}$$

and that

$$I_{10} = \frac{\tilde{I}_{10}}{t^{8/3}(y^2 - t^2)^{\alpha_1}(y + \sqrt{y^2 - t^2})^{\alpha_3}}, \tag{3.75}$$

where

$$\alpha_3 \equiv \frac{91}{24} + \left(2\chi - \epsilon_a\sqrt{\frac{5}{24}}\right)^2 > 0, \tag{3.76}$$

and $\tilde{I}_{10} = \tilde{I}_{10}(t, y)$ is non-zero and finite at $t = 0$, but too complicated to be written out here.

It is very surprising to note that the spacetime is now always singular on the hypersurface $t = 0$, in contrast to the 5-dimensional case in which the spacetime is free of spacetime singularities, as shown explicitly in the Class IIc solutions in the Einstein frame. The strength of the singularities once again depend on $y < 0$ and $y > 0$. In particular, when $t = 0$ and $y > 0$, we find that

$$I_{10}\Big|_{t=0, y>0} \propto \frac{\tilde{I}_{10}}{t^{8/3}}, \tag{3.77}$$

with $\tilde{I}_{10} = -512y^7/9$. But for $t = 0$ and $y < 0$, we find that

$$I_{10}\Big|_{t=0,y<0} = \frac{\tilde{I}_{10}(y - \sqrt{y^2 - t^2})^{\alpha_3}}{t^{8/3+2\alpha_3}(y^2 - t^2)^{\alpha_1}}, \tag{3.78}$$

where $\alpha_3 > 0$, as shown by Eq.(3.76). Therefore, we again have a spacetime singularity at $t = 0$ for $y < 0$, but with more singular strength.

The singular behavior of the spacetime along the hypersurfaces $y = \pm t$ are similar to the last case. In particular, it is singular for $\chi^2 < 3/4$, and corresponding Penrose diagram is given by Fig.3.4. It is interesting to note again that the solution with $1/2 \leq \chi^2 < 3/4$ is not singular in the 5-dimensional effective theory.

When $3/4 \leq \chi^2 < 1$ the corresponding solutions in the 10-dimensional bulk are not singular across the hypersurfaces $y^2 = t^2$, and one must extend the solutions beyond these surfaces. The extension is quite similar to the last case, where two null coordinates were introduced, defined by Eq.(3.73). Then, the Penrose diagram for the extended solutions is given exactly by Fig. 3.3.

When $\chi^2 \geq 1$, the hypersurfaces $y^2 = t^2$ already represent the null infinities, and the corresponding Penrose diagram is given by Fig.3.4.

3.3 Conclusions and Discussing Remarks

According to the singularity theorems [76, 85], 4-dimensional spacetimes are generically singular in the presence of a matter field. In the framework of string/M theory, it is generally hoped that such singularities may disappear when the spacetimes are lifted to high dimensions.

In this Chapter, we have investigated this problem, by first studying the local and global properties of the spacetimes in the low dimensional effective theory, and then lifting them to a corresponding higher dimensional spacetime. We have shown explicitly that spacetime singularities may or may not remain after lifted to higher dimensions, depending on the particular solutions considered. We have also found that there exist cases in which the spacetimes of a low dimensional effective theory do not possess any singularities, but when lifted to string theory, new spacetime singularities arise [86].

These results strongly indicate that spacetime singularities cannot be made disappear by simply lifting the spacetimes to higher dimensions, and new physical mechanisms are needed, in order to solve the singularity problem of the low dimensional theories [87].

CHAPTER FOUR

Colliding Branes and Formation of Spacetime Singularities in 5 Dimensions

As mentioned previously, studies of brane collisions are very mathematically involved. To see clearly how both dimensions and branes affect the formation of spacetime singularities, in the last chapter we assumed that the effects of branes are negligible. In this chapter we shall restrict ourselves to the collision of two 3-branes in a 5-dimensional bulk, and take the backreaction of branes into account.

We present a class of analytic solutions to the 5-dimensional Einstein field equations, which represents the collision of two timelike 3-branes in a 5-dimensional vacuum bulk, and show explicitly that a spacelike singularity always develops after the collision due to the mutual focus of the two branes, when both of them satisfy the energy conditions. If only one of them satisfies the energy conditions, spacetime singularities always exist too, but these singularities may appear either before or after the collision. Non-singular spacetimes can be constructed only in the case where both of the two branes violate the energy conditions.

Specifically, this chapter is organized as follows: in Section 4.1 we first present such solutions, and then study their local and global properties, while in Section 4.2 we present our main conclusions and remarks. Once again, we make extensive use of the Appendices in this chapter also, to preserve the continuity of the main ideas behind this study.

4.1 Colliding Timelike 3-Branes

Let us consider the solutions (Appendix F),

$$ds_5^2 = A^{-2/3}(t,y)\left(dt^2 - dy^2\right) - A^{2/3}(t,y)d\Sigma_0^2, \tag{4.1}$$

where $d\Sigma_0^2 \equiv (dx^2)^2 + (dx^3)^2 + (dx^4)^2,\ \ x^A = \{t, y, x^i\},\ (i = 2, 3, 4)$, and

$$A(t, y) = a\,(t + by)\,H\,(t + by) + b\,(t - ay)\,H\,(t - ay)\,A_0, \tag{4.2}$$

with $a,\ b$ and A_0 being arbitrary constants, and $H(x)$ the Heavside function, defined as in Eq.(2.25).

Without loss of generality, we assume $a \neq -b$ and $A_0 > 0$. Then, it can be shown that the corresponding spacetime is vacuum, except on the hypersurfaces $t = ay$ and $t = -by$, where the non-vanishing components of the Einstein tensor are given by,

$$\begin{aligned} G_{00} &= -ab\left[\frac{a\delta(t-ay)}{A} + \frac{b\delta(t+by)}{A}\right], \\ G_{01} &= ab\left[\frac{\delta(t-ay)}{A} - \frac{\delta(t+by)}{A}\right], \\ G_{11} &= -\left[\frac{b\delta(t-ay)}{A} + \frac{a\delta(t+by)}{A}\right], \\ G_{ij} &= \frac{1}{3}A^{1/3}\delta_{ij}\Big[b(a^2-1)\delta(t-ay) + a(b^2-1)\delta(t+by)\Big], \end{aligned} \tag{4.3}$$

where $\delta(x)$ denotes the Dirac delta function.

These equations can the thought of as being 'activated' on the branes, since the delta functions restrict their extent to the hypersurfaces $\Phi_I = 0$. The quantity $A(t, y)$, takes on different values depending on the signs of $\Phi_I = 0$, once again, determining the dynamics of the interaction of the branes.

As to be explained below, with the proper choice of the free parameters a and b, on each of these two hypersurfaces the spacetime represents a 3-brane filled with a perfect fluid.

The normal vectors to the surfaces $t - ay = 0$ and $t + by = 0$ are given by,

$$\begin{aligned} n_A &\equiv \frac{\partial(t-ay)}{\partial x^A} = \delta^t_A - a\delta^y_A, \\ l_A &\equiv \frac{\partial(t+by)}{\partial x^A} = \delta^t_A + b\delta^y_A, \end{aligned} \tag{4.4}$$

respectively. For which we find

$$
\begin{aligned}
n_A n^A &= -A^{2/3}\left(a^2-1\right), \\
l_A l^A &= -A^{2/3}\left(b^2-1\right).
\end{aligned}
\tag{4.5}
$$

Thus, in order to have these surfaces be timelike, we must choose a and b such that

$$
a^2 > 1, \quad b^2 > 1. \tag{4.6}
$$

Next we introduce the timelike vectors u_A and v_A along each of the two 3-branes,

$$
\begin{aligned}
u_A &= \frac{1}{A_{\Phi_1}^{1/3}(t)\left(a^2-1\right)^{1/2}}\left(a\delta_A^t - \delta_A^y\right), \\
v_A &= \frac{1}{A_{\Phi_2}^{1/3}(t)\left(b^2-1\right)^{1/2}}\left(b\delta_A^t + \delta_A^y\right),
\end{aligned}
\tag{4.7}
$$

The delta functions reduce the form of $A(t,y)$ on each brane appropriately, so that,

$$
\begin{aligned}
\Phi_1 &= 0 \Rightarrow \quad t - ay = 0 \Rightarrow \quad y = \frac{t}{a} \\
\Phi_2 &= 0 \Rightarrow \quad t + by = 0 \Rightarrow \quad y = -\frac{t}{b}
\end{aligned}
\tag{4.8}
$$

and thus,

$$
\begin{aligned}
A_{\Phi_1}(t) &\equiv A(t,y)\big|_{y=t/a} = (a+b)\, tH\left(t + \frac{b}{a}t\right) + A_0, \\
A_{\Phi_2}(t) &\equiv A(t,y)\big|_{y=-t/b} = (a+b)\, tH\left(t + \frac{a}{b}t\right) + A_0,
\end{aligned}
\tag{4.9}
$$

From the 5-dimensional Einstein field equations, $G_{AB} = \kappa T_{AB}$, we obtain

$$
T_{AB} = A_{\Phi_1}^{1/3} T_{AB}^{(\Phi_1)} \delta(\Phi_1) + A_{\Phi_2}^{1/3} T_{AB}^{(\Phi_2)} \delta(\Phi_1), \tag{4.10}
$$

where

$$
\begin{aligned}
T_{AB}^{(\Phi_1)} &= \rho_{\Phi_1} u_A u_B + p_{\Phi_1} \sum_{i=2}^{4} X_A^{(i,\Phi_1)} X_B^{(i,\Phi_1)}, \\
T_{AB}^{(\Phi_2)} &= \rho_{\Phi_2} v_A v_B + p_{\Phi_2} \sum_{i=2}^{4} X_A^{(i,\Phi_2)} X_B^{(i,\Phi_2)},
\end{aligned}
\tag{4.11}
$$

and $X_A^{(i,\Phi_I)}$ are unit vectors, defined as $X_A^{(i,\Phi_I)} \equiv A_a^{1/3}\delta_A^i$ $(i = 2, 3, 4;\ I = 1, 2)$, and

$$\begin{aligned}
\rho_{\Phi_1} &= -3p_{\Phi_1} = -\frac{b\,(a^2-1)}{\kappa A_{\Phi_1}^{2/3}(t)}, \\
\rho_{\Phi_2} &= -3p_{\Phi_2} = -\frac{a\,(b^2-1)}{\kappa A_{\Phi_2}^{2/3}(t)}.
\end{aligned} \tag{4.12}$$

Therefore, the solutions in the present case represent the collision of two timelike 3-branes, moving along, respectively, the line $t - ay = 0$ and the one $t + by = 0$. Each of the two 3-branes supports a perfect fluid. They approach each other as t increases, and collide at point $(t, y) = (0, 0)$, and then move apart.

Depending on the specific values of the free parameters a and b, we have three distinguishable cases: (a) $a,\ b < -1$; (b) $a > 1,\ b < -1$; and (c) $a,\ b > 1$. The case $a < -1,\ b > 1$ can be obtained from Case (b) by exchanging the two free parameters.

In the following let us consider them separately.

4.1.1 *Case A: $a < -1,\ b < -1$*

In this subcase, from Eq.(4.12) we can see that the perfect fluids on both of the two branes satisfy all the three energy conditions, weak, strong, and dominant [76].

To study the solutions further, we divide the spacetime into four regions, as outlined in Eq.(2.26), as shown in Fig. 4.1, with the two 3-branes as their boundaries, which we denote as, Σ_1 and Σ_2, as described in the outline of our model in Chapter 2, Eq.(2.28).

Along the hypersurface Σ_2, we find

$$\begin{aligned}
ds^2\big|_{t=|b|y} &= \frac{b^2-1}{b^2 A_{\Phi_2}^{2/3}(t)}dt^2 - A_{\Phi_2}^{2/3}(t)d\Sigma_0^2 \\
&= d\tau^2 - a_{\Phi_2}^2(\tau)d\Sigma_0^2,
\end{aligned} \tag{4.13}$$

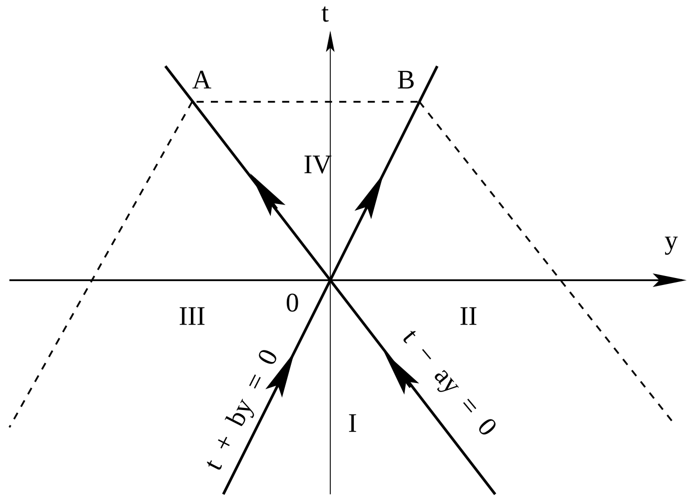


Figure 4.1: The 5-dimensional spacetime in the $(t,\ y)$-plane for $a < -1,\ b < -1$. The two 3-branes approach each other from $t = -\infty$ and collide at $(t, y) = (0, 0)$. Due to their gravitational mutual focus, the spacetime ends up at a spacelike singularity on the hypersurface $A_0 + (a + b)t = 0$ in Region IV, denoted by the horizontal dashed line. The spacetime is also singular along the line $A_0 - |a|(t - |b|y) = 0$ $(A_0 - |b|(t + |a|y) = 0)$ in Region III (II), which is parallel to the 3-brane located on the hypersurface $t + by = 0$ $(t - ay = 0)$.

where

$$
\begin{aligned}
A_{\Phi_2}(t) &= \begin{cases} A_0 - (|a| + |b|)\, t, & t \geq 0, \\ A_0, & t < 0, \end{cases} \\
d\tau &= \frac{\sqrt{b^2 - 1}}{|b| A_{\Phi_2}^{1/3}(t)} dt, \\
a_{\Phi_2}(\tau) &= \begin{cases} a^0_{\Phi_2} \left(\tau_0 - \tau\right)^{1/2}, & t \geq 0, \\ A_0^{1/3}, & t < 0, \end{cases}
\end{aligned}
\tag{4.14}
$$

with $\tau_0 = \tau_0\,(a, b, A_0)$, and $a^0_{\Phi_2} \equiv A_0^{1/3} \tau_0^{-1/2}$. Exchanging the free parameters a and b we can get the corresponding expressions for the brane located on the hypersurface $t - ay = 0$. From these expressions and Eq.(4.12) we can see that the two 3-branes come from $t = -\infty$ with constant energy densities and pressures, for which the spacetime on each of the branes is Minkowski. After they collide at the point $(t, y) = (0, 0)$, they focus each other, where $\dot{a}_{\Phi_1,\Phi_2}(\tau) < 0$, and finally end up at a singularity where $a_{\Phi_1,\Phi_2}(\tau) = 0$, denoted, respectively, by the point A and B in Fig. 4.1.

The spacetime outside the two 3-branes are vacuum, and the function $A(t, y)$ is given by

$$A(t,y) = \begin{cases} A_0 - (|a| + |b|)\, t, & IV, \\ A_0 - |a|\, (t - |b|y)\,, & III, \\ A_0 - |b|\, (t + |a|y)\,, & II, \\ A_0, & I. \end{cases} \tag{4.15}$$

From this expression we can see that the spacetime is Minkowski in Region I and the function $A(t, y)$ vanishes on the hypersurfaces $A_0 - (|a| + |b|)\, t = 0$ in Region IV, $A_0 - |a|\, (t - |b|y) = 0$ in Region III,and $A_0 - |b|\, (t + |a|y) = 0$ in Region II, denoted by the dashed lines in Fig. 4.1. These hypersurfaces actually represent the spacetime singularities.

This can be seen clearly from the Kretschmann scalar,

$$I \equiv R_{ABCD}R^{ABCD} = \frac{8}{9A^{8/3}} \times \begin{cases} (a+b)^4, & IV, \\ a^4\left(b^2 - 1\right)^2, & III, \\ b^4\left(a^2 - 1\right)^2, & II, \\ 0, & I. \end{cases} \tag{4.16}$$

The above analysis shows clearly that, when the matter fields on the two branes satisfy the energy conditions, due to their mutual gravitational focus, a spacelike singularity is always formed after the collision. This is similar to the conclusions obtained numerically in [66, 65, 67, 68].

4.1.2 Case B: $a > 1,\ b < -1$

In this case, Eq.(4.12) shows that the perfect fluid on the brane $t = ay$ satisfies all the three energy conditions, while the one on the brane $t = -by$ does not.

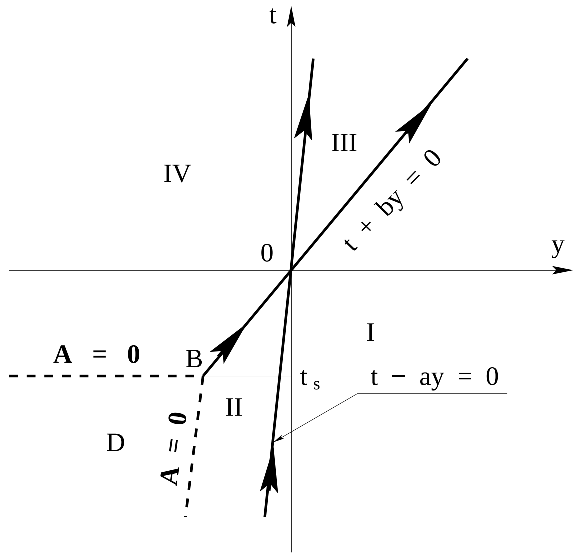


Figure 4.2: The spacetime in the $(t,\ y)$-plane for $a > |b| > 1,\ b < -1$. It is singular along the two half dashed lines, $t = -A_0/(a - |b|),\ y < -A_0/[|b|(a - |b|)]$, and $A_0 - |b|(t - ay) = 0,\ t < -A_0/(a - |b|)$. The 3-brane located on the hypersurface $t + by = 0$ starts to expand from the singular point B, $t = -A_0/(a - |b|)$ and $y = -A_0/[|b|(a - |b|)]$, until the point $(t, y) = (0, 0)$, where it collides with the other brane moving in along the hypersurface $t - ay = 0$. After the collision, it continuously moves forward but with constant energy density and pressure, and the spacetime on the brane becomes flat. The spacetime on the 3-brane located on the hypersurface $t - ay = 0$ is flat before the collision, but starts to expand as $a_{\Phi_1}(\eta) \propto (\eta + \eta_0)^{1/2}$ after the collision. This 3-brane is free of any kind of spacetime singularities.

To study these solutions further, it is found convenient to consider the two cases $a > |b| > 1$ and $|b| > a > 1$ separately.

Case B.1: $a > |b| > 1$. In this case, the two colliding branes divide the whole spacetime into the four regions as shown in Fig. 4.2. Then, we find that

$$A(t, y) = \begin{cases} A_0 + (a - |b|)\, t, & IV, \\ A_0 + a\,(t - |b|y)\,, & III, \\ A_0 - |b|\,(t - ay)\,, & II, \\ A_0, & I. \end{cases} \tag{4.17}$$

Clearly, the spacetime is again Minkowski in Region I, but the function $A(t, y)$ now vanishes only on the hypersurfaces $A_0 + (a - |b|)\, t = 0$ in Region IV, and

$A_0 - |b|\,(t - ay) = 0$ in Region II, denoted by the dashed lines in Fig. 4.2. Similar to the last case, the Kretschmann scalar blows up on these surfaces, so they actually represent the spacetime singularities. As a result, the region $A_0/|b| + ay < t < -A_0/(a - |b|),\ y < 0$, denoted by D in Fig. 4.2, is not part of the whole spacetime. In Region III we have $A(t, y) > 0$, and no any kind of spacetime singularities appears in this region.

Along the hypersurface $t + by = 0$, the metric takes the same form as that given by Eq.(4.13) but now with

$$A_{\Phi_2}(t) = \begin{cases} A_0, & t \geq 0, \\ A_0 + (a - |b|)\,t, & t < 0, \end{cases} \tag{4.18}$$

$$a_{\Phi_2}(\tau) = \begin{cases} A_0^{1/3}, & t \geq 0, \\ a^0_{\Phi_2}\left(\tau + \tau_s\right)^{1/2}, & t < 0, \end{cases} \tag{4.19}$$

where $t = t_s \equiv -A_0/(a - |b|)$ corresponds to $\tau = \tau_s$ and $t = 0$ to $\tau = \tau_0$, with $\tau_0 \equiv (b^2 -)^{1/2} A_0^{2/3}/[2|b|(a - |b|)]$, and $a^0_{\Phi_2} = A_0^{1/3}(\tau_0 + \tau_s)^{-1/2}$.

Thus, in this case the 3-brane located on the hypersurface $t + by = 0$ starts to expand from the singular point $\tau = \tau_s$ and collides with the other incoming 3-brane at the point $(t,\ y) = (0,\ 0)$. After the collision, the 3-brane transfers part of its energy to the one moving along the hypersurface $t - ay = 0$, so that its energy density and pressure remain constant, and whereby the spacetime on this 3-brane becomes Minkowski.

Along the hypersurface $t - ay = 0$, the metric takes the form

$$\begin{aligned} ds^2\big|_{t=ay} &= \frac{a^2 - 1}{a^2 A^{2/3}_{\Phi_1}(t)} dt^2 - A^{2/3}_{\Phi_1}(t) d\Sigma_0^2 \\ &= d\eta^2 - a^2_{\Phi_1}(\eta) d\Sigma_0^2, \end{aligned} \tag{4.20}$$

where

$$
\begin{aligned}
A_{\Phi_1}(t) &= \begin{cases} A_0 + (a - |b|)\, t, & t \geq 0, \\ \\ A_0, & t < 0, \end{cases} \\
d\eta &= \sqrt{\frac{a^2 - 1}{a^2 A^{2/3}_{\Phi_1}(t)}} dt, \\
a_{\Phi_1}(\eta) &= \begin{cases} a^0_{\Phi_1} \left(\eta + \eta_0\right)^{1/2}, & t \geq 0, \\ \\ A_0^{1/3}, & t < 0, \end{cases}
\end{aligned} \tag{4.21}
$$

where $t = 0$ corresponds to $\eta = 0$ and $\eta_0 \equiv 3(a^2-1)^{1/2} A_0^{2/3}/[2a(a-|b|)] > 0$. Thus, in the present case the brane located on the hypersurface $t - ay = 0$ comes from $t = -\infty$ with constant energy density and pressure $\rho_{\Phi_1} = -3p_{\Phi_1} = |b|(a^2 - 1)/(\kappa A_0^{2/3}) > 0$, which satisfies all the three energy conditions. The spacetime on this brane is flat before the collision.

After the collision, the spacetime of the brane starts to expand as $(\eta + \eta_0)^{1/2}$ without the big-bang type of singularities. The expansion rate is the same as that of a radiation-dominated universe in Einstein's theory of 4D gravity, where $a(\eta) \propto \eta^{1/2}$. However, its energy density and pressure now decreases as $\rho_{\Phi_1} = -3p_{\Phi_1} \propto (\eta + \eta_0)^{-1}$, in contrast to $\rho = 3p \propto \eta^{-2}$ in Einstein's 4D gravity [76].

Case B.2: $|b| > a > 1$. In this case, the two colliding branes divide the whole spacetime into the four regions, as shown in Fig. 4.3.

Following a similar analysis as we did in the last subcase one can show that the spacetime now is singular on the half lines $t = A_0/\left(|b| - a\right),\ y < A_0/[|b|\left(|b| - a\right)]$ in Region IV, and $t = A_0/|b| + ay > A_0/\left(|b| - a\right)$ in Region III, denoted by the dashed lines in Fig. 4.3.

