ABSTRACT

Chaos in Dendritic and Circular Julia Sets

Nathan Averbeck, Ph.D.

Advisor: Brian Raines, D.Phil.

We demonstrate the existence of various forms of chaos (including transitive distributional chaos, ω-chaos, topological chaos, and exact Devaney chaos) on two families of abstract Julia sets: the dendritic Julia sets $\mathcal{D}_\tau$ and the "circular" Julia sets $\mathcal{E}_\tau$, whose symbolic encoding was introduced by Stewart Baldwin. In particular, suppose one of the two following conditions hold: either f_c has a Julia set which is a dendrite, or (provided that the kneading sequence of c is Γ-acceptable) that f_c has an attracting or parabolic periodic point. Then, by way of a conjugacy which allows us to represent these Julia sets symbolically, we prove that f_c exhibits various forms of chaos.

Chaos in Dendritic and Circular Julia Sets

by

Nathan Averbeck, B.S., M.A.

A Dissertation

Approved by the Department of Mathematics

Lance L. Littlejohn, Ph.D., Chairperson

Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

Approved by the Dissertation Committee

Brian Raines, D.Phil., Chairperson

Will Brian, D.Phil.

Markus Hunziker, Ph.D.

Alexander Pruss, Ph.D.

David Ryden, Ph.D.

Accepted by the Graduate School
August 2016

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I am deeply indebted to my advisor, Brian Raines, not only for his time and effort but also for his encouragement. I am grateful to my remaining dissertation committee members (Will Brian, Markus Hunziker, Lance Littlejohn, Alexander Pruss, and David Ryden) for their work in reviewing my manuscript. Thanks also to all my teachers at Baylor who helped prepare me to do research.

Without the support of my many Baylor friends, I could not have made it through graduate school. Reeve and Jessie, Tim, James, Chuck, and Curtis played special roles. To the many, many others who have helped me at some point on my journey, I thank you all.

Dedicated to anyone who has taken a long journey

CHAPTER ONE

Preliminaries

1.1 Continuum Theory and Dynamical Systems

A *continuum* is a compact, connected, metric space. An *arc* is any space homeomorphic to $I = [0, 1]$; its *endpoints* are the unique points corresponding to 0 and 1. In continuum theory, a *graph* is a continuum which can be written as the union of finitely many arcs, where each pair of arcs may meet only at one or both endpoints and are otherwise pairwise disjoint. A *tree* is a graph containing no simple closed curve. A space X is *uniquely arcwise connected* if, for any $x, y \in X$, there exists only one arc in X which has x and y as endpoints. A *dendroid* is a uniquely arcwise-connected continuum. A *dendrite* is a locally connected dendroid.[1] The *degree* of a point x in a dendrite X is the number of components of $X - \{x\}$. If $X - \{x\}$ has 3 or more components, then x is a *branch point.* Recall that a *perfect set* has no isolated points, and a *Cantor set* is any compact, perfect, totally disconnected set (which is uncountable, [72]). The Cantor fan, which is the cone over the Cantor set, is an example of a dendroid which is not a dendrite (see [26] for more details). Simple examples of dendrites include arcs and trees. For a more interesting example, the Wazewski dendrite, see [25]; in the Wazewski dendrite, each branch point has infinite degree, and any arc A contains a set of branch points of the dendrite which is dense in A. For definitions of basic topological concepts, including local connectedness, consult [66]. For a thorough discussion of the properties of trees, dendroids, and dendrites, see [67].

Unless otherwise noted, X will always represent a compact metric space with metric d. A *discrete dynamical system* is a nonempty set X with a map $f : X \to X$ (however, X is often assumed to be a compact metric space). We typically refer

[1] The terms "dendrite" and "dendroid" come from the Greek for "tree."

to such a system simply as a dynamical system (there are *continuous dynamical systems*, but we will not consider them). Let $\omega = \mathbb{N} \cup \{0\}$. For $x \in X$, the *forward orbit* of x under f is the set $\mathrm{Orb}^+(x) = \{f^n(x) : n \in \omega\}$, where f^n denotes the composition of f with itself n times (and $f^0(x) = x$). If $f(x) = x$, we say x is a *fixed point* of the function f. If $f^n(x) = x$ for some $n \in \mathbb{N}$, we say x is a *periodic point with period* n; if n is the smallest natural number such that $f^n(x) = x$, we say x has *prime period* n (however, in practical usage, by "period" we often mean "prime period"). We will denote the set of fixed points of f by $\mathrm{Fix}(f)$ and the set of periodic points of f by $\mathrm{Per}(f)$. If x is not periodic, but there exists $n \in \mathbb{N}$ and a periodic point $y \in X$ such that $f^n(x) = y$, we say x is *eventually periodic* or *pre-periodic.* If n is the smallest number such that $f^n(x)$ is periodic, we call n the *pre-period* of x. Note that we consider periodic points to be pre-periodic points with pre-period zero. If $x, y \in X$ such that $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$, we say the pair (x, y) is *proximal.* If, on the other hand, we have $\liminf_{n\to\infty} d(f^n(x), f^n(y)) > 0$, we say that the pair (x, y) is *distal.* If $\limsup_{n\to\infty} d(f^n(x), f^n(y)) = 0$ (implying the limit itself is 0), then we say that the pair (x, y) is *asymptotic.*

Two dynamical systems (X, f) and (Y, g) are *topologically conjugate* if there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. See Figure 1.1. Conjugate systems are, as Robert L. Devaney describes them, "completely equivalent in terms of their dynamics." For example, if x is a periodic point of X with period n, then $h(x)$ is a periodic point of Y with period n, [29, page 47]. However, note that metric properties, such as sensitive dependence on initial conditions (to be defined shortly), are not necessarily preserved by conjugacy, [10]. Sometimes, spaces which seem very different at first glance are related by a conjugacy, which can be helpful if one space is especially convenient to work in. Conjugacies, being homeomorphisms, preserve topological properties.

$$\begin{array}{ccc} X & \xrightarrow[f]{} & X \\ \downarrow h & & \downarrow h \\ Y & \xrightarrow[g]{} & Y \end{array}$$

Figure 1.1. The commutative diagram for h, a conjugacy

The following three properties give an idea how open sets interact under f. We say $f : X \to X$ is *topologically transitive* if, for any nonempty open sets U and V in X, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. We say f is *topologically mixing* if there exists $M \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq M$. We say $f : X \to X$ is *topologically exact* (or *locally eventually onto*) if, for any nonempty open set U in X, there exists $n \in \mathbb{N}$ such that $f^n(U) = X$. (The word "topological" is used to emphasize that these properties are preserved by homeomorphisms. However, often we simply say that f is exact, mixing, or transitive.) Clearly, an exact map is also mixing, and a mixing map is transitive. For a more thorough introduction to dynamical systems, see [22].

For a polynomial f, the set of points whose orbits are not bounded is the *basin of attraction of infinity, U*. The *filled-in Julia set* of f is $K = \mathbb{C} - U$. The *Julia set* of f is $J = \mathrm{Bd}(K)$, where $\mathrm{Bd}(K)$ denotes the boundary of K. Note that for a general (non-polynomial) function f, the definition of the Julia set is more complicated; see [47] for details. By J_c and K_c, we mean the Julia set of $f_c = z^2 + c$ and filled-in Julia set of $f_c = z^2 + c$, respectively. Although we often discuss functions of the form $f_c(z) = z^2 + c$ since their orbits are easier to compute than those of a typical quadratic, note that if $g(z)$ is any quadratic polynomial, then $g(z)$ is conjugate to a unique quadratic of the form $f_c(z) = z^2 + c$, [21, page 77]. Hence, we can describe all quadratic polynomials by considering functions of the form $f_c(z) = z^2 + c$.

The *Mandelbrot set* is the set of points c whose orbits under $f_c(z) = z^2 + c$ are bounded. This set was discovered by Benoit Mandelbrot in 1979 while working for IBM (at roughly the same time, Brooks and Matelski independently discovered the same set; see [23]). The Mandelbrot set can be defined as $\{c \in \mathbb{C} : 0 \in K_c\} = \{c \in \mathbb{C} : c \in K_c\}$, and hence James Gleick describes the Mandelbrot set as a "catalogue of Julia sets," [41, page 222]. In fact, there are points in the Mandelbrot set (known as Misiurewicz points, to be defined in Section 3.3) at which the Mandelbrot set and Julia set of f_c are locally similar. In other words, if c is a Misiurewicz point, and we zoom in far enough on c in either the Mandelbrot set or in the Julia set of f_c, the pictures are similar (up to rotation), [47, pages 101-102]. The Mandelbrot set is illustrated in [61, page 188] and Figure 1.2 below. Mandelbrot's original computer printout can be seen in [41, page 225]. For a more detailed introduction to the Mandelbrot set, see [21].

In 1918 and 1919, Pierre Fatou and Gaston Julia proved the following:

Theorem 1.1. *([21, page 80], citing [35] and [46]) Let Ω denote the set of critical points*[2] *for a polynomial P. Let J and K denote the Julia set of P and filled-in Julia set of P, respectively. Then*

a) $\Omega \subset K$ if and only if J is connected, and

b) $\Omega \cap K = \emptyset$ implies J is a Cantor set.

For a general polynomial, this theorem treats only the extreme cases. However, since a quadratic polynomial has only one critical point, every quadratic polynomial has a Julia set which is either connected or a Cantor set.

1.2 Unimodal Maps

The definitions in this section are inspired by maps on [0,1] which have a single turning point (in the usual sense). An example is the *logistic map*, $f(x) = \mu x(1-x)$, which has its turning point at $x = \frac{1}{2}$. Let C be the set of points which do not eventually escape $[0, 1]$ under iteration by f. We can split the interval into two

[2] For a polynomial P, x is a critical point provided $P'(x) = 0$.

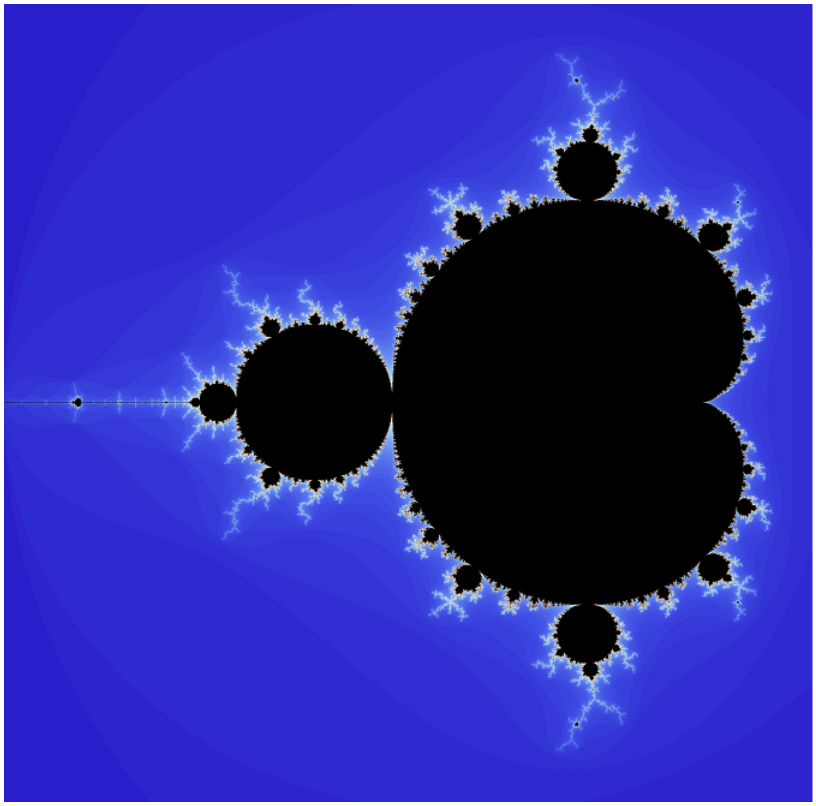

Figure 1.2. The Mandelbrot set

pieces, $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$, which we will represent by 0 and 1, respectively, and use $*$ to represent the turning point. We can assign to each point of C a sequence corresponding to the order in which the point's orbit visits each side. Such a sequence for a point is called the point's itinerary. Suppose $\mu = 2 + \sqrt{5}$. Then, for example, there is a periodic point p (approximately 0.6588) such that p has itinerary $(110)^\infty$. It can be shown that C is a Cantor set and is conjugate to (Σ_2^+, σ), the set of all one-sided infinite binary sequences under the shift map, [29, Theorem 5.6 and subsequent Remark]. Thus, no other point of C has an itinerary of $(110)^\infty$. Fortunately, then, we can come to conclusions about the original space by working in a shift space (where the dynamics are much easier to comprehend). For a closer look at the origins of itinerary theory, see for example the 1973 paper [64] by Metropolis, M.

L. Stein, and P. R. Stein, which considers itineraries of interval maps with a unique maximum (possibly attained by an interval).

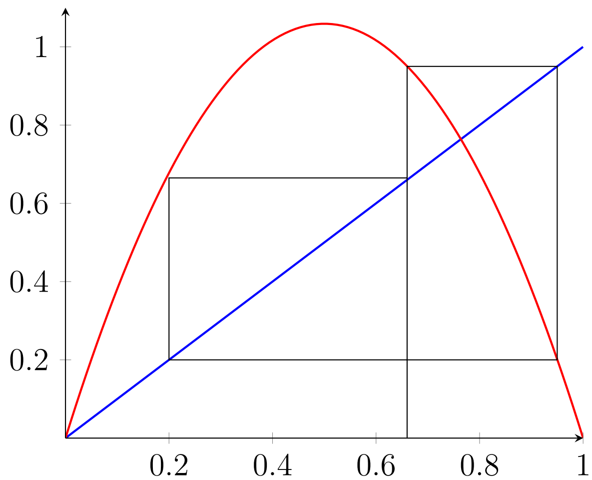

Figure 1.3. The orbit of $p \approx 0.659$ under $f(x) = \mu x(1-x)$, where $\mu = 2 + \sqrt{5}$.

Stewart Baldwin's work, [6, 7], generalizes unimodal maps to dendrites. If X is any dendrite, a point at which f is not locally one-to-one is called a *turning point* of f. A map $f : X \to X$ is called *unimodal* provided it has at most one turning point. For a dendrite X, the *legs* of X with respect to t are the components of $X - \{t\}$. Since X is a dendrite, there are at most countably many legs with respect to t, [6, page 2892]. We enumerate the legs of X with respect to t as follows: we let $L_0 = \{t\}$ and suppose the legs are numbered $L_1, L_2, \ldots$ according to the order in which they are visited by the orbit of t. For any $x \in X$, the *itinerary* of x is the sequence $i(x) = i_0 i_1 i_2 \ldots$ where $i_k = n$ if and only if $f^n(x) \in L_n$. If f has exactly one turning point t, the *kneading sequence* of f is the itinerary of $f(t)$. (It should be noted that sometimes, the kneading sequence is defined as the itinerary of t itself, as in [5,6].) If $x \neq y$ implies that $i(x) \neq i(y)$, then we say f has the *unique itinerary property with respect to* t.

We say a map f is *tentish* provided it is unimodal with turning point t and has the unique itinerary property with respect to t. If the Julia set of $f_c = z^2 + c$ is a dendrite, then $f_c|_{J_c}$ is tentish, [6, page 2895]. In particular, if the Julia set of $f_c|_{J_c}$ is a dendrite, then $f_c|_{J_c}$ has the unique itinerary property.

1.3 Various Notions of Chaos

We begin with two important points. First, the word "chaos" has many different meanings, of which we will only discuss a few of the most widely-known (Li-Yorke, distributional, Devaney, topological, and ω-chaos). Second, chaos is not randomness: chaotic systems may appear unpredictable, but in fact they are deterministic. That is, the same initial conditions always produce the same results.

In 1965, Adler, Konheim, and McAndrew introduced the precursor of the modern notion of topological entropy in [1]. (A measure-theoretic definition of entropy had been studied earlier, such as in the 1959 paper [77].) Later works, such as the 1971 papers [19, 20] by Rufus Bowen, employ a definition of entropy in metric spaces (while the definition of Adler, Konheim, and McAndrew only requires a compact Hausdorff space). Although Bowen's definition requires a metric space, it does not depend on the metric generating the topology, [22, page 38]. On compact metric spaces, the definitions of Adler, Konheim, and McAndrew and Bowen coincide, [20, Remark 4.6].

Let (X, f) be a dynamical system such that f is uniformly continuous (which is guaranteed if f is continuous and X is compact). Fix n and ϵ. The set $A \subset X$ is said to be an (n, ϵ)-*spanning set* if for all $x \in X$ there exists $y \in A$ such that $\max\limits_{0 \leq k \leq n-1} d(f^k(x), f^k(y)) < \epsilon$. Let $\operatorname{span}(n, \epsilon, f)$ be the minimum cardinality of an (n, ϵ)-spanning set. We define the *topological entropy* of f as

$$h(f) = \lim_{\epsilon \to 0} \left(\limsup \frac{1}{n} \log(\operatorname{span}(n, \epsilon, f)) \right).$$

Note that there are several equivalent definitions of entropy; one, for example, involves the notion of (n, ϵ)-*separated* sets. As Brin and Stuck explain, "topological entropy is the exponential growth rate of the number of essentially different orbit segments of length n" [22, page 36]. As the name suggests, topological entropy is preserved by conjugacy, [22, page 39].

We say a dynamical system has *topological chaos* provided it has positive topological entropy, i.e., $h(f) > 0$, which is sometimes written PTE. In 1993, Glasner and Weiss, [40, page 1068], were perhaps the first to define topological chaos this way (though Furstenberg in 1967 spoke of certain systems with entropy 0 as being *deterministic*, [39]).

In 1961, Edward Lorenz discovered the "butterfly effect," so named for Lorenz's idea that a butterfly flapping its wings in Brazil might cause a tornado in Texas, [41]. Lorenz was running a weather forecasting model on his computer when, to save time, he input a number as 0.506 when the computed value was actually 0.506127. After running the simulation for an hour, he found that the outputs corresponding to the inputs 0.506127 and 0.506 had lost any resemblance, [41]. He realized that, in certain systems, slight perturbations in initial conditions can result in drastic differences over time, which leads us to our next definition, introduced by Guckenheimer in 1979 in [42].

Definition 1.2. Let (X, d) be a metric space. The continuous function $f : X \to X$ has *sensitive dependence on initial conditions*, or SDIC, provided there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood U of x, there exists $y \in U$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.

Informally, in a system with SDIC, every point x has a "close" neighbor y such that x and y eventually become separated under iteration. (Note that we do not require every pair of points to become separated under iteration–such as map is called *expansive.*)

By 1986, Devaney had a notion of chaos which is now known as *Devaney chaos.* In Devaney's view, chaos has 3 characteristics: indecomposability (the system cannot be split into parts which do not interact), unpredictability (points which are close initially may eventually behave very differently), and regularity (there still are many points which are periodic).

Definition 1.3. Let X be a metric space. The continuous function $f : X \to X$ is said to be *chaotic in the sense of Devaney* if

1) f is topologically transitive,

2) X has a dense set of periodic points, and

3) f has sensitive dependence on initial conditions.

This is the definition originally given by Devaney. However, in 1992 it was shown that in an infinite metric space X, the first two conditions imply the third condition, [10] (also proved independently in [40]). Conjugacy does not necessarily preserve metric properties such as SDIC;[3] fortunately, as condition 3) is superfluous, Devaney chaos is still a topological property, [10]. Nevertheless, some authors choose to retain all 3 conditions. Devaney chaos can be strengthened by replacing condition 1) with a stronger condition, such as requiring that f be topologically mixing (yielding *mixing Devaney chaos*) or topologically exact (yielding *exact Devaney chaos*). Finally, note that in spaces without any isolated points, condition 1) is equivalent to the existence of a point with a dense orbit, [18].

In 1975, in their now-classic paper *Period Three Implies Chaos*, Tien-Yien Li and James Yorke introduced a notion of chaos which is now referred to as *Li-Yorke chaos*, [56].

Definition 1.4. Let (X, d) be a metric space and let $f : X \to X$ be continuous. Points $x, y \in X$ form a *Li-Yorke chaotic pair* if

1) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$, and

[3] However, on compact spaces, conjugacy does preserve SDIC, [10].

2) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$.

If $S \subset X$ and for any $x, y \in S$, we have that x and y form a chaotic pair, then S is a *Li-Yorke scrambled set*. If S is uncountable, f exhibits *Li-Yorke chaos*.

Li and Yorke's original definition included a third condition (that S not contain any asymptotically periodic points) which was shown later to be superfluous for the compact interval, [48]. In a general compact metric space, since S can have at most one asymptotically periodic point, this third condition is unnecessary, [68]. In 1989, Kuchta and Smital, [48], showed that if (I, f) has a single Li-Yorke pair, then (I, f) has Li-Yorke chaos, and the next year this result was extended to $X = S^1$, the circle, in [49]. In 2014, Kuchta and Smital's result was extended to graph maps, [75].

A potential objection is that the definition of Li-Yorke chaos does not specify how often $f^n(x)$ and $f^n(y)$ must be δ apart. Consider Figure 1.4, which, for a particular pair (x, y), illustrates the distance between $f^n(x)$ and $f^n(y)$ for values of n from 1 to 40. Aside from the relatively infrequent values of n such that $f^n(x)$ and $f^n(y)$ are $\delta = 0.3$ apart, $f^n(x)$ and $f^n(y)$ behave similarly. Assuming the trend continues, Figure 1.4 illustrates a Li-Yorke chaotic pair, even though $f^n(x)$ and $f^n(y)$ are very close most of the time.

