ABSTRACT

Eigenvalue Comparison Theorems For Certain Boundary Value Problems and Positive Solutions for a Fifth Order Singular Boundary Value Problem

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Comparison of smallest eigenvalues for certain two point boundary value problems for a fifth order linear differential equation are first obtained. The results are extended to (2n+1)-order and (3n+2)-order boundary value problems. Methods used for these results involve the theory of u_0-positive operators with respect to a cone in conjunction with sign properties of Green's functions.

Finally, initial results are established for the existence of positive solutions for singular two point boundary value problems for a fifth order nonlinear differential equation. The methods involve application of a fixed point theorem for decreasing operators.

Eigenvalue Comparison Theorems For Certain Boundary Value Problems and Positive Solutions for a Fifth Order Singular Boundary Value Problem

by

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TABLE OF CONTENTS

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To Helen Hudson.
I am still plugging away Grandma!

CHAPTER ONE

Introduction

The initial focus of this dissertation will be on comparing the smallest eigenvalues for the eigenvalue problems,

$$u^{(5)} + \lambda p(t)u = 0, \quad 0 < t < 1, \tag{1.1}$$

and

$$u^{(5)} + \sigma q(t)u = 0, \quad 0 < t < 1, \tag{1.2}$$

with eigenvectors satisfying the boundary conditions,

$$u(0) = u''(0) = u(1) = u''(1) = u^{(4)}(1) = 0, \tag{1.3}$$

where $p(t), q(t)$ are continuous, non-negative functions on $[0, 1]$ and both $p(t)$ and $q(t)$ do not vanish identically on any compact subinterval $[\alpha, \beta]$ of $[0, 1]$. After establishing the sufficient conditions for the comparison of smallest eigenvectors for the above problems, the result will be extended to the $(2n + 1)$-order $(n \geq 2)$ case. Another consideration for a $(3n+2)$-order extension will be made. The final chapter will be devoted to finding positive solutions for a singular fifth order boundary value problem.

In Chapter Two, the comparison results for the eigenvalue problems (1.1), (1.3) and (1.2), (1.3) are derived. Once this is done the results are extended to the $(2n + 1)$-order $(n \geq 2)$ case,

$$(-1)^n u^{(2n+1)} + \lambda p(t)u = 0, \quad 0 < t < 1,$$

and

$$(-1)^n u^{(2n+1)} + \sigma q(t)u = 0, \quad 0 < t < 1,$$

with eigenvectors in each case satisfying the boundary conditions,

$$u^{(2i)}(0) = 0, \quad i = 0, \ldots, n-1,$$
$$u^{(2j)}(1) = 0, \quad j = 0, \ldots, n.$$

The technique for the comparison of these eigenvalues involves the application of sign properties of a Green's function, followed by applications from the theory of u_0-positive operators with respect to a cone in a Banach space. These cone theoretic applications are presented in Krasnosel'skii's book [26] and in the book by Krein and Rutman [27].

Several authors have applied the cone theoretic techniques which are applied here in comparing eigenvalues for boundary conditions other than the two point boundary conditions seen in Chapters Two and Three. Some of the previous work has been devoted to boundary value problems for ordinary differential equations involving conjugate, Lidstone, and right focal conditions. For example, Eloe and Henderson [9] have studied smallest eigenvalue comparisons for a class of two point boundary value problems, and for a class of multi-point boundary value problems [11]. For more work on this field, see, for example, [7, 12, 13, 15, 20, 24, 37, 38]. In addition, Hankerson and Henderson [19] have obtained comparison results for difference equations, and Henderson and Prasad [21] obtained results for Lidstone boundary value problems on time scales.

In Chapter Three, the comparison results for eighth order eigenvalue problems are derived. Once this is done the results are extended to the $(3n+2)$-order case $(n \geq 2)$,

$$u^{(3n+2)} + \lambda p(t)u = 0, \quad 0 < t < 1,$$

and

$$u^{(3n+2)} + \sigma q(t)u = 0, \quad 0 < t < 1,$$

with eigenvectors both satisfying the boundary conditions,

$$u(0) = u(1) = 0,$$
$$u^{(3j+2)}(0) = u^{(3j+2)}(1) = 0, \quad \text{for} \quad j = 0, 1, \ldots, n-1,$$
$$u^{(3k+1)}(1) = 0, \quad \text{for} \quad k = 1, 2, \ldots, n.$$

In Chapter Five, the focus shifts to finding positive solutions for the nonlinear fifth order singular boundary value problem,

$$u^{(5)} = f(x, u), \quad 0 < x < 1, \tag{1.4}$$

$$u(0) = u'''(0) = u(1) = u''(1) = u'''(1) = 0 \tag{1.5}$$

where $f(x, y)$ is singular at $x = 0, 1$, $y = 0$, and may be singular at $y = \infty$. The following conditions are also assumed to hold on f:

(H1) $f(x, y) : (0, 1) \times (0, \infty) \to (0, \infty)$ is continuous, and $f(x, y)$ is decreasing in y, for every x.

(H2) $\lim_{y \to 0^+} f(x, y) = +\infty$ and $\lim_{y \to +\infty} f(x, y) = 0$ uniformly on compact subsets of $(0, 1)$.

The techniques utilized for finding positive solutions involve applying a fixed point theorem by Gatica, Oliker, and Waltman [14] for operators that are decreasing with respect to a cone. Also fundamental to obtaining positive solutions of (1.4), (1.5) is a positivity result by Graef and Yang [16, 17].

Singular boundary value problems for ordinary differential equations, often times of the second order and involving semi-infinite intervals, for which there are positive solutions are often used to model applications, such as, glacial advance and transport of coal slurries down conveyor belts as examples of non-Newtonian fluid theory in studies of pseudoplastic fluids [31], for problems involving draining flows [1, 3], semipositone and positone problems [2], and as models in boundary layer applications, Emden-Fowler boundary value problems, and reaction-diffusion applications [4–6,28]. In addition, there is a large literature for semi-linear boundary value problems for bounded domains Ω in any space of dimension $N > 1$, for second order differential operators (such as the Laplacian -Δ) with nonlinearities $f(x, u)$ which are singular both in u (when u goes to 0) and in x (when $\partial(x) = \partial(x, \Omega)$ goes to zero); see, for example [23] and the references therein.

CHAPTER TWO

Comparison Theory for a Certain Class Of (2n+1)-Order Eigenvalue Problems

2.1 Introduction

This chapter establishes the existence of smallest positive eigenvalues and their comparisons for the fifth order eigenvalue problems,

$$u^{(5)} + \lambda p(t)u = 0, \quad 0 < t < 1, \tag{2.1}$$

and

$$u^{(5)} + \sigma q(t)u = 0, \quad 0 < t < 1, \tag{2.2}$$

with eigenvectors satisfying the boundary conditions,

$$u(0) = u''(0) = u(1) = u''(1) = u^{(4)}(1) = 0. \tag{2.3}$$

In each case, $p(t), q(t) : [0,1] \to [0,\infty)$ are continuous and neither vanishes identically on any non-degenerate compact subinterval of $[0,1]$.

Remark 2.1. At this point, note that if $\lambda > 0$ (respectively $\sigma > 0$), and if u is a nontrivial solution of (2.1), (2.3) (respectively (2.2), (2.3)), then $u^{(5)}(t) \le 0$, $u'''(t) \ge 0$, $u''(t) \le 0,$ and $u(t) \ge 0, 0 \le t \le 1.$

In the ensuing sections, the techniques applied to examine the comparison of eigenvalues involve sign properties of a Green's function along with the theory of u_0-positive operators with respect to a cone in a Banach space. These necessary preliminary definitions and fundamental results from cone theory are as follows.

Definition 2.2. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone, provided

- $\alpha u + \beta u \in \mathcal{P}$, for all u, $v \in \mathcal{P}$ and for all α, $\beta \ge 0$, and

- $(-\mathcal{P})\bigcap\mathcal{P} = \{0\}$.

Definition 2.3. A cone $\mathcal{P}$ is solid if $\mathcal{P}^\circ \neq \emptyset$, where $\mathcal{P}^\circ$ denotes the interior of $\mathcal{P}$. A cone is said to be reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$, where $\mathcal{P} - \mathcal{P}$ denotes the 'difference set of $\mathcal{P}$'.

Remark 2.4. Krasnosel'skii [26] proved that every solid cone is reproducing.

A cone induces a partial ordering on a Banach space and a partial ordering on the bounded linear operators defined on the Banach space.

Definition 2.5. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u,\, v \in \mathcal{B}$, we say $u \leq v$ (with respect to $\mathcal{P}$), if $v - u \in \mathcal{P}$. If both $M,\, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, we say $M \leq N$ (with respect to $\mathcal{P}$), if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

Definition 2.6. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$, and let $M : \mathcal{B} \to \mathcal{B}$ be a bounded linear operator. If there exists $u_0 \in \mathcal{P}\setminus\{0\}$ such that, for each $u \in \mathcal{P}\setminus\{0\}$ there exist constants $k_1(u) > 0$ and $k_2(u) > 0$, such that $k_1u_0 \leq Mu \leq k_2u_0$, then M is said to be u_0-positive (with respect to $\mathcal{P}$).

For the next three results, the first two are in Krasnosel'skii's book [26], and the third result is proved in Keener and Travis [25].

Theorem 2.7. *Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$ and let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $M : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $M : \mathcal{P}\setminus\{0\} \to \mathcal{P}^\circ$, then M is u_0-positive.*

Theorem 2.8. *Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$ and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $M : \mathcal{B} \to \mathcal{B}$ be a compact, linear operator that is u_0-positive. Then M has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.9. *Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$ and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M,\, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators, and assume that at least one of the* *operators is u_0-positive. If $M \le N$, $\lambda u_1 \le M u_1$ for some $u_1 \in \mathcal{P}$ and some $\lambda > 0$, and $N u_2 \le \sigma u_2$ for some $u_2 \in \mathcal{P}$ and some $\sigma > 0$, then $\lambda \le \sigma$. Moreover, $\lambda = \sigma$ implies u_1 is a scalar multiple of u_2.*

2.2 A Comparison Theorem for Certain Fifth Order Boundary Value Problems

In this section the definitions and results stated in the previous section are applied when comparing smallest eigenvalues λ and σ of (2.1), (2.3) and (2.2), (2.3), respectively. First, an appropriate Green's function needs to be constructed to serve the role of a kernel for a compact linear operator that will fulfill the roles in the previous results. In particular, the Green's function for

$$u^{(5)} = 0, \tag{2.4}$$

satisfying the boundary conditions (2.3) is needed. Let $K(t,s)$ be the Green's function for $u''' = 0$, with boundary conditions $u(0) = u(1) = u''(1) = 0$. Graef and Yang [18] have obtained

$$K(t,s) = \frac{1}{2}\begin{cases} t(1-t) - t(1-s)^2, & \text{if } 0 \le t \le s \le 1, \\ t(1-t) - t(1-s)^2 + (t-s)^2, & \text{if } 0 \le s \le t \le 1. \end{cases}$$

Some properties of $K(t,s)$ that will prove useful include

- $K(t,s) > 0$ on $(0,1) \times (0,1)$,
- $\dfrac{\partial}{\partial t} K(0,s) = \frac{1}{2} s(2-s) > 0, \quad 0 < s \le 1,$
- $\dfrac{\partial}{\partial t} K(1,s) = -\frac{1}{2} s^2 < 0, \quad 0 < s \le 1.$

Next, let $H(t,s)$ be the Green's function for $-u'' = 0$, with boundary conditions $u(0) = u(1) = 0$. It is well known that

$$H(t,s) = \begin{cases} t(1-s), & \text{if } 0 \le t \le s \le 1, \\ s(1-t), & \text{if } 0 \le s \le t \le 1. \end{cases}$$

