ABSTRACT

On Quasi-dominant Weights and Hilbert Series of Determinantal Varieties

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The coordinate rings of the classical determinantal varieties are each isomorphic to a classical invariant ring by Weyl's fundamental theorems of invariant theory. Since these rings are Cohen-Macaulay, their Hilbert series are rational functions whose numerator polynomials have nonnegative integer coefficients. In the case of general determinantal varieties, as well as in the case of symmetric determinantal varieties, these numerator polynomials were shown to be equal to the Hilbert series of certain finite-dimensional highest weight modules and were given an explicit combinatorial description. The current work extends these results to the alternating determinantal varieties.

The proof of these results, in all three cases, relies on the fact that the coordinate rings of the determinantal varieties carry the structure of a Wallach representation. The Hilbert series of the Wallach representation is a rational function whose numerator polynomial is given by the Hilbert series of a finite-dimensional highest weight module, and the Hilbert series of the determinantal variety is equal to the Hilbert series of the Wallach representation. T. J. Enright and J. F. Willenbring introduced the more general class of quasi-dominant weights and showed that if L is a unitarizable highest weight module with quasi-dominant highest weight, then the

Hilbert series of L is of the form

$$H_L(t) = R \cdot \frac{H_E(t)}{(1-t)^D},$$

where R is a rational number, E is a finite-dimensional highest weight module, and D is the Gelfand-Kirillov dimension of L. The set of quasi-dominant weights has an interesting characterization in terms of parabolic category $\mathcal{O}$ and Kostant's minimal length coset representatives. We give a new characterization in terms of associated varieties and show that the subset of quasi-dominant weights whose highest weight modules occur in the setting of Howe dual pairs has a nice description in terms of the highest weights of the "Howe dual" representations. Finally, we give some new results on the number of quasi-dominant reduction points.

On Quasi-dominant Weights and Hilbert Series of Determinantal Varieties

by

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CHAPTER ONE

Introduction

1.1 Hilbert Series and Resolutions

Hilbert's famous papers of 1888-1893 changed the course of modern algebra and invariant theory and set the stage for the development of modern algebraic geometry. They established the finite generation of the invariant ring for a wide class of groups by an existential proof, which was the first proof of its type. They also provided a correspondence between the radical ideals of a polynomial ring over an algebraically closed field $\mathbb{K}$ and the algebraic sets (or varieties) of affine n-space over $\mathbb{K}$. Two other landmark results in these papers, both motivated by invariant theory, involve constructions that play an important role in the present work.

Let S be the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$. Given a finitely generated graded S-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$, the function $h_M(i) = \dim M_i$ is called the *Hilbert function of M*. The finite generation of M ensures that these dimensions are finite. Hilbert showed that for large i the Hilbert function agrees with a ploynomial function of degree $\leq n+1$. The *Hilbert series of M*,

$$H_M(t) = \sum_{i \in \mathbb{N}} (\dim M_i) t^i.$$

gives the graded structure of M as a vector space. In representation theory, it serves as a coarse invariant of highest weight modules. Both the Hilbert function and the Hilbert series of M can be calculated by comparing M with free modules, which leads us to another important object of study. A *finite graded free resolution* of a finitely generated graded S-module M is an exact squence

$$0 \to F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

of graded free modules with maps that take the d-th homogeneous component of a module to the d-th homogeneous component of the next module. Hilbert's other result that plays a large role in what follows is his syzygy theorem, which asserts the existence of such a resolution for every finitely generated graded S-module. As a result, $H_M(t)$ can be found by computing $\sum_{j=0}^{m}(-1)^j H_{F_j}(t)$.

1.2 Determinantal Varieties

Let $(G_{\mathbb{R}}, K_{\mathbb{R}})$ be an irreducible Hermitian symmetric pair, and denote the corresponding complexified Lie algebras by $(\mathfrak{g}, \mathfrak{k})$. Then $\mathfrak{g}$ has Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$, and $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ as a $\mathfrak{k}$-module. The closures of the K-orbits on $\mathfrak{p}^+$ form a chain of varieties

$$\{0\} = Y_0 \subset Y_1 \subset \cdots \subset Y_r = \mathfrak{p}^+,$$

where r is the rank of the Hermitian symmetric space $G_{\mathbb{R}}/K_{\mathbb{R}}$. In the case of the classical Hermitian symmetric pairs, the varieties Y_k are the classical determinantal varieties in the space $\mathrm{M}_{p,q}$ of general complex $p \times q$ matrices, the space Sym_n of symmetric $n \times n$ matrices, and the space Alt_n of skew-symmetric (alternating) $n \times n$ matrices, as displayed in Table 1.1.

Table 1.1. The classical determinantal varieties

K	Y_k	r
$\mathrm{S}(\mathrm{GL}_p \times \mathrm{GL}_q)$	$\{x \in \mathrm{M}_{p,q} \mid \mathrm{rk}(x) \leq k\}$	$\min\{p, q\}$
GL_n	$\{x \in \mathrm{Sym}_n \mid \mathrm{rk}(x) \leq k\}$	n
GL_n	$\{x \in \mathrm{Alt}_n \mid \mathrm{rk}(x) \leq 2k\}$	$\lfloor \frac{n}{2} \rfloor$

The coordinate rings $\mathbb{C}[Y_k]$ turn out to be isomorphic to a classical invariant ring by an argument involving Weyl's fundamental theorems. Roger Howe observed that these invariant rings carry the structure of the k-th Wallach representation $L = L(-kC\zeta)$, which is a unitary highest weight module. By a result of Hochster and

Roberts [19], the modules $L(-kC\zeta)$ are graded Cohen-Macaulay algebras generated by elements in degree one, so their Hilbert series can be written as a rational function whose numerator has positive integer coefficients and whose denominator has a pole of order equal to the Krull dimension of L. In fact, they are of the form:

$$H_L(t) = \frac{H_E(t)}{(1-t)^D},$$

where E is a finite-dimensional simple highest weight module of a reduced Hermitian symmetric pair related to L and D is the Gelfand-Kirllov dimension of L. The correspondence between the families L and E in this context is referred to as the *Wonderful Correspondence*. Special cases of the Wonderful Correspondence were discovered by Enright and Willenbring for general and symmetric determinantal varieties [13], and their results were extended by Enright and Hunziker [8]. This result is new in the skew-symmetric determinantal variety case.

1.3 Quasi-dominant Weights

For the more general setting in which the highest weight of L is allowed to be any quasi-dominant weight, a result called the Transfer Theorem gives the Hilbert series of L in the same form as in the Wonderful Correspondence case, except that the numerator polynomial is multiplied by a rational number [13], [8] (cf. [10]). The key ingredient in the proof of this result is a comparison of the finite graded free resolutions of L and E, which have equal length. The existence of these resolutions is guaranteed by the Hilbert Syzygy Theorem, and their explicit description comes from Enright's results on the Lie algebra cohomology of $\mathfrak{p}^+$ [7].

Quasi-dominant weights are exceptional for more reasons than the production of an interesting Hilbert series. Their definition will be given in Section 2.6 in terms of root data, but there are several other equivalent characterizations. For example, the correspondence $L \to E$ mentioned above is a special case of a correspondence $L(\lambda) \to L(\mu)$ induced by equivalences of categories related to infinitesimal blocks in parabolic category $\mathcal{O}^{\mathfrak{p}}$ [11] (cf. [10]). In this context, $L(\mu)$ is finite-dimensional precisely when λ is quasi-dominant. This is also equivalent to μ being the highest weight in its orbit under a certain subgroup of the Weyl group. A new characterization, which is described in Section 4, identifies the quasi-dominant weights as those whose unitary highest weight module has a small associated variety (in the sense of Theorem 4.1.1). Equivalently, the module associated to $L(\lambda)$ via reductive dual pairs has, in some sense, a small highest weight, as described in Section 4.2.

CHAPTER TWO

Background

2.1 Hilbert Series

As in the introduction, let $(G_{\mathbb{R}}, K_{\mathbb{R}})$ be an irreducible Hermitian symmetric pair of noncompact type with Lie algebras $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}})$. Denote the complexified Lie algebras $(\mathfrak{g}, \mathfrak{k})$, and fix a Cartan subalgebra $\mathfrak{h}$ of both $\mathfrak{g}$ and $\mathfrak{k}$. Then $\mathfrak{k}$ has a one-dimensional center $\mathfrak{z}(\mathfrak{k}) \subset \mathfrak{h}$. Take Δ to be the corresponding roots of $\mathfrak{g}$, and take Δ_{c} to be the roots of $\mathfrak{k}$. Call the complementary set Δ_{nc} so that $\Delta = \Delta_{\mathrm{c}} \cup \Delta_{\mathrm{nc}}$. These two sets are also referred to as the compact and noncompact roots, respectively. Fix a positive root system Δ^+, and define $\Delta_{\mathrm{c}}^+ := \Delta_{\mathrm{c}} \cap \Delta^+$ and $\Delta_{\mathrm{nc}}^+ := \Delta_{\mathrm{nc}} \cap \Delta^+$. Then $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ as a $\mathfrak{k}$-module, where the summands in the decomposition are the -1, 0, and $+1$ eigenspaces of $\mathrm{ad} z_0$ for a unique element $z_0 \in \mathfrak{z}(\mathfrak{k})$. This central element acts semisimply on any highest weight $\mathfrak{g}$-module M and gives the following natural grading of M as a $\mathfrak{k}$-module. Define M_0 to be the $\mathfrak{k}$-submodule of M generated by a highest weight vector, and for $i > 0$ define $M_i := \mathfrak{p}^- M_{i-1}$. Let $A_i := \{\sum_{j=1}^{i} \alpha_j \mid \text{each } \alpha_j \in \mathfrak{p}^-\}$. Then M_i is the $(\lambda(z_0) - i)$-eigenspace for the action of z_0 on M and decomposes as the direct sum of weight spaces $M_i = \oplus_{\alpha \in A_i} \mathfrak{g}_{\lambda-\alpha}$. The Hilbert series of a highest weight module M is given by the formal power series

$$H_M(t) = \sum_{i \in \mathbb{N}} \dim(M_i) t^i. \tag{2.1}$$

Example 2.1.1 (Generalized Verma module)**.** Let $\lambda \in \mathfrak{h}^*$ be $\mathfrak{k}$-dominant integral, and denote by F_λ the finite-dimensional simple $\mathfrak{k}$-module with highest weight λ. Induce to a $\mathfrak{q} := \mathfrak{k} \oplus \mathfrak{p}^+$-module by letting $\mathfrak{p}^+$ act by zero. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. Define the *generalized Verma module* $N(\lambda)$ of highest weight λ by

$$N(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F_\lambda.$$

By the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{g}) \cong U(\mathfrak{p}^-) \otimes U(\mathfrak{q})$ as vector spaces. Since the nilradical $\mathfrak{p}^+$ is abelian, we can identify $U(\mathfrak{p}^\pm)$ with the symmetric algebra $S(\mathfrak{p}^\pm)$. In particular, we have the isomorphism of $\mathfrak{k}$-modules

$$N(\lambda) \cong S(\mathfrak{p}^-) \otimes_{\mathbb{C}} F_\lambda.$$

The grading of $N(\lambda)$ is given by $N(\lambda)_i \cong S(\mathfrak{p}^-)^i \otimes F_\lambda$, so that

$$H_{N(\lambda)}(t) = \frac{\dim F_\lambda}{(1-t)^{\dim \mathfrak{p}^-}}.$$

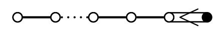
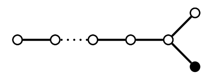
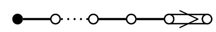
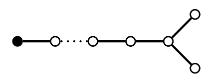
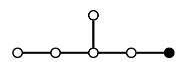
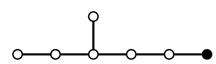
2.2 Organization of Unitarizable Highest Weight Modules

We will obtain the Hilbert series of the classical determinantal varieties by studying the Hilbert series of a special family of unitary highest weight modules. In [17] and [18], Harish-Chandra showed that the nontrivial unitarizable highest weight modules can be classified according to irreducible Hermitian symmetric pairs of noncompact type. The list of all such pairs is given in Table 2.1 with the three families of pairs occurring in a dual pair setting at the top (the dual pair setting will be explained in Section 2.3). The filled in nodes in the Dynkin diagram represent the unique simple noncompact root.

Here we recall the classification of unitarizable highest weight modules given by Enright, Howe and Wallach [14] and discuss an alternate organization of the reduction points in terms of vertices and cones. This cone decomposition, which will be essential to the results communicated in Section 4, was introduced by Davidson, Enright, and Stanke [4] §6.

