ABSTRACT

Glazman-Krein-Naimark Theory, Left-Definite Theory, and the Square of the Legendre Polynomials Differential Operator

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As an application of a general left-definite spectral theory, Everitt, Littlejohn and Wellman, in 2002, developed the left-definite theory associated with the classical Legendre self-adjoint second-order differential operator A in $L^2(-1,1)$ which has the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ as eigenfunctions. As a consequence, they explicitly determined the domain $\mathcal{D}(A^2)$ of the self-adjoint operator A^2. However, this domain, in their characterization, does not contain boundary conditions in its formulation. In fact, this is a general feature of the left-definite approach developed by Littlejohn and Wellman. Yet, the square of the second-order Legendre expression is in the limit-4 case at each endpoint $x = \pm 1$ in $L^2(-1,1)$, so $\mathcal{D}(A^2)$ should exhibit four boundary conditions. In this thesis, we show that this domain can, in fact, be expressed using four separated boundary conditions using the classical GKN (Glazman-Krein-Naimark) theory. In addition, we determine a new characterization of $\mathcal{D}(A^2)$ that involves four *non*-GKN boundary conditions. These new boundary conditions are surprisingly simple and natural, and are equivalent to the boundary conditions obtained from the GKN theory.

Glazman-Krein-Naimark Theory, Left-Definite Theory, and the Square of the Legendre Polynomials Differential Operator

by

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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DEDICATION

For my family

CHAPTER ONE

Introduction

The analytical study of the classical second-order Legendre differential expression

$$\ell[y](x) = -\left((1-x^2)y'(x)\right)'$$

has a long and rich history stretching back to the seminal work of Weyl in 1910 [68] and Titchmarsh in 1940 [64]. Part, if not most, of the reason for the importance of this second-order expression lies in the fact that the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are solutions. More specifically, the Legendre polynomial $y = P_n(x)$, for $n \in \mathbb{N}_0$, is a solution of the eigenvalue equation

$$\ell[y](x) = n(n+1)y(x).$$

In the Hilbert space $L^2(-1,1)$, there is a continuum of self-adjoint operators generated by $\ell[\cdot]$. One such operator A stands out from the rest: this is the Legendre polynomials operator, so named because the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are eigenfunctions of A. We review properties of this operator in chapter six.

In the mid-1970s, Å. Pleijel wrote two papers (see [55] and [56]) on the Legendre expression from a left-definite spectral point of view. Everitt's contribution [19] continued this left-definite study in addition to detailing an in-depth analysis of the Legendre expression in the right-definite setting $L^2(-1,1)$ where he discovered new properties of functions in the domain $\mathcal{D}(A)$ of A. In [42], A. M. Krall and Littlejohn considered properties of the Legendre expression under the left-definite energy norm. In 2000, Vonhoff extended Everitt's results in [66] with an extensive study of $\ell[\cdot]$ in its (first) left-definite setting. In 2002, Everitt, Littlejohn, and Marić [24] published further results in which they gave several equivalent conditions for functions to belong to $\mathcal{D}(A)$; this result is given below in Theorem 5.6. We also refer the reader to the paper [49] by Littlejohn and Zettl where the authors determine all self-adjoint operators, generated by the Legendre expression $\ell[\cdot]$, in the Hilbert spaces $L^2(-1,1)$, $L^2(-\infty,-1)$, $L^2(1,\infty)$, and $L^2(\mathbb{R})$.

Littlejohn and Wellman [47], in 2002, developed a general left-definite theory for an unbounded self-adjoint operator T bounded below by a positive constant in a Hilbert space $H=(V,(\cdot,\cdot))$, where V denotes the underlying (algebraic) vector space and H is the resulting topological space induced by the inner product $(\cdot,\cdot)$. To summarize, the authors construct a continuum of Hilbert spaces $\{H_r=(V_r,(\cdot,\cdot)_r)\}_{r>0}$, forming a Hilbert scale, generated by positive powers of T. The authors called these Hilbert spaces *left-definite spaces*; they are constructed using the Hilbert space spectral theorem (see [58]) for self-adjoint operators.

It is a difficult problem, in general, to explicitly determine the domain of a power of an unbounded operator. However, the authors in [47] prove that $V_r = \mathcal{D}(T^{\frac{r}{2}})$ and $(f,g)_r = (T^{\frac{r}{2}}f, T^{\frac{r}{2}}g)$. Furthermore, in many practical applications, as the authors demonstrate in [47], the computation of the vector spaces V_r and inner products $(\cdot,\cdot)_r$ is surprisingly not difficult. In a subsequent paper, Everitt, Littlejohn, and Wellman [25] applied this theory to the Legendre polynomials operator A. Among other results, the authors explicitly compute the domains of $\mathcal{D}(A^{\frac{n}{2}})$ for each $n\in\mathbb{N}$. Specifically, they proved that

$$\begin{aligned}\mathcal{D}(A^{\frac{n}{2}}) = \{f:(-1,1)\to\mathbb{C} \mid f,f',...,f^{(n-1)}\in AC_{\mathrm{loc}}(-1,1);\\ (1-x^2)^{\frac{n}{2}}f^{(n)}\in L^2(-1,1)\}\quad (n\in\mathbb{N}).\end{aligned} \tag{1.1}$$

In particular, we see that $\mathcal{D}(A^2)$ is explicitly given by

$$B=\{f:(-1,1)\to\mathbb{C}\mid f,f',f'',f'''\in AC_{\mathrm{loc}}(-1,1);(1-x^2)^2f^{(4)}\in L^2(-1,1)\}; \tag{1.2}$$

the reason for using the notation B, instead of $\mathcal{D}(A^2)$, will be made clear shortly. Of course, for $f\in B$, we have $A^2f=\ell^2[f]$, where $\ell^2[\cdot]$ is the square of the Legendre differential expression given by

$$\begin{aligned}\ell^2[y](x) &= \left((1-x^2)^2y''(x)\right)'' - 2\left((1-x^2)y'(x)\right)' \\ &= (1-x^2)^2y^{(4)}(x) - 8x(1-x^2)y'''(x) + (14x^2-6)y''(x) + 4xy'(x).\end{aligned} \tag{1.3}$$

Notice that, curiously, there are no "boundary conditions" given in (1.2). From the Glazman-Krein-Naimark (GKN) theory (see [52]), there should be *four* such boundary conditions. This begs an obvious question: how can we "extract" boundary conditions from the representation of $\mathcal{D}(A^2)$ in (1.2)? In this thesis, we will answer this question. It is interesting that the condition $(1-x^2)^2f^{(4)} \in L^2(-1,1)$ seems to "encode" these boundary conditions. In fact, along the way, we will characterize $\mathcal{D}(A^2)$ in four different ways. Of course, we have the algebraic definition

$$\mathcal{D}(A^2) := \{f \in \mathcal{D}(A) \mid Af \in \mathcal{D}(A)\} \tag{1.4}$$

(we will show that $\mathcal{D}(A^2)$, given in (1.4), is equal to B, defined in (1.2)). We will also prove that $\mathcal{D}(A^2)$ is characterized by GKN boundary conditions associated with a self-adjoint operator S, generated by $\ell^2[\cdot]$, in $L^2(-1,1)$. Specifically, we prove that $\mathcal{D}(A^2)$ is equal to

$$\begin{aligned}\mathcal{D}(S) := \{f : (-1,1) \to \mathbb{C} \mid f, f', f'', f''' \in AC_{\mathrm{loc}}(-1,1); f, \ell^2[f] \in L^2(-1,1); \\ \lim_{x\to\pm1}[f,1]_2(x) = 0; \lim_{x\to\pm1}[f,x]_2(x) = 0\},\end{aligned} \tag{1.5}$$

where $[\cdot,\cdot]_2$ is the sesquilinear form associated with Green's formula and $\ell^2[\cdot]$ in $L^2(-1,1)$; this form will be defined in Section 7.1. In this thesis, we also show that $\mathcal{D}(A^2)$ is equal to

$$\begin{aligned}D := \{f : (-1,1) \to \mathbb{C} \mid f, f', f'', f''' \in AC_{\mathrm{loc}}(-1,1); f, \ell^2[f] \in L^2(-1,1); \\ \lim_{x\to\pm1}(1-x^2)f'(x) = 0; \lim_{x\to\pm1}\left((1-x^2)^2f''(x)\right)' = 0\}.\end{aligned} \tag{1.6}$$

This characterization of $\mathcal{D}(A^2)$ is surprising since the boundary conditions in (1.6) are *not* GKN boundary conditions; we say that D is a GKN-like domain. The boundary conditions in (1.6) are remarkably simple; indeed, they are obtained as limits from each of the two terms in (1.3) minus one derivative.

In [12], the authors first showed the smoothness condition

$$f \in \mathcal{D}(A) \implies f' \in L^2(-1,1). \tag{1.7}$$

As a consequence of our results in this thesis, we are able to generalize (1.7) by proving

$$f \in \mathcal{D}(A^2) \implies f'' \in L^2(-1,1) \text{ and } \ell[f] \in AC[-1,1];$$

see Corollary 7.11 below.

The contents of this thesis are as follows. In chapter two, we discuss the Legendre polynomials in the context of orthogonal polynomials systems and the Legendre expression. In chapter three, we explain GKN theory in general as well as Weyl theory and the method of Frobenius, all of which are essential to the GKN analysis of the Legendre polynomials operator. In chapter four we discuss the maximal domain of the Legendre operator. In chapter five, we apply GKN theory to find all the self-adjoint extensions of the minimal operator associated with the Legendre differential expression, and then focus on the properties of the particular self-adjoint extension that has the Legendre polynomials as eigenfunctions. This gives context for Theorem 5.6, which lists all the known equivalent conditions for a function to be in the domain of the Legendre operator. We also briefly look at the Legendre differential operator in the left-definite Hilbert-Sobolev function space $H_1^2(-1,1)$, which paves the way for the discussion of Littlejohn-Wellman left-definite theory in chapter six. In chapter seven, we define and prove that the four various ways to define the domain of the operator A^2, are all equal; i.e.,

- B, the left-definite domain given in (1.2),
- $\mathcal{D}(A^2)$, the algebraic definition of the domain given in (1.4),

- $\mathcal{D}(S)$, the domain of the self-adjoint operator S (which turns out to be A^2) defined by GKN theory given in (1.5), and

- D, the domain given by non-GKN boundary conditions given in (1.6) which most resembles the original domain for the Legendre polynomials operator, $\mathcal{D}(A)$,

are equivalent. We first define the above four domains in Sections 7.1 and 7.2. Then, a key and indispensable analytic tool used in the proofs of these theorems, called the Chisholm-Everitt (CE) theorem, is discussed in Section 7.3. The proofs of the theorems in Sections 7.4 through 7.6 establish our main result, Theorem 7.12. Finally, in chapter eight, we conjecture a generalization of our main results.

One final remark: to summarize, in this thesis we show that our left-definite characterization (1.2) of $\mathcal{D}(A^2)$ can be rewritten as a GKN domain (Theorem 7.6) and as a GKN-like domain (Theorem 7.12). Presumably, techniques developed in this paper will establish, for $n \in \mathbb{N}$, that the left-definite characterization $\mathcal{D}(A^n)$, given in (1.1), can be expressed as both a GKN domain and a GKN-like domain. However, it is important to note—see (1.1)—that the left-definite theory also explicitly determines the domains $\mathcal{D}(A^{\frac{n}{2}})$ of $A^{\frac{n}{2}}$ for odd, positive, integers n. The GKN theory was not built to handle these operators or domains.

Figure 1.1: The only known image of Adrien-Marie Legendre [16]

CHAPTER TWO

Legendre Polynomials and the Legendre Differential Equation

2.1 Definition and Properties of Orthogonal Polynomials

Let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. A complex-valued linear functional $\mathcal{L}$ defined on the vector space of all polynomials with complex coefficients by $\mathcal{L}[x^n] = \mu_n$, $n \in \mathbb{N}_0$, is called the *moment functional* determined by the *moment sequence* $\{\mu_n\}$. The number μ_n is called the *moment of order* n. A sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials is called an *orthogonal polynomial sequence* with respect to some moment functional $\mathcal{L}$ if for all nonnegative integers m and n,

(i) $p_n(x)$ is a polynomial of degree n

(ii) $\mathcal{L}(p_n p_m) = 0$ when $m \neq n$, and

(iii) $\mathcal{L}(p_n^2) \neq 0$.

If, in addition, we also have $\mathcal{L}(p_n^2) = 1$, $n \in \mathbb{N}_0$, then $\{p_n\}_{n=0}^{\infty}$ is an *orthonormal polynomial sequence.*

Of course, not every sequence of complex numbers determines a moment functional having an orthogonal polynomial sequence. For $n \in \mathbb{N}_0$, let

$$\Delta_n = \det\,(\mu_{i+j})_{i,j=0}^{n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}$$

for $n \in \mathbb{N}_0$. A moment functional $\mathcal{L}$ is *quasi-definite* if $\Delta_n \neq 0$ for all $n \in \mathbb{N}_0$.

A moment function $\mathcal{L}$ is called *positive-definite* if $\mathcal{L}[p(x)] > 0$ for every polynomial $p(x)$ which is non-negative for all real x and not identically 0. We have the following important characterization of positive-definite moment functionals:

Theorem 2.1. *A moment functional $\mathcal{L}$ is positive-definite if and only if its moments are all real and $\Delta_n > 0$ for each $n \in \mathbb{N}_0$.*

Proof. See [11]. $\square$

One of the most important characteristics of orthogonal polynomials in our setting is that they satisfy a three-term recurrence formula. More specifically, we have the following theorem:

Theorem 2.2. *Let $\mathcal{L}$ be a moment functional with orthogonal polynomial sequence $\{p_n(x)\}$. Then there exist constants A_n, B_n, and C_n, where $A_n \neq 0$ and $C_n \neq 0$, such that*

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x)$$

for $n \in \mathbb{N}_0$, where we define $p_{-1}(x) := 0$.

Remarkably, the converse of Theorem 2.2 is true and is known as Favard's theorem. We also have the following, known as Boas' moment theorem.

Theorem 2.3. *Let $\{\mu_n\}$ be an arbitrary sequence of real numbers. Then there is a function φ of bounded variation on $(-\infty, \infty)$ such that for $n \in \mathbb{N}_0$,*

$$\int_{-\infty}^{\infty} x^n d\varphi(x) = \mu_n.$$

Proof. See [9]. $\square$

In the case of a moment functional $\mathcal{L}$ with complex moments, a generalization of Boas' theorem shows that $\mathcal{L}$ can be represented by a complex-valued function of bounded variation. It should be noted that the function φ in Theorem 2.3 is not unique since we can always add a function of bounded variation to φ with the property that all of its moments are zero. In addition, though Boas' theorem is an important theoretical result, its proof is not constructive. In practice it is difficult to find a weight function for a given moment sequence [50].

The following theorem will be referred to below in chapter four.

Theorem 2.4. *Let $\alpha(x)$ be a nondecreasing function which is not constant on the compact interval $[a,b]$. Assume $\{p_n(x)\}$ is an orthogonal polynomial sequence with respect to the distribution $d\alpha(x)$ on $[a,b]$. Then $\{p_n(x)\}$ is a complete orthogonal polynomial sequence in $L^2_\alpha[a,b]$ where*

$$L^2_\alpha[a,b] = \{f : [a,b] \to \mathbb{C} \mid f \text{ is measurable with respect to } \alpha \\ \text{and } \int_a^b |f(x)|^2 d\alpha(x) < \infty\}.$$

Proof. See [63]. □

2.2 The Classical Systems of Orthogonal Polynomials

In 1929, Bochner [10] classified all orthogonal polynomial solutions to the second order equation

$$a_n(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = \lambda y(x), \tag{2.1}$$

where a_2, a_1, and a_0 are polynomials and λ is a parameter independent of x. He observed that if (2.1) has a polynomial solution of degree m, $m = 0, 1, 2$, then a_2, a_1, and a_0 were of degrees at most 2, 1, and 0, respectively. By considering the possible locations of the roots of a_2, Bochner concluded that the only polynomial solutions (up to a linear change of variables) are

(i) the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^\infty$, where $-\alpha, -\beta, -(\alpha+\beta+1) \notin \mathbb{N}$;

(ii) the Laguerre polynomials $\{L_n^{(\alpha)}\}_{n=0}^\infty$, where $-\alpha \notin \mathbb{N}$;

(iii) the Hermite polynomials $\{H_n\}_{n=0}^\infty$;

(iv) the Bessel polynomials $\{y_n^a\}_{n=0}^\infty$, where $-(a+1) \notin \mathbb{N}$;

(v) and $\{x^n\}_{n=0}^\infty$.

(There is clear evidence that Bochner knew of the existence of the Bessel orthogonal polynomial sequence, though these polynomials were not officially discovered until 1948.) The polynomials in (5), however, cannot form an orthogonal polynomial sequence with respect to any moment functional $\mathcal{L}$ since $0 \neq \mathcal{L}(x^2x^2) = \mathcal{L}(xx^3) = 0$. Thus, the only orthogonal polynomials that are solutions to a second order differential equation of the form (2.1) are the classical orthogonal polynomials of Jacobi, Laguerre, and Hermite, together with the Bessel polynomials. We call these four sequences of polynomials the *Bochner-Krall orthogonal polynomials of order 2*.

Another important classification theorem was given by Hahn [29], who showed that if $\{p_n\}_{n=0}^{\infty}$ and $\{p_n'\}_{n=0}^{\infty}$ are orthogonal polynomial sequences with respect to positive-definite moment functionals, then $\{p_n\}_{n=0}^{\infty}$ is (up to a linear change of variable) one of the three classical systems of orthogonal polynomials. It was later observed by Krall [44] and Beale [7] that the only orthogonal polynomial sequences whose derivatives form an orthogonal polynomial sequence with respect to a quasi-definite moment functional are the classical orthogonal polynomials and the Bessel polynomials.

A third characterization of these polynomials was suggested by Tricomi [65] and a complete proof was given by Ebert [17] and Cryer [13]. They proved that the only polynomial sequences that have Rodrigues formulas are the Hermite, Laguerre, Jacobi, and Bessel polynomals. By a *Rodrigues formula* we mean a formula of the form

$$p_n(x) = K_n^{-1}[w(x)]^{-1}D^n[\rho^n(x)w(x)], \quad n \in \mathbb{N}_0,$$

where

(i) K_n is independent of x;

(ii) $\rho(x)$ is a polynomial independent of n;

(iii) $w(x)$ is positive and integrable over some interval (a, b).

Several orthogonal polynomial sequences can be found through *generating functions*. A generating function for $\{p_n\}_{n=0}^{\infty}$ is a function F of two variables such that

$$F(x,w)=\sum_{n=0}^{\infty}a_np_n(x)w^n,$$

where convergence is in some region of the plane $\mathbb{R}^2$ and $\{a_n\}$ is a known sequence of constants.

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}$, where $\alpha>-1$ and $\beta>-1$, are the classical system of orthogonal polynomials for which the Legendre polynomials are a special case. The explicit formula for the n^{th} Jacobi polynomial is

$$P_n^{(\alpha,\beta)}(x)=2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k}\quad (n\in\mathbb{N}_0).$$

The Jacobi polynomials are the eigenfunctions of the differential equation

$$(1-x^2)y''(x)+[\beta-\alpha-(\alpha+\beta+2)x]y'(x)+n(n+\alpha+\beta+1)y(x)=\lambda_ny(x),$$

where $\lambda_n=n(n+\alpha+\beta+1)$. These polynomials are orthogonal on $[-1,1]$ with respect to the weight function $w(x)=(1-x)^\alpha(1+x)^\beta$ and

$$\int_{-1}^{1}P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)w(x)dx=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta_1)\Gamma(n+\alpha+\beta+1)n!}\delta_{mn},$$

where Γ denotes the Gamma function (use of the Γ symbol is also credited to Legendre) and δ_{mn} denotes Dirac's δ-function.

The Rodrigues Formula for the Jacobi polynomials is

$$P_n^{(\alpha,\beta)}(x)=(-2)^{-n}(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}],$$

and a generating function is

$$2^{\alpha+\beta}R^{-1}(1-x+R)^{-\alpha}(1+w+R)^{-\beta}=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)w^n,$$

where $R = (1 - 2xw + w^2)^{\frac{1}{2}}$. Finally, the Jacobi polynomials satisfy the recurrence relation

$$\begin{aligned} 2n(n + \alpha{+}\beta)&(2n + \alpha + \beta - 2)P_n^{(\alpha,\beta)}(x) \\ &= (2n + \alpha + \beta - 1)[(2n + \alpha + \beta)(2n + \alpha + \beta - 2)x + \alpha^2 - \beta^2]P_{n-1}^{(\alpha,\beta)}(x) \\ &- 2(n + \alpha - 1)(n + \beta - 1)(2n + \alpha + \beta)P_{n-2}^{(\alpha,\beta)}(x), \quad n \in \mathbb{N}, \end{aligned}$$

where $P_{-1}^{(\alpha,\beta)}(x) = 0$ and $P_0^{(\alpha,\beta)}(x) = 1$. The Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are the special case of the Jacobi polynomials determined by setting the parameters $\alpha = \beta = 0$, hence their specific properties simplify to:

(i) the explicit formula

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2n - 2k)!}{2^n (n - k)! k! (n - 2k)!} x^{n-2k}, \tag{2.2}$$

where $[\frac{n}{2}]$ denotes the greatest integer less than or equal to $\frac{n}{2}$;

(ii) the differential equation

$$(1 - x^2)y''(x) - 2xy'(x) + n(n + 1)y(x) = 0;$$

(iii) the orthogonality relation

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n + 1}\delta_{nm} \tag{2.3}$$

on $[-1, 1]$ with respect to the weight function $w(x) = 1$, so that

$$||P_n||^2 = \int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n + 1};$$

(iv) the Rodrigues formula

$$P_n(x) = (-2)^{-n}(n!)^{-1}\frac{d^n}{dx^n}[(1 - x^2)^n];$$

(v) the generating function

$$(1 - 2xw + w^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)w^n;$$

(vi) and the recurrence relation

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

where $P_{-1}(x) = 0$ and $P_0(x) = 1$ for $n \in \mathbb{N}$.

Among other places that the Legendre polynomials appear in mathematics and physics, the Legendre polynomial P_n can also be defined by the contour integral

$$P_n(z) = \frac{1}{2\pi i} \oint (1 - 2tz + t^2)^{-\frac{1}{2}} t^{-n-1} dt,$$

where the contour encloses the origin and is traversed in a counterclockwise direction (see [5]).

The Legendre polynomials have "interlacing zeros," as evidenced by the following theorem:

Theorem 2.5. *The zeros of $P_n(x)$ and $P_{n+1}(x)$ mutually separate each other, i.e.*

$$x_{n+1,i} < x_{n,i} < x_{n+1,i+1}, \ \ i = 1, 2, ..., n$$

where $x_{n,i}$ is the ith zero of the nth polynomial.

Proof. See [11]. □

All the zeros of $\{P_n\}_{n=1}^{\infty}$ lie in $(-1, 1)$. Finally, the Legendre polynomials satisfy the Parity Property; i.e.,

$$P_n(-x) = (-1)^n P_n(x).$$

From this we see that P_n is even (odd) if n is even (odd).

2.3 The Legendre Polynomials and the Legendre Differential Equation

Solving Laplace's equation using the method of separation of variables in spherical coordinates with rotational symmetry leads to the eigenvalue problem

$$(1 - x^2)y''(x) - 2xy'(x) + \lambda y(x) = 0 \quad (x \in (-1, 1)), \tag{2.4}$$

known as *Legendre's equation* [14]. We define the classic second-order Legendre differential expression $\ell[\cdot]$ as

$$\ell[y](x) := (1 - x^2)y''(x) - 2xy'(x) \quad (x \in (-1, 1)). \tag{2.5}$$

Since (2.5) can be written

$$\ell[y](x) = -\big((1 - x^2)y'(x)\big)', \tag{2.6}$$

we see that $\ell[\cdot]$ is formally symmetric.

The eigenvalues and associated eigenfunctions are

$$\lambda_n = n(n+1), \quad y_n(x) = P_n(x), \quad n \in \mathbb{N}_0,$$

where $P_n(x)$ denotes the Legendre polynomial of order n [14].

The Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ were first introduced by Adrien-Marie Legendre in 1782 in the context of astronomy, namely, as the coefficients in the expansion of the Newtonian potential

$$\frac{1}{||x-y||} = \frac{1}{\sqrt{r^2 + x^2 - 2rs\cos\gamma}} = \sum_{n=0}^{\infty} \frac{s^n}{r^{n+1}} P_n(\cos\gamma),$$

where r and s are the respective lengths of the vectors x and y and γ is the angle between them [46].

The first several Legendre polynomials appear below.

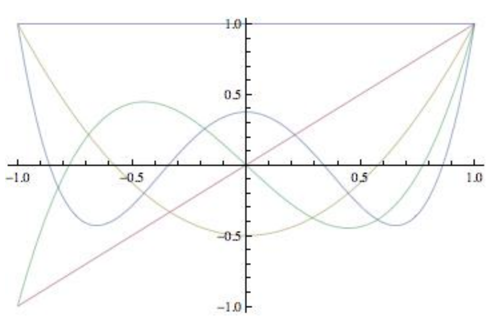

Figure 2.1: Five Legendre polynomials

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

CHAPTER THREE

The GKN Theory of Self-Adjoint Extensions of Symmetric Operators

3.1 Symmetric Operators

Let H be a complex Hilbert space with inner product $(\cdot,\cdot)$ and let the operator $T : \mathcal{D}(T) \to H$ be *densely defined*, i.e., the domain of T, $\mathcal{D}(T)$, is a dense subset of H. In this setting, we now define the *Hilbert-adjoint operator*, T^*, of T. The domain $\mathcal{D}(T^*)$ consists of all $y \in H$ such that the mapping $f_y := x \to (Tx, y)$ is continuous on $\mathcal{D}(T)$. By the Hahn-Banach theorem, f_y has a continuous extension to all of H. Hence, by the Riesz representation theorem, there exists a unique $y^* \in H$ such that $(Tx, y) = (x, y^*)$ for all $x \in \mathcal{D}(T)$. We define $T^*y = y^*$.

