ABSTRACT

Strengthening Pedagogical Content Knowledge in Division of Fractions through a Three-Part Professional Development Series: A Multiple Case Study

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Good teaching matters. The teacher represents the most dominant factor affecting student achievement. Understanding the knowledge teachers bring to the classroom and how to effectively strengthen this knowledge for in-service teachers has been under investigation for the past several decades. Rational numbers represent the most dominant area of K – 8 mathematics content and standards; however, research has revealed significant deficiencies in teachers' understanding of fractions, especially fraction division. The purpose of this case study was to explore how a series of focused professional development workshop-type sessions can impact teacher participants' understanding of and instructional practices for the division of fractions. The study also explored how a workshop model of professional development can be used as an intervention to increase the teachers' pedagogical content knowledge. The participants in this study included seven fifth-grade teachers from the same school district, teaching the *Texas Essential Knowledge and Skills*, and working in the same small Texas town.

Teachers participated in a professional development workshop series that consisted of three similarly formatted workshop sessions which included instruction designed using recommendations from the field of fraction research, discussion, an analysis of student work and incorporated practices that help develop a deeper understanding of fractions by building on students' informal knowledge, creating meaningful contexts, and constructing an understanding based on whole number operations. Data was collected in the form of pre- and post-assessments, interviews, workshop session observations and field notes, written teacher reflections, both a demographic and workshop feedback questionnaire. Each participant was analyzed as a separate case and a cross case analysis was conducted to show common themes.

Results from this case study reveal an increase in teacher participants' understanding of fractions and a positive impact on their instructional practices. In addition, results revealed significant findings related to developing teachers' understanding of the division of fractions, including connecting whole number division with the division of fractions, developing lessons and learning experiences centered on real-world situations and familiar context. This report included implications and recommendations for state agencies, curriculum leaders and professional development facilitators, and teachers. Areas of future research are also identified by the researcher.

Strengthening Pedagogical Content Knowledge in Division of Fractions through a Three-Part Professional Development Series: A Multiple Case Study

by

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CHAPTER ONE

Introduction

Teacher effectiveness is the most dominant factor affecting student achievement. Research suggests that student achievement levels for students with an effective teacher can increase by 53 percentage points in one academic year (Marzano, 2003). This research provides some profound food for thought about how the effectiveness of teachers affects student achievement; however, as additional research also suggests, it is a complex issue.

Teachers represent a wide array of factors that affect their ability to be effective in the classroom. Factors such as teacher preparation, content knowledge, and understanding of pedagogy all contribute to the effectiveness of an educator. Lampert (1990) stated "simply by virtue of having had more education, the teacher represents the most expert knower of mathematics in the classroom and, in this role, has the potential to demonstrate the nature of expertise to those who seek to acquire it" (p. 41).

Content Knowledge, Beliefs, and Teaching Practices

Nonetheless, over the past few decades, there has been a growing body of research that has scrutinized the characteristics of effective teachers. For example, Thompson (1984) studied the relationship between teaching practices and beliefs about mathematics. The results of this study showed that while the two facets were loosely related, no simple relationship existed. In fact, the results of the study showed that a teacher's instructional decisions were based on many factors not just on beliefs alone.

A studied conducted by Charalambous (2015) built on Thompson's (1984) seminal work and explored how a teacher's level of knowledge in and beliefs about mathematics do influence their teaching practices. This study also examined how strengths in one area accommodated for weaknesses in the other. Like Thompson (1984), Charalambous (2015) concluded that an in-depth focus on any one aspect of what teachers bring to the classroom did not provide a complete picture of the factors that influence quality teaching because many factors were at play.

While many studies have focused on connecting a teacher's beliefs to content knowledge, a study conducted by Hill, Schilling, and Ball (2004) provided a lens into elementary school teachers' mathematical knowledge for teaching and examined whether this knowledge resided under one umbrella or whether it was indeed multi-faceted and deserving of separate categories. The researchers also studied how teachers' mathematical knowledge for teaching might more effectively be measured and reported. The findings of this study suggest that mathematics content knowledge consists of more than just the knowledge that a well-educated adult might possess and requires more depth than what can be gleaned from a basic grade-level mathematics textbook. The study also revealed that more research was needed to examine how this knowledge is used and demonstrated in the classroom.