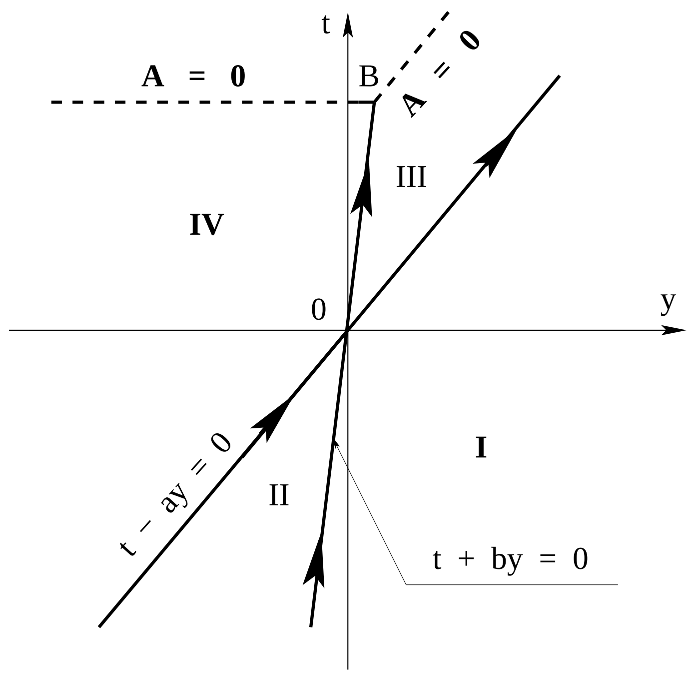


Figure 4.3: The spacetime in the $(t,\ y)$-plane for $|b| > a > 1,\ b < -1$. It is singular along the two half dashed lines where $A = 0$. The spacetime of the 3-brane along $t + by = 0$ is flat before the collision, but collapses to form a spacetime singularity at the point B. The spacetime of the 3-brane along $t - ay = 0$ is contracting from $t = -\infty$ before the collision, but becomes flat after the collision. At the colliding point $(t, y) = (0, 0)$ no any kind of spacetime singularities exists.

Along the hypersurface $t - ay = 0$, the metric takes the form of Eq.(4.20) but now with

$$
\begin{aligned}
A_{\Phi_1}(t) &= \begin{cases} A_0, & t \geq 0, \\ A_0 - (|b| - a)\, t, & t < 0, \end{cases} \\
a_{\Phi_1}(\eta) &= \begin{cases} A_0^{1/3}, & t \geq 0, \\ a^0_{\Phi_1}\, (\eta_0 - \eta)^{1/2}, & t < 0, \end{cases}
\end{aligned}
\tag{4.22}
$$

where $t \leq 0$ corresponds to $\eta \leq 0$ with $\eta_0 \equiv 3(a^2-1)^{1/2} A_0^{2/3}/[2a(|b|-a)] > 0$. Thus, in the present case the brane located on the hypersurface $t - ay = 0$ comes from $t = -\infty$ with energy density and pressure $\rho_{\Phi_1} = -3p_{\Phi_1} \propto (\eta_0 - \eta)^{-1}$, which satisfies all the three energy conditions. The spacetime on this brane is non-flat before the collision and becomes flat after the collision.

Along the line $t + by = 0$, the metric takes the same form as that given by Eq.(4.13) but now with

$$
\begin{aligned}
A_{\Phi_2}(t) &= \begin{cases} A_0 - (|b| - a)\, t, & t \geq 0, \\ A_0, & t < 0, \end{cases} \\
a_{\Phi_2}(\tau) &= \begin{cases} a^0_{\Phi_2} \left(\tau_s - \tau\right)^{1/2}, & t \geq 0, \\ A_0^{1/3}, & t < 0, \end{cases}
\end{aligned} \tag{4.23}
$$

where $t = t_s \equiv A_0/(|b| - a)$ corresponds to $\tau = \tau_s$ and $t = 0$ to $\tau = \tau_0$, with $\tau_0 \equiv (b^2 - 1)^{1/2} A_0^{2/3}/[2|b|(|b| - a)]$. Thus, in this case the 3-brane located on the hypersurface $t + by = 0$ moves in from $t = -\infty$ and has constant energy density and pressure before the collision. After the collision, it collapses to a singularity at $\tau = \tau_s$.

4.1.3 Case C: $a > 1,\ b > 1$

In this subcase, from Eq.(4.12) we can see that both of the two branes violate all the three energy conditions [76]. Dividing the spacetime into the four regions shown in Fig. 4.4, we find that

$$
\begin{aligned}
A(t, y) &= \begin{cases} A_0 + (a + b)\, t, & IV, \\ A_0 + b\,(t - ay)\,, & III, \\ A_0 + a\,(t + by)\,, & II, \\ A_0, & I, \end{cases} \\
A_{\Phi_1}(t) &= \begin{cases} A_0 + (a + b)t, & t \geq 0, \\ A_0, & t < 0, \end{cases} \\
A_{\Phi_2}(t) &= \begin{cases} A_0 + (a + b)t, & t \geq 0, \\ A_0, & t < 0, \end{cases}
\end{aligned} \tag{4.24}
$$

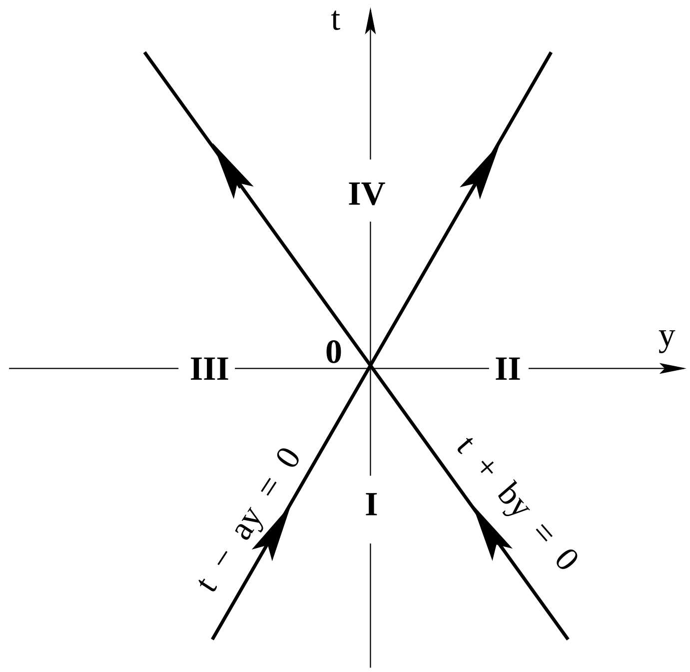


Figure 4.4: The spacetime in the $(t,\ y)$-plane for $a > 1,\ b > 1$. It is free of any kind of spacetime singularities in the whole spacetime, including the two hypersurfaces of the 3-branes. The two 3-branes all come from $t = -\infty$ with constant energy density and pressure. They remain so until the moment right before collision. After the collision, the spacetime on each of the 3-branes is expanding like $a(\tau) \propto \tau^{1/2}$, while their energy densities and pressures decrease like $\rho = -3p \propto \tau^{-1}$.

which are non-zero in the whole spacetime. Thus, in the present case the spacetime is free of any kind of spacetime singularities, and flat in Region I. Before the collision the two branes move in from $t = -\infty$ all with constant energy density and pressure.

After the collision, their energy densities and pressures all decrease like τ^{-1}, while the spacetime on these two branes is expanding like $a(\tau) \propto \tau^{1/2}$, where τ is the proper time on each of the two branes, and $a(\tau)$ their expansion factor.

4.2 Conclusions

In this chapter, we have studied the collision of branes and the formation of spacetime singularities. We have constructed a class of analytic solutions to the 5-dimensional Einstein field equations, which represents such a collision, and found that when both of the two 3-branes satisfy the energy conditions, a spacelike singularity is always developed after the collision, due to their mutual gravitational focus. This is consistent with the results obtained numerically in [66, 65, 67, 68].

When only one of the two branes satisfies the energy conditions, the other brane either starts to expands from a singular point [cf. Fig. 4.2], or comes from $t = -\infty$ and then focuses to a singular point after the collision [cf. Fig. 4.3]. It is interesting to note that in all these three cases the spacetime in Region IV is locally Kasner. As a result, the power-law singularity developed after the brane collision is that of Kasner type.

However, if both of the two colliding 3-branes violate the weak energy condition, no spacetime singularities exist at all in the whole spacetime. Before the collision, the two branes approach each other in a flat background with constant energy densities and pressures. After they collide at $(t, y) = (0, 0)$, they start to expand as $a(\tau) \propto \tau^{1/2}$, where $a(\tau)$ denotes their expansion factor, and τ their proper time. As the branes are expanding, their energy densities and pressures decrease as $\rho,\ p \propto \tau^{-1}$, in contrast to that of $\rho,\ p \propto \tau^{-2}$ in the four-dimensional FRW model. Region IV in this case is also locally Kasner, but the Kasner spacetime singularity is not part of this region [88].

As argued in [71], these singularities may become very mild when the 5-dimensional spacetime is left to higher dimensional spacetimes, ten dimensions in string theory and eleven in M-Theory, a question that is to be considered in the next chapter.

Finally, we would like to note that the solution presented here is purely gravitational, and hence there is no charge associated with the colliding 3-branes. If charges are included, one may wonder whether these 3-branes are stable and ask if the spacetime singularities are still formed after the collision?

CHAPTER FIVE

Colliding Branes and Formation of Spacetime Singularities in String Theory

We are now in position to study the mechanics of colliding branes and the formation of spacetime singularities in string theory. Using the general formulas developed in Chapter 2, we study a particular class of exact solutions first in the 5-dimensional effective theory, and then lift it to the 10-dimensional spacetime.

In general, the 5-dimensional spacetime is singular, due to the mutual focus of the two colliding 3-branes. Non-singular cases also exist, but with the price that both of the colliding branes violate all the three energy conditions, weak, dominant, and strong. After lifted to 10 dimensions, we find that the spacetime remains singular, whenever it is singular in the 5-dimensional effective theory. In the cases where no singularities are formed after the collision, we find that the two 8-branes necessarily violate all the energy conditions.

The organization of this chapter is as follows: In Section 5.1, we apply our model of the NS-NS sector of Type II string theory in the Einstein frame, for a large class of spacetimes, and obtain the explicit field equations both outside the two branes and on the two branes. In Section 5.2, we construct a class of exact solutions in the Einstein frame, in which the potential of the radion field on the two branes take an exponential form, while the matter fields on the two branes are dust fluids. After identifying spacetime singularities both outside and on the branes, we are able to draw the corresponding Penrose diagrams for various cases. In Section 5.3, we study the local and global properties of these solutions in the 5-dimensional string frame, while in Section 5.4 we first lift the solutions to 10 dimensions, and then study the local and global properties of these 10-dimensional solutions in details. Section 5.5 contains a study of a class of 10-dimensional spacetimes. In particular, we divide the Einstein tensor explicitly into three parts, one on each side of a colliding brane, and the other is on the brane. It is remarkable that the part on the brane can be written in the form of an anisotropic fluid. In Section 5.6, we present our main conclusions.

5.1 Colliding Timelike 3-branes in the Einstein Frame

We consider the 5-dimensional spacetime in the Einstein frame described by the metric,

$$\begin{aligned} ds_5^2 &= g_{ab}dx^a dx^b \\ &= e^{2\sigma(t,y)}\left(dt^2 - dy^2\right) - e^{2\omega(t,y)}d\Sigma_0^2, \end{aligned} \tag{5.1}$$

where $d\Sigma_0^2 \equiv \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2$, and $x^0 = t$, $x^1 = y$. Then, the non-vanishing components of the Ricci tensor is given by

$$R_{tt} = -[3\omega_{,tt} + \sigma_{,tt} + 3\omega_{,t}\left(\omega_{,t} - \sigma_{,t}\right) - \sigma_{,yy} - 3\omega_{,y}\sigma_{,y}], \tag{5.2}$$

$$R_{ty} = -3[\omega_{,ty} + \omega_{,t}\omega_{,y} - \omega_{,t}\sigma_{,y} - \omega_{,y}\sigma_{,t}], \tag{5.3}$$

$$R_{yy} = -[3\omega_{,yy} + \sigma_{,yy} + 3\omega_{,y}\left(\omega_{,y} - \sigma_{,y}\right) - \sigma_{,tt} - 3\omega_{,t}\sigma_{,t}], \tag{5.4}$$

$$R_{ij} = \delta_{ij}e^{2(\omega-\sigma)}[\omega_{,tt} + 3{\omega_{,t}}^2 - (\omega_{,yy} + 3{\omega_{,y}}^2)], \tag{5.5}$$

where now $i,\ j = 2, 3, 4$, and $\omega_{,t} \equiv \partial\omega/\partial t$, etc.

We assume that the two colliding 3-branes move along the hypersurfaces given, respectively, by

$$\begin{aligned} \Phi_1(t,y) &= t - ay = 0, \\ \Phi_2(t,y) &= t + by = 0, \end{aligned} \tag{5.6}$$

where a and b are two arbitrary constants, subjected to the constraints,

$$a^2 > 1, \quad b^2 > 1, \tag{5.7}$$

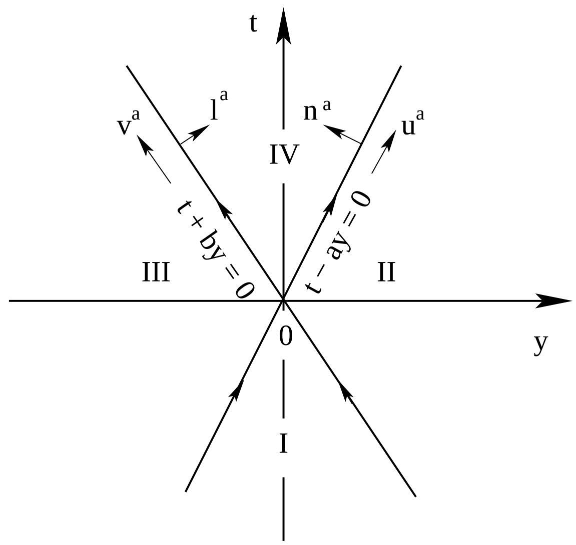


Figure 5.1: The five-dimensional spacetime in the $(t,\ y)$-plane for $a > 1,\ b > 1$. The two 3-branes are moving along the hypersurfaces, Σ_1 and Σ_2, which are defined by Eq.(5.10) in the text. The four regions, $I - IV$, are defined by Eq.(5.8).

in order for the two hypersurfaces to be timelike. The two colliding branes divide the whole spacetime into four regions, $I - IV$, which are defined, respectively, as

$$\begin{aligned}
\text{Region I} &\equiv \{x^a : \Phi_1 < 0,\ \Phi_2 < 0\}\,,\\
\text{Region II} &\equiv \{x^a : \Phi_1 > 0,\ \Phi_2 < 0\}\,,\\
\text{Region III} &\equiv \{x^a : \Phi_1 < 0,\ \Phi_2 > 0\}\,,\\
\text{Region IV} &\equiv \{x^a : \Phi_1 > 0,\ \Phi_2 > 0\}\,,
\end{aligned} \tag{5.8}$$

as shown schematically in Fig. 5.1. In each of these regions, we will define various quantities in these regions, by the following description,

$$F^A \equiv F(t,y)\Big|_{Region\ A}, \tag{5.9}$$

where now $A = I,\ \ II,\ III,\ IV$.

We also define the two hypersurfaces Σ_1 and Σ_2 as,

$$\begin{aligned}
\Sigma_1 &\equiv \{x^a :\ \Phi_1 = 0\}\,,\\
\Sigma_2 &\equiv \{x^a :\ \Phi_2 = 0\}\,.
\end{aligned} \tag{5.10}$$

Then, it can be shown that the normal vectors to each of these two surfaces are given by

$$
\begin{aligned}
n_a &= N\left(\delta^t_a - a\delta^y_a\right), \\
l_a &= L\left(\delta^t_a + b\delta^y_a\right),
\end{aligned}
\tag{5.11}
$$

where

$$
\begin{aligned}
F^{(I)} &\equiv F(t,y)\Big|_{\Phi_I=0}, \\
N &\equiv \frac{e^{\sigma^{(1)}}}{\left(a^2-1\right)^{1/2}}, \\
L &\equiv \frac{e^{\sigma^{(2)}}}{\left(b^2-1\right)^{1/2}},
\end{aligned}
\tag{5.12}
$$

with $F = \{\sigma,\ \omega,\ \phi\}$. We also introduce the two timelike vectors u_c and v_c via the relations,

$$
\begin{aligned}
u_a &= N\left(a\delta^t_a - \delta^y_a\right), \\
v_a &= L\left(b\delta^t_a + \delta^y_a\right).
\end{aligned}
\tag{5.13}
$$

It can be shown that these vectors have the following properties,

$$
\begin{aligned}
n_a n^a &= -1 = l_a l^a, \\
u_a u^a &= +1 = v_a v^a, \\
n_a u^a &= 0 = l_a v^a.
\end{aligned}
\tag{5.14}
$$

Note that for the sake of the reader's convenience, we have repeated our earlier presentation given by Eqs. (2.26)-(2.31), in the above development.

In the following, we shall consider the field equations, (2.19) and (2.21), in Regions $I - IV$ and along the hypersurfaces $\Sigma_{1,2}$, separately. It should be noted that in the above setup, the two 3-branes do not have the Z_2 symmetry considered in Horava-Witten in M-theory [17, 29] and Randall-Sundrum models [41, 42].

5.1.1 *Field Equations in Regions $I - IV$*

In these regions, the field equations of Eqs.(2.19) and (2.21) take the form,

$$R^A_{ab} = \varphi^A_{,a}\varphi^A_{,b}, \tag{5.15}$$

$$\Box^{(A)}\varphi^A = 0, \tag{5.16}$$

$$\Box^{(A)} \equiv g^{(A)\,ab}\nabla^{(A)}_a\nabla^{(A)}_b \tag{5.17}$$

where $\varphi = \kappa_5\phi$, $\nabla^{(A)}_a$ denotes the covariant derivative with respect to $g^{(A)}_{ab}$, and $g^{(A)}_{ab}$ is the metric defined in Region A. From Eq.(5.5) and the fact that $\varphi = \varphi(t, y)$, we find that

$$\omega = \frac{1}{3}\ln\Big[f\,(t+y) + g\,(t-y)\Big], \tag{5.18}$$

where $f\,(t+y)$ and $g\,(t-y)$ are arbitrary functions of their indicated arguments. Note that in writing Eq.(5.18) we dropped the super indices A. In the following we shall adopt this convention, except where confusions may raise.

In the following we consider only the case where

$$f'g' \neq 0, \tag{5.19}$$

where a prime denotes the ordinary derivative with respect the indicated argument. Then, introducing two new variables $\xi_\pm$ via the relations,

$$\xi_\pm(t, y) \equiv f\,(t+y) \pm g\,(t-y)\,, \tag{5.20}$$

we find that Eq.(5.15) yields,

$$M_+ = \frac{1}{2}\xi_+\left({\varphi_+}^2 + {\varphi_-}^2\right), \tag{5.21}$$

$$M_- = \xi_+\varphi_+\varphi_-, \tag{5.22}$$

and

$$M_{++} - M_{--} = -\frac{1}{2}\left({\varphi_+}^2 - {\varphi_-}^2\right), \tag{5.23}$$

where $M_{\pm} \equiv \partial M/\partial \xi_{\pm}$, and

$$M\left(\xi_{+}, \xi_{-}\right) = \sigma + \frac{1}{3}\ln \xi_{+} - \frac{1}{2}\ln\left(4f'g'\right). \tag{5.24}$$

On the other hand, Eq.(5.16) can be cast in the form,

$$\varphi_{++} - \varphi_{--} + \frac{1}{\xi_{+}}\varphi_{+} = 0. \tag{5.25}$$

It should be noted that Eqs.(5.21)-(5.23) and (5.25) are not all independent. In fact, Eq.(5.23) is the integrability condition of Eqs.(5.21) and (5.22), and can be obtained from Eqs.(5.21), (5.22) and (5.25). Therefore, in Regions $I-IV$, the field equations reduce to Eqs. (5.21), (5.22) and (5.25).

To find the solutions, one may first integrate Eq.(5.25) to find φ, and then integrate Eqs.(5.21) and (5.22) to find M. However, Eq.(5.25) has an infinite number of solutions, and the corresponding general solutions of M have not been worked out yet [82]. Once φ and M are known, the metric coefficients σ and ω are then given by

$$\begin{aligned} \sigma &= M - \frac{1}{3}\ln\left(f+g\right) + \frac{1}{2}\ln\left(4f'g'\right), \\ \omega &= \frac{1}{3}\ln\left(f+g\right). \end{aligned} \tag{5.26}$$

5.1.2 Field Equations on the 3-branes

Field Equations on the surface $\Phi_1 = 0$. Across the hypersurface $\Phi_1 = 0$, for any given C^0 function $F(t,y)$, it can be written as [30],

$$F(t,y) = F^{+}(t,y)H\left(\Phi_1\right) + F^{-}(t,y)\left[1 - H\left(\Phi_1\right)\right], \tag{5.27}$$

where F^{+} (F^{-}) denotes the function $F(t,y)$ defined in the region $\Phi_1 > 0$ $(\Phi_1 < 0)$, and $H(x)$ denotes the Heaviside function, defined as

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{5.28}$$

Projecting $F_{,a}$ onto the n_a and u_a directions, we find

$$F_{,a} = F_u u_a - F_n n_a, \tag{5.29}$$

where

$$F_u \equiv u^a F_{,a}, \qquad F_n \equiv n^a F_{,a}. \tag{5.30}$$

Since $[F_u]^- = 0$, from the above expressions we find

$$[F_{,a}]^- = -[F_n]^- n_a, \tag{5.31}$$

where

$$[F_{,a}]^- \equiv \lim_{\Phi_1 \to 0^+} F^+_{,a} - \lim_{\Phi_1 \to 0^-} F^-_{,a}. \tag{5.32}$$

Then, we find that

$$\begin{aligned}
F_{,t} &= F^+_{,t} H(\Phi_1) + F^-_{,t}\left[1 - H(\Phi_1)\right], \\
F_{,y} &= F^+_{,y} H(\Phi_1) + F^-_{,y}\left[1 - H(\Phi_1)\right], \\
F_{,tt} &= F^+_{,tt} H(\Phi_1) + F^-_{,tt}\left[1 - H(\Phi_1)\right] - N\left[F_n\right]^- \delta(\Phi_1), \\
F_{,ty} &= F^+_{,ty} H(\Phi_1) + F^-_{,ty}\left[1 - H(\Phi_1)\right] + aN\left[F_n\right]^- \delta(\Phi_1), \\
F_{,yy} &= F^+_{,yy} H(\Phi_1) + F^-_{,yy}\left[1 - H(\Phi_1)\right] - a^2 N\left[F_n\right]^- \delta(\Phi_1),
\end{aligned} \tag{5.33}$$

where $\delta(\Phi_1)$ denotes the Dirac delta function. Then, we find that the Ricci tensor given by Eqs.(5.2)-(5.5) can be cast in the form,

$$R_{ab} = R^+_{ab} H(\Phi_1) + R^-_{ab}\left[1 - H(\Phi_1)\right] + R^{Im}_{ab}\delta(\Phi_1), \tag{5.34}$$

where R^+_{ab} $\left(R^-_{ab}\right)$ is the Ricci tensor calculated in the region $\Phi_1 > 0$ $(\Phi_1 < 0)$, and R^{Im}_{ab} denotes the Ricci tensor calculated on the hypersurface $\Phi_1 = 0$, which has the following non-vanishing components,

$$\begin{aligned}
R^{Im}_{tt} &= N\left\{3\left[\omega_n\right]^- - \left(a^2 - 1\right)\left[\sigma_n\right]^-\right\}, \\
R^{Im}_{ty} &= -3aN\left[\omega_n\right]^-, \\
R^{Im}_{yy} &= N\left\{3a^2\left[\omega_n\right]^- + \left(a^2 - 1\right)\left[\sigma_n\right]^-\right\}, \\
R^{Im}_{ij} &= N e^{2(\omega^{(1)} - \sigma^{(1)})}\left(a^2 - 1\right)\left[\omega_n\right]^- \delta_{ij}.
\end{aligned} \tag{5.35}$$

On the hypersurface $\Phi_1 = 0$, the metric (5.1) reduces to

$$ds_5^2\Big|_{\Phi_1=0} = g^{(1)}_{\mu\nu} d\xi^{\mu}_{(1)} d\xi^{\nu}_{(1)} = d\tau^2 - a_u^2(\tau) d\Sigma_0^2,$$

where $\xi^{\mu}_{(1)} \equiv \{\tau,\ x^2,\ x^3,\ x^4\}$, and

$$\begin{aligned} d\tau &\equiv \epsilon_\tau \left(\frac{a^2-1}{a^2}\right)^{1/2} e^{\sigma^{(I)}} dt, \\ a_u(\tau) &\equiv e^{\omega^{(1)}}, \end{aligned} \tag{5.36}$$

with $\epsilon_\tau = \pm 1$. Then, we find that

$$\begin{aligned} e^{(1)\ a}_{(\tau)} &\equiv \frac{\partial x^a}{\partial \tau} = \dot{t}\left(\delta^a_t + \frac{1}{a}\delta^a_y\right), \\ e^{(1)\ a}_{(i)} &\equiv \frac{\partial x^a}{\partial \xi^i_{(1)}} = \delta^a_i, \\ \sqrt{\left|\frac{g_4^{(1)}}{g_5}\right|} &= e^{-2\sigma^{(1)}}, \end{aligned} \tag{5.37}$$

where $i = 2,\ 3,\ 4$ and $\dot{t} \equiv dt/d\tau$. The field equations of Eq.(2.19) can now be written as

$$[\omega_n]^- = \frac{\kappa_5^2 e^{-\sigma^{(1)}}}{3\left(a^2-1\right)^{1/2}}\left(\rho^{(1)}_m + V^{(1)}_4\right), \tag{5.38}$$

$$2\left[\omega_n\right]^- + \left[\sigma_n\right]^- = \frac{\kappa_5^2 e^{-\sigma^{(1)}}}{\left(a^2-1\right)^{1/2}}\left(V^{(1)}_4 - p^{(1)}_m\right), \tag{5.39}$$

where in writing the above expressions we assumed that $T^{(m,1)}_{\mu\nu}$ takes the form of a perfect fluid,

$$\begin{aligned} T^{(m,1)}_{\mu\nu} &\equiv \left(\rho^{(1)}_m + p^{(1)}_m\right) w^{(1)}_\mu w^{(1)}_\nu - p^{(1)}_m g^{(1)}_{\mu\nu}, \\ w^{(1)}_\mu &= \delta^\tau_\mu. \end{aligned} \tag{5.40}$$