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots in Δ^+, labeled according to Bourbaki [3]. Define $\Pi_c := \Pi \cap \Delta_c$ and $\Pi_{nc} := \Pi \cap \Delta_{nc}$. Then $|\Pi_{nc}| = 1$. Let $(\ ,\ )$ be the nondegenerate bilinear form on $\mathfrak{h}^*$ induced by the Killing form on $\mathfrak{g}$. For $\alpha \in \Delta$,

Table 2.1. Hermitian symmetric pairs of noncompact type

$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{k}_{\mathbb{R}}$	Dynkin diagram
$\mathfrak{su}(p,q)$	$\mathfrak{s}(\mathfrak{u}(p)\oplus\mathfrak{u}(q))$	
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{u}(n)$	
$\mathfrak{so}^*(2n)$	$\mathfrak{u}(n)$	
$\mathfrak{so}(2n-1,2)$	$\mathfrak{so}(2n-1)\oplus\mathfrak{so}(2)$	
$\mathfrak{so}(2n-2,2)$	$\mathfrak{so}(2n-2)\oplus\mathfrak{so}(2)$	
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10)\oplus\mathbb{R}$	
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6\oplus\mathbb{R}$	

the coroot of α is $\alpha^\vee := 2\alpha/(\alpha,\alpha)$. The fundamental weights $\omega_1,\dots,\omega_n$ are defined by $(\omega_i,\alpha_j^\vee)=\delta_{i,j}$ for $\alpha_j\in\Pi$ with $1\le i,j\le n$. Take β to be the maximal root in Δ, and let ζ be the element of $\mathfrak{h}^*$ that is orthogonal to Δ_{c} and for which $(\zeta,\beta^\vee)=1$. Then ζ is the i-th fundamental root, where α_i is the noncompact simple root. Define

$$P_{\mathrm{c}}^+ := \{\tau\in\mathfrak{h}^* \mid (\tau,\alpha^\vee)\in\mathbb{Z}_{\ge 0} \text{ for } \alpha\in\Pi_{\mathrm{c}} \text{ and } (\tau,\alpha^\vee)=0 \text{ for } \alpha\in\Pi_{\mathrm{nc}}\}.$$

Denote by Λ_{u} the set of all weights that occur as the highest weight of a unitary highest weight module. If $\lambda\in\Lambda_{\mathrm{u}}$, then $\lambda=\tau+u\zeta$ for some $\tau\in P_{\mathrm{c}}^+$ and $u\in\mathbb{R}$. Another parametrization of the line $\tau+\mathbb{R}\zeta$ is useful for describing the set $\Lambda_{\mathrm{u}}\cap(\tau+\mathbb{R}\zeta)$. For a fixed $\tau\in P_{\mathrm{c}}^+$, define

$$\lambda_0(\tau) := \tau-(\tau+\rho,\beta^\vee)\zeta. \tag{2.2}$$

Then $\lambda_0(\tau)$ is the unique point on the line $\tau+\mathbb{R}\zeta$ whose associated highest weight module is a limit of discrete series module, which is equivalent to the property that $(\lambda_0(\tau)+\rho,\beta^\vee)=0$. The classification presented in [14] gives the weights $\lambda\in\Lambda_{\mathrm{u}}\cap(\tau+\mathbb{R}\zeta)$ in the form $\lambda_0+z\zeta$. For a given line $\lambda_0+\mathbb{R}\zeta$, the set of values of

z for which $\lambda_0 + z\zeta \in \Lambda_\mathrm{u}$ has a continuous spectrum and a discrete set. The discrete set is equally spaced and begins at the positive tail end of the continuous spectrum, as illustrated in Figure 2.1.

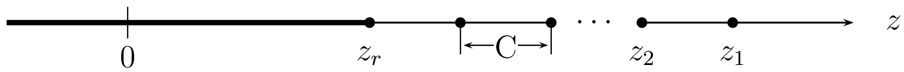


Figure 2.1. The set of unitary values for z

Let $L(\lambda)$ denote the unique irreducible quotient of the generalized Verma module $N(\lambda)$. The values of z in the continuous spectrum give weights $\lambda = \lambda_0 + z\zeta$ for which $N(\lambda) = L(\lambda)$. The values z_ℓ in the discrete set give weights λ for which $N(\lambda)$ is reducible. The weights $\lambda = \lambda_0 + z_\ell\zeta$ are called the reduction points, and the positive integer ℓ is called the *level of reduction* of $L(\lambda)$. Define

$$\Lambda_\mathrm{r} := \{\lambda \in \Lambda_\mathrm{u} \mid N(\lambda) \neq L(\lambda)\},$$

the set of all reduction points. To give formulas for the reduction points, two root systems were introduced that in some sense measure the singularity of τ with respect to the compact roots. Define $\Delta_\mathrm{c}(\tau) := \{\alpha \in \Delta_\mathrm{c} \mid (\tau, \alpha^\vee) = 0\}$ and take Ψ to be the subroot system of Δ generated by $\pm\beta$ and $\Delta_\mathrm{c}(\tau)$. Then $Q(\tau)$ is the simple component of Ψ that contains $-\beta$. If Δ has two root lengths and there are short compact roots α not orthogonal to $Q(\tau)$ for which $(\tau, \alpha^\vee) = 1$, take Φ to be the root system generated by $\pm\beta$, $\Delta_\mathrm{c}(\tau)$, and all such α. Then $R(\tau)$ is the simple component of Φ that contains $-\beta$. If Δ only has one root length or no such α exists, take $R(\tau) = Q(\tau)$. Since these two systems are subsystems of Δ, they both contain compact and non-compact roots and are the root system of a Hermitian symmetric pair with $-\beta$ playing the role of the noncompact simple root. The constant C in Figure 2.1 depends only on the type of Hermitian symmetric pair. The constant r is the real rank of $Q(\tau)$.

Table 2.2. Constants for Hermitian symmetric pairs of noncompact type

$\mathfrak{g}_{\mathbb{R}}$	r	C	$h^\vee$
$\mathfrak{su}(p,q), \quad n=p+q-1$	$\min\{p,q\}$	1	$n+1$
$\mathfrak{sp}(n,\mathbb{R})$	n	$1/2$	$n+1$
$\mathfrak{so}^*(2n)$	$[n/2]$	2	$2n-2$
$\mathfrak{so}(2,2n-2)$	2	$n-2$	$2n-2$
$\mathfrak{so}(2,2n-1)$	2	$n-3/2$	$2n-1$
$\mathfrak{e}_{6(-14)}$	2	3	12
$\mathfrak{e}_{7(-25)}$	3	4	18

The real numbers $z_r, \ldots, z_1$ were determined in [14] for each type of Hermitian symmetric pair, and a uniform description was discovered by Hunziker and Bai in [2]. The latter description utilizes the dual Coxeter number of $\mathfrak{g}$, which is $h^\vee := (\rho, \beta^\vee)+1$. Using this description, the reduction points are the weights $\lambda_0 + z_\ell \zeta$ with

$$z_\ell = h_Q^\vee + \frac{r_R - r_Q}{2} - (\ell-1)C - 1, \quad 1 \le \ell \le r_Q. \tag{2.3}$$

Here r_Q and r_R denote the real rank of $Q(\tau)$ and $R(\tau)$, respectively, and $h_Q^\vee$ is the dual Coxeter number associated to $Q(\tau)$. Translating back to the original τ and letting $\tau + u_\ell\zeta = \lambda_0(\tau) + z_\ell\zeta$ gives the equivalent description

$$u_\ell = h_Q^\vee - h^\vee + \frac{r_R - r_Q}{2} - (\ell-1)C - (\tau, \beta^\vee), \quad 1 \le \ell \le r_Q. \tag{2.4}$$

The reduction points Λ_{r} were reorganized in [4] according to $Q(\tau)$, $R(\tau)$, vertices, and cones in the following manner. From the formulation of $Q(\tau)$ and $R(\tau)$ and the algorithm for computing them, it is apparent that certain reduction points will share the same Q and R. The set of points for which this is true can be divided into subsets according to the level of reduction, $\ell(\tau)$. For a given triple $(Q(\tau), R(\tau), \ell(\tau))$, the set of reduction points sharing the triple can be written $\lambda_v + C_v$ for a vertex

$$\lambda_v = \tau_v + (z_\ell - (\tau_v + \rho, \beta^\vee))\zeta$$

and an associated cone, C_v. For $\lambda = \lambda_0 + z\zeta$, we have $(\lambda + \rho, \beta^\vee) = z$. It follows from (2.3) that preserving $(\lambda + \rho, \beta^\vee)$ preserves the level of reduction, so

$$C_v = \{\gamma \in \mathfrak{h}^* \mid (\gamma, \alpha^\vee) \in \mathbb{Z}_{\geq 0} \text{ for } \alpha \in \Delta_{\mathrm{c}}^+,\ (\gamma, R(\tau_v)) = 0, \text{ and } (\gamma, \beta^\vee) = 0\}.$$

For example, consider the case $\mathfrak{g}_\mathbb{R} = \mathfrak{su}(p, q)$. Here $Q = R = \mathrm{SU}(p', q')$ with $1 \leq p' \leq p$ and $1 \leq q' \leq q$. The cone C_v is given by

$$C_v = \{\sum_{i=p'}^{n-q'} a_i \omega_i \mid a_p = -a_{p'} - \ldots - a_{p-1} - a_{p+1} - \ldots - a_{n-q'}\},$$

where each a_i other than a_p is a nonnegative integer.

2.3 Dual Pair Setting

The rich theory of reductive dual pairs intersects the study of Hermitian symmetric pairs of noncompact type in the cases $\mathfrak{g}_\mathbb{R} = \mathfrak{su}(p, q), \mathfrak{sp}(n, \mathbb{R}), \mathfrak{so}^*(2n)$. In particular, let $K'_\mathbb{R}$ be the compact Lie group associated to $\mathfrak{g}_\mathbb{R}$ in Table 2.3, and denote its complexification by K'. Then there is a duality between the (unitary) irreducible representations of K' and a subset of the unitarizable simple highest weight modules of $\mathfrak{g}$. In fact, every reduction point occurs in this setting for $\mathfrak{g}_\mathbb{R} = \mathfrak{su}(p, q), \mathfrak{sp}(n, \mathbb{R})$. Table 2.3 also gives a set of matrices Z whose coordinate

Table 2.3. Objects involved in Howe duality

$\mathfrak{g}_\mathbb{R}$	$K_\mathbb{R}$	$K'_\mathbb{R}$	$\mathbb{C}[Z]$
$\mathfrak{su}(p, q)$	$\mathrm{U}(p) \times \mathrm{U}(q)$	$\mathrm{U}(k)$	$\mathbb{C}[\mathrm{M}_{p,k} \oplus M_{k,q}]$
$\mathfrak{sp}(n, \mathbb{R})$	$\tilde{\mathrm{U}}(n)$	$\mathrm{O}(k)$	$\mathbb{C}[\mathrm{M}_{n,k}]$
$\mathfrak{so}^*(2n)$	$\mathrm{U}(n)$	$\mathrm{Sp}(k)$	$\mathbb{C}[\mathrm{M}_{n,2k}]$

ring gives a realization of this duality. In the table, $\tilde{\mathrm{U}}(n)$ is the double cover given by $\tilde{\mathrm{U}}(n) := \{(u, s) \in \mathrm{U}(n) \times \mathbb{C} \mid \det u = s^2\}$. The group $K_{\mathbb{R}}$ acts on Z as follows:

$$\mathrm{U}(p) \times \mathrm{U}(q) \circlearrowleft \mathrm{M}_{p,k} \oplus M_{k,q} : ((u_1, u_2), (z_1, z_2)) \mapsto (u_1 z_1, z_2 u_2^{-1}),$$

$$\tilde{\mathrm{U}}(n) \circlearrowleft \mathrm{M}_{n,k} : ((u, s), z) \mapsto uz,$$

$$\mathrm{U}(n) \circlearrowleft \mathrm{M}_{n,2k} : (u, z) \mapsto uz.$$

To extend the action of $K_{\mathbb{R}}$ on $\mathbb{C}[Z]$ to a $(\mathfrak{g}, \mathrm{K}_{\mathbb{R}})$-action, the usual $K_{\mathbb{R}}$-action on $\mathbb{C}[Z]$ is twisted in the following way:

$$\mathrm{U}(p) \times \mathrm{U}(q) \circlearrowleft \mathbb{C}[\mathrm{M}_{p,k} \oplus M_{k,q}] : ((u_1, u_2).f)(z) \mapsto (\det u_1)^{-k} f(u_1^{-1} z_1, z_2 u_2),$$

$$\tilde{\mathrm{U}}(n) \circlearrowleft \mathbb{C}[\mathrm{M}_{n,k}] : ((u, s), z) \mapsto s^{-k} f(u^{-1} z),$$

$$\mathrm{U}(n) \circlearrowleft \mathbb{C}[\mathrm{M}_{n,2k}] : (u, z) \mapsto (\det u)^{-k} f(u^{-1} z).$$

The Lie algebra $\mathfrak{g}$ can be embedded into $\mathcal{D}(Z)^K$, the algebra of K-invariant differential operators on Z with polynomial coefficients. Under this embedding, $\mathbb{C}[Z]$ is a $(\mathfrak{g}, K_{\mathbb{R}})$-module, and the image of $\mathfrak{g}$ generates $\mathcal{D}(Z)^K$ [20] (cf. [10]). Denote by $\widehat{K'}$ the isomorphism classes of irreducible unitary highest weight representations of K'.