The findings from studies in this area support the work of Lee Schulman who, in the 1980s, identified a missing component of teachers' content knowledge (Rowland & Ruthven, 2011). He and his colleagues labeled this component as pedagogical content knowledge, or PCK. This facet of teacher knowledge is described as both content knowledge and the pedagogical knowledge that is necessary for effective teaching.

In another study, Hill, Rowan, and Ball (2005) examined the relationship between teacher knowledge and student achievement when only the content knowledge of a teacher was considered. Their findings revealed a positive relationship between a teacher's content knowledge and student achievement. The results of this study shed new light on and provided a purpose for using professional development to increase the content knowledge of teachers.

Content Knowledge for Teaching- Rational Number Operations

Rational number concepts represent "the most complex and important mathematical ideas children encounter during their presecondary years" (Behr, Lesh, Post, & Silver, 1983); however, past results of the National Assessment of Education Progress, or NAEP, show "children experience significant difficulty learning and applying rational-number concepts" (p. 91). Behr et al. (1983) contend that while difficulty with rote computational tasks may be surprising due to the persistent classroom use of traditional algorithms and procedures, poor performance on these nationwide tests "may be a direct result of this curricular emphasis on procedures rather than the careful development of important foundational understandings" (p. 92).

Rational numbers and operations on them embody the largest percentage of the Kindergarten – Grade 8 *Common Core State Standards for Math* (National Governors Association & Council of Chief State School Officers, 2010) curriculum. Traditionally, the study of rational number operations begins in Kindergarten, with adding and subtracting whole numbers, and continues through the middle grades. Much like the progression of whole number operations, students begin understanding fraction computations with addition and subtraction and then move to understanding the multiplication and division of fractions in grades four and five.

In grade four, students build on the work they did with basic operations in the primary grades and begin to extend that understanding to complete operations with fractions. In the Common Core Standards, fourth grade students represent fractions as the sum of unit fractions and multiply a fraction by a whole number. In grade five, students continue applying their understanding of addition, subtraction, and multiplication to more complex problems. The study of fraction division begins in grade five. By eighth grade, students use the understanding developed in elementary school to form the basis of their understanding of algebraic expressions and equations as eighth graders are expected to perform operations with whole numbers, fractions, and decimals.

Susan Lamon (2012) states "understanding fractions marks only the beginning of the journey toward rational number understanding" (p. xi). Yet, John P. Smith III states, "no area of elementary school mathematics is as mathematically rich, cognitively complicated, and difficult to teach as fractions" (Smith, 2002, p. 3). While the foundation for understanding fractions first appears in the curriculum in early elementary school, most of the understanding surrounding fraction operations is developed within the middle grades' curriculum. Over the years, however, both the local state curriculum, the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012) and the *Common Core State Standards* (NGA Center & CCSSO, 2010) placed the most significant amount of work with fraction operations in the fifth-grade curriculum. This presents a major challenge for fifth-grade teachers, many of whom may have completed their teaching credentials with an emphasis in reading or another non-mathematics area and who are ill-prepared to teach a conceptual-based unit on fraction operations.

Statement of the Problem

There is little debate that good teaching matters. As the results of a study conducted by Wright, Horn, and Sanders (1994) revealed, the factor that most contributes to student learning is the teacher and improving teacher effectiveness can do more to improve education than any other action. In fact, additional research has shown that while students gain an average of 34 percentage points within an academic calendar year, an effective teacher can increase that to 53 percentage points. Conversely, the results of this study also showed that an ineffective teacher can decrease student gain to just 14 percentage points (Marzano, 2002).