In 1994, B. Schweizer and J. Smital, [76], introduced a notion of chaos on I which they called *strong chaos* but is now more commonly referred to as *distributional chaos* or somewhat less commonly as *Schweizer-Smital chaos*. This type of chaos, in contrast to Li-Yorke chaos, requires (roughly speaking) that, as n increases, the percentage of the first n iterates of x and y which are δ apart must sometimes tend to 100% and at other times tend to 0%. Hence, unlike Li-Yorke chaos, there must be "many" n for which $f^n(x)$ and $f^n(y)$ are δ apart. Formally, distributional chaos is defined as follows:

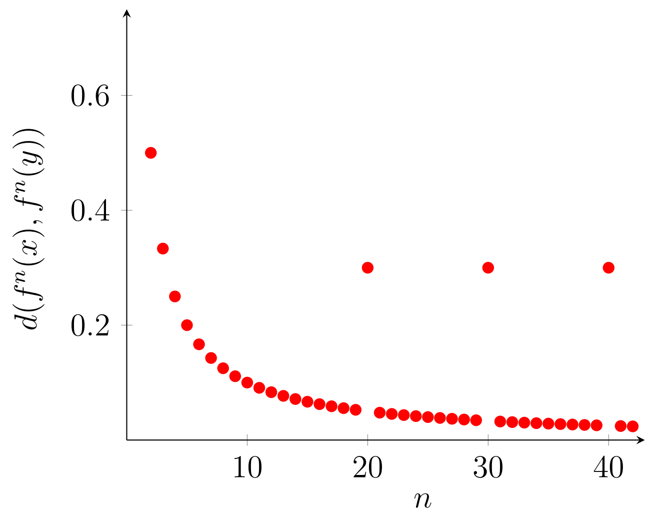

Figure 1.4. An example of a Li-Yorke chaotic pair

Definition 1.5. Let (X, d) be a metric space and let $x, y \in X$. Let

$$\Phi_{x,y}(t,n) = \frac{1}{n} \left| \{0 \le i < n : d(f^i(x), f^i(y)) < t\} \right|$$

and define

$$F_{xy}(t) = \liminf_{n\to\infty} \Phi_{x,y}(t,n) = \liminf_{n\to\infty} \frac{1}{n} \left| \{0 \le i < n : d(f^i(x), f^i(y)) < t\} \right|$$

and

$$F^*_{xy}(t) = \limsup_{n\to\infty} \Phi_{x,y}(t,n) = \limsup_{n\to\infty} \frac{1}{n} \left| \{0 \le i < n : d(f^i(x), f^i(y)) < t\} \right| .$$

If $x \neq y$, the pair (x, y) is a *DC1 scrambled pair* for f provided the following two conditions are met:

1) For all $t > 0$, $F^*_{xy}(t) = 1$, and

2) For some $t > 0$, $F_{xy}(t) = 0$.

Observe that if $F_{xy}(t) = 0$ for some $t > 0$, then $F_{xy}(t') = 0$ holds for all $0 < t' < t$.

If $D \subset X$ is a set such that every $x, y \in D$ form a DC1 scrambled pair, then we say that D is a *DC1 scrambled set* for f. If in addition D is uncountable, then

f exhibits *distributional chaos of type 1*,[4] often abbreviated DC1. Unless otherwise noted, by "distributional chaos" we always mean DC1. Note that every DC1 scrambled pair is a Li-Yorke scrambled pair, so that DC1 implies Li-Yorke chaos. See Figure 1.5 for an illustration of a distributionally chaotic pair.

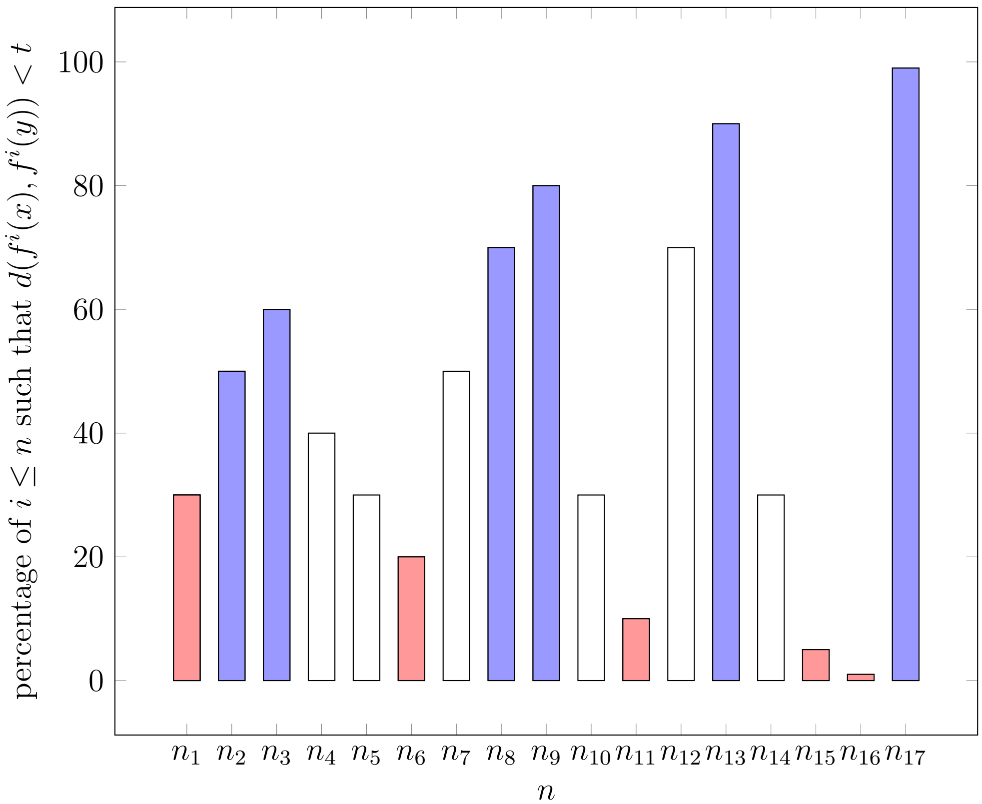

Figure 1.5. An example of a distributionally chaotic pair

The values $n_2, n_3, n_8, n_9, n_{17}$ form part of a subsequence in which $\Phi_{x,y}(t,n) \to 1$, while $n_1, n_6, n_{11}, n_{15}, n_{16}$ form part of a subsequence in which $\Phi_{x,y}(t,n) \to 0$. Note that n_i may be much smaller than n_{i+1} and that the figure only illustrates the situation for a single value of t.

Suppose $f|_D$ exhibits distributional chaos. If there exists $t' > 0$ such that $F_{xy}(t') = 0$ for all distinct points $x, y \in D$, then the chaos is said to be *uniform*.

[4] Some authors define "distributional chaos" differently. See for example [43]. The author of [54] defines Schweizer-Smital chaos as we define DC1, instead reserving the term DC1 for a single DC1 pair.

Suppose the distributional chaos is uniform. If D is dense, the chaos is *dense distributional chaos*; if, additionally, each point of D has a dense orbit, the chaos is *transitive distributional chaos.* Fortunately, a conjugacy between compact metric spaces preserves uniform distributional chaos, [62], as well as transitive distributional chaos and dense distributional chaos, [70, Theorem 13]. We will appeal to these facts multiple times.

Theorem 1.6. *(See [70]) If $(X, f), (Y, g)$ are topologically conjugate dynamical systems acting on compact metric spaces, then f exhibits dense (or transitive) distributional chaos if so does g.*

The idea of weakening the definition of DC1 was introduced in 2004 in [79]. The weakened versions of DC1, namely DC2 and DC3, are defined as follows. If we keep condition 1) of DC1 (i.e. $F^*_{xy}(t) = 1$ for all $t > 0$), but change condition 2) so that $F_{xy}(t) < F^*_{xy}(t)$ for some $t > 0$, then (x, y) is a *DC2 scrambled pair* for f. (If $F_{xy}(t) < 1 = F^*_{xy}(t)$ holds for some $t > 0$, then the inequality holds for all $0 < t' < t$, so this condition is often stated as "$F_{xy}(t) < F^*_{xy}(t)$ for all t in some nondegenerate interval.") If we drop condition 1) and simply require that $F_{xy}(t) < F^*_{xy}(t)$ for some $t > 0$ (and hence, for any $t' < t$), then (x, y) is a *DC3 scrambled pair* for f. However, unless otherwise noted, we will only consider DC1 pairs and DC1 chaos.

A *local dendrite* is a continuum in which every point has a closed neighborhood which is a dendrite. In [2], it is demonstrated that, for a local dendrite X, and a continuous function $f : X \to X$, any Li-Yorke scrambled set of f must be totally disconnected. In particular, this implies that any DC1 scrambled set of a dendrite map is also totally disconnected. Also, it is noteworthy that, for any $n \in \mathbb{N}$ and any compact space X, $f : X \to X$ exhibits distributional chaos if and only if f^n does, [84]. This result was later extended: for any $n \in \mathbb{N}$ and any compact space X, $f : X \to X$ exhibits DC2 if and only if f^n does, [54]. DC1 and DC2 are preserved by conjugacy between compact metric spaces, [79], but not necessarily by semiconjugacy, [73, Example 6]. However, DC3 is not preserved even by conjugacy, [30] (although that author suggests a strengthening of DC3, namely DC2$\frac{1}{2}$, which is preserved by conjugacy).

As many authors have noted, in compact metric spaces, DC1 and DC2 imply Li-Yorke chaos, [9, 68].[5] However, DC3 does not imply Li-Yorke chaos, [82]. In fact, the authors of [82] conjectured that there is a compact DC3 system which has not even a single Li-Yorke pair; the conjecture was subsequently proven in [30]. The authors of [34] give a necessary condition for a compact space to have DC3 pairs (and hence a necessary condition to have DC1 and DC2 pairs).

Let X be a metric space. For $x \in X$, we define $\omega_f(x)$, the ω-*limit set* of x under f, as the set

$$\omega_f(x) = \{y \in X : \text{there exists a sequence } (n_i)_{i=0}^{\infty} \text{ such that } f^{n_i}(x) \to y\}$$

If there is no danger of confusion regarding the function, we simply write $\omega(x)$. A set A is said to be *invariant* under a function f provided $f(A) \subseteq A$ and *strongly invariant* if $f(A) = A$. We will make use of the following lemma in a later section.

Lemma 1.7. *[18, Lemma 2, Chapter 4] The set $\omega_f(x)$ is a nonempty, closed and strongly invariant set.*

If A is a closed, invariant subset of X such that A contains no nonempty proper subset which is also closed and invariant, then A is said to be *minimal* under f. Note that if $x \in A$, a minimal set, then $\omega(x) \subseteq A$ (as A is invariant), and since $\omega(x)$ is a closed invariant subset of a minimal set, we must have that $\omega(x) = A$. The authors of [8] consider DC1 in relation to triangular maps of the square into itself and ask several questions regarding the existence of minimal DC1 systems, some of which are addressed in [71, 85].

[5] It should be noted that [9] defines DC1 (DC2, DC3 respectively) chaos as the existence of a single DC1 (DC2, DC3) chaotic pair.

In 1993 in [55], Shi Hai Li (not to be confused with Tien-Yien Li of Li-Yorke chaos) introduced the following notion, ω-chaos.

Definition 1.8. $S \subset X$ is called an *ω-scrambled set* provided that for any distinct $x, y \in S$, the following three conditions hold:

1) $\omega_f(x) - \omega_f(y)$ is uncountable,

2) $\omega_f(x) \cap \omega_f(y) \neq \emptyset$, and

3) $\omega_f(x) - \mathrm{Per}(f) \neq \emptyset$.

If S is uncountable, f is said to exhibit *ω-chaos.*

For example, the shift map on the space of infinite binary sequences exhibits ω-chaos; in fact, this dynamical system has a perfect ω-chaotic scrambled set, [78]. When Li introduced ω-chaos, he noted that on I, condition 3) is unnecessary, being naturally satisfied by the other two conditions, [55]. Subsequently, in more complicated spaces, attempts have been made to show that one of the three conditions is implied by the others (as was done with Devaney chaos). In particular, in certain hereditarily locally connected spaces (such as dendrites with a discrete set of branch points), condition 3) is implied by the other conditions, [63]. However, Lampart showed in [53] the third condition of ω-chaos is, in general, essential.

Li showed that on I, ω-chaos, Devaney chaos, and PTE are equivalent, [55]. Similarly, Schweizer and Smital showed that DC1, DC2, and PTE are equivalent on I, [76]. Hence, most notions of chaos which we will consider, except Li-Yorke chaos, are equivalent on I.

In a general compact metric space, these notions of chaos are not equivalent. It is clear that DC1 implies Li-Yorke chaos, but the question as to whether or not PTE implies Li-Yorke chaos was open until 2002, when it was answered affirmatively in [17]. In [51], Lampart demonstrates that ω-chaos implies the existence of a Li-Yorke chaotic pair.[6] However, Pikula in [74] demonstrated that a map can be ω-chaotic and yet not be Li-Yorke chaotic. According to Lampart and Oprocha, ω-chaos "is not related to many other notions of chaos known from the literature," [52]. In a compact metric space, it is now known that distributional, topological, and Devaney chaos imply Li-Yorke chaos, [45, 68]. There exist maps which are PTE but not DC1, [79], maps which are DC1 but not PTE, [36, 57], and maps which are topologically transitive but not PTE, [25]. An excellent summary of the relations between various types of chaos on a general compact metric space is given by Oprocha, [68, Fig. 1].

In discussions of chaos, the space under consideration is typically assumed to be compact (but not always). See [83] for an interesting example of DC1 scrambled sets in a non-compact space. See [59] for an example of a noncompact space and a continuous function such that the entire space is a scrambled set. The author of [59] conjectures that if X is compact and $f : X \to X$ is continuous, then the whole space cannot be a scrambled set; this conjecture was disproved in [44].

Finally, note that there are still many more notions of chaos, such as chaos in the sense of Robinson, [37], and chaos in the sense of Wiggins and chaos in the sense of Martelli, [70], but we will not consider them here.

1.4 Applications of Chaos

Chaotic behavior appears in models of natural phenomena, as we have already seen in the weather forecasting model of Edward Lorenz. According to Denton et al., [28], who define chaos as "an aperiodic, seemingly random behavior in a deterministic system[7] that exhibits sensitive dependence on initial conditions," scientists

[6] Note that Lampart's definition of Li-Yorke chaos in a general compact metric space only requires a single chaotic pair.

[7] Some would argue that quantum mechanics rules out truly deterministic systems in nature. See [33].

have observed chaotic behavior in contexts including fluid turbulence, the orbit of Pluto, some chemical reactions, atomic motion, and many others. In [28], Denton et al. explore instances of chaotic behavior arising in cardiology (specifically, cardiac arrhythmia).

Distributional chaos appears in models of human behavior. With the increasing number of cars being driven, it is not surprising that modeling highway traffic is of interest. In [11], Barrachina et al. introduce an interesting model of highway traffic with infinitely many cars, which they call the Infinite Forward-and-Backward Control model. In this model, each car adjusts its acceleration based on the speeds of the cars immediately before and behind it (as in a convoy). The authors demonstrate that under certain conditions, the flow of traffic in this model exhibits distributional chaos, and explain their results as follows:

> "The appearance of this type of chaos means, roughly speaking, that we can pick two vectors of initial speeds for all the cars on the road (from an uncountable set) and, as time goes by, there will be long time intervals in which the vectors of speeds of the cars on the road are very similar for both vectors of initial speeds. On the other hand, there will also be intervals as long as the previous ones in which the vectors of speeds of the cars are quite different depending on which one of these two initial vectors we have chosen," [11, page 213].

These results are comparable to those in [27], whose authors consider a traffic model, called the Quick-Thinking Driver Model, in which drivers adjust their speed based solely on the speed of the car ahead; this model was shown to exhibit distributional and Devaney chaos and to be topologically mixing, [27].

1.5 Dissertation Layout

In Chapter Two, we answer negatively a conjecture of Fu and You, [38], regarding Li-Yorke scrambled sets in shift spaces with an orbit invariant (which is defined in Chapter Two). We then propose an alternate condition, not involving orbit invariants, which guarantees the existence of a scrambled set S: namely, that for each $x \neq y$ in S, the set of $i \in \mathbb{N}$ such that $x_i = y_i$ must be thick but not cofinal in $\mathbb{N}$ (see Chapter Two for these definitions).

In the following chapters, we consider chaos in a family of dendrites and certain self-similar arcwise-connected spaces which contain infinitely many simple closed curves, which we describe as "circular." We make the following conclusions regarding distributional chaos on Julia sets of $f_c(z)$:

- if f_c has a Julia set which is a dendrite, then f_c exhibits transitive distributional chaos, and

- if f_c has an attracting or parabolic periodic point, then provided the kneading sequence of c is Γ-acceptable (see Chapter Four), f_c exhibits transitive distributional chaos.

In Chapter Five, we consider other forms of chaos, including Devaney chaos and ω-chaos. We show that under appropriate assumptions, f_c also exhibits these forms of chaos. Finally, in Chapter Six, we discuss future work, including properties of branch points in $\mathcal{D}_\tau$. We show that the set of branch points is dense in the dendrite $\mathcal{D}_\tau$ (unless the set of branch points is empty, which only occurs when $\mathcal{D}_\tau$ is an arc).

CHAPTER TWO

Li-Yorke Scrambled Sets via Thickness

2.1 Introduction

In this chapter, which comes from the author's work in [4], we answer a question of Fu and You in [38]. Let $S \subseteq X$. The relation γ is called an *orbit invariant* on S for f if the following conditions are met:

(a) $\gamma : \bigcup_{n=0}^{\infty} f^n(S) \to (0,1)$ is a function,

(b) $\gamma|_S$ is injective, and

(c) $\gamma(f^n(x)) = \gamma(x)$ for all $x \in S$ and for all $n \geq 0$.

In other words, γ gives the same value to all the elements of the same orbit but gives different values to elements from different orbits.

Let $A_m = \{0, ..., m-1\}$. For the rest of this dissertation, σ will represent the one-sided shift map (that is, if $x = x_0x_1x_2...$, then $(\sigma(x))_i = x_{i+1}$ for all $i \in \omega$). A *symbolic dynamical system* is a dynamical system (X, σ) in which X is a space of infinite sequences with symbols drawn from A_m. In [38], Fu and You consider what we call the *agreement set* for a pair of points in a symbolic dynamical system. Let $N(x, y, n) = \{i : 0 \leq i \leq n, x_i = y_i\}$. Then the *agreement set* for x and y is the set of integers

$$N(x, y) = \bigcup_{n \in \mathbb{N}} N(x, y, n).$$

Also, let

$$\eta(x, y) = \limsup_{n \to \infty} \frac{1}{n} |N(x, y, n)|.$$

Fu and You prove that if S is a set with an orbit invariant and if $\eta(x, y) = 1$ for all $x \neq y \in S$, then the set S is a Li-Yorke scrambled set, [38, Theorem 3.5]. In the passage following this theorem they conjecture that the condition can be weakened to “there is some $0 \leq \delta < 1$ such that $\eta(x, y) > \delta$ for all $x, y \in S$.” It is not difficult to see that the condition $\eta(x, y) = 1$ in the theorem is used only to guarantee that x and y satisfy $\liminf_{n\to\infty} \rho(f^n(x), f^n(y)) = 0$ of the definition of Li-Yorke scrambled pair. So their conjecture reduces to the following: “Suppose that there is some $0 < \delta < 1$ such that for all $x \neq y \in S$ we have $\delta \leq \eta(x, y) < 1$. Then $\liminf\limits_{n\to\infty} \rho(f^n(x), f^n(y)) = 0$.” Then, in the last section of the paper they ask whether it is possible to state necessary and sufficient conditions for S to be a scrambled set. In this chapter, we give a counter-example to the conjecture of Fu and You and give a condition on $N(x, y)$ that guarantees S is a Li-Yorke scrambled set.

We begin by showing that, for every $0 < \delta < 1$ there is a pair (x, y) with $\delta \leq \eta(x, y) < 1$ such that x and y do not satisfy $\liminf\limits_{n\to\infty} \rho(f^n(x), f^n(y)) = 0$. Thus the conjecture of Fu and You is false. We also construct a pair (x, y) with $\eta(x, y) = 0$ such that, nevertheless, (x, y) is a Li-Yorke chaotic pair. So unless $\eta(x, y) = 1$ there is little that the density of the agreement set indicates for being chaotic. We then give an answer to the question of Fu and You by showing that S is a scrambled set if, and only if, for every $x \neq y \in S$, $N(x, y)$ is *thick* but not *cofinal* in $\mathbb{N}$ (these terms are defined in Section 2.3 below). We include the proof of this result, which is straightforward, for completeness.

2.2 Preliminaries and Examples

Let $A_N = \{0, 1, ..., N-1\}$ with metric d defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Let $\Sigma(A_N) = \prod_{n\in\mathbb{N}} A_N$, which with the Tychonoff topology is compact. We use the metric on $\Sigma(A_N)$ given by

$$\rho(x, y) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}.$$

We argue in Example 1 that the condition $\eta(x,y)=1$ for all $x,y\in S$ cannot be weakened to $\delta\le\eta(x,y)<1$ for all $x,y\in S$ (where $0<\delta<1$). Then in Example 2 we consider a case where $\eta(x,y)=0$ for all $x,y\in S$ and yet S is scrambled.

Example 2.1. Let $\delta\in(0,1)$. Choose $N\in\mathbb{N}$ such that $\frac{1}{N}\le 1-\delta$. Let a be such that $a_n=0$ for all $n\in\mathbb{N}\cup\{0\}$ and let b be such that $b_n=1$ if n is a positive multiple of N and $b_n=0$ otherwise. Let $S=\{a,b\}$. Let $\gamma(a)=\frac{1}{2}$ and $\gamma(b)=\gamma(\sigma(b))=...=\gamma(\sigma^N(b))=\frac{3}{4}$. Then γ is an orbit invariant on S for σ.