It follows from [7] and [8] that the Green's function for (2.4), (2.3) is given by the convolution

$$G_5(t,s) = \int_0^1 H(t,r)K(r,s)\,dr. \tag{2.5}$$

Properties of $G_5(t,s)$ that are useful to us include

- $G_5(t,s) > 0$ on $(0,1)\times(0,1)$,
- $\dfrac{\partial^2}{\partial t^2}G_5(t,s) = K(t,s) < 0$ on $(0,1)\times(0,1)$,
- $G_5(0,s) = G_5(1,s) = 0, \quad 0 < s < 1,$
- $\dfrac{\partial}{\partial t}G_5(0,s) = \int_0^1 (1-r)K(r,s)\,dr > 0, \quad 0 \le s < 1,$
- $\dfrac{\partial}{\partial t}G_5(1,s) = \int_0^1 (-r)K(r,s)\,dr < 0, \quad 0 < s \le 1,$
- $\dfrac{\partial^2}{\partial t^2}G_5(0,s) = K(0,s) = 0, \quad 0 < s < 1,$
- $\dfrac{\partial^2}{\partial t^2}G_5(1,s) = K(1,s) = 0, \quad 0 < s < 1,$
- $\dfrac{\partial^4}{\partial t^4}G_5(1,s) = \dfrac{\partial^2}{\partial t^2}K(1,s) = 0, \quad 0 < s < 1.$

Next, in order to apply the positive cone theory from Section 2.1, a suitable Banach space and cone within the Banach space are introduced. Let the Banach space $\mathcal{B}$ be given by

$$\mathcal{B} := \{u \in \mathrm{C}^{(1)}[0,1] \mid u(0) = u(1) = 0\},$$

equipped with the norm defined by

$$\|u\| := |u'|_0, \text{where}\, |\cdot|_0 := \max_{0\le t\le 1} |\cdot|.$$

It is straightforward that, for each $u \in \mathbb{R}$,

$$|u|_0 \le |u'|_0 \le \|u\|.$$

Let the cone $\mathcal{P} \subset \mathcal{B}$ be defined as

$$\mathcal{P} := \{u \in \mathcal{B} \,|\, u(t) \geq 0,\ 0 \leq t \leq 1\}.$$

By Remark 2.4, in order to show that the cone $\mathcal{P}$ is reproducing, it suffices to show that $\mathcal{P}^\circ \neq \emptyset$. For that purpose we prove the following Lemma.

Lemma 2.10. *The cone $\mathcal{P}$ has nonempty interior, and $\mathcal{Q} := \{v \in \mathcal{B} \,|\, v(t) > 0,\ 0 < t < 1,\ v'(0) > 0,$ and $v'(1) < 0\} \subset \mathcal{P}^\circ$.*

Proof. It is clear that $\mathcal{Q} \subseteq \mathcal{P}$. Next, choose $u \in \mathcal{Q}$. Then $u(t) > 0$ on $(0,1)$, $u'(0) > 0$, and $u'(1) < 0$. Let $B_\varepsilon(u) := \{v \in \mathcal{B} \,|\, \|u - v\| = \max_{0 \leq t \leq 1} |u' - v'| < \varepsilon\}$. It needs to be shown that, for $\varepsilon > 0$ sufficiently small, $B_\varepsilon(u) \subseteq \mathcal{P}$. So, let $\varepsilon_1 > 0$ be such that $u'(0) - \varepsilon_1 > 0$, $\varepsilon_2 > 0$ be such that $u'(1) + \varepsilon_2 < 0$, and $\varepsilon_3 > 0$ such that $\|u(x) - v(x)\| < \varepsilon_3$. Then set $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. Choose $v \in B_{\varepsilon_0}(u)$. It follows that $v'(0) > u'(0) - \varepsilon_0 > 0$, and $v'(1) < u'(1) + \varepsilon_0 < 0$. Then recall that $|u(t) - v(t)| \leq \|u(t) - v(t)\| < \varepsilon_0$, and so $v(x) > 0$ on $(0,1)$. This implies that $v \in \mathcal{Q} \subseteq \mathcal{P}$, and so $B_{\varepsilon_0}(u) \subseteq \mathcal{Q} \subseteq \mathcal{P}$ where $\mathcal{Q}$ is open and $u \in \mathcal{P}^\circ$. Since u is arbitrary and $u \in \mathcal{Q}$, this implies $\mathcal{Q} \subseteq \mathcal{P}^\circ$ as well as $\mathcal{P}^\circ \neq \emptyset$. $\square$

Corollary 2.11. *The cone $\mathcal{P}$ is solid and hence is reproducing.*

Now consider the linear operators $M,\ N : \mathcal{B} \to \mathcal{B}$ defined by

$$Mu(t) := \int_0^1 G_5(t,s)p(s)u(s)\,ds$$

and

$$Nu(t) := \int_0^1 G_5(t,s)q(s)u(s)\,ds.$$

Lemma 2.12. *The operators M and N are compact.*

Proof. The statement will be proven for M, and the argument for N is analagous. Let $L = \max_{0 \leq t \leq 1} p(t)$ and $K = \max_{(t,s) \in [0,1] \times [0,1]} |\frac{\partial}{\partial t} G_5(t,s)|$. Let $\varepsilon > 0$ be given and let

$\delta = \frac{\varepsilon}{KL}$. Then for u, $v \in \mathcal{B}$, with $\|u - v\| < \delta$, we have for each $0 \leq t \leq 1$,

$$\begin{aligned}\|\left(Mu\right)'(t) - \left(Mv\right)'(t)\| &= \left|\int_0^1 \frac{\partial}{\partial t} G_5(t,s) p(s)[u(s) - v(s)]\, ds\right| \\ &\leq \int_0^1 \left|\frac{\partial}{\partial t} G_5(t,s)\right| p(s)\, |[u(s) - v(s)]|\, ds \\ &\leq LK\delta \\ &= \varepsilon.\end{aligned}$$

So, $\|Mu - Mv\| \leq \varepsilon$, and M is continuous. Next, choose $u \in \mathcal{P}$. Since all of $u(t), G_5(t,s)$, and $p(t)$ are nonnegative functions, we have for $0 \leq t \leq 1$,

$$Mu(t) = \int_0^1 G_5(t,s)p(s)u(s)\, ds \geq 0.$$

Moreover, $Mu(0) = Mu(1) = 0$, and so $M : \mathcal{P} \to \mathcal{P}$. Let $\{u_n\}$ be a bounded sequence in $\mathcal{P}$; say, $\|u_n\| \leq K_0$, for all n. Then, for each n, and $0 \leq t \leq 1$

$$\begin{aligned}\left|\left(Mu_n\right)'(t)\right| &\leq \int_0^1 \left|\frac{\partial}{\partial t} G_5(t,s)\right| p(s)|u_n(s)|\, ds \\ &\leq KK_0L,\end{aligned}$$

so $\|\left(Mu_n\right)\| \leq KK_0L$, for all n. From the continuity of $\frac{\partial}{\partial t}G_5(t,s)$ and the absolute continuity of the integral, $\{Mu_n\}$ is equicontinuous. Hence, an application of the Arzela-Ascoli theorem yields that M is compact. □

Lemma 2.13. *The bounded linear operators M and N are u_0-positive (with respect to $\mathcal{P}$).*

Proof. It will be shown that $M : \mathcal{P} \setminus \{0\} \to \mathcal{Q} \subset \mathcal{P}^\circ$, and then Theorem 2.7 yields the conclusion. First, recall from the proof of Lemma 2.12 that $M : \mathcal{P} \to \mathcal{P}$. Now, choose $u \in \mathcal{P} \setminus \{0\}$. Then, by the assumptions on $p(t)$, there exists a compact subinterval $[\alpha, \beta] \subseteq [0,1]$ such that $p(t)u(t) > 0$ on $[\alpha, \beta]$. From the property that $G_5(t,s) > 0$ on $(0,1) \times (0,1)$, it follows that, for $0 < t < 1$,

$$Mu(t) = \int_0^1 G_5(t,s)p(s)u(s)\, ds$$

$$
\begin{aligned}
&\geq \int_\alpha^\beta G_5(t,s)p(s)u(s)\,ds \\
&> 0.
\end{aligned}
$$

Moreover, from the properties that $\frac{\partial}{\partial t}G_5(0,s) > 0$, for $0 < s < 1$ and $\frac{\partial}{\partial t}G_5(1,s) < 0$, for $0 < s < 1$, it follows that

$$
\begin{aligned}
(Mu)'(0) &= \int_0^1 \frac{\partial}{\partial t}G_5(0,s)p(s)u(s)\,ds \\
&\geq \int_\alpha^\beta \frac{\partial}{\partial t}G_5(0,s)p(s)u(s)\,ds \\
&> 0,
\end{aligned}
$$

and similarly,

$$
(Mu)'(1) = \int_0^1 \frac{\partial}{\partial t}G_5(1,s)p(s)u(s)\,ds < 0.
$$

So, $Mu \in \mathcal{Q}$; that is $M : \mathcal{P} \setminus \{0\} \to \mathcal{P}^\circ$. Thus M is u_0-positive, and similarly N is also u_0-positive. $\square$

Remark 2.14. At this point it is important to note that

$$
\Lambda u(t) = Mu(t) = \int_0^1 G_5(t,s)p(s)u(s)\,ds, \quad 0 \leq t \leq 1,
$$

if and only if

$$
u(t) = \frac{1}{\Lambda}\int_0^1 G_5(t,s)p(s)u(s)\,ds, \quad 0 \leq t \leq 1,
$$

if and only if

$$
\begin{gathered}
u^{(5)}(t) = -\frac{1}{\Lambda}p(t)u(t),\ 0 < t < 1, \text{ and} \\
u(0) = u'''(0) = u(1) = u''(1) = u'''(1) = 0.
\end{gathered}
$$

That is, the eigenvalues of (2.1), (2.3) are reciprocals of the eigenvalues of M, and conversely. In analogy, the eigenvalues of (2.2), (2.3) are reciprocals of the eigenvalues of N, and conversely.

Now, in light of the previous lemmas, Theorems 2.8 and 2.9 can now be applied.

Theorem 2.15. *The operator M (and N), has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to belong to $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator which is u_0-positive with respect to $\mathcal{P}$, it follows from Theorem 2.8 that M has an essentially unique eigenvector, $u \in \mathcal{P}$, and associated eigenvalue, Λ, having the properties as in the statement of the theorem. Since $u \neq 0$, then $Mu \in \mathcal{Q} \subset \mathcal{P}^\circ$, and so $u = M\left(\frac{1}{\Lambda u}\right) \in \mathcal{P}^\circ$. $\square$

Theorem 2.16. *Let $p(t) \leq q(t)$ on $[0,1]$. Let Λ and Σ be the eigenvalues of Theorem 2.15 corresponding to M and N, respectively, with corresponding essentially unique eigenvectors, u_1 and u_2, belonging to $\mathcal{P}^\circ$. Then $\Lambda \leq \Sigma$, and $\Lambda = \Sigma$ if, and only if, $p(t) = q(t)$ on $[0,1]$.*

Proof. With $p(t) \leq q(t)$ on $[0,1]$, then for any $u \in \mathcal{P}$ and $t \in [0,1]$,

$$(Nu - Mu)(t) = \int_0^1 G_5(t,s)\,(q(s) - p(s))\,u(s)\,ds \geq 0.$$

So, $Nu - Mu \in \mathcal{P}$, for all $u \in \mathcal{P}$; that is, $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 2.9, $\Lambda \leq \Sigma$.