Similarly, it can be shown that the Klein-Gordon equation (2.21) and the conservation law of the matter fields (2.22) on Σ_1 take, respectively, the forms,

$$[\phi_n]^- = -\frac{e^{-\sigma^{(1)}}}{\left(a^2-1\right)^{1/2}}\frac{\partial V^{(1)}_4(\phi)}{\partial \phi}, \tag{5.41}$$

$$\frac{d\rho^{(1)}_m}{d\tau} + 3H_u\left(\rho^{(1)}_m + p^{(1)}_m\right) = 0, \tag{5.42}$$

where the hubble constant is defined in this case as,

$$H_u \equiv \dot{a}_u / a_u. \tag{5.43}$$

Field Equations on the surface $\Phi_2 = 0$. Following a similar procedure to the last sub-section, one can show that the Ricci tensor across the brane $\Phi_2 = 0$ can be written as

$$R_{ab} = R^{+}_{ab} H\left(\Phi_2\right) + R^{-}_{ab}\left[1 - H\left(\Phi_2\right)\right] + R^{Im}_{ab}\delta\left(\Phi_2\right), \tag{5.44}$$

where R^{+}_{ab} $\left(R^{-}_{ab}\right)$ is now the Ricci tensor calculated in the region $\Phi_2 > 0$ $(\Phi_2 < 0)$, and R^{Im}_{ab} denotes the Ricci tensor calculated on the hypersurface $\Phi_2 = 0$, which has the following non-vanishing components,

$$\begin{aligned}
R^{Im}_{tt} &= L\left\{3\left[\omega_l\right]^{-} - \left(b^2 - 1\right)\left[\sigma_l\right]^{-}\right\}, \\
R^{Im}_{ty} &= 3bL\left[\omega_l\right]^{-}, \\
R^{Im}_{yy} &= L\left\{3b^2\left[\omega_l\right]^{-} + \left(b^2 - 1\right)\left[\sigma_l\right]^{-}\right\}, \\
R^{Im}_{ij} &= Le^{2\left(\omega^{(2)} - \sigma^{(2)}\right)}\left(b^2 - 1\right)\left[\omega_l\right]^{-}\delta_{ij},
\end{aligned} \tag{5.45}$$

where $\omega_l \equiv l^a \omega_{,a}$ etc. On the hypersurface $\Phi_2 = 0$, the metric (5.1) reduces to

$$\left.ds^2_5\right|_{\Phi_2=0} = g^{(2)}_{\mu\nu} d\xi^{\mu}_{(2)} d\xi^{\nu}_{(2)} = d\eta^2 - a^2_v(\eta) d\Sigma^2_0,$$

where $\xi^{\mu}_{(2)} \equiv \{\eta,\ x^2,\ x^3,\ x^4\}$, and

$$\begin{aligned}
d\eta &\equiv \epsilon_\eta \left(\frac{b^2 - 1}{b^2}\right)^{1/2} e^{\sigma^{(2)}} dt, \\
a_v(\eta) &\equiv e^{\omega^{(2)}},
\end{aligned} \tag{5.46}$$

with $\epsilon_\eta = \pm 1$. Then, we find that

$$\begin{aligned}
e^{(2)\ a}_{(\eta)} &\equiv \frac{\partial x^a}{\partial \eta} = t^* \left(\delta^a_t - \frac{1}{b}\delta^a_y\right), \\
e^{(2)\ a}_{(i)} &\equiv \frac{\partial x^a}{\partial \xi^i_{(2)}} = \delta^a_i, \\
\sqrt{\left|\frac{g^{(2)}_4}{g_5}\right|} &= e^{-2\sigma^{(2)}},
\end{aligned} \tag{5.47}$$

where $t^* \equiv dt/d\eta$. Hence, the field equations of Eq.(2.19) can be written as

$$\begin{aligned} [\omega_l]^- &= \frac{\kappa_5^2 e^{-\sigma^{(2)}}}{3\left(b^2-1\right)^{1/2}}\left(\rho_m^{(2)} + V_4^{(2)}\right), & (5.48)\\ 2\left[\omega_l\right]^- + [\sigma_l]^- &= \frac{\kappa_5^2 e^{-\sigma^{(2)}}}{\left(b^2-1\right)^{1/2}}\left(V_4^{(2)} - p_m^{(2)}\right), & (5.49)\end{aligned}$$

where in writing the above equations we had assumed that $T_{\mu\nu}^{(m,2)}$ takes the form,

$$\begin{aligned} T_{\mu\nu}^{(m,2)} &\equiv \left(\rho_m^{(2)} + p_m^{(2)}\right) w_\mu^{(2)} w_\nu^{(2)} - p_m^{(2)} g_{\mu\nu}^{(2)},\\ w_\mu^{(2)} &= \delta_\mu^\eta. & (5.50)\end{aligned}$$

Similarly, it can be shown that the Klein-Gordon equation (2.21) and the conservation law of the matter fields (2.22) on Σ_2 take, respectively, the forms,

$$[\phi_l]^- = -\frac{e^{-\sigma^{(2)}}}{\left(b^2-1\right)^{1/2}}\frac{\partial V_4^{(2)}(\phi)}{\partial\phi}, \quad (5.51)$$

$$\frac{d\rho_m^{(2)}}{d\eta} + 3H_v\left(\rho_m^{(2)} + p_m^{(2)}\right) = 0, \quad (5.52)$$

where $H_v \equiv a_v^*/a_v$, is the effective Hubble expansion factor of the brane $\Phi_2 = 0$.

5.2 Particular Solutions for Colliding Timelike 3-branes in the Einstein Frame

Choosing the potentials $V_4^{(I)}(\phi)$ on the two branes as

$$V_4^{(I)}(\phi) = V_4^{(I,0)} e^{-\alpha\phi}, \quad (5.53)$$

where $V_4^{(I,0)}$'s and α are constants, and that the matter fields on each of the two branes are dust fluids, i.e.,

$$p_m^{(I)} = 0, \quad (5.54)$$

we find a class of solutions, which represents the collision of two timelike 3-branes and is given by

$$\begin{aligned} \sigma &= \left(\chi^2 - \frac{1}{3}\right)\ln\left(X_0 - X\right) + \sigma_0,\\ \omega &= \frac{1}{3}\ln\left(X_0 - X\right) + \omega_0,\\ \phi &= \frac{1}{\alpha}\ln\left(X_0 - X\right) + \phi_0, & (5.55)\end{aligned}$$

where $\chi \equiv \kappa_5/(\sqrt{2}\alpha)$, $A_0,\ \sigma_0,\ \omega_0$ and ϕ_0 are arbitrary constants, and

$$\begin{aligned} X &= b\,(t-ay)\,H\,(\Phi_1) + a\,(t+by)\,H\,(\Phi_2) \\ &= \begin{cases} (a+b)t, & IV, \\ a\,(t+by)\,, & III, \\ b\,(t-ay)\,, & II, \\ 0, & I. \end{cases} \end{aligned} \tag{5.56}$$

The constants a and b are given by

$$\begin{aligned} b\left(a^2-1\right) &= \frac{3\kappa_5^2 V_4^{(1,0)}}{3\chi^2+1}, \\ a\left(b^2-1\right) &= -\frac{3\kappa_5^2 V_4^{(2,0)}}{3\chi^2+1}. \end{aligned} \tag{5.57}$$

When $\alpha = \pm\infty$, the solutions reduces to the ones studied previously in Chapter 4 and in [88]. So, in the rest of this paper we shall consider only the case where $\alpha \neq \pm\infty$. Without loss of generality, we can always set $\sigma_0 = \omega_0 = \phi_0 = 0$, and assume that

$$X_0 > 0. \tag{5.58}$$

by taking advantage of the general convariance of the metric, and the fact that these are arbitrary cosntants. In addition, by ignoring these terms our solutions are somewhat simplified and their physical interpretation is made more accessible.

5.2.1 Spacetimes in Regions $I - IV$

It can be shown that the field equations, Eqs.(5.15) and (5.16) [or Eqs.(5.21), (5.22) and (5.25)], in Regions $I-IV$ are satisfied identically for the above solutions. To study the singular behavior of the spacetime in each of the four regions, we calculate the Ricci scalar, which in the present case is given by

$$\begin{aligned} R &= \kappa_5^2 g^{ab}\phi_{,a}\phi_{,b} \\ &= \frac{\kappa_5^2 B}{\alpha^2\left(X_0-X\right)^{2(\chi^2+2/3)}}, \end{aligned} \tag{5.59}$$

where X is given by Eq.(5.56), and

$$B = \begin{cases} (a+b)^2, & IV, \\ -a^2\left(b^2-1\right), & III, \\ -b^2\left(a^2-1\right), & II, \\ 0, & I. \end{cases} \tag{5.60}$$

5.2.2 Spacetimes on the 3-brane Located at $\Phi_1 = 0$

On the 3-brane located on $\Phi_1 = 0$, the reduced metric takes the form,

$$\left. ds_5^2 \right|_{\Sigma_1} = d\tau^2 - a_u^2(\tau) d^2\Sigma_0, \tag{5.61}$$

where

$$a_u(\tau) = \begin{cases} \left[\beta\left(\tau_s - \tau\right)\right]^{\frac{1}{3\chi^2+2}}, & \Phi_2 > 0, \\ X_0^{1/3}, & \Phi_2 < 0, \end{cases} \tag{5.62}$$

with

$$\begin{aligned} \left.\Phi_2\right|_{\Phi_1=0} &= \frac{a+b}{a} t, \\ \beta &\equiv \frac{|a(a+b)|}{\left(a^2-1\right)^{1/2}} \left(\chi^2 + \frac{2}{3}\right), \\ \tau_s &\equiv \beta^{-1} X_0^{\chi^2+\frac{2}{3}} \end{aligned} \tag{5.63}$$

Note that in writing the above expressions, we had chosen $\epsilon_\tau = \text{sign}(a+b)$. From Eqs.(5.38) and (5.39), on the other hand, we find that

$$\rho_m^{(1)} = \frac{\rho_m^{(1,0)}}{X_0 - X^{(1)}(t)} \begin{cases} \left[\beta\left(\tau_s - \tau\right)\right]^{-\frac{3}{3\chi^2+2}}, & \Phi_2 > 0, \\ X_0^{-1}, & \Phi_2 < 0, \end{cases} \tag{5.64}$$

where

$$\begin{aligned} \rho_m^{(1,0)} &\equiv \frac{b\left(a^2-1\right)}{\kappa_5^2} \left(\frac{2}{3} - \chi^2\right), \\ X^{(1)}(t) &\equiv (a+b)\, tH\left(\Phi_2\right). \end{aligned} \tag{5.65}$$

and from Eqs.(5.55) and (5.56) we find that

$$\phi^{(1)}(\tau) = \begin{cases} \frac{1}{\alpha(3\chi^2+2)} \ln\left[\beta\left(\tau_s - \tau\right)\right], & \Phi_2 > 0, \\ \frac{1}{\alpha} \ln X_0, & \Phi_2 < 0, \end{cases} \tag{5.66}$$

5.2.3 Spacetimes on the 3-brane Located at $\Phi_2 = 0$

Similarly, on the 3-brane located on the hypersurface $\Phi_2 = 0$, the reduced metric takes the form,

$$ds_5^2\Big|_{\Sigma_2} = d\eta^2 - a_v^2(\eta) d^2\Sigma_0, \tag{5.67}$$

where

$$a_v(\eta) = \begin{cases} \left[\gamma\left(\eta_s - \eta\right)\right]^{\frac{1}{3\chi^2+2}}, & \Phi_1 > 0, \\ X_0^{1/3}, & \Phi_1 < 0, \end{cases} \tag{5.68}$$

with $\epsilon_\eta = \mathrm{sign}(a+b)$, and

$$\begin{aligned} \Phi_1\big|_{\Phi_2=0} &= \frac{a+b}{b} t, \\ \gamma &\equiv \frac{|b(a+b)|}{\left(b^2-1\right)^{1/2}} \left(\chi^2 + \frac{2}{3}\right), \\ \eta_s &\equiv \gamma^{-1} X_0^{\chi^2+\frac{2}{3}}. \end{aligned} \tag{5.69}$$

The field equations (5.48) and (5.49), on the other hand, yield

$$\begin{aligned} \rho_m^{(2)} &= \frac{\rho_m^{(2,0)}}{X_0 - X^{(2)}(t)} \\ &= \begin{cases} \left[\gamma\left(\eta_s - \eta\right)\right]^{-\frac{3}{3\chi^2+2}}, & \Phi_1 > 0, \\ X_0^{-1}, & \Phi_1 < 0, \end{cases} \end{aligned} \tag{5.70}$$

where

$$\begin{aligned} \rho_m^{(2,0)} &\equiv -\frac{a\left(b^2-1\right)}{\kappa_5^2} \left(\frac{2}{3} - \chi^2\right), \\ X^{(2)}(t) &\equiv (a+b)\, t H\left(\Phi_1\right). \end{aligned} \tag{5.71}$$

and

$$\phi^{(2)}(\eta) = \begin{cases} \frac{1}{\alpha(3\chi^2+2)} \ln\left[\gamma\left(\eta_s - \eta\right)\right], & \Phi_1 > 0, \\ \frac{1}{\alpha} \ln X_0, & \Phi_1 < 0, \end{cases} \tag{5.72}$$

It is interesting to note that when $\chi^2 = 2/3$, we have $\rho_m^{(I)} = 0$, $(I = 1, 2)$, and the two 3-branes are supported only by the tensions $V_4^{(I)}(\phi)$, which are non-zero for any finite value of α [Recall the conditions (5.7)]. It is also remarkable to note that the presence of these two dust fluids is not essential to the singularity nature of the spacetime both in the bulk and on the branes. So, in the following we shall study the case with $\chi^2 = 2/3$ together with other cases.

To study the above solutions further, let us consider the following cases separately: (a) $a > 1,\ b > 1$; (b) $a > 1,\ b < -1$; (c) $a < -1,\ b > 1$; and (d) $a < -1,\ b < -1$.

5.2.4 *Case A: $a > 1,\ b > 1$*

In this case, from Eq.(5.57) we find that

$$V_4^{(1)}(\phi) > 0, \qquad V_4^{(2)}(\phi) < 0, \tag{5.73}$$

while Eqs.(5.64) and (5.70) show that

$$\begin{aligned} \rho_m^{(1)} &= \begin{cases} \geq 0, & \chi^2 \leq 2/3 \\ < 0, & \chi^2 > 2/3 \end{cases} \\ \rho_m^{(2)} &= \begin{cases} \leq 0, & \chi^2 \leq 2/3 \\ > 0, & \chi^2 > 2/3. \end{cases} \end{aligned} \tag{5.74}$$

From Eq.(5.59) we can also see that the spacetime is singular along the line $X_0 = (a+b)t$ in Region IV, the line $X_0 = a(t+by)$ in Region III, and the line $X_0 = b(t-ay)$ Region II, as shown by Fig. 5.2.

Before the collision $(t < 0)$, the scalar field is constant, $\phi^{(I)} = \phi_1 \equiv (1/\alpha)\ln(X_0)$, and the two potentials $V_4^{(1)}(\phi)$ and $V_4^{(2)}(\phi)$ are non-zero, unlike the dust energy densities $\rho_m^{(I)}$, except for the case when $\chi^2 = 2/3$. In the case $\chi^2 = 2/3$, the dust fluids disappear and the two branes are supported only by tensions, denoted by the two

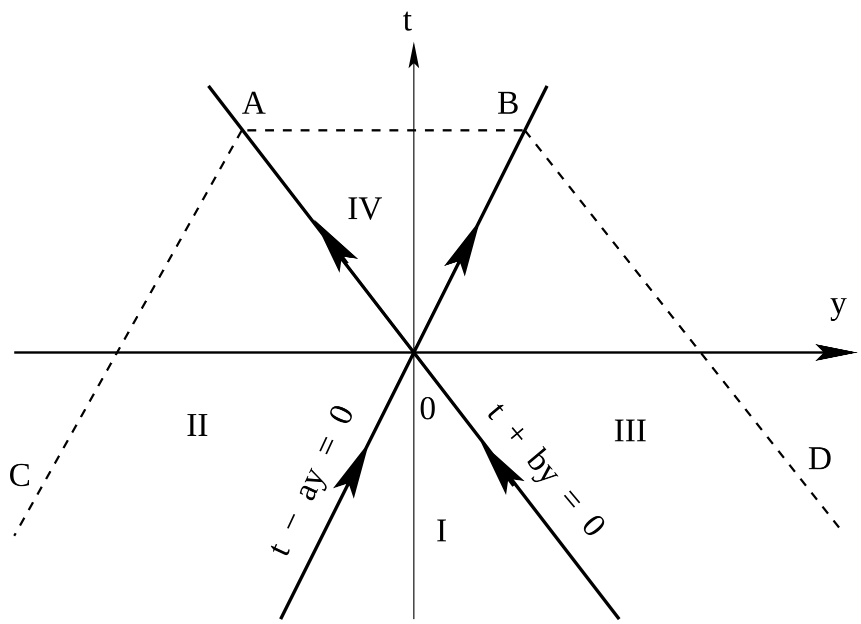


Figure 5.2: The five-dimensional spacetime in the $(t,\ y)$-plane for $a > 1,\ b > 1$. The two 3-branes are moving along the hypersurfaces, $\Sigma_1 : t - ay = 0$ and $\Sigma_2 : t + by = 0$. AB denotes the line $X_0 = (a + b)t$, AC the line $X_0 = b(t - ay)$, and BD the line $X_0 = a(t + by)$. The spacetime is singular along these lines. The four regions, $I - IV$, are defined by Eq.(5.8).

constant potential $V_4^{(1)}(\phi_1)$ and $V_4^{(2)}(\phi_1)$, which have opposite signs, and are quite similar to the case of Randall-Sundrum (RS) branes [41, 42], except for that in the RS model the two branes have Z_2 symmetry, whereas here we do not.

Before the collision, the spacetimes on the two branes are flat, that is, the matter fields on the 3-brane do not curve the 3-branes. However, it does curve the spacetime outside the 3-branes. This is quite similar to the so-called self-tuning mechanism of brane worlds [89, 90, 91, 92].

After the collision, the two 3-branes focus each other and finally a spacetime singularity is developed at, respectively, $\tau = \tau_s$ and $\eta = \eta_s$. The spacetime on the two branes is homogeneous and isotropic, and is described by Eqs.(5.61)-(5.62) and Eqs.(5.67)-(5.68). The corresponding Penrose diagram is given by Fig. 5.3.

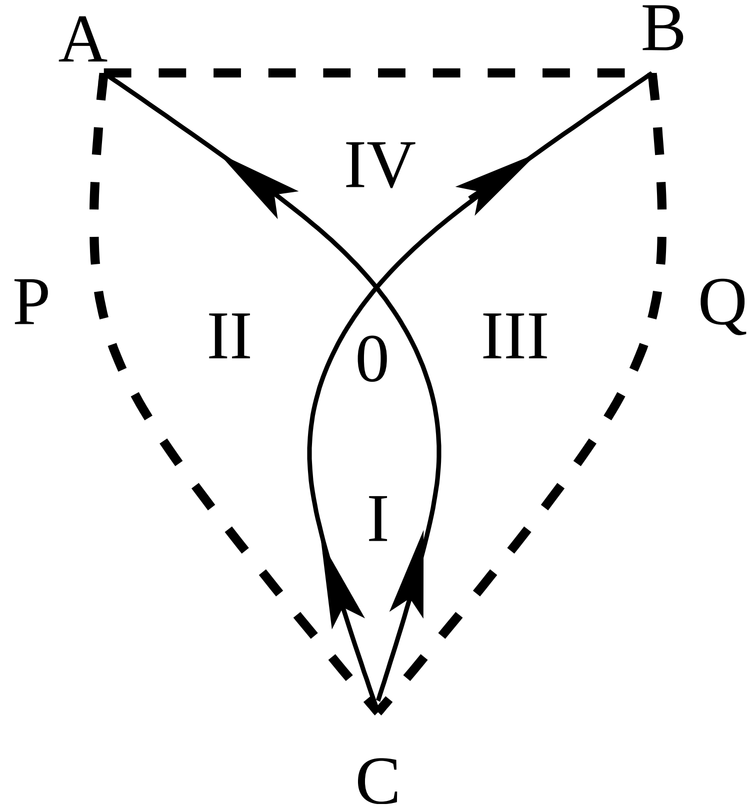


Figure 5.3: The Penrose diagram for $a > 1,\ b > 1$. The spacetime is singular along the straight line AB and the curved lines APC and BQC.

5.2.5 Case B: $a > 1,\ b < -1$

In this case, we find that

$$V_4^{(1)}(\phi) < 0, \qquad V_4^{(2)}(\phi) < 0,$$
$$\rho_m^{(I)} = \begin{cases} \geq 0, & \chi^2 \geq 2/3 \\ < 0, & \chi^2 < 2/3. \end{cases} \tag{5.75}$$

Thus, unlike the last case, now both potentials $V_4^{(I)}(\phi)$ are negative, while the two dust energy densities always have the same sign.

To study the solutions further in this case, we shall consider the two subcases, $a > |b| > 1$ and $|b| > a > 1$, separately.

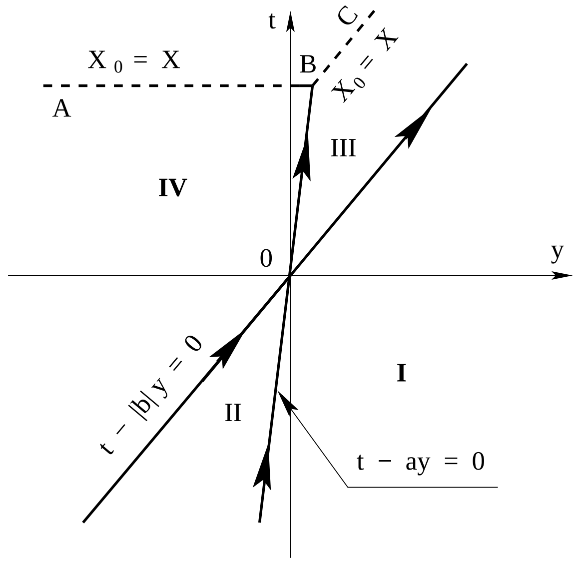


Figure 5.4: The five-dimensional spacetime in the $(t,\ y)$-plane for $a > -b > 1$. The two 3-branes are moving along the hypersurfaces, $\Sigma_1 : t-ay = 0$ and $\Sigma_2 : t-|b|y = 0$. The spacetime is singular along the line AB in Region IV and the line BC in Region III. The spacetime is also singular on the 3-brane at the point B where $\tau = \tau_s$. The four regions, $I - IV$, are defined by Eq.(5.8).

Case B1: $a > -b > 1$. When $a > -b > 1$, we have

$$
\begin{aligned}
\Phi_1 \mid_{\Phi_2=0} &= -\frac{a-|b|}{|b|}t = \begin{cases} < 0, & t > 0, \\ > 0, & t < 0, \end{cases} \\
\Phi_2 \mid_{\Phi_1=0} &= +\frac{a-|b|}{a}t = \begin{cases} > 0, & t > 0, \\ < 0, & t < 0, \end{cases} \\
X_0 - X &= \begin{cases} X_0 - (a-|b|)t, & IV, \\ X_0 - a\left(t-|b|y\right), & III, \\ X_0 + |b|\left(t-ay\right), & II, \\ 0, & I. \end{cases}
\end{aligned} \tag{5.76}
$$

Then, we find that the spacetime is singular along the line $X_0 = (a-|b|)t$ in Region IV, and the line $X_0 = a\left(t-|b|y\right)$ in Region III, as shown in Fig. 5.4.