It is clear that S is not scrambled: for any $n\in\mathbb{N}$, $\sigma^n(a)$ and $\sigma^n(b)$ differ somewhere in the first N symbols and hence $\liminf\limits_{n\to\infty}\rho(\sigma^n(a),\sigma^n(b))>0$. Yet, it is easy to show that $\eta(x,y)=\frac{N-1}{N}$, so $\delta\le\eta(x,y)<1$. Thus, for any $\delta<1$, we can find a set S and an orbit invariant γ on S such that $\delta\le\eta(x,y)<1$ for all $x,y\in S$, and yet S is not scrambled.

Example 2.2. Let $a=0^\infty$ and $b=010^21^20^31^5...0^i1^{2^i-i}...$ Because a and b have arbitrarily long segments of agreement infinitely often, $\liminf\limits_{n\to\infty}\rho(\sigma^n(a),\sigma^n(b))=0$. Also, given any $N\in\mathbb{N}$, there exists $n>N$ such that $a_n\ne b_n$, so $\limsup\limits_{n\to\infty}\rho(\sigma^n(a),\sigma^n(b))>0$. Hence $S=\{a,b\}$ forms a scrambled set.

Now for each integer $n\ge 2$, there exists some $m\in\mathbb{N}$ such that $2^1+...+2^{m-1}\le n<2^1+...+2^m$. Hence, in the first $2^1+...+2^m$ symbols of b, we see that b agrees with a for exactly $1+...+m$ symbols. So

$$\frac{|N(a,b,n)|}{n}\le\frac{1+...+m}{n}\le\frac{m(m+1)/2}{2^{m-1}}=\frac{m(m+1)}{2^m}.$$

Hence $\eta(a,b)=0$. Although S is a scrambled set, $\eta(x,y)=0$ for all $x,y\in S$.

We conclude that, generally speaking, η is not in helpful in detecting scrambled sets (the case $\eta(x,y)=1$ being an exception).

2.3 An Alternate Condition: Thickness

A set $K\subseteq\mathbb{N}$ is *syndetic* if there is some $k\in\mathbb{N}$ such that for every $p\in\mathbb{N}$ there is some $m\in K\cap[p,p+k]$. In other words, if K is syndetic, the gaps in $\mathbb{N}$

between elements of K are bounded by k. A set $T \subseteq \mathbb{N}$ is said to be *thick* if for every syndetic set K, $T \cap K \neq \emptyset$. We say a set $A \subseteq \mathbb{N}$ is *cofinal* in $\mathbb{N}$ provided there is a $k \in \mathbb{N}$ such that for all $n > k$, $n \in A$. See [15] and [16] for further discussion of these notions.

Lemma 2.3. *Let $T \subseteq \mathbb{N}$. Then T is thick if, and only if, for all $n \in \mathbb{N}$ there exists a $p \in \mathbb{N}$ such that $[p, p+n] \subseteq T$.*

Proof. Suppose not. Then there is some $n \in \mathbb{N}$ such that for all $p \in \mathbb{N}$, $[p, p+n] \nsubseteq T$. Let K be the set of points $\{k_i\}_{i\in\mathbb{N}}$ such that $k_i \in [i, i+n] - T$. Then $K \cap T = \emptyset$ but by construction K is syndetic.

Next suppose that for all $n \in \mathbb{N}$ there is some $p \in \mathbb{N}$ such that $[p, p+n] \subseteq T$. Let K be syndetic. Let $k \in \mathbb{N}$ such that $[q, q+k] \cap K \neq \emptyset$ for all $q \in \mathbb{N}$. Then it follows that $K \cap T \neq \emptyset$. $\square$

Lemma 2.4. *If T is a thick set and there exist $n, p \in \mathbb{N}$ such that $[p, p+n] \subset T$ then there exists $p' > p$ such that $[p', p'+n] \subset T$.*

Proof. Note $n+p \in \mathbb{N}$ so there exists $\hat{p}$ such that $\{\hat{p}, ..., \hat{p}+p+n\} \subset T$. If $\hat{p} > p$, then $\{\hat{p}, ..., \hat{p}+n\} \subset T$. If $\hat{p} \leq p$, then $\{p, ..., \hat{p}+p+n\} \subset T$, so $\{p+1, ..., p+1+n\} \subset T$. $\square$

Theorem 2.5. *$N(x, y)$ is thick if, and only if,* $\liminf_{n\to\infty} \rho(\sigma^n(x), \sigma^n(y)) = 0$.

Proof. Suppose $N(x, y)$ is thick. Let $\epsilon > 0$, and choose $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$. Since $N(x, y)$ is thick, there exists $p_0 \in \mathbb{N}$ such that $[p_0, p_0+n] \subset N(x, y)$. Thus $x_{p_0}...x_{p_0+n} = y_{p_0}...y_{p_0+n}$, so $\rho(\sigma^{p_0}(x), \sigma^{p_0}(y)) \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n} < \epsilon$. Continuing inductively, for each p_i there exists a $p_{i+1} > p_i$ such that $[p_{i+1}, p_{i+1}+n] \subset N(x, y)$, so $\rho(\sigma^{p_{i+1}}(x), \sigma^{p_{i+1}}(y)) < \epsilon$. So there exists an increasing sequence $\{p_i\}_{i=0}^{\infty}$ such that $\rho(\sigma^{p_i}(x), \sigma^{p_i}(y)) < \epsilon$ for each $i \in \mathbb{N}$. Hence $\liminf_{n\to\infty} \rho(\sigma^n(x), \sigma^n(y)) = 0$.

Now suppose $\liminf_{n\to\infty} \rho(\sigma^n(x), \sigma^n(y)) = 0$. Then for all $n \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that $x_p...x_{p+n} = y_p...y_{p+n}$. (Otherwise, there exists m such that for all

p, $x_p...x_{m+p} \neq y_p...y_{m+p}$, so $\rho(\sigma^p(x), \sigma^p(y)) \geq \frac{1}{2^m}$ for all p.) So for all n there exists a p such that $\{p, ..., p+n\} \subset N(x, y)$, and thus $N(x, y)$ is thick. $\square$

The following is part of [38, Theorem 3.5].

Theorem 2.6. *(See [38, Theorem 3.5]). Let $S \subseteq \Sigma(N)$ with $|S| \geq 2$ and let $x, y \in S$. If there exists an orbit invariant γ on S for σ, then*

$$\limsup_{n\to\infty} \rho(\sigma^n(x), \sigma^n(y)) > 0.$$

This immediately leads to the following result.

Corollary 2.7. *Let $S \subseteq \Sigma(N)$ with $|S| \geq 2$. If there exists an orbit invariant γ on S for σ, and $N(x, y)$ is thick for all $x, y \in S$, then S is a Li-Yorke scrambled set for σ.*

The following is a modified version of [38, Corollary 3.6].

Corollary 2.8. *Let $S \subseteq \Sigma(N)$. If $N(x, y)$ is thick for all $x, y \in S$ and there exists a surjective orbit invariant γ on S, then S is an uncountable scrambled set for σ.*

Proof. Note

$$\gamma\left(\bigcup_{n=0}^{\infty} \sigma^n(S)\right) = \bigcup_{n=0}^{\infty} \gamma(\sigma^n(S)) = \bigcup_{n=0}^{\infty} \gamma(S) = \gamma(S)$$

Now $\gamma : \bigcup_{n=0}^{\infty} \sigma^n(S) \to (0, 1)$ is surjective, meaning $|S| = |(0, 1)|$. So S is uncountable; by Corollary 2.7, S is a Li-Yorke scrambled set. $\square$

A perhaps simpler notion than the existence of an orbit invariant is that of being cofinal in $\mathbb{N}$.

Theorem 2.9. *Let $S \subseteq \Sigma(N)$. Then S is a Li-Yorke scrambled set if, and only if, for all $x, y \in S$ with $x \neq y$ we have that $N(x, y)$ is thick and not cofinal in $\mathbb{N}$.*

Proof. By Theorem 2.5, $\liminf_{n\to\infty} \rho(\sigma^n(x), \sigma^n(y)) = 0$ if and only if $N(x,y)$ is thick. For the limsup condition notice that

$$\limsup_{n\to\infty} \rho(\sigma^n(x), \sigma^n(y)) > 0$$

is equivalent to the existence of an increasing sequence $\{j_i\}_{i\in\mathbb{N}}$ with $x_{j_i} \neq y_{j_i}$. This is true if, and only if, there is no $k \in \mathbb{N}$ with $\sigma^k(x) = \sigma^k(y)$. This is equivalent to $N(x,y)$ not being cofinal in $\mathbb{N}$. $\square$

CHAPTER THREE

Distributional Chaos in Dendritic Julia Sets

3.1 Introduction

Portions of this chapter have been published previously in [3]. In [32, Theorem 1.1], Downarowicz proved that if a topological dynamical system (X, f) has positive topological entropy, then the system possesses an uncountable DC2 scrambled set. By [81], citing [58], maps of the form $f_c(z) = z^2 + c$ (where $c \in \mathbb{C}$) have topological entropy $\log(2)$ on their Julia sets, independent of the parameter c. It is thus clear that every quadratic Julia set dendrite map exhibits DC2, but in this section, we show that many quadratic Julia set maps in fact exhibit transitive DC1.

We take a symbolic approach to this problem. We consider a family of abstract Julia sets introduced by Stewart Baldwin, [6], which contains copies of all the dendritic Julia sets of complex quadratic polynomials. Each such Julia set is a shift-invariant subspace of a non-Hausdorff itinerary space which is called Λ. The itinerary topology on Λ mimics in many ways the itinerary topology generated by a unimodal map on I.

Section 3.2 introduces abstract Hubbard trees, which share many properties with the dendrites $\mathcal{D}_\tau$ we discuss in this chapter. Section 3.3 discusses situations in which dendritic Julia sets arise, and Section 3.4 explores the definitions for the topology on Λ and for a description of the abstract Julia sets in Λ that contain conjugate copies of every quadratic Julia set which is a dendrite. In Section 3.5, we use this symbolic representation to prove that these systems have transitive distributional chaos of type 1.

3.2 Abstract Hubbard Trees

An *abstract Hubbard tree* is a tree T with a map $f : T \to T$ and a special point called the *critical point*, t, such that all the following conditions hold:

- $f : T \to T$ is continuous and surjective,
- for any point $x \in T$, we have $|f^{-1}(x)| \leq 2$,
- at every point $x \in T$ such that $x \neq t$, the map f is a local homeomorphism onto its image,
- the point t is periodic or preperiodic, but not fixed,
- if x and y are distinct points that are either branch points or points in the orbit of t, then there is an $n \geq 0$ such that $f^n([x, y])$ contains t, and
- all endpoints of T are in the orbit of t.

The trees now known as Hubbard trees were introduced by Douady and Hubbard, [31], and are always embedded in $\mathbb{C}$, in contrast to the abstract Hubbard trees defined above, which might not be realized by any quadratic polynomial, [24]. Bruin and Schleicher give a condition in [24] that identifies which kneading sequences correspond to quadratic polynomials. As we shall see, abstract Hubbard trees have many properties in common with the dendrites $\mathcal{D}_\tau$, to be defined in Section 3.4.

3.3 Dendritic Julia Sets

For a polynomial function, dendritic Julia sets occur, for example, when every finite critical point of the polynomial is pre-periodic but not periodic, [14, page 258], which we will call a *strictly pre-periodic point*. The only critical point of $f_c(z) = z^2 + c$ is 0, so $f_c(z) = z^2 + c$ has a dendritic Julia set when 0 is strictly pre-periodic under f_c, [29, page 294], in which case c is called a *Misiurewicz point*, [80]. (Equivalently, a connected Julia set will be a dendrite provided that $J_c = K_c$, [21].) For instance, 0 is strictly pre-periodic under $f(z) = z^2 - i$, as

$$\begin{aligned} f(0) &= -i \\ f^2(0) &= (-i)^2 - i = -1 - i \\ f^3(0) &= (-1-i)^2 - i = i \\ f^4(0) &= i^2 - i = -1 - i \\ f^5(0) &= i. \end{aligned}$$

Thus, $f(z) = z^2 - i$ has a dendritic Julia set, which is shown in Figure 3.1. Similarly,

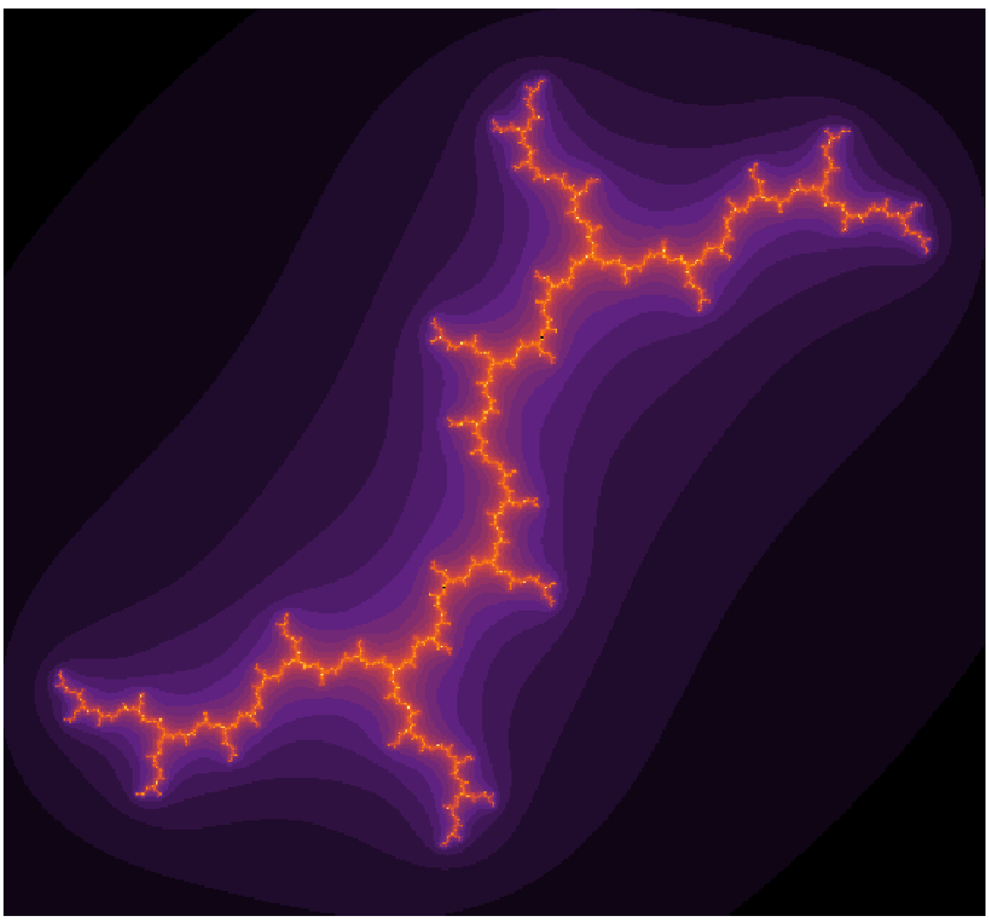

Figure 3.1. The Julia set of $f(z) = z^2 - i$

$f(z) = z^2 + i$ and $f(z) = z^2 - 2$ have dendritic Julia sets, though the latter is just a horizontal line.

According to [80, page 589], citing [65], Misiurewicz points form a countable dense subset of the boundary of the Mandelbrot set, so examples are not hard to find. For example, suppose $f(z) = z^2 + c$ is such that 0 has pre-period 2 and period 3, meaning $f^2(0) = f^5(0)$. Using technology, we can estimate the solutions, one of which is $c \approx -1.2392255553895674976657360 8 - 0.41260218160200354540879140i$. In fact, c is a Misiurewicz point, as

$$\begin{aligned}
f(0) &\approx -1.239 - 0.413i \\
f^2(0) &\approx 0.126 + 0.610i \\
f^3(0) &\approx -1.595 - 0.259i \\
f^4(0) &\approx 1.239 + 0.413i \\
f^5(0) &\approx 0.126 + 0.610i.
\end{aligned}$$

Hence f has a dendritic Julia set, which is pictured in Figure 3.2.

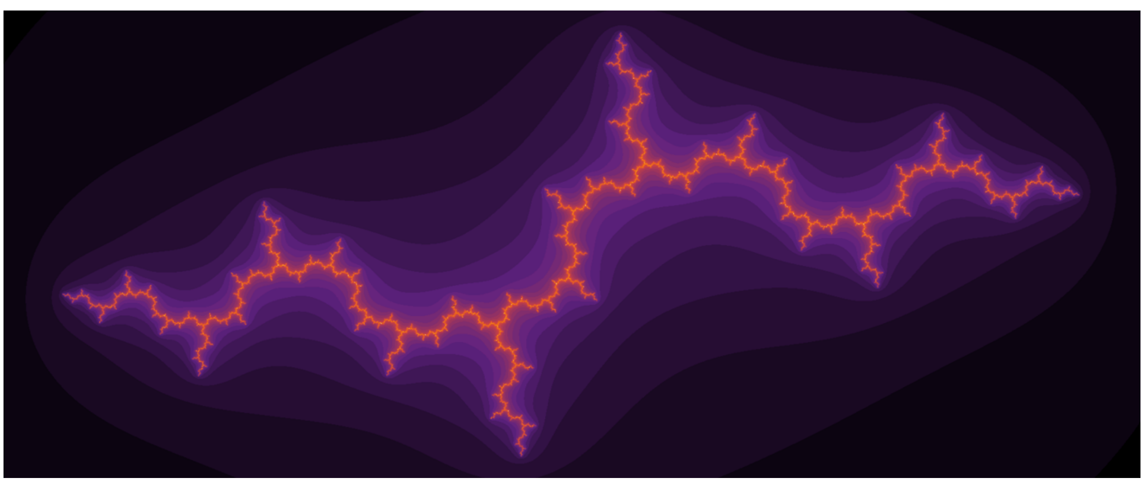

Figure 3.2. The Julia set of $f(z) = z^2 + c$ where $c \approx -1.239 - 0.413i$

Another interesting example is given in [14], where c is the nonzero solution of $(c^2+c)^2+c = \frac{1}{2}\left(1 - \sqrt{1-4c}\right)$, so that $c \approx -1.5436$. The left side of the equation is

$f^3(0)$, and the right side of the equation is one of the fixed points of $f(z) = z^2 + c$. In other words, we are supposing that under $f(z) = z^2 + c$, $z = 0$ is a pre-fixed point with pre-period 3. The Julia set of f_c is pictured in Figure 3.3. This map and its

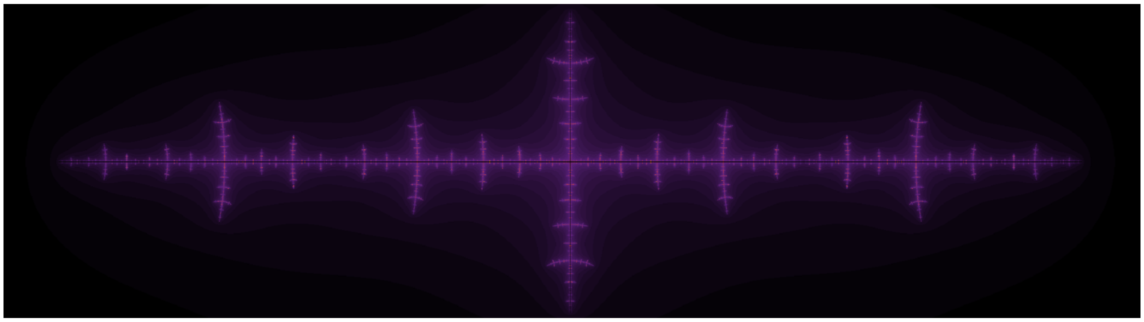

Figure 3.3. The Julia set of $f(z) = z^2 + c$ where $c \approx -1.5436$

Julia set are thoroughly explored in [29, page 293].

3.4 The Set $\mathcal{D}_\tau$

In this section we introduce the symbolic representation of Λ and the dendritic Julia sets due to Baldwin, [6, 7]. Let σ represent the usual one-sided shift map. Let Λ be the product space on $\{0, 1, *\}^\omega$, where each factor space $\{0, 1, *\}$ has topology induced by the basis $\{\{0\}, \{1\}, \{0, 1, *\}\}$. We then see that Λ is a non-Hausdorff space since if two points in Λ disagree only where one has a $*$, the points cannot be separated with open sets. This is consistent with letting 0 and 1 represent open regions, S_0 and S_1, in the dendrite while $*$ represents the unique critical point in $\overline{S_0} \cap \overline{S_1}$. Note that S_0 and S_1 are not necessarily connected.

Definition 3.1. A sequence $\tau \in \Lambda$ is called Λ-*acceptable* if and only if

1) For all $n \in \omega$, we have $\tau_n = *$ if and only if $\sigma^{n+1}(\tau) = \tau$, and

2) For all $n \in \omega$ such that $\sigma^n(\tau) \neq \tau$ there exists $m \in \omega$ such that $* \neq \tau_{m+n} \neq \tau_m \neq *$.

Condition 1) means that either τ is periodic, where $*$ signals the end of the period, or τ is not periodic and $\tau_n \neq *$ for all $n \in \omega$ (though possibly τ is eventually periodic). Condition 2) means that if $\sigma^n(\tau) \neq \tau$, then $\sigma^n(\tau)$ and τ can be separated by open sets–in other words, they differ in a place where neither is $*$. Λ-acceptable sequences are the possible kneading sequences.

Definition 3.2. If τ is Λ-acceptable, then $x \in \Lambda$ is (Λ, τ)-*consistent* if and only if for all $n \in \omega$, $x_n = *$ implies $\sigma^{n+1}(x) = \tau$.

Note that $\alpha \in \{0, 1\}^\omega$ is (Λ, τ)-consistent for any Λ-acceptable τ.

Definition 3.3. The point $x \in \Lambda$ is (Λ, τ)-*admissible* if and only if it is (Λ, τ)-consistent and for all $n \in \omega$ such that $\sigma^n(x) \neq *\tau$, there exists $m > 0$ such that $* \neq x_{m+n} \neq \tau_{m-1} \neq *$ (that is, there is a position where $\sigma^n(x)$ and $*\tau$ differ and neither is a $*$). If τ is Λ-acceptable, let $\mathcal{D}_\tau = \{x \in \Lambda : x \text{ is } (\Lambda, \tau)\text{-admissible}\}$.