Understanding the knowledge needed to be effective in the classroom is a complicated field of research—one that has been under a microscope for decades. After extensive research in the field, Lee Schulman (1986) and Deborah Ball et al. (2005) defined the work that teachers do in the classroom as mathematical knowledge for teaching and separated it into two distinct domains—subject-matter knowledge and pedagogical content knowledge. They further divided the two domains into sub-categories to more accurately describe the tasks of teaching and the knowledge needed to teach mathematics effectively.

Shulman (1986) described pedagogical content knowledge as a blend of subject-matter knowledge and knowledge of pedagogy. He also described this domain as understanding "the most useful forms of representation of those [content area] ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). As will be discussed further in Chapter Two, the work of Deborah Ball and colleagues refined Shulman's initial category and divided it into three distinct categories, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum, in order to "identify mathematical knowledge that is demanded by the work teachers do" and "needed to perform the recurrent tasks of teaching mathematics to students" (Ball, Thames, & Phelps, 2008, p. 399).

Knowledge of content and teaching, or KCT, was defined by Ball et al. (2008) as "coordination between the mathematics at stake and the instructional options at play" (p. 401) and as a combination of mathematical content knowledge and knowledge of teaching. Knowledge of content and students, or KCS, was defined by Hill, Ball, and Schilling (2008) as "an understanding of what makes the learning of specific topics easy or difficult" (p. 375) and is a combination of mathematical content knowledge and knowledge of students. Knowledge of content and curriculum refers to "knowledge of the available instructional materials" (Petrou & Goulding, 2011).

Mathematical Knowledge for Teaching Rational Numbers

Rational numbers represent the most dominant area of K – 8 mathematics content and standards; however, the number of research studies in the field is not equivalent (Lamon, 2007). In studies conducted by Ball (1990), Borko et al. (1992), Koichu, Harel, and Manaster (2013), Ma (2010), and Zhou, Peverly, and Zin (2006) researchers revealed significant deficiencies in the understanding of fractions in both pre-service and in-service teachers. Even more troubling, several of these studies revealed an inability for teachers to effectively communicate their understanding of fractions or to connect representations to mathematical tasks that included numbers without contexts.

While there have been studies conducted with pre-service teachers who are still earning their teaching credentials (Tirosh, 2000; Li & Kulm, 2008; Chinnappan & Forrester, 2014), there is little research to show how to develop facets of mathematical knowledge for teaching with in-service teachers, specifically those who teach fraction operations, especially division of fractions, in the elementary grades. In addition, little research exists indicating what type of professional development is needed to help teachers strengthen their subject matter knowledge and pedagogical content knowledge.

The purpose of this study was to explore how a series of focused professional development workshop-type sessions can impact teacher participants' understanding of and instructional practices for the division of fractions. The study also explored how a workshop model of professional development can be used as an intervention to increase the pedagogical content knowledge of fifth grade math teachers utilizing the *Texas Essential Knowledge and Skills*, or TEKS (Texas Education Agency, 2012). Providing professional development through a workshop model, a model of instruction where teachers are taught by an expert in the field in a classroom setting (Loucks-Horsley, Stiles, Mundry, Love & Hewson, 2010), has been widely used to offer training to in-service teachers. Additionally, research has indicated that when a workshop model is used to increase teacher knowledge and improve classroom teaching practices, it has a positive effect on student achievement (Darling-Hammond, Wei, Andree, Richardson, Orphanos, 2009; Scher & O'Reilly, 2009).

During the professional development series, the researcher focused on two domains of pedagogical content knowledge within the mathematical knowledge for teaching framework: knowledge of content and teaching and knowledge of content and students. To gain a deeper understanding of knowledge of content and teaching, this study was organized to explore how teachers develop an understanding of fraction division— an operation that is most often taught devoid of conceptual connections and using a rule, invert the second fraction and multiply, without a connection to reason (Ball, 1990; Van de Walle & Lovin, 2006, Van de Walle, Karp & Bay-Williams, 2019).