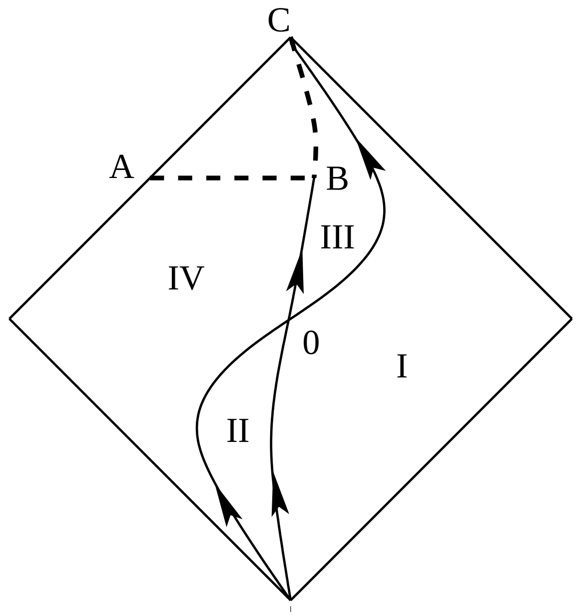


Figure 5.5: The Penrose diagram for $a > -b > 1$. The spacetime is singular along the lines AB and BC.

Before the collision ($t < 0$), the scalar field $\phi^{(1)}$ is constant on the 3-brane located on the hypersurface $\Sigma_1 : t - ay = 0$, so does the dust energy density $\rho_m^{(1)}$. In contrast, both the scalar field $\phi^{(2)}$ and the dust energy density $\rho_m^{(2)}$ are time-dependent on the 3-brane located on $\Sigma_2 : t - |b|y = 0$, and the corresponding spacetime is described by Eqs.(5.67) and (5.68) with $\eta \leq 0$. Note that along the hypersurface Σ_2, we have $\Phi_1 > 0$ for $t < 0$, as shown by Eq.(5.76).

After the collision, the 3-brane along Σ_2 transfers its energy to the one along Σ_1, so that its energy density $\rho_m^{(2)}$ and potential $V_4^{(2)}(\phi)$, as well as the scalar field $\phi^{(2)}$, become constant, while the energy density $\rho_m^{(1)}$ and the scalar field $\phi^{(1)}$ become time-dependent. Because of the mutual focus of the two branes, a spacetime singularity is finally developed at $\tau = \tau_s$, denoted by the point B in Fig. 5.4. Afterwards, the spacetime becomes also singular along the line $X_0 = (a - |b|)t$ in Region IV and the line $X_0 = a(t - |b|y)$ in Region III. It is interesting to note that these singularities are always formed, regardless of the signs of $\rho_m^{(1)}$ and $\rho_m^{(2)}$. In fact, they are formed even when $\rho_m^{(1)}(\chi^2 = 2/3) = 0 = \rho_m^{(2)}(\chi^2 = 2/3)$, as can be seen from Eqs.(5.59), (5.61) and (5.62). This is because the scalar field and the potentials $V_4^{(I)}(\phi)$ are still non-zero, and due the non-linear interaction of the scalar field itself, spacetime singularities are still formed. The corresponding Penrose diagram is given by Fig. 5.5.

Case B2: $-b > a > 1$. When $-b > a > 1$, we have

$$
\begin{aligned}
\Phi_1 \mid_{\Phi_2=0} &= -\frac{|b|-a}{|b|}t = \begin{cases} > 0, & t > 0, \\ < 0, & t < 0, \end{cases} \\
\Phi_2 \mid_{\Phi_1=0} &= -\frac{|b|-a}{a}t = \begin{cases} < 0, & t > 0, \\ > 0, & t < 0, \end{cases} \\
X_0 - X &= \begin{cases} X_0 + (|b|-a)t, & IV, \\ X_0 - a\left(t - |b|y\right), & III, \\ X_0 + |b|\left(t - ay\right), & II, \\ 0, & I. \end{cases}
\end{aligned} \tag{5.77}
$$

Now we find that the spacetime is singular along the line $X_0 = -(|b|-a)t$ in Region IV and the line $X_0 = (a-|b|)t$ in Region III, as shown in Fig. 5.6.

Unlike the last case, now the 3-brane on Σ_1 starts to expand at the singular point B where $\tau = \tau_s$, as shown in Fig. 5.6, and collides with the one on Σ_2 at the moment $\tau = 0$ ($t = 0$). After the collision, its energy density $\rho_m^{(1)}$ the scalar field $\phi^{(2)}$ and the dust energy density $\rho_m^{(2)}$ on Σ_2 become time-dependent, and the corresponding spacetime is described by Eqs.(5.67) and (5.68) with $\eta \in (0, -\infty)$. The corresponding Penrose diagram is given by Fig. 5.7.

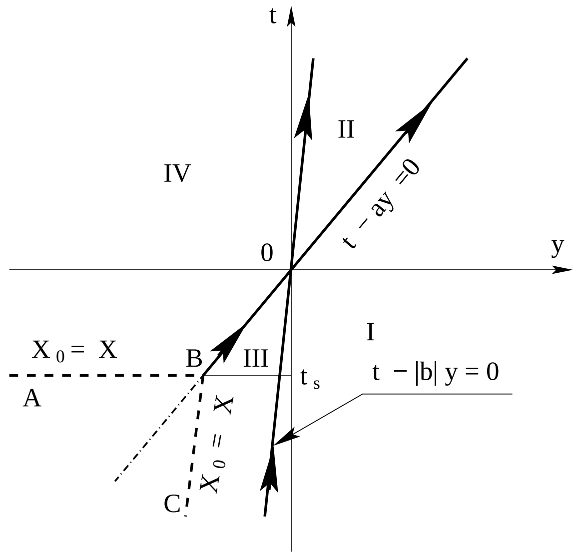


Figure 5.6: The five-dimensional spacetime in the $(t,\ y)$-plane for $-b > a > 1$. The two 3-branes are moving along the hypersurfaces, $\Sigma_1 : t - ay = 0$ and $\Sigma_2 : t - |b|y = 0$. The spacetime is singular along the line AB in Region IV and the line BC in Region III. The spacetime is also singular on the 3-brane at the point B.

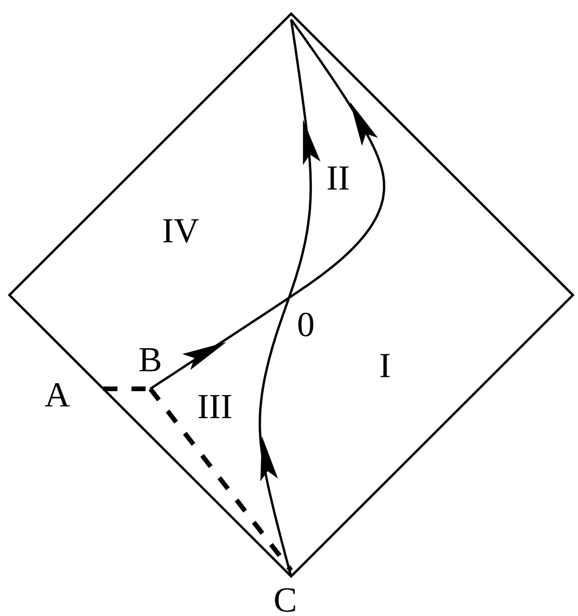


Figure 5.7: The Penrose diagram for $-b > a > 1$. The spacetime is singular along the lines AB and BC.

5.2.6 *Case C:* $a < -1,\ b > 1$

In this case, we find that

$$V_4^{(I)}(\phi) > 0,$$
$$\rho_m^{(I)} = \begin{cases} \geq 0, & \chi^2 \leq 2/3, \\ < 0, & \chi^2 > 2/3, \end{cases} \tag{5.78}$$

where $I = 1,\ 2$. Thus, in contrast to the last case, now both potentials $V_4^{(I)}(\phi)$ are positive, while the two dust energy densities always have the same sign.

Case C1: $-a > b > 1$. When $-a > b > 1$, we have

$$\begin{aligned}
\Phi_1\,|_{\Phi_2=0} &= -\frac{|a|-b}{b}t = \begin{cases} < 0, & t > 0, \\ > 0, & t < 0, \end{cases} \\
\Phi_2\,|_{\Phi_1=0} &= \frac{|a|-b}{|a|}t = \begin{cases} > 0, & t > 0, \\ < 0, & t < 0, \end{cases} \\
X_0 - X &= \begin{cases} X_0 + (|a|-b)t, & IV, \\ X_0 + |a|\left(t + by\right), & III, \\ X_0 - b\left(t + |a|y\right), & II, \\ 0, & I. \end{cases}
\end{aligned} \tag{5.79}$$

Then, the spacetime is singular along the line $X_0 = -(|a|-b)t$ in Region IV, and along the line $X_0 = b\left(t + |a|y\right)$ in Region II, as shown in Fig. 5.8. The corresponding Penrose diagram is given by Fig. 5.9.

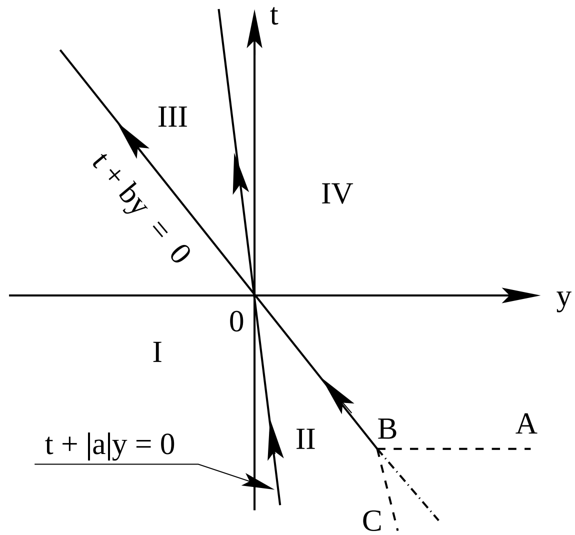


Figure 5.8: The five-dimensional spacetime in the $(t,\ y)$-plane for $-a > b > 1$. The two 3-branes are moving along the hypersurfaces, $\Sigma_1 : t + |a|y = 0$ and $\Sigma_2 : t + by = 0$. The spacetime is singular along the line AB in Region IV and the line BC in Region II. The spacetime is also singular on the 3-brane at the point B where $\eta = \eta_s$.

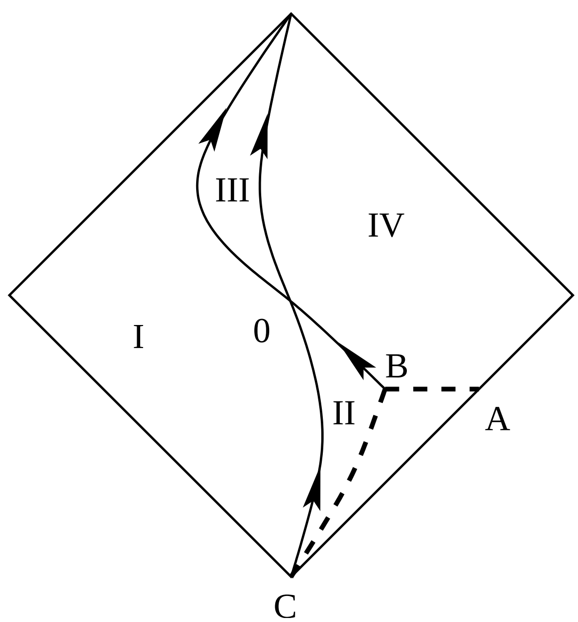


Figure 5.9: The Penrose diagram for $-a > b > 1$. The spacetime is singular along the lines AB and BC.

In this case, we also have

$$
\begin{aligned}
\phi^{(1)}(\tau) &= \begin{cases} \frac{1}{\alpha(3\chi^2+2)} \ln\left[\beta\left(\tau_s - \tau\right)\right], & t > 0, \\ \frac{1}{\alpha} \ln X_0, & t < 0, \end{cases} \\
\phi^{(2)}(\eta) &= \begin{cases} \frac{1}{\alpha} \ln X_0, & t > 0, \\ \frac{1}{\alpha(3\chi^2+2)} \ln\left[\gamma\left(\eta_s - \eta\right)\right], & t < 0, \end{cases} \\
\rho_m^{(1)} &= \begin{cases} \left[\beta\left(\tau_s - \tau\right)\right]^{-\frac{3}{3\chi^2+2}}, & t > 0, \\ X_0^{-1}, & t < 0, \end{cases} \\
\rho_m^{(2)} &= \begin{cases} X_0^{-1}, & t > 0, \\ \left[\gamma\left(\eta_s - \eta\right)\right]^{-\frac{3}{3\chi^2+2}}. & t < 0. \end{cases}
\end{aligned}
\tag{5.80}
$$

Case C2: $b > -a > 1$. When $b > -a > 1$, we have

$$
\begin{aligned}
\Phi_1 \mid_{\Phi_2=0} &= +\frac{b-|a|}{b} t = \begin{cases} > 0, & t > 0, \\ < 0, & t < 0, \end{cases} \\
\Phi_2 \mid_{\Phi_1=0} &= -\frac{b-|a|}{|a|} t = \begin{cases} < 0, & t > 0, \\ > 0, & t < 0, \end{cases} \\
X_0 - X &= \begin{cases} X_0 - (b-|a|)t, & IV, \\ X_0 + |a|\left(t + by\right), & III, \\ X_0 - b\left(t + |a|y\right), & II, \\ 0, & I. \end{cases}
\end{aligned}
\tag{5.81}
$$

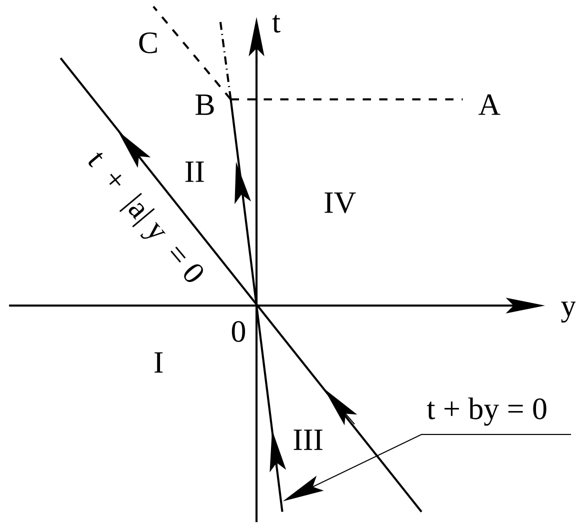


Figure 5.10: The five-dimensional spacetime in the $(t,\ y)$-plane for $b > -a > 1$. The two 3-branes are moving along the hypersurfaces, $\Sigma_1 : t+|a|y = 0$ and $\Sigma_2 : t+by = 0$. The spacetime is singular along the line AB in Region IV and the line BC in Region II. The spacetime is also singular on the 3-brane at the point B where $\eta = \eta_s$.

We also have

$$\begin{aligned}
\phi^{(1)}(\tau) &= \begin{cases} \frac{1}{\alpha}\ln X_0, & t > 0, \\ \frac{1}{\alpha(3\chi^2+2)}\ln\left[\beta\left(\tau_s - \tau\right)\right], & t < 0, \end{cases} \\
\phi^{(2)}(\eta) &= \begin{cases} \frac{1}{\alpha(3\chi^2+2)}\ln\left[\gamma\left(\eta_s - \eta\right)\right], & t > 0, \\ \frac{1}{\alpha}\ln X_0, & t < 0, \end{cases} \\
\rho_m^{(1)} &= \begin{cases} X_0^{-1}, & t > 0, \\ \left[\beta\left(\tau_s - \tau\right)\right]^{-\frac{3}{3\chi^2+2}}, & t < 0, \end{cases} \\
\rho_m^{(2)} &= \begin{cases} \left[\gamma\left(\eta_s - \eta\right)\right]^{-\frac{3}{3\chi^2+2}}. & t > 0, \\ X_0^{-1}, & t < 0. \end{cases}
\end{aligned} \tag{5.82}$$

Then, the spacetime is singular along the line $X_0 = (b-|a|)t$ in Region IV, and along the line $X_0 = b\,(t + |a|y)$ in Region II, as shown in Fig. 5.10. The corresponding Penrose diagram is given by Fig. 5.11.

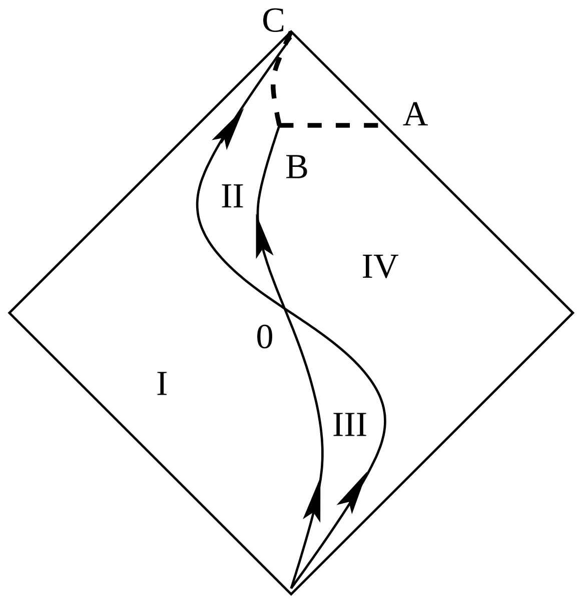


Figure 5.11: The Penrose diagram for $b > -a > 1$. The spacetime is singular along the lines AB and BC.

5.2.7 Case D: $a < -1,\ b < -1$

In this case, we have

$$
\begin{aligned}
V_4^{(1)}(\phi) &< 0, \qquad V_4^{(2)}(\phi) > 0,\\
\rho_m^{(1)} &= \begin{cases} \geq 0, & \chi^2 \geq 2/3,\\[2ex] < 0, & \chi^2 < 2/3, \end{cases}\\
\rho_m^{(2)} &= \begin{cases} \geq 0, & \chi^2 \leq 2/3,\\[2ex] > 0, & \chi^2 > 2/3, \end{cases}
\end{aligned}
\tag{5.83}
$$

and

$$
\begin{aligned}
\Phi_1 \mid_{\Phi_2=0} &= \frac{|a|+|b|}{|b|} t = \begin{cases} >0, & t>0, \\ <0, & t<0, \end{cases} \\
\Phi_2 \mid_{\Phi_1=0} &= \frac{|a|+|b|}{|a|} t = \begin{cases} >0, & t>0, \\ <0, & t<0, \end{cases} \\
X_0 - X &= \begin{cases} X_0 + (|a|+|b|)t, & IV, \\ X_0 + |a|\left(t - |b|y\right), & III, \\ X_0 + |b|\left(t + |a|y\right), & II, \\ 0, & I. \end{cases}
\end{aligned} \tag{5.84}
$$

Then, we find that

$$
\begin{aligned}
\phi^{(1)}(\tau) &= \begin{cases} \frac{1}{\alpha(3\chi^2+2)} \ln\left[\beta\left(\tau_s - \tau\right)\right], & t>0, \\ \frac{1}{\alpha}\ln X_0, & t<0, \end{cases} \\
\phi^{(2)}(\eta) &= \begin{cases} \frac{1}{\alpha(3\chi^2+2)} \ln\left[\gamma\left(\eta_s - \eta\right)\right], & t>0, \\ \frac{1}{\alpha}\ln X_0, & t<0, \end{cases} \\
\rho_m^{(1)} &= \begin{cases} \left[\beta\left(\tau_s - \tau\right)\right]^{-\frac{3}{3\chi^2+2}}, & t>0, \\ X_0^{-1}, & t<0, \end{cases} \\
\rho_m^{(2)} &= \begin{cases} \left[\gamma\left(\eta_s - \eta\right)\right]^{-\frac{3}{3\chi^2+2}}. & t>0, \\ X_0^{-1}, & t<0 \end{cases}
\end{aligned} \tag{5.85}
$$

Note that in the present case, after the collision $t > 0$, we have $\tau,\ \eta < 0$. Thus, in this case the spacetime is free of any kind singularity in all the four regions, as well as on the two branes, as shown in Fig. 5.12. The corresponding Penrose diagram is given by Fig. 5.13.

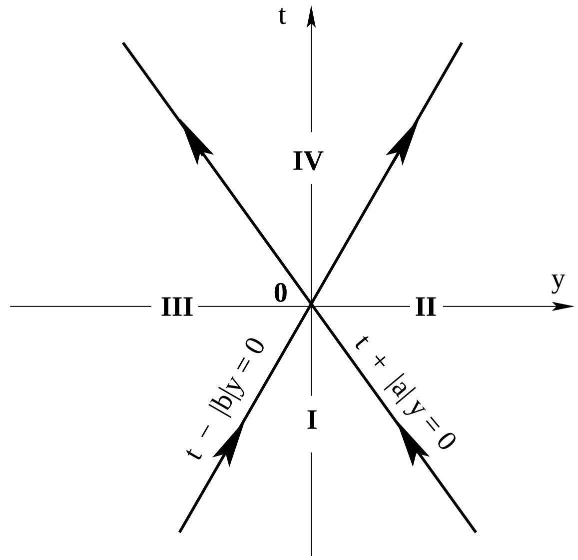


Figure 5.12: The five-dimensional spacetime in the $(t,\ y)$-plane for $a < -1,\ b < -1$. The two 3-branes are moving along the hypersurfaces, $\Sigma_1 : t + |a|y = 0$ and $\Sigma_2 : t - |b|y = 0$. The spacetime is free of any kind of spacetime singularities in the four regions, $I - IV$, as well as on the two 3-branes.

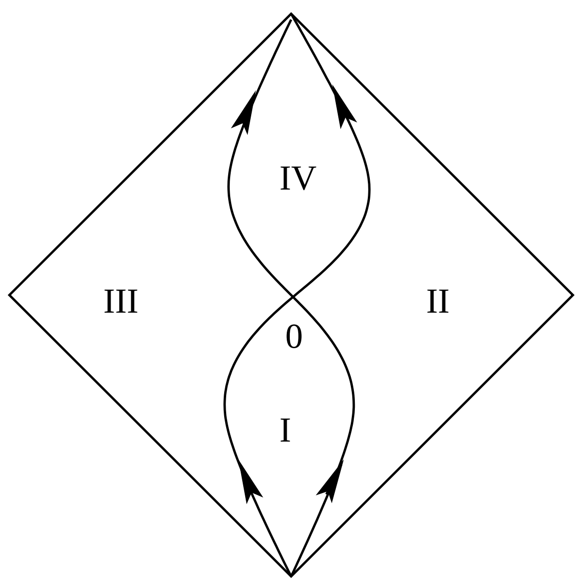


Figure 5.13: The Penrose diagram for $a < -1,\ b < -1$. The spacetime is non-singular in all the regions.

It is interesting to note that when $\chi^2 = 2/3$, the dust fluid on each of the two 3-branes disappears, and the branes are supported only by the tensions, where the brane along Σ_1 has a negative tension, while the one along Σ_1 has a positive tension. It is also interesting to note that, when $\chi^2 \neq 2/3$, both dust fluid are present, but they always have opposite signs, that is, if one satisfies the energy conditions [76], the other one must violate these conditions.

5.3 Colliding 3-branes in the 5-dimensional string frame

The spacetime singularity behavior in general can be quite different in the two frames, due to the conformal transformations of Eq.(2.9), which are often singular. The 5-dimensional spacetime in the string frame is given by

$$\begin{aligned} d^2\hat{s}_5 &\equiv \gamma_{ab}dx^a dx^b \\ &= e^{2\hat{\sigma}(t,y)}\left(dt^2 - dy^2\right) - e^{2\hat{\omega}(t,y)}d\Sigma_0^2, \end{aligned} \tag{5.86}$$

where $d\Sigma_0^2$ is given in Eq.(5.1), and

$$\begin{aligned} \hat{\sigma}(t,y) &\equiv \left(\chi^2 - \epsilon\sqrt{\frac{5}{12}}\,\chi - \frac{1}{3}\right)\ln\left(X_0 - X\right), \\ \hat{\omega}(t,y) &\equiv \left(\frac{1}{3} - \epsilon\sqrt{\frac{5}{12}}\,\chi\right)\ln\left(X_0 - X\right), \\ \hat{\phi}(t,y) &\equiv \left(X_0 - X\right)^{\epsilon\sqrt{\frac{3}{20}}\,\chi}, \end{aligned} \tag{5.87}$$

where $\epsilon = \pm 1$.