The following three theorems are due to Baldwin:

Theorem 3.4. *[7, Theorem 2.4] Let τ be λ-acceptable. Then*

- *$\mathcal{D}_\tau$ is a dendrite.*
- *$\sigma(\mathcal{D}_\tau) = \mathcal{D}_\tau$.*
- *The only point at which $\mathcal{D}_\tau$ is not locally one-to-one is $*\tau$.*
- *D_τ is self-similar in the following sense: if we let $S_i = \{\alpha \in \mathcal{D}_\tau : \alpha_0 = i\}$ for $i = 0, 1$, then $\sigma|_{\overline{S_i}}$ is a homeomorphism from $\overline{S_i} = S_i \cup \{*\tau\}$ to $\mathcal{D}_\tau$.*

Theorem 3.5. *[6, Theorem 2.5] Let $f_c(z) = z^2 + c$. If J_c is a dendrite, then there is a Λ-acceptable τ such that $f_c|_{J_c}$ is conjugate to $\sigma|_{\mathcal{D}_\tau}$.*

Finally, σ is exact on $\mathcal{D}_\tau$.

Theorem 3.6. *[6, Theorem 4.11] If $U \subset D_\tau$ is open, then $\sigma^n(U) = D_\tau$ for some n.*

The dendrite $\mathcal{D}_\tau$ therefore has many of the properties of an abstract Hubbard tree, the exceptions being that $\mathcal{D}_\tau$ is of course typically not a tree, τ might be neither periodic nor eventually periodic, and the endpoints of T might not be in the orbit of τ (except in the case of the arc, $\mathcal{D}_\tau$ has infinitely many endpoints, whereas if τ is periodic, the orbit of τ is of course finite).

Definition 3.7. Let $\mathcal{P}_0 = \{*\tau\}$. For $n \in \mathbb{N}$, let $\mathcal{P}_n = \{a_0 ... a_{n-1} * \tau : a_i \in \{0,1\}\}$. We then define $\mathcal{P}_\omega = \bigcup_{n\in\omega} \mathcal{P}_n$.

Definition 3.8. We define $x{\restriction_n} = x_0 ... x_n$. If, for all $i \leq n$, we have $x_i = y_i$, $x_i = *$, or $y_i = *$, then we say $x{\restriction_n}$ and $y{\restriction_n}$ are *equivalent* and write $x{\restriction_n} \approx y{\restriction_n}$. We write $x{\restriction_n} \simeq y{\restriction_n}$ if either $x{\restriction_n} = y{\restriction_n}$ or there exists a point $z \in \mathcal{P}_\omega$ such that $x{\restriction_n} \approx z{\restriction_n} \approx y{\restriction_n}$.

The following lemma from [13] will be useful in proving the main theorem of this chapter.

Lemma 3.9. *(See [13]) (1) Let $\epsilon > 0$. Then there is a natural number N_ϵ such that if $x, y \in D_\tau$ with $x{\restriction_{N_\epsilon}} \simeq y{\restriction_{N_\epsilon}}$ then $d(x,y) < \epsilon$.*
(2) Let $N \in \mathbb{N}$. Then there is a positive number δ_N such that if $x, y \in D_\tau$ with $d(x,y) < \delta_N$ then $x{\restriction_N} \simeq y{\restriction_N}$.

3.5 Distributional Chaos in $\mathcal{D}_\tau$

In this section we modify the approach of Oprocha in [70, Example 9] to find a transitive distributionally scrambled set in $\mathcal{D}_\tau$. This section is an improvement of the author's own work in [3].

For $x \in \{0,1\}^n$, let $|x|$ denote the length of x, so that $|x_0 ... x_{n-1}| = n$. Let $r_0 = 1$, $m_n = \sum_{i=0}^{n} r_n$, and $r_{n+1} = 2^{n+1} m_n$. Define $f : \omega \to \bigcup_{n\in\omega} \{0,1\}^n$ by letting $f(0)$ be the empty string and

$$f(1) = 0, f(2) = 1, f(3) = 00, f(4) = 10, ...$$

Define $u(0,n) = 0^{r_n - |f(n)|}$ and $u(1,n) = 1^{r_n - |f(n)|}$. Let

$$H = \{z_0 0 z_0 z_1 0 z_0 z_1 z_2 0 ... : z \in \{0,1\}^\omega\}$$

and

$$D = \{u(\alpha_0, 0) f(0) u(\alpha_1, 1) f(1) u(\alpha_2, 2) f(2) ... : \alpha \in H\}$$

and

$$E = \bigcup_{n \in \mathbb{N}} \sigma^n(D).$$

Distinct points in H will agree infinitely often and disagree infinitely often. The points of D are created from points of H by "stretching" them (by repeating the same symbol many times) and then giving them a dense orbit (via f). Also, observe that no point of D or E is periodic or eventually periodic.

Lemma 3.10. *The set E is uncountable.*

Proof. Certainly $\{0,1\}^\omega$ is uncountable, and since the function $g_1 : \{0,1\}^\omega \to H$ defined by $g_1(z) = z_0 0 z_0 z_1 0 z_0 z_1 z_2 0 ...$ is an injection, we see that H is uncountable. Also, the function $g_2 : H \to D$ defined by $g_2(\alpha) = u(\alpha_0, 0) f(0) u(\alpha_1, 1) f(1) u(\alpha_2, 2) ...$ is an injection, meaning D is uncountable. Since $D \subset E$, we see that E is uncountable, as well. $\square$

Lemma 3.11. *If $\tau = *^\infty$, then $\mathcal{D}_\tau = \{\tau\}$.*

Proof. Let $\tau = *^\infty$. Note that $*^\infty$ is Λ-acceptable. Suppose x is (Λ, τ)-admissible. If $x_0 = *$, then since x is (Λ, τ)-consistent, $\sigma(x) = \tau = *^\infty$, implying $x = *^\infty$. If $x_0 \neq *$, then since $\sigma^0(x) = x \neq *\tau$ and x is (Λ, τ)-admissible, there must be a position where the sequences x and $*\tau$ differ and neither is $*$, but this is impossible, as $\tau_n = *$ for all $n \in \omega$. Hence, no such x exists and $\mathcal{D}_\tau = \{*^\infty\}$. $\square$

As a consequence of the above lemma, for the rest of this section, we will exclude the possibility that $\tau = *^\infty$. Note that if $\tau \approx 0^\infty$ (with $\tau \neq *^\infty$) then to satisfy condition 1) of Λ-acceptability, $\tau = (0^k *)^\infty$ (where $k > 0$), but this situation does not satisfy condition 2). Hence, if $\tau \neq *^\infty$ is Λ-acceptable, then $\tau \not\approx 0^\infty$.

Lemma 3.12. *Let $A \subseteq \{0,1\}^\omega$ be such that every $a \in A$ contains arbitrarily long sequences of 0. Suppose $\tau \in \Lambda$ is Λ-acceptable with $\tau \neq *^\infty$ and that τ is periodic. Then $A \subset \mathcal{D}_\tau$.*

Proof. Let $\tau \in \Lambda$ be as in the hypothesis. Now, τ has some period p and $\tau \not\approx 0^\infty$, so $\tau_M = 1$ for some $0 \leq M < p$, which implies $\tau_{M+kp} = 1$ for all $k \in \mathbb{N}$. Let $x \in A$. Then x does not contain $*$ anywhere and hence is (Λ, τ)-consistent. Since x contains arbitrarily long sequences of 0, for any $m \in \omega$, $\sigma^m(x)$ contains a block of $p+1$ zeros, meaning the sequences $*\tau$ and $\sigma^m(x)$ differ in a place where neither is $*$, and hence x is (Λ, τ)-admissible. Thus, $A \subset \mathcal{D}_\tau$. $\square$

In particular, by the above lemma, we see that if $\tau \neq *^\infty$ is Λ-acceptable and periodic, then $E \subseteq \mathcal{D}_\tau$. We now define the set P, which is a set of points not in $\mathcal{D}_\tau$.

Definition 3.13. Let $P = \displaystyle\bigcup_{k \in \omega} \{\beta\tau : \beta \in \{0,1\}^k\}$.

The next lemma shows that excluding P from any set of binary points A is enough to make the rest of A admissible.

Lemma 3.14. *Let $A \subseteq \{0,1\}^\omega$ be uncountable and suppose $\tau \in \Lambda$ is Λ-acceptable. If $\sigma^n(\tau) \neq \tau$ for all $n \in \omega$, then $A - P$ is uncountable and $A - P \subset \mathcal{D}_\tau$.*

Proof. Recall that every point of $\{0,1\}^\omega$ is (Λ, τ)-consistent. We wish to find, and exclude from A, any points which are not (Λ, τ)-admissible. Suppose $x \in A$ such that for some n, $\sigma^n(x) \neq *\tau$ but $\sigma^n(x) \approx *\tau$ (that is, the sequences only differ where one is a $*$). Since $x \in A \subseteq \{0,1\}^\omega$, for all $i \in \omega$ we have $x_i \neq *$. Note that for all $i \in \omega$, $\tau_i \neq *$ (otherwise τ would be periodic). Hence $\sigma^n(x) \approx *\tau$ implies $\sigma^{n+1}(x) = \tau$. So we exclude from A the (countably many) pre-images of τ. Thus, $A - P \subset \mathcal{D}_\tau$. As A is uncountable and P is countable, $A - P$ is uncountable. $\square$

In particular, if $\tau \neq *^\infty$ is Λ-acceptable but not periodic, we apply the preceding lemma to E and conclude that $E - P$ is uncountable and $E - P \subseteq \mathcal{D}_\tau$.

Observe that E is invariant under σ. It follows that $E - P$ is also invariant under σ (if $x \in E - P$ but $\sigma(x) \notin E - P$, so that $\sigma(x) \in P$, then $\sigma(x)$ has form $b_1 ... b_n \tau$ for $b_i \in \{0, 1\}$. Thus x has the form $b_0 b_1 ... b_n \tau$, meaning $x \in P$, a contradiction). So, for all $n \in \omega$, $\sigma^n(E - P) \subset E - P \subset \mathcal{D}_\tau$; if τ is Λ-acceptable such that $E \subset \mathcal{D}_\tau$, then $\sigma^n(E) \subset E \subset \mathcal{D}_\tau$ for all $n \in \omega$.

Lemma 3.15. *Every (Λ, τ)-admissible point of E has an orbit which is dense in $\mathcal{D}_\tau$.*

Proof. Consider $x \in E \cap \mathcal{D}_\tau$. Let $\epsilon > 0$. There exists $N_\epsilon \in \mathbb{N}$ such that $x\restriction_{N_\epsilon} \simeq y\restriction_{N_\epsilon}$ implies $d(x, y) < \epsilon$. Let $a \in \mathcal{D}_\tau$. If a does not contain $*$, then by the construction of E, there exists $n \in \mathbb{N}$ such that $\sigma^n(x)\restriction_{N_\epsilon} = a\restriction_{N_\epsilon}$. If a does contain $*$, there exists $n \in \mathbb{N}$ such that, for all $0 \le i \le N_\epsilon$, $\sigma^n(x)_i = a_i$ whenever $a_i \neq *$ and $\sigma^n(x)_i = 1$ otherwise. So in either case, there exists $n \in \mathbb{N}$ such that $\sigma^n(x)\restriction_{N_\epsilon} \simeq a\restriction_{N_\epsilon}$, so that $d(\sigma^n(x), a) < \epsilon$ and hence $\sigma^n(x) \in B_\epsilon(a)$, and thus $\mathrm{Orb}^+(x)$ is dense in $\mathcal{D}_\tau$. □

Corollary 3.16. *Suppose τ is Λ-acceptable. The set $E - P$ is dense in $\mathcal{D}_\tau$. Additionally, if τ is periodic, E is dense in $\mathcal{D}_\tau$.*

Proof. Regardless of whether τ is periodic or not, we know $E - P \subset \mathcal{D}_\tau$. Let $x \in E - P$, so that x has a dense orbit in $\mathcal{D}_\tau$. As $\sigma^n(E - P) \subset E - P$ for all $n \in \omega$, $\mathrm{Orb}^+(x) \subset E - P$. Since $\mathrm{Orb}^+(x)$ is dense in $\mathcal{D}_\tau$, so is $E - P$. If τ is periodic, then $E \subset \mathcal{D}_\tau$, allowing us to conclude E is also dense in $\mathcal{D}_\tau$. □

Theorem 3.17. *If $\tau \in \Lambda$ is Λ-acceptable with $\tau \neq *^\infty$, then $\sigma|_{\mathcal{D}_\tau}$ exhibits transitive distributional chaos.*

Proof. If τ is periodic, then let $E' = E$. If τ is non-periodic, let $E' = E - P$. In either case, by the previous lemmas, $E' \subset \mathcal{D}_\tau$ and E' is uncountable.

Let $\epsilon > 0$ and let $x, y \in E'$ be distinct. By Lemma 3.9, there exists $N_\epsilon \in \mathbb{N}$ such that if $x \restriction_{N_\epsilon} \simeq y \restriction_{N_\epsilon}$, then $d(x,y) < \epsilon$. Certainly, then, $x \restriction_{N_\epsilon} = y \restriction_{N_\epsilon}$ implies $d(x,y) < \epsilon$, so

$$\frac{1}{n}\left|\{0 \le i < n : d(\sigma^i(x), \sigma^i(y)) < \epsilon\}\right| \;\ge\; \frac{1}{n}\left|\{0 \le i < n : \sigma^i(x)\restriction_{N_\epsilon} = \sigma^i(y)\restriction_{N_\epsilon}\}\right|.$$

Now, points in E' are generated by shifting points of D, so there exist $a, b \in D$ such that $x = \sigma^s(a)$ and $y = \sigma^t(b)$. Without loss of generality, suppose $s \ge t$ and let $k = s - t$. By the construction of E, there exist some $\alpha, \beta \in H$ such that $a = u(\alpha_0, 0)f(0)u(\alpha_1, 1)f(1)u(\alpha_2, 2)...$ and $b = u(\beta_0, 0)f(0)u(\beta_1, 1)f(1)u(\beta_2, 2)...$, meaning there exists a sequence v_j such that $\alpha_{v_j} = 0 = \beta_{v_j}$. Then $u(\alpha_{v_j}, v_j) = u(\beta_{v_j}, v_j)$ for all j. Recall that $|u(\alpha_{v_j}, v_j)| = r_{v_j} - |f(v_j)|$. Thus, a and b can be expressed as follows:

$$\begin{aligned}
a &= \underbrace{\alpha_0}_{r_0-|f(0)|}\underbrace{}_{|f(0)|}\underbrace{\alpha_1...\alpha_1}_{r_1-|f(1)|}\underbrace{0}_{|f(1)|}\underbrace{\alpha_2...\alpha_2}_{r_2-|f(2)|}\underbrace{1}_{|f(2)|}\underbrace{\alpha_3...\alpha_3}_{r_3-|f(3)|}\underbrace{00}_{|f(3)|}...\\
b &= \underbrace{\beta_0}_{r_0-|f(0)|}\underbrace{}_{|f(0)|}\underbrace{\beta_1...\beta_1}_{r_1-|f(1)|}\underbrace{0}_{|f(1)|}\underbrace{\beta_2...\beta_2}_{r_2-|f(2)|}\underbrace{1}_{|f(2)|}\underbrace{\beta_3...\beta_3}_{r_3-|f(3)|}\underbrace{00}_{|f(3)|}...
\end{aligned}$$

Recall that $m_n = \sum_{i=0}^n r_n$ and note that if we shift a or b by m_n for any n, we have

$$\begin{aligned}
\sigma^{m_n}(a) &= \underbrace{\alpha_{r_{n+1}}...\alpha_{r_{n+1}}}_{r_{n+1}-|f(n+1)|} f(n+1) \underbrace{\alpha_{r_{n+2}}...\alpha_{r_{n+2}}}_{r_{n+2}-|f(n+2)|} f(n+2)...\\
\sigma^{m_n}(b) &= \underbrace{\beta_{r_{n+1}}...\beta_{r_{n+1}}}_{r_{n+1}-|f(n+1)|} f(n+1) \underbrace{\beta_{r_{n+2}}...\beta_{r_{n+2}}}_{r_{n+2}-|f(n+2)|} f(n+2)...
\end{aligned}$$

Recall that $\alpha_{v_j} = \beta_{v_j}$ for all j. Hence, for $m_{v_j-1} \le i < m_{v_j}$, the first symbols of $\sigma^i(a)$ and $\sigma^i(b)$ agree. However, the agreement given by $f(n+1)$ is fragile in that $f(n+1)$ and $\sigma(f(n+1))$ may differ in many places, so we will restrict our attention to those i such that $m_{v_j-1} \le i < m_{v_j} - |f(v_j)|$.

Let N be the smallest natural number such that $r_{v_N-1} - t \ge 0$ and $r_{v_N} - |f(v_N)| - s - N_\epsilon \ge r_{v_N-1} - t$. (Such N exists. Although t, s, and N_ϵ may be very large, they are constant. The expression $|f(v_n)| + s + N_\epsilon$ increases very slowly compared to r_{v_n}, as $|f(v_n)| \leq v_n$ and $r_{v_n} = 2^{v_n} m_{v_n - 1} > v_n - 1)$. In particular, these conditions guarantee that for $j > N$, the following conditions hold:

$$\begin{aligned} m_{v_j - 1} - t &\geq 0 \\ r_{v_j} - |f(v_j)| - k &\geq 0 \\ m_{v_j} - |f(v_j)| - k &\geq 0 \\ m_{v_j} - s - |f(v_j)| - N_\epsilon &\geq 0. \end{aligned}$$

These facts will be useful below.

Now $x = \sigma^s(a)$ and $y = \sigma^t(b)$. The loss of guaranteed agreement, or *offset*, from shifting a more than b is represented by k. Note that $\sigma^i(x)$ and $\sigma^i(y)$ still have an offset of k. Then for all $j > N$, observe that if $i = m_{v_j - 1} - t$, then

$$\begin{aligned} \sigma^i(x) &= \sigma^{m_{v_j-1} - t}(x) \\ &= \sigma^{m_{v_j-1} - t}(\sigma^s(a)) \\ &= \sigma^{m_{v_j-1} + k}(a) \\ \sigma^i(y) &= \sigma^{m_{v_j-1} - t}(y) \\ &= \sigma^{m_{v_j-1} - t}(\sigma^t(b)) \\ &= \sigma^{m_{v_j-1}}(b). \end{aligned}$$

Thus, when $i = m_{v_j - 1} - t$, we have the following picture:

$$\begin{aligned} \sigma^i(x) &= \underbrace{\alpha_{v_j} \ldots\ldots \alpha_{v_j}}_{r_{v_j} - |f(v_j)| - k} f(v_j) \underbrace{\alpha_{v_j+1} \ldots \alpha_{v_j+1}}_{r_{v_j+1} - |f(v_j+1)|} f(v_j + 1) \ldots \\ \sigma^i(y) &= \underbrace{\beta_{v_j} \ldots\ldots\ldots\ldots \beta_{v_j}}_{r_{v_j} - |f(v_j)|} f(v_j) \underbrace{\beta_{v_j+1} \ldots \beta_{v_j+1}}_{r_{v_j+1} - |f(v_j+1)|} f(v_j + 1) \ldots \end{aligned}$$

where $\alpha_{v_j} = \beta_{v_j}$. The first $r_{v_j} - |f(v_j)| - k$ symbols agree, so the first disagreement between $\sigma^i(x)$ and $\sigma^i(y)$ does not appear until we shift $\sigma^i(x)$ and $\sigma^i(y)$ by $r_{v_j} - |f(v_j)| - k$, meaning $i = m_{v_j - 1} - t + r_{v_j} - |f(v_j)| - k = m_{v_j} - s - |f(v_j)|$.