In their work surrounding big ideas and essential understandings of rational numbers, The National Council of Teachers of Mathematics states, “computation with rational numbers is an extension of computation with whole numbers but introduces some new ideas and processes” (NCTM, 2010, p. 8). Research on fraction computations reveals that the most effective way to teach fraction operations is to build on students’ existing knowledge (Sharp & Adams, 2002) and connect it to whole number operations. Specifically, for the division of fractions, the literature suggests that fraction division is the same as whole number division but requires some different interpretations (NCTM, 2010); therefore, workshop sessions were centered on the development of fraction division via contextual situations and connections to whole number division. In addition, teachers worked toward developing the understanding necessary to think more deeply about fraction operations through conceptual lessons and activities.

Lamon (2007) states that teachers need “to know the particular ingredients of understanding that can facilitate children’s construction of knowledge about that content” (p. 646). To support this notion, the initial professional development session focused on the exploration of the big ideas of fractions as these objectives reflect essential understandings needed for students to be successful understanding fractions. While unofficial, the following ideas represent the most widely discussed aspects of understanding fractions and appear frequently in the literature.

1. Fractions are rational numbers (Lamon, 2007, 2012; NCTM, 2010), written as $\frac{a}{b}$, that represent the non-negative quotient of two integers, a and b, where b does not equal zero.
2. Conceptualizing the unit, or the whole, is key to understanding and operating with fractions (Cramer, Monson, Whitney, Leavitt & Wyberg, 2010; Lamon, 2007, 2012; NCTM, 2010; Neagoy, 2017; Van de Walle et al., 2019).
3. Equal partitioning of the whole (Confrey, 2012; Lamon, 2007, 2012; Smith, 2002) is essential to understanding the concept of fractions.

Research suggests that students can make connections with fraction concepts based on their own experiences and informal work with fractions (Mack, 1990; Mack & Campbell, 1993; Sharp & Adams, 2002; Vukovic et al., 2014). In order to develop this way of thinking about fraction instruction with the teacher participants, workshop sessions were designed based on the idea of developing an understanding of fractions using meaningful contexts, as well as, concrete and visual models (Cramer et al., 2010; Roddick, Silvas-Centeno, 2007; Sharp, Garofalo & Adams, 2002; Smith, 2002; Van de Walle & Lovin, 2006; Van de Walle et al., 2019) to create meaning with fractions. Building on this informal understanding, teacher participants explored the big ideas of whole number division and how they connect to the division of fractions (Ball, 1990; Mack, 1990, 1995; Mack & Campbell, 1993; Sharp & Adams, 2002; Smith, 2002; Van de Walle & Lovin, 2006). Following whole number explorations, participants used contextual situations to attach meaning to traditional algorithms for computations with fractions (Mack, 1995; Smith, 2002; Van de Walle & Lovin, 2006).

The sequence of instruction provided via the professional development series was designed using recommendations from the field of fraction research. Sharp, Garofalo, & Adams (2002) and Van de Walle & Lovin (2006) recommend several practices for the development of meaningful fraction algorithms using a problem-based approach. These recommendations formed the basis of the conceptual framework that was used to guide the workshop series for teacher participants. The framework was supported by existing studies and work highlighting the development of fraction algorithms and computations with fractions.

As an additional component of the professional development series, teachers participated in discussions and analysis of student work to further develop their knowledge of content and students. Because it is important for teachers to be able to identify and diagnose student errors, a portion of the professional development experience was devoted to help participants become more familiar with student errors in fraction division situations, as well as, help them address the issue with solid communication about fraction division and conceptual examples in response.

Workshop sessions also focused on teaching practices that help develop a deeper understanding of fractions by building on students' informal knowledge, creating meaningful contexts, and constructing an understanding based on whole number operations. In addition, teachers engaged in a variety of learning experiences, discussed and analyzed student work, and prepared lessons and activities to use in the classroom.

Based on the research in the field of teacher knowledge, the researcher hypothesized that teacher participants' pedagogical content knowledge would be positively influenced by targeted professional development. The researcher also expected to observe an increase in the teacher participants' understanding of fractions. As a result, the researcher anticipated that student achievement would be positively influenced by the teachers' participation in the workshop series, as well as, their instructional practices. It was assumed that some teachers may not experience much growth because their beliefs prevent them from making instructional decisions aligned with learning gleaned from the professional development experience (Charalambous, 2015). This research study explored the following research questions:

Primary research question:

1.) In what ways does the professional development series impact teacher participants' understanding of the division of fractions?