5.3.1 The Spacetime Singularities in Regions $I - IV$

To study the spacetime singularities in Regions $I - IV$, let us consider the quantity,

$$\hat{\phi}_{,a}\hat{\phi}^{,a} = \frac{3\chi^2 B}{20\left(X_0 - X\right)^{\frac{4}{5} + \left(\sqrt{\frac{8}{15}} - \epsilon\sqrt{2}\,\chi\right)^2}}, \tag{5.88}$$

where B is given by Eq.(5.60). Comparing the above expression with Eq.(5.59), we find that the spacetime in Regions $I - IV$ is singular in the string frame whenever it is singular in the Einstein frame, although the strength of the singularity is different, as can be seen clearly from the following expression,

$$20\frac{\hat{\phi}_{,a}\hat{\phi}^{,a}}{R} = \frac{3}{2}\left(X_0 - X\right)^{\epsilon\chi\sqrt{\frac{64}{15}}}, \tag{5.89}$$

In particular, if $\epsilon\alpha > 0$ the singularity in the Einstein frame is stronger, and if $\epsilon\alpha < 0$ it is the other way around.

5.3.2 The Spacetime on the 3-brane $\Phi_1 = 0$

On the hypersurface $\Phi_1 = 0$, we have $t = ay$, and so the metric (5.86) reduces to

$$d^2\hat{s}_5 \mid_{t=ay} = d\hat{\tau}^2 - a_u^2\left(\hat{\tau}\right) d\Sigma_0^2, \tag{5.90}$$

where

$$\begin{aligned} a_u(\hat{\tau}) &= \begin{cases} a_0\left(\hat{\tau}_s - \hat{\tau}\right)^{\Delta}, & \Phi_2 > 0, \\ a_0\hat{\tau}_s^{\Delta}, & \Phi_2 < 0, \end{cases} \\ \hat{\phi}^{(1)}(\hat{\tau}) &= \begin{cases} \left[\hat{\beta}\left(\hat{\tau}_s - \hat{\tau}\right)\right]^{\epsilon\sqrt{\frac{3}{20}}\frac{\chi}{\delta}}, & \Phi_2 > 0, \\ \left(\hat{\beta}\hat{\tau}_s\right)^{\epsilon\sqrt{\frac{3}{20}}\frac{\chi}{\delta}}, & \Phi_2 < 0, \end{cases} \end{aligned} \tag{5.91}$$

with

$$\begin{aligned} X_0 - X^{(1)} &= \begin{cases} \left[\hat{\beta}\left(\hat{\tau}_s - \hat{\tau}\right)\right]^{\frac{1}{\delta}}, & \Phi_2 > 0, \\ X_0, & \Phi_2 < 0, \end{cases} \\ \Phi_2 \mid_{\Phi_1=0} &= \frac{a+b}{a}t, \qquad \hat{\beta} \equiv \frac{|a(a+b)|}{\sqrt{a^2-1}}\delta, \\ \hat{\tau}_s &\equiv \hat{\beta}^{-1}X_0^{\delta}, \qquad a_0 \equiv \hat{\beta}^{\Delta}. \\ \delta &\equiv \left(\sqrt{\frac{5}{48}} - \epsilon\chi\right)^2 + \frac{9}{16} > 0, \\ \Delta &\equiv \frac{1}{\delta}\left(\frac{1}{3} - \epsilon\chi\sqrt{\frac{5}{12}}\right). \end{aligned} \tag{5.92}$$

Note that in writing the above expressions, we had chosen $\epsilon_{\hat{\tau}} = \text{sign}(a+b)$. To study the spacetime singularity on the brane, we calculate the Ricci scalar, which now is given by

$$R_{u\ \ \lambda}^{(4)\lambda} = \frac{3\Delta\left(2-\Delta\right)}{2a_0\left(\hat{\tau}_s - \hat{\tau}\right)^{\Delta+2}}, \tag{5.93}$$

where

$$\begin{aligned} \Delta+2 &= \frac{1}{\delta}\left[2\left(\epsilon\chi - \sqrt{\frac{15}{64}}\right)^2 + \frac{115}{96}\right] > 0, \\ \Delta-2 &= -\frac{1}{\delta}\left[2\left(\epsilon\chi - \sqrt{\frac{5}{48}}\right)^2 + \frac{19}{24}\right] < 0. \end{aligned} \tag{5.94}$$

5.3.3 The Spacetime on the 3-brane $\Phi_2 = 0$

Similarly, on the 3-brane located on the hypersurface $\Phi_2 = 0$, we have $t = -by$ and so the metric (5.86) reduces to

$$d^2\hat{s}_5 \mid_{t=-by} = d\hat{\eta}^2 - a_v^2\left(\hat{\eta}\right)d\Sigma_0^2, \tag{5.95}$$

where

$$\begin{aligned} a_v(\hat{\eta}) &= \begin{cases} a_0\left(\hat{\eta}_s - \hat{\eta}\right)^{\Delta}, & \Phi_1 > 0, \\ a_0\hat{\eta}_s^{\Delta}, \Phi_1 < 0, \end{cases} \\ \hat{\phi}^{(2)}(\hat{\eta}) &= \begin{cases} \left[\hat{\gamma}\left(\hat{\eta}_s - \hat{\eta}\right)\right]^{\epsilon\sqrt{\frac{3}{20}}\frac{\chi}{\delta}}, & \Phi_1 > 0, \\ \left(\hat{\gamma}\hat{\eta}_s\right)^{\epsilon\sqrt{\frac{3}{20}}\frac{\chi}{\delta}}, & \Phi_1 < 0, \end{cases} \end{aligned} \tag{5.96}$$

with

$$\begin{aligned} X_0 - X^{(2)} &= \begin{cases} \left[\hat{\gamma}\left(\hat{\eta}_s - \hat{\eta}\right)\right]^{\frac{1}{\delta}}, & \Phi_1 > 0, \\ X_0, & \Phi_1 < 0, \end{cases} \\ \Phi_1\Big|_{\Phi_2=0} &= \frac{a+b}{b}t, \qquad \hat{\gamma} \equiv \frac{|a(a+b)|}{\sqrt{b^2-1}}\delta, \\ \hat{\eta}_s &\equiv \hat{\gamma}^{-1}X_0^{\delta}, \end{aligned} \tag{5.97}$$

but now we have $a_0 \equiv \hat{\gamma}^{\Delta}$ and $\epsilon_{\hat{\eta}} = \text{sign}(a+b)$. For the metric (5.95), we also find that

$$R_{v\ \ \lambda}^{(4)\lambda} = \frac{3\Delta\left(2-\Delta\right)}{2a_0\left(\hat{\eta}_s - \hat{\eta}\right)^{\Delta+2}}. \tag{5.98}$$

From Eqs.(5.93) and (5.98) we can see that the spacetime on each of the branes is not singular when $\Delta = 0$ or $\chi = \epsilon\sqrt{\frac{4}{15}}$. As a matter of fact, in this case the spacetime on each of the two branes is flat. Thus, in the following we need to consider only the case $\chi \neq \epsilon\sqrt{\frac{4}{15}}$.

From Eqs.(5.91)-(5.94) and Eqs.(5.96)-(5.98), it can be shown that the spacetime singularities on each of the two branes are similar to these in the Einstein frame. For example, for the case $a > 1,\ b > 1$, it is singular at $\hat{\tau} = \hat{\tau}_s$ and $\hat{\eta} = \hat{\eta}_s$, which correspond to, respectively, the point A and B in Fig. 5.3. Similarly, the spacetime is free from any kind of singularities for the case $a < -1,\ b < -1$, and the corresponding Penrose diagram is also given by Fig. 5.13.

5.4 Colliding 3-branes in the 10-dimensional Spacetimes

Lifting the metric to 10-dimensions, it is given by Eq.(5.1), which can be cast in the form,

$$\begin{aligned} d^2\hat{s}_{10} &\equiv \gamma_{ab}dx^a dx^b + \hat{\phi}^2\left(x^c\right)\hat{\gamma}_{ij}\left(z^k\right)dz^i dz^j \\ &= e^{2\hat{\sigma}(t,y)}\left(dt^2 - dy^2\right) - e^{2\hat{\omega}(t,y)}d\Sigma_0^2 - \hat{\phi}^2\left(t,y\right)d\Sigma_z^2, \end{aligned} \tag{5.99}$$

where $\hat{\sigma},\ \hat{\omega}$ and $\hat{\phi}$ are given by Eq.(5.87), and $d\Sigma_z^2 \equiv -\sum_{i,j=1}^{5}\hat{\gamma}_{ij}\left(z^k\right)dz^i dz^j$. Then, it can be shown that the spacetime in Regions $I - IV$ is vacuum,

$$R_{AB}^{(A)} = 0, \tag{5.100}$$

where $A = I, ..., IV$, as it is expected. To study the singular behavior of the spacetime in these regions, we calculate the Kretschmann scalar, which in the present case is given by

$$\begin{aligned} I_{10} &\equiv R_{ABCD}R^{ABCD} \\ &= \frac{B^2 I_{10}^{(0)}}{\left(X_0 - X\right)^{\left(2\chi - \epsilon\sqrt{\frac{5}{12}}\right)^2 + \frac{9}{4}}}, \end{aligned} \tag{5.101}$$

where B is given by Eq.(5.60), and

$$\begin{aligned} I_{10}^{(0)} &\equiv \frac{1}{45}\Big[\Big(720\chi^6 + 1287\chi^4 + 200\chi^2 + 40\Big) \\ &\quad -312\epsilon\sqrt{\frac{5}{3}}\chi^3\Big(2+3\chi^2\Big)\Big]. \end{aligned} \tag{5.102}$$

It can be shown that $I_{10}^{(0)}$ is non-zero for any given χ. Then, comparing the expression of Eq.(5.101) with Eq.(5.59), we find that the lifted 10-dimensional spacetime has a similar singular behavior as that in the 5-dimensional spacetime in the Einstein frame. In particular, it is also singular on the hypersurface $X_0 - X = 0$.

5.4.1 The Spacetime on the 3-brane $\Phi_1 = 0$

On the hypersurface $t = ay$, the metric (5.99) reduces to

$$d^2\hat{s}_5 \,|_{t=ay} = d\hat{\tau}^2 - a_u^2\left(\hat{\tau}\right)d\Sigma_0^2 - b_u^2\left(\hat{\tau}\right)d\Sigma_z^2, \tag{5.103}$$

where $a_u\left(\hat{\tau}\right)$ and $b_u\left(\hat{\tau}\right) \equiv \hat{\phi}^{(1)}\left(\hat{\tau}\right)$ are given by Eqs.(5.91) and (5.92). On the 8-brane, the Einstein tensor has distribution given by Eqs.(5.124) and (5.125). Inserting Eq.(5.87) into Eq.(5.92), and noticing that $\hat{\psi} \equiv \ln\left(\hat{\phi}\right)$, we find

$$\begin{aligned} \hat{\rho}_u &= \frac{b\left(a^2-1\right)}{\left[X_0 - X^{(1)}(t)\right]^{\mu}}, \\ \hat{p}_u^Z &= -\frac{b\left(a^2-1\right)}{\left[X_0 - X^{(1)}(t)\right]^{\mu}}\left[\left(\chi - \epsilon\sqrt{\frac{4}{15}}\right)^2 + \frac{2}{5}\right], \\ \hat{p}_u^X &= -\frac{b\left(a^2-1\right)}{\left[X_0 - X^{(1)}(t)\right]^{\mu}}\left(\chi^2 + \frac{1}{3}\right), \end{aligned} \tag{5.104}$$

where $X^{(1)}(t)$ is given by Eq.(5.92), and

$$\mu \equiv 2\left(\chi - \epsilon\sqrt{\frac{5}{48}}\right)^2 + \frac{1}{8}. \tag{5.105}$$

Clearly, whenever $X_0 - X^{(1)}(t) = 0$, the spacetime on the 8-brane is singular.

5.4.2 The Spacetime on the 3-brane $\Phi_2 = 0$

On the hypersurface $t = -by$, the metric (5.99) reduces to

$$d^2\hat{s}_5 \,|_{t=-by} = d\hat{\eta}^2 - a_v^2\left(\hat{\eta}\right)d\Sigma_0^2 - b_v^2\left(\hat{\eta}\right)d\Sigma_z^2, \tag{5.106}$$

where $a_v(\hat{\eta})$ and $b_v(\hat{\eta}) \equiv \hat{\phi}^{(2)}(\hat{\eta})$ are given by Eqs.(5.96) and (5.97). On this 8-brane, the Einstein tensor has distribution given by Eqs.(5.127) and (5.128), which in the present case yield,

$$
\begin{aligned}
\hat{\rho}_v &= \frac{a\left(b^2-1\right)}{\left[X_0 - X^{(2)}(t)\right]^{\mu}}, \\
\hat{p}_v^Z &= -\frac{a\left(b^2-1\right)}{\left[X_0 - X^{(2)}(t)\right]^{\mu}}\left[\left(\chi - \epsilon\sqrt{\frac{4}{15}}\right)^2 + \frac{2}{5}\right], \\
\hat{p}_v^X &= -\frac{a\left(b^2-1\right)}{\left[X_0 - X^{(2)}(t)\right]^{\mu}}\left(\chi^2 + \frac{1}{3}\right),
\end{aligned}
\tag{5.107}
$$

where $X^{(2)}(t)$ is given by Eq.(5.97). Thus, the spacetime on this 8-brane is also singular whenever $X_0 - X^{(2)}(t) = 0$.

When $a > 1$ and $b > 1$, from Eqs.(5.104) and (5.107) it can be shown that both of the weak and dominant energy conditions [76] are satisfied by the matter fields on the two 8-branes, provided that

$$
\begin{cases}
\sqrt{\frac{4}{15}} - \sqrt{\frac{3}{5}} \le \chi \le \sqrt{\frac{2}{3}}, & \epsilon = +1, \\
-\sqrt{\frac{2}{3}} \le \chi \le \sqrt{\frac{3}{5}} - \sqrt{\frac{4}{15}}, & \epsilon = -1,
\end{cases}
\tag{5.108}
$$

but the strong energy condition is always violated. When $a > 1$ and $b < -1$, the matter field on the 8-brane $\Phi_1 = 0$ violates all the three energy conditions, while the one on the 8-brane $\Phi_2 = 0$ satisfies the weak and dominant energy conditions, provided that the conditions (5.108) holds, but violates the strong one.

When $a < -1$ and $b > 1$, it is the other way around, that is, the matter field on the 8-brane $\Phi_1 = 0$ satisfies the weak and dominant energy conditions, provided that the conditions (5.108) holds, but violates the strong one, while the one on the 8-brane $\Phi_2 = 0$ violates all the three energy conditions.

When $a < -1$ and $b < -1$, the matter fields on the two 8-branes all violate the three energy conditions. However, in all these four cases, the spacetime singular behavior is similar to the corresponding 5-dimensional cases in the Einstein frame.

In particular, in the first three cases the spacetime in the four regions and on the 8-branes are always singular, and the corresponding Penrose diagrams are given, respectively, by Figs. 5.3, 5.5, 5.7, 5.9, and 5.11, but now each point in these figures now represents a 8-dimensional spatial space.

In the last case, in which the matter fields on the two 8-branes violate all the energy conditions, the spacetime is free of any kind of spacetime singularities, either in Regions $I-IV$ or on the two 8-branes, and the corresponding Penrose diagram is given by Fig. 5.13. Therefore, all the above results seemingly indicate that violating the energy conditions is a necessary condition for spacetimes of colliding branes to be non-singular.

5.5 Gravitational Field Equations in the 10-dimensional Bulk and on the 8-branes

For the metric,

$$d^2\hat{s}_{10} = e^{2\hat{\sigma}(t,y)}\left(dt^2 - dy^2\right) - e^{2\hat{\omega}(t,y)}d\Sigma_0^2 - \hat{\phi}^2\left(t,y\right)d\Sigma_z^2, \tag{5.109}$$

where

$$d\Sigma_0^2 \equiv \sum_{p=2}^{4}\left(dx^p\right)^2, \quad d\Sigma_z^2 \equiv \sum_{i=1}^{5}\left(dz^i\right)^2, \tag{5.110}$$

the non-vanishing components of the Einstein tensor are given by,

$$\begin{aligned}
G^{(10)}_{tt} &= 3\hat{\omega}_{,t}\left(\hat{\sigma}_{,t} + \hat{\omega}_{,t}\right) + 5\hat{\psi}_{,t}\left(\hat{\sigma}_{,t} + 3\hat{\omega}_{,t} + 2\hat{\psi}_{,t}\right)\\
&\quad -3\hat{\omega}_{,yy} - 5\hat{\psi}_{,yy} - 15\hat{\psi}_{,y}\left(\hat{\omega}_{,y} + \hat{\psi}_{,y}\right)\\
&\quad +\hat{\sigma}_{,y}\left(3\hat{\omega}_{,y} + 5\hat{\psi}_{,y}\right) - 6\hat{\omega}^2_{,y},
\end{aligned} \tag{5.111}$$

$$\begin{aligned}
G^{(10)}_{ty} &= -3\hat{\omega}_{,ty} - 5\hat{\psi}_{,ty}\\
&\quad +3\left(\hat{\sigma}_{,t}\hat{\omega}_{,y} + \hat{\sigma}_{,y}\hat{\omega}_{,t} - \hat{\omega}_{,t}\hat{\omega}_{,y}\right)\\
&\quad +5\left(\hat{\sigma}_{,t}\hat{\psi}_{,y} + \hat{\sigma}_{,y}\hat{\psi}_{,t} - \hat{\psi}_{,t}\hat{\psi}_{,y}\right),
\end{aligned} \tag{5.112}$$

$$
\begin{aligned}
G^{(10)}_{yy} &= -3\hat{\omega}_{,tt} - 5\hat{\psi}_{,tt} - 15\hat{\psi}_{,t}\left(\hat{\omega}_{,t} + \hat{\psi}_{,t}\right) \\
&\quad +\hat{\sigma}_{,t}\left(3\hat{\omega}_{,t} + 5\hat{\psi}_{,t}\right) - 6\hat{\omega}^2_{,t} \\
&\quad +3\hat{\omega}_{,y}\left(\hat{\sigma}_{,y} + \hat{\omega}_{,y}\right) \\
&\quad +5\hat{\psi}_{,y}\left(\hat{\sigma}_{,y} + 3\hat{\omega}_{,y} + 2\hat{\psi}_{,y}\right),
\end{aligned} \tag{5.113}
$$

$$
\begin{aligned}
G^{(10)}_{pq} &= \delta_{pq} e^{2(\hat{\omega}-\hat{\sigma})}\Big\{\hat{\sigma}_{,yy} + 2\hat{\omega}_{,yy} + 5\hat{\psi}_{,yy} \\
&\quad +5\hat{\psi}_{,y}\left(2\hat{\omega}_{,y} + 3\hat{\psi}_{,y}\right) + 3\hat{\omega}^2_{,y} \\
&\quad -\Big[\hat{\sigma}_{,tt} + 2\hat{\omega}_{,tt} + 5\hat{\psi}_{,tt} \\
&\quad +3\hat{\omega}^2_{,t} + 5\hat{\psi}_{,t}\left(2\hat{\omega}_{,t} + 3\hat{\psi}_{,t}\right)\Big]\Big\},
\end{aligned} \tag{5.114}
$$

$$
\begin{aligned}
G^{(10)}_{ij} &= \delta_{ij} e^{2\left(\hat{\psi}-\hat{\sigma}\right)}\Big\{\hat{\sigma}_{,yy} + 3\hat{\omega}_{,yy} + 4\hat{\psi}_{,yy} \\
&\quad +2\hat{\psi}_{,y}\left(6\hat{\omega}_{,y} + 5\hat{\psi}_{,y}\right) + 6\hat{\omega}^2_{,y} \\
&\quad -\Big[\hat{\sigma}_{,tt} + 3\hat{\omega}_{,tt} + 4\hat{\psi}_{,tt} \\
&\quad -10\hat{\psi}^2_{,t} + 6\hat{\omega}_{,t}\left(\hat{\omega}_{,t} + 2\hat{\psi}_{,t}\right)\Big]\Big\},
\end{aligned} \tag{5.115}
$$

where $p,\ q = 2,\ 3,\ 4$ and $i,\ j = 1, ...,\ 5$, and $\hat{\psi} \equiv \ln\left(\hat{\phi}\right)$.

5.5.1 Field Equations on the Hypersurface $\Phi_1 = 0$

Following Section 5.1.2, it can be shown that the derivatives of any given function $F(t, y)$, which is C^0 across the hypersurface $\Phi_1 = 0$ and at least C^2 in the regions $\Phi_1 > 0$ and $\Phi_1 < 0$, are given by Eq.(5.33) but now with N being replaced by $\hat{N}$, and n_a and u_a by, respectively, $\hat{n}_a$ and $\hat{u}_a$, where

$$
\begin{aligned}
\hat{n}_a &= \hat{N}\left(\delta^t_a - a\delta^y_a\right), \\
\hat{u}_a &= \hat{N}\left(a\delta^t_a - \delta^y_a\right), \\
\hat{N} &\equiv \frac{e^{\hat{\sigma}^{(1)}}}{\left(a^2-1\right)^{1/2}}.
\end{aligned} \tag{5.116}
$$

Hence, Eq.(5.115) can be cast in the form,

$$
G^{(10)}_{ab} = G^{(10)+}_{ab} H\left(\Phi_1\right) + G^{(10)-}_{ab}\left[1 - H\left(\Phi_1\right)\right] + G^{(10)Im}_{ab}\delta\left(\Phi_1\right), \tag{5.117}
$$

where $G_{ab}^{(10)+}$ $\left(G_{ab}^{(10)-}\right)$ is the Einstein tensor calculated in the region $\Phi_1 > 0$ $(\Phi_1 < 0)$, and $G_{ab}^{(10)Im}$ denotes the distribution of the Einstein tensor on the hypersurface $\Phi_1 = 0$, which has the following non-vanishing components,

$$G_{tt}^{(10)Im} = a^2 \hat{N} \left(3 \left[\hat{\omega}_n \right]^- + 5 \left[\hat{\psi}_n \right]^- \right), \tag{5.118}$$

$$G_{ty}^{(10)Im} = -a\hat{N} \left(3 \left[\hat{\omega}_n \right]^- + 5 \left[\hat{\psi}_n \right]^- \right), \tag{5.119}$$

$$G_{yy}^{(10)Im} = \hat{N} \left(3 \left[\hat{\omega}_n \right]^- + 5 \left[\hat{\psi}_n \right]^- \right), \tag{5.120}$$

$$G_{pq}^{(10)Im} = -\delta_{pq} \hat{N}^{-1} e^{2\hat{\omega}^{(1)}} \left(\left[\hat{\sigma}_n \right]^- + 2 \left[\hat{\omega}_n \right]^- + 5 \left[\hat{\psi}_n \right]^- \right), \tag{5.121}$$

$$G_{ij}^{(10)Im} = -\delta_{ij} \hat{N}^{-1} e^{2\hat{\psi}^{(1)}} \left(\left[\hat{\sigma}_n \right]^- + 3 \left[\hat{\omega}_n \right]^- + 4 \left[\hat{\psi}_n \right]^- \right). \tag{5.122}$$

Introducing the unit vectors,

$$X_a^{(p)} = e^{\hat{\omega}^{(1)}} \delta_a^p, \quad Z_a^{(i)} = e^{\hat{\psi}^{(1)}} \delta_a^i, \tag{5.123}$$

we find that Eq.(5.118) can be cast in the form,

$$G_{ab}^{(10)Im} = \kappa_{10}^2 \left(\hat{\rho}_u \hat{u}_a \hat{u}_b + \hat{p}_u^X \sum_{p=2}^{4} X_a^{(p)} X_b^{(p)} + \hat{p}_u^Z \sum_{i=1}^{5} Z_a^{(i)} Z_b^{(i)} \right), \tag{5.124}$$

where

$$\begin{aligned} \hat{\rho}_u &= \frac{1}{\hat{N}\kappa_{10}^2} \left(3 \left[\hat{\omega}_n \right]^- + 5 \left[\hat{\psi}_n \right]^- \right), \\ \hat{p}_u^X &= \frac{1}{\hat{N}\kappa_{10}^2} \left(\left[\hat{\sigma}_n \right]^- + 2 \left[\hat{\omega}_n \right]^- + 5 \left[\hat{\psi}_n \right]^- \right), \\ \hat{p}_u^Z &= \frac{1}{\hat{N}\kappa_{10}^2} \left(\left[\hat{\sigma}_n \right]^- + 3 \left[\hat{\omega}_n \right]^- + 4 \left[\hat{\psi}_n \right]^- \right). \end{aligned} \tag{5.125}$$

5.5.2 Field Equations on the hypersurface $\Phi_2 = 0$

Similarly, it can be shown that, crossing the hypersurface $\Phi_2 = 0$, Eq.(5.115) can be cast in the form,

$$G_{ab}^{(10)} = G_{ab}^{(10)+} H(\Phi_2) + G_{ab}^{(10)-} \left[1 - H(\Phi_2) \right] + G_{ab}^{(10)Im} \delta(\Phi_2), \tag{5.126}$$

but now $G^{(10)+}_{ab}$ $\left(G^{(10)-}_{ab}\right)$ is the Einstein tensor calculated in the region $\Phi_2 > 0$ $(\Phi_2 < 0)$, and $G^{(10)Im}_{ab}$ denotes the distribution of the Einstein tensor on the hypersurface $\Phi_2 = 0$, which can be written in the form,

$$G^{(10)Im}_{ab} = \kappa^2_{10}\left(\hat{\rho}_v \hat{v}_a \hat{v}_b + \hat{p}^X_v \sum_{p=2}^{4} X^{(p)}_a X^{(p)}_b + \hat{p}^Z_v \sum_{i=1}^{5} Z^{(i)}_a Z^{(i)}_b\right), \tag{5.127}$$

where

$$\begin{aligned}
\hat{\rho}_v &= \frac{1}{\hat{L}\kappa^2_{10}}\left(3\left[\hat{\omega}_l\right]^- + 5\left[\hat{\psi}_l\right]^-\right), \\
\hat{p}^X_v &= \frac{1}{\hat{L}\kappa^2_{10}}\left(\left[\hat{\sigma}_l\right]^- + 2\left[\hat{\omega}_l\right]^- + 5\left[\hat{\psi}_l\right]^-\right), \\
\hat{p}^Z_v &= \frac{1}{\hat{L}\kappa^2_{10}}\left(\left[\hat{\sigma}_l\right]^- + 3\left[\hat{\omega}_l\right]^- + 4\left[\hat{\psi}_l\right]^-\right),
\end{aligned} \tag{5.128}$$

and

$$\begin{aligned}
X^{(p)}_a &= e^{\hat{\omega}^{(2)}}\delta^p_a, \qquad Z^{(i)}_a = e^{\hat{\psi}^{(2)}}\delta^i_a, \\
\hat{l}_a &= \hat{L}\left(\delta^t_a + b\delta^y_a\right), \qquad \hat{v}_a = \hat{L}\left(b\delta^t_a + \delta^y_a\right), \\
\hat{L} &\equiv \frac{e^{\hat{\sigma}^{(2)}}}{\left(b^2-1\right)^{1/2}}.
\end{aligned} \tag{5.129}$$

5.6 Conclusions

In this chapter, we have applied our model to the case $D = 5 = d$ for a class of spacetimes. We first obtained explicitly the field equations both outside and on the 3-branes in terms of distributions. We then considered a class of exact solutions that represents the collision of two 3-branes in the Einstein frame, and studied their local and global properties in detail.