Theorem 2.3.1 (Howe duality). *Let* $\Sigma := \{\sigma \in \widehat{K'} \mid \mathrm{Hom}_{K'}(V_\sigma, \mathbb{C}[Z]) \neq 0\}$. *Then, as a* $K' \times (\mathfrak{g}, K_{\mathbb{R}})$*-module,*

$$\mathbb{C}[Z] = \bigoplus_{\sigma \in \Sigma} V_\sigma \otimes V^\sigma,$$

where $V^\sigma := \mathrm{Hom}_{K'}(V_\sigma, \mathbb{C}[Z])$ *is a simple highest weight module of* $\mathfrak{g}$ *with* $V^\sigma \not\cong V^{\sigma'}$ *whenever* $\sigma \neq \sigma'$. *Furthermore, each module* V^σ *is a unitary highest weight module, i.e., the* $(\mathfrak{g}, K_{\mathbb{R}})$*-module* V^σ *is the Harish–Chandra module of a unitary highest weight representation of* $G_{\mathbb{R}}$. □

Proof. Since $\mathbb{C}[Z]$ is a simple $\mathcal{D}(Z)$-module, the result follows from a general duality theorem of Goodman and Wallach [16], §4.2,4.6. □

The reflection $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ is the linear map given by $s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha$, and the Weyl group $\mathcal{W}$ is the group generated by the simple reflections s_α with $\alpha \in \Pi$. The length of a Weyl group element w is denoted $\ell(w)$ and is defined to be the length of the shortest expression of w in terms of simple reflections. Let $\mathcal{W}_{\mathrm{I}}$ be the subgroup of $\mathcal{W}$ generated by s_α with $\alpha \in \Pi_{\mathrm{c}}$, and for $w \in \mathcal{W}$ define

$$\Phi_w := \Phi^+ \cap w\Phi^-,$$

the positive roots sent to negative roots by w^{-1}. Kostant [25] showed that the set

$${}^{\mathrm{I}}\mathcal{W} := \{w \in \mathcal{W} \mid \Phi_w \subseteq \Delta^+_{\mathrm{nc}}\}$$

is the set of minimal length coset representatives for $\mathcal{W}_{\mathrm{I}}\backslash\mathcal{W}$, the right cosets of $\mathcal{W}_{\mathrm{I}}$.

Remark 2.4.1. The notation for the sets $\mathcal{W}_{\mathrm{I}}$ and ${}^{\mathrm{I}}\mathcal{W}$ is motivated by the more general setting of parabolic subalgebras, where I is an arbitrary subset of Π. In the setting of Hermitian symmetric pairs, the subset I is always Π_{c}.

The set ${}^{\mathrm{I}}\mathcal{W}$ is a poset via the Bruhat ordering, which is defined as follows. For $v, w \in {}^{\mathrm{I}}\mathcal{W}$, let $v \to w$ indicate the existence of a root α for which $vs_\alpha = w$ and $\ell(w) = \ell(v) + 1$. Then $v \leq w$ in the Bruhat ordering precisely when there is a chain $v = w_1 \to w_2 \to \cdots \to w_m = w$ in ${}^{\mathrm{I}}\mathcal{W}$. If, in the definition, we change vs_α to $s_\alpha v$, we obtain the same ordering. There is also a notion of weak Bruhat ordering, which is identical to the Bruhat ordering with the caveat that $\alpha \in \Pi$. However, the Bruhat ordering and the weak Bruhat ordering coincide in the context of Hermitian symmetric pairs of noncompact type.

The set of noncompact positive roots, Δ^+_{nc}, is also a poset via the ordering induced by the usual ordering on weights: $\mu \leq \lambda$ if and only if $\mu - \lambda$ is a sum of positive roots or zero. The Hasse diagrams of Δ^+_{nc} were used by Jakobsen in his classification of the unitarizable highest weight modules, and he showed that each Hasse diagram is two-dimensional and can be drawn on a square lattice ([22] Lemma 4.1). These Hasse diagrams have corresponding generalized Young diagrams obtained by rotating the Hasse diagram 90° clockwise and fattening the nodes as in Table 2.4.

The numbers attached to the edges of the Hasse diagram give the index of the simple root that is added to the (noncompact root represented by the) bottom node in order to obtain the (noncompact root represented by the) top node. Opposite edges of a given square are understood to represent the same simple root. To understand the numbers in the boxes of the Young diagram, first notice that there is a natural bijection between ${}^{\mathrm{I}}\mathcal{W}$ and Δ^+_{nc} given by

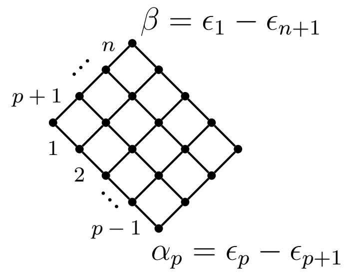
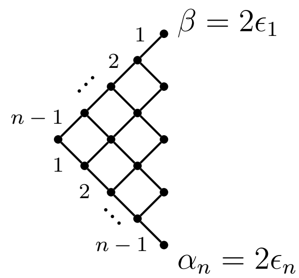
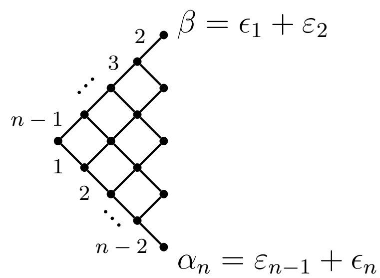
$$w \leftrightarrow \beta, \quad \text{where } \Phi_w = \{\gamma \in \Delta^+_{\mathrm{nc}} \mid \gamma \leq \beta\}. \tag{2.5}$$

In general, ${}^{\mathrm{I}}\mathcal{W}$ and Δ^+_{nc} are not isomorphic as posets, i.e. their Hasse diagrams are not isomorphic. However, in the notation of (2.5), we can use the Young diagram of β to obtain a canonical reduced expression for w (in terms of $\ell(w)$ simple reflections) in the following way. Suppose $\beta \in \Delta^+_{\mathrm{nc}}$. Then there exist unique $v, w \in {}^{\mathrm{I}}\mathcal{W}$ for which $\Phi_w = \Phi_v \dot{\cup} \beta$. It follows that $\ell(w) = \ell(v) + 1$ and $w = vs_\alpha$ for $\alpha = v^{-1}\beta \in \Pi$. Given $w \in {}^{\mathrm{I}}\mathcal{W}$, the corresponding $\beta \in \Delta^+_{\mathrm{nc}}$ is represented by a node in the Hasse diagram of Δ^+_{nc}. Fill in the corresponding box of the Young diagram with the index of the simple root $v^{-1}\beta$. Once all the boxes have been filled in, the portion of the Young diagram corresponding to Φ_w yields a reduced expression for any $w \in {}^{\mathrm{I}}\mathcal{W}$. There is a second canonical reduced expression for w in terms of noncompact positive roots that is sometimes more convenient to use than the one just described. This is made explicit in the following proposition.

Proposition 2.4.2 (cf. Enright-Hunziker-Pruett [10]). *Let $w \in {}^{\mathrm{I}}\mathcal{W}$ and write*

$$\Phi_w = \{\beta_1, \beta_2, \ldots, \beta_l\}$$

Table 2.4. Hasse diagrams and generalized Young diagrams for Δ^+_{nc}

$\mathfrak{g}_{\mathbb{R}}$	Hasse diagram of Δ^+_{nc}	Young diagram
$\mathfrak{su}(p,q)$		$\begin{array}{\|c\|c\|c\|c\|c\|}\hline p & p-1 & \cdots & 2 & 1 \\ \hline p+1 & p & \cdots & 3 & 2 \\ \hline \vdots & \vdots & \cdots & \vdots & \vdots \\ \hline n & n-1 & \cdots & q+1 & q \\ \hline\end{array}$
$\mathfrak{sp}(n,\mathbb{R})$		$\begin{array}{ccccc} n & n-1 & \cdots & 2 & 1 \\ & n & n-1 & \cdots & 2 \\ & & \ddots & \cdots & \vdots \\ & & & n & n-1 \\ & & & & n \end{array}$
$\mathfrak{so}^*(2n)$		$\begin{array}{ccccc} n & n-2 & \cdots & 2 & 1 \\ & n-1 & n-2 & \cdots & 2 \\ & & \ddots & \cdots & \vdots \\ & & & n & n-2 \\ & & & & n-1 \end{array}$

so that for $1 \le j \le l$, *the set* $\{\beta_1, \beta_2, \ldots, \beta_j\}$ *is a lower order ideal of* Δ^+_{nc}. *Then*

$$w = s_{f(\beta_1)} s_{f(\beta_2)} \ldots s_{f(\beta_l)}$$

is a reduced expression for w. *Furthermore,* $w = s_{\beta_l} s_{\beta_{l-1}} \ldots s_{\beta_1}$. $\square$

Note that in the case $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n)$ the bottom right box of the Young diagram is n if n is even and is $n-1$ if n is odd.

Example 2.4.3. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(3, \mathbb{R})$ and take w to be the longest element of ${}^{\mathrm{I}}\mathcal{W}$. The Hasse diagram and generalized Young diagram of ${}^{\mathrm{I}}\mathcal{W}$ are given in Figure 2.2.

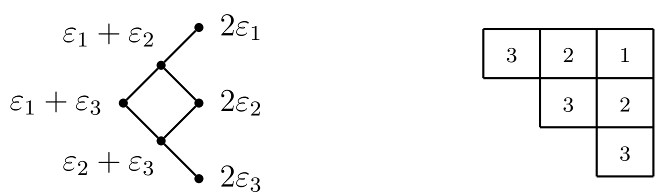


Figure 2.2. Hasse diagram and Young diagram of the longest element in ${}^{\mathrm{I}}\mathcal{W}$ for $\mathfrak{sp}(3, \mathbb{R})$

It follows that $w = s_3 s_2 s_1 s_3 s_2 s_3 = s_{2\varepsilon_1} s_{\varepsilon_1+\varepsilon_2} s_{2\varepsilon_2} s_{\varepsilon_1+\varepsilon_3} s_{\varepsilon_2+\varepsilon_3} s_3$. Note that s_1 and s_3 commute (as do $s_{2\varepsilon_2}$ and $s_{\varepsilon_1+\varepsilon_3}$), so the ordering of the noncompact roots in Proposition 2.4.2 is canonical up to the order of orthogonal roots that have the same length.

Example 2.4.4. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(8)$ and again let w to be the longest element of ${}^{\mathrm{I}}\mathcal{W}$. The associated diagrams of ${}^{\mathrm{I}}\mathcal{W}$ are then the ones displayed in Figure 2.3.

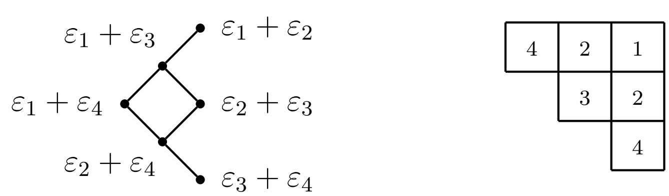


Figure 2.3. Hasse diagram and Young diagram of the longest element in ${}^{\mathrm{I}}\mathcal{W}$ for $\mathfrak{so}^*(8)$

It follows that $w = s_4 s_2 s_1 s_3 s_2 s_4 = s_{\varepsilon_1+\varepsilon_2} s_{\varepsilon_1+\varepsilon_3} s_{\varepsilon_2+\varepsilon_3} s_{\varepsilon_1+\varepsilon_4} s_{\varepsilon_2+\varepsilon_4} s_4$. Here s_1 and s_3 commute, as do $s_{\varepsilon_2+\varepsilon_3}$ and $s_{\varepsilon_1+\varepsilon_4}$.

2.5 Reduced Hermitian Symmetric Pairs and Resolutions

Every $\mathfrak{k}$-dominant weight λ gives rise to a reduced Hermitian symmetric pair (G_λ, K_λ) in the following way. Define $\Xi_\lambda := \{\alpha \in \Delta \mid (\lambda+\rho, \alpha^\vee) = 0\}$ and

$$\Omega_\lambda := \{\gamma \in \Delta^+_{\mathrm{nc}} \mid (\gamma, \alpha) = 0 \ \forall \alpha \in \Xi_\lambda; \gamma \text{ is short if } \Xi_\lambda \text{ contains a long root}\}. \quad (2.6)$$

Define $\Delta^+_{\lambda,nc} := \{\alpha \in \Omega_\lambda \mid (\lambda+\rho, \alpha^\vee) \in \mathbb{Z}_{>0}\}$. Take $\mathcal{W}_\lambda$ to be the subgroup of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ generated by the reflections s_γ for which $\gamma \in \Delta^+_{\lambda,nc}$. Then $\Delta_\lambda := \{\alpha \in \Delta \mid s_\alpha \in \mathcal{W}_\lambda\}$ is an abstract root system with Weyl group $\mathcal{W}_\lambda$. Let $\Delta^+_\lambda := \Delta_\lambda \cap \Delta^+$, $\Delta_{\lambda,\mathrm{c}} := \Delta_\lambda \cap \Delta_{\mathrm{c}}$, and $\Delta^+_{\lambda,\mathrm{c}} := \Delta_{\lambda,\mathrm{c}} \cap \Delta^+$. Take ρ (respectively, ρ_λ) to be half the sum of the positive roots in Δ (respectively, Δ_λ). Denote by $\mathfrak{g}_\lambda$ the semisimple part of the reductive Lie algebra with root system Δ_λ and Cartan subalgebra $\mathfrak{h}$, and set $\mathfrak{k}_\lambda$ to be the reductive Lie subalgebra of $\mathfrak{g}_\lambda$ with Cartan subalgebra $\mathfrak{g}_\lambda \cap \mathfrak{h}$ and root system $\Delta_{\lambda,\mathrm{c}}$. Then $(\mathfrak{g}_\lambda, \mathfrak{k}_\lambda)$ is the pair of complexified Lie algebras of an irreducible Hermitian symmetric pair of noncompact type (G_λ, K_λ).