Turning to the last part, if $p(t) = q(t)$ on $[0,1]$, then trivially $\Lambda = \Sigma$. On the other hand, if $p(t) \neq q(t)$, then there exists a subinterval $[c,d] \subseteq [0,1]$ such that $p(t) < q(t)$ on $[c,d]$. Then, by arguments as in the proof of Lemma 2.13, $(N-M)u_1 \in \mathcal{Q} \subseteq \mathcal{P}$, and so there exists an $\varepsilon > 0$ such that $(N-M)u_1 - \varepsilon u_1 \in \mathcal{P}^\circ$. We have

$$\Lambda u_1 + \varepsilon u_1 = Mu_1 + \varepsilon u_1 \leq Nu_1,$$

or $(\Lambda + \varepsilon)\,u_1 \leq Nu_1$. Since $N \leq N$ and $Nu_2 = \Sigma u_2$, we have from Theorem 2.9 that $\Lambda + \varepsilon \leq \Sigma$, or $\Lambda < \Sigma$. $\square$

In light of Remark 2.15 and Theorems 2.15 and 2.16, the main result concerning the existence and comparison of smallest positive eigenvalues for (2.1), (2.3) and (2.2), (2.3) can be stated.

Theorem 2.17. *Let $p(t) \leq q(t)$ on $[0,1]$. Then, there exist smallest positive eigenvalues, λ and σ, of* (2.1), (2.3) *and* (2.2), (2.3), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problem, and the eigenvectors correpsonding to λ and σ may be chosen belonging to $\mathcal{P}^\circ$. Finally, $\lambda \geq \sigma$, and $\lambda = \sigma$ if, and only if, $p(t) = q(t)$ on $[0,1]$.*

2.3 A Comparison Theorem for Certain $(2n+1)$-Order Boundary Value Problems $(n \geq 2)$

In this section, for $n \geq 2$, the comparisons of smallest eigenvalues are examined for the (2n+1)-order eigenvalue problems,

$$(-1)^n u^{(2n+1)} + \lambda p(t) u = 0, \tag{2.6}$$

and

$$(-1)^n u^{(2n+1)} + \sigma q(t) u = 0, \tag{2.7}$$

with eigenvectors for both satisfying the boundary conditions

$$\begin{aligned} u^{(2i)}(0) = 0, \quad i = 0, ..., n-1, \\ u^{(2j)}(1) = 0, \quad j = 0, ..., n. \end{aligned} \tag{2.8}$$

In each case, $(-1)^n p(t)$, $(-1)^n q(t) : [0,1] \to [0,\infty)$ are continuous and neither vanishes identically on any compact subinterval of [0,1]. Proceeding in the same fashion as in the previous section, to compare the smallest eigenvalues for (2.6), (2.8) and (2.7), (2.8), largest eigenvalues of equivalent integral problems will be examined. Here the appropriate Green's function needs to be constructed for

$$(-1)^n u^{(2n+1)} = 0 \tag{2.9}$$

that satisfies (2.7). Let $H(t,s)$ and $K(t,s)$ be the Green's functions as defined in the previous section. Then let

$$G_{2n+1}(t,s) = \int_0^1 H(t,r)G_{2n-1}(r,s)\,dr, \quad n \geq 3, \tag{2.10}$$

where

$$G_5(t,s) = \int_0^1 H(t,r)K(r,s)\,dr,$$

which is the Green's function found in Section 2.2. Then note that G_{2n+1} has the following properties that are useful:

- $(-1)^n G_{2n+1}(t,s) > 0 \quad \text{on} \quad (0,1)\times(0,1)$,
- $(-1)^n \frac{\partial^2}{\partial t^2} G_{2n+1}(t,s) < 0 \quad \text{on} \quad (0,1)\times(0,1)$,
- $(-1)^n \frac{\partial^{2i}}{\partial t^{2i}} G_{2n+1}(0,s) = 0 \quad \text{for} \quad i = 0,...,n-1, \quad 0 \leq s < 1$,
- $(-1)^n \frac{\partial^{2j}}{\partial t^{2j}} G_{2n+1}(1,s) = 0 \quad \text{for} \quad j = 0,...,n, \quad 0 < s \leq 1$,
- $(-1)^n \frac{\partial^{2i+1}}{\partial t^{2i+1}} G_{2n+1}(0,s) > 0 \quad \text{for} \quad i = 0,...,n-1, \quad 0 \leq s < 1$,
- $(-1)^n \frac{\partial^{2i+1}}{\partial t^{2i+1}} G_{2n+1}(1,s) < 0 \quad \text{for} \quad i = 0,...,n-1, \quad 0 < s \leq 1$.

At this point, as was seen in Section 2.2, it is necessary to define a suitable Banach space and a cone within that Banach space. These are defined as follows,

$$\mathcal{B} := \{u \in \mathrm{C}^{(1)}[0,1] \,|\, u(0) = u(1) = 0\},$$

equipped with the norm defined by

$$\|u\| := |u'|_0, \text{ where } |\cdot|_0 := \max_{0\leq t\leq 1} |\cdot|.$$

As in the previous section, it is straightforward to show that, for each $u \in \mathcal{B}$,

$$|u|_0 \leq |u'|_0 = \|u\|.$$

Then, define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} := \{u \in \mathcal{B} \,|\, (-1)^n u(t) \geq 0,\, 0 \leq t \leq 1\}.$$

Lemma 2.18. *The cone $\mathcal{P}$ has nonempty interior, and $\mathcal{Q} := \{v \in \mathcal{B} \,|\, (-1)^n v(t) > 0,\ 0 < t < 1,\ (-1)^n v'(0) > 0,$ and $(-1)^n v'(1) < 0\} \subset \mathcal{P}^\circ$.*

The proof for Lemma 2.18 is nearly identical to that for Lemma 2.10.

Corollary 2.19. *The cone $\mathcal{P}$ is solid and hence reproducing.*

In the same manner as the fifth order case, it suffices to seek eigenvalues of the linear operators $M, N : \mathcal{B} \to \mathcal{B}$ defined by

$$Mu(t) = \int_0^1 G_{2n+1}(t,s)p(s)u(s)\,ds$$

and

$$Nu(t) = \int_0^1 G_{2n+1}(t,s)q(s)u(s)\,ds.$$

Also, by a similar argument to the one used in Section 2.2, M and N are compact operators.

Lemma 2.20. *The bounded linear operators M and N are u_0-positive (w.r.t $\mathcal{P}$).*

Proof. It suffices to show that $M : \mathcal{P} \setminus \{0\} \to \mathcal{Q} \subset \mathcal{P}^\circ$. Then Theorem 2.7 yields the conclusion.

First, choose $u \in \mathcal{P}$. Since each of $(-1)^n u(t)$,$(-1)^n G_{2n+1}(t,s)$, and $(-1)^n p(t)$ is a nonnegative functions, for $0 \le t \le 1$,

$$(-1)^n Mu(t) = (-1)^n \int_0^1 G_{2n+1}(t,s)p(s)u(s)\,ds \ge 0.$$

Moreover, $Mu(0) = Mu(1) = 0$, thus $M : \mathcal{P} \to \mathcal{P}$. Now, choose $u \in \mathcal{P} \setminus \{0\}$. Then, by the assumptions on $p(t)$, there exists a compact subinterval $[\alpha, \beta] \subseteq [0,1]$ such that $(-1)^n p(t)u(t) > 0$ on $[\alpha, \beta]$. From the property that $(-1)^n G_{2n+1}(t,s) > 0$ on $(0,1) \times (0,1)$, it follows that, for $0 < t < 1$,

$$(-1)^n Mu(t) = \int_0^1 (-1)^n G_{2n+1}(t,s)p(s)u(s)\,ds$$

$$\geq \int_\alpha^\beta (-1)^n G_{2n+1}(t,s)p(s)u(s)\,ds$$
$$> 0.$$

Moreover, recalling that $(-1)^n \frac{\partial}{\partial t} G_{2n+1}(0,s) > 0$ and $(-1)^n \frac{\partial}{\partial t} G_{2n+1}(1,s) < 0$ for $0 < s < 1$, it follows that

$$\begin{aligned}(-1)^n (Mu)'(0) &= \int_0^1 (-1)^n \frac{\partial}{\partial t} G_{2n+1}(0,s)p(s)u(s)\,ds \\ &\geq \int_\alpha^\beta (-1)^n \frac{\partial}{\partial t} G_{2n+1}(0,s)p(s)u(s),\,ds \\ &> 0,\end{aligned}$$

and similarly,

$$\begin{aligned}(-1)^n (Mu)'(1) &= \int_0^1 (-1)^n \frac{\partial}{\partial t} G_{2n+1}(1,s)p(s)u(s)\,ds \\ &\leq \int_\alpha^\beta (-1)^n \frac{\partial}{\partial t} G_{2n+1}(1,s)p(s)u(s)\,ds \\ &< 0.\end{aligned}$$

So $(-1)^n Mu \in \mathcal{Q}$; that is $M : \mathcal{P} \setminus \{0\} \to \mathcal{P}^\circ$. Thus Theorem 2.7 yields the conclusion that M is u_0-positive. Similarly N is also u_0-positive. $\square$

Remark 2.21. This is a good place to point out that

$$\Lambda u(t) = Mu(t) = \int_0^1 G_{2n+1} p(s)u(s)\,ds, \quad 0 \leq t \leq 1,$$

if and only if

$$u(t) = \frac{1}{\Lambda} \int_0^1 G_{2n+1}(t,s)p(s)u(s)\,ds, \quad 0 \leq t \leq 1,$$

if and only if

$$(-1)^n u^{(2n+1)}(t) = -\frac{1}{\Lambda} p(t)u(t), \quad 0 < t < 1, \quad \text{and}$$

$$u^{(2i)}(0) = 0, \quad i = 0, ..., n-1,$$

$$u^{(2j)}(1) = 1, \quad j = 0, .., n.$$

That is, eigenvalues of (2.6), (2.8) are reciprocals of the eigenvalues of M, and conversely. In analogy, the eigenvalues of (2.7), (2.8) are reciprocals of the eigenvalues of N, and conversely.

Theorem 2.22. *The operator M (and N), has one eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue with an essentially unique eigenvector that can be chosen to belong to $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator which is u_0-positive with respect to $\mathcal{P}$, it follows from Theorem 2.8 that M has an essentially unique eigenvector, $u \in \mathcal{P}$, and associated eigenvalue, Λ, having the properties as in the statement of the theorem. Since $u \neq 0$, then $(-1)^n Mu \in \mathcal{Q} \subset \mathcal{P}^\circ$, and so $(-1)^n u = (-1)^n M\left(\frac{1}{\Lambda u}\right) \in \mathcal{P}^\circ$. □

Theorem 2.23. *Let $(-1)^n p(t) \leq (-1)^n q(t)$ on $[0,1]$. Let Λ and Σ be the eigenvalues from the previous theorem corresponding to M and N, respectively, with corresponding, essentially unique eigenvectors, u_1 and u_2 belonging to $\mathcal{P}^\circ$. Then $\Lambda \leq \Sigma$ and $\Lambda = \Sigma$ if and only if $p(t) = q(t)$ on $[0,1]$.*

Proof. With $(-1)^n p(t) \leq (-1)^n q(t)$ on [0,1], then for any $u \in \mathcal{P}$ and $t \in [0,1]$,

$$(-1)^n \left(Nu - Mu\right)(t) = \int_0^1 (-1)^n G_{2n+1}(t,s)\left(q(s) - p(s)\right) u(s)\, ds \geq 0.$$

So, $(Nu - Mu) \in \mathcal{P}$ for all $u \in \mathcal{P}$; that is $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 2.9, $\Lambda \leq \Sigma$.