We have found, among other things, the collision in general ends up with the formation of spacetime singularities, due to the mutual focus of the colliding branes, although non-singular spacetime also exist, with the price that both of the two branes violate all the energy conditions, weak, strong and dominant. Similar conclusions hold also in the 5-dimensional string frame.

After lifting the solutions to 10-dimensional spacetimes, we have found that the corresponding solutions represent the collision of two timelike 8-branes without Z_2 symmetry. In some cases the two 8-branes satisfy the weak and dominant energy conditions, while in other case, they do not. But, in all these cases the strong energy condition is always violated. The formation of spacetime singularities due to the mutual focus of the two colliding branes occurs in general, although the non-singular cases also exist with the price that both of the two branes violate all the three energy conditions. The spacetime singular behavior is similar in the 5-dimensional effective theory to that of 10-dimensional string theory.

At this point, we have concluded a first round of investigations. We have obtained a baseline of results that we will be using in our future work, as we slowly increase the complexity of our model. In the next chapter we report on our current research and ideas for future projects.

CHAPTER SIX

Conclusions and Remarks

6.1 Concluding Remarks

In this work we presented the first steps into a greater investigation of braneworld scenarios. Starting from a simple model described by the NS-NS sector, appropriate for string brane gas cosmology, we set the dilaton and form field to zero. We analyzed the nature of the spacetimes in the string and einstein frames and found that one must always explicitly identify the singularities present. We concluded that singularities present in one of the frames, are not always present in the other. We found that the opposite is also true. A frame free of singularities may in fact develop them, when viewed from a higher or lower dimensional frame [86].

We then constructed a class of exact solutions, describing the collision of two timelike 3-branes in a five-dimensional Einstein background. We found that spacelike singularities generically develop after the collision, due to the mutual focus of the two branes. Non-singular spacetimes can be constructed only in the case where both of the two branes violate the energy conditions [88]. Therefore the formation of spacetime singularities is closely related to the nature of the matter present on each brane, and their conformity with the accepted energy conditions [76].

In the final chapter we combined these conclusions and explored a string gas collision scenario, in the eleven-dimensional NS-NS sector of Type II string theory, without Z_2 reflection symmetry. We arrive at the effective five-dimensional string and einstein frames by toroidal compactification. There we find that, for both frames, the collision of the 3-branes results in the formation of spacetime singularities. In some specific cases the singularities can be avoided, but at the price of both the branes violating all energy conditions, weak, string and dominant. When the solutions are lifted to the ten-dimensional spacetime, we have found the notable result that the corresponding timelike 8-branes always violate the strong energy condition. Here, the collisions also result in spacetime singularities, where singularity free spacetimes are again only produced when both branes violate all the three energy conditions. Thus the spacetime singular behavior is similar in the five-dimensional effective theory to that of ten- dimensional string theory.

6.2 Prospects and Future Work

As we pointed out in the introduction, brane cosmologies are a very new field and one should tread carefully, while developing an understanding of the theory. Even then, exact solutions are very hard to come by and often one must resort to numerical methods to solve the appropriate equations. Here we have succeeded in obtaining some of the very few analytical results in string cosmology, by establishing the characteristics of a string theory appropriate to brane collisions and also by studying the fundamentals of brane collisions in a 'toy' five-dimensional model. We are now ready to proceed by adding to the complexity and hopefully to the richness of our results.

In a first step, we are currently exploring the spacetime in the NS-NS sector with non-zero dilatonic field. This is a first step towards a full blown analytical description of Type II string theory, in the context of brane collisions. Once successful, we will attempt to include the form field. There is a great amount of literature on possible uses/applications of the dilaton to cosmology. The same does not hold true for the form field, as one finds that it is often divergent, leading to unphysical cosmologies. By slowly adding stringy effects to our model however, we aim to understand what exactly the effect of such form fields are, and hopefully devise ways of dealing with them.

On the other hand, we have been concurrently working on colliding branes in M-theory, a field that is central to brane cosmologies and is receiving widespread attention. In short, in our model we consider the spacetimes,

$$ds^2_{11} = V^{-2/3}\gamma_{\alpha\beta}dx^\alpha dx^\beta - V^{1/3}\Omega_{ab}dz^a dz^b \tag{6.1}$$

where $d\Omega_3^2 = d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi d\psi^2)$, is the metric of the unit 3-sphere, $ds^2_{CY,6} = \Omega_{ab}dz^a dx^b$, denotes the Calabi-Yau 3-fold, and V is the Calabi-Yau volume modulus, that measures the deformation of the CY space, and depends only on the coordinates z^a. The effective five-dimensional Horawa-Witten action is given by [29, 70, 71, 72]

$$\begin{aligned} S_5 &= \frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-\gamma}\left(R[\gamma] + \frac{1}{2}(\nabla\phi)^2 - 6\alpha^2 e^{-2\phi}\right) + \\ &\quad + \sum_{i=1}^{2}\epsilon_i\frac{6\alpha}{\kappa_5^2}\int_{M_5^{(i)}}\sqrt{-g^{(i)}}e^{-\phi} \end{aligned} \tag{6.2}$$

where in turn $\epsilon_1 = -\epsilon_2 = 1$, represent the positions of the two branes, $\phi = ln(V)$ is the scalar field, $\kappa_5^2 = \kappa_{11}^2/V_{CY,6}$, with $V_{CY,6} = \int_x \sqrt{\Omega}$ representing the volume of the Calabi-Yau space, α is a constant related to the internal four-form that is needed for the dimensional reduction and that is a result of the source terms in the $11D$ Bianchi identity and finally, $g^{(i)}$ are the reduced metrics on the two boundaries $M_4^{(i)}$.

We have been successful in producing brane collision scenarios, similar to the cases explored in the present work. However, we believe that maintaining the uniform theme of collisions in the NS-NS sector of Type II string theory, is more instructive in outlining our focus and research objectives.

In short, we have found an increased number of diverse cases, which in general concur with the results of this work, where spacetimes tend to be free of singularities, when the energy conditions are not satisfied on the branes. A major difference is that the branes are found to have energy conditions that evolve dynamically throughout the collisions, and are often seen to evolve from one type of energy condition to another, even after the collision has taken place [93]. This opens up a wide range of exciting cosmological implications that we are eager to address in future work.

APPENDICES

APPENDIX A

The Equations of General Relativity

In this section we provide a concise reference to the equations of General Relativity. In the body of this work we assume normalized natural units, $c = \hbar = 1$ and we will use $(\ )_{,a} \equiv \partial(\ )/\partial x^a$ to represent partial, and $(\ )_{;a}$ to represent covariant differentiation.

A *tensor* is defined as an invariant property of spacetime, if it transforms according to,

$$g_{ab}(x'(x)) = \frac{\partial x^\mu}{\partial x'^a}\frac{\partial x^\nu}{\partial x'^b} g_{\mu\nu}(x). \tag{A.1}$$

A *metric* describes the spacetime interval between two points,

$$ds^2 = g_{ab} dx^a dx^b, \tag{A.2}$$

where $x^a = \{x^0, x^1, ..., x^{d-1}\}$, where d is the number of dimensions in the spacetime, and $g_{ab} = g_{ba}$. 4-dimensional flat spaces are called *Minkowski spacetimes*, and have a metric given by,

$$\begin{aligned} ds^2 &= \eta_{ab} dx^a dx^b \\ &= dt^2 - dx^2 - dy^2 - dz^2. \end{aligned} \tag{A.3}$$

The tensor η_{ab} has form,

$$\eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{A.4}$$

In this work we will adopt the Landau-Lifshitz sign convention, where our metric has signs $(+,-,-,-)$. Although this is merely a matter of convention, it is important to note as the Riemann tensor and Einstein equation are expressed differently under other conventions.

A metric expresses qualitatively the homogeneities and isotropies of the spacetime. For example, in a 4-dimensional spacetime, these can be identified as follows:

Homogeneous and Isotropic:

$$ds^2 = dt^2 - a(t)\sum_{i=1}^{3} dx_i^2, \tag{A.5}$$

Inhomogeneous and Isotropic:

$$ds^2 = dt^2 - a(t,r)\sum_{i=1}^{3} dx_i^2, \tag{A.6}$$

Homogeneous and Anisotropic:

$$ds^2 = dt^2 - a_1(t)dx_1^2 - a_2(t)dx_2^2 - a_3(t)dx_3^2, \tag{A.7}$$

Inhomogeneous and Anisotropic:

$$ds^2 = dt^2 - a_1(t,r)dx_1^2 - a_2(t,r)dx_2^2 - a_3(t,r)dx_3^2, \tag{A.8}$$

where $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$.

The *Christoffel symbol*, describes the covariant derivatives of the unit vectors in our coordinate frame,

$$\Gamma^a_{bc} = \frac{1}{2}\, g^{ab}\, (g_{cd,b} + g_{bd,c} - g_{bc,d}). \tag{A.9}$$

The *Riemann Tensor* defines the curvature of spacetime, thus is a basic notion for describing gravitational fields,

$$R^a_{\ bcd} = \Gamma^a_{\ db,c} - \Gamma^a_{\ cb,d} + \Gamma^e_{\ bd}\Gamma^a_{\ ec} - \Gamma^e_{\ bc}\Gamma^a_{\ ed}. \tag{A.10}$$

The *Weyl Tensor* or *Conformal Tensor* has the property that it is trace-free, thus vanishes under any pair of contractions, $C^{a}{}_{bad} = 0$, and is defined in d-dimensions as [77],

$$\begin{aligned} C_{abcd} &= R_{abcd} + \frac{1}{d-2}\left(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} - g_{bd}R_{ca}\right) \\ &\quad - \frac{1}{(d-1)(d-2)}(g_{ac}g_{db} - g_{ad}g_{cb})R. \end{aligned} \tag{A.11}$$

The *Ricci Tensor* contains encoded curvature information and is zero for flat spacetimes, and results from contracting the first and third indices of the Riemann tensor,

$$R_{ab} = R^{e}{}_{aeb} = g^{ed}R_{daeb}. \tag{A.12}$$

The *Ricci Scalar* results from another contraction if the Ricci tensor with the metric,

$$R = g^{ab}R_{ab}. \tag{A.13}$$

The *Einstein field equation*, expresses how the geometry of the spacetime is affected by the presence of any matter fields,

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab}, \tag{A.14}$$

where G_{ab} is the *Einstein Tensor*, T_{ab} is the energy-momentum, or stress-energy tensor, that describes the matter content of the spacetime and κ, is the gravitational coupling constant, given by,

$$\kappa = \frac{8\pi G}{c^4}. \tag{A.15}$$

The *Energy momentum tensor* takes various forms, depending on the nature of the matter field present.

In *vacuum*, we have,

$$T_{ab} = 0. \tag{A.16}$$

For a dust-like fluid, with four-velocity $u^a = {}^{dx^a}/_{d\tau}$ and proper time τ,

$$T_{ab} = \rho\, u_a u_b, \tag{A.17}$$

where ρ, denotes the energy density of the fluid, and p, is its pressure.

For a perfect fluid,

$$T_{ab} = (\rho + p) u_a u_b - p g_{ab}, \tag{A.18}$$

For a *massive scalar field*, ϕ,

$$T_{ab} = \phi_{;a}\phi_{;b} - \frac{1}{2} g_{ab} \left(\phi_{;c}\phi_{;d} g^{cd} + m^2\phi^2\right) \tag{A.19}$$

APPENDIX B

Toroidal Compactification of the NS-NS Action

B.1 The NS-NS Action

The NS-NS sector of the string effective actions, contains a dilaton field, a graviton and a 2-form potential and is common to both the type II and heterotic theories [12].

In Kaluza-Klein dimensional reduction, the universe is viewed as a product space $M_{D+d} = M_D \times M_d$, where,

- $M_D(x^\mu)$, (external space) has metric $\gamma_{\mu\nu}(x^\lambda)$, with $\lambda \in (0, D-1)$,
- $M_d(z^a)$, (internal space) has metric $\hat{h}_{ab}(z^c)$, with $c \in (D, D+d-1)$,

The design of this spacetime is such, that for matter fields independent of the coordinates z^a there are no matter sources present in the bulk, and the resulting space is Ricci flat. K_d can then be taken to be a d-dimensional torus, which topologically is the Cartesian product of d circles,

$$T^d = S^1 \times S^1 \times ... \times S^1 \tag{B.1}$$

and is metrically flat, since $R_{T^d} = 0$, as described above.

A very popular compactification scheme is the Calabi-Yau SU(n) and corresponding n-form, holonomies [94, 95] that is used in Horawa-Witten M-Theory [17, 18]. In the present scheme however, we will not follow this route albeit, most of our conclusions can be adapted to other settings when the only dynamically important variable is the volume of the internal space.

When higher-dimensional metrics are compactified on a circle, they split into a lower-dimensional metric tensor, a 1-form potential (gauge field) and a 0-form potential (scalar field). It can be shown [96, 97], that the form of the $(D+d)$-dimensional NS-NS action, compactified on T^d, is,

$$S_{D+d} = -\frac{1}{2\kappa^2_{D+d}} \int d^{D+d}x \sqrt{|\hat{g}_{D+d}|}\, e^{-\hat{\Gamma}} \Big\{ \hat{R}_{{}_{D+d}}[\hat{g}] + (\hat{\nabla}\hat{\Gamma})^2 - \frac{1}{12}\hat{H}^2 \Big\}, \tag{B.2}$$

where $\hat{g}_{D+d} \equiv \hat{g} = det(g_{AB})$, and $(\hat{\nabla}\hat{\Gamma})^2 = \hat{g}^{AB}(\hat{\nabla}_A\hat{\Gamma})(\hat{\nabla}_B\hat{\Gamma})$. The dilatonic scalar field, $\hat{\Gamma}$, and vector form field, $\hat{H}$, are the products of the toroidal compactification scheme.

The spacetime we are considering has metric,

$$\begin{aligned} d\hat{s}^2_{D+d} &= \hat{g}_{AB}dx^A dx^B \\ &= \gamma_{\mu\nu}(x^\lambda)dx^\mu dx^\nu + \Phi^2(x^\lambda)\hat{\gamma}_{ab}(z^c)dz^a dz^b, \end{aligned} \tag{B.3}$$

where $\hat{\gamma}^{ab}\hat{\gamma}_{ac} = \delta^b_c$, and thus $\hat{\gamma} = det(\hat{\gamma}_{ab}) = 1$.

B.2 The Reduced Ricci Scalar

In this first investigation, we will set the dilaton $\hat{\Gamma}$ and form field $\hat{H}$ to zero,

$$\hat{\Gamma} = \hat{H} = 0, \tag{B.4}$$

where the action takes the simple form,

$$S_{D+d} = -\frac{1}{2\kappa^2_{D+d}} \int d^{D+d}x \, \sqrt{|\hat{g}|}\; \hat{R}_{D+d}[\hat{g}] \tag{B.5}$$

It can be shown the $(D+d)$-dimensional Ricci scalar $\hat{R}[\hat{g}]$, has form [98],

$$\hat{R}_{D+d}[\hat{g}] = R_D[\gamma] + \frac{1}{4}(\nabla_\mu h^{ab})(\nabla^\mu h_{ab}) + \nabla_\mu(\ln\sqrt{h})\nabla^\mu(\ln\sqrt{h}) - 2\frac{(\Box\sqrt{h})}{\sqrt{h}}. \tag{B.6}$$

where in the case of Eq.(B.3), we have,

$$\begin{aligned} h_{ab} &= \Phi^2\hat{\gamma}_{ab}, \\ h^{ab} &= \Phi^{-2}\hat{\gamma}^{ab}, \\ h &= det(h_{ab}) = \Phi^{2d}\hat{\gamma} = \Phi^{2d}. \end{aligned} \tag{B.7}$$

where h is the determinant of the internal metric. We incrementally construct the terms, using the property, $\partial_\mu \ln h = h^{ab}\partial_\mu h_{ab}$ and adhering carefully to the dimensionality of each term. We have,

$$\begin{aligned}
\frac{1}{4}(\nabla_\mu h^{ab})(\nabla^\mu h_{ab}) &= \frac{1}{4}\cdot(\nabla_\mu \Phi^{-2}\hat{\gamma}^{ab})(\nabla^\mu \Phi^2 \hat{\gamma}_{ab}) \\
&= \frac{1}{4}\cdot\frac{-2}{\Phi^3}\,\hat{\gamma}^{ab}(\nabla_\mu\Phi)\cdot 2\Phi\,\hat{\gamma}_{ab}(\nabla^\mu\Phi) \\
&= -d\frac{(\nabla_\mu\Phi)(\nabla^\mu\Phi)}{\Phi^2}
\end{aligned} \tag{B.8}$$

where $\hat{\gamma}_{ab}\hat{\gamma}^{ab} = \delta^a_a = d$, since the internal space has d-dimensions. Next,

$$\begin{aligned}
\nabla_\mu(\ln\sqrt{h})\nabla^\mu(\ln\sqrt{h}) &= \nabla_\mu\Big[\ln(\Phi^{2d})^{1/2}\Big]\nabla^\mu\Big[\ln(\Phi^{2d})^{1/2}\Big] \\
&= d^2\frac{(\nabla_\mu\Phi)(\nabla^\mu\Phi)}{\Phi^2}
\end{aligned} \tag{B.9}$$

And finally,

$$2\frac{(\Box\sqrt{h})}{\sqrt{h}} = 2\frac{(\Box\Phi^d)}{\Phi^d} \tag{B.10}$$

where,

$$\begin{aligned}
\Box\Phi^d &= g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi^d \\
&= g^{\mu\nu}\nabla_\mu(d\Phi^{d-1}\nabla_\nu\Phi) \\
&= d(d-1)\;\Phi^{d-2}\;(\nabla_\mu\Phi)(\nabla^\mu\Phi)\;+\;d\;\Phi^{d-1}(\Box\Phi) \\
\frac{(\Box\Phi^d)}{\Phi^d} &= d(d-1)\;\frac{(\nabla_\mu\Phi)(\nabla^\mu\Phi)}{\Phi^2} + d\;\frac{(\Box\Phi)}{\Phi}
\end{aligned} \tag{B.11}$$

Finally, we can combine Eq.(B.6)-(B.11) to find the Ricci scalar in our model,

$$\hat{R}_{D+d}[\hat{g}] = R_D[\gamma] - d(d-1)\;\frac{(\nabla_\mu\Phi)(\nabla^\mu\Phi)}{\Phi^2} - 2d\;\frac{(\Box\Phi)}{\Phi} \tag{B.12}$$

B.3 The Conformal String Frame

By integrating out d-dimensions, it can be shown that the effective D-dimensional action in the string frame, takes form,

$$S^{eff.}_{D,S} = -\frac{1}{2\kappa_D^2}\int d^Dx\sqrt{|\gamma|}\;e^{-\phi}\Big\{R_D[\gamma] + (\nabla_\mu\varphi)(\nabla^\mu\varphi) + \frac{1}{4}(\nabla_\mu h^{ab})(\nabla^\mu h_{ab})\Big\} \tag{B.13}$$

with,

$$\varphi = \hat{\Gamma} - \frac{1}{2}\ln h \tag{B.14}$$

where, φ is a 'shifted' dilaton field, as it is often described in the literature [99], that comes from the extra dimensions and describes the change of the extra dimensional volume.

Since we have assumed $\hat{\Gamma} = 0$ in Eq.(B.4) and by our definition of h_{ab} in Eq.(B.7), we have,

$$\varphi = -\frac{1}{2}\ln h = -\frac{1}{2}\ln(\Phi^{2d}\gamma) = -d\ln\Phi, \tag{B.15}$$

and so

$$e^{-\varphi} = \exp\left\{\frac{1}{2}\ln\Phi^{2d}\right\} = \Phi^d, \tag{B.16}$$

and

$$\begin{aligned} \nabla_\mu\varphi &= -d\frac{(\nabla_\mu\Phi)}{\Phi}, \\ (\nabla_\mu\varphi)(\nabla^\mu\varphi) &= d^2\frac{(\nabla_\mu\Phi)(\nabla^\mu\Phi)}{\Phi^2}. \end{aligned} \tag{B.17}$$

Combining the above equations, we arrive at the graviton sector of the dimensionally reduced string action,

$$S^{eff.}_{D,S} = -\frac{1}{2\kappa_D^2}\int d^Dx\sqrt{|\gamma|}\;\Phi^d\Big\{R_D[\gamma] + d(d-1)\frac{(\nabla_\mu\Phi)(\nabla^\mu\Phi)}{\Phi^2}\Big\}, \tag{B.18}$$

which we call the *String Frame*. We can directly compare this to Eq.(B.12), where we notice the inclusion of the rescaled dilatonic scalar, and the vanishing boundary term.

B.4 The Conformal Einstein Frame

We now apply a conformal Weyl transformation, of type,

$$g_{\mu\nu} = \Omega^2\gamma_{\mu\nu}. \tag{B.19}$$

where,

$$\Omega^2 = \exp\Big\{-\frac{2}{D-2}\varphi\Big\} = \Phi^{\frac{2d}{D-2}}, \tag{B.20}$$

Then the action can be written as [12],

$$S_D^{eff.} = -\frac{1}{2\kappa_D^2}\int d^Dx\sqrt{|g|}\,\Big\{R_D[g] - \frac{1}{2}(\nabla\bar{\phi})^2 + \frac{1}{4}(\nabla_\mu h_{ab})(\nabla^\mu h^{ab})\Big\}. \tag{B.21}$$

with

$$\bar{\phi} = \sqrt{\frac{2}{D-2}}\,\varphi = -d\,\sqrt{\frac{2}{D-2}}\ln\Phi. \tag{B.22}$$

Again, we systematically determine the closed form of the action,

$$\begin{aligned}\frac{1}{2}(\nabla_\mu\bar{\phi})^2 &= \frac{1}{2}\Big(-d\sqrt{\frac{2}{D-2}}\frac{1}{\Phi}\nabla_\mu\Phi\Big)^2\\ &= \frac{d^2}{D-2}\frac{(\nabla_\mu\Phi)(\nabla^\mu\Phi)}{\Phi^2}\end{aligned} \tag{B.23}$$

So using Eq.(B.8) and Eq.(B.23), and setting,

$$\phi = \pm\frac{1}{\kappa_D}\left[\frac{d(D+d-2)}{D-2}\right]^{1/2}\ln\Phi, \tag{B.24}$$

we arrive at,

$$S_{D,E}^{eff.} = -\frac{1}{2\kappa_D^2}\int d^Dx\sqrt{|g|}\,\big\{R_D\,[g] - \kappa_D^2\,(\nabla\phi)^2\big\}, \tag{B.25}$$

which is called the *Einstein Frame*.