Secondary research questions:

2a.) How does the professional development series impact teacher participants' instructional practices?

2b.) What aspects of the workshop model of professional development are most effective for teacher learning?

Purpose of the Study

The purpose of this study was to explore how teacher participants' understanding of the division of fractions and their instructional practices could be further developed through a series of focused professional development workshop sessions and used as an intervention to increase the pedagogical content knowledge of fifth-grade math teachers, utilizing the state curriculum standards, the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012), or TEKS, as a guide.

Discussed further in Chapter Two, research supports the use of the following teaching practices for the development of meaningful fraction algorithms: 1.) help students develop fraction concepts through personal contexts and contextual tasks (Burns, 2000; Mack, 1990; Siegler et al., 2010; Sharp et al., 2002; Van de Walle & Lovin, 2006; Van de Walle et al., 2019); 2.) encourage students to use invented algorithms, procedures, and estimation to solve problems (Neagoy, 2017; Siegler et al., 2010; Sharp et al., 2002; Van de Walle & Lovin, 2006; Van de Walle et al., 2019); 3.) allow students to explore the operations using their own language, pictures, and models (Roddick and Silvas-Centeno, 2007; Sharp et al., 2002; Van de Walle & Lovin, 2006).

Through a workshop model of professional development, this study was designed to explore and describe how the use of the teaching practices, as recommended above, impacted the teacher participants' understanding of fraction division and their level of pedagogical content knowledge. What does it mean to possess an understanding of the division of fractions, instead of just knowing the algorithm, and how can a workshop model of professional development be used to influence teachers' knowledge of it?

Teaching Practices for Fraction Computations

Research suggests that students come to school with an informal understanding of fractions based on their own lived experiences and social situations with other children, such as fair-sharing (Empson, 1995; Mack, 1990; Mack & Campbell, 1993; Smith, 2002; Vukovic et al., 2014). Research also suggests that children can use this "intuitive knowledge" to tackle more complex problems that are well-beyond grade-level expectations (Leinhardt, 1988, p. 121). Intuitive knowledge is defined as "applied, real-life, circumstantial knowledge, which can be either correct or incorrect" (Leinhardt, 1988, p. 120). Both Gunderson and Gunderson (1957) and Leinhardt (1988) showed that second grade students were able to solve problems with fractions when the problems were presented in a meaningful context. This supports the recommendation by Sharp et al. (2002) that students' understanding of fractions in meaningful contexts can be used to promote the development of procedural knowledge of operations with fractions.

As children use their informal understanding of fractions to solve fraction problems, research suggests they be encouraged to develop invented procedures to document the processes they are using to solve problems. While these invented algorithms and procedures often do not match the traditional ones (Mack, 1993), they serve an important purpose—they allow children to make sense of the work they are doing in a way that is meaningful for them. NCTM (2014) states that when students build procedural knowledge from conceptual understandings, they ensure they, themselves, understand and can provide a mathematical foundation for the procedures they are utilizing, demonstrate using strategies flexibly, reflect on which strategies are most effective in certain situations, and make generalizations regarding whether specific approaches can be used to solve a specific group of problems. In addition to using informal methods, Petit, Laird, Marsden, and Ebby (2016), Siegler et al. (2010), Van de Walle and Lovin (2006), and Van de Walle et al. (2019) recommend allowing estimation to play a key role in the development of computational strategies for fractions urging that "estimation keeps the focus on the meaning of the numbers and the operations, encourages reflective thinking, and helps build informal number sense with fractions" (Van de Walle & Lovin, 2006, p. 161).