Now, if $p(t) = q(t)$ on [0,1], then trivially $\Lambda = \Sigma$. On the other hand, if $p(t) \neq q(t)$, then there exists a subinterval $[c,d] \subseteq [0,1]$ such that $(-1)^n p(t) < (-1)^n q(t)$ on [c,d]. Then, by an argument similar to the one in the proof of Lemma 2.13 $(N - M)u_1 \in Q \subseteq \mathcal{P}^\circ$, and so there exists and $\varepsilon > 0$ such that $(N - M)u_1 - \varepsilon u_1 \in \mathcal{P}^\circ$. (Here u_1 is still the eigenvector associated with M.) Thus we have,

$$\Lambda u_1 + \varepsilon u_1 = Mu_1 + \varepsilon u_1 \leq Nu_1$$

or

$$(\Lambda + \varepsilon)u_1 \leq Nu_1.$$

Since $N \leq N$ and $Nu_2 = \Sigma u_2$ it follows from Theorem 2.9 that $\Lambda + \varepsilon \leq \Sigma$, or $\Lambda < \Sigma$. $\square$

Combining Remark 2.21, Theorem 2.22, and Theorem 2.23, a concluding theorem can now be stated concerning the existence and comparison of smallest positive eigenvalues for (2.5), (2.7) and (2.6), (2.7).

Theorem 2.24. *Let* $0 \leq (-1)^n p(t) \leq (-1)^n q(t)$ *on* $[0,1]$*. Then, there exist smallest positive eigenvalues,* λ *and* σ*, of* $(2.5), (2.7)$ *and* $(2.6), (2.7)$*, respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problem, and the eigenvectors corresponding to* λ *and* σ *may be chosen belonging to* $\mathcal{P}^\circ$*. Finally,* $\lambda \geq \sigma$*, and* $\lambda = \sigma$ *if and only if* $p(t) = q(t)$ *on* $[0,1]$*.*

CHAPTER THREE

Comparison Theory for a Certain Class of (3n+2)-Order Boundary Value Problems

3.1 Introduction

This chapter first establishes the existence of smallest positive eigenvalues and their comparisons for the eighth order eigenvalue problems,

$$-u^{(8)} + \lambda p(t)u = 0, \quad 0 < t < 1, \tag{3.1}$$

and

$$-u^{(8)} + \sigma q(x)u = 0, \quad 0 < t < 1, \tag{3.2}$$

with eigenvectors satisfying the boundary conditions,

$$\begin{gathered} u(0) = u''(0) = u^{(5)}(0) = 0, \\ u(1) = u''(1) = u^{(4)} = u^{(5)} = u^{(7)} = 0. \end{gathered} \tag{3.3}$$

In each case, $p(t), q(t) : [0,1] \to [0,\infty)$ are continuous and neither vanishes identically on any non-degenerate compact subinterval of [0,1]. In the following sections, the same techniques as those used in Chapter Two will be employed. Namely, the sign properties of an appropriate Green's function along with the theory of u_0-positive operators with respect to a cone in a Banach space are applied. The arguments will again make use of Theorem 2.7, Theorem 2.8, and Theorem 2.9. Then, after establishing the comparison for the eighth order case a more general result will be shown for the (3n+2)-order problem.

3.2 A Comparison Theorem for Certain Eighth Order Boundary Value Problems

In this section, the previously stated definitions and results are applied when comparing smallest eigenvalues λ and σ of (3.1), (3.3) and (3.2), (3.3), respectively. First, as mentioned in this chapter's introduction an appropriate Green's function must be constructed and its sign properties determined. In particular, the Green's function for

$$-u^{(8)} = 0 \tag{3.4}$$

satisfying the boundary conditions (3.3) is needed. Let $K(t,s)$, $H(t,s)$, and $G_5(t,s)$ be as defined in Section 2.2. Then by [7] and [8] it follows that the Green's function for (3.4), (3.3) is given by the convolution,

$$G_8(t,s) = \int_0^1 G_5(t,r)K(r,s)\,dr. \tag{3.5}$$

Properties of $G_8(t,s)$ that are useful include,

- $G_8(t,s) > 0$ on $(0,1)\times(0,1)$,
- $\frac{\partial^2}{\partial t^2}G_8(t,s) = \int_0^1 K(t,r)K(r,s)\,dr\ < 0$,
- $G_8(0,s) = G_8(1,s) = 0$,
- $\frac{\partial}{\partial t}G_8(0,s) = \int_0^1 (1-\alpha)K(\alpha,r)K(r,s)\,dr\ > 0$,
- $\frac{\partial}{\partial t}G_8(1,s) = \int_0^1 -\alpha K(\alpha,r)K(r,s)\,dr\ < 0$,
- $\frac{\partial^2}{\partial t^2}G_8(0,s) = \int_0^1 K(0,r)K(r,s)\,dr = 0, \quad 0 \le s < 1$,
- $\frac{\partial^2}{\partial t^2}G_8(1,s) = \int_0^1 K(1,r)K(r,s)\,dr = 0, \quad 0 < s \le 1$,
- $\frac{\partial^4}{\partial t^4}G_8(1,s) = \int_0^1 \frac{\partial^2}{\partial t^2}K(1,s)K(r,s)\,dr = 0, \quad 0 < s \le 1$,
- $\frac{\partial^5}{\partial t^5}G_8(0,s) = K(0,s) = 0, \quad 0 \le s < 1$,
- $\frac{\partial^5}{\partial t^5}G_8(1,s) = K(1,s) = 0, \quad 0 < s \le 1$,
- $\frac{\partial^7}{\partial t^7}G_8(1,s) = \frac{\partial^2}{\partial t^2}K(1,s) = 0.$

Now, in order to apply the positive cone theory from Section 2.1, a suitable Banach space and cone within the Banach space are introduced. Let the Banach space $\mathcal{B}$ be given by

$$\mathcal{B} := \{u \in \mathrm{C}^{(1)}[0,1] \,|\, u(0) = u(1) = 0\},$$

equipped with the norm defined by

$$\|u\| := |u'|_0, \text{where}\, |\cdot|_0 := \max_{0\le t\le 1} |\cdot|.$$

It is straightforward that, for each $u \in \mathbb{R}$,

$$|u|_0 \le |u'|_0 \le \|u\|.$$

Let the cone $\mathcal{P} \subset \mathcal{B}$ be defined as

$$\mathcal{P} := \{u \in \mathcal{B} \,|\, u(t) \ge 0,\ 0 \le t \le 1\}.$$

Lemma 3.1. *The cone $\mathcal{P}$ has nonempty interior, and $\mathcal{Q} := \{v \in \mathcal{B}|v(t) > 0,\, 0 < t < 1,\, v'(0) > 0,\, and\, v'(1) < 0\} \subset \mathcal{P}^\circ$.*

Proof. See the proof of Lemma 2.10. □

Corollary 3.2. *The cone $\mathcal{P}$ is solid and hence reproducing.*

Now consider the linear operators $M,\, N\, :\, \mathcal{B} \to \mathcal{B}$ defined by

$$Mu(t) := \int_0^1 G_8(t,s)p(s)u(s)\,ds$$

and

$$Nu(t) := \int_0^1 G_8(t,s)q(s)u(s)\,ds.$$

Remark 3.3. For a typical argument that both M and N are compact operators, see Lemma 2.12.

Lemma 3.4. *The bounded linear operators M and N are u_0-positive (with respect to $\mathcal{P}$).*

Proof. It will be shown that $M : \mathcal{P} \setminus \{0\} \to \mathcal{Q} \subset \mathcal{P}^\circ$, and then Theorem 2.7 yields the conclusion. First, recall from the proof of Lemma 2.12 that $M : \mathcal{P} \to \mathcal{P}$. Now, choose $u \in \mathcal{P} \setminus \{0\}$. Then, by the assumptions on $p(t)$, there exists a compact subinterval $[\alpha, \beta] \subseteq [0,1]$ such that $p(t)u(t) > 0$ on $[\alpha, \beta]$. From the property that $G_8(t,s) > 0$ on $(0,1) \times (0,1)$, it follows that, for $0 < t < 1$,

$$\begin{aligned} Mu(t) &= \int_0^1 G_8(t,s)p(s)u(s)\,ds \\ &\geq \int_\alpha^\beta G_8(t,s)p(s)u(s)\,ds \\ &> 0. \end{aligned}$$

Moreover, from the properties that $\frac{\partial}{\partial t}G_8(0,s) > 0$, for $0 < s < 1$ and $\frac{\partial}{\partial t}G_8(1,s) < 0$, for $0 < s < 1$, it follows that

$$\begin{aligned} (Mu)'(0) &= \int_0^1 \frac{\partial}{\partial t}G_8(0,s)p(s)u(s)\,ds \\ &\geq \int_\alpha^\beta \frac{\partial}{\partial t}G_8(0,s)p(s)u(s)\,ds \\ &> 0, \end{aligned}$$

and similarly,

$$(Mu)'(1) = \int_0^1 \frac{\partial}{\partial t}G_8(1,s)p(s)u(s)\,ds < 0.$$

So, $Mu \in \mathcal{Q}$; that is $M : \mathcal{P} \setminus \{0\} \to \mathcal{P}^\circ$. Thus M is u_0-positive, and similarly N is also u_0-positive. □

Remark 3.5. At this point it is important to note that

$$\Lambda u(t) = Mu(t) = \int_0^1 G_8(t,s)p(s)u(s)\,ds, \quad 0 \leq t \leq 1,$$

if and only if

$$u(t) = \frac{1}{\Lambda}\int_0^1 G_8(t,s)p(s)u(s)\,ds, \quad 0 \leq t \leq 1,$$

if and only if

$$u^{(8)}(t) = -\frac{1}{\Lambda}p(t)u(t),\ 0 < t < 1, \text{ and}$$

$$u(0) = u''(0) = u^{(5)}(0) = 0,$$
$$u(1) = u''(1) = u^{(4)} = u^{(5)} = u^{(7)} = 0.$$

That is, the eigenvalues of (3.1), (3,3) are reciprocals of the eigenvalues of M, and conversely. In analogy, the eigenvalues of (3.2), (3.3) are reciprocals of the eigenvalues of N, and conversely.

Now, in light of the previous lemmas, Theorems 2.8 and 2.9 can now be applied.

Theorem 3.6. *The operator M (and N), has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to belong to $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator which is u_0-positive with respect to $\mathcal{P}$, it follows from Theorem 2.8 that M has an essentially unique eigenvector, $u \in \mathcal{P}$, and associated eigenvalue, Λ, having the properties as in the statement of the theorem. Since $u \neq 0$, then $Mu \in \mathcal{Q} \subset \mathcal{P}^\circ$, and so $u = M\left(\frac{1}{\Lambda u}\right) \in \mathcal{P}^\circ$. □

Theorem 3.7. *Let $p(t) \leq q(t)$ on $[0,1]$. Let Λ and Σ be the eigenvalues of Theorem 3.6 corresponding to M and N, respectively, with corresponding essentially unique eigenvectors, u_1 and u_2, belonging to $\mathcal{P}^\circ$. Then $\Lambda \leq \Sigma$, and $\Lambda = \Sigma$ if, and only if, $p(t) = q(t)$ on $[0,1]$.*

Proof. With $p(t) \leq q(t)$ on $[0,1]$, then for any $u \in \mathcal{P}$ and $t \in [0,1]$,

$$(Nu - Mu)(t) = \int_0^1 G_5(t,s)\,(q(s) - p(s))\,u(s)\,ds \geq 0.$$

So, $Nu - Mu \in \mathcal{P}$, for all $u \in \mathcal{P}$; that is, $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 2.9, $\Lambda \leq \Sigma$.