Note that in the body of our work, we have used $\hat{\Phi}$ for Φ.

APPENDIX C

The Einstein-Hilbert Action

In traditional classical mechanics, the calculus of variations is held to the highest regard, both due to the mathematical elegance of the theory, as well as the sheer power of the technique. Put forth by Johann Bernoulli, in his *Acta Eruditorum* in June 1696, variational methods have been used to provide a holistic description of the energy of a system and its dynamics in the fields of Classical theory, Quantum Field Theory as well as in General Relativity.

C.1 The Einstein-Hilbert Action

The Einstein-Hilbert action which gives rise to the vacuum Einstein equations is given by,

$$\begin{aligned} S[g] &= \int d^4x \, \mathcal{L}_G \\ &= \int d^4x \, \frac{1}{2\kappa} R\sqrt{-g} \end{aligned} \tag{C.1}$$

where

- $g = \frac{1}{c^2} det(g_{ab})$, is the determinant of the metric g_{ab}, of the spacetime,
- R, is the Ricci scalar,
- $\kappa = \frac{8\pi G}{c^4}$, is a universal constant related to the coupling strength of gravity.
- $\mathcal{L}_G$, is the Lagrangian density in $Joules/m^3$.

To derive the field equations we must include the Lagrange density, $\mathcal{L}_m$, of the matter fields present,

$$S = \int \left[\frac{1}{2\kappa} R\sqrt{-g} + \mathcal{L}_m \sqrt{-g} \right] d^4x \tag{C.2}$$

C.2 The Action Principle

The action principle states that, the variation of the action with respect to the inverse metric g^{ab} is zero,

$$\frac{\delta S}{\delta g^{ab}} = 0 \tag{C.3}$$

Varying the Einstein-Hilbert action, we find,

$$\begin{aligned}
\frac{\delta S}{\delta g^{ab}} &= \int \left[\frac{1}{2\kappa} \frac{\delta(\sqrt{-g}R)}{\delta g^{ab}} + \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} \right] d^4x = 0 \\
\delta S &= \int \left[\frac{1}{2\kappa} \frac{\delta(\sqrt{-g}R)}{\delta g^{ab}} + \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} \right] \delta g^{ab} d^4x = 0 \\
0 &= \int \left[\frac{1}{2\kappa} \left(R\frac{\delta\sqrt{-g}}{\delta g^{ab}} + \sqrt{-g}\frac{\delta R}{\delta g^{ab}} \right) + \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} \right] \delta g^{ab} d^4x \\
0 &= \int \left[\frac{1}{2\kappa} \left(\frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{ab}} + \frac{\delta R}{\delta g^{ab}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} \right] \sqrt{-g}\, \delta g^{ab} d^4x \\
0 &= \int \left[\frac{1}{2\kappa} \left(\frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{ab}} + \frac{\delta R}{\delta g^{ab}} \right) + \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} \right] d^4x
\end{aligned} \tag{C.4}$$

To continue further, we must determine the variation of the components of Eq.(C.4). The second term will give rise to the energy momentum tensor, as we will see later.

C.3 Variation of the Riemann Tensor

To evaluate the variation of the Ricci scalar R, we will need to find the variation of the Riemann and Ricci tensors first, because,

$$\begin{aligned}
R &= g^{ab}R_{ab} \\
\delta R &= \delta(g^{ab}R_{ab}) \\
\delta R &= R_{ab}\delta g^{ab} + g^{ab}\delta R_{ab}
\end{aligned} \tag{C.5}$$

Thus the variation of the Ricci tensor, δR_{ab} is needed. The itself Ricci tensor is defined as,

$$\begin{aligned}
R_{ab} &= R^c{}_{acb} \\
\delta R_{ab} &= \delta R^c{}_{acb}
\end{aligned} \tag{C.6}$$

and therefore the variation of the Riemann tensor, $\delta R^{c}{}_{acb}$ is required. The variation of the Riemann tensor follows from its definition, in terms of the metric connection, or Christoffel symbol of the second kind,

$$\begin{aligned}
R^{a}{}_{bcd} &= \Gamma^{a}{}_{db,c} - \Gamma^{a}{}_{cb,d} + \Gamma^{a}{}_{ce}\Gamma^{e}{}_{db} - \Gamma^{a}{}_{de}\Gamma^{e}{}_{cb} \\
\delta R^{a}{}_{bcd} &= \delta\Gamma^{a}{}_{db,c} - \delta\Gamma^{a}{}_{cb,d} + \delta\Gamma^{a}{}_{ce}\Gamma^{e}{}_{db} + \Gamma^{a}{}_{ce}\delta\Gamma^{e}{}_{db} - \delta\Gamma^{a}{}_{de}\Gamma^{e}{}_{cb} - \Gamma^{a}{}_{de}\delta\Gamma^{e}{}_{cd} \quad \text{(C.7)}
\end{aligned}$$

Motivated by the first two terms in this equation, we consider the covariant derivative of the variation of the connection $\delta\Gamma^{a}{}_{bc}$,

$$\nabla_{d}(\delta\Gamma^{a}{}_{bc}) = \delta\Gamma^{a}{}_{bc,d} + \Gamma^{a}{}_{de}\delta\Gamma^{e}{}_{bc} - \Gamma^{e}{}_{bd}\delta\Gamma^{a}{}_{ec} - \Gamma^{e}{}_{dc}\delta\Gamma^{a}{}_{be} \quad \text{(C.8)}$$

We notice that many of the terms that appear in the variation of the Riemann tensor, are also components of the covariant derivative of the variation of the connection. By careful choice of indices, we find,

$$\begin{aligned}
\nabla_{c}(\delta\Gamma^{a}{}_{bd}) - \nabla_{d}(\delta\Gamma^{a}{}_{bc}) &= \delta\Gamma^{a}{}_{bd,e} + \Gamma^{a}{}_{ce}\delta\Gamma^{e}{}_{bd} - \Gamma^{e}{}_{bc}\delta\Gamma^{a}{}_{ed} - \Gamma^{e}{}_{dc}\delta\Gamma^{a}{}_{be} \\
&\quad -\delta\Gamma^{a}{}_{bc,d} - \Gamma^{a}{}_{de}\delta\Gamma^{e}{}_{bc} + \Gamma^{e}{}_{bd}\delta\Gamma^{a}{}_{ec} + \Gamma^{e}{}_{dc}\delta\Gamma^{a}{}_{be} \\
&= \delta R^{a}{}_{bcd}. \quad \text{(C.9)}
\end{aligned}$$

Namely,

$$\delta R^{a}{}_{bcd} = \nabla_{c}(\delta\Gamma^{a}{}_{bd}) - \nabla_{d}(\delta\Gamma^{a}{}_{bc}). \quad \text{(C.10)}$$

By contracting the first and third indices in the Riemann tensor, we arrive at the corresponding variation of the Ricci tensor,

$$\delta R_{ab} = \delta R^{c}{}_{bcd} = \nabla_{c}(\delta\Gamma^{c}{}_{ab}) - \nabla_{b}(\delta\Gamma^{c}{}_{ac}) \quad \text{(C.11)}$$

We can now write the variation of the Ricci scalar by substituting into Eq.(C.5),

$$\begin{aligned}
\delta R &= R_{ab}\delta g^{ab} + g^{ab}\left[\nabla_{c}(\delta\Gamma^{c}{}_{ab}) - \nabla_{b}(\delta\Gamma^{c}{}_{ac})\right] \\
&= R_{ab}\delta g^{ab} + \nabla_{c}(g^{ab}\delta\Gamma^{c}{}_{ab}) - \nabla_{b}(g^{ab}\delta\Gamma^{c}{}_{ac}) \\
&= R_{ab}\delta g^{ab} + \nabla_{d}(g^{ab}\delta\Gamma^{d}{}_{ab}) - \nabla_{d}(g^{ad}\delta\Gamma^{c}{}_{ac}) \\
&= R_{ab}\delta g^{ab} + \nabla_{d}\left(g^{ab}\delta\Gamma^{d}{}_{ab} - g^{ad}\delta\Gamma^{c}{}_{ac}\right) \quad \text{(C.12)}
\end{aligned}$$

where we switched the internal indices c, in the second term and b, in the third to d. Keeping in mind that this term will be integrated, we recall the divergence theorem,

$$\int_V (\nabla \cdot \mathbf{f}) dV = \oint_S \mathbf{f} \cdot d\mathbf{A}. \tag{C.13}$$

From the first term in the integral of Eq.(C.4) where we now substitute Eq.(C.12), we have:

$$\begin{aligned}
&\int \frac{1}{2\kappa} \left(\frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{ab}} + \frac{\delta R}{\delta g^{ab}} \right) d^4x = \\
&\int \frac{1}{2\kappa} \left(\frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{ab}} + \frac{R_{ab}\delta g^{ab}}{\delta g^{ab}} + \frac{\nabla_d \left(g^{ab}\delta\Gamma^d{}_{ab} - g^{ad}\delta\Gamma^c{}_{ac} \right)}{\delta g^{ab}} \right) d^4x = \\
&\int \frac{1}{2\kappa} \left(\frac{R}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g^{ab}} + R_{ab} \right) d^4x
\end{aligned} \tag{C.14}$$

where by application of the divergence theorem, the last term became a surface integral, that vanishes at infinity as a boundary term.

C.4 Variation of the Determinant of the Metric

Next, we seek the variation of $\sqrt{-g}$,

$$\frac{\delta\sqrt{-g}}{\delta g^{ab}} = \frac{1}{2} \frac{1}{\sqrt{-g}} \frac{\delta(-g)}{\delta g^{ab}} = -\frac{1}{2\sqrt{-g}} \frac{\delta(g)}{\delta g^{ab}} \tag{C.15}$$

We will need to calculate $\frac{\delta(g)}{\delta g^{ab}}$. To do this first consider a square matrix, $A = (a_{ij})$, with inverse,

$$b^{ij} = \frac{1}{a} (A^{ij})', \tag{C.16}$$

where,

- $a = det(A)$, is the determinant of the matrix A,
- $(A^{ij})'$, is the cofactor of a_{ij}.

Expanding the determinant along the i^{th} row, we get,

$$a = \sum_{j=1}^{n} a_{ij} \, (A^{ij})' \tag{C.17}$$

Assuming einstein summation and varying the result, we find,

$$\begin{aligned}\delta a &= \delta a_{ij}\,(A^{ij})' \\ \frac{\delta a}{\delta a_{ij}} &= (A^{ij})' \\ \frac{\delta a}{\delta a_{ij}} &= ab^{ij}\end{aligned} \tag{C.18}$$

Assuming now that $a(a_{ij}(x^k))$, we expand the derivative so that,

$$\frac{\delta a}{\delta x^k} = \frac{\delta a}{\delta a_{ij}}\frac{\delta a_{ij}}{\delta x^k} = ab^{ij}\,\frac{\delta a_{ij}}{\delta x^k} \tag{C.19}$$

In the case where $a_{ij} = g_{ab}$, we have,

$$\begin{aligned}\delta g &= gg^{ab}\delta g_{ab} \\ \delta g_{ab} &= \frac{\delta g}{gg^{ab}}\end{aligned} \tag{C.20}$$

Note that the derivative/variation, is with respect to the covariant metric g_{ab} and not the contravariant form g^{ab} that we need. We notice however that,

$$\delta(g^{ab}g_{ab}) = \delta(\delta^a_b) = \delta(4) = 0. \tag{C.21}$$

Another way of writing this, using Eq.(C.20), is

$$\begin{aligned}\delta g^{ab}g_{ab} + g^{ab}\delta g_{ab} &= 0 \\ \delta g^{ab}g_{ab} &= -g^{ab}\delta g_{ab} \\ \delta g^{ab}g_{ab} &= -g^{ab}\frac{\delta g}{gg^{ab}} \\ \delta g &= -gg_{ab}\delta g^{ab}\end{aligned} \tag{C.22}$$

Namely,

$$\delta g = gg^{ab}\delta g_{ab} = -gg_{ab}\delta g^{ab} \tag{C.23}$$

Now keeping in mind that,

$$\frac{\delta g}{\delta g^{ab}} = -gg_{ab} \tag{C.24}$$

We perform the variation of the determinant of the metric,

$$\frac{\delta\sqrt{-g}}{\delta g^{ab}} = -\frac{1}{2}\frac{1}{\sqrt{-g}}(-g g_{ab}) = -\frac{1}{2}\sqrt{-g}\; g_{ab} \tag{C.25}$$

Namely,

$$\frac{\delta\sqrt{-g}}{\delta g^{ab}} = -\frac{1}{2}\sqrt{-g}\; g_{ab} \tag{C.26}$$

This is an important rule for differentiating a determinant that appears often in studies that explore the equations of a system by variation of the appropriate action. As we can see here, the formal derivation of this rule involves many intricate steps, which we will make good use of in this work.

C.5 The Curvature Terms

We now have everything we need to evaluate the equation of motion, Eq.(C.4), via Eq.(C.14), by first evaluating,

$$\frac{R}{\sqrt{-g}}\left(-\frac{1}{2}\sqrt{-g}\; g_{ab}\right) = -\frac{1}{2}g_{ab}R \tag{C.27}$$

Therefore (suppressing the d^4x), we can write,

$$\begin{aligned} &\int \frac{1}{2\kappa}\left(\frac{R}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{ab}} + \frac{\delta R}{\delta g^{ab}}\right) + \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} = \\ &\int \frac{1}{2\kappa}\left(R_{ab} - \frac{1}{2}g_{ab}R\right) + \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ab}} = 0 \end{aligned} \tag{C.28}$$

Thus, we have derived the geometrical part of the Einstein equation, which we shall denote G_{ab},

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R \tag{C.29}$$

We expect that the other side of the equation will provide us with information on the energy content of this geometry. In the next section we see how this arises explicitly.

C.6 The Energy Momentum Tensor

To understand the right hand side of the equation of motion, Eq.(C.4), we must go back to definitions; consider the general action that involves curvature and mass Lagrangian,

$$S = -\frac{1}{2\kappa}\int_V \left(\mathcal{L}_G - 2\kappa\mathcal{L}_m\right) dV. \tag{C.30}$$

Varying we get,

$$\begin{aligned}
\int_V \left(\frac{\mathcal{L}_G}{\delta g_{ab}} - 2\kappa\frac{\mathcal{L}_m}{\delta g_{ab}}\right) dV &= 0 \\
-(\sqrt{-g})G^{ab} - 2\kappa(\sqrt{-g})T^{ab} &= 0 \\
G^{ab} &= \kappa T^{ab} \qquad \text{(C.31)}
\end{aligned}$$

In this expression we *define* the energy momentum tensor as the variation of the matter field Lagrangian. Comparing with Eq.(C.4), we can explicitly define T^{ab},

$$T_{ab} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\left[\sqrt{-g}\mathcal{L}_m\right]}{\delta g^{ab}} \tag{C.32}$$

From our definition, we have now identified the energy momentum tensor, of our matter field.

C.7 The Einstein Field Equations

We can finally combine everything,

$$R_{ab} - \frac{1}{2}g_{ab}R = \kappa T_{ab} \tag{C.33}$$

into the *Einstein Field Equations*, as derived from the Einstein-Hilbert action.

APPENDIX D

The Hyperspace Equations

Consider for example, a 5D spacetime with metric,

$$ds_5^2 = e^{A(t,y)}[dt^2 - dy^2] - e^{B(t,y)} d\Omega_3^2 \tag{D.1}$$

where $d\Omega_3^2 = d\theta^2 + sin^2\theta\,(d\phi^2 + sin^2\phi\; d\psi^2)$, is the metric on a unit 3-sphere. We define a hypersurface on the 5D manifold by the general equation,

$$u = \alpha t + \beta y = 0. \tag{D.2}$$

The nornal to the hypersurface u is,

$$n_a = \frac{\partial u}{\partial \chi^a}, \tag{D.3}$$

where $a = 0, 1, ..., 4$ and $\chi^a = (t, y, \theta, \phi, \psi)$. Thus,

$$n_a = \alpha\delta_a^t + \beta\delta_a^y. \tag{D.4}$$

Our $u = 0$ hypersurface must be timelike, for it to be a viable brane. Thus the normal must be spacelike, as described by Fig.D.1. Recall that,

$$\begin{aligned} n_a n^a &< 0, \quad \text{spacelike vector} \\ n_a n^a &> 0, \quad \text{timelike vector} \\ n_a n^a &= 0, \quad \text{null vector} \end{aligned} \tag{D.5}$$

From Eq.(D.1) and Eq.(D.4) we find,

$$\begin{aligned} n_\alpha n_\beta g^{\alpha\beta} &= n_\alpha n_\beta g^{\alpha\beta} \\ &= n_t^2 g^{tt} + n_y^2 g^{yy} \\ &= n_t^2 g^{tt} - n_y^2 g^{tt} \\ &= g^{tt}(\alpha^2 - \beta^2) < 0 \end{aligned} \tag{D.6}$$

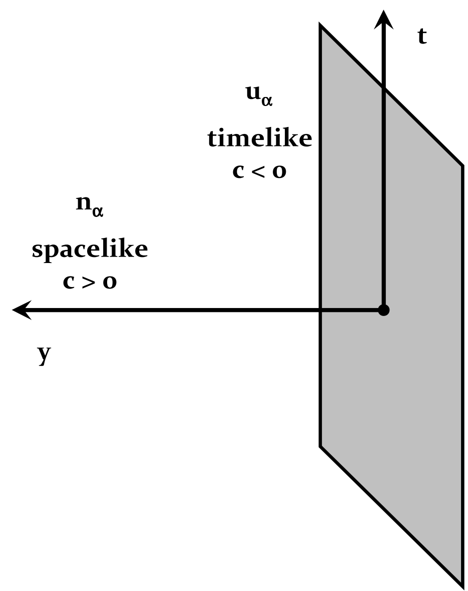


Figure D.1: The timelike vector constrained to the hypersurface can also be thought of as the direction of the four-velocity of free falling frame along any geodesics on the brane. In other words, any matter fields present will have timelike velocities, and will thus travel slower than the speed of light, as required by General Relativity. The normal to the hypersurface must thus be spacelilke, thus defining an appropriate geometrical frame for the study of brane properties.

Thus we have four possible cases,

$$|\alpha| < |\beta|, \tag{D.7}$$

corresponding to the solutions,

$$\begin{aligned} \pm\alpha &< \beta \\ \pm\alpha &< -\beta \end{aligned} \tag{D.8}$$

By fixing $\alpha = 1$ we can have $\beta > 1$ or $\beta < -1$. Given these options we can construct two characteristic hypersurfaces, that will represent the 3-branes in our model,

$$\begin{aligned} u &= t - ay = 0 \\ v &= t + by = 0 \end{aligned} \tag{D.9}$$

Note the change from greek to latin parameters here, as we consider a new solution. According to this outline, the requirement that the normal vectors we have selected in our work are spacelike dictate that,

$$\begin{aligned} n_a &= \frac{\partial u}{\partial \chi^a} = \delta^t_a - a\delta^y_a \\ l_a &= \frac{\partial v}{\partial \chi^a} = \delta^t_a + b\delta^y_a \end{aligned} \tag{D.10}$$

and so (using $g^{tt} = -g^{yy}$),

$$\begin{aligned} n_a n_b g^{ab} < 0 \quad &\Longrightarrow \quad a^2 > 1 \\ l_a l_b g^{ab} < 0 \quad &\Longrightarrow \quad b^2 > 1 \end{aligned} \tag{D.11}$$

Thus $a > 1$, or $a < -1$ and $b > 1$, or $b < -1$.

APPENDIX E

The Algebraic Invariants of the Riemann Tensor

A set of fourteen scalars can be constructed out of the Riemann tensor, that represent the curvature invariants of the spacetime. Narlikar and Karmarkar [79] first constructed these acalars in 1948. Their original work is often not readily available and so I present them here with some comments,

$$
\begin{aligned}
I_1 &= R^k_k && \text{(E.1)}\\
I_2 &= R^k_i\, R^i_k && \text{(E.2)}\\
I_3 &= R^k_i\, R^i_j\, R^j_k && \text{(E.3)}\\
I_4 &= R^k_i\, R^i_j\, R^j_\delta\, R^\delta_k && \text{(E.4)}\\
J_1 &= A_{abcd}\, g^{ac}\, g^{bd} && \text{(E.5)}\\
J_2 &= B_{abcd}\, g^{ac}\, g^{bd} && \text{(E.6)}\\
J_3 &= E_{abcd}\, g^{ac}\, g^{bd} && \text{(E.7)}\\
J_4 &= F_{abcd}\, g^{ac}\, g^{bd} && \text{(E.8)}\\
K_1 &= C_{abcd}\, R^{ac}\, R^{bd} && \text{(E.9)}\\
K_2 &= A_{abcd}\, R^{ac}\, R^{bd} && \text{(E.10)}\\
K_3 &= \bar{D}_{abcd}\, R^{ac}\, R^{bd} && \text{(E.11)}\\
K_4 &= C_{abcd}\, Q^{ac}\, Q^{bd} && \text{(E.12)}\\
K_5 &= A_{abcd}\, Q^{ac}\, Q^{bd} && \text{(E.13)}\\
K_6 &= D_{abcd}\, Q^{ac}\, Q^{bd} && \text{(E.14)}
\end{aligned}
$$

where C is the Weyl tensor, and

$$
\begin{aligned}
A_{abcd} &= C_{abef}\, C_{ghcd}\, g^{eg} g^{fh} && \text{(E.15)}\\
B_{abcd} &= C_{abef}\, A_{ghcd}\, g^{eg} g^{fh} && \text{(E.16)}\\
D_{abcd} &= B_{abcd} - \tfrac{1}{2}\, J_2\, (g_{ac} g_{bd} - g_{ad} g_{bc}) - \tfrac{1}{4}\, J_1\, C_{abcd} && \text{(E.17)}\\
\bar{D}_{abcd} &= (J_3)^{1/2} D_{abcd} && \text{(E.18)}\\
E_{abcd} &= C_{abef}\, D_{ghcd}\, g^{eg} g^{fh} && \text{(E.19)}\\
F_{abcd} &= C_{abef}\, E_{ghcd}\, g^{eg} g^{fh} && \text{(E.20)}\\
Q^k_l &= R^k_i\, R^i_l && \text{(E.21)}
\end{aligned}
$$

The units for all scalar invariants (sometimes called quadratic invariants) is L^{-4}.

APPENDIX F

The Einstein Equations for Branes Colliding in Vacuum

F.1 Vacuum Einstein Solution

Consider the general 5-dimensional metric with form,

$$ds_5^2 = dt^2 - e^{2F(t)}dy^2 - e^{2G(t)}d\Sigma_0^2 \tag{F.1}$$

where $x^A = \{t, y, x^2, x^3, x^4\}$ and $d\Sigma_0^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. Using the Maple symbolic computational program, we can calculate the non-zero Einstein tensors,

$$G_{00} = 3G'(F' + G'), \tag{F.2}$$

$$G_{11} = -3e^{2F}(2G'^2 + G''), \tag{F.3}$$

$$G_{22} = -e^{2G}(F'^2 + F'' + 2F'G' + 3G'^2 + 2G''). \tag{F.4}$$

For vacuum, the energy-momentum tensor is $T_{ab} = 0$, thus

$$0 = F' + G', \tag{F.5}$$

$$0 = 2G'^2 + G'', \tag{F.6}$$

$$0 = F'^2 + F'' + 2F'G' + 3G'^2 + 2G''. \tag{F.7}$$

Setting $u = G'$, $u' = G''$ in Eq.(F.6), we find,

$$\begin{aligned} 2u^2 + u' &= 0, \\ \int \frac{du}{u^2} &= -2\int dt, \\ u &= \frac{1}{2}\frac{1}{t + c_0}, \end{aligned} \tag{F.8}$$

where c_i, will denote integration constants. We can now find G, by substituting back,

$$\begin{aligned} G &= \int \frac{dt}{t + c_0}, \\ G &= \frac{1}{2}\ln[c_1(t + c_0)]. \end{aligned} \tag{F.9}$$

And using Eq.(F.5) and Eq.(F.8) we can find F by integrating,

$$\begin{aligned} F' &= -G', \\ F' &= -u, \\ F &= -\frac{1}{2}\int \frac{dt}{t+c_0} \\ F &= -\frac{1}{2}\ln[c_2(t+c_0)] \end{aligned} \tag{F.10}$$

Substituting back into the metric Eq.(F.1), we find,

$$\begin{aligned} ds_5^2 &= dt^2 - \frac{1}{t+c_0}dy^2 - (t+c_0)d\Sigma_0^2, \\ ds_5^2 &= \frac{1}{t+c_0}\left[(t+c_0)dt^2 - dy^2 - (t+c_0)^2 d\Sigma_0^2,\right] \end{aligned} \tag{F.11}$$

where we have taken advantage of the fact that the metric is invariant by the addition of a constant. We want to simplify this expression for the metric, further, so make the further substitution,

$$\begin{aligned} d\bar{t}^2 &= (t+c_0)dt^2, \\ \int d\bar{t} &= \int (t+c_0)^{1/2}dt, \\ \bar{t} &= \frac{2}{3}(t+c_0)^{3/2}, \\ t+c_0 &= \left(\frac{3\bar{t}}{2}\right)^{2/3}. \end{aligned} \tag{F.12}$$

Substituting Eq.(F.12) into Eq.(F.11), we arrive at,

$$ds_5^2 = t^{-2/3}(dt^2 - dy^2) - t^{2/3}d\Sigma_0^2, \tag{F.13}$$

where we have dropped the ‘bar’ notation and absorbed all constants into the coordinates.