Example 2.5.1. Suppose $\mathfrak{g}_\mathbb{R} = \mathfrak{so}^*(2n)$ and $\lambda = -2k\omega_n$, the highest weight of the k-th Wallach representation. Then $\lambda + \rho = (n-k-1, n-k-2, \ldots, -k)$ in Euclidean coordinates, and the positive roots orthogonal to $\lambda+\rho$ are $\Xi^+_\lambda = \{\varepsilon_{n-2k} + \varepsilon_n, \varepsilon_{n-2k+1} + \varepsilon_{n-1}, \ldots, \varepsilon_{n-k-1} + \varepsilon_{n-k+1}\}$. Since $\varepsilon_{n-k} = 0$ and $\varepsilon_i \in \mathbb{Z}_{>0}$ for $1 \le i \le n-2k-1$, the sets Ω_λ and $\Delta^+_{\lambda,\mathrm{nc}}$ are equal and are given by:

$$\Delta^+_{\lambda,\mathrm{nc}} = \{\varepsilon_i + \varepsilon_j \mid 1 \le i < j \text{ with } j \le n-2k-1 \text{ or } j = n-k\}.$$

The noncompact real form of $\mathfrak{g}_\lambda$ is $\mathfrak{g}_{\lambda,\mathbb{R}} \cong \mathfrak{so}^*(2(n-2k))$, and the Hasse diagram of ${}^{\mathrm{E}}\mathcal{W}_\lambda$ is displayed in Figure 2.4.

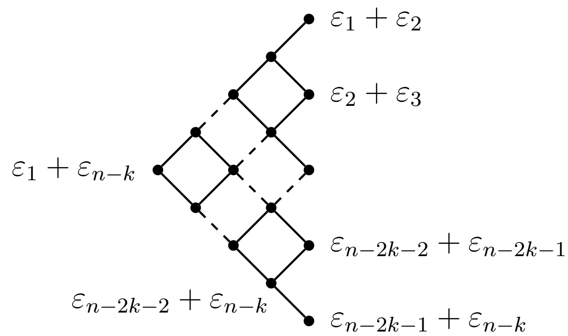


Figure 2.4. Hasse diagram of ${}^{\mathrm{E}}\mathcal{W}_\lambda$ for $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n)$ with $\lambda = -2k\omega_n$

Define subsets of the Weyl group by ${}^{j}\mathcal{W} := \{w \in \mathcal{W} \mid \ell(w) = j\}$ and ${}^{\mathrm{I},j}\mathcal{W} := {}^{\mathrm{I}}\mathcal{W} \cap {}^{j}\mathcal{W}$. Recall the "dot action" of $w \in \mathcal{W}$ on $\gamma \in \mathfrak{h}^*$ given by $w.\gamma = w(\gamma+\rho) - \rho$. In 1977, Lepowsky generalized a result of Bernstien-Gelfand-Gelfand by proving the following result.

Theorem 2.5.2 (Lepowsky [26]). *Let $\lambda \in \mathfrak{h}^*$ be $\mathfrak{g}$-dominant, and denote by E_λ the finite-dimensional simple highest weight module with highest weight λ. Then there exists an exact sequence of $\mathfrak{g}$-modules, $0 \to C_p \to \cdots \to C_1 \to C_0 \to E_\lambda \to 0$, with*

$$C_j := \bigoplus_{w} N(w.\lambda),$$

where $p = |\Delta_{nc}^+|$ and the sum is over elements of ${}^{\mathrm{I},j}\mathcal{W}$. $\square$

For $w \in \mathcal{W}_\lambda$, denote by $\ell_\lambda(w)$ the length of w with respect to the simple system $\Pi_\lambda \subset \Delta_\lambda^+$. Note that, in general, $\ell_\lambda(w) \neq \ell(w)$. Define ${}^{j}\mathcal{W}_\lambda := \{w \in \mathcal{W}_\lambda \mid \ell_\lambda(w) = j\}$ and ${}^{\mathrm{E},j}\mathcal{W}_\lambda := {}^{\mathrm{E}}\mathcal{W}_\lambda \cap {}^{j}\mathcal{W}_\lambda$, where Π_λ / E is the unique noncompact root in Π_λ. For any $\mathfrak{k}$-integral γ, let $[\gamma]^+$ denote the unique $\mathfrak{k}$-dominant element in the $\mathcal{W}_{\mathrm{I}}$-orbit of γ. Define the "plus-dot" action of $\mathcal{W}$ on $\mathfrak{h}^*$ by $w \stackrel{+}{.} \gamma = [w(\gamma+\rho)]^+ - \rho$. In 2003, Enright and Willenbring used Enright's results on Lie algebra cohomology to obtain resolutions for unitarizable highest weight modules that occur in a dual pair setting. This work was extended by Enright and Hunziker in 2004 to obtain resolutions for all unitarizable highest weight modules.

Theorem 2.5.3 (Enright-Hunziker [8], cf. Enright-Willenbring [13])**.** *Let $L(\lambda)$ be the unitarizable highest weight module with highest weight λ. Then there exists an exact sequence of $\mathfrak{g}$-modules, $0 \to C^{\lambda}_{p_\lambda} \to \cdots \to C^{\lambda}_1 \to C^{\lambda}_0 \to L(\lambda) \to 0$, with*

$$C^{\lambda}_j := \bigoplus_{w} N(w \dot{+} \lambda),$$

where $p_\lambda = |\Delta^{+}_{\lambda,nc}|$ and the sum is over ${}^{\mathrm{E},j}\mathcal{W}_\lambda$. □

2.6 The Transfer Theorem

In order to obtain homogeneous maps of degree 0 in the resolutions above, the grading of $N(w.\lambda)$ (respectively, $N(w \dot{+} \lambda)$) is shifted so that the i-th homogeneous component of the original grading is the $(\lambda(z_0) - w.\lambda(z_0) + i)$-th homogeneous component (respectively, the $(\lambda(z_0) - w \dot{+} \lambda(z_0) + i)$-th homogeneous component) of the new grading. For a given Weyl group element w, define $d_w := (\lambda - w \dot{+} \lambda)(z_0) = (\lambda + \rho - [w(\lambda + \rho)]^{+})(z_0)$. Since z_0 is $\mathfrak{k}$-central, $d_w = (\lambda + \rho - w(\lambda + \rho))(z_0)$.

Proposition 2.6.1 (cf. Enright-Hunziker [8])**.** *Suppose $L = L(\lambda)$ is a unitarizable highest weight module. Then the Hilbert series of L is*

$$H_L(t) = \frac{1}{(1-t)^p} \sum_{w} (-1)^{\ell_\lambda(w)} \dim(F_{w \dot{+} \lambda}) t^{d_w},$$

where $p = |\Delta^{+}_{nc}|$ and the sum is over $w \in {}^{\mathrm{E}}\mathcal{W}_\lambda$. □

Proof. Apply the Euler-Poincaré method to the resolution in Theorem 2.5.3. □

The following definition plays a central role in the remainder of this work.

Definition 2.6.2. A $\mathfrak{k}$-dominant integral weight λ is called *quasi-dominant* if

$$(\lambda + \rho, \gamma^{\vee}) \notin \mathbb{Z}_{\leq 0} \quad \text{for all } \gamma \in \Omega_\lambda.$$

When λ is quasi-dominant, $\mu = (\lambda + \rho)|_{\mathfrak{g}_\lambda \cap \mathfrak{h}} - \rho_\lambda$ is Δ^{+}_{λ}-dominant.

Proposition 2.6.3 (cf. Enright-Hunziker [8])**.** *Suppose λ is quasi-dominant and $E = E_\mu$ is the finite-dimensional simple $\mathfrak{g}_\lambda$-module with highest weight $\mu = (\lambda + \rho)|_{\mathfrak{g}_\lambda \cap \mathfrak{h}} - \rho_\lambda$. Then the Hilbert series of E is*

$$H_E(t) = \frac{1}{(1-t)^{p_\lambda}} \sum_w (-1)^{\ell_\lambda(w)} \dim(F_{\mathfrak{k}_\lambda, w.\mu}) t^{d_w},$$

where $p_\lambda = |\Delta^+_{\lambda,nc}|$ and the sum is over $w \in {}^{\mathrm{E}}\mathcal{W}_\lambda$. □

Proof. Apply the Euler-Poincaré method to the resolution in Theorem 2.5.2 and use the fact that $z_0 \in \mathfrak{g}_\lambda \cap \mathfrak{h}$, where $\mu + \rho_\lambda = \lambda + \rho$. □

These two propositions point toward a relationship between the Hilbert series of $L(\lambda)$ and that of E_μ. It is shown in [8] that the ratio $\dim(F_{\mathfrak{k},w\dagger\lambda})/\dim(F_{\mathfrak{k}_\lambda,w.\mu})$ is independent of $w \in {}^{\mathrm{E}}\mathcal{W}_\lambda$. This immediately gives the following theorem.

Theorem 2.6.4 (The Transfer Theorem – Enright-Hunziker [8] p. 623, cf. Enright-Willenbring [13])**.** *Suppose the irreducible highest weight representation L is unitarizable with quasi-dominant highest weight λ. Let E be the finite-dimensional simple $\mathfrak{g}_\lambda$-module with highest weight $\mu = \lambda + \rho - \rho_\lambda$. Then*

$$H_L(t) = R \cdot \frac{H_E(t)}{(1-t)^D},$$

where $R = \dim(F_{\mathfrak{k},\lambda})/\dim(F_{\mathfrak{k}_\lambda,\mu})$ and $D = |\Delta^+_{nc}| - |\Delta^+_{\lambda,nc}|$. □

It is worth noting here that D is the Gelfand-Kirillov dimension of L and that the Bernstein degree of L is $R \cdot H_E(1)$, which is R times the dimension of E as a vector space.

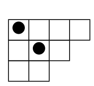
CHAPTER THREE

Hilbert Series of Determinantal Varieties

3.1 The Wonderful Correspondence

Let the constant C and the fundamental weight ζ be defined as in Section 2.2. The simple highest weight module $L(-kC\zeta)$, $1 \leq k \leq r-1$, is refered to as the k-th Wallach representation [28]. In 2003, Enright and Willenbring discovered some examples of Wallach representations that give $R = 1$ in the Transfer Theorem [12]. In 2004, Enright and Hunziker showed that for the cases $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p,q), \mathfrak{sp}(n,\mathbb{R}), \mathfrak{so}^*(2n)$, the k-th Wallach representation gives $R = 1, 1, \binom{n-k-1}{n}^{-1}$, respectively. The first result of this work is to show that one can modify the choice of the finite-dimensional $\mathfrak{g}_\lambda$-module E in the case $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n)$ to obtain $R = 1$ for each Wallach representation. The modification is motivated by the isomorphism of posets between the positive noncompact roots of $\mathfrak{so}^*(2m)$ and those of $\mathfrak{sp}(m-1,\mathbb{R})$. In the proof of this result it will be convenient to use Frobenius notation for integer partitions, so this notation is introduced here.

Let $\nu = (n_1, n_2, \ldots)$ be a partition and let $\nu' = (n_1', n_2', \ldots)$ be its dual partition, whose Young diagram is the transpose of the Young diagram of ν. The length r of the diagonal of the Young diagram of ν is called the *Frobenius rank* (or *Durfee rank*) of ν. For $1 \leq i \leq r$, define $a_i = n_i - i + 1$ and $b_i = n_i' - i + 1$. Then $(a_1, \ldots, a_r \mid b_1, \ldots, b_r)$ is called the *Frobenius symbol* of ν. The partition ν is uniquely determined by its Frobenius symbol and we will, by abuse of notation, write $\nu = (a_1, \ldots, a_r \mid b_1, \ldots, b_r)$. For example,

 $= (4,3,2) = (4,2 \mid 3,2).$

Here the diagonal boxes are filled with dots for emphasis.