An operator T is *Hermitian* if, for all $x, y \in \mathcal{D}(T)$ where $\mathcal{D}(T)$ is not necessarily dense in H, $(Tx, y) = (x, Ty)$. If in addition to being Hermitian T has a dense domain, then T is called *symmetric*. The following theorem, proved by Hellinger and Toeplitz, shows that an unbounded Hermitian operator T cannot be defined on all of H.

Theorem 3.1. *If a linear operator T is defined on all of a complex Hilbert space H and satisfies $(Tx, y) = (x, Ty)$ for all $x, y \in H$, then T is bounded.*

Proof. See [45]. □

Because the domain of an unbounded symmetric operator T is a proper subspace of H, it makes sense to study symmetric extensions (if they exist) of the operator T in H. The next theorem gives relationships between the adjoint of a symmetric operator and the adjoint of its symmetric extension. We write $S \subset T$ when the operator T is an extension of the operator S, meaning that $\mathcal{D}(S) \subseteq \mathcal{D}(T)$ and $S[f] = T[f]$ for $f \in \mathcal{D}(S)$.

Theorem 3.2. *Suppose that T is a densely defined operator on a Hilbert space H.*

(i) T is symmetric if and only if $T \subset T^*$.

(ii) If S is an extension of T, then T^* is an extension of S^*, i.e., $T \subset S$ implies $S^* \subset T^*$.

(iii) If T is symmetric, then every symmetric extension S of T satisfies $T \subset S \subset S^* \subset T^*$.

A densely defined operator with the property that $T = T^*$ is *self-adjoint*. Property (i) of Theorem 3.2 shows that every self-adjoint operator is symmetric, but the converse is not true; however, in the case of a bounded operator $T : H \to H$, the concepts of symmetry and self-adjointness are identical. From property (iii) of Theorem 3.2, we see that the most general symmetric extension in H (in particular, the most general self-adjoint extension) of a symmetric operator T, is a suitably chosen restriction of the adjoint T^* of T. We characterize the domains of self-adjoint extensions of a general unbounded operator S below.

3.2 Weyl Theory

Suppose $a_k : (a, b) \to \mathbb{C}$ is such that $a_k \in C^k(a, b)$, $k = 0, 1, ..., n$. Let

$$L[y] = \sum_{k=0}^{n} a_k y^{(k)}, \quad y \in C^n(a, b). \tag{3.1}$$

The *Lagrange adjoint* (or *formal adjoint*) of $L[\cdot]$ is the differential expression

$$L^*[y] = \sum_{k=0}^{n} (-1)^k (\overline{a}_k y)^{(k)}.$$

The expression $L[\cdot]$ is *formally symmetric* if $L[y] = L^*[y]$ for all $y \in C^n(a, b)$. Theorem 3.3 determines a general form for all formally symmetric differential operators with real-valued coefficients.

Theorem 3.3. *Suppose $L[\cdot]$ is given as in (3.1) with each coefficient $a_k \in C^k(a,b)$ being real-valued. If $L[\cdot]$ is formally symmetric, then n is necessarily even and $L[\cdot]$ may be written as*

$$L[y] = \sum_{k=0}^{n/2} (-1)^k (b_k y^{(k)})^{(k)}.$$

Proof. See [15]. □

Since the domain of L consists of all functions y with n derivatives and the domain of L^* consists of all functions y such that $(\overline{a}_k y)^{(k)}$ exists for $k = 0, 1, ..., n$, it follows that $\mathcal{D}(L) \subseteq \mathcal{D}(L^*)$. Therefore formal symmetry corresponds to the usual definition of symmetry from the previous section.

Weyl studied the solutions to the *Sturm-Liouville differential equation*

$$D_\lambda[y](x) := [p(x)y'(x)]' + [\lambda w(x) - q(x)]y(x) = 0 \tag{3.2}$$

on the interval (a,b) where

(i) p, p', q and w are real-valued and continuous on (a,b);

(ii) $p(x) > 0$ and $w(x) > 0$ in (a,b); and

(iii) $\lambda \in \mathbb{C}$.

Note that if we define

$$\ell[y] := \frac{1}{w}\big(- (py')' + qy\big),$$

then, by Theorem 3.3, $w\ell[y]$ is formally symmetric and the equation $\ell[y] = \lambda y$ is fully equivalent to $D_\lambda y = 0$. A solution of (3.2) is a function $y \in C^2(a,b)$ such that $D_\lambda[y](x) = 0$ for every $x \in (a,b)$.

The following two theorems proved by Weyl are essential in extending symmetric operators.

Theorem 3.4. *Let* $\lambda_0 \in \mathbb{C}$ *(possibly real) and let* $x_0 \in (a, b)$. *Suppose*

$$\int_{x_0}^{b} |y(x)|^2 w(x) dx < \infty$$

for all solutions $y(x)$ *of* $D_{\lambda_0}[y] = 0$. *Then*

$$\int_{x_0}^{b} |v(x)|^2 w(x) dx < \infty$$

for all solutions $v(x)$ *of* $D_\lambda[y] = 0$ *for all* $\lambda \in \mathbb{C}$. *A similar result holds at the other endpoint* a *of* (a, b).

Proof. See [32]. □

Theorem 3.5. *Let* $x_0 \in (a, b)$ *and let* $\lambda \in \mathbb{C}$ *with* $\mathrm{Im}(\lambda) \neq 0$. *Then*

(i) there exists at least one solution $u(x) \not\equiv 0$ *of* $D_\lambda[y] = 0$ *such that*

$$\int_{x_0}^{b} |u(x)|^2 w(x) dx < \infty;$$

(ii) if there exists at least one solution $\hat{u}(x)$ *of* $D_\lambda[y] = 0$ *with*

$$\int_{x_0}^{b} |\hat{u}(x)|^2 w(x) dx = \infty,$$

then for every solution $v(x)$ *of* $D_\lambda[y] = 0$ *which satisfies*

$$\int_{x_0}^{b} |v(x)|^2 w(x) dx < \infty$$

we have

$$\lim_{x \to b} p(x)[v'(x)\overline{v}(x) - v(x)\overline{v}'(x)] = 0;$$

(iii) if

$$\int_{x_0}^{b} |u(x)|^2 w(x) dx < \infty$$

for every solution $u(x)$ *of* $D_\lambda[y] = 0$ *then there exists a fundamental system* $u_1(x), u_2(x)$ *of* $D_\lambda[y] = 0$ *and a circle* $|\xi - \xi_0| = r_0$ *in the complex plane with center* ξ_0 *and radius* $r_0 > 0$ *such that for* $w(x) = \xi u_1(x) + u_2(x)$, *we have*

$$\lim_{x \to b} p(x)[w'(x)\overline{w}(x) - w(x)\overline{w}'(x)] = 0$$

for all ξ *which lie on the circle.*

Corresponding statements are true at the endpoint a of (a, b).

Proof. See [32]. □

Weyl's proof of Theorem 3.5 is geometric and involves a series of contracting circles, hence, in view of his second theorem, we say that at $x = b$ the *limit-circle case* with respect to λ occurs if for λ

$$\int_{x_0}^{b} |u(x)|^2 w(x) dx < \infty$$

for every solution $u(x)$ of $D_\lambda[y] = 0$. We say that at $x = b$ the *limit point case* with respect to λ occurs if for λ there exists only one linearly independent solution $u(x)$ of $D_\lambda[y] = 0$ for which

$$\int_{x_0}^{b} |u(x)|^2 w(x) dx = \infty$$

with corresponding terminology at the endpoint a of (a, b). The next theorem, known as Weyl's alternative, shows that these definitions are independent of λ.

Theorem 3.6 (Weyl's Alternative). *The occurrence of the limit circle case and the limit point case, respectively, is independent of λ.*

Proof. See [32]. □

We now discuss the von Neumann-Stone theory of symmetric extensions of symmetric operators; the standard reference in this case is [15].

3.3 Extensions of General Symmetric Operators

Suppose $T : \mathcal{D}(T) \to H$ is a densely defined symmetric operator. Let

$$D_+ = \{f \in \mathcal{D}(T^*) \mid T^* f = if\}$$

and

$$D_- = \{f \in \mathcal{D}(T^*) \mid T^* f = -if\}$$

where $i = \sqrt{-1}$. Then the spaces D_+ and D_- are respectively called the *positive* and *negative deficiency spaces* of T. We call the dimensions $n_+ = \dim D_+$ and $n_- = \dim D_-$ the *positive* and *negative deficiency indices* of T.

The *graph* $\mathcal{G}(T)$ of an operator T is the subset of $H \oplus H$ consisting of all points of the form (x, Tx) with $x \in \mathcal{D}(T)$. If $\mathcal{G}(T)$ is a closed subset of $H \oplus H$ in the inner product defined by

$$\big((x_1, x_2), (y_1, y_2)\big) = (x_1, y_1) + (x_2, y_2)$$

then T is a *closed operator*. (Equivalently, T is closed if whenever $\{x_n\} \subset \mathcal{D}(T)$ satisfies $x_n \to x$ and $Tx_n \to y$, then $x \in \mathcal{D}(T)$ and $Tx = y$.) We say that T_1 is a *minimal closed linear extension* of T if whenever S is a closed linear extension of T, we have $T_1 \subset S$. In this case we call T_1 the *closure* of T and write $\overline{T} = T_1$. T is *closable* or is said to *admit a closure* if such a closed linear extension exists. With regard to self-adjoint operators, we have the following theorem.

Theorem 3.7. *Let $T : \mathcal{D}(T) \to H$ be a symmetric operator.*

(i) $\overline{T}$ exists (that is, T admits a closure) and is a uniquely defined symmetric operator.

(ii) The operators T and $\overline{T}$ have the same closed extensions.

(iii) A self-adjoint operator is closed.

Proof. See [15]. □

Since we are concerned with self-adjoint extensions of symmetric operators T, by the above theorem we can concentrate on the consideration of closed symmetric extensions of $\overline{T}$.

If $x, y \in \mathcal{D}(T^*)$, define

$$(x, y)^* := (x, y) + (T^*x, T^*y).$$

We see then that $(x, y)^*$ is simply the inner product that $\mathcal{D}(T^*)$ inherits from the previously defined inner product on $H \oplus H$ if we identify $\mathcal{D}(T^*)$ with $\mathcal{G}(T^*)$ via the map $x \to (x, T^*x)$. Then it can be verified that $\mathcal{D}(T^*)$ is a complete Hilbert space under the inner product $(x, y)^*$. The following theorem gives a decomposition of the Hilbert space $\mathcal{D}(T^*)$ as a direct sum of closed orthogonal subspaces.

Theorem 3.8. *Suppose $T : \mathcal{D}(T) \to H$ is a symmetric operator. Then*

(i) $\mathcal{D}(\overline{T}), D_+,$ and D_- are closed, mutually orthogonal subspaces of $\mathcal{D}(T^*)$ in the inner product $(x, y)^*$, and

(ii) $\mathcal{D}(T^*) = \mathcal{D}(\overline{T}) \oplus D_+ \oplus D_-$.

Proof. See [15]. $\square$

The representation of $\mathcal{D}(T^*)$ in property (ii) of the above theorem is known as *von Neumann's formula.* Since by Theorem 3.7 every closed symmetric extension S of a symmetric operator T satisfies $\overline{T} \subset S \subset S^* \subset T^*$, we see from von Neumann's formula that the space $D_+ \oplus D_-$ plays a central role in the search for self-adjoint extensions of the operator T. The theorem and corollary below give us the connection between this space and the domains of closed symmetric extensions of T.

Theorem 3.9. *Let $T : \mathcal{D}(T) \to H$ be a symmetric operator. Let $\mathcal{G}'$ be a closed subspace of $D_+ \oplus D_-$ and $\mathcal{G} = \mathcal{D}(\overline{T}) \oplus \mathcal{G}'$.*

(i) The space $\mathcal{G}$ is the domain of a closed symmetric extension of T if and only if $\mathcal{G}'$ is the graph of an isometric transformation mapping a subspace of D_+ onto a subspace of D_-.

(i) The restriction of T^* to $\mathcal{G}$ is self-adjoint if and only if $\mathcal{G}'$ is the graph of an isometric transformation mapping D_+ onto all of D_-.

Proof. See [15]. $\square$

Corollary 3.10. *(i) A symmetric operator T has self-adjoint extensions if and only if its deficiency indices n_+ and n_- are equal.*

(ii) If $n_+ = n_- = 0$, the only self-adjoint extension of T is its closure $\overline{T} = T^*$.

Proof. This corollary is an immediate consequence of Theorems 3.7, 3.8, and 3.9 and the fact that two Hilbert spaces are isometrically isomorphic if and only if they have the same dimension [50]. □

We note that although symmetric operators with unequal deficiency indices do not have self-adjoint extensions, it is still possible for such operators to have symmetric extensions (for an example, see [50]).

3.4 Extensions of Symmetric Differential Operators

Let (a, b) be an open interval of $\mathbb{R}$ with $\infty \leq a < b \leq \infty$. In this section, the Hilbert space H is the Lebesgue space $L^2(a, b)$. By Theorem 3.3, every formally symmetric differential expression $L[\cdot]$ of order $2n$ with coefficients $a_k : (a, b) \to \mathbb{R}$ and $a_k \in C^k(a, b)$ for $k = 0, 1, ..., n$ and $n \in \mathbb{N}$ is of the form

$$L[y](x) = \sum_{k=0}^{n} (-1)^k \big(a_k(x) y^{(k)}(x)\big)^{(k)}, \quad x \in (a, b). \tag{3.3}$$

In this section we assume that

$$\frac{1}{a_n}, a_{n-1}, ..., a_1, a_0 \in L_{\text{loc}}(a, b).$$

The endpoint a is a *regular point* of $L[\cdot]$ and $L[\cdot]$ is *regular* at a if $a > -\infty$ and there exists an $\varepsilon > 0$ such that

$$\frac{1}{a_n}, a_{n-1}, ..., a_1, a_0 \in L_{\text{loc}}(a, a + \varepsilon).$$

Otherwise the endpoint a is a *singular point* of $L[\cdot]$ and $L[\cdot]$ is *singular* at a. We use similar terminology at the endpoint b. The expression $L[\cdot]$ is said to be *regular* if both a and b are regular points, otherwise $L[\cdot]$ is *singular*. Since the Legendre operator is a singular expression in $(-1,1)$, we assume here that $L[\cdot]$ is singular though much of the theory applies equally well to the regular case, as described in Naimark (see [52]). We note here that Naimark considers symmetric differential expressions with fewer differentiability requirements. Indeed, the less restrictive hypotheses assumed by Naimark and other authors lead them to the concept of the quasi-derivative, which we define below in the context of defining the sesquilinear form. However, since the coefficients of the Legendre expression are polynomials, we assume as above that $a_k \in C^k(a,b)$.

The *maximal operator* $\mathcal{L}$ generated by the expression $L[\cdot]$ is defined by

$$\begin{aligned} \mathcal{L}[y] &:= L[y] \\ \mathcal{D}(\mathcal{L}) &:= \{y : (a,b) \to \mathbb{C} \mid y^{(k)} \in AC_{\text{loc}}(a,b), k = 0,1,...,2n-1; \\ &\qquad\qquad y, L[y] \in L^2(a,b)\}, \end{aligned} \tag{3.4}$$

where $L[\cdot]$ is given by (3.3). Note that the term "maximal" is appropriate because the space $\mathcal{D}(\mathcal{L})$ is the largest possible subspace for which $\mathcal{L}$ can be defined as an operator from $L^2(a,b)$ into $L^2(a,b)$.

For ease of notation in the definition of the sesquilinear form below, we now define the concept of the quasi-derivative. The kth *quasi-derivative* $y^{[k]}$ of a function y is defined as

$$\begin{aligned} y^{[0]} &= y = y^{(0)} \\ y^{[k]} &= y^{(k)}, \; k = 0,1,...,n-1 \\ y^{[n]} &= a_n y^{(n)} \\ y^{[n+k]} &= a_{n-k} y^{(n-k)} - \left[y^{[n+k-1]}\right]', \; k = 1,2,...,n-1 \end{aligned}$$

where, as above,

$$\frac{1}{a_n}, a_{n-1}, ..., a_1, a_0 \in L_{\text{loc}}(a,b)$$

and $y^{(k)}$ denotes the usual derivative. Using the quasi-derivative, we define the *sesquilinear (symplectic) form* $[y,z](\cdot)$ of two functions y and z by

$$[y,z] = \sum_{k=1}^{n} \left\{ y^{[k-1]}\overline{z}^{[2n-k]} - y^{[2n-k]}\overline{z}^{[k-1]} \right\} \tag{3.5}$$

where $\overline{z}$ denotes the usual complex conjugate of z and $y^{[k]}$ denotes the kth quasi-derivative of y. We note that without using quasi-derivatives, the sesquilinear form takes the form

$$[y,z] = \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{k+j} \left\{ (a_k \overline{z}^{(k)})^{(k-j)} y^{(j-1)} - (a_k y^{(k)})^{(k-j)} \overline{z}^{(j-1)} \right\}.$$

For $f, g \in \mathcal{D}(\mathcal{L})$ and any compact subinterval $[\alpha, \beta] \subset (a,b)$, the following formula can be easily verified by integration by parts:

$$\int_{\alpha}^{\beta} \{ \ell[f]\overline{g} - \ell[\overline{g}]f \}\, dx = [f,g](x)\Big|_{\alpha}^{\beta}, \tag{3.6}$$

where $[f,g](\cdot)$ is the sesquilinear form defined in (3.5). Notice that for all $f, g \in \mathcal{D}(\mathcal{L})$ and $a < x < b$, $[g,f](x) = -\overline{[f,g]}(x)$. By the definition of $\mathcal{D}(\mathcal{L})$ and Hölder's inequality, we have the following theorem.

Theorem 3.11. *The limits*

$$[f,g](b) := \lim_{x \to b^-} [f,g](x) \text{ and } [f,g](a) := \lim_{x \to a^+} [f,g](x)$$

both exist and are finite for all $f, g \in \mathcal{D}(\mathcal{L})$.

Proof. See [52]. □

Equation (3.6) is known as *Green's formula* for $L[\cdot]$, which is essential in the determination of all self-adjoint extensions in $L^2(a,b)$ of the *minimal operator* generated by $L[\cdot]$, which we now discuss.

Define a restriction $\mathcal{L}_0'$ of the maximal operator $\mathcal{L}$ by

$$\mathcal{L}_0'[y] := L[y]$$

$$\mathcal{D}(\mathcal{L}_0') := \{ y \in \mathcal{D}(\mathcal{L}) \mid y \text{ has compact support in } (a,b) \}.$$

For reasons that will soon become clear, we call $\mathcal{L}_0'$ the *pre-minimal operator.* It can be shown that the operator $\mathcal{L}_0'$ is symmetric (see [52]). Hence, by Theorem 3.7, $\mathcal{L}_0'$ admits a closure in $L^2(a,b)$ which is also symmetric. Let

$$\mathcal{L}_0 := \overline{\mathcal{L}}_0'.$$

Then $\mathcal{L}_0$ is called the *minimal operator* generated by $L[\cdot]$. Since $\mathcal{D}(\mathcal{L}_0)$ is dense in $L^2(a,b)$, the adjoint operator $\mathcal{L}_0$ exists. The following theorem states the relationship between the maximal operator $\mathcal{L}$ and the minimal operator $\mathcal{L}_0$.

Theorem 3.12. *$\mathcal{L}_0^* = \mathcal{L}$ and $\mathcal{L}^* = \mathcal{L}_0$.*

Proof. See [52]. □

In general, if a densely defined operator has a "large" domain, its adjoint will have a "small" domain, as described in property (ii) of Theorem 3.2. Therefore, since the maximal operator has the largest possible domain, its adjoint (the minimal operator) has the minimally small domain. Since $\mathcal{L}$ and $\mathcal{L}_0$ have the same form as the operator L, we call the operator with the maximal domain the *maximal operator* $\mathcal{L}$ and the operator with the minimal domain the *minimal operator* $\mathcal{L}_0$. It follows that to find self-adjoint operators generated by $L[\cdot]$, we need to either look at extensions of $\mathcal{D}(\mathcal{L}_0)$ or restrictions of $\mathcal{D}(\mathcal{L})$.

To determine whether a function $f \in \mathcal{D}(\mathcal{L})$ is in the minimal domain $\mathcal{D}(\mathcal{L}_0)$, we have the following theorem which makes use of the sesquilinear form.

Theorem 3.13. *The minimal domain, i.e., the domain of the minimal operator $\mathcal{L}_0$, is given by*

$$\mathcal{D}(\mathcal{L}_0) = \{f \in \mathcal{D}(\mathcal{L}) \mid [f,g](x)\big|_a^b = 0 \text{ for all } g \in \mathcal{D}(\mathcal{L})\}.$$

Proof. See [52]. □

From Corollary 3.10 and the following theorem, due to the equality of the positive and negative deficiency indices of $\mathcal{L}_0$, the minimal operator does have self-adjoint extensions in $L^2(a, b)$.

Theorem 3.14. *The deficiency indices of the operator $\mathcal{L}_0$ have the form (m, m) where $0 \leq m \leq 2n$ (and $2n$ is the order of the differential expression $L[\cdot]$).*

Proof. See [52]. $\square$

In fact, as shown by Glazman by means of actual examples, m can take on each value between 0 and $2n$. On a side note, in the case of an operator $L[\cdot]$ with one singular endpoint and one regular endpoint, the range of the deficiency indices is limited to $n \leq m \leq 2n$ [52].

In Section 3.2 above we defined the terms limit-point and limit-circle for second-order formally symmetric differential expressions. We generalize this terminology to a differential operator $L[\cdot]$ or order $2n$ by using the term *limit-m* at the endpoint a if there exist exactly m linearly independent solutions to $L[y] = \lambda[y]$ that belong to $L^2(a, x_0)$ for some $x_0 \in (a, b)$. In particular, if $n = 1$, the original terms limit-point and limit-circle apply in place of limit-1 and limit-2.

3.5 Glazman-Krein-Naimark (GKN) Theory

We turn now to the calculation of the deficiency indices for the operator $\mathcal{L}_0$ when there are two singular endpoints, which is the case for the Legendre differential operator at ± 1.

Before stating the GKN theorem, we introduce the following definition.

Definition 3.15. Suppose M_1 and M_2 are subspaces of a vector space V such that $M_1 \subset M_2$. Let $\{x_1, x_2, ..., x_n\} \subseteq M_2$. We say that $\{x_1, x_2, ..., x_n\}$ is *linearly independent modulo M_1* if

$$\sum_{i=1}^{n} \alpha_i x_i \in M_1 \text{ implies } \alpha_i = 0, \;\; i = 1, 2, ..., n.$$

The dimension of M_2 modulo M_1 is the maximum number of vectors in M_2 that are linearly independent modulo M_1. If this dimension is $n \leq \infty$, then we write that $\dim M_2 = n \bmod (\dim M_1)$.

The following theorem characterizes all self-adjoint extensions of $\mathcal{L}_0$.

Theorem 3.16 (GKN). *Suppose the deficiency indicies of $\mathcal{L}_0$ are $n_+ = n_- := m$.*

(i) Let S be a self-adjoint extension of $\mathcal{L}_0$ (in $L^2(a, b)$). Then there exist $w_1, w_2, ..., w_m \in \mathcal{D}(S)$ such that $\{w_1, w_2, ..., w_m\}$ is a set which is linearly independent modulo $\mathcal{D}(\mathcal{L}_0)$ where

(a) $Sx = \mathcal{L}x = Lx$;

(b) $\mathcal{D}(S) = \{x \in \mathcal{D}(\mathcal{L}) \mid [x, w_j]\big|_a^b = 0, j = 1, 2, ..., m\}$; and

(c) $[w_i, w_j]\big|_a^b = 0$ for $i, j = 1, 2, ..., m$. (These are called Glazman symmetry conditions.)

(ii) Conversely, suppose $\{w_1, w_2, ..., w_m\} \subseteq \mathcal{D}(\mathcal{L})$ are such that they are linearly independent modulo $\mathcal{D}(\mathcal{L}_0)$ and satisfy the Glazman symmetry conditions in (c) above. Then with S as defined in (a) and (b), S is a self-adjoint extension of $\mathcal{L}_0$.

Proof. See [52]. □

In chapter five, we use the GKN theorem (3.16) to find all self-adjoint extensions of the minimal operator associated with the Legendre differential expression below, and then find the particular self-adjoint extension which has the Legendre polynomials as eigenfunctions.

3.6 The Method of Frobenius

In this section we describe the method of Frobenius, which is a technique for finding n linearly independent solutions in the form of generalized power series for certain types of ordinary differential equations of order n. We use this method to determine whether a Sturm-Liouville expression (3.2) is limit-point or limit-circle, which then determines the deficiency indices of the expression. The method of Frobenius can be generalized for a higher order differential equation up to limit-$2n$ as discussed above.

Consider the differential equation given by

$$M[y](z) := a_2(z)y''(z) + a_1(z)y'(z) + a_0(z)y(z) = 0 \tag{3.7}$$

where each a_k, $k = 0, 1, 2$, is analytic in some open neighborhood $N(a)$ of $a \in \mathbb{C}$. We say that $z = a$ is a regular point of $M[\cdot]$ if $a_2(a) \neq 0$. The point $z = a$ is a *regular singular point* of $M[\cdot]$ if

$$\frac{a_k(z)}{a_2(z)} \text{ has a pole of order } \leq 2 - k \text{ at } z = a \quad (k = 0, 1),$$

otherwise, $z = a$ is called an *irregular singular point* of $M[\cdot]$.

If $z = a$ is a regular singular point of $M[\cdot]$, then

$$P_k(z) = \frac{(z-a)^{2-k}a_k(z)}{a_2(z)} \quad (k = 0, 1)$$

is analytic in $N(a)$. Then (3.7) may be rewritten as

$$M[y](z) = \frac{a_2(z)}{(z-a)^2}[(z-a)^2y''(z) + (z-a)P_1(z)y'(z) + P_0(z)y(z)].$$

The expression

$$(z-a)^2y''(z) + (z-a)P_1(z)y'(z) + P_0(z)y(z) \tag{3.8}$$

is called the *canonical form* of $M[y](z)$.