F.2 Branes

Proceeding now with developing the brane collision mechanics, we define the hypersurfaces as outlined in the general description of our model, in Chapter 2.1,

$$\Phi_1 = t - ay, \tag{F.14}$$

$$\Phi_2 = t + by. \tag{F.15}$$

Adding and subtracting by parts, it is easy to show that,

$$t = \frac{a\Phi_2 + b\Phi_1}{a+b}, \tag{F.16}$$

$$y = \frac{\Phi_2 - \Phi_1}{a+b}. \tag{F.17}$$

Now for a trick, that is at the center of our approach to solving brane collision models analytically. We define an 'advancing time' variable, $T(\Phi_1, \Phi_2)$ and promote the hypersurfaces Φ_i to Heaviside functions, where $i = 1, 2$, as shown below,

$$T(\Phi_1, \Phi_2) = t + t_0, \tag{F.18}$$

$$\Phi_1 = \Phi_1 H(\Phi_1), \tag{F.19}$$

$$\Phi_2 = \Phi_2 H(\Phi_1). \tag{F.20}$$

Using Eq.(F.18) with Eq.(F.16), the 'time advancement' process gives,

$$T(\Phi_1, \Phi_2) = \frac{a\Phi_2 + b\Phi_1}{a+b} + t_0,$$

$$T(\Phi_1, \Phi_2) = \frac{1}{a+b}\Big[a\Phi_2 + b\Phi_1 + (a+b)t_0\Big]. \tag{F.21}$$

Now we redefine, $\bar{t}_0 = (a+b)t_0$ and substitute Eq.(F.19) and Eq.(F.20), into Eq.(F.21), and arrive at,

$$T(t, y) = \frac{A(t, y)}{a+b}, \tag{F.22}$$

where

$$A(\Phi_1, \Phi_2) = a(\Phi_2)H(\Phi_2) + b(\Phi_1)H(\Phi_1) + (a+b)t_0, \tag{F.23}$$

$$A(t, y) = a(t+by)H(t+by) + b(t-ay)H(t-ay) + A_0, \tag{F.24}$$

using Eq.(F.14) and Eq.(F.15) and setting $A_0 = (a + b)t_0$. By the inherent gauge freedom of the metric, we can find a new solution to Eq.(F.13), by the gauge choice,

$$t = \frac{A(t, y)}{a + b} \tag{F.25}$$

Thus

$$ds_5^2 = A^{-2/3}(t, y)(dt^2 - dy^2) - A^{2/3}(t, y)d\Sigma_0^2, \tag{F.26}$$

where again the general covariant properties of the metric have been used to absorb any constants into the units and simplify our expression.

F.3 The Einstein Tensor for Colliding Branes

We now solve the metric of Eq.(F.26) for the Einstein tensors and find,

$$\begin{aligned} G_{00} &= -\frac{A_{,yy}}{A}, \\ G_{01} &= -\frac{A_{,ty}}{A}, \\ G_{11} &= -\frac{A_{,tt}}{A}, \\ G_{ij} &= \frac{1}{3}A^{1/3}(A_{,yy} - A_{,tt}). \end{aligned} \tag{F.27}$$

The derivatives of $A(t, y)$ are given by,

$$\begin{aligned} A_{,t} &= bH(\Phi_1) + aH(\Phi_2), \\ A_{,y} &= ab\,[H(\Phi_2) - H(\Phi_1)], \\ A_{,tt} &= b\delta(\Phi_1) + a\delta(\Phi_2), \\ A_{,ty} &= ab\,[\delta(\Phi_2) - \delta(\Phi_1)], \\ A_{,yy} &= ab\,[a\delta(\Phi_1) + b\delta(\Phi_2)]. \end{aligned} \tag{F.28}$$

Substituting Eqs.(F.28) into Eqs.(F.27), we find,

$$
\begin{aligned}
G_{00} &= -\frac{ab}{A}\Big[a\delta(\Phi_1) + b\delta(\Phi_2)\Big] \\
G_{01} &= \frac{ab}{A}\Big[\delta(\Phi_1) - \delta(\Phi_2)\Big] \\
G_{11} &= -\frac{1}{A}\Big[b\delta(\Phi_1) + a\delta(\Phi_1)\Big] \\
G_{ij} &= -\frac{A^{1/3}}{3}\delta_{ij}\Big[b\delta(\Phi_1)(a^2-1) + a\delta(\Phi_2)(b^2-1)\Big] \qquad \text{(F.29)}
\end{aligned}
$$

F.4 Geometrics

We now define the orthonormal coordinates appropriate to describing the brane interactions.

The normal vectors to the surfaces $t - ay = 0$ and $t + by = 0$ are given by,

$$
\begin{aligned}
n_A &\equiv \frac{\partial(t-ay)}{\partial x^A} = \delta^t_A - a\delta^y_A, \\
l_A &\equiv \frac{\partial(t+by)}{\partial x^A} = \delta^t_A + b\delta^y_A. \qquad \text{(F.30)}
\end{aligned}
$$

The four-velocity of the matter fields along the brane will always be perpendicular to all directions normal to the brane. Using coordinate subscripts for clarity, for the brane Φ_1, we have,

$$
\begin{aligned}
u_A n^A &= 0, \\
u_A n^A = u_A n_B\, g^{AB} &= 0, \\
u_t n_t\, g^{tt} + u_y n_y\, g^{yy} &= 0, \\
u_t A^{2/3} + u_y(-a)(-A^{2/3}) &= 0, \\
u_t &= -a u_y. \qquad \text{(F.31)}
\end{aligned}
$$

The normalisation condition for the four-velocity can be used to find u_A, as follows,

$$
\begin{aligned}
u_A u^A &= 1, \\
u_A u^A = u_A n_B\, g^{AB} &= 1, \\
u_t^2\, g^{tt} + u_y^2\, g^{yy} &= 1, \\
u_t^2\, A^{2/3} + u_y^2(-A^{2/3}) &= 1, \\
A^{2/3} u_y^2 (a^2 - 1) &= 1, \\
u_y &= \frac{1}{A^{1/3}(a^2-1)^{1/2}}
\end{aligned}
\tag{F.32}
$$

where we have chosen the positive solution, comming out of the square root. Comparing with Eq.(F.31),

$$
u_t = -\frac{a}{A^{1/3}(a^2-1)^{1/2}}. \tag{F.33}
$$

We follow the same reasoning for the brane Φ_2. Summarily, we find,

$$
\begin{aligned}
u_A &= \frac{1}{A_{\Phi_1}^{1/3}(t)\,(a^2-1)^{1/2}}(a\delta_A^t - \delta_A^y), \\
v_A &= \frac{1}{A_{\Phi_2}^{1/3}(t)\,(b^2-1)^{1/2}}(b\delta_A^t + \delta_A^y),
\end{aligned}
\tag{F.34}
$$

where we have used the equations of the hypersurfaces,

$$
\begin{aligned}
\Phi_1 = 0 \;&\rightarrow\; y = \frac{t}{a}, \\
\Phi_2 = 0 \;&\rightarrow\; y = -\frac{t}{b},
\end{aligned}
\tag{F.35}
$$

to write $A(t,y)|_{\Phi_i} = A(t)$.

F.5 The Brane in the Synchronous Gauge

The metric parameter $A(t,y)$ is designed in such a way, so that is describes a vacuum solution to the Einstein equations, before the branes collide, where the spacetime is Minkowski. After the brane collision however, the Heaviside function is 'activated' and the branes are allowed to interact and exchange energy. We will now examine what the form of the metric is during their interaction time. We will use Φ_1 as an example.

On the hypersurface $\Phi_1 = 0$ it is true that,

$$\begin{aligned} t &= ay, \\ dy &= \frac{1}{a} dt. \end{aligned} \tag{F.36}$$

Using this relationship, we can transform the metric, Eq.(F.23) to the synchronous gauge. First consider,

$$\begin{aligned} A^{-2/3}(dt^2 - dy^2) &= \\ A^{-2/3}\left(dt^2 - \frac{1}{a^2}dt^2\right) &= \\ A^{-2/3}\left(\frac{a^2-1}{a^2}\right)dt^2 &= d\tau^2 \end{aligned} \tag{F.37}$$

Namely,

$$d\tau = \left(\frac{a^2-1}{a^2}\right)^{1/2} \frac{1}{A^{1/3}} dt. \tag{F.38}$$

Therefore, Eq.(F.23) becomes,

$$\begin{aligned} A &= a\Phi_2 H(\Phi_2) + A_0 \\ A &= a(t + by)H(\Phi_2) + A_0 \\ A &= (a+b)tH(\Phi_2) + A_0 \end{aligned} \tag{F.39}$$

And so,

$$A = \begin{cases} (a+b)t + A_0 & , \Phi_2 > 0 \\ A_0 & , \Phi_2 < 0 \end{cases} \tag{F.40}$$

Now we can describe the spacetime explicitly, after the branes collide, where we are in region $\Phi_2 > 0$, and

$$\begin{aligned} d\tau &= \left(\frac{a^2-1}{a^2}\right)^{1/2} \frac{dt}{[(a+b)t + A_0]^{1/3}} \\ \tau &= \left(\frac{a^2-1}{a^2}\right)^{1/2} \int \frac{dt}{[(a+b)t + A_0]^{1/3}} \end{aligned} \tag{F.41}$$

Letting,

$$\begin{aligned}
x &= (a+b)t + A_0, \\
dx &= (a+b)dt, \\
dt &= \frac{dx}{(a+b)}.
\end{aligned} \tag{F.42}$$

We get,

$$\begin{aligned}
\tau &= \frac{(a^2-1)^{1/2}}{a(a+b)} \int \frac{dx}{x^{1/3}}, \\
\tau &= \frac{(a^2-1)^{1/2}}{a(a+b)} \frac{3}{2} x^{2/3} + x_0, \\
\tau &= \frac{3(a^2-1)^{1/2}}{2a(a+b)} [(a+b)t + A_0]^{2/3}] + \tau_0, \\
\tau - \tau_0 &= \Delta[(a+b)t + A_0]^{2/3}], \\
(a+b)t + A_0 &= \left(\frac{\tau-\tau_0}{\Delta}\right)^{3/2},
\end{aligned} \tag{F.43}$$

where,

$$\Delta = \frac{3(a^2-1)^{1/2}}{2a(a+b)} \tag{F.44}$$

We can now write the metric in the sychronous gauge, using Eq.(F.38), as,

$$ds_5^2|_{\Phi_1=0} = d\tau^2 - A^{2/3} d\Sigma_0^2, \tag{F.45}$$

where,

$$A(t) = (a+b)t + A_0 = \left(\frac{\tau-\tau_0}{\Delta}\right)^{3/2} \tag{F.46}$$

From the form above, it is clear that we have a Friendman-Robertson-Walker metric,

$$ds_5^2|_{\Phi_1=0} = d\tau^2 - a(\tau) d\Sigma_0^2, \tag{F.47}$$

where the scale factor $a(\tau)$ (that should not be confused with the parameter a) is,

$$a(\tau) = A^{1/3} = \left(\frac{\tau-\tau_0}{\Delta}\right)^{1/2} \tag{F.48}$$

For $\tau_0 = 0$ and $a_0 = \Delta^{-1/2}$, we have,

$$a(\tau) = a_0 \tau^{1/2} \tag{F.49}$$

Summarily, we have found that our solution admits a FRW metric that develops after the collision, with equations

$$\begin{aligned} ds_4^2 &= ds_5^2|_{\Phi_1=0} = d\tau^2 - a(\tau)d\Sigma_0^2 \\ a(\tau) &= a_0\tau^{1/2} \end{aligned} \tag{F.50}$$

F.6 Perfect Fluid Energy Momentum Tensor on the Branes

In this model, we are exploring the possibility of a viable universe being produced from the brane collision. We saw in the previous section that we expect a Friedmann-Robertson-Walker type solution, which admits a perfect fluid matter field, which in the FRW 4-dimensional spacetime has the familiar form,

$$T_{ab} = (\rho + p)u_a u_b - p g_{ab}. \tag{F.51}$$

In our case however, we will have to examine if there are any changes in this equation when we examine the spacetime hypersurfaces Φ_i that represent the branes. We start from the very definition of the energy momentum tensor, as derived in Appendix C, Eq.(C.32),

$$T_{ab}^{(5)} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\left[\sqrt{-g}\mathcal{L}_m^{(5)}\right]}{\delta g^{ab}} \tag{F.52}$$

The equivalent form of the equation on the brane is,

$$T_{ab}^{(4)} \equiv -\frac{2}{\sqrt{-\gamma}}\frac{\delta\left[\sqrt{-g}\mathcal{L}_m^{(5)}\right]}{\delta g^{ab}}\delta(\Phi_i) \tag{F.53}$$

By comparison, we can define,

$$T_{ab}^{(4)} = \sqrt{\frac{g}{\gamma}}\,T_{ab}^{(5)}\delta(\Phi_i) \tag{F.54}$$

In this case, the general form of the energy momentum tensor of *any* fluid on the brane with reduced metric g_{ab} (external space), embedded in γ_{ab} (internal space) is,

$$T_{ab}^{(i)} = \delta(\Phi_i)C[g]\left[\rho_{\Phi_i}(u_A u_B) + p(X_a X_b + Y_a Y_b + Z_a Z_b)\right] \tag{F.55}$$

where

- $C[g] = \sqrt{-\frac{g}{\gamma}}$, is a metric scaling term, with g, γ the determinants of the external and internal spaces,

- $X_a X^a = -1$, are spacial vectors, that define the external space on the brane.

In our particular case, for Eq.(F.26) we find,

$$\begin{aligned} C[g] &= A^{1/3}, \\ X_a &= A^{1/3}\delta_a^2, \\ Y_a &= A^{1/3}\delta_a^3, \\ Z_a &= A^{1/3}\delta_a^4. \end{aligned} \tag{F.56}$$

Substituting Eq.(F.34) and Eq.(F.56) into Eq.(F.55), and for brane Φ_1 we have, (where A is now just a function of time)

$$\begin{aligned} T_{ab}^{(1)} &= A^{1/3}\Big[\rho_{\Phi_1}\frac{1}{A^{2/3}(a^2-1)}(a\delta_a^t - \delta_a^y)(a\delta_b^t - \delta_b^y) \\ &\quad + p_{\Phi_1}A^{2/3}(\delta_a^2\delta_b^2 + \delta_a^3\delta_b^3 + \delta_a^4\delta_b^4)\Big], \\ T_{ab}^{(1)} &= \rho_{\Phi_1}\frac{1}{A^{1/3}\,(a^2-1)}\Big(a^2\delta_a^t\delta_b^t - a\delta_a^t\delta_b^y - a\delta_a^y\delta_b^t + \delta_a^y\delta_b^y\Big) \\ &\quad + p_{\Phi_1}A\ \delta_{ij}\delta_a^i\delta_b^j. \end{aligned} \tag{F.57}$$

Picking out components, we find,

$$\begin{aligned} T_{00}^{(1)} &= \frac{a^2}{(a^2-1)A^{1/3}}\ \rho_{\Phi_1}, \\ T_{01}^{(1)} &= -\frac{a}{(a^2-1)A^{1/3}}\ \rho_{\Phi_1}, \\ T_{11}^{(1)} &= \frac{1}{(a^2-1)A^{1/3}}\ \rho_{\Phi_1}, \\ T_{22}^{(1)} &= A\ p_{\Phi_1}, \end{aligned} \tag{F.58}$$

Now we can proceed with solving the Einstein equation. Comparing terms by term between Eq.(F.29) and Eq.(F.58), where we have set $\kappa = 1$,

$$\begin{aligned}
\frac{a^2}{(a^2-1)A^{1/3}}\rho_{\Phi_1} &= -\frac{a^2 b}{A}, \\
-\frac{a}{(a^2-1)A^{1/3}}\rho_{\Phi_1} &= \frac{ab}{A}, \\
\frac{1}{(a^2-1)A^{1/3}}\rho_{\Phi_1} &= -\frac{b}{A}, \\
A p_{\Phi_1} &= \frac{b(a^2-1)}{3}A^{1/3}.
\end{aligned} \tag{F.59}$$

We can now derive the physical and dynamical properties of the perfect fluid that develops after the collision on the two branes,

$$\begin{aligned}
\rho_{\Phi_1} &= -\frac{b(a^2-1)}{A^{2/3}}, \\
p_{\Phi_1} &= -\frac{b(a^2-1)}{3A^{2/3}}.
\end{aligned} \tag{F.60}$$

Where we can clearly see that the *equation of state of the fluid, that develops after the collision on brane* Φ_1 *is*,

$$p_{\Phi_1} = -\frac{1}{3}\rho_{\Phi_1} \tag{F.61}$$

F.7 The Spacetime Before the Collision

For $\Phi_2 < 0$ we have,

$$A = A_0 \tag{F.62}$$

Thus the metric in Eq.(F.50), takes form,

$$ds_5^2|_{\Phi_1} = d\tau^2 - A_0^{2/3} d\Sigma_0^2 \tag{F.63}$$

This solution also corresponds to a static FRW spacetime.

F.8 The Spacetime After the Collision

In Einstein theory, the perfect fluid solution to the FRW metric results in the cosmological equations,

$$
\begin{aligned}
H^2 &= \frac{8\pi G}{3}\rho, && \text{(F.64)}\\
\dot{\rho} + 3H(\rho + p) &= 0, && \text{(F.65)}
\end{aligned}
$$

where $H = \dot{a}/a$, is the Hubble parameter and G, is the gravitational constant. For a perfect fluid with equation of state $p = \omega\rho$, we have the solution for Eq.(F.65),

$$
\begin{aligned}
\dot{\rho} + 3H(1+\omega)\rho &= 0,\\
\frac{\dot{\rho}}{\rho} &= -3(1+\omega)\frac{\dot{a}}{a},\\
\ln\rho &= -3(1+\omega)\ln a + \ln\rho_0,\\
\rho &= a^{-3(1+\omega)}\rho_0, && \text{(F.66)}
\end{aligned}
$$

where ρ_0 is an integration constant. Now Eq.(F.64) can be integrated by quadratures,

$$
\begin{aligned}
H &= \sqrt{\frac{8\pi G}{3}\rho_0}\, a^{-\frac{3(1+\omega)}{2}},\\
\frac{\dot{a}}{a} &= \sqrt{\frac{8\pi G}{3}\rho_0}\, a^{-\frac{3(1+\omega)}{2}},\\
a^{\frac{3\omega+1}{2}}da &= \sqrt{\frac{8\pi G}{3}\rho_0}\, dt,\\
a &= a_0\tau^{\frac{2}{3(\omega+1)}} && \text{(F.67)}
\end{aligned}
$$

To recover our previously derived result, Eq.(F.61), we set $\omega = 1/3$,

$$p_{rad} = \frac{1}{3}\rho_{rad}, \tag{F.68}$$

which is the equation of state for radiation [78]. This gives us,

$$a(\tau) = a_0\tau^{\frac{1}{2}}. \tag{F.69}$$

Therefore, we conclude that the spacetime on the brane, after the collision is expanding, at a rate that corresponds to a radiation dominated era.

As an example, we will consider the case where $a > |b| > 1$. The hypersurfaces Σ_1 and Σ_2 that represent the trajectories of the two branes divide the space into four regions, as shown in Fig. 2.3.

In each of these, the quantity $A(t, y)$, takes on the following form,

$$A(t,y) = \begin{cases} A_0 & \text{, Region I} \\ A_0 - |b|\Phi_1 & \text{, Region II} \\ a(\Phi_2 - |b|\Phi_1 + A_0 & \text{, Region III} \\ a\Phi_2 + A_0 & \text{, Region IV} \end{cases} \tag{F.70}$$

where, we should recall that,

$$\Phi_1 = t - ay \qquad \Phi_2 = t - |b|y. \tag{F.71}$$

Now we consider the spacetimes in each region.

Region I: The metric becomes,

$$\begin{aligned} ds_I^2 &= A_0^{-2/3}(dt^2 - dy^2) - A_0^{2/3} d\Sigma_0^2, \\ ds_I^2 &= dt'^2 - dy^2 - d\Sigma_0'^2, \end{aligned} \tag{F.72}$$

where we have absorbed the constants into the coordinates, using the covariant properties of the metric. The resulting space is Minkowski flat and free of singularities.

Region II: We now find that,

$$ds_{II}^2 = A^{-2/3}(dt^2 - dy^2) - A^{2/3} d\Sigma_0^2, \tag{F.73}$$

where $A = A_0 - |b|t + a|b|y$. To determine the presence of any singularities in this region, we calculate the Kretschmann scalar,

$$\begin{aligned} I &= R_{abcd}R^{abcd} \\ I &= \frac{8b^4(a^2-1)}{9A^{8/3}}. \end{aligned} \tag{F.74}$$

The spacetime is singular at $A = 0$, which corresponds to,

$$\begin{aligned} A &= 0 \\ A_0 - |b|t + a|b|y &= 0 \\ t &= \frac{A_0}{|b|} + ay \end{aligned} \tag{F.75}$$

This is a timelike singular hypersurface, that geometrically corresponds to a line parallel to $\Phi_1 = 0$, on the (t,y)-plane, and displaced to the left by $A_0/|b|$.

Region III: here we have,

$$ds^2_{III} = A^{-2/3}(dt^2 - dy^2) - A^{2/3}d\Sigma_0^2, \tag{F.76}$$

where $A = A_0 + (a - |b|)t$. The Kretschmann scalar is,

$$I = \frac{8(a-|b|)^4}{A^{8/3}}. \tag{F.77}$$

Thus the singularities in this region are also located at $A = 0$,

$$\begin{aligned} A_0 + (a - |b|)t &= 0 \\ t &= -\frac{A_0}{a - |b|} \end{aligned} \tag{F.78}$$

which is a spacelike singularity parallel to the $y - axis$.

The singularities in regions II and III intersect at a point P, with spacetime coordinates,

$$\begin{aligned} t_P &= -\frac{A_0}{a - |b|} \\ y_P &= -\frac{A_0}{|b|(a - |b|)} \end{aligned} \tag{F.79}$$

The brane $\Phi_2 = 0$, intersects this singularity, and appears to originate from it, when we examine its trajectory throughout the (t,y)-plane.

Region IV: In this case, the metric becomes,

$$ds_{IV}^2 = A^{-2/3}(dt^2 - dy^2) - A^{2/3} d\Sigma_0^2, \tag{F.80}$$

where $A = A_0 + at - a|b|y$. The Kretschmann scalar is,

$$I = \frac{8(a - |b|)^4}{A^{8/3}}. \tag{F.81}$$

Thus the singularities in this region are also located at $A = 0$, here also. However,

$$A = a\Phi_2 + A_0 \tag{F.82}$$

and since $a > 0$, $A_0 > 0$ and $\Phi_2 > 0$ in region IV, there is no way to make $A = 0$. Therefore this region is free of singularities.

Hypersurface Σ_1*:* Having derived the equation of state of the fluid that develops on the branes, in Eq.(F.60), we can examine the nature of the fluid by checking if they satisfy the energy equations, of Eq.(2.35)-(2.37).

In the case we are considering, we have $a > |b| > 1$, and therefore,

$$\begin{aligned} \rho_{\Phi_1} &= -\frac{b(a^2 - 1)}{A^{2/3}} < 0, \\ p_{\Phi_1} &= -\frac{b(a^2 - 1)}{3A^{2/3}} < 0. \end{aligned} \tag{F.83}$$

Therefore none of the energy conditions are satisfied, at any time before, or after the collision, on the brane Φ_1.

This analysis can be applied to the Φ_2 brane also, with similar results. This methodology is used throughout this paper.

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