Theorem 3.1.1. *Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n)$ and set $\lambda = -2k\zeta$, $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$, so that the simple highest weight module $L = L(\lambda)$ is the k-th Wallach representation. Choose $\mathfrak{g}'_{\mathbb{R}} = \mathfrak{sp}(n-2k-1, \mathbb{R})$ and let E be the finite-dimensional $\mathfrak{g}'$-module of highest weight $\mu = k\zeta'$. The Hilbert series of L is then*

$$H_L(t) = \frac{H_E(t)}{(1-t)^D},$$

where $D = |\Delta^+_{nc}| - |\Delta'^+_{nc}|$. □

Proof. Let $w \in {}^{\mathrm{E}}\mathcal{W}_\lambda$ and $w' \in {}^{I'}\mathcal{W}'$ be the elements whose Young diagrams are $(a_1, \ldots, a_r|1^r)$ in Frobenius notation. Set $m = n - 2k - 1$, and by abuse of notation write $[\alpha]$ for the reflection s_α. We will show that $\dim F_{\mathfrak{k}, w^+_\cdot\lambda} = \dim F_{\mathfrak{k}', w.\mu}$, which completes the proof when combined with the resolutions given in Theorem 2.5.2 and the fact that $(\lambda + \rho)(z_0) = (\mu + \rho')(z_0)$.

In Euclidean coordinates, $\lambda + \rho = (n-k-1, n-k-2, \ldots, -k)$. By Proposition 2.4.2, $w(\lambda+\rho) = R_r \ldots R_2 R_1(\lambda+\rho)$, with $R_1 = [\varepsilon_{m+1-a_1} + \varepsilon_{n-k}] \ldots [\varepsilon_{m-1} + \varepsilon_{n-k}][\varepsilon_m + \varepsilon_{n-k}]$ and $R_i = [\varepsilon_{m+2-i-a_i} + \varepsilon_{m+2-i}] \ldots [\varepsilon_{m-i} + \varepsilon_{m+2-i}][\varepsilon_{m+1-i} + \varepsilon_{m+2-i}]$ for $2 \leq i \leq r$. It follows that $R_1(\lambda + \rho) = (n-k-1, \ldots, \widehat{a_1 + k}, \ldots, k+1, 0, k, \ldots, 1, -(a_1 + k), -1, \ldots, -k)$, where the symbol $\widehat{}$ denotes omission, and

$$R_i \ldots R_2 R_1(\lambda + \rho) = (*, \ldots, *, 0, -a_i - k, \ldots, -a_2 - k, k, \ldots, -a_1 - k, -1, \ldots, -k).$$

Here the entries $*$ are in $\{n-k-1, \ldots, k+1\} \backslash \{a_1+k, \ldots a_i+k\}$ and are in decreasing order. In the setting of the theorem, the operation $[\cdot]^+ : \mathfrak{h}^* \to \mathfrak{h}^*$ orders the entries of the pertinent weight in decreasing fashion, so

$$[R_i \ldots R_2 R_1(\lambda + \rho)]^+ = (*, \ldots, *, -a_i - k, -a_{i-1} - k, \ldots, -a_1 - k),$$

with the entries $*$ decreasing and in the set $\{n-k-1, \ldots, -k\} \backslash \{a_1 + k, \ldots, a_i + k\}$.

Now we will make a similar calculation for the weight $w'(\mu + \rho')$, where $w' \in {}^{I'}\mathcal{W}'$ has Young diagram $(a_1, \ldots, a_r|1^r)$ as above. In Euclidean coordinates, $\mu +$

$\rho' = (n-k-1, \dots, k+1)$. Here we have $w'(\mu+\rho') = R_r \dots R_2 R_1(\mu+\rho')$ with $R_i = [\varepsilon_{m+2-i-a_i} + \varepsilon_{m+1-i}] \dots [\varepsilon_{m-i} + \varepsilon_{m+1-i}][2\varepsilon_{m+1-i}]$. A quick calculation shows that $R_1(\mu+\rho') = (n-k-1, \dots, \widehat{a_1+k}, \dots, k+1, -(a_1+k))$ and

$$R_i \dots R_2 R_1(\mu+\rho') = (*, \dots, *, -a_i - k, -a_{i-1} - k, \dots, -a_1 - k),$$

with the entries $*$ in decreasing order and in $\{n-k-1, \dots, k+1\} \backslash \{a_1+k, \dots, a_i+k\}$.

Since $w^\dagger\lambda + \rho$ differs from $w'.\mu + \rho'$ by its additional $2k+1$ entries $k, \dots, -k$, the Weyl dimension formula gives $\dim F_{\mathfrak{k}, w^\dagger\lambda} = \dim F_{\mathfrak{k}', w'.\mu} \cdot ABC/D$, where

$$A = \frac{[(n-2k-1)\dots(n-1)][(n-2k-2)\dots(n-2)]\dots[1\dots(2k+1)]}{[a_1\dots(a_1+2k)][a_2\dots(a_2+2k)]\dots[a_r\dots(a_r+2k)]},$$

$$B = (2k)!(2k-1)!\dots 1!,$$

$$C = [(a_r+2k)\dots(a_1+2k)][(a_r+2k-1)\dots(a_1+2k-1)]\dots[a_r\dots a_1],$$

$$D = (n-1)!(n-2)!\dots(n-2k-1)!.$$

After some refactoring, it is evident that $ABC/D = 1$, which proves the theorem. $\square$

Example 3.1.2. Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(12)$, and consider the first Wallach representation, $L(-2\omega_6)$. In this case, $\lambda = -2\omega_6$ is quasi-dominant, the real Lie algebra $\mathfrak{g}'_{\mathbb{R}}$ is $\mathfrak{sp}(3, \mathbb{R})$, and the finite domensional simple $\mathfrak{g}'$-module E has highest weight $\mu = \omega_3$. The resolutions of L and E are identical when the highest weights $w^\dagger\lambda$ and $w'.\mu$ are represented by the Young diagrams of w and w', as given in Figure 3.1.

$$\begin{array}{c}
\qquad\qquad\qquad\qquad\qquad N_{\ydiagram{3}} \\
0 \to N_e \to N_{\ydiagram{1}} \to N_{\ydiagram{2}} \to \quad \oplus \quad \to N_{\ydiagram{3,1+1}} \to N_{\ydiagram{3,1+2}} \to N_{\ydiagram{3,1+2,2+1}} \to M \to 0 \\
\qquad\qquad\qquad\qquad\qquad N_{\ydiagram{2,1+1}}
\end{array}$$

Figure 3.1. Resolution of $M = L(-2\omega_6)$ and $M = E_{\omega_3}$

The Hilbert series of L (respectively, E) is presented below to illustrate the Euler-Poincaré method. When calculating the degree of the numerators, it is helpful to note that the decomposition of $\mathfrak{g}$ as a $\mathfrak{k}$-module under the adjoint action of $z_0 \in \mathfrak{z}(\mathfrak{k})$ implies $\varepsilon_i(z_0) = 1/2$ for all i. As a result, $d_w = \frac{1}{2}|\lambda+\rho-w(\lambda+\rho)| = \frac{1}{2}|\mu+\rho'-w'(\mu+\rho')|$, where $|\lambda|$ sums up the Euclidean coordinates of λ.

$$\tfrac{1}{(1-t)^{15}} - \tfrac{15t^2}{(1-t)^{15}} + \tfrac{35t^3}{(1-t)^{15}} - (\tfrac{21t^4+21t^5}{(1-t)^{15}}) + \tfrac{35t^6}{(1-t)^{15}} - \tfrac{15t^7}{(1-t)^{15}} + \tfrac{t^9}{(1-t)^{15}} = H_L(t)$$

$$\tfrac{1}{(1-t)^{6}} - \tfrac{15t^2}{(1-t)^{6}} + \tfrac{35t^3}{(1-t)^{6}} - (\tfrac{21t^4+21t^5}{(1-t)^{6}}) + \tfrac{35t^6}{(1-t)^{6}} - \tfrac{15t^7}{(1-t)^{6}} + \tfrac{t^9}{(1-t)^{6}} = H_E(t)$$

It is now easy to verify that these Hilbert series agree with Theorem 3.1.1. Namely,

$$H_L(t) \;=\; \frac{H_E(t)}{(1-t)^{15-6}} \;=\; \frac{1+6t+6t^2+t^3}{(1-t)^9}.$$

The new correspondence between L and E in the case that L is the k-th Wallach representation in a dual pair setting is given in Table 3.1.

Table 3.1. The Wonderful Correspondence

$\mathfrak{g}_{\mathbb{R}}$	$-kC\zeta$	$\mathfrak{g}'_{\mathbb{R}}$	$k\zeta'$
$\mathfrak{su}(p,q)$	$-k\omega_p$	$\mathfrak{su}(p-k,q-k)$	$k\omega'_{p-k}$
$\mathfrak{sp}(n,\mathbb{R})$	$-\frac{k}{2}\omega_n$	$\mathfrak{so}^*(2(n-k+1))$	$k\omega'_{n-k+1}$
$\mathfrak{so}^*(2n)$	$-2k\omega_n$	$\mathfrak{sp}(n-2k-1,\mathbb{R})$	$k\omega'_{n-2k-1}$

Corollary 3.1.3 (cf. A-Hunziker-Willenbring [1]). *Let L be the k-th Wallach representation of highest weight $-kC\zeta$, $1 \le k \le r-1$, and let E be the finite-dimensional $\mathfrak{g}'$-representation of highest weight $k\zeta'$ as given in Table 3.1. Then*

$$H_L(t) = \frac{H_E(t)}{(1-t)^D},$$

where $D = |\Delta^+_{nc}| - |\Delta'^+_{nc}|$. □

3.2 Determinantal Varieties

We will now follow the work of Enright and Hunziker [8] (cf. [10]) to obtain an explicit description of the Hilbert series of the classical determinantal varieties.

Recall from Table 2.3 the set of matrices Z, which in the case $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p,q)$ (respectively $\mathfrak{sp}(n,\mathbb{R}), \mathfrak{so}^*(2n)$) is defined to be $\mathrm{M}_{p,k} \oplus \mathrm{M}_{k,q}$ (respectively $\mathrm{M}_{n,k}, \mathrm{M}_{2n,k}$). Define a polynomial map $\pi : Z \to \mathfrak{p}^+$ by

$$\pi : \mathrm{M}_{p,k} \oplus \mathrm{M}_{k,q} \to \mathrm{M}_{p,q}, \ (z_1, z_2) \mapsto z_1 z_2,$$

$$\pi : \mathrm{M}_{n,k} \to \mathrm{Sym}_n, \ z \mapsto zz^t,$$

$$\pi : \mathrm{M}_{n,2k} \to \mathrm{Alt}_n, \ z \mapsto zJz^t,$$

where J is the $2k \times 2k$ block diagonal matrix whose blocks have the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then π is $K_{\mathbb{R}}$-equivariant and constant on $K'_{\mathbb{R}}$-orbits. The varieties

$$Y_k := \pi(Z) \subset \mathfrak{p}^+$$

are the classical determinantal varieties. For $K' = \mathrm{GL}_k$ (respectively $\mathrm{O}_k, \mathrm{Sp}_{2k}$), the determinantal variety Y_k is the set of matrices in $\mathfrak{p}^+$ whose rank is less than or equal to k (respectively $k, 2k$). The coordinate rings $\mathbb{C}[Y_k]$ carry the structure of the simple unitarizable highest weight module $L(-kC\zeta)$, the k-th Wallach representation.

Proposition 3.2.1 (cf. Enright-Hunziker-Pruett [10])**.** *Set $\lambda = -kC\zeta$. Then*

$$\mathbb{C}[Y_k] \cong L(\lambda) \otimes F_{-\lambda}$$

as a $K_{\mathbb{R}}$-module. □

Proof. Notice that $\mathbb{C}[Z]^{K'} = V^{\mathrm{triv}}$ is a highest weight module of $\mathfrak{g}$ by Theorem 2.3.1. The highest weight vector is the constant function $1 \in \mathbb{C}[Z]^{K'}$. By the definition of the twisted $K_{\mathbb{R}}$-action on $\mathbb{C}[Z]^{K'}$, the highest weight is $\lambda = -kC\zeta$. A straightforward argument of classical invariant theory that makes use of Weyl's First Fundamental Theorem shows the induced map $\pi^* := f \circ \pi : \mathbb{C}[Y_k] \to \mathbb{C}[Z]^{K'}$ to be an isomorphism of algebras. As a $K_{\mathbb{R}}$-module, $\mathbb{C}[Y_k]$ is isomorphic to the tensor product of $L(\lambda)$ with the one-dimensional highest weight module $F_{-\lambda}$. □

Tensor the resolution of $L(\lambda)$ in Theorem 2.5.3 to obtain the following minimal graded free resolution of $\mathbb{C}[Y_k]$, viewed as an $S = S(\mathfrak{p}^-)$-module.