Turning to the other part, if $p(t) = q(t)$ on $[0, 1]$, then trivially $\Lambda = \Sigma$. On the other hand, if $p(t) \neq q(t)$, then there exists a subinterval $[c, d] \subseteq [0, 1]$ such that $p(t) < q(t)$ on $[c, d]$. Then, by arguments as in the proof of Lemma 2.13, $(N - M)u_1 \in \mathcal{Q} \subseteq \mathcal{P}$, and so there exists an $\varepsilon > 0$ such that $(N - M)u_1 - \varepsilon u_1 \in \mathcal{P}^\circ$. We have

$$\Lambda u_1 + \varepsilon u_1 = Mu_1 + \varepsilon u_1 \leq Nu_1,$$

or $(\Lambda + \varepsilon)\, u_1 \leq Nu_1$. Since $N \leq N$ and $Nu_2 = \Sigma u_2$, we have from Theorem 2.9 that $\Lambda + \varepsilon \leq \Sigma$, or $\Lambda < \Sigma$. $\square$

In light of Remark 3.5 and Theorems 3.6 and 3.7, the main result concerning the existence and comparison of smallest positive eigenvalues for (3.1), (3.3) and (3.2), (3.3) can be stated.

Theorem 3.8. *Let* $p(t) \leq q(t)$ *on* $[0, 1]$. *Then, there exist smallest positive eigenvalues,* λ *and* σ, *of* (3.1), (3.3) *and* (3.2), (3.3), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problem, and the eigenvectors correpsonding to* λ *and* σ *may be chosen belonging to* $\mathcal{P}^\circ$. *Finally,* $\lambda \geq \sigma$, *and* $\lambda = \sigma$ *if, and only if,* $p(t) = q(t)$ *on* $[0, 1]$.

3.3 A Comparison Theorem for Certain $(3n + 2)$*-Order Boundary Value Problems* $(n \geq 2)$

In this section, for $n \geq 2$, the comparisons of smallest eigenvalues are examined for the (3n+2)-order eigenvalue problems,

$$u^{(3n+2)} + \lambda p(t)u = 0, \tag{3.6}$$

and

$$u^{(3n+2)} + \sigma q(t)u = 0, \tag{3.7}$$

with eigenvectors for both satisfying the boundary conditions,

$$\begin{gathered} u(0) = u(1) = 0, \\ u^{(3j+2)}(0) = u^{(3j+2)}(1) = 0 \quad \text{for} \quad j = 0, 1, \ldots, n-1, \\ u^{(3k+1)}(1) = 0 \quad \text{for} \quad k = 1, 2, \ldots, n. \end{gathered} \tag{3.8}$$

In each case, $p(t)$, $q(t) : [0,1] \to [0,\infty)$ are continuous and neither vanishes identically on any compact subinterval of [0,1]. Following the same pattern used in Section 3.2, an appropriate Green's function and its sign properties need to be determined. In particular the Green's function for

$$u^{(3n+2)} = 0 \tag{3.9}$$

satisfying (3.8) needs to be constructed. Let $H(t,s)$, $K(t,s)$ be the Green's functions as defined in Section 2.2. Then let

$$G_{(3n+2)}(t,s) = \int_0^1 G_{(3n-1)}(t,r)K(r,s)\,dr, \quad n \geq 3, \tag{3.10}$$

where

$$G_8(t,s) = \int_0^1 H(t,r)K(r,s)\,dr,$$

which is the Green's function constructed in Section 3.2. Then note that $G_{(3n+2)}$ has the following properties:

- $G_{(3n+2)}(t,s) > 0$ on $(0,1) \times (0,1)$,
- $\frac{\partial^2}{\partial t^2} G_{(3n+2)}(t,s) > 0$ on $(0,1) \times (0,1)$,
- $G_{(3n+2)}(0,s) = (-1)^n G_{(3n+2)}(1,s) = 0$,
- $\frac{\partial}{\partial t} G_{(3n+2)}(0,s) > 0, \quad 0 \leq s < 1$,
- $\frac{\partial}{\partial t} G_{(3n+2)}(1,s) < 0, \quad 0 < s \leq 1$,
- $\frac{\partial^{3j}}{\partial t^{3j}} G_{(3n+2)}(0,s) > 0$ for $j = 1, \ldots, n, \quad 0 \leq s < 1$,

- $\frac{\partial^{3j}}{\partial t^{3j}} G_{(3n+2)}(1, s) < 0$ for $j = 1, \ldots, n, \quad 0 < s \leq 1,$

- $G_{(3n+2)}(t, s)$ satisfies the boundary conditions (3.8).

Now, following the pattern established in Section 3.2 it is necessary to define a suitable Banach space and a cone within that Banach space. These are defined as follows,

$$\mathcal{B} := \{u \in \mathrm{C}^{(1)}[0, 1] \,|\, u(0) = u(1) = 0\},$$

equipped with the norm defined by

$$\|u\| := |u'|_0, \text{ where } |\cdot|_0 := \max_{0 \leq t \leq 1} |\cdot|.$$

As in the previous sections, it is straightforward to show that, for each $u \in \mathcal{B}$,

$$|u|_0 \leq |u'|_0 = \|u\|.$$

Then, define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} := \{u \in \mathcal{B} \,|\, u(t) \geq 0,\, 0 \leq t \leq 1\}.$$

Lemma 3.9. *The cone $\mathcal{P}$ has nonempty interior, and $\mathcal{Q} := \{v \in \mathcal{B} \,|\, v(t) > 0,\, 0 < t < 1,\, v'(0) > 0, \textit{ and } v'(1) < 0\} \subset \mathcal{P}^{\circ}$.*

The proof for Lemma 3.9 is nearly identical to that for Lemma 2.10.

Corollary 3.10. *The cone $\mathcal{P}$ is solid and hence reproducing.*

In the same manner as the eighth order case, it suffices to seek eigenvalues of the linear operators $M, N : \mathcal{B} \to \mathcal{B}$ defined by

$$Mu(t) = \int_0^1 G_{(3n+2)}(t, s) p(s) u(s)\, ds$$

and

$$Nu(t) = \int_0^1 G_{(3n+2)}(t, s) q(s) u(s)\, ds.$$

Also, by a similar argument to the one used in Section 2.2, M and N are compact operators.

Lemma 3.11. *The bounded linear operators M and N are u_0-positive (w.r.t $\mathcal{P}$).*

Proof. It suffices to show that $M : \mathcal{P} \setminus \{0\} \to \mathcal{Q} \subset \mathcal{P}^{\circ}$. Then Theorem 2.7 yields the conclusion.

First, choose $u \in \mathcal{P}$. Because $u(t)$, $G_{3n+2}(t,s)$ and $p(t)$ are nonnegative functions, for $0 \leq t \leq 1$,

$$Mu(t) = \int_0^1 G_{(3n+2)}(t,s)p(s)u(s)\,ds \geq 0.$$

Moreover, $Mu(0) = Mu(1) = 0$, thus $M : \mathcal{P} \to \mathcal{P}$. Now, choose $u \in \mathcal{P} \setminus \{0\}$. Then, by the assumptions on $p(t)$, there exists a compact subinterval $[\alpha, \beta] \subseteq [0,1]$ such that $p(t)u(t) > 0$ on $[\alpha, \beta]$. From the property that $G_{(3n+2)}(t,s) > 0$ on $(0,1) \times (0,1)$, it follows that, for $0 < t < 1$,

$$\begin{aligned} Mu(t) &= \int_0^1 G_{(3n+2)}(t,s)p(s)u(s)\,ds \\ &\geq \int_\alpha^\beta G_{(3n+2)}(t,s)p(s)u(s)\,ds \\ &> 0. \end{aligned}$$

Moreover, recalling that $\frac{\partial}{\partial t}G_{(3n+2)}(0,s) > 0$ and $\frac{\partial}{\partial t}G_{(3n+2)}(1,s) < 0$ for $0 < s < 1$, it follows that

$$\begin{aligned} (Mu)'(0) &= \int_0^1 \frac{\partial}{\partial t}G_{(3n+2)}(0,s)p(s)u(s)\,ds \\ &\geq \int_\alpha^\beta \frac{\partial}{\partial t}G_{(3n+2)}(0,s)p(s)u(s),\,ds \\ &> 0, \end{aligned}$$

and similarly,

$$\begin{aligned} (Mu)'(1) &= \int_0^1 \frac{\partial}{\partial t}G_{(3n+2)}(1,s)p(s)u(s)\,ds \\ &\leq \int_\alpha^\beta \frac{\partial}{\partial t}G_{(3n+2)}(1,s)p(s)u(s)\,ds \\ &< 0. \end{aligned}$$

So $Mu \in \mathcal{Q}$; that is $M : \mathcal{P} \setminus \{0\} \to \mathcal{P}^\circ$. Thus Theorem 2.7 yields the conclusion that M is u_0-positive. Similarly N is also u_0-positive. $\square$

Remark 3.12. This is a good place to point out that

$$\Lambda u(t) = Mu(t) = \int_0^1 G_{(3n+2)} p(s)u(s)\,ds, \quad 0 \le t \le 1,$$

if and only if

$$u(t) = \frac{1}{\Lambda}\int_0^1 G_{(3n+2)}(t,s)p(s)u(s)\,ds, \quad 0 \le t \le 1,$$

if and only if

$$u^{(3n+2)}(t) = \frac{1}{\Lambda}p(t)u(t), \quad 0 < t < 1, \quad \text{and}$$

$$u(0) = u(1) = 0,$$

$$u^{(3j+2)}(0) = u^{(3j+2)}(1) = 0, \quad j = 0, 1, ..., n-1,$$

$$u^{(3k+1)}(1) = 0, \quad k = 1, 2, .., n.$$

That is, eigenvalues of (3.6), (3.8) are reciprocals of the eigenvalues of M, and conversely. In analogy, the eigenvalues of (3.7), (3.8) are reciprocals of the eigenvalues of N, and conversely.

Theorem 3.13. *The operator M (and N), has one eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue with an essentially unique eigenvector that can be chosen to belong to $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator which is u_0-positive with respect to $\mathcal{P}$, it follows from Theorem 2.8 that M has an essentially unique eigenvector, $u \in \mathcal{P}$, and associated eigenvalue, Λ, having the properties as in the statement of the theorem. Since $u \neq 0$, then $Mu \in \mathcal{Q} \subset \mathcal{P}^\circ$, and so $u = M\left(\frac{1}{\Lambda u}\right) \in \mathcal{P}^\circ$. $\square$

Theorem 3.14. *Let $p(t) \le q(t)$ on $[0,1]$. Let Λ and Σ be the eigenvalues from the previous theorem corresponding to M and N, respectively, with corresponding, essentially unique eigenvectors, u_1 and u_2 belonging to $\mathcal{P}^\circ$. Then $\Lambda \le \Sigma$ and $\Lambda = \Sigma$ if and only if $p(t) = q(t)$ on $[0,1]$.*

Proof. With $p(t) \leq q(t)$ on [0,1], then for any $u \in \mathcal{P}$ and $t \in [0,1]$,

$$(Nu - Mu)(t) = \int_0^1 G_{(3n+2)}(t,s)\,(q(s) - p(s))\,u(s)\,ds \geq 0$$

So, $(Nu - Mu) \in \mathcal{P}$ for all $u \in \mathcal{P}$; that is $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 2.9, $\Lambda \leq \Sigma$.