In the same way that children should be encouraged to invent their own algorithms and procedures as they are constructing an understanding using their informal and intuitive knowledge of fractions, students should also be allowed to create their own language, pictures, and models to express their thinking. In an article written by Cramer, Monson, Whitney, Leavitt, and Wyberg (2010), the authors describe a series of fraction lessons conducted with students where they were able to construct, without assistance from their teacher, a method of fraction division using common denominators and pictures. This supports the recommendation by both Sharp et al. (2002) and Van de Walle and Lovin (2006) that pictures "allow crucial issues, such as the importance of common denominators, to become clear at a conceptual level" (Sharp et al., 2002, p. 27) and models provide a familiar setting when students begin working with the more standard algorithms.

A survey of current research in the development of fraction computations demonstrates a lack of a cohesive framework to help teachers develop the skill with their students; therefore, as mentioned earlier, the three recommendations above were used to frame the lessons and activities employed to guide teacher participants through operational thinking surrounding fraction division during the professional development experience and was utilized to emphasize development of operational thinking from these foundational understandings.

Significance of the Study

Much of the prevailing research in teacher knowledge examines the relationship between a teacher's subject matter knowledge and their beliefs or teaching practices (Thompson, 1984; Charalambous, 2015). Several of these same studies also compared teacher knowledge to student achievement (Ball, Hill, & Bass, 2005 and Hill et al., 2005). The findings of these studies indicated that a teacher's instructional decisions were based on more than their beliefs or the content knowledge they possessed. Other studies examined additional aspects of teacher knowledge and revealed the need to better understand how teachers process their instructional decisions so that changes can be made to teacher certification programs and in-service professional development (Hill et al., 2004).

While the field of educational research has explored the interplay of teachers' content knowledge with a variety of teacher behaviors, "scholarly evidence about what PCK is, and how it relates to students' mathematical outcomes, is actually quite thin" (Hill et al., 2004, p. 373). In addition, the field lacks research that examines how a teacher's pedagogical content knowledge in specific areas of mathematics can be developed through grade-level and content-focused professional development experiences.

Furthermore, research has indicated a need for teachers, American teachers, to have a deeper understanding of fractions (Ball, 1990; Borko et al., 1992; Koichu et al., 2013; Ma, 2010; Zhou et al., 2006) as studies in this area revealed several deficits and inconsistencies in fraction understanding and the ability to communicate it accurately and effectively. Several studies have explored how professional development can be used to change teacher knowledge in the area of fractions (Chinnappan & Forrester, 2014; Tirosh, 2000); however, few have been conducted with in-service teachers. Like Ma (2010) and Zhou et al. (2006) stated and as suggested by Hill et al. (2005), Reeder and Utley (2017), and Siegler et al. (2010), these studies revealed a need for teachers to receive professional development while in the classroom so that they will be more effective with their students.

The focus of this study was fifth-grade teachers teaching a specific set of state standards, which, after being removed from the middle grades' curriculum, were implemented four years ago. Until recent years, educator certification requirements, in states like Texas, for fifth grade teachers was nothing more than what their first through fourth grade counterparts obtained. In fact, former President of The National Council of Teachers of Mathematics, Lee V. Stiff (2000), states, "in most states, teachers in grades K-6 are not mathematics specialist, nor have states asked them to be" (paragraph 2). He goes on to say, "40 percent of elementary school and middle grades teachers of mathematics report that they do not feel qualified to teach the content that they teach." (Stiff, 2000, paragraph 2).

According to the Texas Education Agency's (2019) website, in the state of Texas, teachers that choose an EC-6 Core Subjects certification, a general certification for elementary education, is certified to teach early childhood through grade six. A more specified certification for math teachers, Mathematics (Grades 4-8), is also available for educators interested in teaching mathematics in grades four through eight. This means a teacher in grade five may not have had the additional coursework used to prepare for the specialized mathematics certification; they could in fact be a generalist, with no specialized focus area, who received less-specialized instruction across all content areas. Thus, the type of certification obtained may not allow a teacher to be adequately prepared to teach middle grades mathematics content and skills. In addition, the local governing agency has done little widespread work to address this need once teachers have been placed in the classroom and are working with students as curriculum changes are generally addressed at the district level.