Theorem 3.2.2 (Enright-Hunziker [8] Thm. 31, cf. Enright-Hunziker-Pruett [10] Thm. 7.6)**.** *Let $\lambda = -kC\zeta$. Then there exists an exact sequence of S-modules, $0 \to S \otimes F_{p_\lambda} \to \ldots \to S \otimes F_1 \to S \to \mathbb{C}[Y_k] \to 0$, with*

$$F_j := \bigoplus_{w} F_{w \cdot \lambda} \otimes F_{-\lambda},$$

where $p_\lambda = |\Delta^+_{\lambda,nc}|$ and the sum is over $w \in {}^{\mathrm{E},j}\mathcal{W}_\lambda$. □

In order to calculate the Hilbert series of the coordinate rings of determinantal varieties, we will need a result that uses Harish-Chandra's strongly orthogonal noncompact roots $\gamma_1, \ldots, \gamma_r$. To derive this set, let γ_1 be the unique simple root in Δ^+_{nc} and for $2 \leq i \leq r$ take γ_i to be the lowest root in Δ^+_{nc} orthognal to $\gamma_1, \ldots, \gamma_{i-1}$. For example, in the case $\mathfrak{su}(p,q)$, $\gamma_1 = \alpha_p = \varepsilon_p - \varepsilon_{p+1}$. The roots in Δ^+_{nc} that are not orthogonal to $\varepsilon_p - \varepsilon_{p+1}$ are the ones connected to it by a straight line in the Hasse diagram of Δ^+_{nc}. Thus, $\gamma_2 = \alpha_p + \alpha_{p-1} + \alpha_{p+1} = \varepsilon_{p-1} - \varepsilon_{p+2}$, and in general $\gamma_i = \varepsilon_{p+1-i} - \varepsilon_{p+i}$ for $1 \leq i \leq r$.

Theorem 3.2.3 (Enright-Hunziker-Wallach [15] Thm. 3.1)**.** *Let $(\mathfrak{g}, \mathfrak{k})$ be an irreducible complexified Hermitian symmetric pair, and let $E_{k\zeta}$ be the finite-dimensional $\mathfrak{g}$-module with highest weight $k\zeta$, where ζ is the fundamental weight that is orthogonal to the compact roots. Then, as a $\mathfrak{k}$-module,*

$$E_{k\zeta} \otimes F_{-k\zeta} = \bigoplus_{k \geq m_1 \geq \ldots \geq m_r \geq 0} F_{-m_1\gamma_1 - \ldots - m_r\gamma_r},$$

where $\gamma_1 < \gamma_2 < \ldots < \gamma_r$ are Harish-Chandra's strongly orthogonal noncompact roots with γ_1 being the noncompact simple root. □

For $m \geq 1$ and $\nu = (\nu_1, \ldots, \nu_m)$ an integer partition with at most m parts, let $F^{(m)}_\nu$ denote the simple $\mathrm{GL}(m, \mathbb{C})$-module with highest weight $\nu_1\varepsilon_1 + \ldots + \nu_m\varepsilon_m$. Then Theorems 3.2.2 and 3.2.3 combine to give the following Hilbert series.

Corollary 3.2.4 (cf. Enright-Hunziker [8] Thm. 23, Thm. 32)**.** *Let* $Y_k = \{x \in \mathrm{M}_{p,q} \,|\, \mathrm{rk}\, x \leq k\}$ *with* $1 \leq k \leq \min\{p,q\} - 1$*. Then*

$$H_{\mathbb{C}[Y_k]}(t) = \frac{1}{(1-t)^{k(p+q-k)}} \sum_{\nu} (\dim F_{\nu}^{(p-k)})(\dim F_{\nu}^{(q-k)}) t^{|\nu|},$$

where the sum is over all partitions ν *whose Young diagram fits inside a rectangle of size* $\min\{p-k, q-k\} \times k$*.* □

Proof. Since $\mathfrak{g}'_{\mathbb{R}} = \mathfrak{su}(p-k, q-k)$, we have $r = \min\{p-k, q-k\}$. Harish-Chandra's strongly orthogonal roots are $\gamma_i = \varepsilon_{p-k+1-i} - \varepsilon_{p-k+i}, 1 \leq i \leq r$, and $\mathfrak{k}' = \mathfrak{s}(\mathfrak{u}(p-k) \oplus \mathfrak{u}(q-k))$, so Weyl's dimension formula gives the product of dimensions in the statement of the corollary. Recall that the (-1)-eigenspace of $\mathrm{ad}(z_0)$ is $\mathfrak{p}^-$. It follows that $F_{-m_1\gamma_1 - \ldots - m_r\gamma_r}$ is contained in the $|\nu|$-eigenspace of $\mathrm{ad}(z_0)$. □

Corollary 3.2.5 (cf. Enright-Hunziker [8] Thm. 24, Thm. 32)**.** *Let* $Y_k = \{x \in \mathrm{Sym}_n \,|\, \mathrm{rk}\, x \leq k\}$ *with* $1 \leq k \leq n-1$*. Then*

$$H_{\mathbb{C}[Y_k]}(t) = \frac{1}{(1-t)^{k(2n-k+1)/2}} \sum_{\nu} (\dim F_{\nu}^{(n-k+1)}) t^{|\nu|/2},$$

where the sum is over all partitions ν *whose Young diagram has only columns of even length and fits inside a rectangle of size* $(n-k+1) \times k$*.* □

Proof. In this case $\mathfrak{g}'_{\mathbb{R}} = \mathfrak{so}^*(2(n-k+1)), r = \lfloor \frac{n-k+1}{2} \rfloor$, and the strongly orthogonal roots are $\gamma_1 = \varepsilon_{n-k} + \varepsilon_{n-k+1}, \gamma_2 = \varepsilon_{n-k-2} + \varepsilon_{n-k-1}, \ldots, \gamma_r = \varepsilon_1 + \varepsilon_2$. Furthermore, $\mathfrak{k}' = \mathfrak{u}(2n-k+1)$, and $\varepsilon_i(z_0) = 1/2$ for $1 \leq i \leq n-k+1$. □

Corollary 3.2.6 (cf. A-Hunziker-Willenbring [1])**.** *Let* $Y_k = \{x \in \mathrm{Alt}_n \,|\, \mathrm{rk}\, x \leq 2k\}$ *with* $1 \leq k \leq \lfloor n/2 \rfloor - 1$*. Then*

$$H_{\mathbb{C}[Y_k]}(t) = \frac{1}{(1-t)^{k(2n-2k-1)}} \sum_{\nu} (\dim F_{\nu}^{(n-2k-1)}) t^{|\nu|/2},$$

where the sum is over all partitions ν *whose Young diagram has only rows of even length and fits inside a rectangle of size* $(n-2k-1) \times 2k$*.* □

Proof. Here $\mathfrak{g}'_{\mathbb{R}} = \mathfrak{sp}(n-2k-1,\mathbb{R}), r = n-2k-1$, and the strongly orthogonal roots are $\gamma_1 = 2\varepsilon_r, \gamma_2 = 2\varepsilon_{r-1}, \ldots, \gamma_r = 2\varepsilon_1$. The Lie algebra $\mathfrak{k}'$ is $\mathfrak{u}(n-2k-1)$, and $\varepsilon_i(z_0) = 1/2$ for $1 \leq i \leq n-2k-1$. $\square$

Example 3.2.7. Let $Y_1 = \{x \in \mathrm{Alt}_6 \mid \operatorname{rk} x \leq 2\}$. Then

$$\begin{aligned}
H_{\mathbb{C}[Y_1]}(t) &= \frac{1}{(1-t)^9}(\dim F^3_e + \dim F^3_{\ydiagram{2}}t + \dim F^3_{\ydiagram{2,2}}t^2 + \dim F^3_{\ydiagram{2,2,2}}t^3) \\
&= \frac{1+6t+6t^2+t^3}{(1-t)^9} \\
&= 1 + 15t + 105t^2 + 490t^3 + 1764t^4 + \cdots
\end{aligned}$$

Example 3.2.8. Let $Y_1 = \{x \in \mathrm{Sym}_8 \mid \operatorname{rk} x \leq 1\}$. Then

$$\begin{aligned}
H_{\mathbb{C}[Y_1]}(t) &= \frac{1}{(1-t)^8}(\dim F^8_e + \dim F^8_{\ydiagram{1,1}}t + \dim F^8_{\ydiagram{1,1,1,1}}t^2 + \dim F^8_{\ydiagram{1,1,1,1,1,1}}t^3 + \dim F^8_{\ydiagram{1,1,1,1,1,1,1,1}}t^4) \\
&= \frac{1+28t+35t^2+28t^3+t^4}{(1-t)^8} \\
&= 1 + 36t + 295t^2 + 1436t^3 + 5175t^4 + \cdots
\end{aligned}$$

CHAPTER FOUR

Classification of Quasi-dominant Reduction Points

4.1 Classification

Denote the set of all quasi-dominant weights by Λ_q. Here we give a description of $\Lambda_r^+ := \Lambda_q \cap \Lambda_r$, the quasi-dominant reduction points. Define $k(\lambda)$ to be the nonnegative integer $-\frac{(\lambda,\beta^\vee)}{C}$, and define a new root system $S(\tau)$ by pruning the branches on the extended Dynkin diagram of Δ according to Table 4.1. For $\mathfrak{g}_\mathbb{R} \neq \mathfrak{su}(p,q)$ and $\lambda = \lambda_v + \sum a_i\omega_i$, let $s = \sum a_j$ with the sum over coefficients a_j for which $\omega_j \neq \zeta$. For $\mathfrak{g}_\mathbb{R} = \mathfrak{su}(p,q)$ and $\lambda = \lambda_v + \sum a_i\omega_i$, let $s' = \sum_{j=p'}^{p-1} a_j$ and $s'' = \sum_{j=p+1}^{n+1-q'} a_j$. The root system $Q(\tau)$ is listed in Table 4.1 when helpful.

Table 4.1. The root system $S(\tau)$

$\mathfrak{g}_\mathbb{R}$	$Q(\tau)$	$S(\tau)$
$\mathfrak{su}(p,q)$		$\mathrm{SU}(p-s'', q-s')$
$\mathfrak{sp}(n,\mathbb{R})$		$\mathrm{Sp}(n-s,\mathbb{R})$
$\mathfrak{so}^*(2n)$		$\mathrm{SO}^*(2(n-s))$
$\mathfrak{so}(2,2n-1)$		$\mathrm{SO}(2,2n-1)$
$\mathfrak{so}(2,2n-2)$		$\mathrm{SO}(2,2n-2)$ if $s=0$
–		$\mathrm{SU}(1,n-s)$ if $s\geq 1$
$\mathfrak{e}_{6(-14)}$	E_{III}	E_{III}
$\mathfrak{e}_{7(-25)}$	E_{VII}	E_{VII}
–	$\mathrm{SO}(2,10)$	E_{VII} if $s=0$
–	–	$\mathrm{SO}(2,10)$ if $s=1$
–	–	$\mathrm{SU}(1,7-s)$ if $s\geq 2$

Theorem 4.1.1. *Suppose $L = L(\lambda)$ is a unitarizable highest weight module and $\lambda = \lambda_0(\tau) + z_\ell\zeta$ is a reduction point. Then λ is quasi-dominant if and only if*

$$k(\lambda) \leq r_{S(\tau)} - 1,$$

where $r_{S(\tau)}$ is the real rank of the root system $S(\tau)$ given in Table 4.1. $\square$

Proof. The Transfer Theorem implies that if $\lambda \in \Lambda_q \cap \Lambda_r$ then $\mathrm{GK}\dim L(\lambda) < \dim \mathfrak{p}^+$. The work in Joseph [23] and Bai-Hunziker [2] then implies that the only choices of the triple $(\mathfrak{g}_\mathbb{R}, Q(\tau), R(\tau))$ having quasi-dominant reduction points are the ones listed in Table 4.2. For these cases, we consider a reduction point $\lambda \in \lambda_v + C_v$ and observe the conditions that determine quasi-dominance.

Case 1 ($\mathfrak{g}_\mathbb{R} = \mathfrak{su}(p,q)$). In this case, it suffices to take $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ and $\rho = (n, n-1, \ldots, 0)$. We have $Q = R = \mathrm{SU}(p', q')$ for integers $1 \leq p' \leq p$ and $1 \leq q' \leq q$, so $\tau_v = \omega_{p'} + \omega_{n+1-q'}$. From (2.2) and (2.3), one obtains

$$\lambda_0(\tau_v) = \omega_{p'} + \omega_{n+1-q'} - (n+2)\omega_p \ \text{ and } \ z_\ell = p' + q' - \ell.$$

It follows from the defintion of C_v given in Section 2.2 that a reduction point λ, written in Euclidean coordinates, has the form:

$$\lambda = (\underbrace{-k, \ldots, -k}_{p'}, \underbrace{-k-1-s'_1, \ldots, -k-1-s'_{p-p'}}_{p-p'}, \underbrace{1+s''_1, \ldots, 1+s''_{q-q'}}_{q-q'}, \underbrace{0, \ldots, 0}_{q'}),$$

with $k = k(\lambda) = p+q-p'-q'+\ell-1$, $s'_i = a_{p'}+\cdots+a_{p'+i-1}$, and $s''_i = a_{p+i}+\cdots+a_{p+q-q'}$. Let $(\lambda+\rho)_i$ be the i^{th} coordinate of $\lambda+\rho$ and let Θ be the intersection of the first p coordinates of $\lambda+\rho$ with the last q coordinates of $\lambda+\rho$. Recalling the definition given in (2.6), one finds

$$\Omega_\lambda = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \leq p, \ \ p+1 \leq j \leq p+q, \ \text{ and } \ (\lambda+\rho)_i, (\lambda+\rho)_j \notin \Theta\}.$$

Note that $(\lambda+\rho)_{p'-\ell+1} = q'$ and $(\lambda+\rho)_{p+q-q'+\ell} = q'-\ell$ are not in Θ, so two necessary conditions for quasi-dominance are $(\lambda+\rho)_1 \geq (\lambda+\rho)_{p+1}$ and $(\lambda+\rho)_p \geq (\lambda+\rho)_{p+q}$.