Now, if $p(t) = q(t)$ on [0,1], then trivially $\Lambda = \Sigma$. On the other hand, if $p(t) \neq q(t)$, then there exists a subinterval $[c,d] \subseteq [0,1]$ such that $p(t) < q(t)$ on [c,d]. Then, by an argument similar to the one in the proof of Lemma 2.13 $(N-M)u_1 \in Q \subseteq \mathcal{P}^\circ$, and so there exists an $\varepsilon > 0$ such that $(N-M)u_1 - \varepsilon u_1 \in \mathcal{P}^\circ$. (Here u_1 is still the eigenvector associated with M.) Thus we have,

$$\Lambda u_1 + \varepsilon u_1 = Mu_1 + \varepsilon u_1 \leq Nu_1$$

or

$$(\Lambda + \varepsilon)u_1 \leq Nu_1.$$

Since $N \leq N$ and $Nu_2 = \Sigma u_2$ it follows from Theorem 2.9 that $\Lambda + \varepsilon \leq \Sigma$, or $\Lambda < \Sigma$. $\square$

Combining Remark 3.12, Theorem 3.13, and Theorem 3.14, a concluding theorem can now be stated concerning the existence and comparison of smallest positive eigenvalues for (3.6), (3.8) and (3.7), (3.8).

Theorem 3.15. *Let $0 \leq p(t) \leq q(t)$ on $[0,1]$. Then, there exist smallest positive eigenvalues, λ and σ, of (3.6), (3.8) and (3.7), (3.8), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problem, and the eigenvectors corresponding to λ and σ may be chosen belonging to $\mathcal{P}^\circ$. Finally, $\lambda \geq \sigma$, and $\lambda = \sigma$ if and only if $p(t) = q(t)$ on $[0,1]$.*

CHAPTER FOUR

Positive Solutions for a Singular Fifth Order Boundary Value Problem

4.1 Introduction

Much theoretical interest has been given to singular boundary value problems for ordinary differential equations. For several of these studies, see [3, 32–35, 39, 40]. In this chapter, the methods used involve the application of a fixed point theorem by Gatica, Oliker, and Waltman [14] for operators that are decreasing with respect to a cone. This method has been used to obtain positive solutions for other singular boundary value problems by Eloe and Henderson [10], Henderson and Yin [22], Maroun [29, 30], and Singh [36]. Fundamental to obtaining positive solutions of the singular problem of this chapter is a positivity result by Graef and Yang [16, 17].

This chapter will establish the existence of positive solutions for the singular fifth order boundary value problem.

$$u^{(5)} = f(x, u), \quad 0 < x < 1, \tag{4.1}$$

$$u(0) = u'''(0) = u(1) = u''(1) = u'''(1) = 0, \tag{4.2}$$

where $f(x, y)$ is singular at $x = 0, 1$, $y = 0$, and may be singular at $y = \infty$.

The following conditions are assumed to hold on f:

(H1) $f(x, y) : (0, 1) \times (0, \infty) \to (0, \infty)$ is continuous, and $f(x, y)$ is decreasing in y, for every x.

(H2) $\lim\limits_{y \to 0^+} f(x, y) = +\infty$ and $\lim\limits_{y \to +\infty} f(x, y) = 0$ uniformly on compact subsets of $(0, 1)$.

The strategy used in this chapter will be to convert the problem (4.1), (4.2) into an integral equation problem, then define a sequence of decreasing integral operators associated with a sequence of perturbed integral equations. Applications of a Gatica, Oliker and Waltman fixed point theorem [14] yield a sequence of fixed points of the integral operators. A solution of (4.1), (4.2) is then obtained from a subsequence of the fixed points.

4.2 Definitions, Cone Properties and the Gatica, Oliker and Waltman Fixed Point Theorem

In this section, the definitions and properties of Banach space cones are provided, and the statement of the fixed point theorem on which this chapter's main result depends.

Definition 4.1. Let $(\mathcal{B}, \|\cdot\|)$ be a real Banach space. A nonempty closed $\mathcal{K} \subset \mathcal{B}$ is called a cone if the following hold:

(i) $\alpha u + \beta v \in \mathcal{K}$, for all $u, v \in \mathcal{K}$, and for all $\alpha, \beta \in [0, \infty)$.

(ii) $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.

Definition 4.2. Given a cone $\mathcal{K}$, a partial order, $\leq$, is induced on $\mathcal{B}$ by $x \leq y$, for $x, y \in \mathcal{B}$ if, and only if, $y - x \in \mathcal{K}$.

Definition 4.3. If $x, y \in \mathcal{B}$ with $x \leq y$, let $\langle x, y\rangle$ denote the closed order interval between x and y and be defined by, $\langle x, y\rangle := \{z \in \mathcal{B} \,|\, x \leq z \leq y\}$.

Definition 4.4. A cone $\mathcal{K}$ is normal in $\mathcal{B}$ provided there exists $\delta > 0$ such that $\|e_1 + e_2\| \geq \delta$, for all $e_1, e_2 \in \mathcal{K}$ with $\|e_1\| = \|e_2\| = 1$.

Remark 4.5. If $\mathcal{K}$ is a normal cone in $\mathcal{B}$, then closed order intervals are norm bounded.

Now the statement is given of the Gatica, Oliker and Waltman [14] fixed point theorem on which the main result of this chapter depends.

Theorem 4.6 (Gatica, Oliker, and Waltman). *Let $\mathcal{B}$ be a Banach space, $\mathcal{K}$ a normal cone, $\mathcal{J}$ a subset of $\mathcal{K}$ such that, if $x, y \in \mathcal{J}$, $x \leq y$, then $\langle x, y\rangle \subseteq \mathcal{J}$, and let*

$T : \mathcal{J} \to \mathcal{K}$ be a continuous decreasing mapping which is compact on any closed order interval contained in $\mathcal{J}$. Suppose there exists $x_0 \in \mathcal{J}$ such that T^2x_0 is defined, and furthermore, Tx_0 and T^2x_0 are order comparable to x_0.

Then T has a fixed point in $\mathcal{J}$ provided that, either

(I) $Tx_0 \le x_0$ and $T^2x_0 \le x_0$, or $x_0 \le Tx_0$ and $x_0 \le T^2x_0$, or

(II) The complete sequence of iterates $\{T^n x_0\}_{n=0}^{\infty}$ is defined, and there exists $y_0 \in \mathcal{J}$ such that $y_0 \le T^n x_0$, for every n.

4.3 Properties of Positive Solutions

Now in order to use Theorem 4.6 some preliminaries must be established. Consider the Banach space $(\mathcal{B}, \|\cdot\|)$ defined by

$$\mathcal{B} := \{u : [0,1] \to \mathbb{R} \,|\, u \, \text{is continuous}\}, \quad \|u\| := \sup_{0\le x\le 1} |u(x)|.$$

Also, define a cone $\mathcal{K} \subset \mathcal{B}$ by

$$\mathcal{K} := \{u \in \mathcal{B} \,|\, u(x) \ge 0 \text{ on } [0,1]\}.$$

Then, observe that if $y(x)$ is a solution of (4.1)-(4.2), then

$$y^{(5)}(x) \le 0,\ y'''(x) \ge 0,\ y''(x) \le 0, y(x) \ge 0, \text{ and } y(x) \text{ is concave.}$$

Next, define $g(x) : [0,1] \to [0, \frac{3}{4}]$ by

$$g(x) := \min\{1-x,\, 3x\},$$

and for $\theta > 0$, define

$$g_\theta(x) := \theta g(x).$$

Then notice that,

$$\max_{0\le x\le 1} g(x) = \frac{3}{4} \text{ and } \max_{0\le x\le 1} g_\theta(x) = \frac{3\theta}{4}.$$

In addition to the assumptions (H1) and (H2), the following will be an additional assumption needed hereafter:

(H3) $\int_0^1 f(x, g_\theta(x))\, dx \; < \infty$, for all $\theta > 0$.

The following theorem due to Graef and Yang [16, 17] will be used extensively during the process of achieving the main result.

Theorem 4.7 (Graef and Yang). *Let $u(x) \in C^{(3)}[0,1]$. If $u(x)$ satisfies the boundary conditions* (4.2) *is such that $u''' \geq 0$ on $[0,1]$, then*

$$u(x) \geq \min\{1-x,\, 3x\} \sup_{0 \leq x \leq 1} |u(x)|. \tag{4.3}$$

Due to this theorem then, for each positive solution $u(x)$ of (4.1), (4.2), there exists a $\theta > 0$ such that

$$g_\theta(x) \leq u(x), \quad 0 \leq x \leq 1.$$

In particular, with $\theta = \sup\limits_{0 \leq x \leq 1} |u(x)|$, then

$$u(x) \geq \min\{1-x,\, 3x\}\theta = g_\theta(x), \quad 0 \leq x \leq 1.$$

Now let $D \subset \mathcal{K}$ be defined by $D := \{v \in \mathcal{K} \,|\, \text{there exists } \theta(v) > 0 \text{ such that } g_\theta(x) \leq v(x),\ 0 \leq x \leq 1\}$.

Observe that, for each $v \in D$ and $\frac{1}{8} \leq x \leq \frac{5}{8}$,

$$v(x) \geq g_\theta(x) = \min\{1-x,\, 3x\}\theta \geq \frac{3}{8}\theta, \tag{4.4}$$

and for each positive solution $u(x)$ of (4.1)-(4.2),

$$u(x) \geq g(x) \sup_{0 \leq x \leq 1} |u(x)| \geq \frac{3}{8} \sup_{0 \leq x \leq 1} |u(x)|, \; \frac{1}{8} \leq x \leq \frac{5}{8}. \tag{4.5}$$

Next, a Green's function for $-u^{(5)} = 0$ satisfying (4.2) is needed to play the role of a kernel for certain compact operators meeting the requirements of Theorem (4.6). It will be constructed in the following way.

Let $K(x,s)$ be the Green's function for $u''' = 0$, with boundary conditions $u(0) = u(1) = u''(1) = 0$. Graef and Yang [18] have obtained

$$K(x,s) = \frac{1}{2} \begin{cases} x(1-x) - x(1-s)^2, & \text{if } 0 \leq x \leq s \leq 1, \\ x(1-x) - x(1-s)^2 + (x-s)^2, & \text{if } 0 \leq s \leq x \leq 1. \end{cases}$$

Some properties of $K(x,s)$ that will prove useful include

- $K(x,s) > 0$ on $(0,1)\times(0,1)$,
- $\dfrac{\partial}{\partial x}K(0,s) = \frac{1}{2}s(2-s) > 0, \quad 0 < s \le 1,$
- $\dfrac{\partial}{\partial x}K(1,s) = -\frac{1}{2}s^2 < 0, \quad 0 < s \le 1.$

Next, let $H(x,s)$ be the Green's function for $-u'' = 0$, with boundary conditions $u(0) = u(1) = 0$. It is well known that

$$H(x,s) = \begin{cases} x(1-s), & \text{if } 0 \le x \le s \le 1,\\[2ex] s(1-x), & \text{if } 0 \le s \le x \le 1. \end{cases}$$

Some properties of $H(x,s)$ that will prove useful include

- $H(x,s) > 0$ on $(0,1)\times(0,1)$,
- $\dfrac{\partial}{\partial x}H(0,s) = 1-s > 0, \quad 0 \le s \le 1,$
- $\dfrac{\partial}{\partial x}H(1,s) = -s < 0, \quad 0 \le s \le 1.$

It follows from [7] and [8] that the Green's function $G(x,s)$ for $-u^{(5)} = 0$, satisfying the boundary conditions (4.2) is formed by the convolution

$$G(x,s) = \int_0^1 K(x,r)H(r,s)\,dr,$$

and properties which prove useful include

- $G(x,s) > 0$ on $(0,1)\times(0,1)$ and continuous on $[0,1]\times[0,1]$,
- $G(0,s) = 0$, $0 < s \le 1$, and $G(1,s) = \frac{\partial^2}{\partial t^2}G(1,s) = 0$ for $0 \le s < 1$,
- $\frac{\partial^2}{\partial x^2}G(x,s)$ is continuous as a function of t on $[0,s]$ and $[s,1]$,
- $\frac{\partial}{\partial x}G(0,s) = \int_0^1 \frac{r(2-r)}{2}H(r,s)\,dr\ > 0$ for $0 < s < 1$,

- $\frac{\partial}{\partial x}G(1,s) = -\frac{1}{2}\int_0^1 r^2 H(r,s)\,dr\ < 0$ for $0 < s < 1$.