The method of Frobenius shows that (3.8) always has a solution of the form

$$y_1(z) = \sum_{k=0}^{\infty} a_k(z-a)^{k+r_1}$$

for some value $r_1 \in \mathbb{C}$. In fact, $r = r_1$ is a root of the *indicial equation*, which is given by

$$r(r-1) + rP_1(a) + P_0(a) = 0. \tag{3.9}$$

The following is a precise statement of the method of Frobenius for second-order differential equations.

Theorem 3.17. *Consider the second-order differential equation (3.8) and suppose $r = r_1$ and $r = r_2$ are the roots of the indicial equation (3.9) with $\mathrm{Re}(r_1) \geq \mathrm{Re}(r_2)$.*

(i) If $r_1 - r_2 \notin \mathbb{N}_0$, then (3.8) has a basis of solutions $\{y_1, y_2\}$ of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1)(x-a)^{n+r_1} \quad (a_0(r_1) \neq 0)$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n(r_2)(x-a)^{n+r_2} \quad (b_0(r_2) \neq 0).$$

(ii) If $r_1 = r_2$, then (3.8) has a basis of solutions $\{y_1, y_2\}$ of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1)(x-a)^{n+r_1} \quad (a_0(r_1) \neq 0)$$
$$y_2(x) = \log(x-a)y_1(x) + \sum_{n=0}^{\infty} b_n(r_2)(x-a)^{n+r_2} \quad (b_0(r_2) \neq 0).$$

(iii) If $r_1 - r_2 \in \mathbb{N}$, then (3.8) has a basis of solutions $\{y_1, y_2\}$ of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1)(x-a)^{n+r_1} \quad (a_0(r_1) \neq 0)$$
$$y_2(x) = k\log(x-a)y_1(x) + \sum_{n=0}^{\infty} b_n(r_2)(x-a)^{n+r_2} \quad (b_0(r_2) \neq 0)$$

for some $k \in \mathbb{C}$.

Note that if the roots of the indicial equation involve a parameter, all possible values of the parameter must be considered in order to determine the form of the basis of solutions [62]. For further details of the method described above, see the appendix of [50].

CHAPTER FOUR

Properties and Restrictions of the Maximal Domain

In this chapter we study the Legendre differential expression

$$\begin{aligned}\ell[y](x) &:= (1-x^2)y''(x) - 2xy'(x) + ky(x) \\ &= -\big((1-x^2)y'(x)\big)' + ky(x) \quad (x \in (-1,1)),\end{aligned} \tag{4.1}$$

where $k \geq 0$ is a fixed constant. Note that $k = 0$ in the original definition of Legendre's equation as written in (2.5). As we will see below, for the spectral analysis of the Legendre operator, it is essential to have $k > 0$.

The classical Legendre polynomials satisfy the second-order differential equation

$$\ell[y] = (\lambda_n + k)y, \tag{4.2}$$

where $\lambda_n = n(n+1)$, and hence, they are eigenfunctions of $\ell[\cdot]$ as defined in (4.1). An explicit formula for these polynomials is given in (2.2). The Legendre polynomials are orthogonal in the space $L^2(-1,1)$ with explicit orthogonality relationship given in (2.3). We investigate the self-adjoint operator in $L^2(-1,1)$ having the Legendre polynomials as eigenfunctions, which we refer to as the "Legendre polynomials operator." The original study of the Legendre differential equation in the right-definite case is due to Titchmarsh (see [64]) who began his investigation in 1941. Early in the 1950s, Glazman analyzed the right-definite problem using an operator approach.

We also study the operator $\ell[\cdot]$ in the Hilbert space $H_1^2(-1,1)$ defined by

$$H_1^2(-1,1) := \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{loc}(-1,1); f, (1-x^2)^{\frac{1}{2}} f' \in L^2(-1,1)\}$$

with inner product

$$(f,g)_1 := \int_{-1}^{1} \left\{(1-x^2)f'(x)\overline{g}'(x) + kf(x)\overline{g}(x)\right\} dx,$$

which is the left-definite boundary problem. As a consequence, we establish the orthogonality relationship

$$(P_n, P_m)_1 = \big(n(n+1)+k\big)\frac{2}{2n+1}\delta_{nm}. \tag{4.3}$$

In 1980, Everitt [19] published a study of the right- and left-definite problems for the Legendre differential equation which utilizes the theory found in Titchmarsh [64]. The work of Everitt has been extended by Onyango-Otieno [54], who used the Titchmarsh approach to study the right- and left-definite problems for the classical differential equations of Jacobi, Laguerre, and Hermite.

4.1 Properties of the Maximal Domain of $\ell[\cdot]$

The maximal domain $\Delta_{1,\max}$ of $\ell[\cdot]$ in $L^2(-1,1)$ is defined to be

$$\Delta_{1,\max} := \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); f, \ell[f] \in L^2(-1,1)\}. \tag{4.4}$$

Since $C_0^\infty \subset \Delta_{1,\max}$, it follows that $\Delta_{1,\max}$ is dense in $L^2(-1,1)$.

For $f, g \in \Delta_{1,\max}$ and $[a,b] \subset (-1,1)$, we have (as in (3.6)) Green's formula

$$\int_a^b \{\ell[f](x)\overline{g}(x) - f(x)\ell[\overline{g}](x)\}\, dx = [f,g]_1(x)\Big|_a^b$$

where $[f,g]_1(\cdot)$ is the skew-symmetric sesquilinear form defined by

$$[f,g]_1(x) := -(1-x^2)[f'(x)\overline{g}(x) - \overline{g}'(x)f(x)] \tag{4.5}$$

and *Dirichlet's formula*

$$\int_a^b \big\{(1-x^2)f'(x)\overline{g}'(x) + kf(x)\overline{g}(x)\big\}\, dx = (1-x^2)f'(x)\overline{g}(x)\Big|_a^b + \int_a^b \ell[f](x)\overline{g}(x)dx. \tag{4.6}$$

Note that by definition of $\Delta_{1,\max}$, the limits as $x \to \pm 1$ of $[f,g](x)$ exist and are finite for all functions $f, g \in \Delta_{1,\max}$.

To find the deficiency indices n_+ and n_- of $\ell[\cdot]$, we solve

$$\ell[y](x) = -\big((1-x^2)y'(x)\big)' = 0,$$

since by Weyl's Alternative (see 3.6), the occurrence of the limit-point and limit-circle case is independent of k in

$$\ell[y] = ky. \tag{4.7}$$

We note that $y_1(x) = 1$ is a solution. To find y_2, we set

$$y_2'(x) = \frac{1}{1-x^2}$$

and use the method of partial fractions to integrate and find that

$$y_2(x) = \frac{1}{2}\log\frac{1+x}{1-x}.$$

Noting that both y_1 and y_2 are L^2 near $x = \pm 1$, we see that $\ell[y]$ has two L^2 solutions near each endpoint, and since $\ell[y]$ is a second-order differential equation, we calculate the deficiency index to be $(2, 2)$ [68].

On the other hand, since both ± 1 are regular singular endpoints of (2.4), we can also use the method of Frobenius to give the general form of two linearly independent solutions of

$$\ell[y] = 0. \tag{4.8}$$

Although the Frobenius solutions are superfluous in this case since we already solved (4.7), we list them because in later chapters we will be working with powers of the Legendre differential expression which necessarily have higher orders, where it will not be possible to find explicit solutions to the analogous differential equations. The indicial equation of (4.8) is $r^2 = 0$, therefore the Frobenius solutions to (4.8) are of the form

$$\hat{y}_1(x) = \sum_{n=0}^{\infty} a_n(x-1)^n, \quad a_0 \neq 0$$

and

$$\hat{y}_2(x) = \log|1-x| \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} b_n(x-1)^n, \quad b_0 \neq 0, \tag{4.9}$$

where all the series converge for $|x-1|<2$. There exist corresponding solutions at the regular singular endpoint $x=-1$.

We now list the properties of the maximal domain $\Delta_{1,\max}$.

Theorem 4.1. *Let $f,g\in\Delta_{1,\max}$. Then*

(i) $1\in\Delta_{1,\max}$ *and* $\lim_{x\to\pm1}[f,1](x)=\lim_{x\to\pm1}-(1-x^2)f'(x)$.

(ii) If $h_\pm\in C^2(-1,1)$ are defined by

$$h_+(x)=\begin{cases}-\frac{1}{2}\log(1-x^2) & \text{for } x \text{ near } 1\\ 0 & \text{for } x \text{ near } -1\end{cases}$$

and

$$h_-(x)=\begin{cases}0 & \text{for } x \text{ near } 1\\ \frac{1}{2}\log(1-x^2) & \text{for } x \text{ near } -1\end{cases},$$

then $h_\pm\in\Delta_{1,\max}$. Furthermore,

$$\lim_{x\to+1}[f,h_+](x)=\lim_{x\to+1}\left\{\frac{1}{2}(1-x^2)\log(1-x^2)f'(x)+xf(x)\right\}$$

and

$$\lim_{x\to-1}[f,h_-](x)=\lim_{x\to-1}\left\{-\frac{1}{2}(1-x^2)\log(1-x^2)f'(x)-xf(x)\right\}.$$

(iii) $\lim_{x\to\pm1}[f,g](x)=\lim_{x\to\pm1}\left\{[f,1](x)\overline{g}(x)-[\overline{g},1](x)f(x)\right\}$.

Proof. This theorem is an immediate consequence of definition (4.5). □

We remark that the above theorem is as strong as possible in the sense that the solution $\hat{y}_2$, given in (4.9), of (4.8) is in $\Delta_{1,\max}$ but $\lim_{x\to\pm1}\hat{y}_2(x)$ do not exist.

We next consider properties of a restriction $\mathcal{D}_1$ of $\Delta_{1,\max}$ and then show that $\mathcal{D}_1$ is the domain of the self-adjoint operator in $L^2(-1,1)$ having the Legendre polynomials as a complete set of orthogonal eigenfunctions.

Define, for $f \in \Delta_{1,\max}$ and $x \in (-1,1)$,

$$\Lambda[f](x) := \int_0^x \{\ell[f](t) - kf(t)\}\,dt - f'(0) = -(1-x^2)f'(x). \tag{4.10}$$

By definition of $\Delta_{1,\max}$, given in (4.4),

$$\Lambda'[f] \in L^2(-1,1) \tag{4.11}$$

whenever $f \in \Delta_{1,\max}$. Hence, by defining $\Lambda[f](\pm 1) := \lim_{x \to \pm 1} \Lambda[f](x)$, we have then that $\Lambda[f] \in AC[-1,1]$.

Let $e_\pm \in C^2[-1,1]$ have the properties

$$e_+(x) = \begin{cases} 1 & \text{for } x \text{ near } 1 \\ 0 & \text{for } x \text{ near } -1 \end{cases}$$

and

$$e_-(x) = \begin{cases} 0 & \text{for } x \text{ near } 1 \\ 1 & \text{for } x \text{ near } -1 \end{cases}.$$

Note that $e_\pm \in \Delta_{1,\max}$. Define a restriction $\mathcal{D}_1$ of $\Delta_{1,\max}$ by

$$\mathcal{D}_1 := \{f \in \Delta_{1,\max} \mid [f,e_+](1) = [f,e_-](-1) = 0\}. \tag{4.12}$$

An important characterization of $\mathcal{D}_1$ is given in the next lemma.

Lemma 4.2. *Let $f \in \Delta_{1,\max}$. Then $f \in \mathcal{D}_1$ if and only if* $\Lambda[f](\pm 1) = 0$.

Proof. By property (i) of the theorem above and by the definition of $\Lambda[\cdot]$, we have the identities $[f,e_+](1) = \Lambda[f](1)$ and $[f,e_-](-1) = \Lambda[f](-1)$ whenever $f \in \Delta_{1,\max}$. This lemma is now a direct consequence of the definition of $\mathcal{D}_1$ given in (4.12). $\square$

The next theorem lists properties of functions in $\mathcal{D}_1$.

Theorem 4.3. *Let $f, g \in \mathcal{D}_1$. Then*

(i) $f' \in L^2(-1,1)$*;*

(ii) $f \in AC[-1,1]$*;*

(iii) $\lim_{x\to\pm1}(1-x^2)f'(x)g(x) = 0$*; and*

(iv) $\lim_{x\to\pm1}[f,g](x) = 0$.

Proof. Property (i) of this theorem was first proved in 1988 by Everitt and Marić [27]. We give their proof of this property below.

Let $f \in \mathcal{D}_1$. Then $\Lambda[f](1) = 0$ by the lemma above. Furthermore, by (4.11), $\Lambda'[f] \in L^2(-1,1)$. Hence, using the definition of $\Lambda[\cdot]$ in (4.10), we have the representation

$$f'(x) = -\frac{\Lambda[f](x)}{(1-x^2)} = \frac{1}{1-x^2}\int_x^1 \Lambda'[f](t)dt$$

for $x \in [0,1)$. Since

$$\int_0^x \frac{1}{(1-t^2)^2}dt \int_x^1 1^2 dt \leq K$$

for some constant $K > 0$ and for all $x \in [0,1)$, we have $f' \in L^2[0,1)$ by the CE Theorem [12]. (A statement of the CE Theorem can be found in Section 7.3.) Similarly, $f' \in L^2(-1,0]$ and property (i) is proved.

Property (ii) is now an immediate consequence of property (i).

Assume $f, g \in \mathcal{D}_1$. From property (i) of this theorem and Hölder's inequality, we see that $(1-x^2)f'(x)\overline{g}'(x) \in L(-1,1)$. Hence, we can now see from Dirichlet's formula (4.6) that $\lim_{x\to\pm1}(1-x^2)f'(x)\overline{g}(x)$ exist and are finite. If, for instance,

$$(1-x^2)|f'(x)\overline{g}(x)| \geq c$$

for x near 1, then $|f'(x)\overline{g}(x)| \geq \dfrac{c}{1-x^2}$ when x is close to 1. This would indicate that $|f'(x)\overline{g}(x)| \notin L^2(-1,1)$. Therefore, it must be the case that $\lim_{x\to+1} f'(x)\overline{g}(x) = 0$. Similarly, $\lim_{x\to-1}(1-x^2)f'(x)\overline{g}(x) = 0$. Thus, property (iii) is proved. Property (iv) follows from property (iii) and the definition of the sesquilinear form $[\cdot,\cdot]$ given in (4.5). $\square$

We note that $\mathcal{D}_1$ is a proper subspace of $\Delta_{1,\max}$. For example, the solution $\hat{y}_2$ given in (4.8) of (4.9) is an element of $\Delta_{1,\max}$ which is not in $\mathcal{D}_1$. Furthermore, property (i) of this theorem is as strong as possible in the sense that $f(x) = \int_0^x \log(1-t^2)dt$, $x \in (-1,1)$ is in $\mathcal{D}_1$ but $f'' \notin L^2(-1,1)$.

In 2001, Everitt, Littlejohn, and Marić extended Theorem 4.3 to include even more equivalent conditions. We discuss this below in Section 5.3 (see Theorem 5.6).

4.3 The Right-Definite Problem

The space $\mathcal{D}_1$ studied in the above section has the property that $\mathcal{D}_1 \subset L^2(-1,1)$; hence, we can define an operator T_1 in $L^2(-1,1)$ by

$$T_1[f](x) := \ell[f](x), \quad x \in (-1,1)$$
$$\mathcal{D}(T_1) := \mathcal{D}_1$$

Theorem 4.4. *T_1 is self-adjoint in $L^2(-1,1)$.*

Proof. From (3.13), the minimal domain of the differential expression $\ell[\cdot]$ in $L^2(-1,1)$ is

$$\mathcal{D}_{\min}(T_1) = \left\{ y \in \Delta_{1,\max} \middle| [y,z]\Big|_{-1}^{1} = 0 \text{ for all } z \in \Delta_{1,\max} \right\}.$$

Now if the linear combination $\alpha_+ e_+ + \alpha_- e_-$ for $\alpha_\pm \in \mathbb{C}$ of the functions $e_\pm$ defined at the beginning of the previous section is in $\mathcal{D}_{\min}(T_1)$, then

$$0 = [\alpha_+ e_+ + \alpha_- e_-, h_+](1) = \alpha_+$$

and

$$0 = [\alpha_+ e_+ + \alpha_- e_-, h_-](-1) = \alpha_-,$$

where $h_\pm \in \Delta_{1,\max}$ are the functions defined in property (ii) of the Theorem 4.1. Therefore, $e_\pm$ are linearly independent modulo $\mathcal{D}_{\min}(T_1)$. The functions $e_\pm$ also satisfy the Glazman symmetry conditions from the GKN Theorem by property (i) of Theorem 4.1. Hence, by the GKN Theorem, T_1 is self-adjoint in $L^2(-1,1)$.

If $f, g \in \mathcal{D}(T_1) = \mathcal{D}_1$, then by Dirichlet's formula (4.6) and property (iii) of the previous theorem,

$$(T_1[f], g) = \int_{-1}^{1} \ell[f](t)\overline{g}(t)dt = \int_{-1}^{1} \left\{(1-t^2)f'(t)\overline{g}'(t) + kf(t)\overline{g}(t)\right\} dt. \tag{4.13}$$

Hence, if we let $f = g$, equation (4.13) becomes

$$\begin{aligned} (T_1[f], f) &= \int_{-1}^{1} \left\{(1-t^2)|f'(t)|^2 + k|f(t)|^2\right\} dt \\ &= \int_{-1}^{1} (1-t^2)|f'(t)|^2 dt + k(f,f) \\ &\geq k(f,f). \end{aligned} \tag{4.14}$$

The inequality in (4.14) holds because $(1-x^2)|f'(x)|^2 \geq 0$ for $x \in (-1,1)$. Since (4.14) is valid for all $f \in \mathcal{D}(T_1)$, we conclude that $T_1[\cdot]$ is bounded below by kI in $L^2(-1,1)$. □

Because T_1 is bounded below by kI, the number $0 \in \rho(T_1)$, the resolvent set of T_1, as long as $k > 0$. Consequently, the resolvent operator $R_0(T_1)$ exists and is a bounded operator from $L^2(-1,1)$ onto $\mathcal{D}_1$. We will utilize this operator when considering the left-definite problem in the next section.

The next theorem completely characterizes the spectrum of T_1, with proof based on the previous theorem.

Theorem 4.5. *(i) The Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are a complete set of orthogonal eigenfunctions for the operator T_1 in $L^2(-1,1)$.*

(ii) The spectrum of T_1 in $L^2(-1,1)$ is given by

$$\sigma(T_1) = \{n(n+1) + k \mid n \in \mathbb{N}\},$$

i.e., T_1 has a discrete spectrum which is bounded below and all eigenvalues are simple.

Proof. By construction, the Legendre polynomials are eigenfunctions of $T_1[\cdot]$ with corresponding eigenvalues $\{n(n+1)+k \mid n \in \mathbb{N}\}$ as above. Furthermore, by Theorem 2.4, $\{P_n\}_{n=0}^{\infty}$ is a complete orthogonal polynomial sequence in $L^2(-1,1)$.

We proved in Theorem 4.4 that the operator T_1 is self-adjoint, therefore its residual spectrum is empty. Since the set of eigenvalues of T_1, $\{n(n+1)+k \mid n \in \mathbb{N}\}$, has no finite accumulation points, it follows that the continuous spectrum of T_1 is also empty (see [57]). Hence the spectrum of T_1 contains only the eigenvalues of T_1, so the statement in part (ii) of the theorem follows. □

In the next chapter, we discuss the properties of A, the particular self-adjoint operator that has the Legendre polynomials as eigenfunctions.

CHAPTER FIVE

The Legendre Polynomials Self-Adjoint Operator A

In this chapter, we show how the GKN theorem can be applied to the Legendre expression $\ell[\cdot]$ to produce all of the self-adjoint extensions in $L^2(-1,1)$ of the associated minimal operator, including that extension having the Legendre polynomials as a complete set of eigenfunctions. We note that in this classical second-order case, the symmetry factor f for the expression is identical with the orthogonalizing weight function (which in the Legendre case is $w(x) = 1$) for the associated orthogonal polynomials, the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$. Consequently, the GKN theory of self-adjoint extensions of the minimal operator $\mathcal{L}_0$ will yield, as a special case, that self-adjoint extension having the corresponding orthogonal polynomials as eigenfunctions [23].

In Section 4.1, we found the solutions y_1 and y_2 for (4.7) and calculated the deficiency indices to be $(2,2)$. By von Neumann's formula (see (3.8)), we have that $\dim(D_+ \oplus D_-) = 4$. By the first isomorphism theorem from algebra,

$$\mathcal{D}(\mathcal{L})/\mathcal{D}(\mathcal{L}_0) \cong D_+ \oplus D_-. \tag{5.1}$$

We now construct a basis for the space (5.1). Define $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \subseteq \mathcal{D}(\mathcal{L})$ by

$$\varphi_1(x) = \begin{cases} 0 & \text{near } x = -1 \\ 1 & \text{near } x = 1 \end{cases},$$

$$\varphi_2(x) = \begin{cases} 1 & \text{near } x = -1 \\ 0 & \text{near } x = 1 \end{cases},$$

$$\varphi_3(x) = \begin{cases} 0 & \text{near } x = -1 \\ \frac{1}{2}\log\frac{1+x}{1-x} & \text{near } x = 1 \end{cases},$$

$$\varphi_4(x) = \begin{cases} \frac{1}{2}\log\frac{1+x}{1-x} & \text{near } x=-1 \\ 0 & \text{near } x = 1 \end{cases}.$$

To show that the set $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ is linearly independent modulo $\mathcal{D}(\mathcal{L}_0)$, suppose that

$$\sum_{k=1}^{4} \alpha_k \varphi_k \in \mathcal{D}(\mathcal{L}_0).$$

We want to show that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. By definition, we have that $y \in \mathcal{D}(\mathcal{L}_0)$ if and only if $[y,z]\big|_{-1}^{1} = [y,z](1) - [y,z](-1) = 0$ for all $z \in \mathcal{D}(\mathcal{L})$. Hence

$$\sum_{k=1}^{4} \alpha_k [\varphi_k, \varphi_i]\big|_{-1}^{1} = 0 \text{ for } i = 1,2,3,4. \tag{5.2}$$

In the case of the Legendre differential equation, we calculate the sesquilinear form (3.5) to be

$$[y,z](x) = (1-x^2)\big(y(x)\overline{z}'(x) - y'(x)\overline{z}(x)\big).$$

From the definition of the φ_k, we immediately see that

$$[\varphi_1,\varphi_2]\big|_{-1}^{1} = [\varphi_3,\varphi_4]\big|_{-1}^{1} = [\varphi_1,\varphi_4]\big|_{-1}^{1} = [\varphi_2,\varphi_3]\big|_{-1}^{1} = 0.$$

Additionally,

$$[\varphi_1,\varphi_3](1) = \lim_{x\to 1}(1-x^2)(\varphi_1\varphi_3' - \varphi_1'\varphi_3) = \lim_{x \to 1}(1-x^2)\left(\frac{1}{1-x^2} - 0\right) = 1.$$

Since both φ_1 and φ_3 vanish near -1, $[\varphi_1,\varphi_3](-1) = 0$. Hence $[\varphi_1,\varphi_3]\big|_{-1}^{1} = 1$.

Let $i = 1$ in (5.2). Since $[y,z](x) = -\overline{[z,y]}(x)$,

$$0 = \cancelto{0}{\alpha_1[\varphi_1,\varphi_1]\big|_{-1}^{1}} + \cancelto{0}{\alpha_2[\varphi_2,\varphi_1]\big|_{-1}^{1}} + \alpha_3[\varphi_3,\varphi_1]\big|_{-1}^{1} + \cancelto{0}{\alpha_4[\varphi_4,\varphi_1]\big|_{-1}^{1}} = -\alpha_3,$$

hence $\alpha_3 = 0$. Similarly, we determine that $\alpha_1 = \alpha_2 = \alpha_4 = 0$ and conclude that $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ is linearly independent modulo $\mathcal{D}(\mathcal{L}_0)$, i.e., $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ is a basis for $\mathcal{D}(\mathcal{L})/\mathcal{D}(\mathcal{L}_0)$. Referring back to the GKN theorem (3.16), this means that our boundary conditions w_1 and w_2 can be written as

$$w_1 = \sum_{i=1}^{4} c_i\varphi_i \text{ where } (c_1, c_2, c_3, c_4) \in \mathbb{C}^4, \tag{5.3}$$

$$w_2 = \sum_{i=1}^{4} k_i \varphi_i \text{ where } (k_1, k_2, k_3, k_4) \in \mathbb{C}^4. \tag{5.4}$$

These two functions must satisfy the Glazman symmetry conditions (as given in (i) (c) of (3.16))

$$[w_1, w_1]\big|_{-1}^{1} = [w_1, w_2]\big|_{-1}^{1} = [w_2, w_1]\big|_{-1}^{1} = [w_2, w_2]\big|_{-1}^{1} = 0,$$

giving us the system of equations

$$\begin{aligned} c_1\overline{c_3} - \overline{c_1}c_3 - c_2\overline{c_4} + \overline{c_2}c_4 &= 0 \\ c_1\overline{k_3} - \overline{k_1}c_3 - c_2\overline{k_4} + \overline{k_2}c_4 &= 0 \\ c_3\overline{k_1} - \overline{k_3}c_1 - c_4\overline{k_2} + \overline{k_4}c_2 &= 0 \\ k_1\overline{k_3} - \overline{k_1}k_3 - k_2\overline{k_4} + \overline{k_2}k_4 &= 0. \end{aligned} \tag{5.5}$$

Hence, every self-adjoint operator S in $L^2(-1, 1)$ generated by the Legendre operator $\ell[y](x) = -\big((1 - x^2)y'(x)\big)'$ has the form

$$\begin{aligned} Sy &= \ell[y] \\ \mathcal{D}(S) &= \{y \in \mathcal{D}(\mathcal{L}) \mid [y, w_i]\big|_{-1}^{1} = 0, i = 1, 2\}, \end{aligned}$$

where w_1 and w_2 are defined in (5.3) and (5.4) and satisfy the conditions listed in (5.5).