Hence for i such that $m_{v_j-1} - t \leq i < m_{v_j} - s - |f(v_j)|$, the first symbol of $\sigma^i(x)$ and $\sigma^i(y)$ match. Thus, the first N_ϵ symbols of $\sigma^i(x)$ and $\sigma^i(y)$ will match whenever i is such that $m_{v_j-1} - t \leq i < m_{v_j} - s - |f(v_j)| - N_\epsilon$. For all such i, we thus have $\sigma^i(x)\restriction_{N_\epsilon} = \sigma^i(y)\restriction_{N_\epsilon}$. For clarity in the inequality below, let $B_1 = m_{v_j-1} - t$ and $B_2 = m_{v_j} - s - |f(v_j)| - N_\epsilon$. Thus,

$$\begin{aligned}
\frac{\left|\{0 \leq i < m_{v_j} : \sigma^i(x)\restriction_{N_\epsilon} = \sigma^i(y)\restriction_{N_\epsilon}\}\right|}{m_{v_j}} &\geq \frac{|\{B_1 \leq i < B_2 : \sigma^i(x)\restriction_{N_\epsilon} = \sigma^i(y)\restriction_{N_\epsilon}\}|}{m_{v_j}} \\
&= \frac{B_2 - B_1}{m_{v_j}} \\
&= \frac{m_{v_j} - m_{v_j-1} - |f(v_j)| - (s-t) - N_\epsilon}{m_{v_j}} \\
&= \frac{r_{v_j} - |f(v_j)| - k - N_\epsilon}{m_{v_j}} \\
&= \frac{r_{v_j}}{r_{v_j} + m_{v_j-1}} - \frac{|f(v_j)| + k + N_\epsilon}{m_{v_j}} \\
&= \frac{2^{v_j} m_{v_j-1}}{(2^{v_j}+1) m_{v_j-1}} - \frac{|f(v_j)| + k + N_\epsilon}{m_{v_j}} \\
&= \frac{2^{v_j}}{2^{v_j}+1} - \frac{|f(v_j)| + k + N_\epsilon}{m_{v_j}} \longrightarrow 1
\end{aligned}$$

since $\dfrac{|f(v_j)| + k + N_\epsilon}{m_{v_j}} < \dfrac{v_j + k + N_\epsilon}{r_{v_j}} = \dfrac{v_j + k + N_\epsilon}{2^{v_j} m_{v_j-1}} \longrightarrow 0$. So for any $x, y \in E'$, and any $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} |\{0 \leq i < n : d(\sigma^i(x), \sigma^i(y)) < \epsilon\}| = 1.$$

Now, there exists a subsequence μ_j such that $\alpha_{\mu_j} \neq \beta_{\mu_j}$, so that $u(\alpha_{\mu_j}, \mu_j)$ and $u(\beta_{\mu_j}, \mu_j)$ disagree in every place. By Lemma 3.9, for any $M \in \mathbb{N}$, there exists a δ_M such that $d(x,y) < \delta_M$ implies $x\restriction_M \simeq y\restriction_M$. Take $M = 1$ and recall that $x\restriction_1$ has length 2. Thus,

$$\frac{1}{m_{\mu_j}} \left|\{i < m_{\mu_j} : d(\sigma^i(x), \sigma^i(y)) < \delta_1\}\right| \leq \frac{1}{m_{\mu_j}} \left|\{i < m_{\mu_j} : \sigma^i(x)\restriction_1 \simeq \sigma^i(y)\restriction_1\}\right|.$$

Let N be the smallest natural number such that

$$r_{\mu_N - 1} - t \geq 0 \text{ and}$$

$$r_{\mu_N} - |f(\mu_N)| - s \quad \geq \quad r_{\mu_N - 1} - t.$$

Then for all $j > N$, observe that if $i = m_{\mu_j - 1} - t$, then $\sigma^i(x) = \sigma^{m_{\mu_j - 1} + k}(a)$ and $\sigma^i(y) = \sigma^{i+t}(b) = \sigma^{m_{\mu_j - 1}}(b)$. Thus, when $i = m_{\mu_j - 1} - t$, we have the following picture:

$$\begin{aligned}
\sigma^i(x) &= \underbrace{\alpha_{\mu_j} \ldots\ldots \alpha_{\mu_j}}_{r_{\mu_j} - |f(\mu_j)| - k} f(\mu_j) \underbrace{\alpha_{\mu_j+1} \ldots \alpha_{\mu_j+1}}_{r_{\mu_j+1} - |f(\mu_j+1)|} f(\mu_j + 1) \ldots \\
\sigma^i(y) &= \underbrace{\beta_{\mu_j} \ldots\ldots\ldots\ldots \beta_{\mu_j}}_{r_{\mu_j} - |f(\mu_j)|} f(\mu_j) \underbrace{\beta_{\mu_j+1} \ldots \beta_{\mu_j+1}}_{r_{\mu_j+1} - |f(\mu_j+1)|} f(\mu_j + 1) \ldots
\end{aligned}$$

where $\alpha_{\mu_j} \neq \beta_{\mu_j}$ for all j. Hence for i such that $m_{v_j - 1} - t \leq i < m_{\mu_j} - s - |f(\mu_j)|$, the first symbol of $\sigma^i(x)$ and $\sigma^i(y)$ disagree. Hence, for all i such that $m_{\mu_j - 1} - t \leq i < m_{\mu_j} - s - |f(\mu_j)| - 1$, we see that $\sigma^i(x) \restriction_1$ and $\sigma^i(y) \restriction_1$ differ in both symbols. If $\sigma^i(x) \restriction_1 \simeq \sigma^i(y) \restriction_1$, then there exists a pre-critical point $z \in \mathcal{D}_\tau$ such that $\sigma^i(x) \restriction_1 \approx z \approx \sigma^i(y) \restriction_1$. However, this forces $z_0 = z_1 = *$. Because z must be (Λ, τ)-admissible, $\sigma^1(z) = \tau$, so that $\tau_0 = z_1 = *$. Since τ is Λ-acceptable, $\sigma^1(\tau) = \tau$, so that $\tau = *^\infty$, a contradiction. Therefore,

$$\{m_{\mu_j - 1} - t \leq i < m_{\mu_j} - s - |f(\mu_j)| - 1 : \sigma^i(x) \restriction_1 \simeq \sigma^i(y) \restriction_1\} = \emptyset.$$

Hence, letting $B = m_{\mu_j - 1} - t$ for clarity, we have

$$\begin{aligned}
\frac{\left|\{i < m_{\mu_j} : \sigma^i(x) \restriction_1 \simeq \sigma^i(y) \restriction_1\}\right|}{m_{\mu_j}} &\leq \frac{|\{i < B : \sigma^i(x) \restriction_1 \simeq \sigma^i(y) \restriction_1\}|}{m_{\mu_j}} + \frac{s + 1 + |f(\mu_j)|}{m_{\mu_j}} \\
&\leq \frac{B}{m_{\mu_j}} + \frac{s + 1 + \mu_j}{m_{\mu_j}} \\
&= \frac{m_{\mu_j - 1} - t}{m_{\mu_j}} + \frac{s + 1 + \mu_j}{m_{\mu_j}} \\
&= \frac{m_{\mu_j - 1}}{r_{\mu_j} + m_{\mu_j - 1}} + \frac{k + 1 + \mu_j}{m_{\mu_j}} \\
&= \frac{m_{\mu_j - 1}}{2^{\mu_j} m_{\mu_j - 1} + m_{\mu_j - 1}} + \frac{k + 1 + \mu_j}{m_{\mu_j}} \\
&= \frac{1}{2^{\mu_j} + 1} + \frac{k + 1 + \mu_j}{m_{\mu_j}} \longrightarrow 0.
\end{aligned}$$

So for any $x, y \in E'$,

$$\liminf_{n\to\infty} \frac{1}{n}|\{i < n : d(\sigma^i(x), \sigma^i(y)) < \delta_1\}| = 0.$$

This completes the proof that E' is an uncountable DC1 scrambled set for σ, and the chaos is uniform as $F_{xy}(\delta_1) = 0$ for all $x, y \in E'$. As E' is dense in $\mathcal{D}_\tau$, and each $x \in E'$ has an orbit which is dense in $\mathcal{D}_\tau$, we see that $\sigma|_{\mathcal{D}_\tau}$ exhibits transitive distributional chaos. $\square$

Conjugacy preserves transitive distributional chaos between compact metric spaces, [70, Theorem 13], and thus we have the following corollary.

Corollary 3.18. *Consider $f_c = z^2 + c$. If the Julia set of f_c is a dendrite, then f_c exhibits transitive distributional chaos.*

CHAPTER FOUR

Distributional Chaos in Circular Julia Sets

4.1 Introduction

Portions of this chapter have previously been published in [3]. Let z_0 be a periodic point of $f(z) = z^2 + c$ of period n and let $\lambda = (f^n)'(z)$ (which, by application of the chain rule, equals $\prod_{j=0}^{n-1} f'(z_j)$, where $z_j = f^j(z)$). We call λ the *multiplier* or *eigenvalue* of the cycle. The point z_0 is called an *attracting* point of f if $|\lambda| < 1$, a *super-attracting* point if $\lambda = 0$, a *repelling* point if $|\lambda| > 1$, and *indifferent* or *neutral* if $|\lambda| = 1$. The point z_0 is a *parabolic* point if $|\lambda| = 1$ and $|(f^n)'(z)|$ has an argument which is a rational multiple of 2π, [7]. In this chapter, we consider certain quadratic Julia sets generated by maps with an attracting or parabolic periodic point, which are sometimes called "circular" Julia sets because they contain many simple closed curves. These Julia sets are naturally represented in a non-Hausdorff itinerary space called Γ, which has a topology similar to the itinerary topology generated by angle-doubling on the unit circle. We are especially interested in those Julia sets corresponding to values of c which belong to the interior or cusp of a cardioid within the Mandelbrot set, for in this case there exists a conjugacy between the Julia set of $f_c(z) = z^2 + c$ and the space $\mathcal{E}_\tau$, [7, page 215]. The space $\mathcal{E}_\tau$ is defined in Section 4.3.

4.2 "Circular" Julia Sets

In Section 3.3, we saw that if the critical point of $f_c = z^2 + c$ is strictly pre-periodic, then the Julia set of f_c is a dendrite. If instead the critical point tends to an attracting point that is either fixed or periodic, the Julia set of f_c will be the closure of one or infinitely many simple closed curves, [29, pages 292, 294]. If $0 < |c| < \frac{1}{4}$, the Julia set of f_c is a simple closed curve which contains no smooth arcs, though this condition can be relaxed to simply requiring that f_c (with $c \neq 0$) have an attracting fixed point, [29, pages 291, 292]. (If $c = 0$, the Julia set of f_c is the unit circle, which is smooth.) Circular Julia sets can also occur when f_c has a periodic point which is parabolic, [7]. For example, the point $z = 0$ is a periodic

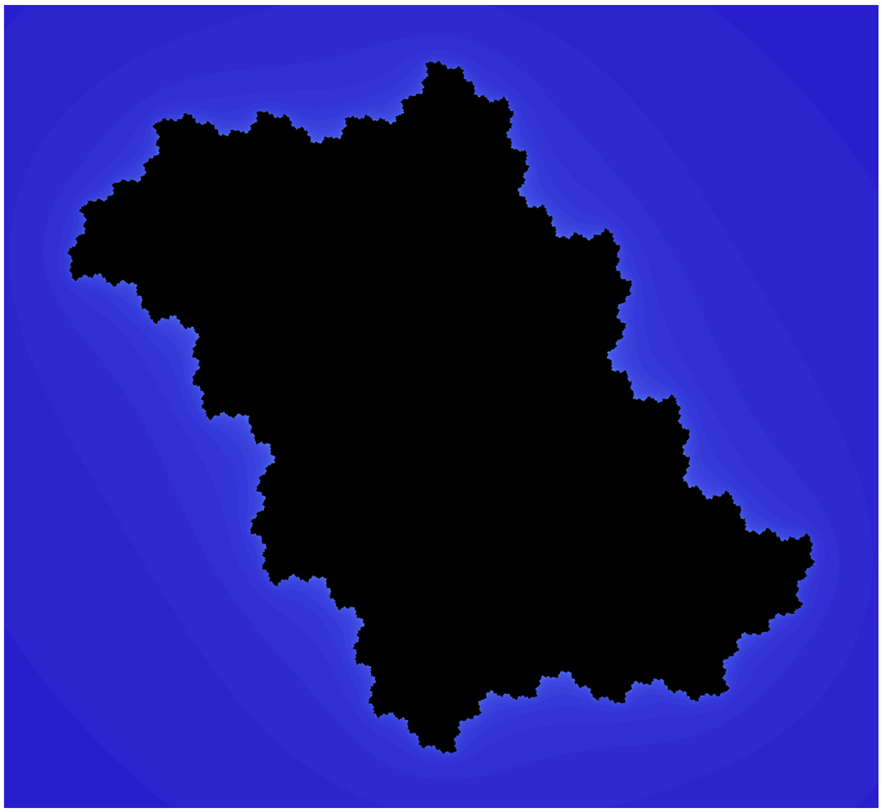

Figure 4.1. The Julia set of $f(z) = z^2 + 0.5i$

point for $f(z) = z^2 - 1$ since

$$f(0) = -1, \quad f(-1) = 0, \quad f(0) = -1, \quad \ldots$$

and $z = 0$ is an attracting point since $(f^n)'(0) = 0$. Hence we expect a circular Julia set. See Figure 4.2. As another example, consider $f(z) = z^2 - \frac{3}{4}$, which has $z = -\frac{1}{2}$ as a fixed point. The multiplier for $z = -\frac{1}{2}$ is thus $\lambda = f'(-\frac{1}{2}) = -1$ which has argument π. Hence, $z = -\frac{1}{2}$ is a parabolic fixed point. See Figure 4.3.

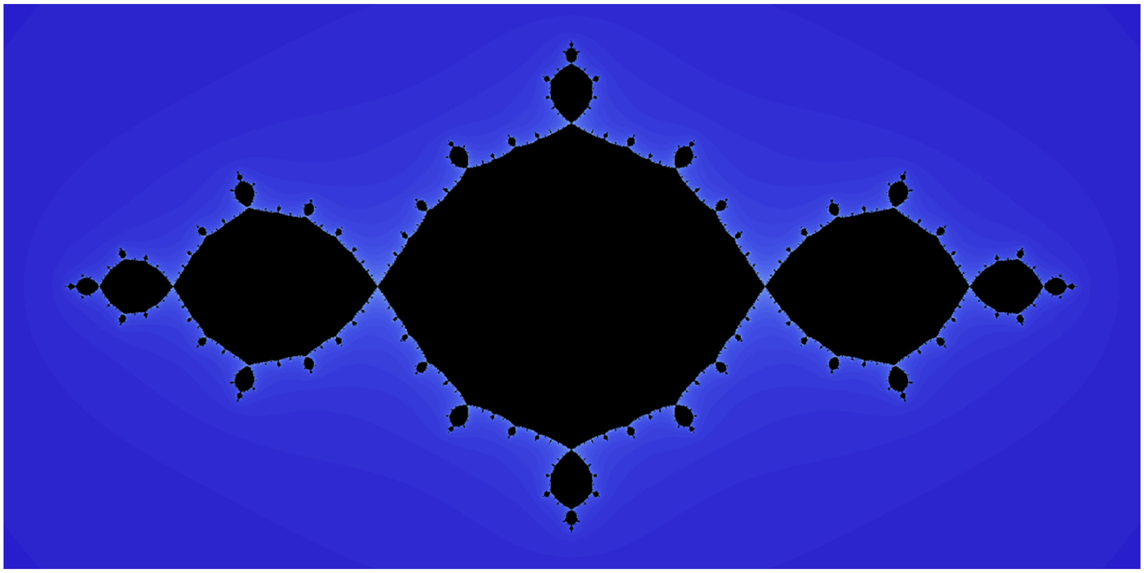

Figure 4.2. The Julia set of $f(z) = z^2 - 1$

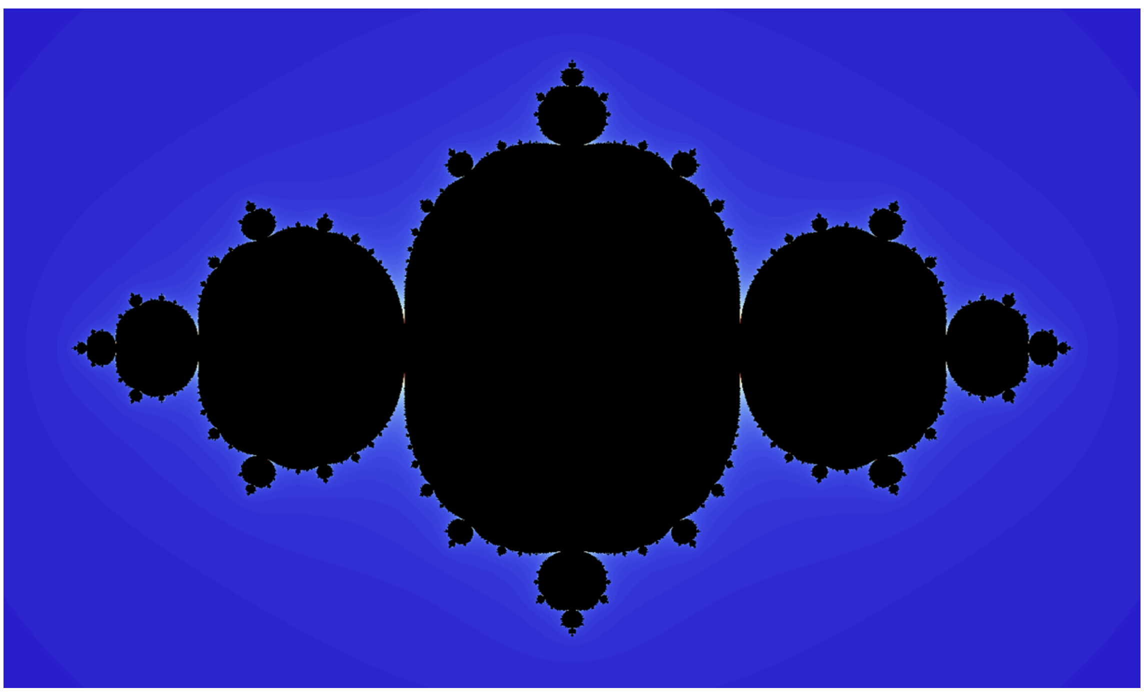

Figure 4.3. The Julia set of $f(z) = z^2 - \frac{3}{4}$

4.3 The Set $\mathcal{E}_\tau$

We begin this section with a description of Baldwin's symbolic representation of certain quadratic Julia sets generated by maps of the form $f_c = z^2 + c$ with an attracting or parabolic periodic point, [7]. In this setting we again use two symbols, 0 and 1, to correspond with two open regions S_0 and S_1, but now, as in the case for angle-doubling on the circle, there are two points in $\overline{S_0} \cap \overline{S_1}$, which we represent by $*$ and $\#$. The point $*$ will be periodic and $\#$ is the unique preimage of $*$ that is not in the orbit of $*$.

Specifically, for $\alpha \in \{0,1,*,\#\}^{\leq\omega}$, and $i \in \{0,1\}$, define $s_i(\alpha)$ as follows:

$$(s_i(\alpha))_j = \begin{cases} \alpha_j, & \text{if } \alpha_j \in \{0,1\}, \\ i, & \text{if } a_j = *, \\ 1-i, & \text{if } a_j = \#. \end{cases}$$

Let $K(\alpha) = \{\alpha, s_0(\alpha), s_1(\alpha)\}$ and let $\Gamma = \{0,1,*,\#\}^\omega$ with topology given by the basis $\{B_\alpha : \alpha \in \{0,1,*,\#\}^{<\omega}$, where $B_\alpha = \{\beta \in \Gamma : \beta\restriction_{|\alpha|} \in K(\alpha)\}\}$. Note that if α contains a $*$ or $\#$, then $K(\alpha)$ contains 3 elements, making Γ non-Hausdorff.

Let $p \in \mathbb{N}$ and let $\tau = (\alpha*)^\infty$, where $\alpha \in \{0,1\}^{p-1}$. Then, for any $n \in \mathbb{N}$, the sequence τ is called an *n-tupling* provided $n > 1$ divides p and there exists a sequence β of length $\frac{p}{n}$ such that β^∞ is either $s_0(\tau)$ or $s_1(\tau)$. For example, $\tau = (00100*)^\infty$ is a 2-tupling, since $s_1(\tau) = (001)^\infty$.

Definition 4.1. A sequence $\tau \in \Gamma$ is Γ-*acceptable* if it satisfies the following conditions:

1) $\tau = (\beta*)^\infty$ for some $\beta \in \{0,1\}^{<\omega}$, and

2) for all $n \in \omega$, either $K(\sigma^n(\tau)) \cap K(\tau) = \emptyset$ or $K(\sigma^n(\tau)) = K(\tau)$.

Γ-acceptable sequences are precisely those sequences in Γ of the form $(\alpha*)^\infty$ which are not n-tuplings for any $n \in \mathbb{N}$, [7, Lemma 3.8].

Definition 4.2. If τ is Γ-acceptable and $\alpha \in \Gamma$, we say that α is (Γ,τ)-*consistent* provided that $\alpha_n \in \{*,\#\}$ if and only if $\sigma^{n+1}(\alpha) = \tau$.

Definition 4.3. If α is (Γ,τ)-consistent, we say that α is (Γ,τ)-*admissible* provided that for all $n \in \omega$, either $K(\sigma^n(\alpha))$ is disjoint from both $K(*\tau)$ and $K(\#\tau)$ or $K(\sigma^n(\alpha))$ is equal to $K(*\tau)$ or $K(\#\tau)$. We let $\mathcal{E}_\tau = \{\alpha \in \Gamma : \alpha \text{ is } (\Gamma,\tau)\text{-admissible}\}$.

The following is due to Baldwin:

Theorem 4.4. *[7] Let τ be Γ-acceptable. Then $\mathcal{E}_\tau$ is a locally connected shift-invariant compact metric space on which σ is exactly two-to-one.*

The following result will be helpful in proving the main result of this section:

Lemma 4.5. *[12, Theorem 22] Let τ be a Γ-acceptable sequence. Then the following hold:*

1) for $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that if $x, y \in \mathcal{E}_\tau$ and $x{\restriction}_{N_\epsilon} \sim y{\restriction}_{N_\epsilon}$ then $d(x, y) < \epsilon$, and

2) for $N \in \mathbb{N}$ there exists δ_N such that if $x, y \in \mathcal{E}_\tau$ with $d(x, y) < \delta_N$, then $x{\restriction}_N \sim y{\restriction}_N$.

4.4 Distributional Chaos in $\mathcal{E}_\tau$

In this section, by again modifying the approach in [70, Example 9], we construct a DC1 scrambled set in $\mathcal{E}_\tau$. This section is an improvement of the author's own work in [3].

Lemma 4.6. *If τ is Γ-acceptable, and $A \subset \{0, 1\}^\omega$ contains only points which are not eventually periodic, then $A \subset \mathcal{E}_\tau$.*

Proof. Since τ is Γ-acceptable, τ is periodic. Let $\alpha \in A$. Now $\sigma^n(\alpha) \neq \tau$ for all $n \in \omega$ (since τ contains $*$ but $\sigma^n(\alpha)$ does not), so α is (Γ, τ)-consistent. Next, note that $\sigma^n(\alpha) = s_0(\sigma^n(\alpha)) = s_1(\sigma^n(\alpha))$, so $K(\sigma^n(\alpha)) = \{\sigma^n(\alpha)\}$ for all $n \in \omega$. Now $*\tau$, $s_0(*\tau)$, and $s_1(*\tau)$ are all periodic, whereas $\sigma^n(\alpha)$ is not periodic, so $K(\sigma^n(\alpha)) \cap K(*\tau) = \emptyset$. Similarly $\#\tau$, $s_0(\#\tau)$, and $s_1(\#\tau)$ are eventually periodic, whereas $\sigma^n(\alpha)$ is not eventually periodic, so $K(\sigma^n(\alpha)) \cap K(\#\tau) = \emptyset$, and thus $A \subset \mathcal{E}_\tau$. $\square$

In particular, the preceding lemma shows that $E \subset \mathcal{E}_\tau$ (recall that E was defined in Section 3.5). Since $\mathcal{E}_\tau$ is shift-invariant, and $E \subset \mathcal{E}_\tau$, we see that $\sigma^n(E) \subset \mathcal{E}_\tau$ for all $n \in \omega$.