Now we define an integral operator $T : D \to \mathcal{K}$ by

$$(Tu)(x) := \int_0^1 G(x,s)f(s,u(s))\,ds, \quad u \in D$$

Next it needs to be shown that T is well-defined on D and decreasing and that $T : D \to D$.

First, let $v, u \in D$ be given, with $v(x) \le u(x)$. Then, there exists $\theta > 0$ such that $g_\theta(x) \le v(x)$. By assumptions $(H1)$ and $(H3)$, and the first property of the Green's function listed above,

$$\begin{aligned} 0 &\le \int_0^1 G(x,s)f(x,u(s))\,ds \\ &\le \int_0^1 G(x,s)f(x,v(s))\,ds \\ &\le \int_0^1 G(x,s)f(x,g_\theta(s))\,ds \\ &< \infty. \end{aligned}$$

Therefore, T is well-defined on D and T is a decreasing operator.

Next, for $v \in D$, let $w(x) := (Tv)(x) = \int_0^1 G(x,s)f(s,v(s))\,ds \ge 0$, $0 \le x \le 1$. Then by another property of the above Green's function, $w'''(x) = H(x,s)f(s,v(s)) > 0$, $0 < x < 1$, and $w(0) = w(1) = w''(1) = 0$, which imply $w''(x) \le 0$, or that $w(x)$ is concave. Moreover, by (4.7), $w = Tv \in D$. So, $T : D \to D$.

Remark 4.8. It is well-known that $Tu = u$ if, and only if, u is a solution of (4.1), (4.2). Therefore, a solution to (4.1), (4.2) belonging to D needs to be found. It follows from (4.4), (4.5), in the context of the Banach space $\mathcal{B}$, that for each positive solution $u(x)$ of (4.1), (4.2),

$$u(x) \ge g(x)\|u\| \ge \frac{3}{8}\|u\|, \quad \frac{1}{8} \le x \le \frac{5}{8}. \tag{4.6}$$

4.4 Bounds on Norms of Solutions

In this section, it will be shown that solutions of (4.1), (4.2) have *a priori* upper and lower bounds on their norms.

Lemma 4.9. *If f satisfies $(H1)-(H3)$, then there exists an $S > 0$ such that $\|u\| \leq S$, for any solution u of* (4.1), (4.2) *in D.*

Proof. Assume the conclusion is false. Then there exists a sequence $\{u_m\}_{m=1}^{\infty}$ of solutions of (4.1), (4.2) in D such that $u_m(x) > 0$, for all $0 < x < 1$, and

$$\|u_m\| \leq \|u_{m+1}\| \text{ and } \lim_{m\to\infty} \|u_m\| = \infty.$$

From (4.5) or (4.6),

$$u_m(x) \geq \frac{3}{8}\|u_m\|, \quad \frac{1}{8} \leq x \leq \frac{5}{8}.$$

So,

$$\lim_{m\to\infty} u_m = \infty \text{ uniformly on } \left[\frac{1}{8}, \frac{5}{8}\right].$$

Next, let

$$M := \max\{G(x,s) \mid (x,s) \in [0,1]\times[0,1]\}.$$

From (H2), there exists $m_0 \in \mathbb{N}$ such that, for each $m \geq m_0$ and $\frac{1}{8} \leq x \leq \frac{5}{8}$,

$$f(x, u_m(x)) \leq \frac{2}{M}.$$

Let $\theta := \|u_{m_0}\|$. Then, for $m \geq m_0$,

$$u_m(x) \geq g_{\|u_m\|}(x) \geq g_{\|u_{m_0}\|}(x), \quad 0 \leq x \leq 1.$$

So, for $m \geq m_0$ and $0 \leq x \leq 1$,

$$\begin{aligned}
u_m(x) &= Tu_m(x) \\
&= \int_0^1 G(x,s) f(s, u_m(s))\, ds \\
&= \int_0^{\frac{1}{8}} G(x,s) f(s, u_m(s))\, ds + \int_{\frac{1}{8}}^{\frac{5}{8}} G(x,s) f(s, u_m(s))\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{5}{8}}^{1} G(x,s) f(s,u_m(s))\,ds \\
&\le \int_0^{\frac{1}{8}} G(x,s) f(s,u_m(s))\,ds + \int_{\frac{5}{8}}^{1} G(x,s) f(s,u_m(s))\,ds + \int_{\frac{1}{8}}^{\frac{5}{8}} M\cdot \frac{2}{M}\,ds \\
&\le \int_0^{\frac{1}{8}} G(x,s) f(s,g_\theta(s))\,ds + \int_{\frac{5}{8}}^{1} g(x,s) f(s,g_\theta(s))\,ds + 1 \\
&\le M\int_0^1 f(s,g_\theta(s))\,ds + 1,
\end{aligned}
$$

which contradicts $\lim_{m\to\infty}\|u_m\| = \infty$. Therefore, there exists an $S > 0$ such that $\|u\| \le S$, for any solution $u \in D$ of (4.1), (4.2). $\square$

Now it will be shown that there are positive *a priori* lower bounds on the solution norms.

Lemma 4.10. *If f satisfies $(H1)-(H3)$, then there exists an $R > 0$ such that $\|u\| \ge R$, for any solution u of* (4.1), (4.2) *in D.*

Proof. Again, assume the conclusion to the lemma is false. Then there exists a sequence $\{u_m\}_{m=1}^{\infty}$ of solutions of (4.1), (4.2) in D such that $u_m(x) > 0$, for $0 < x < 1$, and

$$\|u_m\| \ge \|u_{m+1}\| \text{ and } \lim_{m\to\infty} u_m = 0.$$

In particular,

$$\lim_{m\to\infty} u_m(x) = 0 \text{ uniformly on } [0,1].$$

Next, define

$$\overline{m} := \min\left\{ G(x,s) \,|\, (x,s) \in \left[\frac{1}{8},\frac{5}{8}\right] \times \left[\frac{1}{8},\frac{5}{8}\right] \right\} > 0.$$

From $(H2)$, $\lim_{y\to 0^+} f(x,y) = \infty$ uniformly on compact subsets of $(0,1)$, and so, there exists a $\delta > 0$ such that, for $\frac{1}{8} \le x \le \frac{5}{8}$ and $0 < y < \delta$,

$$f(x,y) > \frac{2}{\overline{m}}.$$

Also, there exists $m_0 \in \mathbb{N}$ such that, for $m \geq m_0$ and $0 < x < 1$,

$$0 < u_m(x) < \frac{\delta}{2}.$$

So, for $m \geq m_0$, and $\frac{1}{8} \leq x \leq \frac{5}{8}$,

$$\begin{aligned}
u_m(x) &= Tu_m(x) \\
&= \int_0^1 G(x,s) f(s, u_m(s))\, ds \\
&\geq \int_{\frac{1}{8}}^{\frac{5}{8}} G(x,s) f(s, u_m(s))\, ds \\
&\geq \overline{m} \int_{\frac{1}{8}}^{\frac{5}{8}} f(s, u_m(s))\, ds \\
&\geq \overline{m} \int_{\frac{1}{8}}^{\frac{5}{8}} f(s, \frac{\delta}{2})\, ds \\
&\geq \overline{m} \int_{\frac{1}{8}}^{\frac{5}{8}} \frac{2}{\overline{m}}\, ds \\
&= 1.
\end{aligned}$$

This contradicts $\lim_{m\to\infty} u_m(x) = 0$ uniformly on $[0,1]$. Therefore, there exists an $R > 0$ such that $R \leq \|u\|$ for any solution $u \in D$ of (4.1), (4.2). □

So, as a result of the previous lemmas, there exist $0 < R < S$ such that, for each solution $u \in D$ of (4.1), (4.2),

$$R \leq \|u\| \leq S.$$

4.5 Existence of Positive Solutions

A sequence of operators, $\{T_m\}_{m=1}^{\infty}$, each of which is defined on all of $\mathcal{K}$ will be constructed. Then it will be shown, via applications of Theorem 4.6, that each T_m has a fixed point $\phi_m \in \mathcal{K}$, for every m. Finally, a subsequence of $\{\phi_m\}_{m=1}^{\infty}$ will be extracted that converges to a fixed point of T.

Theorem 4.11. *If f satisfies $(H1)-(H3)$, then (4.1), (4.2) has at least one positive solution $u \in D$.*

Proof. For each $m \in \mathbb{N}$, let

$$u_m(x) := T(m) = \int_0^1 G(x,s)f(s,m)\,ds, \quad 0 \le x \le 1.$$

Since f is decreasing with respect to its second component, we have

$$0 < u_{m+1}(x) < u_m(x), \quad \text{for } 0 < x < 1,$$

and by $(H2)$, $\lim_{m\to\infty} u_m(x) = 0$ uniformly on $[0,1]$.

Next, define $f_m(x,y) : (0,1) \times [0,\infty) \to (0,\infty)$ by

$$f_m(x,y) := f(x, \max\{y, u_m(x)\}).$$

Then, f_m is continuous and f_m does not have the singularity at $y = 0$ possessed by f. In addition, for $(x,y) \in (0,1) \times (0,\infty)$,

$$f_m(x,y) \le f(x,y) \text{ and } f_m(x,y) \le f(x, u_m(x)).$$

Now, define a sequence of operators, $T_m : \mathcal{K} \to \mathcal{K}$, for $\phi \in \mathcal{K}$ and $0 \le x \le 1$, by

$$T_m\phi(x) := \int_0^1 G(x,s)f_m(s,\phi(s))\,ds.$$

Arguments similar to those used in a previous chapter show that T_m is a compact operator on $\mathcal{K}$. Furthermore,

$$\begin{aligned} T_m(0) &= \int_0^1 G(x,s)f_m(s,0)\,ds \\ &= \int_0^1 G(x,s)f(s,\max\{0,u_m(s)\})\,ds \\ &= \int_0^1 G(x,s)f(s,u_m(s))\,ds \\ &> 0. \end{aligned}$$

and

$$T_m^2(0) = T_m\Big(\int_0^1 G(x,s)f_m(s,0)\,ds\Big) \ge 0.$$

By Theorem 4.6, with $\mathcal{J} = \mathcal{K}$ and $x_0 = 0$, T_m has a fixed point in $\mathcal{K}$, for each m. That is, for each m, there exists $\phi_m \in \mathcal{K}$ such that

$$T_m\phi_m(x) = \phi_m(x), \quad 0 \le x \le 1.$$

So, for each $m \ge 1$, ϕ_m satisfies the boundary conditions (4.2), and also,

$$\begin{aligned} T_m\phi_m(x) &= \int_0^1 G(x,s) f_m(s, \phi_m(s))\, ds \\ &\le \int_0^1 G(x,s) f(s, u_m(s))\, ds \\ &= T u_m(x). \end{aligned}$$

That is, for each $0 \le x \le 1$ and for each m, $\phi_m(x) = T_m\phi_m(x) \le T u_m(x)$.