We now focus our attention on determining the self-adjoint extension(s) S of $\ell[\cdot]$ in $L^2(-1, 1)$ that have the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ as eigenfunctions. Notice that if S is such an extension, then we must necessarily have

$$[w_i, P_0]\big|_{-1}^{1} = 0, \quad i = 1, 2,$$

where w_1 and w_2 are given by (5.3) and (5.4). Since $P_0(x) \equiv 1$, these conditions yield $c_3 = c_4$ and $k_3 = k_4$. In addition, since $P_1(x) = x$, we must also have

$$[w_i, x]\big|_{-1}^{1} = 0, \quad i = 1, 2,$$

which yields $c_3 = -c_4$ and $k_3 = -k_4$, forcing $c_3 = c_4 = k_3 = k_4 = 0$. Therefore,

$$w_1 = c_1\varphi_1 + c_2\varphi_2 = \begin{cases} c_2 & \text{near } x = -1 \\ c_1 & \text{near } x = 1 \end{cases},$$

$$w_2 = k_1\varphi_1 + k_2\varphi_2 = \begin{cases} k_2 & \text{near } x = -1 \\ k_1 & \text{near } x = 1 \end{cases}$$

where (c_1, c_2) and (k_1, k_2) are linearly independent vectors in $\mathbb{C}_2$. However, it is easy to see that

$$[w_1, y]\big|_{-1}^{1} = c_1[1, y](1) - c_2[1, y](-1)$$
$$[w_2, y]\big|_{-1}^{1} = k_1[1, y](1) - k_2[1, y](-1)$$

for all $y \in \mathcal{D}(S)$. From these conditions, it is clear that $[y, w_i]\big|_{-1}^{1} = 0$ for $i = 1, 2$ if and only if $[y, 1](1) = [y, 1](-1) = 0$. Consequently, we have the following theorem.

Theorem 5.1. *The self-adjoint operator S in $L^2(-1, 1)$ which extends the minimal operator $\mathcal{L}_0$ generated by the Legendre differential expression $\ell[y]$ and has the Legendre polynomials as eigenfunctions is given by*

$$S[y] = \ell[y]$$
$$\mathcal{D}(S) = \{y \in \mathcal{D}(\mathcal{L}) \mid [y, 1](1) = [y, 1](-1) = 0\}.$$

Furthermore, the spectrum of S is

$$\sigma(S) = \{n(n + 1) + k \mid n \in \mathbb{N}_0\}.$$

Proof. Details about the spectrum can be found in [50]. □

Note that by the definition of the sesquilinear form in (4.5), we have that

$$[1, y](1) = \lim_{x\to 1}(1 - x^2)y'(x) = 0$$
$$[1, y](-1) = \lim_{x\to -1}(1 - x^2)y'(x) = 0,$$

hence the operator S given by the GKN theorem is the desired Legendre polynomials operator $A : \mathcal{D}(A) \subset L^2(-1,1) \to L^2(-1,1)$, which we define by

$$\begin{aligned} Af(x) &:= \ell[f](x) \quad (a.e.\ x \in (-1,1)) \\ f &\in \mathcal{D}(A) \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \mathcal{D}(A) &:= \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); f, \ell[f] \in L^2(-1,1); \\ &\qquad\qquad \lim_{x\to\pm1} (1-x^2)f'(x) = 0\} \\ &= \{f \in \Delta_{1,\max} \mid \lim_{x\to\pm1}(1-x^2)f'(x) = 0\}. \end{aligned} \tag{5.7}$$

We now turn to the study of this operator in the appropriately defined right- and left-definite spaces.

5.1 Hilbert Function Spaces

The two Hilbert function spaces involved in this study of the Legendre differential expression are

(i) the right-definite space $L^2(-1,1)$, and

(ii) the left-definite space $H_1^2(-1,1)$.

The space $L^2(-1,1)$ is the classic integrable-square space of equivalence classes of Lebesgue measurable functions $f : (-1,1) \to \mathbb{C}$ such that $\int_{-1}^{1} |f(x)|^2 dx < \infty$ with inner product

$$(f,g) := \int_{-1}^{1} f(x)\overline{g}(x)dx \quad (f,g \in L^2(-1,1)).$$

The space $H_1^2(-1,1)$ is defined by

$$H_1^2(-1,1) := \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}(-1,1); (1-x^2)^{\frac{1}{2}} f' \in L^2(-1,1)\} \tag{5.8}$$

with inner product

$$(f,g)_1 := \int_{-1}^{1} \{(1-x^2)f'(x)\overline{g}'(x) + kf(x)\overline{g}(x)\}dx, \tag{5.9}$$

where $k > 0$ is the constant given in the definition of $\ell[\cdot]$ in (2.6). We note that the definition of $H_1^2(-1,1)$ may be simplified to read

$$H_1^2(-1,1) = \{f : (-1,1) \to \mathbb{C} \mid f \in AC_{\text{loc}}(-1,1); (1-x^2)^{\frac{1}{2}} f' \in L^2(-1,1)\}. \quad (5.10)$$

Indeed, if $f \in AC_{\text{loc}}(-1,1)$ and $(1-x^2)^{\frac{1}{2}} f' = g \in L^2(-1,1)$, then

$$f(x) = f(x) + \int_0^x \frac{g(t)}{(1-t^2)^{\frac{1}{2}}} dt \quad (x \in [0,1));$$

an application of Hölder's inequality now gives

$$|f(x)|^2 \leq K |\log(1-x)| \text{ as } x \to 1^-$$

for some $K > 0$; hence $f \in L^2(0,1)$. Similarly $f \in L^2(-1,0)$ and so $f \in L^2(-1,1)$.

The space $H_1^2(-1,1)$ is actually a Hilbert space of functions rather than a space of equivalence classes as in $L^2(-1,1)$; the null element of $H_1^2(-1,1)$ is the zero function on $(-1,1)$. The proof that the vector space $H_1^2(-1,1)$ is complete in the norm derived from the inner product (5.9) is given in [50] and [6].

It is well known (see [63]) that the set of Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ is a complete, orthogonal set in $L^2(-1,1)$. In fact, the Legendre polynomials also form a complete orthogonal set in $H_2^1(-1,1)$ (see [50] and [6] for proof). It can be seen that the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are orthogonal in $H_1^2(-1,1)$ through well-known properties of the first derivative of the Gegenbauer polynomials $\{P_n^{(1,1)}\}_{n=0}^{\infty}$; see [63]. Indeed, as we found above, we have the orthogonality relationship (4.3).

See [24] for the proof of the following

Lemma 5.2. *For all $\lambda \in \mathbb{C}$, the following properties hold for the solution base*

$$\{y_{1,+}(\cdot,\lambda), y_{2,+}(\cdot,\lambda)\}$$

of the Legendre differential equation (2.6):

$$\begin{array}{ll} y_{1,+}(\cdot,\lambda) \in L^2(0,1) & y_{2,+}(\cdot,\lambda) \in L^2(0,1) \\ y_{1,+}(\cdot,\lambda) \in H_1^2(0,1) & y_{2,+}(\cdot,\lambda) \notin H_1^2(0,1). \end{array}$$

There are corresponding results for the solution base $\{y_{1,-}(\cdot,\lambda), y_{2,-}(\cdot,\lambda)\}$ *and the spaces* $L^2(-1,0)$ *and* $H_1^2(-1,0)$.

5.2 The Legendre Differential Operator A in $L^2(-1,1)$

This section is based on the general GKN theory of self-adjoint differential operators generated by real, Lagrange symmetric (formally self-adjoint) differential expressions in Hilbert spaces; see [52]. Applications of this theory to the classical second-order differential equations having orthogonal polynomial solutions can be found, for example, in the theses of Loveland [50] and Onyango-Otieno [54].

The maximal operator $T_{1,\max} : \Delta_{1,\max} \subset L^2(-1,1) \to L^2(-1,1)$ generated by the differential expression $\ell[\cdot]$, given in (2.6), is defined by

$$\Delta_{1,\max} := \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); f, \ell[f] \in L^2(-1,1)\} \tag{5.11}$$

and

$$T_{1,\max} f = \ell[f] \quad (f \in \Delta_{1,\max}).$$

The Green's formula (3.6) shows that the limits

$$[f,g](-1) := \lim_{x\to -1^+} [f,g](x) \text{ and } [f,g](1) := \lim_{x\to 1^-} [f,g](x)$$

both exist and are finite for all $f, g \in \Delta_{1,\max}$.

The minimal operator $T_{1,\min} : \mathcal{D}(T_{1,\min}) \subset L^2(-1,1) \to L^2(-1,1)$ is then defined by

$$\mathcal{D}(T_{1,\min}) := \{f \in \Delta_{1,\max} \mid [f,g]_1(x)\big|_\alpha^\beta = 0 \text{ for all } g \in \Delta_{1,\max}\}$$

and

$$T_{1,\min} f = \ell[f] \quad (f \in \mathcal{D}(T_{1,\min})).$$

From [52], it is known that these linear differential operators have the properties:

(i) $T_{1,\min}$ is closed and symmetric in $L^2(-1,1)$;

(ii) $T_{1,\min}^{*} = T_{1,\max}$ and $T_{1,\max}^{*} = T_{1,\min}$ so that $T_{1,\max}$ is closed in $L^2(-1,1)$;

(iii) the deficiency indices (n_+, n_-) of $T_{1,\min}$ are $(2,2)$.

Properties (i) and (ii) follow from the general theory; property (iii) follows from the result that the differential expression $\ell[\cdot]$ is in the limit-circle condition in $L^2(-1,1)$ at both endpoints ± 1 of the interval $(-1,1)$; in turn this result follows from the properties in $L^2(-1,1)$ of the solutions $\{y_{r,+}(\cdot,\lambda), y_{r,-}(\cdot,\lambda)\}_{r=1}^{2}$ of the differential equation $\ell[y] = \lambda y$ on $(-1,1)$; see [24].

Any self-adjoint operator $\tilde{T}$ in $L^2(-1,1)$, generated by $\ell[\cdot]$ is, from the GKN theory, an extension of $T_{1,\min}$ and a restriction of $T_{1,\max}$; that is

$$T_{1,\min} \subset \tilde{T} = \tilde{T}^{*} \subset T_{1,\max}.$$

The domain $\mathcal{D}(\tilde{T})$ of such an operator $\tilde{T}$ is determined from the GKN boundary conditions involving the symplectic form $[\cdot,\cdot](\cdot)$, defined in (4.5), and the maximal domain $\Delta_{1,\max}$; see [52].

Here we are concerned only with the Legendre differential operator, say A, given by

$$Af = \ell[f] \quad (f \in \mathcal{D}(T)), \tag{5.12}$$

where the domain $\mathcal{D}(A)$ is defined by the GKN separated boundary conditions

$$\mathcal{D}(A) := \{f \in \Delta_{1,\max} \mid \lim_{x \to -1^+} [f,1](x) = \lim_{x \to 1^-} [f,1](x) = 0\};$$

equivalently, the boundary conditions take the explicit form found in (5.7). The spectral properties of the self-adjoint operator A are known and are quoted as

Lemma 5.3. *For the operator A, we have the following properties:*

(i) the spectrum $\sigma(A)$ of A is discrete and simple and is given by

$$\sigma(A) = \{\lambda_n \mid n \in \mathbb{N}_0\} \text{ where } \lambda_n = n(n+1);$$

(ii) the operator A is bounded below by I, where I is the identity operator in $L^2(-1,1)$;

(iii) the eigenvectors of A are the eigenfunctions $\{P_n\}_{n=0}^{\infty}$, the Legendre polynomials;

(iv) $\{P_n\}_{n=0}^{\infty}$ is a complete orthogonal set in $L^2(-1,1)$.

Proof. See [1] and [19] for the proofs of (i), (ii), and (iii) and [63] for the proof of (iv). □

We now list some additional properties of the domain $\mathcal{D}(A)$; the proofs can be found in [24].

Theorem 5.4. *Let $\mathcal{D}(A) \subset L^2(-1,1)$ be defined as in (5.7) above. Then for all $f, g \in \mathcal{D}(A)$,*

(i) $(1-x^2)^{\frac{1}{2}} f' \in L^2(-1,1)$ and hence $\mathcal{D}(A) \subset H_1^2(-1,1)$, the vector space of all functions defined by (5.8);

(ii) $\lim_{x\to\pm 1}(1-x^2)f'(x)\overline{g}(x) = 0$;

(iii) $(Af,g) = \int_{-1}^{1} \big((1-x^2)f'(x)\overline{g}'(x) + kf(x)\overline{g}(x)\big)\,dx = (f,g)_1$, where $(\cdot,\cdot)_1$ is the inner product defined in (5.9).

Proof. See [24]. □

Corollary 5.5. *The result in Theorem 6.3 (ii) extends to give*

$$\lim_{x\to\pm 1}(1-x^2)f'(x)\overline{g}(x) = 0$$

for all $f \in \mathcal{D}(A)$ and for all $g \in H_1^2(-1,1)$. Consequently, we obtain the extended Dirichlet identities

$$(Af,g) = (f,g)_1 \quad (f \in \mathcal{D}(A), g \in H_1^2(-1,1))$$

$$(f,Ag) = (f,g)_1 \quad (f \in H_1^2(-1,1), g \in \mathcal{D}(A)).$$

Proof. See [24]. □

5.3 Equivalent Properties of the Domain $\mathcal{D}(A)$

The following theorem, shown by Everitt, Littlejohn, and Marić in [24], lists several equivalent conditions for a function to belong to $\mathcal{D}(A)$. Note the surprising, and remarkable, equivalence of conditions (ii) and (iii), and (ii) and (v), below; parts (ii) and (v) will be of particular use to use in this thesis.

Theorem 5.6. *Let $f \in \Delta_{1,\max}$ where $\Delta_{1,\max}$ is given in (4.4). The following conditions are equivalent:*

(i) $f \in \mathcal{D}(A)$

(ii) $f' \in L^2(-1,1)$;

(iii) $f' \in L^1(-1,1)$;

(iv) f *is bounded on* $(-1,1)$;

(v) $f \in AC[-1,1]$;

(vi) $(1-x^2)^{\frac{1}{2}} f' \in L^2(-1,1)$;

(vii) $(1-x^2)f'' \in L^2(-1,1)$.

Proof. This theorem is proved in its entirety in [24], though some of the above properties were proven in [1], [19], and [35]. Here, we prove that $\mathcal{D}(A) = \mathcal{B}_1$ where

$$\mathcal{B}_1 := \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); (1-x^2)f'' \in L^2(-1,1)\},$$

which is a slightly stronger statement than "(i) if and only if (vii)." This will set the stage for the $n = 2$ case proven in chapter seven.

To show that $\mathcal{D}(A) \subseteq \mathcal{B}_1$, let $f \in \mathcal{D}(A)$. Then $\ell[f] \in L^2(-1,1) \subseteq L^1(-1,1)$. Hence, for $0 < x < 1$,

$$f'(x) = \frac{1}{1-x^2} \int_x^1 \ell[f](t)dt, \tag{5.13}$$

since

$$\lim_{x\to 1^-}(1-x^2)f'(x)=0.$$

Using the CE Theorem (see Section 7.3), we have that $1 \in L^2(x,1)$,

$$\frac{1}{1-t^2} \in L^2(0,x),$$

and

$$\begin{aligned}
K^2(x) &= \left(\int_0^x \frac{1}{(1-t^2)^2}dt\right)\left(\int_x^1 1^2 dt\right) \\
&\leq C\int_0^x \frac{1}{(1-t)^2}dt \cdot (1-x) \\
&= C\left[\frac{-1}{1-t}\bigg|_0^x\right](1-x) \\
&= C\left[\frac{-1}{1-x}+1\right](1-x) = -C + C(1-x) \\
&< \infty,
\end{aligned}$$

Hence $K(x)$ is bounded on $(0,1)$. We therefore obtain from the CE Theorem and (5.13) that $f' \in L^2(0,1)$. Similarly, $f' \in L^2(-1,0)$, so $f' \in L^2(-1,1)$, implying that $2xf' \in L^2(-1,1)$. Hence

$$(1-x^2)f''(x) = -\ell[f](x) + 2xf'(x) \in L^2(-1,1).$$

Therefore $f \in \mathcal{B}_1$, so $\mathcal{D}(A) \subseteq \mathcal{B}_1$.

To show that $\mathcal{B}_1 \subseteq \mathcal{D}(A)$, let $f \in \mathcal{B}_1$. Then $(1-x^2)f'' \in L^2(-1,1)$. Now, for $0 < x < 1$,

$$f'(x) = \int_0^x \frac{1}{1-t^2}\cdot(1-t^2)f''(t)dt + f'(0). \tag{5.14}$$

Since $1 \in L^2(x,1)$,

$$\frac{1}{1-t^2} \in L^2(0,x),$$

and

$$K^2(x) = \int_0^x \frac{1}{(1-t^2)^2}dt \cdot \int_x^1 1^2 dt$$

is bounded on $(0,1)$, we observe from the CE Theorem and (5.14) that

$$f' \in L^2(0,1). \tag{5.15}$$

A similar argument shows $f' \in L^2(-1,0)$ and so it follows that

$$f' \in L^2(-1,1) \tag{5.16}$$

and hence

$$f \in AC[-1,1] \subseteq L^2(-1,1). \tag{5.17}$$

From (5.15), it follows that

$$2xf' \in L^2(0,1). \tag{5.18}$$

Consequently, this implies that

$$(1-x^2)f''(x) - 2xf' \in L^2(0,1);$$

i.e.

$$\ell[f] \in L^2(-1,1). \tag{5.19}$$

We now show that

$$\lim_{x\to\pm 1}(1-x^2)f'(x) = 0.$$

For $0 < x < 1$, we compute that

$$-\int_0^x \ell[f](t)dt = (1-x^2)f'(x) - f'(0). \tag{5.20}$$

Since $\ell[f] \in L^2(-1,1) \subseteq L^1(-1,1)$, we know that

$$\lim_{x\to 1^-} -\int_0^x \ell[f](t)dt$$

exists and is finite. Hence (5.20) implies that

$$\lim_{x\to 1}(1-x^2)f'(x) \tag{5.21}$$

exists and is finite. Suppose

$$\lim_{x\to 1}(1-x^2)f'(x) = c,$$

where $c \neq 0$. Without loss of generality, suppose $c > 0$ and f is real-valued. Then, there exists $x^* \in (0,1)$ such that

$$(1-x^2)f'(x) \geq C \text{ on } [x^*, 1)$$

where $C := \frac{c}{2}$. Then

$$f'(x) \geq \frac{C}{1-x^2} \text{ on } [x^*, 1).$$

However, this contradicts the fact, from (5.16), that $f' \in L^2(-1,1)$. Hence $c = 0$. Similarly, we can show that

$$\lim_{x \to -1} (1-x^2)f'(x) = 0.$$

Hence $f \in \mathcal{D}(A)$, so $\mathcal{B}_1 \subseteq \mathcal{D}(A)$, showing that $\mathcal{D}(A) = \mathcal{B}_1$. □

5.4 Other Self-Adjoint Operators in $L^2(-1,1)$

Recalling our discussion in Section 5.2 of the GKN description of all self-adjoint extensions in $L^2(-1,1)$ of the minimal operator $T_{1,\min}$, each of these self-adjoint extensions has the property that it is a restriction of the maximal operator $T_{1,\max}$; in particular, if $\tilde{T}$ is such an extension, $f \in \Delta_{1,\max}$ for all $f \in \mathcal{D}(\tilde{T})$. We now give the following theorem.

Theorem 5.7. *Suppose that $\tilde{T} \neq A$ is a self-adjoint extension in $L^2(-1,1)$ of the Legendre minimal operator $T_{1,\min}$; here A is the Legendre differential operator defined in (5.6). Then there exists $f \in \mathcal{D}(\tilde{T})$ such that*

$$(1-x^2)f'' \notin L^2(-1,1) \text{ and } f' \notin L^2(-1,1).$$

Proof. See [24]. □

5.5 The Legendre Differential Operator $\hat{S}$ in $H_1^2(-1,1)$

We now define the self-adjoint differential operator $\hat{S}$, the so-called left-definite operator, generated by the differential expression $\ell[\cdot]$ in the left-definite Hilbert-Sobolev function space $H_1^2(-1,1)$.

Definition 5.8. The linear operator $\hat{S} : \mathcal{D}(\hat{S}) \subset H_1^2(-1,1) \to H_1^2(-1,1)$ is given by

$$\mathcal{D}(\hat{S}) := \{f \in \mathcal{D}(A) \mid Af \in H_1^2(-1,1)\}$$
$$\hat{S}f = Af = \ell[f] \quad (f \in \mathcal{D}(\hat{S}));$$

here, A is the Legendre operator defined (5.6). We call $\hat{S}$ the left-definite operator associated with the pair $(L^2(-1,1), A)$.

A few historical remarks on this left-definite operator are in order. As mentioned in the Introduction, the first definition of the left-definite Legendre differential operator in $H_1^2(-1,1)$ is due to Pleijel in [55] and [56]. Pleijel first observed the Legendre differential expression (2.6) is limit-point at $x = \pm 1$ in this setting; see also [8] for a generalization of the limit-point/limit-circle theory in a left-definite context. Pleijel's work on this subject was followed by Everitt [19] in 1980 who used a different approach to study the left-definite operator in $H_1^2(-1,1)$; at the time, however, it was not clear that Everitt's left-definite operator was a differential operator generated by the Legendre differential expression (this has since been proven; see [50].) Our definition above of $\hat{S}$ was first recorded in the 1988 unpublished manuscript [27] of Everitt and Marić. In 1993, Krall and Littlejohn [42] independently used this same definition and gave the first known proof of self-adjointness of $\hat{S}$ in $H_1^2(-1,1)$; this proof differs from the proof in [24]. In 2000, Vonhoff [66] presented yet another new approach to the left-definite theory of the Legendre expression (2.6); we show below in Theorem 5.11 that his left-definite operator $S_{\max}$ is also identical to our $\hat{S}$. Also in 2000, Arvesú, Littlejohn, and Marcellán [6] defined the left-definite Legendre operator in still a different way using the general left-definite theory of self-adjoint, bounded below operators A in a Hilbert space H, developed earlier by Littlejohn and Wellman in [47] and discussed below in chapter six. More specifically (as we will see) in [47], they construct with the aid of the Hilbert space spectral theorem, a continuum of left-definite Hilbert spaces $\{(V_r, (\cdot,\cdot)_r)\}_{r>0}$ and left-definite operators

$\{A_r\}_{r>0}$ associated with (H, A). In particular, in [6], the authors show that their left-definite operator A_1 has domain V_3, which they explicitly construct. We show below that A_1 is identical to the operator $\hat{S}$ defined above.

Theorem 5.9. *Let the operator* $\hat{S}$ *be as given in (5.8).*

(i) $\hat{S}$ *is closed in* $H_1^2(-1,1)$*; in fact,*

(ii) $\hat{S}$ *is symmetric in* $H_1^2(-1,1)$*; in fact,*

(ii) $\hat{S}$ *is self-adjoint in* $H_1^2(-1,1)$*.*

Proof. See [24]. □

Theorem 5.10. *The self-adjoint operator* $\hat{S}$ *in* $H_1^2(-1,1)$ *is unique in the following sense: if* S' *is another self-adjoint operator in* $H_1^2(-1,1)$ *with*

(i) $\mathcal{D}(S') \subset \mathcal{D}(A)$*, where* A *is the Legendre differential operator defined in Section 6.2,*

(ii) $S'f = Af$ *for all* $f \in \mathcal{D}(S')$*,*

then $\hat{S} = S'$*.*

Proof. See [24]. □

In [66], Vonhoff defines the left-definite Legendre operator $S_{\max}$ (his notation) as

$$\begin{aligned} \mathcal{D}(S_{\max}) = \{f : (-1,1) \to \mathbb{C} \mid f, f', \ell[f] \in AC_{\text{loc}}(-1,1); \\ f, (1-x^2)^{\frac{1}{2}} f', (1-x^2)^{\frac{1}{2}} (\ell[f])', \ell[f] \in L^2(-1,1)\} \\ S_{\max} f = Af \quad (f \in \mathcal{D}(S_{\max})), \end{aligned} \tag{5.22}$$

and then proves that $S_{\max}$ is self-adjoint in $H_1^2(-1,1)$. Since $H_1^2(-1,1) \subset L^2(-1,1)$, we see that

$$\mathcal{D}(S_{\max}) = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); f, \ell[f] \in H_1^2(-1,1)\}.$$

Comparing this formulation of $\mathcal{D}(S_{\max})$ with the definition of the domain of the maximal operator $\Delta_{1,\max}$ in (5.11), it is indeed appropriate for Vonhoff to call his operator $S_{\max}$ the maximal operator in $H_1^2(-1,1)$ generated by the Legendre expression $\ell[\cdot]$.

We now prove

Theorem 5.11. *$\hat{S} = S_{\max}$, where $S_{\max}$ is defined in (5.22).*

Proof. See [24]. $\square$

Theorem 5.12. *The self-adjoint operator $\hat{S}$ in $H_1^2(-1,1)$ has a discrete, simple spectrum $\sigma(\hat{S})$ given by*

$$\sigma(\hat{S}) = \{n(n+1) + k \mid n \in \mathbb{N}_0\}$$

with the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ as eigenfunctions; that is, the self-adjoint operators $\hat{S}$ and T have the same spectrum and the same eigenfunctions.

Proof. See [24]; refer also to Theorem 4.12 in light of the equivalence in Theorem 5.11. $\square$

Theorem 5.13. *The deficiency indices of the self-adjoint operator $\hat{S}$ are $n_+ = n_- = 0$.*

Proof. See [24]. $\square$

We conclude this chapter with a different characterization of the domain $\mathcal{D}(\hat{S})$ of the left-definition operator $\hat{S}$ discussed above which is proven in [24] by Everitt, Littlejohn, and Mariċ.

Theorem 5.14. *Let*

$$\mathcal{D} := \{f : (-1,1) \to \mathbb{C} \mid f, f', f'' \in AC_{\text{loc}}(-1,1); (1-x^2)^{\frac{3}{2}} f^{(3)} \in L^2(-1,1)\}.$$

Then $\mathcal{D}(\hat{S}) = \mathcal{D}$ where $\mathcal{D}(\hat{S})$, the domain of the left-definite operator $\hat{S}$, is defined in (5.8). Furthermore, this result is best possible in the sense that there exists $g \in \mathcal{D}(\hat{S})$ such that $(1-x^2)^{\frac{3}{2}} g^{(3)} \notin L^p(-1,1)$ for any $p > 2$ and where g is independent of p.