Definition 4.7. Let $\mathcal{R}_n = \{\alpha \in \mathcal{E}_\tau : \alpha_i \in \{*, \#\}$ for some $i \leq n\}$ and let $\mathcal{R}_\omega = \bigcup_{n\in\omega} \mathcal{R}_n$. Define $x{\restriction}_n \sim y{\restriction}_n$ if and only if there exists $z \in \mathcal{R}_\omega$ such that $\{x{\restriction}_n, y{\restriction}_n\} \subseteq \{z{\restriction}_n, s_0(z){\restriction}_n, s_1(z){\restriction}_n\}$.

Lemma 4.8. *If $\tau \neq *^\infty$ is Γ-acceptable, then $\tau \not\approx 0^\infty$.*

Proof. Every Γ-acceptable sequence is Λ-acceptable, [6, Proposition 3.6], and for any Λ-acceptable sequence τ besides $*^\infty$, $\tau \not\approx 0^\infty$. $\square$

The following lemma helps us build up (Γ, τ)-admissible sequences from existing ones.

Lemma 4.9. *[7, Proposition 3.10] Let $\tau \in \Gamma$ be Γ-acceptable. Then $i\tau$ is (Γ, τ)-admissible if and only if $i \in \{*, \#\}$, and if $\alpha \neq \tau$ is (Γ, τ)-admissible, then $i\alpha$ is (Γ, τ)-admissible if and only if $i \in \{0, 1\}$.*

The following lemma shows that if τ is Γ-acceptable, and β is a binary sequence, then $\beta{*}\tau$ is (Γ, τ)-admissible, provided $\beta*$ does not contain a period of τ (which is only possible at the end of β, as β does not contain $*$).

Lemma 4.10. *Suppose $\tau \neq *^\infty$ is Γ-acceptable with period p and that $\beta \in \{0, 1\}^k$ for some $k \in \mathbb{N}$. If $|\beta *| < p$ or $\beta_{k-p+2}...\beta_k * \tau \neq \tau$, then $\beta{*}\tau$ is (Γ, τ)-admissible and $\beta{*}\tau \in \mathcal{R}_\omega$.*

Proof. By Lemma 4.9, $*\tau$ is (Γ, τ)-admissible. Now, for any j such that $0 \leq j \leq k$, we see that $\beta_j...\beta_k * \tau \neq \tau$, since otherwise $p = k - j + 2$, implying $\beta_{k-p+2}...\beta_k * \tau = \tau$. Thus, by repeated application of the second part of Lemma 4.9, $\beta * \tau$ is (Γ, τ)-admissible. Since $\beta{*}\tau$ contains $*$, $\beta{*}\tau \in \mathcal{R}_\omega$. $\square$

From [7, Proposition 3.20], $\mathcal{E}_{*\infty}$ is homeomorphic to the unit circle S^1, and $\sigma|_{\mathcal{E}_{*\infty}}$ is conjugate to the squaring map $f_0(z) = z^2$ on S^1. From [60, Theorem 2.2], if f is a continuous map from S to itself, then f exhibits distributional chaos[1] if and only if f has positive topological entropy. Since the squaring map on S^1 has positive topological entropy, [58], it exhibits distributional chaos and hence so does $\sigma|_{\mathcal{E}_{*\infty}}$. Consequently, for the rest of this chapter, we will assume that $\tau \neq *^\infty$.

Lemma 4.11. *Suppose $\epsilon > 0$ and that $N_\epsilon \in \mathbb{N}$ is such that, for any $x, y \in \mathcal{E}_\tau$, we have $x\restriction_{N_\epsilon} \sim y\restriction_{N_\epsilon}$ only if $d(x, y) < \epsilon$. Let $u, v \in \mathcal{E}_\tau \cap \{0,1\}^\omega$. If $u\restriction_{N_\epsilon} = v\restriction_{N_\epsilon}$, then $d(u, v) < \epsilon$.*

Proof. Suppose $u, v \in \mathcal{E}_\tau \cap \{0,1\}^\omega$ such that $u\restriction_{N_\epsilon} = v\restriction_{N_\epsilon}$. Let $z = u_0u_1...u_{N_\epsilon}(1 - \tau_{p-2}) * \tau$. By Lemma 4.9, $z \in \mathcal{R}_\omega$, and since $\{u\restriction_{N_\epsilon}, v\restriction_{N_\epsilon}\} \subset \{z\restriction_{N_\epsilon}, s_0(z)\restriction_{N_\epsilon}, s_1(z)\restriction_{N_\epsilon}\}$ we have $u\restriction_{N_\epsilon} \sim v\restriction_{N_\epsilon}$, which implies $d(u, v) < \epsilon$. □

Lemma 4.12. *Suppose τ is Γ-acceptable. Every point of E has an orbit which is dense in $\mathcal{E}_\tau$.*

Proof. Consider $x \in E \subset \mathcal{E}_\tau$. Let $\epsilon > 0$. There exists $N_\epsilon \in \mathbb{N}$ such that $x\restriction_{N_\epsilon} \sim y\restriction_{N_\epsilon}$ implies $d(x, y) < \epsilon$. Let $a \in \mathcal{E}_\tau$. If a does not contain $*$, then by the construction of E, there exists $n \in \mathbb{N}$ such that $\sigma^n(x)\restriction_{N_\epsilon} = a\restriction_{N_\epsilon}$. If a does contain $*$, there exists $n \in \mathbb{N}$ such that, for all $0 \le i \le N_\epsilon$, $\sigma^n(x)_i = a_i$ whenever $a_i \neq *$ and $\sigma^n(x)_i = 1$ otherwise. So in either case, there exists $n \in \mathbb{N}$ such that $\sigma^n(x)\restriction_{N_\epsilon} \sim a\restriction_{N_\epsilon}$, so that $d(\sigma^n(x), a) < \epsilon$, and thus $\sigma^n(x) \in B_\epsilon(a)$. Therefore, $\mathrm{Orb}^+(x)$ is dense in $\mathcal{E}_\tau$. □

Corollary 4.13. *Suppose τ is Γ-acceptable. The set E is dense in $\mathcal{D}_\tau$.*

Proof. Let $x \in E$. Then x has an orbit which is dense in $\mathcal{E}_\tau$. As $\sigma^n(E) \subset E$ for all $n \in \omega$, $\mathrm{Orb}^+(x) \subset E$. Since $\mathrm{Orb}^+(x)$ is dense in $\mathcal{E}_\tau$, so is E. □

[1] It should be noted that in [60], "distributional chaos" is defined as existence of a single distributionally chaotic pair.

Theorem 4.14. *If $\tau \in \Gamma$ is Γ-acceptable and $\tau \neq *^\infty$, then $\mathcal{E}_\tau$ contains an uncountable DC1 scrambled set. Further, the distributional chaos is transitive.*

Proof. We know $\tau = (\alpha *)^\infty$ for some $\alpha \in \{0,1\}^{<\omega}$. Now τ is periodic, so let its period be p. The set E is uncountable, and by Lemma 4.6, $E \subset \mathcal{E}_\tau$.

Let $\epsilon > 0$ and let $x, y \in E$ be distinct. By Theorem 4.5, there exists $N_\epsilon \in \mathbb{N}$ such that if $u, v \in \mathcal{E}_\tau$ with $u\restriction_{N_\epsilon} \sim v\restriction_{N_\epsilon}$, then $d(u,v) < \epsilon$. By Lemma 4.11, if $u\restriction_{N_\epsilon} = v\restriction_{N_\epsilon}$, then $d(u,v) < \epsilon$. Hence,

$$\frac{1}{n}\left|\{0 \le i < n : d(\sigma^i(x), \sigma^i(y)) < \epsilon\}\right| \ \ge \ \frac{1}{n}\left|\{0 \le i < n : \sigma^i(x)\restriction_{N_\epsilon} = \sigma^i(y)\restriction_{N_\epsilon}\}\right|.$$

Of course, the map σ treats binary sequences the same in $\mathcal{D}_\tau$ and $\mathcal{E}_\tau$, and therefore

$$\limsup_{n\to\infty} \frac{1}{n}\left|\{0 \le i < n : \sigma^i(x)\restriction_{N_\epsilon} = \sigma^i(y)\restriction_{N_\epsilon}\}\right| = 1$$

as in the proof of Theorem 3.17. Hence, for any $x, y \in E$, and any $\epsilon > 0$, we have that $F^*_{xy}(\epsilon) = 1$.

Now, there exists a subsequence μ_j such that $\alpha_{\mu_j} \neq \beta_{\mu_j}$, so that $u(\alpha_{\mu_j}, \mu_j)$ and $u(\beta_{\mu_j}, \mu_j)$ disagree in every place. By Lemma 4.5, for any $M \in \mathbb{N}$, there exists a δ_M such that $d(x,y) < \delta_M$ implies $x\restriction_M \sim y\restriction_M$. Take $M = 1$. Thus,

$$\frac{1}{m_{\mu_j}}\left|\{0 \le i < m_{\mu_j} : d(\sigma^i(x), \sigma^i(y)) < \delta_1\}\right| \le \frac{1}{m_{\mu_j}}\left|\{0 \le i < m_{\mu_j} : \sigma^i(x)\restriction_1 \sim \sigma^i(y)\restriction_1\}\right|.$$

Let N be the smallest natural number such that

$$\begin{aligned} r_{\mu_N - 1} - t &\ \ge\ 0 \text{ and} \\ r_{\mu_N} - |f(\mu_N)| - s &\ \ge\ r_{\mu_N - 1} - t. \end{aligned}$$

Then for all $j > N$, observe that if $i = m_{\mu_j - 1} - t$, then $\sigma^i(x) = \sigma^{m_{\mu_j - 1} + k}(a)$ and $\sigma^i(y) = \sigma^{m_{\mu_j - 1}}(b)$. Thus, when $i = m_{\mu_j - 1} - t$, we have the following picture:

$$\sigma^i(x) \ = \ \underbrace{\alpha_{\mu_j} \ldots\ldots \alpha_{\mu_j}}_{r_{\mu_j} - |f(\mu_j)| - k} f(\mu_j) \underbrace{\alpha_{\mu_j+1} \ldots \alpha_{\mu_j+1}}_{r_{\mu_j+1} - |f(\mu_j+1)|} f(\mu_j + 1) \ldots$$

$$\sigma^i(y) = \underbrace{\beta_{\mu_j}..............\beta_{\mu_j}}_{r_{\mu_j}-|f(\mu_j)|} f(\mu_j) \underbrace{\beta_{\mu_j+1}...\beta_{\mu_j+1}}_{r_{\mu_j+1}-|f(\mu_j+1)|} f(\mu_j+1)...$$

where $\alpha_{\mu_j} \neq \beta_{\mu_j}$ for all j. Hence for i such that $m_{v_j-1} - t \leq i < m_{\mu_j} - s - |f(\mu_j)|$, the first symbol of $\sigma^i(x)$ and $\sigma^i(y)$ disagree. Hence, for all such i such that $m_{\mu_j-1} - t \leq i < m_{\mu_j} - s - |f(\mu_j)| - 1$, we see that $\sigma^i(x)\restriction_1$ and $\sigma^i(y)\restriction_1$ differ in both symbols. Thus $\sigma^i(x)\restriction_1 \sim \sigma^i(y)\restriction_1$ implies the existence of $z \in \mathcal{R}_\omega$ such that

$$\{\sigma^i(x)\restriction_1, \sigma^i(y)\restriction_1\} \subset \{z\restriction_1, s_0(z)\restriction_1, s_1(z)\restriction_1\}.$$

Since $\sigma^i(x)\restriction_1$ and $\sigma^i(y)\restriction_1$ are binary and differ in both symbols, we must have $z_0, z_1 \notin \{0,1\}$ (otherwise $\sigma^i(x)\restriction_1$ and $\sigma^i(y)\restriction_1$ would agree in their first or second symbols, respectively). Then $z_0, z_1 \in \{*, \#\}$, and as z is (Γ, τ)-consistent, we see that $\sigma(z) = \tau$. Thus $z_1 = \tau_0 \in \{*, \#\}$. Since τ is Γ-acceptable, τ cannot contain $\#$, meaning $\tau_0 = *$, so that $\tau = (*)^\infty$, a contradiction. Therefore,

$$\{m_{\mu_j-1} - t \leq i < m_{\mu_j} - s - |f(\mu_j)| - 1 : \sigma^i(x)\restriction_1 \sim \sigma^i(y)\restriction_1\} = \emptyset.$$

Hence, letting $B = m_{\mu_j-1} - t$ for clarity, we have

$$\begin{aligned}
\frac{\left|\{i < m_{\mu_j} : \sigma^i(x)\restriction_1 \sim \sigma^i(y)\restriction_1\}\right|}{m_{\mu_j}} &\leq \frac{|\{i < B : \sigma^i(x)\restriction_1 \sim \sigma^i(y)\restriction_1\}|}{m_{\mu_j}} + \frac{s+1+|f(\mu_j)|}{m_{\mu_j}} \\
&\leq \frac{B}{m_{\mu_j}} + \frac{s+1+\mu_j}{m_{\mu_j}} \\
&= \frac{1}{2^{\mu_j}+1} + \frac{k+1+\mu_j}{m_{\mu_j}} \longrightarrow 0
\end{aligned}$$

as in Theorem 3.17. So for any $x, y \in E$,

$$\liminf_{n\to\infty} \frac{1}{n} |\{i < n : d(\sigma^i(x), \sigma^i(y)) < \delta_1\}| = 0.$$

This completes the proof that E is an uncountable DC1 scrambled set for σ, and the chaos is uniform as $F_{xy}(\delta_1) = 0$ for all $x, y \in E$. As E is dense in $\mathcal{D}_\tau$, and each $x \in E$ has an orbit which is dense in $\mathcal{E}_\tau$, we see that $\sigma|_{\mathcal{E}_\tau}$ exhibits transitive distributional chaos. $\square$

The following theorem is due to Baldwin.

Theorem 4.15. *[7, Theorem 4.11] Let $c \in \mathbb{C}$ and suppose that f_c has an attracting or parabolic periodic point. Let θ be one of the external angles of the parameter c and let $\tau = \tau^\theta$ be the kneading sequence of c. Suppose that τ is Γ-acceptable. Then $f_c|_{J_c}$ is conjugate to $\sigma|_{\mathcal{E}_\tau}$.*

The following corollary is immediate.

Corollary 4.16. *Let $c \in \mathbb{C}$ and suppose that $f_c(z) = z^2 + c$ has an attracting or parabolic periodic point. Let θ be one of the external angles of the parameter c and let $\tau = \tau^\theta$ be the kneading sequence of c. If $\tau \neq *^\infty$ is Γ-acceptable, then f_c exhibits transitive distributional chaos.*

CHAPTER FIVE

Other Forms of Chaos in $\mathcal{D}_\tau$ and $\mathcal{E}_\tau$

5.1 Exact Devaney Chaos

We begin with Devaney chaos, which requires a dense set of periodic points, a transitive map, and sensitive dependence on initial conditions. For recent developments on Devaney chaos in shift spaces, see for example [69], which discusses the computational complexity of Devaney chaos-detection algorithms in the context of shift spaces.

Lemma 5.1. *If τ is Λ-acceptable, then periodic points are dense in $\mathcal{D}_\tau$.*

Proof. Let $\epsilon > 0$. Suppose τ is not periodic. There exists $N_\epsilon \in \mathbb{N}$ such that for all $x, y \in \mathcal{D}_\tau$, we have $d(x,y) < \epsilon$ whenever $x{\restriction_{N_\epsilon}} \simeq y{\restriction_{N_\epsilon}}$. Let $x \in \mathcal{D}_\tau$. If $x \in \{0,1\}^\omega$, let $y = (x_0 ... x_{N_\epsilon})^\infty$. If $x \notin \{0,1\}^\omega$, for $0 \leq i \leq N_\epsilon$ let $z_i = x_i$ if $x_i \neq *$ and $z_i = 0$ otherwise; let $y = (z_0 ... z_{N_\epsilon})^\infty$. In either case, y is (Λ, τ)-consistent since $y \in \{0,1\}^\omega$. Since $\tau \in \{0,1\}^\omega$ is not periodic, it contains no $*$. Now, $\sigma^n(y)$ is periodic for all $n \in \omega$, so we see that $\tau \not\approx \sigma^n(y)$ for all $n \in \omega$, and thus $*\tau \not\approx \sigma^n(y)$ for all $n \in \omega$. Therefore, $y \in \mathcal{D}_\tau$. Clearly $x{\restriction_{N_\epsilon}} \simeq y{\restriction_{N_\epsilon}}$ so $d(x,y) < \epsilon$.

Suppose τ is periodic with period p. If $x \in \{0,1\}$, let $y = (x_0 ... x_{N_\epsilon} 0 ... 0)^\infty$, where N_ϵ is followed by $p+1$ zeros. If $x \notin \{0,1\}^\omega$, for $0 \leq i \leq N_\epsilon$ let $z_i = x_i$ if $x_i \neq *$ and $z_i = 0$ otherwise. Let $y = (z_0 ... z_{N_\epsilon} 0 ... 0)^\infty$, where z_{N_ϵ} is followed by $p+1$ zeros. In either case, y is (Λ, τ)-consistent and $y \in \mathcal{D}_\tau$ (each period of $*\tau$ contains 1, so the length of $p+1$ zeros in y must disagree with $*\tau$ somewhere that $*\tau$ does not have $*$). So $x{\restriction_{N_\epsilon}} \simeq y{\restriction_{N_\epsilon}}$, and thus $d(x,y) < \epsilon$.

Thus for any Λ-acceptable τ, periodic points are dense in $\mathcal{D}_\tau$. $\square$

Lemma 5.2. *If τ is Γ-acceptable, then periodic points are dense in $\mathcal{E}_\tau$.*

Proof. Since τ is Γ-acceptable, we know τ is periodic with some period p. Let $\epsilon > 0$ and let $x \in \mathcal{E}_\tau$. There exists $N_\epsilon \in \mathbb{N}$ such that for all $x, y \in \mathcal{E}_\tau$, we have $d(x,y) < \epsilon$ whenever $x\restriction_{N_\epsilon} \sim y\restriction_{N_\epsilon}$. If $x \in \{0,1\}$, let $y = (x_0...x_{N_\epsilon}0...0)^\infty$, where N_ϵ is followed by $p+1$ zeros. If $x \notin \{0,1\}^\omega$, for $0 \leq i \leq N_\epsilon$ let $z_i = x_i$ if $x_i \neq *$ and $z_i = 0$ otherwise, and then let $y = (z_0...z_{N_\epsilon}0...0)^\infty$, where x_{N_ϵ} is followed by $p+1$ zeros. In either case, y is (Γ, τ)-consistent. Now, by Lemma 4.8, each period of $*\tau$ contains the symbol 1 at least once, so the length of $p+1$ zeros in y must disagree with $*\tau$ somewhere that $*\tau$ does not have $*$. For the same reason, y must disagree with $\#\tau$ somewhere that $\#\tau$ does not have $*$ or $\#$. Thus for any n, $K(\sigma^n(y)) = \{\sigma^n(y)\}$ is disjoint from $K(*\tau)$ and $K(\#\tau)$. Therefore $y \in \mathcal{E}_\tau$ and $x\restriction_{N_\epsilon} \sim y\restriction_{N_\epsilon}$, and thus $d(x,y) < \epsilon$. So for any Γ-acceptable τ, periodic points are dense in $\mathcal{E}_\tau$. □

Theorem 5.3. *The shift map σ exhibits exact Devaney chaos on $\mathcal{D}_\tau$ and $\mathcal{E}_\tau$.*

Proof. By Theorem 3.6, σ is exact on $\mathcal{D}_\tau$ (and hence topologically transitive). Similarly, basic open sets in $\mathcal{E}_\tau$ are defined by sets of points with the first n coordinates restricted, so that for each such basic open set U there exists an n such that $\sigma^n(U) = \mathcal{E}_\tau$. Hence σ is exact, and thus topologically transitive, on $\mathcal{E}_\tau$ as well. By the above lemmas, periodic points are dense in $\mathcal{D}_\tau$ and $\mathcal{E}_\tau$. By [10], in any infinite metric space, topological transitivity and a dense set of periodic points imply sensitive dependence on initial conditions. □

Corollary 5.4. *The map σ exhibits topological chaos on $\mathcal{D}_\tau$ and $\mathcal{E}_\tau$.*

Proof. On compact spaces with more than one point, exact maps have positive topological entropy, [50, Lemma 1]. □

Devaney chaos is preserved by conjugacy, [10], and exactness is a topological property, so we have the following corollary.

Corollary 5.5. *Consider $f_c = z^2 + c$. If the Julia set of f_c is a dendrite, then f_c exhibits exact Devaney chaos.*

5.2 ω-chaos

Before turning our attention to ω-chaos in $\mathcal{D}_\tau$, we need a result on ω-limit sets. In $\{0,1\}^\omega$, we can prove that $y \in \omega(x)$ if and only if for all $n \in \omega$, $y_0 ... y_n$ occurs infinitely often in $x_0 x_1 ...$ The situation in $\mathcal{D}_\tau$ differs somewhat due to the fact that two points can differ early and yet be close if they are both equivalent to a pre-critical point.