Using similar arguments as those used in the proofs of Lemma 4.9 and Lemma 4.10, there exist $R > 0$ and $S > 0$ such that

$$R \le \|\phi_m\| \le S, \quad \text{for every}\, m.$$

Next, let $\theta := R$. Since ϕ_m belongs to $\mathcal{K}$ and is a fixed point of T_m, the conditions on Theorem 4.7 hold. So, for every m and $0 \le x \le 1$,

$$\phi_m(x) \ge g(x)\|\phi_m\| \ge g(x) \cdot R = g_\theta(x).$$

So, the sequence $\{\phi_m\}_{m=1}^{\infty}$ is contained in the closed order interval $\langle g_\theta, S\rangle$, and therefore, the sequence is contained in D. Since T is a compact mapping, it can be assumed that $\lim\limits_{m\to\infty} T\phi_m$ exists; let the limit be ϕ^*.

In order to complete the proof, it suffices to show that

$$\lim_{m\to\infty} (T\phi_m(x) - \phi_m(x)) = 0$$

uniformly on $[0,1]$. It will follow that $\phi^* \in \langle g_\theta, S\rangle$.

So, let $\varepsilon > 0$ be given, and choose $0 < \delta < \frac{1}{2}$ such that

$$\int_0^\delta f(s, g_\theta(s))\, ds + \int_{1-\delta}^1 f(s, g_\theta(s))\, ds < \frac{\varepsilon}{2M}$$

where as before $M := \max\{G(x,s) \,|\, (x,s) \in [0,1] \times [0,1]\}$. Then, there exists m_0 such that, for $m \geq m_0$ and for $\delta \leq x \leq 1-\delta$,

$$u_m(x) \leq g_\theta(x) \leq \phi_m(x).$$

So, for $m \geq m_0$ and for $\delta \leq x \leq 1-\delta$,

$$f_m(x, \phi_m(x)) = f(x, \max\{\phi_m(x), u_m(x)\}) = f(x, \phi_m(x)).$$

Then, for $m \geq m_0$ and $0 \leq x \leq 1$,

$$\begin{aligned}
|T\phi_m(x) - \phi_m(x)| &= |T\phi_m(x) - T_m\phi_m(x)| \\
&= \left| \int_0^1 G(x,s)[f(s,\phi_m(s)) - f_m(s,\phi_m(s))]\, ds \right| \\
&= \Bigg| \int_0^\delta G(x,s)[f(s,\phi_m(s)) - f_m(s,\phi_m(s))]\, ds \\
&\quad + \int_{1-\delta}^1 G(x,s)[f(s,\phi_m(s)) - f_m(s,\phi_m(s))]\, ds \Bigg| \\
&\leq M \int_0^\delta [f(s,\phi_m(s)) + f_m(s,\phi_m(s))]\, ds \\
&\quad + M \int_{1-\delta}^1 [f(s,\phi_m(s)) + f_m(s,\phi_m(s))]\, ds \\
&\leq M \int_0^\delta [f(s,\phi_m(s)) + f(s,\phi_m(s))]\, ds \\
&\quad + \int_{1-\delta}^1 [f(s,\phi_m(s)) + f(s,\phi_m(s))]\, ds \\
&= 2M \left[\int_0^\delta f(s,\phi_m(s))\, ds + \int_{1-\delta}^1 f(s,\phi_m(s))\, ds \right] \\
&\leq 2M \left[\int_0^\delta f(s,g_\theta(s))\, ds + \int_{1-\delta}^1 f(s,g_\theta(s))\, ds \right] \\
&< 2M \cdot \frac{\varepsilon}{2M} \\
&= \varepsilon.
\end{aligned}$$

So, for $m \geq m_0$,

$$\|T\phi_m - \phi_m\| < \varepsilon.$$

That is, $\lim_{m\to\infty}(T\phi_m(x) - \phi_m(x)) = 0$ uniformly on $[0,1]$. Hence, for $0 \leq x \leq 1$,

$$\begin{aligned}
T\phi^*(x) &= T(\lim_{m\to\infty} T\phi_m(x)) \\
&= T(\lim_{m\to\infty} \phi_m(x)) \\
&= \lim_{m\to\infty} T\phi_m(x) \\
&= \phi^*(x),
\end{aligned}$$

and ϕ^* is a desired positive solution for (4.1), (4.2) belonging to D. $\square$

BIBLIOGRAPHY

[1] R. P. Agarwal and D. O'Regan. Singular problems on the infinite interval modelling phenomena in draining flows. *IMA J. Appl. Math.*, 66(6):621–635, 2001.

[2] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong. *Positive solutions of differential, difference and integral equations.* Kluwer Academic Publishers, Dordrecht, 1999.

[3] F. Bernis and L. A. Peletier. Two problems from draining flows involving third-order ordinary differential equations. *SIAM J. Math. Anal.*, 27(2):515–527, 1996.

[4] L. E. Bobisud, D. O'Regan, and W. D. Royalty. Existence and nonexistence for a singular boundary value problem. *Appl. Anal.*, 28(4):245–256, 1988.

[5] L. E. Bobisud, D. O'Regan, and W. D. Royalty. Solvability of some nonlinear boundary value problems. *Nonlinear Anal.*, 12(9):855–869, 1988.

[6] A. Callegari and A. Nachman. Some singular, nonlinear differential equations arising in boundary layer theory. *J. Math. Anal. Appl.*, 64(1):96–105, 1978.

[7] J. M. Davis, P. W. Eloe, and J. Henderson. Comparison of eigenvalues for discrete Lidstone boundary value problems. *Dynam. Systems Appl.*, 8(3-4):381–388, 1999.

[8] P. W. Eloe. Nonlinear eigenvalue problems for higher order Lidstone boundary value problems. *Electron. J. Qual. Theory Differ. Equ.*, No. 2:8 pp. (electronic), 2000.

[9] P. W. Eloe and J. Henderson. Comparison of eigenvalues for a class of two-point boundary value problems. *Appl. Anal.*, 34(1-2):25–34, 1989.

[10] P. W. Eloe and J. Henderson. Singular nonlinear boundary value problems for higher order ordinary differential equations. *Nonlinear Anal.*, 17(1):1–10, 1991.

[11] P. W. Eloe and J. Henderson. Comparison of eigenvalues for a class of multi-point boundary value problems. In *Recent trends in differential equations*, volume 1 of *World Sci. Ser. Appl. Anal.*, pages 179–188. World Sci. Publ., River Edge, NJ, 1992.

[12] P. W. Eloe and J. Henderson. Focal points and comparison theorems for a class of two-point boundary value problems. *J. Differential Equations*, 103(2):375–386, 1993.

[13] P. W. Eloe and J. Henderson. Focal point characterizations and comparisons for right focal differential operators. *J. Math. Anal. Appl.*, 181(1):22–34, 1994.

[14] J. A. Gatica, V. Oliker, and P. Waltman. Singular nonlinear boundary value problems for second-order ordinary differential equations. *J. Differential Equations*, 79(1):62–78, 1989.

[15] R. D. Gentry and C. C. Travis. Comparison of eigenvalues associated with linear differential equations of arbitrary order. *Trans. Amer. Math. Soc.*, 223:167–179, 1976.

[16] J. R. Graef and B. Yang. Existence and nonexistence of positive solutions of a nonlinear third order boundary value problem. In *The 8th Colloquium on the Qualitative Theory of Differential Equations*, volume 8 of *Proc. Colloq. Qual. Theory Differ. Equ.*, pages 9–13. Electron. J. Qual. Theory Differ. Equ., Szeged, 2008.

[17] J. R. Graef and B. Yang. Upper and lower estimates of the positive solutions of a higher order boundary value problem. *J. Appl. Math. Comput.*, 41(1-2):321–337, 2013.

[18] J. R. Graef and B. Yang. Upper and lower estimates of the positive solutions of a higher order boundary value problem. *J. Appl. Math. Comput.*, 41(1-2):321–337, 2013.

[19] D. Hankerson and J. Henderson. Comparison of eigenvalues for n-point boundary value problems for difference equations. In *Differential equations (Colorado Springs, CO, 1989)*, volume 127 of *Lecture Notes in Pure and Appl. Math.*, pages 203–208. Dekker, New York, 1991.

[20] D. Hankerson and A. Peterson. Comparison of eigenvalues for focal point problems for nth order difference equations. *Differential Integral Equations*, 3(2):363–380, 1990.

[21] J. Henderson and K. R. Prasad. Comparison of eigenvalues for Lidstone boundary value problems on a measure chain. *Comput. Math. Appl.*, 38(11-12):55–62, 1999.

[22] J. Henderson and W. Yin. Singular $(k, n-k)$ boundary value problems between conjugate and right focal. *J. Comput. Appl. Math.*, 88(1):57–69, 1998.

[23] J. Hernández, F. J. Mancebo, and J. M. Vega. Positive solutions for singular nonlinear elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(1):41–62, 2007.

[24] E. R. Kaufmann. Comparison of eigenvalues for eigenvalue problems of a right disfocal operator. *Panamer. Math. J.*, 4(4):103–124, 1994.

[25] M. S. Keener and C. C. Travis. Positive cones and focal points for a class of nth-order differential equations. *Trans. Amer. Math. Soc.*, 237:331–351, 1978.

[26] M. A. Krasnosel'skii. *Positive solutions of operator equations*. Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron. P. Noordhoff Ltd. Groningen, 1964.

[27] M. G. Krein and M. A. Rutman. *Linear Operators Leaving a Cone Invariant in a Banach Space*. American Mathematical Society Translations, Series 1. Providence, 1962.

[28] C. D. Luning and W. L. Perry. Positive solutions of negative exponent generalized Emden-Fowler boundary value problems. *SIAM J. Math. Anal.*, 12(6):874–879, 1981.

[29] M. Maroun. Positive solutions to a third-order right focal boundary value problem. *Comm. Appl. Nonlinear Anal.*, 12(3):71–82, 2005.

[30] M. Maroun. Positive solutions to an nth order right focal boundary value problem. *Electron. J. Qual. Theory Differ. Equ.*, pages 4–17, 2007.

[31] A. Nachman and A. Callegari. A nonlinear singular boundary value problem in the theory of pseudoplastic fluids. *SIAM J. Appl. Math.*, 38(2):275–281, 1980.

[32] D. O'Regan. Positive solutions to singular and nonsingular second-order boundary value problems. *J. Math. Anal. Appl.*, 142(1):40–52, 1989.

[33] D. O'Regan. Existence of positive solutions to some singular and nonsingular second order boundary value problems. *J. Differential Equations*, 84(2):228–251, 1990.

[34] D. O'Regan. Some new results for second order boundary value problems. *J. Math. Anal. Appl.*, 148(2):548–570, 1990.

[35] W. B. Qu, Z. X. Zhang, and J. D. Wu. Positive solutions to a singular second order three-point boundary value problem. *Appl. Math. Mech.*, 23(7):759–770, 2002.

[36] P. Singh. A second-order singular three-point boundary value problem. *Appl. Math. Lett.*, 17(8):969–976, 2004.

[37] E. C. Tomastik. Comparison theorems for second order nonselfadjoint differential systems. *SIAM J. Math. Anal.*, 14(1):60–65, 1983.

[38] C. C. Travis. Comparison of eigenvalues for linear differential equations. *Proc. Amer. Math. Soc.*, 96:437–442, 1986.

[39] Z. Wei. Positive solutions of singular sublinear second order boundary value problems. *Systems Sci. Math. Sci.*, 11(1):82–88, 1998.

[40] X. Yang. Positive solutions for nonlinear singular boundary value problems. *Appl. Math. Comput.*, 130(2-3):225–234, 2002.