CHAPTER SIX

Littlejohn-Wellman Left-Definite Theory with Applications to the Legendre Polynomials Operator

6.1 Introduction and Motivation

In this chapter, we describe Littlejohn-Wellman left-definite theory as shown by the authors in [47], which proves that if A is a self-adjoint operator in a Hilbert space $H = (V, (\cdot,\cdot))$ that is bounded below by a positive constant k, i.e., if

$$(Ax, x) \geq k(x, x) \quad (x \in \mathcal{D}(A)),$$

then there is a continuum of unique Hilbert spaces $\{H_r\}_{r>0}$ (called *left-definite Hilbert spaces*) and operators $\{A_r\}_{r>0}$ in H_r (called *left-definite operators*), with each A_r being a unique self-adjoint restriction of A in H_r. Littlejohn and Wellman explicitly determine these Hilbert spaces H_r, together with their inner products $(\cdot,\cdot)_r$, as specific vector subspaces of H. Moreover, the authors are able to explicitly specify the domains of each operator A_r as certain left-definite spaces, and show that the spectrum of each A_r is identical with the spectrum of A. The key result that allows for a determination of these spaces and operators is the classical Hilbert space spectral theorem.

Each of these Hilbert spaces and associated inner products can be viewed as a generalization of a *left-definite* Hilbert space and Dirichlet inner product, respectively, from the theory of self-adjoint differential operators. However, the results summarized here apply to *arbitrary* self-adjoint operators in a Hilbert space that are bounded below. It is the case, however, that the original motivation stems from the study of certain differential equations of the form

$$s[y](t) = \lambda w(t)y(t) \quad (t \in I), \tag{6.1}$$

where $s[\cdot]$ is a Lagrangian symmetric differential expression of order $2n$ given by

$$s[y](t) := \sum_{j=0}^{n}(-1)^j(b_j(t)y^{(j)}(t))^{(j)} \quad (t \in I). \tag{6.2}$$

Here $I = (a, b)$ is an open interval of the real line $\mathbb{R}$, $w(t) > 0$ for $t \in I$, and each coefficient $b_j(t)$ is positive and infinitely differentiable on I. Such equations arise in the functional analytic study of differential equations having orthogonal polynomial solutions [47].

Since one particular setting for the spectral study of $s[\cdot]$ is the Hilbert space $L^2(I; w)$, defined by

$$L^2(I; w) = \left\{ f : I \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_I |f(t)|^2 w(t)dt < \infty \right\}$$

with inner product

$$(f, g) = \int_a^b f(t)\overline{g}(t)w(t)dt,$$

$L^2(I, w)$ is referred to as the classic *right-definite* spectral setting for $w^{-1}s[\cdot]$, due the appearance of the w on the right-hand side of $s[y](t) = \lambda w(t)y(t)$.

For functions $f, g \in \Delta_{\max}$, the maximal domain of $w^{-1}s[\cdot]$ in $L^2(I; w)$, we have Green's formula (as similarly defined in (3.6))

$$\int_a^b s[f](t)\overline{g}(t)dt = \int_a^b f(t)\overline{s[g]}(t)dt + [f, g](t)\Big|_{t=a}^{t=b} \quad (f, g \in \Delta_{\max}), \tag{6.3}$$

where $[\cdot, \cdot]$ is the skew-symmetric sesquilinear form for $s[\cdot]$. The related Dirichlet's formula (similarly defined in (4.6)) is the central motivating factor for the work presented in [47] and is defined here as

$$\int_a^b s[f](t)\overline{g}(t)dt = \sum_{j=0}^{n}\int_a^b b_j(t)f^{(j)}(t)\overline{g}^{(j)}(t)dt + \{f, g\}(t)\Big|_{t=a}^{t=b} \quad (f, g \in \Delta_{\max}), \tag{6.4}$$

where $\{\cdot, \cdot\}$ is another bilinear form, closely related to the $[\cdot, \cdot]$ given in (6.3).

There are two well-known operators generated by $w^{-1}s[\cdot]$ in $L^2(I; w)$, the minimal and maximal operators $T_{\min}$ and $T_{\max}$ defined (see chapter three) by

$$T_{\min}f := w^{-1}s[f] \quad (f \in \Delta_{\min}) \quad \text{and} \quad T_{\max}f := w^{-1}s[f] \quad (f \in \Delta_{\max}).$$

As discussed above, these operators are adjoints of each other, $T_{\min}$ is symmetric in $L^2(I;w)$, and the GKN Theorem of self-adjoint extensions of symmetric differential operators then determines, through appropriate boundary conditions, the various self-adjoint extensions (restrictions) A of $T_{\min}$ ($T_{\max}$).

To continue the motivation for Littlejohn-Wellman left-definite theory, suppose $A : \mathcal{D}(A) \subset L^2(I;w) \to L^2(I;w)$ is a self-adjoint extension of $T_{\min}$ such that

$$(Af, g) = \int_a^b s[f](t)\overline{g}(t)dt = \sum_{j=0}^{n} \int_a^b b_j(t) f^{(j)}(t)\overline{g}^{(j)}(t)dt \quad (f, g \in \mathcal{D}(A)); \tag{6.5}$$

that is to say, for all $f, g \in \mathcal{D}(A)$, the evaluation of the Dirichlet form $\{f, g\}(t)\Big|_{t=a}^{t=b}$ in (6.4) is zero (of course, such an A may or may not exist, in general). Furthermore, suppose that $b_0(t) \geq k > 0$ for all $t \in I$, where k is a positive constant. Then, from (6.5) and our assumed positivity of the coefficients b_j on (a, b), we find that A satisfies

$$(Af, f) \geq k(f, f) \quad (f \in \mathcal{D}(A)). \tag{6.6}$$

Moreover, $s[\cdot]$ generates, through (6.5), a Sobolev space H_1 with inner product (called the *Dirichlet inner product*)

$$(f, g)_1 := \sum_{j=0}^{n} \int_a^b b_j(t) f^{(j)}(t)\overline{g}^{(j)}(t)dt \quad (f, g \in H_1); \tag{6.7}$$

for physical reasons, the norm generated from this inner product is also called the *energy norm* (see [51]). More specifically, H_1 is defined to be the closure of $\mathcal{D}(A)$ in the topology generated by the norm $||\cdot||_1 = (\cdot,\cdot)_1^{\frac{1}{2}}$. Observe that, from (6.5) and (6.7), we have

$$(Af, g) = (f, g)_1 \quad (f, g \in \mathcal{D}(A)). \tag{6.8}$$

Since the inner product $(\cdot,\cdot)_1$ is generated from the *left-hand* side of (6.1), the literature refers to H_1 as the *left-definite* setting for $w^{-1}s[\cdot]$ and calls H_1 the *left-definite* Hilbert space associated with the expression $w^{-1}s[\cdot]$ (actually, H_1 is the *first* left-definite space associated with A, as there is a continuum of left-definite Hilbert spaces associated with such an operator A).

It is possible to extend the identity in (6.8) to obtain

$$(Af, g) = (f, g)_1 \quad (f \in \mathcal{D}(A), g \in H_1). \tag{6.9}$$

From the inequality (6.6), it follows that $0 \in \rho(A)$, the resolvent set of A. Consequently, we see that $R_0(A) = A^{-1}$ is a bounded operator from H_1 onto $\mathcal{D}(A)$. Furthermore, from the inclusion

$$\mathcal{D}(A) \subset H_1 \subset L^2(I; w)$$

and (6.9), it follows that the operator $B : H_1 \to H_1$ defined by

$$Bf = R_0(A)f \quad (f \in \mathcal{D}(B) := H_1),$$

is an invertible, self-adjoint operator. The inverse of B, denoted here by A_1, is also a self-adjoint operator. In the literature, A_1 is called the *left-definite operator* associated with A, though it is more appropriate to name A_1 the *first* left-definite operator associated with A. In fact, Littlejohn and Wellman construct a continuum of left-definite self-adjoint operators $\{A_r\}_{r>0}$ associated with the original operator A, with each A_r being a unique self-adjoint restriction of A in H_r.

To emphasize the starting point of [47], the authors begin with a self-adjoint operator A that is bounded below in H by a positive constant. In the theory of differential operators, A corresponds to a right-definite operator generated from the differential expression $w^{-1}s[\cdot]$ as given in (6.1) and (6.2) in $L^2(I; w)$. However, it is possible that the differential expression $w^{-1}s[\cdot]$ is not right-definite (for example, the function w may be signed on I) and yet $s[\cdot]$ is left-definite (that is, each coefficient $b_j > 0$ on I). This approach is taken by Kong et al. in [36] in their left-definite study of the classic, regular Sturm-Liouville equation on I

$$-(py')' + qy = \lambda wy.$$

The history of left-definite spectral theory as it relates to differential operators can be traced to the work of Weyl [68] who, in his landmark analysis of second-order Sturm-Liouville diferential equations, coined the term *polare-Eigenwertaufgabe* for the study of second-order equations in the left-definite setting. The terminology left-definite (actually, the German *links-definit*) first appeared in the literature in 1965 in a paper by Schäfke and Schneider [59]. In his book [34], Kamke uses the term *F-definit* in his study of the differential equation $Fy = \lambda Gy$ (he also uses *G-definit* for his right-definite study of this equation). In [53], [60], and [61], Niessen and Schneider considered general left-definite singular systems and left-definite s-hermitian problems. In recent years there have been several additional papers dealing with theory and specific examples of left-definite operators, all within the framework of differential operators. Important results related to second-order equations have been obtained by Krall ([38], [40], [43], and [39]), Krall and Littlejohn [37], and Hajmirzaahmad [30]. Left-definite results for higher-order differential equations have been obtained by Loveland [50], Everitt and Littlejohn [22], Everitt *et al.* ([20], [41], [21], [26]), Wellman [67], Vonhoff [66], and Littlejohn and Wellman [47].

6.2 The Definition of a Left-Definite Space and Operator

Let V be a vector space over the complex field $\mathbb{C}$ with inner product $(\cdot,\cdot)$ and norm $||\cdot||$. The resulting inner product space is denoted by $(V,(\cdot,\cdot))$. Suppose V_r (the subscripts will be made clear shortly) is a vector subspace (i.e., a linear manifold) of V and let $(\cdot,\cdot)_r$ and $||\cdot||_r$ denote the respective inner product, possibly different from $(\cdot,\cdot)$, and an associated norm on V_r.

Definition 6.1. Let $H = (V,(\cdot,\cdot))$ be a Hilbert space. Suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number $k > 0$, i.e.,

$$(Ax, x) \geq k(x, x) \quad (x \in \mathcal{D}(A)).$$

Let $H_1 = (V_1,(\cdot,\cdot)_1)$, where V_1 is a subspace of V and $(\cdot,\cdot)_1$ is an inner product on

V_1. Then H_1 is a *left-definite space* associated with the pair (H, A) if each of the following conditions hold:

(i) H_1 is a Hilbert space,

(ii) $\mathcal{D}(A)$ is a subspace of V_1,

(iii) $\mathcal{D}(A)$ is dense in H_1,

(iv) $(x, x)_1 \geq k(x, x) \quad (x \in V_1)$,

(v) $(x, y)_1 = (Ax, y) \quad (x \in \mathcal{D}(A), y \in V_1)$.

Given a self-adjoint operator A that is bounded below by a positive constant, it is not clear that a left-definite space H_1 exists for the pair (H, A). The existence and uniqueness of this Hilbert space however was proven by Littlejohn and Wellman in [47] as will be seen in the theorems below.

If A is a self-adjoint operator in H that is bounded below by a positive number k, then with assistance from the spectral theorem (see below), A^r is a self-adjoint operator bounded below by $k^r I$ for each $r > 0$. Consequently we extend the previous definition to a continuum of left-definite spaces associated with (H, A).

Definition 6.2. Let $H = (V, (\cdot, \cdot))$ be a Hilbert space. Suppose that $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number $k > 0$, i.e.,

$$(Ax, x) \geq k(x, x) \quad (x \in \mathcal{D}(A)).$$

Let $r > 0$. If there exists a subspace V_r of V and an inner product $(\cdot, \cdot)_r$ on V_r such that $H_r = (V_r, (\cdot, \cdot)_r)$ is a left-definite space associated with the pair (H, A^r), we call H^r an *rth left-definite space associated with the pair* (H, A) if each of the following conditions hold:

(i) H_r is a Hilbert space,

(ii) $\mathcal{D}(A^r)$ is a subspace of V_r,

(iii) $\mathcal{D}(A^r)$ is dense in H_r,

(iv) $(x,x)_r \geq k^r(x,x) \quad (x \in V_r)$,

(v) $(x,y)_r = (A^r x, y) \quad (x \in \mathcal{D}(A^r), y \in V_r)$.

For each $r > 0$, H_r exists and is unique as will be seen below. Additionally, though the r^{th} left-definite space H_r appears to depend on H, A, and the positive number k satisfying (iv) above, each of the left-definite spaces H_r is independent of k as will be also be seen below. We now define a left-definite operator associated with A.

Definition 6.3. Let $H = (V, (\cdot,\cdot))$ be a Hilbert space. Suppose $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number $k > 0$. Let $r > 0$ and suppose H_r is the r^{th} left-definite space associated with (H, A). If there exists a self-adjoint operator $A_r : H_r \to H_r$ that is a restriction of A, i.e.,

$$A_r x = Ax, \quad x \in \mathcal{D}(A_r) \subset \mathcal{D}(A),$$

we call such an operator an r^{th} *left-definite operator associated with* (H, A).

6.3 Main Theorems

If A is a self-adjoint operator that is bounded below by a positive number $k > 0$, then for $r > 0$ there exists a unique left-definite operator A_r in H_r associated with (H, A), as will also be considered below.

Theorem 6.4. *Suppose A is a self-adjoint operator in the Hilbert space $H = (V, (\cdot,\cdot))$ that is bounded below by kI, where $k > 0$. Let $r > 0$. Define $H_r = (V_r, (\cdot,\cdot)_r)$ by*

$$V_r = \mathcal{D}(A^{r/2}) \text{ and } (x,y)_r = (A^{r/2}x, A^{r/2}y) \quad (x, y \in V_r).$$

Then H_r is an r^{th} left-definite space associated with the pair (H, A) in the sense of the above definition. Moreover, suppose $H_r = (V_r, (\cdot,\cdot)_r)$ and $H_r' = (V_r', (\cdot,\cdot)_r')$ are r^{th} left-definite spaces associated with the pair (H, A). Then $V_r = V_r'$ and $(x, y)_r = (x, y)_r'$ for all $x, y \in V_r = V_r'$; i.e. $H_r = H_r'$. Consequently, $H_r = (V_r, (\cdot,\cdot)_r)$, as defined above, is the unique r^{th} left-definite Hilbert space associated with (H, A).

Proof. See [47]. □

Theorem 6.5. *Suppose A is a self-adjoint operator in a Hilbert space H that is bounded below by kI for some $k > 0$. For $r > 0$, let $H_r = (V_r, (\cdot,\cdot)_r)$ be the r^{th} left-definite space associated with (H, A). Then there exists a unique left-definite operator A_r in H_r associated with (H, A). More specifically, if there exists a self-adjoint operator $\tilde{A}_r : H_r \to H_r$ such that $\tilde{A}_r x = Ax$ for all $x \in \mathcal{D}(\tilde{A}_r) \subset \mathcal{D}(A)$, then $A_r = \tilde{A}_r$. Furthermore,*

$$\mathcal{D}(A_r) = V_{r+2},$$

and A_r is bounded below by kI in H_r.

Proof. See [47]. □

The following corollary is an immediate consequence of Theorems 6.4 and 6.5. It emphasizes the fact that, set-wise, the domain $\mathcal{D}(A^r)$ of the r^{th} power of A is given by V_{2r} and, in particular, the first and second left-definite spaces associated with A are, respectively, the domain of the positive square root of A and the domain of A. Furthermore, it describes explicitly the domain of the r^{th} left-definite operator in terms of the domain of a certain power of A. Note that the domains of the first and second left-definite operators, A_1 and A_2, are respectively given by $\mathcal{D}(A^{\frac{3}{2}})$ and $\mathcal{D}(A^2)$.

Corollary 6.6. *Suppose A is a self-adjoint operator in the Hilbert space H that is bounded below by kI, where $k > 0$. For each $r > 0$, let $H_r = (V_r, (\cdot,\cdot)_r)$ and*

A_r *denote, respectively, the* r^{th} *left-definite space and the* r^{th} *left-definite operator associated with* (H, A)*. Then*

(i) $\mathcal{D}(A^r) = V_{2r}$*; in particular,* $\mathcal{D}(A^{\frac{1}{2}}) = V_1$ *and* $\mathcal{D}(A) = V_2$*;*

(ii) $\mathcal{D}(A_r) = \mathcal{D}(A^{\frac{r+2}{2}})$*, in particular,* $\mathcal{D}(A_1) = \mathcal{D}(A^{\frac{3}{2}})$ *and* $\mathcal{D}(A_2) = \mathcal{D}(A^2)$*.*

The next theorem describes the triviality of left-definite theory for bounded operators and the richness of the theory for unbounded operators.

Theorem 6.7. *Let* $H = (V, (\cdot,\cdot))$ *be a Hilbert space. Suppose* $A : \mathcal{D}(A) \subset H \to H$ *is a self-adjoint operator that is bounded below by* kI *for some* $k > 0$*. For each* $r > 0$*, let* $H_r = (V_r, (\cdot,\cdot)_r)$ *and* A_r *denote the* r^{th} *left-definite space and* r^{th} *left-definite operator, respectively, associated with* (H, A)*.*

(1) If A *is bounded, then for each* $r > 0$*,* $V = V_r$*,* $(\cdot,\cdot)$ *is equivalent to* $(\cdot,\cdot)_r$*, and* $A = A_r$*.*

(2) If A *is unbounded, then*

(i) V_r *is a proper subspace of* V*;*

(ii) V_s *is a proper subspace of* V_r *whenever* $0 < r < s$*;*

(iii) the inner products $(\cdot,\cdot)$ *and* $(\cdot,\cdot)_s$ *are not equivalent for any* $s > 0$*;*

(iv) the inner products $(\cdot,\cdot)_r$ *and* $(\cdot,\cdot)_s$ *are not equivalent for any* $r, s > 0$*,* $r \neq s$*;*

(v) $\mathcal{D}(A_r)$ *is a proper subspace of* $\mathcal{D}(A)$ *for each* $r > 0$*;*

(vi) $\mathcal{D}(A_s)$ *is a proper subspace of* $\mathcal{D}(A_r)$ *whenever* $0 < r < s$*.*

Since, for each $m > 0$, A^m is a self-adjoint operator that is bounded below in H by $k^m I$, we see from the above theorems that there are a continua of left-definite spaces $\{(H^m)_r\}_{r>0}$ and left-definite operators $\{(A^m)_r\}_{r>0}$ associated with the pair

(H, A^m). Furthermore, since A_m is a self-adjoint operator that is bounded below by kI in H_m, there are continua of left-definite spaces $\{(H_m)_r)\}_{r>0}$ and left-definite operators $\{(A_m)_r\}_{r>0}$ associated with the pair $\{H, A^m\}$. The following questions naturally arise:

(1) What is the relationship, if any, between the three continua of the left-definite spaces $\{H_r\}_{r>0}$, $\{(H^m)_r\}_{r>0}$, and $\{(H_m)_r\}_{r>0}$?

(2) Since for fixed $m > 0$, $(A_r)^m$, which is the m^{th} power of the r^{th} left-definite operator A_r associated with (H, A), is a self-adjoint restriction of A^m. What is the relationship, if any, between the continuum of left-definite operators $\{(A^m)_r\}_{r>0}$ associated with the pair (H, A^m) and the continuum of the left-definite operators $\{(A_r)^m\}_{r>0}$? In particular, is $(A_r)^m$ a left-definite operator associated with (H, A^m), or in other words, is $(A_r)^m \in \{(A_s)^m\}_{s>0}$?

(3) For fixed $m > 0$, what is the relationship, if any, between the continuum of left-definite operators $\{(A_m)_r\}_{r>0}$ associated with the pair (H_m, A_m) and the continuum of the left-definite operators $\{A_r\}_{r>0}$ associated with (H, A)?

Each of these questions is answered in the following theorem. In summary, when exploring the relationship between the various expressions of left-definite operators and spaces, we see that in fact the original spaces $\{H_r\}_{r>0}$ and operators $\{A_r\}_{r>0}$ already encompass all of the left-definite spaces and left-definite operators associated with (H, A^m) and (H_m, A_m).

Theorem 6.8. *Suppose A, H, $\{H_r\}_{r>0}$, and $\{A_r\}_{r>0}$ are as in the above theorems. Fix $m > 0$. For each $r > 0$, let $(H^m)_r = ((V^m)_r, (\cdot,\cdot)_r^m)$ and $(A^m)_r$ respectively denote the r^{th} left-definite space and the r^{th} left-definite operator associated with (H, A). Then*

(i) $(H^m)_r = H_{mr}$.

(ii) $(A_r)^m = (A^m)_{r/m}$ *with* $\mathcal{D}((A_r)^m) = V_{2m+r}$. *Equivalently,* $(A^m)_r = (A_{mr})^m$

with $\mathcal{D}((A^m)_r) = V_{2m+mr}$*, i.e., the* r^{th} *left-definite operator associated with the pair* (H, A^m) *is the* m^{th} *power of the* $(mr)^{th}$ *left-definite operator associated with* (H, A).

Furthermore, let $(H_m)_r = ((V_m)_r, (\cdot,\cdot)_{m,r})$ *and* $(A_m)_r$ *denote the respective* r^{th} *left-definite space and* r^{th} *left definite operator associated with* (H_m, A_m)*. Then*

(iii) $(H_m)_r = H_{m+r}$.

(iv) $(A_m)_r = A_{m+r}$ *with* $\mathcal{D}((A_m)_r) = V_{m+r+2}$*, i.e., the* r^{th} *left-definite operator associated with* (H_m, A_m) *is the* $(m+r)^{th}$ *left-definite operator associated with* (H, A).

Proof. See [47]. □

With regard to the spectra of the left-definite operators $\{A_r\}_{r>0}$, we have the following two theorems.

Theorem 6.9. *For each* $r > 0$*, let* A_r *denote the* r^{th} *left-definite operator associated with the self-adjoint operator* A *that is bounded below by* kI *where* $k > 0$*. Then*

(i) The point spectra of A *and* A_r *coincide; i.e.* $\sigma_p(A_r) = \sigma_p(A)$.

(ii) The continuous spectra of A *and* A_r *coincide; i.e.,* $\sigma_c(A_r) = \sigma_c(A)$.

(iii) The resolvents of A *and* A_r *coincide; i.e.,* $\rho(A) = \rho(A_r)$.

Proof. See [47]. □

Theorem 6.10. *If* $\{\varphi_n\}_{n=0}^{\infty}$ *is a complete orthogonal set of eigenfunctions of* A *in* H*, then for each* $r > 0$*,* $\{\varphi_n\}_{n=0}^{\infty}$ *is a complete set of orthogonal eigenfunctions of the* r^{th} *left-definite operator* A_r *in the* r^{th} *left-definite space* H_r.

Proof. See [47]. □

Finally, we have the following theorem which describes the relationship between eigenfunctions of A and its left-definite operator A_r.

Theorem 6.11. *If $\{\varphi_n\}_{n=0}^{\infty}$ is a complete orthogonal set of eigenfunctions of A in H, then for each $r > 0$, $\{\varphi_n\}_{n=0}^{\infty}$ is a complete orthogonal set of eigenfunctions of the r^{th} left-definite operator A_r in the r^{th} left-definite space H_r.*

Proof. See [47]. □

6.4 *The Spectral Theorem*

If A is a self-adjoint operator in a Hilbert space H with inner product $(\cdot,\cdot)$, it is well known (see [58]) that there exists a unique operator-valued set function $E : \mathscr{B} \to B(H)$, where $\mathscr{B}$ is the σ-algebra of Borel subsets of $\mathbb{R}$ and $B(H)$ is the Banach algebra of bounded linear operators on H, called the *spectral resolution of the identity*, having the following properties:

(i) $E(\emptyset) = 0$ and $E(\mathbb{R}) = I$.

(ii) $E(\Delta)$ is idempotent; that is, $(E(\Delta))^2 = E(\Delta)$ for all $\Delta \in \mathscr{B}$.

(iii) $E(\Delta)$ is self-adjoint in H for all $\Delta \in \mathscr{B}$.

(iv) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2) = E(\Delta_2)E(\Delta_1)$ for all $\Delta_1, \Delta_2 \in \mathscr{B}$.

(v) $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathscr{B}$ with $\Delta_1 \cap \Delta_2 = \emptyset$.

(vi) For each $x, y \in H$, the mapping

$$E_{x,y} : \mathscr{B} \to \mathbb{C} \tag{6.10}$$

defined by $E_{x,y}(\Delta) := (E(\Delta)x, y)$ is a complex, regular Borel measure. Since $E(\Delta)$ is a self-adjoint projection for each $\Delta \in \mathscr{B}$, it follows that $||E(\Delta)|| \leq 1$.

A spectral family (see [45] or [58]) for a self-adjoint operator A is a one-parameter family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of bounded operators in H satisfying:

(i) E_λ is self-adjoint and idempotent for each $\lambda \in \mathbb{R}$.

(ii) For $\lambda < \mu$, $E_\mu - E_\lambda$ is a positive operator.

(iii) $\lim_{\lambda\to\infty} E_\lambda x = x$ for each $\Delta \in H$.

(iv) $\lim_{\lambda\to-\infty} E_\lambda x = 0$ for each $\Delta \in H$.

(v) For each $\lambda \in \mathbb{R}$ and $x \in H$,

$$E_{\lambda+0}x := \lim_{\mu\to\lambda^+} E_\mu x = E_\lambda x. \tag{6.11}$$

A connection between (6.10) and (6.11) lies in the following lemma.