Lemma 5.6. *Let $x = (x_0, x_1, ...)$ and $y = (y_0, y_1, ...)$ be points of $\mathcal{D}_\tau \cap \{0,1\}^\omega$. If, for all $n \in \omega$, $y_0 ... y_n$ occurs infinitely often in x, then $y \in \omega(x)$. If τ is not periodic, then the converse holds.*

Proof. Suppose for all $n \in \omega$ that $y_0 ... y_n$ occurs infinitely often in $(x_0, ...)$. Let $\epsilon > 0$. For each $i \in \omega$, let n_i be the smallest natural such that 1) $(\sigma^{n_i}(x))_j = y_j$ for all $j \in \{0, ..., i\}$, and 2) $n_i > n_k$ for all $k \in \omega$ such that $k < i$. (In other words, $\{n_i\}_{n=0}^\infty$ is a strictly increasing sequence). Recall that there exists N_ϵ such that if $x\restriction_{N_\epsilon} \simeq y\restriction_{N_\epsilon}$ then $d(x,y) < \epsilon$. For n_i sufficiently large, we have that $\sigma^{n_i}(x)\restriction_{N_\epsilon} \simeq y\restriction_{N_\epsilon}$, so $d(\sigma^{n_i}(x), y) < \epsilon$. Thus there is a sequence $\{n_k\}$ such that $\sigma^{n_k}(x) \to y$ as $n_k \to \infty$, so $y \in \omega(x)$.

Suppose τ is not periodic. Suppose $y \in \omega(x)$ but, to the contrary, that for some $n \in \omega$, we have $y_0 ... y_n$ occurring in x only finitely many times. So for some k, $y_0 ... y_n$ does not occur in $\sigma^k(x)$. For $j \geq k$, $\sigma^j(x)$ and y disagree somewhere in the first $n+1$ symbols, so we cannot have $\sigma^j(x)\restriction_n = y\restriction_n$. Next, note that if we have $\sigma^j(x)\restriction_n \approx \alpha * \tau\restriction_n \approx y\restriction_n$ for some $\alpha \in \{0,1\}^{<\omega}$, then $|\alpha| < n+1$ (since otherwise $\sigma^j(x)\restriction_n = y\restriction_n$). Now $y \in \omega(x)$, so for any $m > n$, there exists a j such that $\sigma^j(x) \in B_{\delta_m}(y)$, so that $\sigma^j(x)\restriction_m \simeq y\restriction_m$. This implies $\sigma^j(x)\restriction_m \approx \beta * \tau\restriction_m \approx y\restriction_m$ for some $\beta \in \{0,1\}^{<\omega}$ with $|\beta| < n+1$ (not $m+1$). Now, there are only finitely many possibilities for β. For one such possibility, we must have $y\restriction_{m_i} \approx \beta * \tau\restriction_{m_i}$ for each m_i in some subsequence $\{m_i\}_{i=0}^\infty$. But this is impossible, as it makes y not

(Λ, τ)-admissible. Therefore, if $y \in \omega(x)$, then for all $n \in \omega$, $y_0 ... y_n$ occurs infinitely often in x. □

The following theorem tells us that $\sigma : \{0,1\}^\omega \to \{0,1\}^\omega$ exhibits ω-chaos.

Theorem 5.7. *[72, Theorem 13] Let (X, f) be a dynamical system and let L and $m > 0$ be such that (L, f^m) is conjugate with (Σ_2^+, σ). Then there exists a Cantor set $C \subset L$ which is ω-scrambled for f.*

In fact, in the above theorem, a stronger condition holds: instead of $\omega(x) - \omega(y)$ simply being uncountable for each $x, y \in C$, we have that $\omega(x) - \omega(y)$ contains an infinite minimal set for each $x, y \in C$, [72, Remark 3]. (Note that if a minimal set is not the orbit of a single periodic point, then it is uncountable, [72]. Hence, these sets are uncountable.) Distinct minimal sets must be pairwise disjoint, for if A and B are minimal and $x \in A \cap B$, then $\omega(x) = A$ and $\omega(x) = B$. The following lemma is elementary but is included for completeness.

Lemma 5.8. *Suppose M is a minimal set in a compact metric space X and that $a, b \in M$, $x \in X$. If $a \in \omega(x)$, then $b \in \omega(x)$.*

Proof. Let $\epsilon > 0$. As any point of a minimal set generates the minimal set, we have that $b \in \omega(a) = M$. Hence, for some $N \in \mathbb{N}$, $\sigma^N(a) \in B_{\epsilon/2}(b)$. By the uniform continuity of σ^N, there exists δ such that for any $\alpha, \beta \in X$, we have that $d(\alpha, \beta) < \delta$ implies $d(\sigma^N(\alpha), \sigma^N(\beta)) < \frac{\epsilon}{2}$. Since $a \in \omega(x)$, there exists $L \in \mathbb{N}$ such that $\sigma^L(x) \in B_\delta(a)$, so $d(\sigma^{N+L}(x), \sigma^N(a)) < \frac{\epsilon}{2}$. By the triangle inequality, $\sigma^{N+L}(x) \in B_\epsilon(b)$. Hence $b \in \omega(x)$. □

Next, we show that if τ is not periodic, then minimal sets in $\{0,1\}^\omega$ which are carried over to $\mathcal{D}_\tau$ are still minimal.

Lemma 5.9. *Suppose τ is not periodic. If M is minimal in $\{0,1\}^\omega$, then M is minimal in $\mathcal{D}_\tau$.*

Proof. We begin by noting that a σ-invariant subset of $\{0,1\}^\omega$ is still σ-invariant in $\mathcal{D}_\tau$, since σ treats binary sequences the same in either space. Next, we will show that M is closed in $\mathcal{D}_\tau$. Suppose $x \in \mathcal{D}_\tau$ is an accumulation point of M in $\mathcal{D}_\tau$ but $x \notin M$. We show that this supposition leads to a contradiction in both cases.

Case 1: $x \in \{0,1\}^\omega$. Then, for each n there exists $y \in M$ such that $y \in B_{\delta_n}(x)$, so that $x\restriction_n \simeq y\restriction_n$. Now, there exists some $j \in \omega$ such that x and any $y \in M$ differ in the first j symbols (otherwise x would have been an accumulation point of M in $\{0,1\}^\omega$). So for each $n > j$, and each $y \in M$ such that $x\restriction_n \simeq y\restriction_n$, there exists some $\gamma * \tau$ such that $x\restriction_n \approx \gamma * \tau\restriction_n \approx y\restriction_n$, where the $*$ appears in the first j symbols. So for any $n > j$, there exist $y \in M$ and $\gamma * \tau$ such that $x\restriction_{n+j} \approx \gamma * \tau\restriction_{n+j} \approx y\restriction_{n+j}$. Let $|\gamma| = k$. Then (since $k < j$) we have $x_{k+1}...x_{n+k+1} = \tau_0...\tau_n$. Although k may change depending on γ, since the $*$ appears in the first j symbols, there are only finitely many possibilities for $\gamma * \tau$. Hence, for some $k' < j$, we have $x_{k'+1}...x_{n+k'+1} = \tau_0...\tau_n$ for arbitrarily large n. But this is impossible, as x would have the form $x_0...x_{k'}\tau$ (where $x_i \in \{0,1\}$) and thus x would not be (Λ, τ)-admissible.

Case 2: $x = \beta * \tau$ for some β of finite length, say k. Then by the uniform continuity of σ^{k+1}, we see that $\sigma^{k+1}(\beta * \tau) = \tau$ is an accumulation point of M as well. However, as $\tau \in \{0,1\}^\omega$, this situation is impossible, by Case 1.

Therefore, M is closed and σ-invariant in $\mathcal{D}_\tau$, but we must show that M is minimal in $\mathcal{D}_\tau$. If there is a nonempty, proper, closed, σ-invariant subset of M in $\mathcal{D}_\tau$, say N, then N must not be closed in $\{0,1\}^\omega$. Hence, N must lack one of its accumulation points in $\{0,1\}^\omega$, call it z. But each accumulation point of N in $\{0,1\}^\omega$ is an accumulation point of N in $\mathcal{D}_\tau$, and hence $z \in N$ since N is closed in $\mathcal{D}_\tau$. Thus, we have a contradiction, so M is closed, invariant, and contains no strictly smaller closed, nonempty, σ-invariant set. Thus, M is minimal. $\square$

Theorem 5.10. *If τ is Λ-acceptable but not periodic, then $\sigma : \mathcal{D}_\tau \to \mathcal{D}_\tau$ exhibits ω-chaos.*

Proof. By Lemma 5.7 and the following remark, (Σ_2^+, σ) contains an uncountable collection of uncountable, pairwise disjoint, minimal sets, which we will call $\{M_\lambda\}_{\lambda \in \Lambda}$. Now, for each $\lambda \in \Lambda$, there is an x_λ which generates M_λ. For each such x_λ, let

$$y_\lambda = x_0 0 x_0 x_1 00 ... x_0 ... x_n 0^{n+1} ...$$

If this procedure happens to create points of the form $\beta\tau$ for some $\beta \in \{0,1\}^{<\omega}$, we simply discard these countably many points. By Lemma 3.14, the remnant consists of uncountably many (Λ, τ)-admissible points. Thus, we may suppose that y_λ is (Λ, τ)-admissible for each $\lambda \in \Lambda$.

Now, each M_λ is uncountable but also minimal, meaning for any $x \in M_\lambda$ we have $\omega(x) = M_\lambda$. However, $\omega(0^\infty) = \{0^\infty\}$, so 0^∞ cannot generate any of the sets M_λ. Hence, for all $\lambda \in \Lambda$, $0^\infty \notin M_\lambda$.

It is clear that for all λ, $\omega(y_\lambda)$ is not totally periodic (as $x_\lambda \in \omega(y_\lambda)$, and x_λ is certainly not periodic, since otherwise $\omega(x_\lambda) = M_\lambda$ would be finite). Also, for any y_{λ_1} and y_{λ_2}, we have $0^\infty \in \omega(y_{\lambda_1}) \cap \omega(y_{\lambda_2})$. Finally, we will show that $\omega(y_{\lambda_1}) - \omega(y_{\lambda_2}) \supset M_{\lambda_1}$, which is uncountable. This we accomplish by showing that the orbit of y_{λ_2} does not approach any point of M_{λ_1}.

For some $x = x_0 x_1 x_2 ...$ and $x' = x'_0 x'_1 x'_2 ...$, we have that $y_{\lambda_1} = x_0 0 x_0 x_1 00 ...$ and $y_{\lambda_2} = x'_0 0 x'_0 x'_1 00 ...$, where $x \notin \omega(x')$. Suppose to the contrary there exists some $a \in M_{\lambda_1} = \omega(x)$ such that $a \in \omega(y_{\lambda_1}) \cap \omega(y_{\lambda_2})$, so that $a \in \omega(y_{\lambda_2})$. Then by Lemma 5.8, $x \in \omega(y_{\lambda_2})$. As $0^\infty \notin \omega(x)$, there exists $k \in \mathbb{N}$ such that x and x' never contain k-many of the symbol 0 in a row. Let $\epsilon > 0$ be small enough that the orbit of x' does not meet $B_\epsilon(x)$. By Lemma 3.9, there exists N_ϵ such that $a\upharpoonright_{N_\epsilon} \simeq b\upharpoonright_{N_\epsilon}$ implies $d(a,b) < \epsilon$ for any $a, b \in \mathcal{D}_\tau$. Now, by Lemma 5.6, $x_0 ... x_{N_\epsilon + 2k}$ appears infinitely often in y_{λ_2}. Hence, there exists n such that $\sigma^n(y_{\lambda_2})\upharpoonright_{N_\epsilon+2k} = x_0 ... x_{N_\epsilon+2k}$. Since $x_0 ... x_{N_\epsilon+2k}$ appears infinitely often in y_{λ_2}, we will assume n is large enough so that in y_{λ_2}, the blocks of 0 between occurrences of $x'_0 ... x'_n$ are longer than $N_\epsilon + 2k$. The point x does not contain more than k zeros in a row, so for some $j \leq k$ we have that $\sigma^j(x)$ begins with a 1. Hence, because $\sigma^n(y_{\lambda_2})\restriction_{N_\epsilon+2k} = x_0...x_{N_\epsilon+2k}$, we have that $\sigma^{n+j}(y_{\lambda_2})$ also begins with a 1. Now, $\sigma^{n+j}(y_{\lambda_2})\restriction_{N_\epsilon+2k-j} = \sigma^j(x)\restriction_{N_\epsilon+2k-j}$. Hence, for some $i \in \omega$, $\sigma^{n+j}(y_{\lambda_2})\restriction_0 = x'_i$, and since n is large, we see that $\sigma^{n+j}(y_{\lambda_2})\restriction_{N_\epsilon+2k-j} = x'_i...x'_{N_\epsilon+2k-j+i}$. Thus $\sigma^j(x)\restriction_{N_\epsilon} = x'_i...x'_{N_\epsilon+i}$ so that $x'_ix'_{i+1}... \in B_\epsilon(\sigma^j(x))$.

Now, if $\epsilon' < \epsilon$, we can repeat the above argument to get that $x'_{i'}x'_{i'+1}... \in B_{\epsilon'}(\sigma^{j'}(x))$ for some $i' \in \omega$ and some $j' \leq k$. But as there are finitely many possibilities for $j' \leq k$, we see that one of $x, \sigma(x), ..., \sigma^k(x)$ is a limit point for the orbit of x'. By Lemma 5.8, this means $x \in \omega(x')$, which is a contradiction. Thus $\omega(y_{\lambda_1}) - \omega(y_{\lambda_2}) \supset M_{\lambda_1}$, which is uncountable. Therefore, $\sigma : \mathcal{D}_\tau \to \mathcal{D}_\tau$ exhibits ω-chaos. $\square$

Let $f : X \to X$ and $p : Y \to Y$ be continuous, where X and Y are compact metric spaces. Recall that a semiconjugacy $h : X \to Y$ is a surjectve, continuous map such that $h \circ f = p \circ h$. From [55], we have the following:

Theorem 5.11. *[55, Theorem 3.2] Let X and Y be compact metric spaces. Let $f : X \to X$ and $p : Y \to Y$ be continuous. Suppose f is countable-to-one semiconjugate to p with semiconjugacy $h : X \to Y$. If there is an uncountable ω-scrambled set $S(p)$ in Y such that $\bigcap_{y\in S(p)} \omega_p(y) \neq \emptyset$, then there is an uncountable ω-scrambled set $S(f)$ in X such that $\bigcap_{x\in S(f)} \omega_f(x) \neq \emptyset$.*

In Theorem 5.10, we constructed an ω-scrambled set S such that $0^\infty \in \bigcap_{x\in S} \omega_\sigma(x)$, so that $\bigcap_{x\in S} \omega_\sigma(x) \neq \emptyset$. Hence, we have the following corollary.

Corollary 5.12. *Consider $f_c = z^2 + c$. If f_c is conjugate to $\mathcal{D}_\tau$ with τ non-periodic, then f_c exhibits ω-chaos.*

CHAPTER SIX

Future Work

6.1 Introduction

Our future work involves a couple goals. First, we hope to better understand the structure of $\mathcal{D}_\tau$ for a given Λ-admissible sequence τ; a natural place to start is by exploring properties of branch points in $\mathcal{D}_\tau$, the primary topic of Section 6.2. The main result of this section is that branch points are dense in $\mathcal{D}_\tau$ (provided $\mathcal{D}_\tau$ is not an arc, i.e., $\mathcal{D}_\tau$ has at least one branch point). We also seek to classify spaces on which the conditions for ω-chaos can be weakened, which we discuss in Section 6.3.

6.2 Branch Points in the Dendrite $\mathcal{D}_\tau$

Recall that dendrites are uniquely arcwise-connected. If X is a dendrite, let $[x_1, ..., x_n]$ denote the smallest subcontinuum of X containing the points $x_1, ..., x_n$. (Thus, $[x_1, ..., x_n]$ is the union of all arcs with endpoints in the set $\{x_1, ..., x_n\}$.) Recall that a point $x \in \mathcal{D}_\tau$ has degree n if $\mathcal{D}_\tau - \{x\}$ has exactly n components. We call these components the *branches* of x. The following lemma demonstrates that the degree of a point in the dendrite $\mathcal{D}_\tau$ cannot increase.

Lemma 6.1. *If x has finite degree n, then $\sigma(x)$ has degree as most n. In particular, if x is a non-branch point, then $\sigma(x)$ is another non-branch point.*

Proof. Suppose $x \in \mathcal{D}_\tau$ has finite degree n but, to the contrary, that $\sigma(x)$ has degree $m > n$. We begin by observing that if there is a point in $\mathcal{D}_\tau$ of the form $*\sigma(x)$, then $*\sigma(x)$ is the unique critical point of $\mathcal{D}_\tau$. Then $\sigma(x) = \tau$, forcing $x = *\tau$ by (Λ, τ)-admissibility.

Case 1: $x \neq *\tau$. Thus, there is no point of the form $*\sigma(x)$. Let $a_1, a_2, ..., a_m$ be in different branches of $\sigma(x)$. Without loss of generality, we may assume that

$x \in S_0$. Now $\overline{S_0} - \{x\}$ has n components, $A_1, ..., A_n$, which are path connected. (If x happens to be an endpoint, the following proof is still valid, but $\overline{S_0} - \{x\}$ will have only one component.) Hence $\sigma(A_1), ..., \sigma(A_n)$ are path connected. Since $\sigma(\overline{S_0}) = \mathcal{D}_\tau$, there exist $\alpha_1, ..., \alpha_m \in S_0$ such that $\sigma(\alpha_1) = a_1, ..., \sigma(\alpha_m) = a_m$. Observe that $\alpha_1, ..., \alpha_m$ must all be distinct as $a_1, ..., a_m$ are all distinct and σ is a function. Two of $\alpha_1, ..., \alpha_m$ must be in the same A_i for some $1 \leq i \leq n$; without loss of generality, suppose $\alpha_1, \alpha_2 \in A_1$. Note $[\alpha_1, \alpha_2] \subset A_1$ since A_1 is path connected. Now x is the only point of S_0 which maps to $\sigma(x)$, so no point of A_1 maps to $\sigma(x)$. Since A_1 contains a path connecting α_1 and α_2, $\sigma([\alpha_1, \alpha_2])$ contains a path connecting a_1 and a_2. Any path connecting a_1 and a_2 contains $\sigma(x)$. Thus A_1 must contain the only preimage of $\sigma(x)$, i.e. x, a contradiction. See Figure 6.1 for an illustration of the case $n = 3$ and $m = 4$.

Case 2: $x = *\tau$. Let $a_1, a_2, ..., a_m$ be in different branches of $\sigma(x)$. Now $\mathcal{D}_\tau - \{x\}$ has n components, $A_1, ..., A_n$, which are path connected. Hence $\sigma(A_1), ..., \sigma(A_n)$ are path connected. Since $\sigma(\mathcal{D}_\tau) = \mathcal{D}_\tau$, there exist $\alpha_1, ..., \alpha_m \in \mathcal{D}_\tau - \{x\}$ such that $\sigma(\alpha_1) = a_1, ..., \sigma(\alpha_m) = a_m$. Two of $\alpha_1, ..., \alpha_m$ must be in the same A_i for some $1 \leq i \leq n$; without loss of generality, suppose $\alpha_1, \alpha_2 \in A_1$. Thus $[\alpha_1, \alpha_2] \subset A_1$. By the (Λ, τ)-admissibility rule, x is the only point of $\mathcal{D}_\tau$ which maps to $\sigma(x) = \tau$, so no point of A_1 maps to $\sigma(x)$. Any path connecting a_1 and a_2 contains $\sigma(x)$, but again this a contradiction, since $\sigma(A_1)$ is pathwise-connected but $\sigma(x) \notin \sigma(A_1)$.

In either case we get a contradiction, so the degree of x is greater than or equal to the degree of $\sigma(x)$. □

The following lemma shows that if $*\tau$ is not a branch point, then forward images of $*\tau$ will be endpoints and hence will have degree less than $*\tau$.

Lemma 6.2. *If $*\tau$ is not a branch point, then $\sigma^n(*\tau)$ is an endpoint for all $n > 0$.*

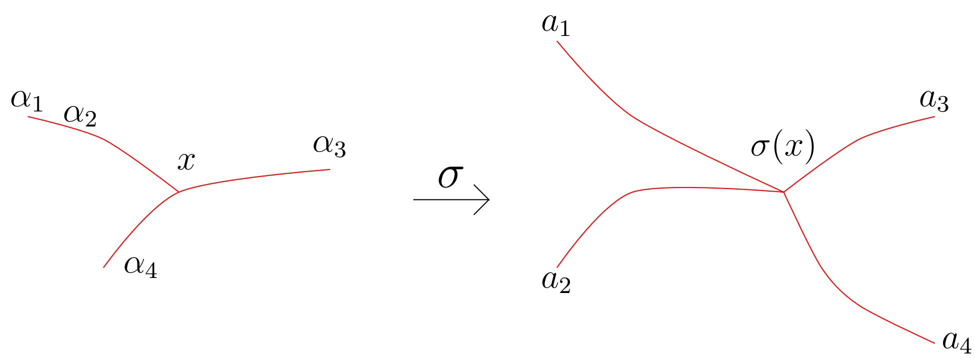

Figure 6.1. Branch points in Lemma 6.1 (the case $n = 3$, $m = 4$)

Proof. Note $\sigma(S_0 \bigcup \{*\tau\}) = \mathcal{D}_\tau$, and $\sigma(*\tau)$ has a unique preimage in $\mathcal{D}_\tau$, so $\sigma(S_0) = \mathcal{D}_\tau - \{\sigma(*\tau)\}$. Since $*\tau$ is not a branch point, S_0 is pathwise-connected, and so is $\sigma(S_0) = \mathcal{D}_\tau - \{\sigma(*\tau)\}$. Hence $\sigma(*\tau)$ is an endpoint with degree 1, implying $\sigma(*\tau) \neq *\tau$.