Since $(\lambda+\rho)_{p+q-q'} > (\lambda+\rho)_{p'}$ and $(\lambda+\rho)_{p'+1} < (\lambda+\rho)_{p+q-q'+1}$, the two conditions are sufficient, so that λ is quasi-dominant precisely when $-k+n \geq s''+q$ and $-k-1-s'+q \geq 0$. This is equivalent to

$$k(\lambda) \leq \min\{p-s'', q-s'\} - 1.$$

Case 2 ($\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n,\mathbb{R})$)**.** Here $Q = \mathrm{Sp}(q,\mathbb{R})$ and $R = \mathrm{Sp}(r,\mathbb{R})$ for integers $1 \leq q \leq r \leq n$, so $\tau_v = \omega_q + \omega_r$. From (2.2) and (2.3), one obtains

$$\lambda_0(\tau_v) = \omega_q - \omega_r - (n+2)\omega_n \text{ and } z_\ell = \frac{1}{2}(r+q+1-\ell).$$

It follows that a reduction point λ has the form:

$$\lambda = \Big(\underbrace{-\frac{k}{2},\dots,-\frac{k}{2}}_{q},\underbrace{-\frac{k}{2}-1,\dots,-\frac{k}{2}-1}_{r-q},\underbrace{-\frac{k}{2}-2-s_1,\dots,-\frac{k}{2}-2-s_{n-r}}_{n-r}\Big),$$

with $k = k(\lambda) = 2n-q-r+\ell-1$ and $s_i = a_r + \cdots + a_{r+i-1}$. Let Θ be the set of coordinates $(\lambda+\rho)_i$ for which $-(\lambda+\rho)_i$ is also a coordinate of $\lambda+\rho$. Note that these entries could be positive, negative, or zero. Since $\rho = (n, n-1, \dots, 1)$,

$$\Omega_\lambda = \begin{cases} \varepsilon_i + \varepsilon_j \mid 1 \leq i \leq j \leq n \text{ and } (\lambda+\rho)_i, (\lambda+\rho)_j \notin \Theta \\ \varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq n \text{ and } (\lambda+\rho)_i, (\lambda+\rho)_j \notin \Theta \end{cases}$$

if Ξ_λ does not contain (respectively, does contain) a long root. An immediate necessary and sufficient condition for quasi-dominance is that all the negative entries of $\lambda+\rho$, except possibly $\frac{1}{2}(q+\ell-r-1)$, are in Θ. Note that $(\lambda+\rho)_r \leq 0$, so $-(\lambda+\rho)_r + 1 > 0$. Since $-(\lambda+\rho)_r + 1 \notin \Theta$ and $-(\lambda+\rho)_r + 1 < -(\lambda+\rho)_n$, quasi-dominance of λ coincides with $-(\lambda+\rho)_n \leq (\lambda+\rho)_1$. It follows that λ is quasi-dominant if and only if

$$k(\lambda) \leq n-s-1.$$

Case 3 ($\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n), n \geq 4$)**.** In this case, the root system $Q = R$ is either $\mathrm{SO}^*(2p)$ for an integer $3 \leq p \leq n$ or $\mathrm{SU}(1, n-1)$ with n even. We begin with

$Q = R = \mathrm{SO}^*(2p)$, where $\tau_v = \omega_p$ and

$$\lambda_0(\tau_v) = \begin{cases} \omega_p - (2n-1)\omega_n & \text{if } 3 \le p \le n-2 \\ \omega_{n-1} - (2n-2)\omega_n & \text{if } p = n-1 \\ -(2n-3)\omega_n & \text{if } p = n \end{cases}.$$

This formula for $\lambda_0(\tau_v)$ is not uniform because of the nature of the fundamental weights. However, z_ℓ is always $2p - 2\ell - 1$, and $\lambda \in \Lambda_r$ has the uniform description:

$$\lambda = (\underbrace{-k, \dots, -k}_{p}, \underbrace{-k-1-s_1, \dots, -k-1-s_{n-p}}_{n-p}),$$

with $k = k(\lambda) = n - p + \ell - 1$ and $s_i = a_p + \cdots + a_{p+i-1}$. Let Θ^* be the set of nonzero coordinates $(\lambda+\rho)_i$ for which $-(\lambda+\rho)_i$ is also a coordinate of $\lambda+\rho$. The Weyl vector is $\rho = (n-1, n-2, \dots, 0)$, so one obtains

$$\Omega_\lambda = \{\varepsilon_i + \varepsilon_j \mid 1 \le i < j \le n \text{ and } (\lambda+\rho)_i, (\lambda+\rho)_j \notin \Theta^*\}.$$

Note that the first p entries of $\lambda+\rho$ are decreasing by 1 and contain 0, so a sufficient condition for quasi-dominance is $(\lambda+\rho)_1 \ge -(\lambda+\rho)_n$. Since the entry 0 is not in Θ^*, the condition is necessary, so λ is quasi-dominant exactly when

$$k(\lambda) \le \left\lfloor \frac{n-s}{2} \right\rfloor - 1.$$

Now consider $Q = R = \mathrm{SU}(1, n-1)$ with n even, where $\tau_v = \omega_1$, $\lambda_0(\tau_v) = \omega_1 - (2n-2)\omega_n$, and $z_1 = n-1$. A reduction point λ has the form:

$$\lambda = \left(-k + \frac{a_1+1}{2}, -k - \frac{a_1+1}{2}, \dots, -k - \frac{a_1+1}{2}\right),$$

with $k = \frac{n}{2} - 1$. After calculating Ξ_λ, we have

$$\Omega_\lambda = \{\varepsilon_i + \varepsilon_j \mid 1 \le i < j \le n \text{ and } (\lambda+\rho)_i, (\lambda+\rho)_j \notin \Theta^*\}.$$

Because $(\lambda+\rho)_2 < -(\lambda+\rho)_n < (\lambda+\rho)_1$ and $-(\lambda+\rho)_n \notin \Theta^*$, a necessary condition for λ to be quasi-dominant is $(\lambda+\rho)_2 = -(\lambda+\rho)_{n-1}$. This condition is equivalent to $a_1 = 0$, which is also a sufficient condition. Quasi-dominance is characterized by

$$k(\lambda) \leq \left\lfloor \frac{n-s}{2} \right\rfloor - 1.$$

Case 4 ($\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(2, 2n-1), n \geq 2$)**.** We begin with the setting in which the root systems are $Q = \mathrm{SU}(1, n-1)$, $R = \mathrm{SO}(2, 2n-1)$. Here $\tau_v = \omega_n$,

$$\lambda_0(\tau_v) = \omega_n - (2n-1)\omega_1 \ \text{ and } \ z_\ell = \frac{3}{2}\ell - (\ell - 2)n - 2.$$

A reduction point λ has the form $\lambda = (1-n, 1/2, \ldots, 1/2)$. In this case $k(\lambda) = 1$ and $\rho = (n - 1/2, n - 3/2, \ldots, 1/2)$. Upon calculating Ξ_λ we have $\Omega_\lambda = \Delta^+_{\mathrm{nc}} = \{\varepsilon_1 \pm \varepsilon_j \mid 2 \leq j \leq n\} \cup \{\varepsilon_1\}$. It follows that λ is quasi-dominant, and $k(\lambda) \leq 2 - 1$ is satisfied.

In the case $Q = R = \mathrm{SO}(2, 2n-1)$, we have $\tau_v = 0$,

$$\lambda_0(\tau_v) = -(2n-2)\omega_1 \ \text{ and } \ z_\ell = \frac{3}{2}\ell - (\ell - 3)n - \frac{7}{2}.$$

A reduction point λ has the form $\lambda = ((3/2 - n)k, 0, \ldots, 0)$, with $k = \ell - 1$. Here $\Xi_\lambda = \emptyset$, so $\Omega_\lambda = \Delta^+_{\mathrm{nc}}$. It follows that λ is quasi-dominant for both $\ell = 1$ and $\ell = 2$, which both satisfy $k(\lambda) \leq 2 - 1$.

Case 5 ($\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}(2, 2n-2), n \geq 3$)**.** The case $Q = R = \mathrm{SU}(1, n-1)$ occurs twice, as described by

$$\lambda_0(\tau_v) = \begin{cases} \omega_{n-1} - (2n-2)\omega_1 & \text{if } \ \tau_v = \omega_{n-1} \\ \omega_n - (2n-2)\omega_1 & \text{if } \ \tau_v = \omega_n \end{cases} \quad \text{and} \quad z_1 = n - 1.$$

A reduction point λ has the form:

$$\lambda = \begin{cases} \left(\frac{3}{2} - n - \frac{a_{n-1}}{2}, \frac{1}{2} + \frac{a_{n-1}}{2}, \ldots, \frac{1}{2} + \frac{a_{n-1}}{2}, -\frac{1}{2} - \frac{a_{n-1}}{2}\right) & \text{if } \ \tau_v = \omega_{n-1} \\ \left(\frac{3}{2} - n - \frac{a_n}{2}, \frac{1}{2} + \frac{a_n}{2}, \ldots, \frac{1}{2} + \frac{a_n}{2}\right) & \text{if } \ \tau_v = \omega_n \end{cases}.$$

In this setting $k(\lambda) = 1$ and $\rho = (n-1, n-2, \ldots, 0)$, from which it follows that

$$\Omega_\lambda = \begin{cases} \{\varepsilon_1 - \varepsilon_n\} & \text{if } \tau_v = \omega_{n-1} \text{ and } a_{n-1} = 0 \\ \Delta_{\mathrm{nc}}^+ & \text{if } \tau_v = \omega_{n-1} \text{ and } a_{n-1} \neq 0 \\ \{\varepsilon_1 + \varepsilon_n\} & \text{if } \tau_v = \omega_n \text{ and } a_n = 0 \\ \Delta_{\mathrm{nc}}^+ & \text{if } \tau_v = \omega_n \text{ and } a_n \neq 0 \end{cases}.$$

It follows that λ is quasi-dominant if and only if $a_i = 0$ for $i = n-1, n$.

For $Q = R = \mathrm{SO}(2, 2n-2)$, we have $\tau_v = 0$,

$$\lambda_0(\tau_v) = -(2n-3)\omega_1 \text{ and } z_\ell = -(\ell - 3)(n-2) + 1.$$

If λ is a reduction point, then it has the form $\lambda = (-(n-2)k, 0, \ldots, 0)$, where $k = \ell - 1$. A quick calculation gives

$$\Omega_\lambda = \begin{cases} \Delta_{\mathrm{nc}}^+ = \{\varepsilon_1 \pm \varepsilon_j \mid 2 \leq j \leq n\} & \text{if } \ell = 1 \\ \{\varepsilon_1 + \varepsilon_{n-1}\} & \text{if } \ell = 2 \end{cases}.$$

It follows that λ is quasi-dominant for $\ell = 1$ and $\ell = 2$.

Case 6 ($\mathfrak{g}_{\mathbb{R}} = \mathfrak{e}_{6(-14)}$)**.** The quasi-dominant reduction points are $\lambda = 0, -3\omega_1$, as given in [9]. These are the highest weights of the Wallach representations, and $k(\lambda) = 0, 1$, respectively.

Case 7 ($\mathfrak{g}_{\mathbb{R}} = \mathfrak{e}_{7(-25)}$)**.** The quasi-dominant reduction points are $\lambda = 0, -4\omega_7, -8\omega_7$, and $\omega_6 - 10\omega_7$, as given in [9]. The first three of these weights are the highest weights of the Wallach representations, where $Q = \mathrm{E}_{VII}$. For these weights $k(\lambda) = 0, 1, 2$, respectively. For the fourth weight, $Q = \mathrm{SO}(2, 10)$. A general weight in this setting gives $k(\lambda) = \ell + 1$, and it turns out that λ is quasi-dominant if and only if $\ell = 1$ and $s = 0$.