Lemma 6.12. *Suppose E is a spectral resolution of the identity in the sense of (6.10). For $\lambda \in \mathbb{R}$, define $E_\lambda = E(-\infty, \lambda]$. Then $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is a spectral family in the sense of (6.11).*

Proof. See [47]. □

As mentioned earlier, the Hilbert-space spectral theorem plays a key role in proving the existence and uniqueness of the left-definite spaces $\{H_r\}_{r>0}$ and the left-definite operators $\{A_r\}_{r>0}$ associated with the pair (H, A), where A is a self-adjoint operator in H that is bounded below by kI for some $k > 0$. In the development of these left-definite spaces and operators, the spectral resolution of the identity E of A is used rather than the one-parameter family. The properties of the spectrum $\sigma(A_r)$ and the resolvent set $\rho(A_R)$ of each left-definite operator A_r however are more easily seen through the spectral family rather than the spectral resolution of the identity. With regards to the spectra we have the following theorem.

Theorem 6.13. *Suppose $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is a spectral family, satisfying the conditions of (6.11) of a self-adjoint operator A. For $\lambda_0 \in \mathbb{R}$, we have:*

(i) $\lambda_0 \in \sigma_p(A)$ (the point spectrum) if and only if $E_{\lambda_0} \neq E_{\lambda_0-0}$.

(ii) $\lambda_0 \in \sigma_c(A)$ *(the continuous spectrum) if and only if* $E_{\lambda_0} = E_{\lambda_0 - 0}$ *and* $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ *is not constant on any neighborhood of* λ_0 *in* $\mathbb{R}$*.*

(iii) $\lambda_0 \in \rho(A)$ *(the resolvent set) if and only if there exists* $\varepsilon > 0$ *such that* $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ *is constant on* $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$*.*

Proof. See [45] or [58]. □

We are now in the position to state the spectral theorem in a Hilbert space (see [58] for proof).

Theorem 6.14 (The Spectral Theorem). *Let* A *be a (bounded or unbounded) self-adjoint operator in a Hilbert space* $H = (V, (\cdot, \cdot))$*. Let* E *be the spectral resolution of the identity associatd with* A*. Then, for each* $r > 0$*, the self-adjoint operator* A^r *has (densely defined) domain* $\mathcal{D}(A^r)$ *given by*

$$\mathcal{D}(A^r) = \left\{ x \in H \,\middle|\, \int_{\mathbb{R}} \lambda^{2r} dE_{x,x} < 0 \right\},$$

and is characterized by the identities

$$(A^r x, y) = \int_{\mathbb{R}} \lambda^r dE_{x,y} \quad (x \in \mathcal{D}(A^r), y \in H)$$

and

$$||A^r x||^2 = \int_{\mathbb{R}} \lambda^{2r} dE_{x,x} \quad (x \in \mathcal{D}(A^r)).$$

Conversely, suppose $F : \mathscr{B} \to B(H)$ *is a spectral resolution of the identity. Then, there exists a unique self-adjoint operator* $\tilde{A}$ *in* H *with (densely defined) domain*

$$\mathcal{D}(\tilde{A}) = \left\{ x \in H \,\middle|\, \int_{\mathbb{R}} \lambda^2 dF_{x,x} < 0 \right\}$$

that is characterized by

$$(\tilde{A}x, y) = \int_{\mathbb{R}} \lambda dF_{x,y} \quad (x \in \mathcal{D}(\tilde{A}), y \in H)$$

and

$$||\tilde{A}x||^2 = \int_{\mathbb{R}} \lambda^2 dF_{x,x} \quad (x \in \mathcal{D}(\tilde{A})).$$

Moreover, in this theorem we can replace the interval $\mathbb{R}$ of integration in each of the above integrals with the spectrum of the self-adjoint operator. In particular, for a self-adjoint operator A that is bounded below by kI for $k > 0$, we can replace the interval of integration $\mathbb{R}$ with $[k, \infty)$ since in this case the spectrum $\sigma(A) \subset [k, \infty)$ (see [58]).

6.5 Left-Definite Theory and the Legendre Polynomials Operator

We see from Littlejohn-Wellman left-definite theory as described above that the domain of the (first) left-definite Legendre operator A_1 is given by

$$\mathcal{D}(A_1) = V_3 = \{f : (-1,1) \to \mathbb{C} \mid f, f', f'' \in AC_{\text{loc}}(-1,1); \\ (1-x^2)^{3/2} f^{(3)} \in L^2(-1,1)\}.$$

More importantly, we have that

$$\mathcal{D}(A) = V_2 = \{f : (-1,1) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}(-1,1); (1-x^2) f'' \in L^2(-1,1)\},$$

which gives a new characterization of the domain of the classical self-adjoint operator A which is equivalent to the GKN domain that includes boundary conditions. We are similarly given that

$$\mathcal{D}(A^2) = \{f : (-1,1) \to \mathbb{C} \mid f, f', f'', f''' \in AC_{\text{loc}}(-1,1); \\ (1-x^2)^2 f^{(4)} \in L^2(-1,1)\},$$

the left-definite characterization for the domain of the square of the Legendre polynomials operator. Using left-definite theory we can generalize that for each $n \in \mathbb{N}$,

$$\mathcal{D}(A^n) = \{f : (-1,1) \to \mathbb{C} \mid f, f', ..., f^{(2n-1)} \in AC_{\text{loc}}(-1,1); \\ (1-x^2)^n f^{(2n)} \in L^2(-1,1)\}.$$

6.6 The Legendre-Stirling Numbers

The spaces $\{H_r\}_{r>0}$ and inner products $(\cdot, \cdot)_r$ are determined from the powers A^r of A, hence we can only determine these spaces and operators when $r \in \mathbb{N}$.

In [25], the authors showed that, for $n \in \mathbb{N}$, the n^{th} composite power of the Legendre differential expression $\ell[y]$ in Lagrangian-symmetric form is explicitly given by

$$\ell^n[y](x) = \sum_{j=1}^{n} (-1)^j \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_1 \left((1-x^2)^j y^{(j)}(x)\right)^{(j)} \tag{6.12}$$

where the numbers

$$\left\{ \begin{matrix} n \\ j \end{matrix} \right\}_1 := \sum_{r=0}^{j} (-1)^{r+j} \frac{(2r+1)(r^2+r)^n}{(r+j+1)!(j-r)!} > 0 \tag{6.13}$$

are the so-called Legendre-Stirling numbers, a subject of current study in combinatorics (for example, see [2], [3], [4], [18], and [28].)

In general, for $n \in \mathbb{N}$, the n^{th} left-definite inner product is

$$(f,g)_n := \sum_{j=1}^{n} \left\{ \begin{matrix} n \\ j \end{matrix} \right\}_1 \int_{-1}^{1} (1-x^2)^j f^{(j)}(x) \overline{g}^{(j)}(x) dx.$$

The natural question to ask given the combinatorial appearance of their definition is, what do the Legendre-Stirling numbers count? First, take two copies of each positive integer between 1 and n, i.e.,

$$1_1, 1_2, 2_1, 2_2, ..., n_1, n_2.$$

Then, for positive integers $p, q \leq n$ and $i, j \in \{1, 2\}$, assume that $p_i > q_j$ if $p > q$. Next, we define two rules on how to fill $j+1$ "boxes" with the numbers $\{1_1, 1_2, 2_1, 2_2, ..., n_1, n_2\}$:

(i) The "zero box" is the only box that may be empty and it may not contain both copies of any of the numbers.

(ii) The other j boxes are indistinguishable, each is non-empty, and for each such box, the smallest element in the box must contain both copies of this smallest number, but no other elements can have both copies in that box.

Given the above two rules, we have the following

Theorem 6.15 (Andrews and Littlejohn, 2009). *For $n, j \in \mathbb{N}_0$ and $j \leq n$, the Legendre-Stirling number $\left\{ {n \atop j} \right\}_1$ is the number of different distributions according to the above two rules.*

Table 6.1: Some Legendre-Stirling numbers

j/n	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$j=1$	1	2	4	8	16	32	64
$j=2$	–	1	8	52	320	1936	11648
$j=3$	–	–	1	20	292	3824	47824
$j=4$	–	–	–	1	40	1092	25664
$j=5$	–	–	–	–	1	70	3192
$j=6$	–	–	–	–	–	1	112
$j=7$	–	–	–	–	–	–	1

For example, from the table above we would write

$$\begin{aligned}\ell^4[y](x) = &- 8[(1-x^2)y'(x)]' + 52[(1-x^2)^2 y''(x)]'' \\ &- 20[(1-x^2)^3 y'''(x)]''' + [(1-x^2)^4 y^{(4)}(x)]^{(4)}.\end{aligned}$$

CHAPTER SEVEN

The Square of the Legendre Polynomials Operator

The square $A^2 : \mathcal{D}(A^2) \subset L^2(-1,1) \to L^2(-1,1)$ of the Legendre polynomials operator A in $L^2(-1,1)$ is *algebraically* defined by

$$A^2 f := \ell^2[f] \tag{7.1}$$

for $f \in \mathcal{D}(A^2)$, where $\mathcal{D}(A^2)$ is defined in (1.4), and where

$$\begin{aligned} \ell^2[y](x) &:= \left((1-x^2)^2 y''(x)\right)'' - 2\left((1-x^2)y'(x)\right)' \\ &= (1-x^2)^2 y^{(4)}(x) - 8x(1-x^2)y'''(x) + (14x^2-6)y''(x) + 4xy'(x). \end{aligned} \tag{7.2}$$

By standard results from functional analysis (specifically, the Hilbert space spectral theorem), it can be shown that A^2 is a self-adjoint operator in $L^2(-1,1)$, the spectrum of A^2 is given by $\sigma(A^2) = \{n^2(n+1)^2 \mid n \in \mathbb{N}_0\}$, and the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are eigenfunctions of A^2.

It is natural to ask whether we can explicitly describe the functions in the domain $\mathcal{D}(A^2)$ similarly to how we characterize elements in $\mathcal{D}(A)$ as in (5.7) (or by Theorem 5.6). In the next section, we identify A^2 with a self-adjoint operator S obtained through an application of the GKN theory.

7.1 A GKN Self-Adjoint Operator Generated by the Square of the Legendre Differential Expression

The maximal domain $\Delta_{2,\max}$ in $L^2(-1,1)$ associated with the square of the Legendre expression $\ell^2[\cdot]$, defined in (7.2), is given by

$$\begin{aligned} \Delta_{2,\max} := \{f : (-1,1) \to \mathbb{C} \mid\ & f, f', f'', f''' \in AC_{\text{loc}}(-1,1); \\ & f, \ell^2[f] \in L^2(-1,1)\}. \end{aligned} \tag{7.3}$$

The sesquilinear form $[\cdot,\cdot]_2(\cdot) : \Delta_{2,\max} \times \Delta_{2,\max} \times (-1,1)$, associated with $\ell^2[\cdot]$, is defined by

$$\begin{aligned}[f,g]_2(x) :&= \left((1-x^2)^2 f''(x)\right)' \overline{g}(x) - \left((1-x^2)\overline{g}''(x)\right)' f(x) \\ &\quad - (1-x^2)^2 f''(x)\overline{g}'(x) + (1-x^2)^2 f'(x)\overline{g}''(x) \\ &\quad - 2(1-x^2) f'(x)\overline{g}(x) + 2(1-x^2) f(x)\overline{g}'(x) \quad (x \in (-1,1)).\end{aligned} \tag{7.4}$$

For $f, g \in \Delta_{2,\max}$ and $[\alpha,\beta] \subset (-1,1)$, Green's formula for $\ell^2[\cdot]$ is given by

$$\int_\alpha^\beta \ell^2[f](x)\overline{g}(x)dx - \int_\alpha^\beta f(x)\overline{\ell^2[g]}(x) = [f,g]_2(x)\big|_\alpha^\beta. \tag{7.5}$$

By definition of $\Delta_{2,\max}$ and Hölder's inequality, we see that the limits

$$[f,g]_2(\pm 1) := \lim_{x\to\pm 1} [f,g]_2(x)$$

exist and are finite for all $f, g \in \Delta_{2,\max}$. Clearly

$$P_n \in \Delta_{2,\max} \quad (n \in \mathbb{N}_0)$$

where P_n is the n^{th} degree Legendre polynomial. In particular, the functions 1 and x belong to $\Delta_{2,\max}$.

The endpoints $x = \pm 1$ are both regular singular points, in the sense of Frobenius, of $\ell^2[\cdot]$ (see Section 3.6 and [33] for explanation of the method of Frobenius.). The Frobenius indicial equation, at either endpoint, is given by

$$r^2(r-1)^2 = 0.$$

It follows, from the general Weyl theory, that each endpoint is in the limit-4 case so the deficiency index of the minimal operator $T_{2,\min}$, generated by $\ell^2[\cdot]$, in $L^2(-1,1)$ is $(4,4)$. Consequently, each self-adjoint operator, generated by $\ell^2[\cdot]$, in $L^2(-1,1)$ is determined by restricting $\Delta_{2,\max}$ to four boundary conditions of the form

$$[f, f_j]_2(1) - [f, f_j]_2(-1) = 0, \tag{7.6}$$

where $[\cdot,\cdot]_2$ is given in (7.4) and where $\{f_1, f_2, f_3, f_4\} \subset \Delta_{2,\max}$ is linearly independent modulo the minimal domain $\Delta_{2,\min}$ defined by

$$\Delta_{2,\min} := \{f \in \Delta_{2,\max} \mid [f,g]_2\big|_{-1}^{1} = 0 \text{ for all } g \in \Delta_{2,\max}\},$$

We now identify a particular self-adjoint operator restriction S of $T_{\max}$, generated by $\ell^2[\cdot]$, having the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ as a complete set of eigenfunctions.

For $j = 1, 2, 3, 4$, define $f_j \in \Delta_{2\,\max} \cap C^4[-1,1]$ by

$$\begin{aligned} f_1(x) &= \begin{cases} 0 & \text{near } x=-1 \\ 1 & \text{near } x=1 \end{cases}, \quad & f_2(x) &= \begin{cases} 1 & \text{near } x=-1 \\ 0 & \text{near } x=1 \end{cases}, \\ f_3(x) &= \begin{cases} 0 & \text{near } x=-1 \\ x & \text{near } x=1 \end{cases}, \quad & f_4(x) &= \begin{cases} x & \text{near } x=-1 \\ 0 & \text{near } x=1 \end{cases}. \end{aligned} \tag{7.7}$$

Proposition 7.1. *The functions $\{f_1, f_2, f_3, f_4\}$ defined in (7.7) are linearly independent modulo $\Delta_{2,\min}$.*

Proof. Calculations show that the functions $\log(1 \pm x)$ and $(1 \pm x)\log(1 \pm x)$ belong to $\Delta_{2,\max}$, so by defining the functions $g_j \in \Delta_{2,\max} \cap C^4(-1,1)$ for $j = 1, 2, 3, 4$ as

$$g_1(x) = \begin{cases} 0 & \text{near } x=-1 \\ \log(1-x) & \text{near } x=1 \end{cases}, \; g_3(x) = \begin{cases} 0 & \text{near } x=-1 \\ (1-x)\log(1-x) & \text{near } x=1 \end{cases},$$

$$g_2(x) = \begin{cases} \log(1+x) & \text{near } x=-1 \\ 0 & \text{near } x=1 \end{cases}, \; g_4(x) = \begin{cases} (1+x)\log(1+x) & \text{near } x=-1 \\ 0 & \text{near } x=1 \end{cases},$$

we see that

Table 7.1: Calculation of some sesquilinear forms

$[f_1, g_j]_2(\pm 1)$	$[f_2, g_j]_2(\pm 1)$	$[f_3, g_j]_2(\pm 1)$	$[f_4, g_j]_2(\pm 1)$
$[f_1, g_1]_2(1) = 0$	$[f_2, g_1]_2(1) = 0$	$[f_3, g_1]_2(1) = -4$	$[f_4, g_1]_2(1) = 0$
$[f_1, g_2]_2(1) = 0$	$[f_2, g_2]_2(1) = 0$	$[f_3, g_2]_2(1) = 0$	$[f_4, g_2]_2(1) = 0$
$[f_1, g_3]_2(1) = 4$	$[f_2, g_3]_2(1) = 0$	$[f_3, g_3]_2(1) = 4$	$[f_4, g_3]_2(1) = 0$
$[f_1, g_4]_2(1) = 0$	$[f_2, g_4]_2(1) = 0$	$[f_3, g_4]_2(1) = 0$	$[f_4, g_4]_2(1) = 0$
$[f_1, g_1]_2(-1) = 0$	$[f_2, g_1]_2(-1) = 0$	$[f_3, g_1]_2(-1) = 0$	$[f_4, g_1]_2(-1) = 0$
$[f_1, g_2]_2(-1) = 0$	$[f_2, g_2]_2(-1) = 0$	$[f_3, g_2]_2(-1) = 0$	$[f_4, g_2]_2(-1) = -4$
$[f_1, g_3]_2(-1) = 0$	$[f_2, g_3]_2(-1) = 0$	$[f_3, g_3]_2(-1) = 0$	$[f_4, g_3]_2(-1) = 0$
$[f_1, g_4]_2(-1) = 0$	$[f_2, g_4]_2(-1) = -4$	$[f_3, g_4]_2(-1) = 0$	$[f_4, g_4]_2(-1) = 4$

Suppose that

$$\sum_{j=1}^{4} \alpha_j f_j \in \Delta_{2,\min};$$

then, by definition of $\Delta_{2,\min}$, we see that

$$\left[\sum_{j=1}^{4} \alpha_j f_j, g\right]_2 \Bigg|_{-1}^{1} = 0 \quad (g \in \Delta_{2,\max}),$$

where $[\cdot,\cdot]_2$ is the sesquilinear form defined in (7.4). A calculation shows that

$$0 = \left[\sum_{j=1}^{4} \alpha_j f_j, g_1\right]_2 \Bigg|_{-1}^{1} = -4\alpha_3,$$

so $\alpha_3 = 0$. Similarly, we find that

$$0 = \left[\sum_{j=1}^{4} \alpha_j f_j, g_2\right]_2 \Bigg|_{-1}^{1} = 4\alpha_4,$$

so $\alpha_4 = 0$;

$$0 = \left[\sum_{j=1}^{4} \alpha_j f_j, g_3\right]_2 \Bigg|_{-1}^{1} = 4\alpha_1 + 4\alpha_3 = 4\alpha_1,$$

so $\alpha_1 = 0$; and

$$0 = \left[\sum_{j=1}^{4} \alpha_j f_j, g_4\right]_2 \Bigg|_{-1}^{1} = 4\alpha_2 - 4\alpha_4 = 4\alpha_2,$$

so $\alpha_2 = 0$. Consequently, we see that $\{f_1, f_2, f_3, f_4\}$ is linearly independent modulo $\Delta_{2,\min}$, completing the proof. $\square$

We note that the four functions $\{g_1, g_2, g_3, g_4\}$ defined above are determined by a basis of solutions of $\ell^2[y] = 0$ at both $x = \pm 1$ found by the standard method of Frobenius (see Section 3.6). Brown, Littlejohn and McCormack developed a program in Mathematica called `Frobenius` which computes the coefficients of the Frobenius solutions for arbitrary ordinary differential equations at a regular singular endpoint. Using this program, a basis of solutions at the regular singular endpoint $x = 1$, valid for $-1 < x < 1$, is $\{y_1, y_2, y_3, y_4\}$, is given by

$$
\begin{aligned}
y_1(x) &= (x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 - \frac{1}{32}(x-1)^4 + \frac{1}{80}(x-1)^5 - \frac{1}{192}(x-1)^6 \\
&\quad + \frac{1}{448}(x-1)^7 - \frac{1}{1024}(x-1)^8 + \cdots , \\
y_2(x) &= 3(x-1) - (x-1)^2 + \frac{13}{36}(x-1)^3 - \frac{9}{64}(x-1)^4 + \frac{23}{400}(x-1)^5 - \frac{7}{288}(x-1)^6 \\
&\quad + \frac{33}{3136}(x-1)^7 - \frac{19}{4096}(x-1)^8 + \cdots \\
&\quad + \log|x-1| \left((x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 - \frac{1}{32}(x-1)^4 + \frac{1}{80}(x-1)^5 \right. \\
&\quad \left. - \frac{1}{192}(x-1)^6 + \frac{1}{448}(x-1)^7 - \frac{1}{1024}(x-1)^8 + \cdots \right), \\
y_3(x) &= \frac{1}{2} + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{24}(x-1)^3 - \frac{1}{64}(x-1)^4 + \frac{1}{160}(x-1)^5 \\
&\quad - \frac{1}{384}(x-1)^6 + \frac{1}{896}(x-1)^7 + \cdots , \text{ and} \\
y_4(x) &= 3 - 3(x-1) + \frac{3}{8}(x-1)^2 - \frac{1}{12}(x-1)^3 + \frac{3}{128}(x-1)^4 - \frac{3}{400}(x-1)^5 \\
&\quad + \frac{1}{384}(x-1)^6 - \frac{3}{3136}(x-1)^7 + \cdots \\
&\quad + 3\log|x-1| \left(\frac{1}{2} + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{24}(x-1)^3 - \frac{1}{64}(x-1)^4 \right. \\
&\quad \left. + \frac{1}{160}(x-1)^5 - \frac{1}{384}(x-1)^6 + \frac{1}{896}(x-1)^7 - \cdots \right).
\end{aligned}
$$

Similarly, a basis of solutions at the endpoint $x = -1$, valid for $-1 < x < 1$ is $\{z_1, z_2, z_3, z_4\}$, is given by

$$
\begin{aligned}
z_1(x) &= (x+1) + \frac{1}{4}(x+1)^2 + \frac{1}{12}(x+1)^3 + \frac{1}{32}(x+1)^4 + \frac{1}{80}(x+1)^5 + \frac{1}{192}(x+1)^6 \\
&\quad + \frac{1}{448}(x+1)^7 + \frac{1}{1024}(x+1)^8 + \cdots, \\
z_2(x) &= 3(x+1) + (x+1)^2 + \frac{13}{36}(x+1)^3 + \frac{9}{64}(x+1)^4 + \frac{23}{400}(x+1)^5 + \frac{7}{288}(x+1)^6 \\
&\quad + \frac{33}{3136}(x+1)^7 + \frac{19}{4096}(x+1)^8 + \cdots \\
&\quad + \log|x+1| \left((x+1) + \frac{1}{4}(x+1)^2 + \frac{1}{12}(x+1)^3 + \frac{1}{32}(x+1)^4 + \frac{1}{80}(x+1)^5 \right. \\
&\quad \left. + \frac{1}{192}(x+1)^6 + \frac{1}{448}(x+1)^7 + \frac{1}{1024}(x+1)^8 + \cdots \right), \\
z_3(x) &= \frac{1}{2} - \frac{1}{2}(x+1) - \frac{1}{8}(x+1)^2 - \frac{1}{24}(x+1)^3 - \frac{1}{64}(x+1)^4 - \frac{1}{160}(x+1)^5 \\
&\quad - \frac{1}{384}(x+1)^6 - \frac{1}{896}(x+1)^7 - \cdots, \text{ and} \\
z_4(x) &= 3 + 3(x+1) + \frac{3}{8}(x+1)^2 + \frac{1}{12}(x+1)^3 + \frac{3}{128}(x+1)^4 + \frac{3}{400}(x+1)^5 \\
&\quad + \frac{1}{384}(x+1)^6 + \frac{3}{3136}(x+1)^7 + \cdots \\
&\quad + 3\log|x+1| \left(\frac{1}{2} - \frac{1}{2}(x+1) - \frac{1}{8}(x+1)^2 - \frac{1}{24}(x+1)^3 - \frac{1}{64}(x+1)^4 \right. \\
&\quad \left. - \frac{1}{160}(x+1)^5 - \frac{1}{384}(x+1)^6 - \frac{1}{896}(x+1)^7 - \cdots \right).
\end{aligned}
$$

Recalling from Green's formula that

$$[f, g]_2(\pm 1) := \lim_{x \to \pm 1} [f, g]_2(x),$$

it is clear that the boundary conditions

$$[f, f_1]_2(1) = [f, f_3]_2(1) = [f, f_2]_2(-1) = [f, f_4]_2(-1) = 0$$

are equivalent to the boundary conditions

$$[f, 1]_2(\pm 1) = [f, x]_2(\pm 1) = 0.$$

We are now in a position to the define the operator S which we show later (see Section 7.4) to be equal to the operator A^2, given in (7.1) and (1.4). Indeed, let

$S : \mathcal{D}(S) \subset L^2(-1,1) \to L^2(-1,1)$ be defined by

$$\begin{aligned} Sf &= \ell^2[f] := \ell[\ell[f]] \\ f &\in \mathcal{D}(S), \end{aligned} \tag{7.8}$$

where the domain $\mathcal{D}(S)$ of S is defined in (1.5) or more succinctly as

$$\mathcal{D}(S) := \{f \in \Delta_{2,\max} \mid [f,1]_2(\pm 1) = [f,x]_2(\pm 1) = 0\}.$$

By the GKN theorem, S is self-adjoint in $L^2(-1,1)$. Moreover, notice that for $f \in \Delta_{2,\max}$,

$$[f,1]_2(x) = \big((1-x^2)^2 f''(x)\big)' - 2(1-x^2)f'(x) \tag{7.9}$$

and

$$\begin{aligned} [f,x]_2(x) &= \big((1-x^2)^2 f''(x)\big)' x - (1-x^2)^2 f''(x) \\ &\quad - 2x(1-x^2)f'(x) + 2(1-x^2)f(x) \\ &= x[f,1]_2(x) - (1-x^2)^2 f''(x) + 2(1-x^2)f(x). \end{aligned} \tag{7.10}$$

From (7.9) and (7.10), it is easy to see that the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ satisfy

$$[P_n, 1]_2(\pm 1) = [P_n, x]_2(\pm 1) = 0,$$

that is to say, the Legendre polynomials $\{P_n\}_{n=0}^{\infty} \subset \mathcal{D}(S)$. Moreover,

$$\ell^2[P_n] = \ell[\ell[P_n]] = n(n+1)\ell[P_n] = n^2(n+1)^2 P_n \quad (n \in \mathbb{N}_0).$$

From [52] and standard results in spectral theory, the following result holds.