Suppose there exists k such that, for each $0 < i \leq k$, we have that $\sigma^i(*\tau)$ is an endpoint and hence has degree 1 (implying $\sigma^i(*\tau) \neq *\tau$). By Lemma 6.1, $\sigma^{k+1}(*\tau)$ has degree 1 and hence is an endpoint. So $\sigma^n(*\tau)$ is an endpoint for all $n > 0$. $\square$

The following lemma will show that, in general, the degree of a branch point cannot decrease.

Lemma 6.3. *If $x \neq *\tau$ has finite degree n, then $\sigma(x)$ has degree at least n. In particular, the image of a non-critical branch point in $\mathcal{D}_\tau$ is another branch point in $\mathcal{D}_\tau$.*

Proof. Let x be non-critical. If x has degree 1, then of course $\sigma(x)$ has degree at least 1. Suppose x has degree $n \geq 2$. Without loss of generality, we may assume $x \in S_0$. There exist points $\alpha_1, \alpha_2, ..., \alpha_n$ which are in different branches of x, and since S_0 is open, we may assume $\alpha_1, \alpha_2, ..., \alpha_n \in S_0$. Note that $\sigma(\alpha_1), ..., \sigma(\alpha_n)$ are distinct since $\sigma|_{\overline{S_0}}$ is a homeomorphism. If $\sigma(x)$ has degree less than n, say m, two of $\sigma[\alpha_1, x]$, $\sigma[\alpha_2, x]$, ..., $\sigma[\alpha_n, x]$ must meet at a point besides $\sigma(x)$. But this is a contradiction, as $[\alpha_1, \alpha_2, ..., \alpha_n] \subseteq S_0$ and $\sigma|_{\overline{S_0}}$ is a homeomorphism. See Figure 6.2. $\square$

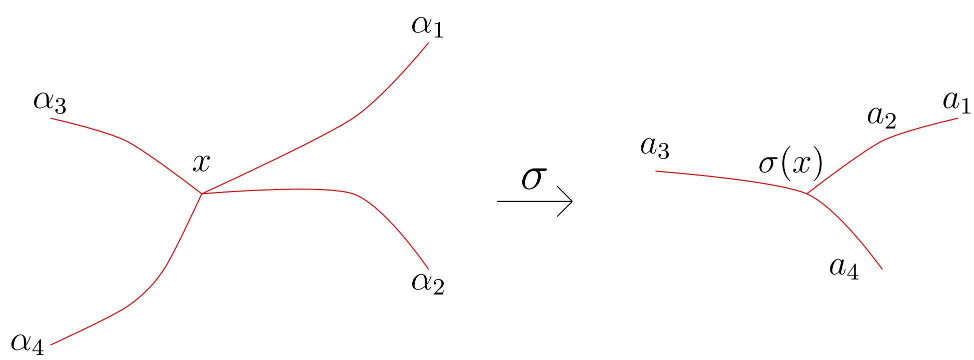

Figure 6.2. Branch points in Lemma 6.3 (the case $n = 4$, $m = 3$)

Thus, the preceding lemmas show that σ preserves the degree of a non-critical point. We have demonstrated that the degree of the critical point in $\mathcal{D}_\tau$ may decrease or remain unchanged, but it cannot increase. As we now show, in the case that $*\tau$ is periodic, σ does preserve the degree of $*\tau$.

Lemma 6.4. *If $*\tau$ is periodic with degree k, then $\sigma^n(*\tau)$ has degree k for all $n \in \omega$.*

Proof. Suppose $*\tau$ is periodic with period p. If $\sigma(*\tau)$ has degree k, then since σ preserves the degree of non-critical points, $\sigma(*\tau), ..., \sigma^{p-1}(*\tau)$ all have degree k. Since $\sigma^{p-1}(*\tau)$ is not critical, $\sigma^p(*\tau) = *\tau$ has degree k. $\square$

Lemma 6.5. *If $*\tau$ is periodic, then $\sigma^n(*\tau)$ is a branch point for all $n \in \omega$.*

Proof. Note that if $*\tau$ is not a branch point, then $*\tau$ is an endpoint and hence not periodic (otherwise, by Lemma 6.4, $*\tau$ would have degree 1). Thus, taking the contrapositive of this statement, we have the result for $n = 1$. Thus $\sigma(*\tau)$ has degree $k > 2$. By Lemma 6.4, all images of $\sigma(*\tau)$ (including $*\tau$ itself, since $*\tau$ is periodic) have degree k. $\square$

We now prove that unless $\mathcal{D}_\tau$ is an arc (which for example happens when $\mathcal{D}_\tau$ is conjugate to the Julia set of $f(z) = z^2 - 2$), $\mathcal{D}_\tau$ has a dense set of branch points.

Theorem 6.6. *Let B denote set of branch points of $\mathcal{D}_\tau$. If $B \neq \emptyset$, B is dense in $\mathcal{D}_\tau$.*

Proof. Suppose U is open in $\mathcal{D}_\tau$ but contains no branch points. The map σ is exact, so for some $n \in \mathbb{N}$, $\sigma^n(U) = \mathcal{D}_\tau$. Let b be a branch point of $\mathcal{D}_\tau$. There exists a point $x \in U$ such that $\sigma^n(x) = b$. But this is impossible, as the degree of x is less than or equal to 2, and thus the degree of $\sigma^m(x)$ is less than or equal to 2 for any $m \in \mathbb{N}$. □

6.3 The ω-FTP Property

A continuum X is said to be *completely regular* provided that every non-degenerate subcontinuum of X has non-empty interior, [63]. Examples of completely regular continua include graphs, trees, and dendrites with a discrete set of branch points. A compact metric space is said to have the *ω-FTP property* provided that for any continuous function $f : X \to X$, the set $\omega_f(x)$ is finite whenever $\omega_f(x) \subset \mathrm{Per}(f)$, [63]. A hereditarily locally connected continuum has the ω-FTP property if and only if it is completely regular, [63]. We now show that $\mathcal{D}_\tau$, if it is not an arc, does not have the ω-FTP property.

Lemma 6.7. *If $\mathcal{D}_\tau$ is not an arc, $\mathcal{D}_\tau$ is not completely regular.*

Proof. Note that any non-degenerate arc $[a, b]$ in $\mathcal{D}_\tau$ contains a branch point, c (otherwise (a, b) would be open but not containing a branch point, contradicting Theorem 6.6). Now, for any $\epsilon > 0$, $B_\epsilon(c) \not\subset [a, b]$ (as c is a branch point). Hence $\mathcal{D}_\tau$ is not completely regular. □

Corollary 6.8. *If $\mathcal{D}_\tau$ is not an arc, does not have the ω-FTP property.*

Proof. By [63, Theorem 1.4], $\mathcal{D}_\tau$ has ω-FTP property if and only if it is completely regular. □

An interesting topic of research is those spaces on which the third condition for ω-chaos (that is, for each $x \in S$, $\omega(x)$ is not totally periodic) is superfluous, being naturally satisfied by the space. For instance, Li noted that on the compact interval, condition 3) is not needed, [55]. In spaces with the ω-FTP property, such as completely regular hereditarily locally connected continua, the failure of condition 3) implies the failure of condition 1). Hence, on these spaces, condition 3) is not needed.

Note that, if we are merely concerned with ω-chaos, the ω-FTP property may be stronger than necessary. A possible long-term goal is characterizing spaces on which the ω-limit set being totally periodic implies it is countably infinite (which we may denote ω-CTP). If a space has this ω-CTP property, any totally periodic ω-limit set is still countable, failing condition 1) of ω-chaos. Hence, in a space with the ω-CTP property, condition 3) of ω-chaos is unnecessary.

BIBLIOGRAPHY

[1] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.*, 114:309–319, 1965.

[2] Ghassen Askri and Issam Naghmouchi. Topological size of scrambled sets for local dendrite maps. *Topology Appl.*, 164:95–104, 2014.

[3] Nathan Averbeck and Brian E. Raines. Distributional chaos in dendritic and circular Julia sets. *J. Math. Anal. Appl.*, 428(2):951–958, 2015.

[4] Nathan Averbeck and Brian E. Raines. Chaotic pairs for shift maps. *Houston J. Math.*, to appear.

[5] Stewart Baldwin. Continuous itinerary functions on dendroids. *Topology Proc.*, 30(1):39–58, 2006. Spring Topology and Dynamical Systems Conference.

[6] Stewart Baldwin. Continuous itinerary functions and dendrite maps. *Topology Appl.*, 154(16):2889–2938, 2007.

[7] Stewart Baldwin. Julia sets and periodic kneading sequences. *J. Fixed Point Theory Appl.*, 7(1):201–222, 2010.

[8] F. Balibrea and J. Smital. Strong distributional chaos and minimal sets. *Topology Appl.*, 156(9):1673–1678, 2009.

[9] F. Balibrea, J. Smital, and M. Štefankova. The three versions of distributional chaos. *Chaos Solitons Fractals*, 23(5):1581–1583, 2005.

[10] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey. On Devaney's definition of chaos. *Amer. Math. Monthly*, 99(4):332–334, 1992.

[11] Xavier Barrachina, J. Alberto Conejero, Marina Murillo-Arcila, and Juan B. Seoane-Sepúlveda. Distributional chaos for the forward and backward control traffic model. *Linear Algebra Appl.*, 479:202–215, 2015.

[12] Andrew D. Barwell, Jonathan Meddaugh, and Brian E. Raines. Shadowing and ω-limit sets of circular Julia sets. *Ergodic Theory Dynam. Systems*, 35(4):1045–1055, 2015.

[13] Andrew D. Barwell and Brian E. Raines. The ω-limit sets of quadratic Julia sets. *Ergodic Theory Dynam. Systems*, 35(2):337–358, 2015.

[14] Alan F. Beardon. *Iteration of rational functions*, volume 132 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Complex analytic dynamical systems.

[15] Vitaly Bergelson. Ergodic Ramsey theory—an update. In *Ergodic theory of* $\mathbf{Z}^d$ *actions (Warwick, 1993–1994)*, volume 228 of *London Math. Soc. Lecture Note Ser.*, pages 1–61. Cambridge Univ. Press, Cambridge, 1996.

[16] Vitaly Bergelson. Minimal idempotents and ergodic Ramsey theory. In *Topics in dynamics and ergodic theory*, volume 310 of *London Math. Soc. Lecture Note Ser.*, pages 8–39. Cambridge Univ. Press, Cambridge, 2003.

[17] Francois Blanchard, Eli Glasner, Sergii Kolyada, and Alejandro Maass. On Li-Yorke pairs. *J. Reine Angew. Math.*, 547:51–68, 2002.

[18] L. S. Block and W. A. Coppel. *Dynamics in one dimension*, volume 1513 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.

[19] Rufus Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.*, 153:401–414, 1971.

[20] Rufus Bowen. Periodic points and measures for Axiom A diffeomorphisms. *Trans. Amer. Math. Soc.*, 154:377–397, 1971.

[21] Bodil Branner. The Mandelbrot set. In *Chaos and fractals (Providence, RI, 1988)*, volume 39 of *Proc. Sympos. Appl. Math.*, pages 75–105. Amer. Math. Soc., Providence, RI, 1989.

[22] Michael Brin and Garrett Stuck. *Introduction to dynamical systems*. Cambridge University Press, Cambridge, 2002.

[23] Robert Brooks and J. Peter Matelski. The dynamics of 2-generator subgroups of PSL(2, $\mathbf{C}$). In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 65–71. Princeton Univ. Press, Princeton, N.J., 1981.

[24] Henk Bruin and Dierk Schleicher. Admissibility of kneading sequences and structure of Hubbard trees for quadratic polynomials. *J. Lond. Math. Soc. (2)*, 78(2):502–522, 2008.

[25] Jakub Byszewski, Fryderyk Falniowski, and Dominik Kwietniak. Transitive dendrite map with zero entropy. Pre-print. abs/1503.03035.

[26] J. J. Charatonik, W. J. Charatonik, and S. Miklos. Confluent mappings of fans. *Dissertationes Math. (Rozprawy Mat.)*, 301:86, 1990.

[27] J. Alberto Conejero, Marina Murillo-Arcila, and Juan B. Seoane-Sepúlveda. Linear chaos for the Quick-Thinking-Driver model. *Semigroup Forum*, 92(2):486–493, 2016.

[28] Timothy Denton, George Diamond, Richard Helfant, Steve Khan, and Hrayr Karagueuzian. Fascinating rhythm: A primer on chaos theory and its applications to cardiology. *Am. Heart J.*, 120(6):1419–1440, 1990.

[29] R. L. Devaney. *An introduction to chaotic dynamical systems.* Studies in Nonlinearity. Westview Press, Boulder, CO, 2003. Reprint of the second (1989) edition.

[30] Jana Dolezelova-Hantakova. The two versions of the weakest form of distributional chaos. in press.

[31] A. Douady and J. H. Hubbard. *Étude dynamique des polynômes complexes. Parties I et II (in French)*, volume 84-2 and 85-4 of *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay].* Université de Paris-Sud, Département de Mathématiques, Orsay, 1984 and 1985. With the collaboration of P. Lavaurs, Tan Lei and P. Sentenac.

[32] T. Downarowicz. Positive topological entropy implies chaos DC2. *Proc. Amer. Math. Soc.*, 142(1):137–149, 2014.

[33] Brigitte Falkenburg and Friedel Weinert. Indeterminism and determinism in quantum mechanics. In *Compendium of Quantum Physics*, pages 307–311. Springer Berlin Heidelberg, Providence, RI, 2009.

[34] Qinjie Fan, Gongfu Liao, and Hui Wang. Compact systems with DC3 pairs. *Internat. J. Modern Phys. B*, 25(27):3641–3646, 2011.

[35] P. Fatou. Sur les équations fonctionnelles. *Bull. Soc. Math. France*, 47:161–271, 1919.

[36] Gian Luigi Forti, Luigi Paganoni, and Jaroslav Smital. Dynamics of homeomorphisms on minimal sets generated by triangular mappings. *Bull. Austral. Math. Soc.*, 59(1):1–20, 1999.

[37] Xin-Chu Fu, Yibin Fu, Jinqiao Duan, and Robert S. Mackay. Chaotic properties of subshifts generated by a nonperiodic recurrent orbit. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 10(5):1067–1073, 2000.

[38] Xinchu Fu and Yuncheng You. Chaotic sets of shift and weighted shift maps. *Nonlinear Anal.*, 71(5-6):2141–2152, 2009.

[39] Harry Furstenberg. Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory*, 1:1–49, 1967.

[40] Eli Glasner and Benjamin Weiss. Sensitive dependence on initial conditions. *Nonlinearity*, 6(6):1067–1075, 1993.

[41] James Gleick. *Chaos: Making a New Science.* Viking Penguin Inc., New York, 1987.

[42] J. Guckenheimer. Sensitive dependence to initial conditions for one-dimensional maps. *Comm. Math. Phys.*, 70(2):133–160, 1979.

[43] Juan Luis Garcia Guirao and Marek Lampart. Relations between distributional, Li-Yorke and ω chaos. *Chaos Solitons Fractals*, 28(3):788–792, 2006.

[44] Wen Huang and Xiangdong Ye. Homeomorphisms with the whole compacta being scrambled sets. *Ergodic Theory Dynam. Systems*, 21(1):77–91, 2001.

[45] Wen Huang and Xiangdong Ye. Devaney's chaos or 2-scattering implies Li-Yorke's chaos. *Topology Appl.*, 117(3):259–272, 2002.

[46] G. Julia. Mémoires sur l'iteration des fonction rationnelles. *J. Math. Pures Appl.*, 8:47–245, 1918.

[47] Linda Keen. Julia sets. In *Chaos and fractals (Providence, RI, 1988)*, volume 39 of *Proc. Sympos. Appl. Math.*, pages 57–74. Amer. Math. Soc., Providence, RI, 1989.

[48] M. Kuchta and J. Smital. Two-point scrambled set implies chaos. In *European Conference on Iteration Theory (Caldes de Malavella, 1987)*, pages 427–430. World Sci. Publ., Teaneck, NJ, 1989.

[49] Milan Kuchta. Characterization of chaos for continuous maps of the circle. *Comment. Math. Univ. Carolin.*, 31(2):383–390, 1990.

[50] Dominik Kwietniak and Michał Misiurewicz. Exact Devaney chaos and entropy. *Qual. Theory Dyn. Syst.*, 6(1):169–179, 2005.

[51] M. Lampart. Two kinds of chaos and relations between them. *Acta Math. Univ. Comenian. (N.S.)*, 72(1):119–127, 2003.

[52] M. Lampart and P. Oprocha. Shift spaces, ω-chaos and specification property. *Topology Appl.*, 156(18):2979–2985, 2009.

[53] Marek Lampart. Necessity of the third condition from the definition of omega chaos. *Appl. Math. Inf. Sci.*, 9(5):2303–2307, 2015.

[54] Risong Li. A note on the three versions of distributional chaos. *Commun. Nonlinear Sci. Numer. Simul.*, 16(4):1993–1997, 2011.

[55] Shi Hai Li. ω-chaos and topological entropy. *Trans. Amer. Math. Soc.*, 339(1):243–249, 1993.

[56] T-Y. Li and J.A. Yorke. Period three implies chaos. *American Mathematical Monthly*, 82:985–992, 1975.

[57] Gongfu Liao and Qinjie Fan. Minimal subshifts which display Schweizer-Smital chaos and have zero topological entropy. *Sci. China Ser. A*, 41(1):33–38, 1998.

[58] M. Yu. Lyubich. Entropy of analytic endomorphisms of the Riemann sphere. *Funktsional. Anal. i Prilozhen.*, 15(4):83–84, 1981.

[59] Jiehua Mai. Continuous maps with the whole space being a scrambled set. *Chinese Sci. Bull.*, 42(19):1603–1606, 1997.

[60] M. Málek. Distributional chaos for continuous mappings of the circle. *Ann. Math. Sil.*, 13:205–210, 1999.

[61] Benoit Mandelbrot. *The Fractal Geometry of Nature*. W. H. Freeman and Company, 1982.

[62] F. Martinez-Gimenez, P. Oprocha, and A. Peris. Distributional chaos for backward shifts. *J. Math. Anal. Appl.*, 351:607–615, 2009.

[63] Habib Marzougui and Issam Naghmouchi. On totally periodic ω-limit sets. Pre-print. abs/1406.4401.

[64] N. Metropolis, M. L. Stein, and P. R. Stein. On finite limit sets for transformations on the unit interval. *J. Combinatorial Theory Ser. A*, 15:25–44, 1973.

[65] John Milnor. Self-similarity and hairiness in the Mandelbrot set. In *Computers in geometry and topology (Chicago, IL, 1986)*, volume 114 of *Lecture Notes in Pure and Appl. Math.*, pages 211–257. Dekker, New York, 1989.

[66] James R. Munkres. *Topology*. Prentice-Hall, Inc., 2000.

[67] Sam B. Nadler, Jr. *Continuum theory*, volume 158 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1992. An introduction.

[68] Piotr Oprocha. Relations between distributional and Devaney chaos. *Chaos*, 16(3):033112, 5, 2006.

[69] Piotr Oprocha. Algorithmic approach to Devaney chaos in shift spaces. *Fund. Inform.*, 87(3-4):435–446, 2008.

[70] Piotr Oprocha. Distributional chaos revisited. *T. Am. Math. Soc.*, 69(9):4901–4925, 2009.

[71] Piotr Oprocha. Minimal systems and distributionally scrambled sets. *Bull. Soc. Math. France*, 140(3):401–439, 2012.

[72] Piotr Oprocha. Transitivity, two-sided limit shadowing property and dense ω-chaos. *J. Korean Math. Soc.*, 51(4):837–851, 2014.

[73] Piotr Oprocha and Pawel Wilczynski. Distributional chaos via semiconjugacy. *Nonlinearity*, 20(11):2661–2679, 2007.

[74] Rafał Pikuła. On some notions of chaos in dimension zero. *Colloq. Math.*, 107(2):167–177, 2007.

[75] Sylvie Ruette and L'ubomir Snoha. For graph maps, one scrambled pair implies Li-Yorke chaos. *Proc. Amer. Math. Soc.*, 142(6):2087–2100, 2014.

[76] B. Schweizer and J. Smital. Measures of chaos and a spectral decomposition of dynamical systems on the interval. *T. Am. Math. Soc.*, 344:737–754, 1994.

[77] Ja. G. Sinai. On the concept of entropy of a dynamical system. *Dokl. Akad. Nauk SSSR*, 124:768–771, 1959.

[78] Jaroslav Smital and Marta Štefánková. Omega-chaos almost everywhere. *Discrete Contin. Dyn. Syst.*, 9(5):1323–1327, 2003.

[79] Jaroslav Smital and Marta Stefankova. Distributional chaos for triangular maps. *Chaos Solitons Fractals*, 21(5):1125–1128, 2004.

[80] Lei Tan. Similarity between the Mandelbrot set and Julia sets. *Comm. Math. Phys.*, 134(3):587–617, 1990.

[81] Giulio Tiozzo. Topological entropy of quadratic polynomials and dimension of sections of the Mandelbrot set. *Adv. Math.*, 273:651–715, 2015.

[82] Hui Wang, Fengchun Lei, and Lidong Wang. DC3 and Li-Yorke chaos. *Appl. Math. Lett.*, 31:29–33, 2014.

[83] Hui Wang, Gongfu Liao, and Qinjie Fan. A note on the map with the whole space being a scrambled set. *Nonlinear Anal.*, 70(6):2400–2402, 2009.

[84] Lidong Wang, Songmei Huan, and Guifeng Huang. A note on Schweizer-Smital chaos. *Nonlinear Anal.*, 68(6):1682–1686, 2008.

[85] Xinxing Wu and Peiyong Zhu. A minimal DC1 system. *Topology Appl.*, 159(1):150–152, 2012.