$\square$

Table 4.2. Cone decomposition for quasi-dominant reduction points

$\mathfrak{g}_{\mathbb{R}}$	$Q(\tau)$	C_v
$\mathfrak{su}(p,q)$	$\mathrm{SU}(p',q')$	$\{\sum_{i=p'}^{n+1-q'} a_i\omega_i \mid a_p = -\sum_{i=p'}^{p-1} a_i - \sum_{i=p+1}^{n+1-q'} a_i\}$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathrm{Sp}(q,\mathbb{R})$	$\{\sum_{i=r}^{n} a_i\omega_i \mid a_n = -\sum_{i=r}^{n-1} a_i\}$
$\mathfrak{so}^*(2n)$	$\mathrm{SO}^*(2p), 3 \leq p \leq n$	$\{\sum_{i=p}^{n} a_i\omega_i \mid a_n = -2\sum_{i=p}^{n-2} a_i - a_{n-1}\}$
–	$\mathrm{SU}(1,n-1), n$ even	$\{a_1\omega_1 + a_n\omega_n \mid a_n = -a_1\}$
$\mathfrak{so}(2,2n-1)$	$\mathrm{SU}(1,n-1)$	$\emptyset$
–	$\mathrm{SO}(2,2n-1)$	$\emptyset$
$\mathfrak{so}(2,2n-2)$	$\mathrm{SU}(1,n-1)$	$\{a_1\omega_1 + a_n\omega_n \mid a_1 = -a_n\}$
$\mathfrak{so}(2,2n-2)$	$\mathrm{SU}(1,n-1)$	$\{a_1\omega_1 + a_{n-1}\omega_{n-1} \mid a_1 = -a_{n-1}\}$
–	$\mathrm{SO}(2,2n-2)$	$\emptyset$
$\mathfrak{e}_{6(-14)}$	E_{III}	$\emptyset$
$\mathfrak{e}_{7(-25)}$	$\mathrm{SO}(2,10)$	$\{a_6\omega_6 + a_7\omega_7 \mid a_7 = -2a_6\}$
–	E_{VII}	$\emptyset$

Corollary 4.1.2. *There are only finitely many quasi-dominant weights in each cone. In particular, each Hermitian symmetric pair has finitely many quasi-dominant reduction points.* $\square$

For a fixed triple $(\mathfrak{g}_{\mathbb{R}}, Q(\tau), R(\tau))$, there exists a unique line $\lambda_0(\tau_v) + z\zeta$ containing all the vertices λ_v. The line has a distinct vertex for each level of reduction. If $R(\tau) \neq \Delta$, each vertex has a corresponding cone of infinite length whose dimension is $\operatorname{rk} G - \operatorname{rk} R(\tau)$. The set of quasi-dominant reduction points can be viewed geometrically as integral points in polytopes contained in these cones. In each case, the polytope contains the vertex of the cone and is determined by either one or two hyperplanes that separate the polytope from the rest of the cone.

Example 4.1.3. For the triple $(\mathfrak{sp}(n, \mathbb{R}), \mathrm{Sp}(n-3, \mathbb{R}), \mathrm{Sp}(n-3, \mathbb{R}))$, quasi-dominant reduction points are characterized by $\sum_{i=1}^{3} a_{n-i} \leq n - \ell - 6$. The set Λ_r^+ can be represented by the integral points in the 3-simplex determined by the hyperplane $\sum_{i=1}^{3} a_{n-i} = n - \ell - 6$, as in Figure 4.1.

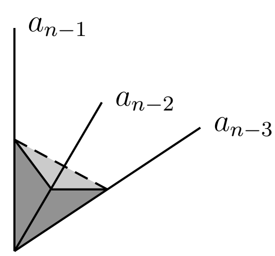


Figure 4.1. Simplex containing Λ_r^+ for the triple $(\mathfrak{sp}(n, \mathbb{R}), \mathrm{Sp}(n-3, \mathbb{R}), \mathrm{Sp}(n-3, \mathbb{R}))$

4.2 Quasi-dominance in Terms of Reductive Dual Pairs

The set of quasi-dominant reduction points also has a nice description in terms of reductive dual pairs. Let $\Lambda_r(k)$ denote the set of reduction points for which $k(\lambda) = k$. Recall the notation for dual pairs given in Theorem 2.3.1. The sets Σ and the highest weights of V^σ were explicitly given in Kashiwara–Vergne [24] for $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p, q)$ and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$, and we will follow their conventions.

Quasi-dominant Highest Weights for $\mathfrak{su}(p,q)$

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p,q)$ and assume that $k \le p \le q$. Following Kashiwara–Vergne,

$$\begin{aligned}\Sigma = &\{(n_1,\dots,n_i,0,\dots,0,-m_j,\dots,-m_1) \mid n_1 \ge \cdots \ge n_i > 0,\\ &\quad m_1 \ge \cdots \ge m_j > 0,\ 0 \le i \le q,\ 0 \le j \le p,\ i+j \le k\}.\end{aligned}$$

If $\sigma = (n_1,\dots,n_i,0,\dots,0,-m_j,\dots,-m_1) \in \Sigma$ and $V^\sigma = L(\lambda)$, then

$$\lambda = (\underbrace{-k,\dots,-k,-m_j-k,\dots,-m_1-k}_{p},\underbrace{n_1,\dots,n_i,0,\dots,0}_{q}). \tag{4.1}$$

Proposition 4.2.1. *Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p,q)$ and $0 \le k < \min\{p,q\}$. If $\lambda \in \Lambda_r(k)$ is of the form* (4.1), then λ is quasi-dominant if and only if $n_1 \le p-k$ and $m_1 \le q-k$. □

Proof. In the language of Theorem 4.1.1, $m_1 = s'+1$ and $n_1 = s''+1$. □

Corollary 4.2.2. *Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p,q)$ and $0 \le k < \min\{p,q\}$. Then*

$$|\Lambda_r^+(k)| = \binom{p+q-k}{k}.$$

□

Proof. Counting the number of elements in $\Lambda_r^+(k)$ is equivalent to counting the number of k-tuples $(n_1,\dots,n_i,m_1,\dots,m_j)$ for which $0 < n_i \le \cdots \le n_1$, $0 \le m_j \le \cdots \le m_1$, $0 \le i \le k$, $j = k-i$, $n_1 \le p-k$, and $m_1 \le q-k$. Fix i. The number of i-tuples $(n_1,\dots,n_i)$ with $0 < n_i \le \cdots \le n_1$ and $n_1 \le p-k$ is equal to the number of paths in the (x,y)-plane from $(0,1)$ to $(i,p-k)$ with unit-length steps only going up or to the right. The total number of steps in each path is $i+p-k-1$, so the total number of paths is $\binom{i+p-k-1}{i} = \binom{i+p-k-1}{p-k-1}$. Similarly, for each of these i-tuples there are $\binom{q-i}{k-i} = \binom{q-i}{q-k}$ j-tuples $(m_1,\dots,m_j)$ with $0 \le m_j \le \cdots \le m_1$ and $m_1 \le q-k$. It follows that $|\Lambda_r^+(k)| = \sum_{i=0}^{k}\binom{i+p-k-1}{p-k-1}\binom{q-i}{q-k}$, which is also equal to $\binom{p+q-k}{k}$. □

The number of quasi-dominant reduction points for a fixed $0 \le k \le r-1$ has an interesting distribution. Figure 4.2 shows this distribution for $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(40,50)$.

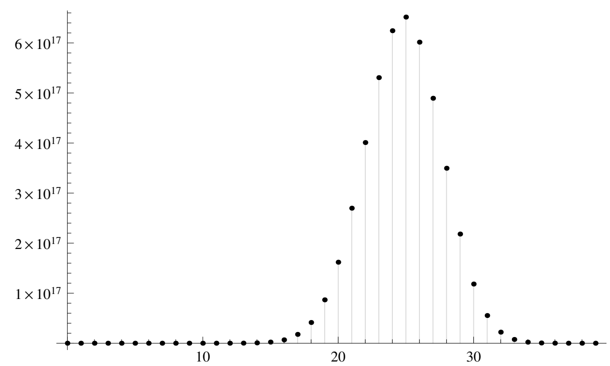


Figure 4.2. The numbers $|\Lambda_r^+(k)|$ for $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(40,50)$

Corollary 4.2.3. *For* $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p,p)$,

$$|\Lambda_r^+| = F_{2p+1},$$

where F_n denotes the n-th Fibonacci number. □

Quasi-dominant Highest Weights for $\mathfrak{sp}(n,\mathbb{R})$

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n,\mathbb{R})$ and assume that $k \le n$. Following Kashiwara–Vergne, $\Sigma = \widehat{\mathrm{O}(k)}$. First consider the case when k is odd. Then $-1 \in \mathrm{O}(k)$ and $\mathrm{O}(k) \cong \mathrm{SO}(k)\times\{\pm 1\}$. Write $k = 2l+1$. Using the same conventions as in Kashiwara–Vergne, we have $\Sigma = \{(m_1,\ldots,m_l;\varepsilon) \mid m_1 \ge \cdots \ge m_l \ge 0,\ \varepsilon = \pm 1\}$. If $\sigma = (m_1,\ldots,m_l;\varepsilon)$ with $\varepsilon = (-1)^{m_1+\cdots+m_l}$ and $V^\sigma = L(\lambda)$, then

$$\lambda = \Big(-\frac{k}{2},\ldots,-\frac{k}{2},-m_l-\frac{k}{2},\ldots,-m_1-\frac{k}{2}\Big). \tag{4.2}$$

If $\sigma = (m_1,\ldots,m_j,0\ldots,0;\varepsilon)$ with $m_j \ne 0$ and $\varepsilon = (-1)^{m_1+\cdots+m_j+1}$, then

$$\lambda = \Big(-\frac{k}{2},\ldots,-\frac{k}{2},\underbrace{-\frac{k}{2}-1,\ldots,-\frac{k}{2}-1,-m_j-\frac{k}{2},\ldots,-m_1-\frac{k}{2}}_{k-j}\Big). \tag{4.3}$$

Proposition 4.2.4. *Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$ and $k = 2l+1 < n$. If $\lambda \in \Lambda_r(k)$ is of the form (4.2) or (4.3), then λ is quasi-dominant if and only if $m_1 \leq n-k+1$.* □

Proof. The description of λ in the proof of Theorem 4.1.1 shows that $m_1 = 2+s$. □

Since the proofs of several several of the remaining corollaries in this section are very similar to the proof of Corollary 4.2.2, they will be omitted.

Corollary 4.2.5. *Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$ and $k = 2l+1 < n$. Then*

$$|\Lambda_r^+(k)| = 2\binom{n-l}{l}.$$

□

Next consider the case when k is even. In this case, $O(k) \cong SO(k) \ltimes \mathbb{Z}_2$. Write $k = 2l$. Using the conventions of Kashiwara–Vergne, we have $\Sigma = \{(m_1, \ldots, m_l)_\pm \mid m_1 \geq \cdots \geq m_l \geq 0\}$. If $m_l \neq 0$, then $(m_1, \ldots, m_l)_+ = (m_1, \ldots, m_l)_-$, otherwise $(m_1, \ldots, m_l)_+ \neq (m_1, \ldots, m_l)_-$. If $\sigma = (m_1, \ldots, m_l)_+$, then

$$\lambda = \Big(-\frac{k}{2}, \ldots, -\frac{k}{2}, -m_l - \frac{k}{2}, \ldots, -m_1 - \frac{k}{2} \Big). \tag{4.4}$$

If $\sigma = (m_1, \ldots, m_j, 0 \ldots)_-$ with $m_j \neq 0$, then

$$\lambda = \Big(-\frac{k}{2}, \ldots, -\frac{k}{2}, \underbrace{-\frac{k}{2} - 1, \ldots, -\frac{k}{2} - 1, -m_j - \frac{k}{2}, \ldots, -m_1 - \frac{k}{2}}_{k-j} \Big). \tag{4.5}$$

Proposition 4.2.6. *Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$ and $k = 2l < n$. If $\lambda \in \Lambda_r(k)$ is of the form (4.4) or (4.5), then λ is quasi-dominant if and only if $m_1 \leq n-k+1$.* □

Corollary 4.2.7. *Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$ and $k = 2l < n$. Then*

$$|\Lambda_r^+(k)| = \binom{n-l+1}{l} + \binom{n-l}{l-1}.$$

□

Corollary 4.2.8. *For $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$,*

$$|\Lambda_r^+| = F_{n+4} - (n+3).$$

□

Quasi-dominant Highest Weights for $\mathfrak{so}^*(2n)$

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n)$ and assume that $k \leq \lfloor n/2 \rfloor$. This case was not studied in Kashiwara–Vergne, but it is well-known (see [20] or [13]) that $k \leq \lfloor n/2 \rfloor$ implies $\Sigma = \widehat{\mathrm{Sp}(k)}$. Furthermore, using the standard conventions, $\Sigma = \{(m_1, \ldots, m_k \mid m_1 \geq \cdots \geq m_k \geq 0\}$. If $\sigma = (m_1, \ldots, m_k)$ and $V^\sigma = L(\lambda)$, then

$$\lambda = (-k, \ldots, -k, -m_k - k, \ldots, -m_1 - k). \tag{4.6}$$

Proposition 4.2.9. *Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n)$ and $k < \lfloor n/2 \rfloor$. If $\lambda \in \Lambda_r(k)$ is of the form (4.6), then λ is quasi-dominant if and only if $m_1 \leq n - 2k - 1$.* □

Proof. From the description of λ in the proof of Theorem 4.1.1, $m_1 = 1 + s$. □

Corollary 4.2.10. *Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n)$ and $k < \lfloor n/2 \rfloor$. Then*

$$|\Lambda_r^+(k)| = \binom{n-k-1}{k} + \delta_{n,2k+2},$$

where $\delta_{i,j}$ is the Kronecker delta. □

Corollary 4.2.11. *Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{so}^*(2n)$. Then*

$$|\Lambda_r^+| = F_n + (-1)^n.$$

□

Remark 4.2.12. It is interesting that the bounds on m_1 and n_1 in the results of this section are the same numbers listed in Table 3.1 for the Wonderful Correspondence.

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