Theorem 7.2. *The operator S, defined in (7.8) and (1.5), is an unbounded self-adjoint operator in $L^2(-1,1)$. The Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ form a complete set of (orthogonal) eigenfunctions of S in $L^2(-1,1)$. The spectrum $\sigma(S)$ of S is discrete and given explicitly by*

$$\sigma(S) = \{n^2(n+1)^2 \mid n \in \mathbb{N}_0\}.$$

7.2 Statements of the Main Theorems

We prove four main theorems in this chapter.

Theorem 7.3. *Let $\mathcal{D}(A^2)$ and $\mathcal{D}(S)$ be given, respectively, as in (1.4) and (1.5). Then*

$$\mathcal{D}(A^2) = \mathcal{D}(S).$$

Proof. See Section 7.4. □

Theorem 7.4. *Let B and $\mathcal{D}(S)$ be given, respectively, as in (1.2) and (1.5). Then*

$$B = \mathcal{D}(S).$$

Proof. See Section 7.5. □

Theorem 7.5. *Let $\mathcal{D}(S)$ and D be given, respectively, as in (1.5) and (1.6). Then*

$$D = \mathcal{D}(S).$$

Proof. See Section 7.6. □

From these three theorems, we obtain our main result, namely

Theorem 7.6. *Let $\Delta_{2,\max}$, given in (7.3), be the maximal domain of the formal square $\ell^2[\cdot]$ of the Legendre differential expression defined by*

$$\ell^2[y](x) = \left((1-x^2)^2 y''(x)\right)'' - 2\left((1-x^2)y'(x)\right)' \quad (x \in (-1,1)) \tag{7.11}$$

and let $[\cdot,\cdot]_2$ be the associated sesquilinear form for $\ell^2[\cdot]$ given in (7.4). Define the operator $T : \mathcal{D}(T) \subset L^2(-1,1) \to L^2(-1,1)$ by

$$(Tf)(x) = \ell^2[f](x) \quad (a.e.\ x \in (-1,1))$$

$$f \in \mathcal{D}(T) := \mathcal{D}(A^2),$$

where $\mathcal{D}(A^2)$, algebraically defined in (1.4), is the domain of the square of the Legendre polynomials operator A defined in (5.7). That is to say, T is the square of the classical Legendre polynomials operator A, given in (5.6) and (5.7). Then the following statements are equivalent:

(i) $f \in \mathcal{D}(T)$;

(ii) $f, f', f'', f''' \in AC_{\mathrm{loc}}(-1,1)$ *and* $(1-x^2)^2 f^{(4)} \in L^2(-1,1)$;

(iii) $f \in \Delta_{2,\max}$ *and* $[f,1]_2(\pm 1) = [f,x]_2(\pm 1) = 0$;

(iv) $f \in \Delta_{2,\max}$ *and* $\lim_{x\to\pm 1}(1-x^2)f'(x) = \lim_{x\to\pm 1}\left((1-x^2)^2 f''(x)\right)' = 0$.

Moreover, T is a self-adjoint operator in $L^2(-1,1)$ having the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ as a complete set of eigenfunctions in $L^2(-1,1)$ and having discrete spectrum $\sigma(T)$ explicitly given by

$$\sigma(T) = \{n^2(n+1)^2 \mid n \in \mathbb{N}_0\}.$$

7.3 A Key Integral Inequality

A key result in our analysis below is the following operator inequality established by Chisholm and Everitt in [12]. Both the theorem and its corollary are referred to as the "CE Theorem."

Theorem 7.7. *Let (a,b) be an open interval of the real line (bounded or unbounded) and let w be a positive Lebesgue measurable function that is positive a.e. $x \in (a,b)$. Suppose $\varphi, \psi : (a,b) \to \mathbb{C}$ satisfy the conditions*

(i) $\varphi, \psi \in L^2_{\mathrm{loc}}((a,b); w)$;

(ii) there exists $c \in (a,b)$ *such that* $\varphi \in L^2((a,c]; w)$ *and* $\psi \in L^2([c,b); w)$;

(iii) for all $[\alpha,\beta] \subset (a,b)$,

$$\int_\alpha^\beta |\varphi(x)|^2\, w(x)dx > 0 \text{ and } \int_\alpha^\beta |\psi(x)|^2\, w(x)dx > 0.$$

Define the linear operators $A, B : L^2((a,b); w) \to L^2_{\mathrm{loc}}((a,b); w)$ by

$$(Af)(x) = \varphi(x)\int_x^b \psi(t)f(t)w(t)dt \quad (t \in (a,b); f \in L^2((a,b); w)),$$

and

$$(Bf)(x) = \psi(x) \int_a^x \varphi(t) f(t) w(t) dt \quad (t \in (a,b); f \in L^2((a,b);w)).$$

Let $K : (a,b) \to (0,\infty)$ *be given by*

$$K(x) = \left(\int_a^x |\varphi(t)|^2 w(t) dt \right)^{\frac{1}{2}} \left(\int_x^b |\psi(t)|^2 w(t) dt \right)^{\frac{1}{2}} \quad (t \in (a,b)),$$

and define $K \in [0,\infty]$ *by*

$$K := \sup_{x \in (a,b)} K(x). \tag{7.12}$$

Then a necessary and sufficient condition that A *and* B *are both bounded operators from* $L^2((a,b);w)$ *into* $L^2((a,b);w)$ *is that*

$$0 < K < \infty.$$

Moreover, the following inequalities hold

$$||Af|| \leq 2K||f|| \quad (f \in L^2((a,b);w)) \tag{7.13}$$

$$||Bg|| \leq 2K||g|| \quad (g \in L^2((a,b);w)) \tag{7.14}$$

where the number K *is defined by (7.12). In general, the number* $2K$ *appearing in both (7.13) and (7.14) is best possible for these inequalities to hold.*

Corollary 7.8. *Suppose the functions* φ *and* ψ *are as in the above theorem. Let* $g \in L^2((a,b);w)$. *Define*

$$g_1(x) = \varphi(x) \int_x^b \psi(x) g(x) w(x) dx \quad (x \in (a,b)),$$
$$g_2(x) = \psi(x) \int_z^x \varphi(x) g(x) w(x) dx \quad (x \in (a,b)).$$

If $K < \infty$, *where* K *is defined as in the previous theorem, then* $g_r \in L^2((a,b);w)$ *for* $r = 1, 2$.

We note that Theorem 7.7, proven by Chisholm and Everitt in 1970, was extended in 1999 by Chisholm, Everitt, and Littlejohn to the spaces $L^p((a,b);w)$

and $L^q((a,b);w)$ where $p,q>1$ are conjugate indices; see [12]. Both Theorem 7.7 and its generalization in [12] have seen several applications including a new proof of the classical Hardy integral inequality [31] and numerous applications to orthogonal polynomials. Several more applications of the CE Theorem will be given in this paper. Indeed, the Theorem 7.7 proves to be an indispensable tool in our analysis below.

7.4 Proof of Theorem 7.3

We now prove Theorem 7.3, namely, that $\mathcal{D}(A^2)=\mathcal{D}(S)$, where $\mathcal{D}(A^2)$ is defined in (1.4) and $\mathcal{D}(S)$ is given in (1.5). Throughout this section, we assume that f is a real-valued function on $(-1,1)$.

Proof. We first show that $\mathcal{D}(S)\subset\mathcal{D}(A^2)$. Let $f\in\mathcal{D}(S)$. We know that

(i) $f,f',f'',f'''\in AC_{\text{loc}}(-1,1)$;

(ii) $f\in L^2(-1,1)$;

(iii) $\ell^2[f]\in L^2(-1,1)$, where $\ell^2[\cdot]$ is defined by (7.2);

(iv) $[f,1]_2(\pm1)=0$, where $[\cdot,1]_2(\cdot)$ is given in (7.9);

(v) $[f,x]_2(\pm1)=0$, where $[\cdot,x]_2(\cdot)$ is given in (7.10).

Taking into account the definition of $\mathcal{D}(A)$ in (5.7) and $\mathcal{D}(A^2)$ in (1.4), we need to show that

(a) $f,f'\in AC_{\text{loc}}(-1,1)$;

(b) $f\in L^2(-1,1)$;

(c) $\ell[f]=-\left((1-x^2)f'\right)'=-(1-x^2)f''+2xf'\in L^2(-1,1)$; in fact, we will show that $\ell[f]\in AC[-1,1]$;

(d) $\lim_{x\to\pm 1}(1-x^2)f'(x) = 0$;

(e) $\ell[f]$, $\ell'[f] \in AC_{\mathrm{loc}}(-1,1)$;

(f) $\ell^2[f] \in L^2(-1,1)$;

(g) $\lim_{x\to\pm 1}(1-x^2)\ell'[f] := \lim_{x\to\pm 1}(1-x^2)\left((1-x^2)f'''(x) - 4xf''(x) - 2f'(x)\right) = 0.$

Clearly, (a), (b), and (f) are satisfied by (i), (ii), and (iii). As for (g), note that

$$\begin{aligned}
-(1-x^2)\ell'[f](x) &= (1-x^2)\big((1-x^2)f'''(x) - 4xf''(x) - 2f'(x)\big)\\
&= (1-x^2)^2 f'''(x) - 4x(1-x^2)f''(x) - 2(1-x^2)f'(x)\\
&= \big((1-x^2)^2 f''(x)\big)' - 2(1-x^2)f'(x)\\
&= [f,1]_2(x),
\end{aligned} \tag{7.15}$$

so (g) follows from (iv) above. Moreover, by (i) and the fact that the product of a polynomial and a function $g \in AC_{\mathrm{loc}}(-1,1)$ also belongs to $AC_{\mathrm{loc}}(-1,1)$, we see that (e) follows, as

$$\ell[f](x) = -(1-x^2)f''(x) + 2xf'(x)$$

and

$$\ell'[f](x) = -(1-x^2)f'''(x) + 4xf''(x) + 2f'(x).$$

To show (c) note that, by (iii),

$$\ell^2[f](x) = \ell[\ell[f]](x) = -\left((1-x^2)\ell'[f](x)\right)' \in L^2(-1,1). \tag{7.16}$$

We now apply the CE Theorem on the interval $[0,1)$ with $\psi(x) = 1$, $\varphi(x) = \frac{1}{1-x^2}$, and $w(x) = 1$; note that $\varphi \in L^2\left(0,\frac{1}{2}\right]$ and $\psi \in L^2\left[\frac{1}{2},1\right)$. A calculation shows that

$$\begin{aligned}
K^2(x) &= \int_0^x \frac{dt}{(1-t^2)^2} \cdot \int_x^1 dt \quad (x \in (0,1))\\
&= \left.\frac{1}{1-t^2}\right|_0^x \cdot (x-1)\\
&= -\frac{1}{1+x},
\end{aligned}$$

hence K is finite. Therefore, we see from Theorem 7.7 that

$$\begin{aligned}\varphi(x)\int_x^1 \psi(t)\ell^2[f](t)w(t)dt &= \frac{1}{1-x^2}\int_x^1 \ell^2[f](t)dt \\ &= -\frac{1}{1-x^2}\int_x^1 \left((1-x^2)\ell'[f](x)\right)' dt \\ &= -\frac{1}{1-x^2}(1-x^2)\ell'[f](x)\Big|_x^1 \\ &= \frac{1}{1-x^2}\left((1-x^2)\ell'[f](x) - \lim_{x\to 1}(1-x^2)\ell'[f](x)\right) \\ &\in L^2[0,1),\end{aligned} \tag{7.17}$$

By (iv) and (7.15), we know

$$\lim_{x\to 1}(1-x^2)\ell'[f](x) = 0.$$

Hence (7.17) simplifies to

$$\ell'[f] \in L^2[0,1).$$

A similar application of the CE Theorem on $(-1,0]$ reveals that $\ell'[f] \in L^2(-1,0]$ and thus we see that

$$\ell'[f] \in L^2(-1,1).$$

It follows that

$$\ell[f] \in AC[-1,1] \subset L^2(-1,1),$$

establishing (c). It remains to show that (d) holds. To this end, observe from (2.6) and (7.2) that

$$\left((1-x^2)^2 f''(x)\right)'' = \ell^2[f](x) - 2\ell[f](x).$$

Consequently, from (c) and (f),

$$\left((1-x^2)^2 f''(x)\right)'' \in L^2(-1,1),$$

from which we see that

$$\left((1-x^2)^2 f''(x)\right)',\ (1-x^2)^2 f''(x) \in AC[-1,1].$$

In particular, we see that the limits

$$\lim_{x\to\pm 1}(1-x^2)^2 f''(x) \tag{7.18}$$

and

$$\lim_{x\to\pm 1}\left((1-x^2)^2 f''(x)\right)' \tag{7.19}$$

exist and are finite. Moreover, from (iv), (v), and (7.10), we see that

$$0 = \lim_{x\to\pm 1}\left(x[f,1]_2(x) - [f,x]_2(x)\right) = \lim_{x\to\pm 1}\left((1-x^2)^2 f''(x) - 2(1-x^2)f(x)\right). \tag{7.20}$$

Thus, from (7.18), we can say that

$$\lim_{x\to\pm 1}(1-x^2)f(x) := r$$

exists and is finite. We claim that $r = 0$; to show this, we deal with the limit as $x \to 1$; a similar proof can be made as $x \to -1$. Suppose to the contrary that $r \neq 0$; without loss of generality, suppose $r > 0$. Then there exists $x^* > 0$ such that

$$(1-x^2)f(x) \geq \frac{r}{2} \text{ for } x \in [x^*, 1).$$

In this case, however,

$$\infty > \int_{-1}^{1} |f(x)|^2 dx \geq \int_{x^*}^{1} |f(x)|^2 dx \geq \left(\frac{r}{2}\right)^2 \int_{x^*}^{1} \frac{dx}{(1-x^2)^2} = \infty,$$

contradicting (ii). Hence it follows that

$$\lim_{x\to\pm 1}(1-x^2)f(x) = 0.$$

Consequently, we see from (7.19), that

$$\lim_{x\to\pm 1}(1-x^2)^2 f''(x) = 0 \tag{7.21}$$

and hence

$$\lim_{x\to\pm 1}(1-x)^2 f''(x) = 0. \tag{7.22}$$

We are now in position to prove part (d). We show that

$$\lim_{x\to 1}(1-x^2)f'(x) = 0; \tag{7.23}$$

a similar argument establishes the limit as $x \to -1$. Let $\varepsilon > 0$. From (7.22), there exists $x^* \in (0,1)$ such that

$$|(1-x)^2 f''(x)| < \frac{\varepsilon}{2} \text{ for } x \in [x^*, 1).$$

Integrating this inequality over $[x^*, x] \subset [x^*, 1)$ yields

$$\frac{\varepsilon}{2(1-x^*)} + f'(x^*) - \frac{\varepsilon}{2(1-x)} < f'(x) < \frac{\varepsilon}{2(1-x)} + f'(x^*) - \frac{\varepsilon}{2(1-x^*)} \text{ for } x \in [x^*, 1).$$

Multiplying this inequality by $(1-x^2)$ yields

$$\begin{aligned}(1-x^2)&\left(f'(x^*) + \frac{\varepsilon}{2(1-x^*)}\right) - \frac{\varepsilon(1+x)}{2}\\ &< (1-x^2)f'(x) < \frac{\varepsilon(1+x)}{2} + (1-x^2)\left(f'(x^*) - \frac{\varepsilon}{2(1-x^*)}\right).\end{aligned}$$

Letting $x \to 1$, we obtain

$$-\varepsilon \le \lim_{x\to 1}(1-x^2)f'(x) \le \varepsilon,$$

and this establishes (7.23). This completes the proof that $\mathcal{D}(S) \subset \mathcal{D}(A^2)$.

We now show that $\mathcal{D}(A^2) \subset \mathcal{D}(S)$. Let $f \in \mathcal{D}(A^2)$. Then $f \in \mathcal{D}(A)$ so

$$f, f' \in AC_{\text{loc}}(-1,1) \tag{7.24}$$

and

$$f \in L^2(-1,1). \tag{7.25}$$

Moreover, since $\ell[f] \in \mathcal{D}(A)$, it follows that

$$\ell^2[f] = \ell[\ell[f]] \in L^2(-1,1), \tag{7.26}$$

$$\ell[f] = -(1-x^2)f'' + 2xf' \in AC_{\text{loc}}(-1,1), \tag{7.27}$$

and

$$\ell'[f] = -(1-x^2)f''' + 4xf' + 2f' \in AC_{\text{loc}}(-1,1). \tag{7.28}$$

It is well known that if $f, g \in AC_{\text{loc}}(1,-1)$, then (a)$'$ $f+g \in AC_{\text{loc}}(-1,1)$;

(b)$'$ $fg \in AC_{\text{loc}}(-1,1)$;

(c)$'$ If $g>0$ on $(-1,1)$, then $\frac{f}{g} \in AC_{\text{loc}}(-1,1)$.

In particular, from (7.24) and (b)$'$ we see that $2xf' \in AC_{\text{loc}}(-1,1)$. Combining this with (a)$'$ and (7.27), we obtain $(1-x^2)f'' \in AC_{\text{loc}}(-1,1)$. Since $1-x^2>0$ on $(-1,1)$ we infer from (c)$'$ that

$$f'' \in AC_{\text{loc}}(-1,1). \tag{7.29}$$

Continuing, $-4xf''-2f' \in AC_{\text{loc}}(-1,1)$ so from (a)$'$ and (7.28), we have that $(1-x^2)f''' \in AC_{\text{loc}}(-1,1)$ and it follows that

$$f''' \in AC_{\text{loc}}(-1,1). \tag{7.30}$$

By definition of $\mathcal{D}(A)$ and the fact that $\ell[f] \in \mathcal{D}(A)$, we see that

$$\lim_{x\to\pm1}(1-x^2)\ell'[f](x)=0;$$

consequently, in view of (7.15), we see that

$$0=\lim_{x\to\pm1}[f,1]_2(x)=\lim_{x\to\pm1}\left(\left((1-x^2)^2f''(x)\right)'-2(1-x^2)f'(x)\right). \tag{7.31}$$

Furthermore, since $f\in\mathcal{D}(A)$, we have

$$\lim_{x\to\pm1}(1-x^2)f'(x)=0, \tag{7.32}$$

so from (7.31), we see that

$$\lim_{x\to\pm1}\left((1-x^2)^2f''(x)\right)''=0.$$

To finish the proof, we need to show that

$$\begin{aligned}
0&=[f,x]_2(\pm1)\\
&=\lim_{x\to\pm1}\left(\left((1-x^2)^2f''(x)\right)'x-(1-x^2)^2f''(x)-2x(1-x^2)f'(x)+2(1-x^2)f(x)\right)\\
&=\lim_{x\to\pm1}\left(-(1-x^2)^2f''(x)+2(1-x^2)f(x)\right)\ \text{by (7.31).}
\end{aligned} \tag{7.33}$$

We note again, from Green's formula (7.5), that the limits in (7.33) exist and are finite. Since $f \in \mathcal{D}(A)$, we see from Theorem 5.6, part (v) that $f \in AC[-1, 1]$ and hence

$$\lim_{x\to\pm 1}(1-x^2)f(x) = 0. \tag{7.34}$$

Thus, proving (7.33) reduces to showing

$$\lim_{x\to\pm 1}(1-x^2)^2 f''(x) = 0.$$

We show that

$$\lim_{x\to 1}(1-x^2)^2 f''(x) = 0; \tag{7.35}$$

a similar argument will show

$$\lim_{x\to -1}(1-x^2)^2 f''(x) = 0.$$

Suppose to the contrary that

$$\lim_{x\to 1}(1-x^2)^2 f''(x) = c \neq 0;$$

without loss of any generality, we can suppose that $c > 0$. Then there exists $x^* \in (0, 1)$ such that

$$(1-x^2)^2 f''(x) \geq r := \frac{c}{2} \text{ on } [x^*, 1);$$

that is,

$$f''(x) \geq \frac{R}{(1-x)^2} \text{ on } [x^*, 1)$$

for some $R > 0$. Integrating this inequality over $[x, x^*] \subset [x^*, 1)$ yields

$$f'(x) \geq R\int_{x^*}^{x} \frac{dt}{(1-t)^2} + f'(x^*) = \frac{R}{1-x} + f'(x^*) - \frac{R}{1-x^*}.$$

Consequently,

$$(1-x^2)f'(x) \geq R(1+x) + \left(f'(x^*) - \frac{R}{1-x^*}\right)(1-x^2) \to 2R > 0 \text{ (as } x \to 1),$$

contradicting (7.32). It follows that (7.35) holds and this proves (7.33). A similar proof establishes $[f,x]_2(-1) = 0$ which proves (7.33). Combining (7.24), (7.25), (7.26), (7.29), (7.30), (7.31) and (7.33), we see that $f \in \mathcal{D}(A^2)$ implies $f \in \mathcal{D}(S)$. This completes the proof of the theorem. □

7.5 Proof of Theorem 7.4

In order to prove Theorem 7.4, we first need to establish three preliminary facts, the first of which is the following result.

Lemma 7.9. *If $f \in \mathcal{D}(S)$, then*

$$\frac{1}{1-x^2}\left((1-x^2)^2 f''(x)\right)' \in L^2(-1,1). \tag{7.36}$$

Proof. Let $f \in \mathcal{D}(S) = \mathcal{D}(A^2)$ so $f' \in L^2(-1,1)$, $[f,1]_2(\pm 1) = 0$, and $\ell^2[f] \in L^2(-1,1)$. We apply the CE Theorem on $[0,1)$ with $\psi(x) = 1$, $\varphi(x) = -\frac{1}{1-x^2}$, and $w(x) = 1$. These functions satisfy the conditions of this theorem on $[0,1)$ so

$$-\frac{1}{1-x^2}\int_x^1 \ell^2[f](t)dt \in L^2(0,1).$$

However, using (7.9) and (7.11), a calculation shows

$$\begin{aligned}
-\frac{1}{1-x^2}\int_x^1 \ell^2[f](t)dt &= -\frac{1}{1-x^2}\int_x^1 \left[\left((1-t^2)^2 f''(t)\right)'' - 2\left((1-t^2)f'(t)\right)'\right]dt \\
&= -\frac{1}{1-x^2}\left[\lim_{x\to 1}\left(\left((1-x^2)^2 f''(x)\right)' - 2(1-x^2)f'(x)\right)\right] \\
&\quad + \frac{1}{1-x^2}\left[\left((1-x^2)^2 f''(x)\right)' - 2(1-x^2)f'(x)\right] \\
&= -\frac{1}{1-x^2}\left[\lim_{x\to 1}[f,1]_2(x) - \left((1-x^2)^2 f''(x)\right)' + 2(1-x^2)f'(x)\right] \\
&= \frac{1}{1-x^2}\left((1-x^2)^2 f''(x)\right)' - 2f'(x).
\end{aligned}$$

A similar calculation shows that

$$\frac{1}{1-x^2}\left((1-x^2)^2 f''(x)\right)' - 2f'(x) \in L^2(-1,0]$$

and hence

$$\frac{1}{1-x^2}\left((1-x^2)^2 f''(x)\right)' - 2f'(x) \in L^2(-1,1).$$

Since $f' \in L^2(-1,1)$, we see, by linearity, that

$$\frac{1}{1-x^2}\left((1-x^2)^2 f''(x)\right)' \in L^2(-1,1).$$

□

Lemma 7.10. *For $f \in \mathcal{D}(S)$, we have*

$$\lim_{x\to\pm 1}(1-x^2)^2 f''(x) = 0. \tag{7.37}$$

Proof. Let $f \in \mathcal{D}(S) = \mathcal{D}(A^2)$. Since $f \in \mathcal{D}(A)$, we have $f \in AC[-1,1]$, so

$$\lim_{x\to\pm 1}(1-x^2)f(x) = 0. \tag{7.38}$$

Furthermore, we have

$$0 = \lim_{x\to\pm 1}[f,1]_2(x) = \lim_{x\to\pm 1}\left(\left((1-x^2)^2 f''(x)\right)' - 2(1-x^2)f'(x)\right). \tag{7.39}$$

Consequently, from (7.10), (7.38), and (7.39), we find that

$$\begin{aligned} 0 = \lim_{x\to\pm 1}[f,x]_2(x) &= \lim_{x\to\pm 1}\left(x[f,1]_2(x) - (1-x^2)^2 f''(x) + 2(1-x^2)f(x)\right) \\ &= -\lim_{x\to\pm 1}(1-x^2)^2 f''(x). \end{aligned}$$

□

The last preliminary result is the following theorem. Since $\mathcal{D}(S) = \mathcal{D}(A^2)$, this next result generalizes the well-known result for $\mathcal{D}(A)$ established in Theorem 5.6, part (v).

Theorem 7.11. *If $f \in \mathcal{D}(S)$, then*

$$f'' \in L^2(-1,1).$$

Moreover,

$$pf'' \in L^2(-1,1) \tag{7.40}$$

for any bounded, Lebesgue measurable function p, including any polynomial.

Proof. Once we establish $f'' \in L^2(-1,1)$, the statement in (7.40), for any bounded measurable function, follows clearly. Let $f \in \mathcal{D}(S)$. We prove that $f'' \in L^2(0,1)$; a similar proof will establish $f'' \in L^2(-1,0)$ and prove the theorem. We again use the CE Theorem with $\psi(x) = 1 - x^2$, $\varphi(x) = \frac{1}{(1-x^2)^2}$, and $w(x) = 1$ on $[0,1)$. Indeed, from the CE Theorem and (7.36), we find that

$$-\frac{1}{(1-x^2)^2}\int_x^1 (1-t^2)\left(\frac{1}{1-t^2}\left((1-t^2)^2 f''(t)\right)'\right) dt \in L^2(0,1).$$

However, from Lemma 7.9,

$$\begin{aligned}
-\frac{1}{(1-x^2)^2}&\int_x^1 (1-t^2)\left(\frac{1}{1-t^2}\left((1-t^2)^2 f''(t)\right)'\right) dt \\
&= -\frac{1}{(1-x^2)^2}\left(\lim_{x\to 1}(1-x^2)f''(x) - (1-x^2)^2 f''(x)\right) \\
&= f''(x).
\end{aligned}$$

□

We are now in position to prove Theorem 7.4, specifically $B = \mathcal{D}(S)$, where B is defined in (1.2) and $\mathcal{D}(S)$ is given in (1.5).

Proof. We first prove that $B \subset \mathcal{D}(S)$. Let $f \in B$. We assume that f is real-valued on $(-1,1)$. We begin by showing, using the CE Theorem, that the condition

$$(1-x^2)^2 f^{(4)} \in L^2(-1,1)$$

implies the two conditions

$$(1-x^2)f''' \in L^2(-1,1) \tag{7.41}$$

and

$$f'' \in L^2(-1,1). \tag{7.42}$$

Regarding (7.41), we will show

$$(1-x^2)f''' \in L^2(0,1); \tag{7.43}$$

a similar proof will yield

$$(1 - x^2)f''' \in L^2(-1, 0), \tag{7.44}$$

and, together, they establish (7.41). Since $(1 - x^2)^2 f^{(4)} \in L^2(0, 1)$, we use the CE Theorem on $[0, 1)$ with

$$\varphi(x) = \frac{1}{(1 - x^2)^2}, \ \psi(x) = 1 - x^2, \text{ and } w(x) = 1 \quad (x \in [0, 1)).$$

It follows that

$$(1 - x^2)f'''(x) = (1 - x^2) \int_0^x \frac{1}{(1 - t^2)^2}(1 - t^2)^2 f^{(4)}(t)dt + f'''(0)(1 - x^2) \in L^2(0, 1).$$

To see (7.42), we apply the CE Theorem once again on $[0, 1)$ to prove that

$$f'' \in L^2(0, 1);$$

a similar argument will show that $f'' \in L^2(-1, 0)$. To this end, let

$$\varphi(x) = \frac{1}{1 - x^2}, \ \psi(x) = 1, \text{ and } w(x) = 1 \quad (x \in [0, 1)).$$

In this case, we see that

$$f''(x) = \int_0^x \frac{1}{1 - t^2} \left((1 - t^2)f'''(t)\right) dt + f''(0) \in L^2(0, 1).$$

Consequently, we see that

$$f, f' \in AC[-1, 1] \subset L^2(-1, 1).$$

Moreover, it is clear that $g(x)(1 - x^2)f'''(x)$, $g(x)f''(x)$, and $g(x)f'(x)$ all belong to $L^2(-1, 1)$ for any bounded, measurable function g on $(-1, 1)$. Hence

$$\ell^2[f](x) = (1-x^2)^2 f^{(4)}(x) - 8x(1-x^2)f'''(x) + (14x^2-6)f''(x) + 4xf'(x) \in L^2(-1, 1).$$

It remains to show that

$$\lim_{x \to \pm 1} [f, 1]_2(x) = \lim_{x \to \pm 1} [f, x]_2(x) = 0. \tag{7.45}$$

Since $1, x \in \Delta_{2,\max}$, we see from Green's formula in (7.5) that the limits in (7.45) both exist and are finite. Now $f' \in AC[-1,1]$ so

$$\lim_{x\to\pm1}(1-x^2)f'(x) = 0.$$

Consequently,

$$\begin{aligned}\lim_{x\to\pm1}[f,1]_2(x) &= \lim_{x\to\pm1}\Big(\big((1-x^2)^2f''(x)\big)' - 2(1-x^2)f'(x)\Big)\\ &= \lim_{x\to\pm1}\big((1-x^2)^2f''(x)\big)'.\end{aligned}$$

We claim that

$$\lim_{x\to1}\big((1-x^2)^2f''(x)\big)' = 0; \tag{7.46}$$

a similar proof will establish

$$\lim_{x\to-1}\big((1-x^2)^2f''(x)\big)' = 0.$$

Suppose to the contrary that

$$\lim_{x\to1}\big((1-x^2)^2f''(x)\big)' = c \neq 0;$$

we can assume that $c > 0$. It follows that there exists $x^* \in (0,1)$ such that

$$\big((1-x^2)^2f''(x)\big)' \geq r := \frac{c}{2} > 0 \quad (x \in [x^*,1)). \tag{7.47}$$

Note that since

$$\big((1-x^2)^2f''(x)\big)' = (1-x^2)^2f'''(x) - 4x(1-x^2)f''(x), \tag{7.48}$$

we see that the inequality in (7.47) can be rewritten as

$$(1-x^2)f'''(x) - 4xf''(x) \geq \frac{r}{1-x^2} \text{ for } x \in [x^*,1). \tag{7.49}$$

However, from (7.41) and (7.42), we know that

$$(1-x^2)f''' - 4xf'' \in L^2(-1,1)$$

so the inequality in (7.49) is not possible. Hence (7.46) is established and thus

$$\lim_{x \to \pm 1} [f, 1]_2(x) = 0.$$

We now show that

$$\lim_{x \to \pm 1} [f, x]_2(x) = 0. \tag{7.50}$$

Since the argument for $x \to -1$ mirrors the proof for $x \to 1$, we will only show that

$$\lim_{x \to 1} [f, x]_2(x) = 0.$$

Now, since $f \in AC[-1, 1]$, we see that

$$\lim_{x \to 1} (1 - x^2) f(x) = 0;$$

moreover, using (7.46),

$$\begin{aligned}
\lim_{x \to 1} [f, x]_2(x) &= \lim_{x \to 1} \left(x[f, 1]_2(x) - (1 - x^2)^2 f''(x) + 2(1 - x^2) f(x) \right) \\
&= - \lim_{x \to 1} (1 - x^2)^2 f''(x).
\end{aligned}$$

Suppose that

$$\lim_{x \to 1} (1 - x^2)^2 f''(x) = d \neq 0;$$

we can assume that $d > 0$. Then, with a possibly different x^* as given in the above argument, there exists an $x^* \in (0, 1)$ with

$$(1 - x^2)^2 f''(x) \geq d' := \frac{d}{2} \quad (x \in [x^*, 1)).$$

Hence

$$f''(x) \geq \frac{d'}{(1 - x^2)^2} \quad (x \in [x^*, 1)).$$

However, this implies that $f'' \notin L^2(0, 1)$, contradicting (7.42). Thus (7.50) is established and this completes the proof that $B \subset \mathcal{D}(S)$.

We now prove that $\mathcal{D}(S) \subset B$. Let $f \in \mathcal{D}(S)$. We need only to show that

$$(1 - x^2)^2 f^{(4)} \in L^2(-1, 1). \tag{7.51}$$

Since by Theorem 7.10 we know that $f'' \in L^2(-1,1)$, we see that $gf'' \in L^2(-1,1)$ for any bounded, measurable function g on $(-1,1)$. In particular, it is the case that

$$4xf'' \in L^2(-1,1) \tag{7.52}$$

and

$$(14x^2 - 6)f'' \in L^2(-1,1). \tag{7.53}$$

By (7.36) (see also (7.48)),

$$(1-x^2)f'''(x) - 4xf''(x) = \frac{1}{1-x^2}\left((1-x^2)^2 f''(x)\right)' \in L^2(-1,1). \tag{7.54}$$

By linearity, it follows from (7.52) and (7.54) that

$$(1-x^2)f''' \in L^2(-1,1).$$

Consequently, $g(1-x^2)f''' \in L^2(-1,1)$ for every bounded, measurable function g on $(-1,1)$;in particular,

$$8x(1-x^2)f''' \in L^2(-1,1). \tag{7.55}$$

Furthermore, since $f' \in L^2(-1,1)$, it follows that

$$4xf'(x) \in L^2(-1,1). \tag{7.56}$$

Finally, since $\ell^2[f] \in L^2(-1,1)$, we see from (1.3), (7.53), (7.55), and (7.56) that

$$(1-x^2)^2 f^{(4)} = \ell^2[f] + 8x(1-x^2)f''' - (14x^2-6)f'' - 4xf' \in L^2(-1,1).$$

This establishes (7.51) and proves $\mathcal{D}(S) \subset B$. This completes the proof of Theorem 7.4. $\square$

7.6 Proof of Theorem 7.5

We now prove Theorem 7.5, namely $\mathcal{D}(S) = D$, where $\mathcal{D}(S)$ is given in (1.5) and D is defined in (1.6).

Proof. Since functions f in both $\mathcal{D}(S)$ and D satisfy the "maximal domain" conditions $f^{(j)} \in AC_{\mathrm{loc}}(-1,1)$ $(j = 0,1,2,3)$, $f \in L^2(-1,1)$, and $\ell^2[f] \in L^2(-1,1)$, we need only to prove that the other properties in their definitions hold.

We first show that $\mathcal{D}(S) \subset D_2$. Let $f \in \mathcal{D}(S) = \mathcal{D}(A^2)$. Then $f \in \mathcal{D}(A)$ so

$$\lim_{x\to\pm 1}(1-x^2)f'(x) = 0. \tag{7.57}$$

Moreover,

$$\begin{aligned} 0 &= [f,1]_2(\pm 1) \\ &= \lim_{x\to\pm 1}\Big(\big((1-x^2)^2 f''(x)\big)' - 2(1-x^2)f'(x)\Big) \\ &= \lim_{x\to\pm 1}\big((1-x^2)^2 f''(x)\big)'. \end{aligned} \tag{7.58}$$

The identities in (7.57) and (7.58) prove that $f \in D$, hence $\mathcal{D}(S) \subset D$.

We now prove that $D \subset \mathcal{D}(S)$. Let $f \in D$. Clearly,

$$[f,1]_2(\pm 1) = \lim_{x\to\pm 1}\Big(\big((1-x^2)^2 f''(x)\big)' - 2(1-x^2)f'(x)\Big) = 0, \tag{7.59}$$

so we need to show that

$$\lim_{x\to\pm 1}[f,x]_2(\pm 1) = 0. \tag{7.60}$$

We remark that the limits in (7.60) exist (by Green's formula) and are finite.

We claim that $\ell'[f] \in L^2(-1,1)$. To see this, recall the two representations of $\ell^2[\cdot]$: the one given in (1.3) and the one given in (7.15). Since $\ell^2[f] \in L^2(-1,1)$, we apply the CE Theorem on $[0,1)$ with $\varphi(x) = (1-x^2)^{-1}$, $\psi(x) = 1$, and $w(x) = 1$ to obtain

$$\frac{1}{1-x^2}\int_x^1 \ell^2[f](t)dt \in L^2(0,1).$$

However, from (1.3) and (7.15), we see that

$$\begin{aligned} &\frac{1}{1-x^2}\int_x^1 \ell^2[f](t)dt \\ &\qquad = \frac{1}{1-x^2}\left(\lim_{x\to 1}\Big(\big((1-x^2)^2 f''(x)\big)' - 2(1-x^2)f'(x)\Big) + (1-x^2)\ell'[f](x)\right) \\ &\qquad = \ell'[f](x) \text{ by (7.59);} \end{aligned}$$

a similar calculation shows that $\ell'[f] \in L^2(-1,0)$, hence $\ell[f] \in AC[-1,1] \subset L^2(-1,1)$. We again apply the CE Theorem on $[0,1)$ with $\varphi(x) = (1-x^2)^{-1}$, $\psi(x) = 1$, and $w(x) = 1$ to obtain

$$\frac{1}{1-x^2}\int_x^1 \ell[f](t)dt \in L^2(0,1).$$

Another calculation shows that

$$\begin{aligned}\frac{1}{1-x^2}\int_x^1 \ell[f](t)dt &= -\frac{1}{1-x^2}\int_x^1 \big((1-t^2)f'(t)\big)'\,dt \\ &= -\frac{1}{1-x^2}\Big(\lim_{x\to 1}(1-x^2)f'(x) - (1-x^2)f'(x)\Big) \\ &= f'(x) \text{ by definition of } D;\end{aligned}$$

a similar argument shows that $f' \in L^2(-1,0)$. Hence

$$f' \in L^2(-1,1). \tag{7.61}$$

Thus, $f \in AC[-1,1]$ and

$$\lim_{x\to\pm 1}(1-x^2)f(x) = 0. \tag{7.62}$$

From (7.59) and (7.62), we see that

$$\begin{aligned}\lim_{x\to\pm 1}[f,x]_2(x) &= \lim_{x\to\pm 1}\big(x[f,1](x) - (1-x^2)^2 f''(x) + 2(1-x^2)f(x)\big) \\ &= -\lim_{x\to\pm 1}(1-x^2)^2 f''(x).\end{aligned}$$

To establish (7.60), it now suffices to prove that

$$\lim_{x\to\pm 1}(1-x^2)^2 f''(x) = 0. \tag{7.63}$$

Since the proof as $x \to -1$ is similar to the proof that $x \to 1$, we will only show that

$$\lim_{x\to 1}(1-x^2)^2 f''(x) = 0.$$

By way of contradiction, suppose that

$$\lim_{x\to 1}(1-x^2)^2 f''(x) = c \neq 0;$$

without loss of generality, we may assume that $c > 0$. Then there exists $x^* \in (0,1)$ such that

$$f''(x) \geq \frac{c}{2(1-x^2)^2} \geq \frac{c}{8(1-x)^2} \text{ for } x \in [x^*, 1).$$

Integrating this inequality over $[x^*, x] \subset [x^*, 1)$ yields

$$f'(x) \geq \frac{c}{8(1-x)} + f'(x^*) - \frac{c}{8(1-x^*)} \quad (x \in [x^*, 1)).$$

However, this contradicts (7.61). It follows that (7.63) holds and this, in turn, establishes (7.60). Consequently, $D \subset \mathcal{D}(S)$ and this completes the proof of the theorem. □

As revealed in the proofs of Theorems 7.3, 7.4, 7.5, and 7.10, we have the following interesting result.

Corollary 7.12. *If $f \in \mathcal{D}(A^2) = \mathcal{D}(S) = B = D$, then*

(i) $f'' \in L^2(-1,1)$ so $f, f' \in AC[-1,1]$;

(ii) $\ell'[f] \in L^2(-1,1)$ and $\ell[f] \in AC[-1,1]$.

We end this section with an important remark. As discussed in Section 7.1, the minimal operator $T_{2,\min}$ in $L^2(-1,1)$ generated by $\ell^2[\cdot]$ has deficiency index $(4,4)$. From the GKN Theorem (see [52]), GKN boundary conditions for any self-adjoint extension of $T_{2,\min}$ in $L^2(-1,1)$ are restrictions of the maximal domain $\Delta_{2,\max}$ and have the appearance (see (7.6))

$$[f, f_j]_2(1) - [f, f_j]_2(-1) = 0 \quad (f \in \Delta_{2,\max},\ j = 1, 2, 3, 4),$$

where $\{f_j\}_{j=1}^4 \subset \Delta_{2,\max}$ are linearly independent modulo the minimal domain $\Delta_{2,\min}$. Taking into account $[\cdot,\cdot]_2$, defined in (7.4), it is clear that the boundary conditions given in (1.6) are not GKN boundary conditions.

CHAPTER EIGHT

The n^{th} Power of the Legendre Polynomials Operator

Recall $\ell^n[\cdot]$, the n^{th} composite power of the Legendre differential expression $\ell[\cdot]$ for $n \in \mathbb{N}$ given in (6.12). The expression in (6.12) is the key in generating the domain $\mathcal{D}(A^n)$ of A^n given in (1.1).

We give the following

Conjecture 8.1. *Let A denote the Legendre polynomials self-adjoint operator defined in (5.6) and (5.7). For $n \in \mathbb{N}$, let $\ell^n[\cdot]$ be given as in (6.12) and let $[\cdot,\cdot]_n$ be the sesquilinear form associated with the maximal domain $\Delta_{n,\max}$ of $\ell^n[\cdot]$ in $L^2(-1,1)$. Then $A_n = B_n = C_n = D_n$, where*

(i) $A_n := \mathcal{D}(A^n)$,

(ii) $$B_n := \{f : (-1,1) \to \mathbb{C} \mid f, f', ..., f^{(2n-1)} \in AC_{\text{loc}}(-1,1); \\ (1-x^2)^n f^{(2n)} \in L^2(-1,1)\},$$

(iii) $$C_n := \{f : (-1,1) \to \mathbb{C} \mid f, f', ..., f^{(2n-1)} \in AC_{\text{loc}}(-1,1); f, \ell^n[f] \in L^2(-1,1); \\ [f, x^j]_n(\pm 1) = 0 \textit{ for } j = 0,1,2,...,n-1\}, \textit{ and}$$

(iv) $$D_n := \{f : (-1,1) \to \mathbb{C} \mid f, f', ..., f^{(2n-1)} \in AC_{\text{loc}}(-1,1); f, \ell^n[f] \in L^2(-1,1); \\ \lim_{x \to \pm 1} \left((1-x^2)^j y^{(j)}(x)\right)^{(j-1)} = 0 \textit{ for } j = 1,2,...,n\}.$$

By repeated applications of the CE Theorem, it is not difficult to establish that if $f \in B_n$, then $f^{(n)} \in L^2(-1,1)$; this result, proven below in the proof of Theorem 8.2, generalizes Theorem 5.6, part (iii) ($n = 1$) and Corollary 7.11, part (i) ($n = 2$).

We remark that, in (iii) above, we can replace the monomials $\{x^j\}_{j=0}^{n-1}$ by the Legendre polynomials $\{P_j\}_{j=0}^{n-1}$. One of the difficulties in our efforts to try and prove this conjecture lies in the fact that the corresponding sesquilinear form $[\cdot,\cdot]_n$, associated with the n^{th} power $\ell^n[\cdot]$, is unwieldy at the present time.

Theorem 8.2. *Let B_n and D_n be defined as in Conjecture 8.1 above. Then $B_n \subseteq D_n$.*

Proof. To show that $B_n \subseteq D_n$, let $f \in B_n$. Then $(1-x^2)^n f^{(2n)} \in L^2(-1,1)$. We first show by using the CE Theorem that this implies $(1-x^2)^{n-1} f^{(2n-1)} \in L^2(-1,1)$.

Lemma 8.3. $(1-x^2)^n f^{(2n)} \in L^2(-1,1) \implies (1-x^2)^{n-1} f^{(2n-1)} \in L^2(-1,1)$.

Proof. Since f is absolutely continuous, so are all its derivatives, hence

$$\int_0^x \frac{1}{(1-t^2)^n}(1-t^2)^n f^{(2n)}(t)dt = f^{(2n-1)}(x) - f^{(2n-1)}(0),$$

by letting

$$\varphi(t) = \frac{1}{(1-t^2)^n}$$

and noting that

$$(1-t^2)^n f^{(2n)} \in L^2(-1,1),$$

so

$$f^{(2n-1)}(x) = f^{(2n-1)}(0) + \int_0^x \frac{1}{(1-t^2)^n}(1-t^2)^n f^{(2n)}(t)dt.$$

Multiply both sides by $(1-x^2)^{n-1}$ to get

$$(1-x^2)^{n-1} f^{(2n-1)}(x) = \\ (1-x^2)^{n-1} f^{(2n-1)}(0) + (1-x^2)^{n-1} \int_0^x \frac{1}{(1-t^2)^n}(1-t^2)^n f^{(2n)}(t)dt.$$

Note that $(1-x^2)^{n-1} f^{(2n-1)}(0) \in L^2(-1,1)$. So

$$\psi(x) = (1-x^2)^{n-1} \text{ and } \varphi(x) = \frac{1}{(1-x^2)^n}, \quad x \in [0,1).$$

Also,

$$\begin{aligned}
K^2(x) &= \int_0^x \frac{dt}{(1-t^2)^{2n}} \int_x^1 (1-t^2)^{2n-2} dt \\
&\le C \int_0^x \frac{dt}{(1-t)^{2n}} \int_x^1 (1-t)^{2n-2} dt \\
&= C \frac{-1}{(1-t)^{2n-1}} \bigg|_0^x (1-t)^{2n-1} \bigg|_x^1 \\
&= C \left(\frac{-1}{(1-x)^{2n-1}} + 1 \right) (-(1-x)^{2n-1}) \\
&= C(1-(1-x)^{2n-1}) \\
&\le C
\end{aligned}$$

for some $0 < C < \infty$ since $x \in [0,1)$ and $n \in \mathbb{N}$. By the CE Theorem, since $K(x)$ is bounded, we have that $(1-x^2)^{n-1} f^{(2n-1)} \in L^2(-1,1)$. $\square$

Lemma 8.4. *$(1-x^2)^n f^{(2n)} \in L^2(-1,1) \implies (1-x^2)^{n-j} f^{(2n-j)} \in L^2(-1,1)$ for $j = 1, 2, ..., n$.*

Proof. We see from Lemma 7 that the statement is true for $j = 1$. Assume that it is true for $j-1$, i.e., that

$$(1-x^2)^n f^{(2n)} \in L^2(-1,1) \implies (1-x^2)^{n-j+1} f^{(2n-j+1)} \in L^2(-1,1).$$

As above, we can write

$$f^{(2n-j)}(x) = f^{(2n-j)}(0) + \int_0^x \frac{1}{(1-t^2)^{n-j+1}} (1-t^2)^{n-j+1} f^{(2n-j+1)}(t) dt.$$

Multiply both sides by $(1-x^2)^{(n-j)}$ to get

$$\begin{aligned}
&(1-x^2)^{(n-j)} f^{(2n-j)}(x) = \\
&\quad (1-x^2)^{(n-j)} f^{(2n-j)}(0) + (1-x^2)^{(n-j)} \int_0^x \frac{1}{(1-t^2)^{n-j+1}} (1-t^2)^{n-j+1} f^{(2n-j+1)}(t) dt.
\end{aligned}$$

Since $(1-x)^{n-j} f^{(2n-j)}(0) \in L^2(-1,1)$, we can use the CE Theorem with

$$\psi(x) = (1-x^2)^{n-j} \text{ and } \varphi(x) = \frac{1}{(1-x^2)^{n-j+1}}$$

as

$$\begin{aligned}
K^2(x) &= \int_0^x \frac{dt}{(1-t^2)^{2n-2j+2}} \int_x^1 (1-t^2)^{2n-2j} dt \\
&\leq C \int_0^x \frac{dt}{(1-t)^{2n-2j+2}} \int_x^1 (1-t)^{2n-2j} dt \\
&= C \frac{-1}{(1-t)^{2n-2j+1}} \bigg|_0^x (1-t)^{2n-2j+1} \bigg|_x^1 \\
&= C \left(\frac{-1}{(1-x)^{2n-2j+1}} + 1 \right) (-(1-x)^{2n-2j+1}) \\
&= C(1-(1-x)^{2n-2j+1}) \\
&\leq C
\end{aligned}$$

for some $0 < C < \infty$ since $x \in [0,1)$ and $n \in \mathbb{N}$, $j \in \{1, 2, ...n\}$. Hence

$$(1-x^2)^{(n-j)} f^{(2n-j)}(x) \in L^2(-1,1),$$

completing the proof by induction. $\square$

When $j = n$ in Lemma 8, we see that $f^{(n)} \in L^2(-1,1)$. Therefore

$$f^{(n-1)} \in AC[-1,1] \subseteq L^2(-1,1).$$

In this way, we get that

$$f, f', ..., f^{(n)} \in L^2(-1,1),$$

which implies that

$$\ell^n[f](x) = \sum_{j=1}^{n} (-1)^j \begin{Bmatrix} n \\ j \end{Bmatrix}_1 \left((1-x^2)^j f^{(j)}(x)\right)^{(j)} \in L^2(-1,1),$$

showing that $f \in \Delta_n$. It remains to show that

$$\lim_{x \to \pm 1} \left((1-x^2)^j f^{(j)}(x)\right)^{(j-1)} = 0 \text{ for } j = 1, 2, ..., n.$$

To again proceed by induction, we first see that this is true when $j = 1$ and $j = 2$ from above. Assume that

$$\lim_{x \to \pm 1} \left((1-x^2)^{j-1} f^{(j-1)}\right)^{(j-2)} = 0.$$

We wish to show that

$$\lim_{x\to\pm1}\left((1-x^2)^j f^{(j)}\right)^{(j-1)}=0.$$

Suppose instead that

$$\lim_{x\to1}\left((1-x^2)^j f^{(j)}(x)\right)^{(j-1)}=c,$$

where $c\neq 0$. Without loss of generality, suppose that $c>0$ and f is real-valued. Then there exists $x^*\in(0,1)$ such that

$$((1-x^2)^j f^{(j)}(x))^{(j-1)}\geq\frac{c}{2}\text{ on }[x^*,1).$$

Since

$$\begin{aligned}((1-x^2)^j f^{(j)}(x))^{(j-1)}&=\sum_{k=0}^{j-1}\binom{j-1}{k}\left((1-x^2)^j\right)^{(k)}f^{(2j-1-k)}(x)\\&=\sum_{k=0}^{j-1}\binom{j-1}{k}\left[\sum_{i=0}^{k}\binom{k}{i}\left(\frac{j!}{(j-i)!}\right)^2(1-x^2)^{j-i}\right]f^{(2j-1-k)}(x)\\&=(1-x^2)\sum_{k=0}^{j-1}\binom{j-1}{k}\left[\sum_{i=0}^{k}\binom{k}{i}\left(\frac{j!}{(j-i)!}\right)^2(1-x^2)^{j-i-1}\right]f^{(2j-1-k)}(x)\end{aligned}$$

and $j-i-1\geq 0$, we can rewrite the inequality as

$$\sum_{k=0}^{j-1}\binom{j-1}{k}\left[\sum_{i=0}^{k}\binom{k}{i}\left(\frac{j!}{(j-i)!}\right)^2(1-x^2)^{j-i}\right]f^{(2j-1-k)}(x)\geq\frac{c}{2}$$

$$(1-x^2)\sum_{k=0}^{j-1}\binom{j-1}{k}\left[\sum_{i=0}^{k}\binom{k}{i}\left(\frac{j!}{(j-i)!}\right)^2(1-x^2)^{j-i-1}\right]f^{(2j-1-k)}(x)\geq\frac{c}{2}$$

$$\sum_{k=0}^{j-1}\binom{j-1}{k}\left[\sum_{i=0}^{k}\binom{k}{i}\left(\frac{j!}{(j-i)!}\right)^2(1-x^2)^{j-i-1}\right]f^{(2j-1-k)}(x)\geq\frac{c}{2(1-x^2)}.$$

However, since we showed above that

$$f,f',...,f^{(n)}\in L^2(-1,1),$$

this means that the left-hand side of the inequality is in $L^2(-1,1)$, giving us a contradiction. Hence $c=0$. We can similarly show that

$$\lim_{x\to-1}\left((1-x^2)^j f^{(j)}\right)^{(j-1)}=0.$$

Hence $f\in D_n$, so $B_n\subseteq D_n$. □

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