Methodology

A qualitative design was appropriate for this study because the focus was to determine how teacher knowledge was impacted with and improved via a professional development workshop, a “natural setting” with which teachers are familiar and usually comfortable and where the researcher was the “key instrument” as the principal observer and collector of multiple forms of data, such as performance tasks, observations, and interviews. This comprehensive data collection process allowed for a more complete picture of each teacher participants’ understanding and development over the course of the professional development series (Creswell, 2018, p. 45).

Participants in the study included seven fifth-grade teachers from the same school district, teaching the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012), and working in the same small Texas town. An exploratory multiple case study research design was used to provide a narrative of each teacher participants’ experiences during the professional development series using examples and observations to demonstrate how, or if, they made impactful changes in their thinking and understanding of division with fractions.

The context of the study was framed by the professional development cohort; however, the study was bounded by curriculum standards, school district, grade level, and participation in the series. The unit of analysis reflected traditional case studies and focused on each individual teacher participant as the case (Yin, 2017). This allowed the researcher to fully depict the story of each teacher's journey through the workshop series and illustrate how their knowledge and thinking evolved, or lack thereof, through the professional development experience.

Multiple forms of data were collected throughout the workshop series to address the research questions guiding the research design and study methodology. These data points include:

- a baseline fraction content knowledge test
- an initial division of fractions performance task to establish each participant's level of pedagogical content knowledge
- follow-up interviews after the initial performance task to probe erroneous or unpredicted responses and gain additional information
- workshop observations and field notes
- workshop reflections
- a final division of fractions performance task, including follow-up interviews
- demographics questionnaire
- workshop feedback questionnaire

A complete discussion of each of the data collection tools, as well as, how they align to this study's research questions is provided in Chapter Three.

After data collection, data was coded, and pattern matching was employed to look for categories and themes. A constant comparison method was then used to analyze the cases for similarities and differences regarding the conceptual development of fraction division and overall changes in knowledge from the pre-workshop to post-workshop performance task. A cross-case analysis was also conducted to generalize about changes among the cohort, inform others how a similar workshop series may be used with teachers in other districts across the state, and offer recommendations for changes that might improve the effectiveness of the intervention.

Both the pre-workshop and post-workshop performance tasks included mathematical tasks designed to assess the essential tasks of teaching which define the domains of knowledge of content and teaching (KCT) and knowledge of content and student (KCS). As identified by Ball et al. (2008) and Hill et al. (2008), the tasks include:

- Sequencing content for instruction (KCT)
- Selecting examples (KCT)
- Evaluating the advantages and disadvantages of certain representations (KCT)
- Determining the most effective examples to use (KCS)
- Conceptions and misconceptions, including common errors, surrounding a topic (KCS)
- Understanding what students find easy and difficult about a topic (KCS)

Definitions

Knowledge of content and students (KCS): a category within the mathematical knowledge for teaching domain of pedagogical content knowledge that refers to understanding how students learn mathematics and how to teach mathematics where teachers anticipate how students will think about a concept and what may cause confusion, select the most appropriate examples to use, and understand the misconceptions and erroneous thinking surrounding the concept (Ball et al., 2008); this category also includes understanding the facets of each concept or skill that make it easy, or difficult, for students to learn (Hill et al., 2008)

Knowledge of content and teaching (KCT): a category within the mathematical knowledge for teaching domain of pedagogical content knowledge that is a blend of understanding the important elements of teaching and the knowledge of mathematics content, which includes understanding the design of mathematics instruction, how content is sequenced during instruction, and how to appropriately select examples and representations, as well as, understanding the advantages and disadvantages of each selection (Ball et al., 2008)

Mathematical knowledge for teaching (MKT): an over-arching term that encompasses both subject-matter knowledge and pedagogical content knowledge and refers to all the mathematical knowledge teachers need to carry out the tasks of mathematics teaching and the demands of each task (Ball et al., 2008)

Pedagogical content knowledge (PCK): a domain of mathematical knowledge for teaching (Ball et al., 2008) that is thought to be a special blend that “goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching” (Schulman, 1986, p. 9), which includes the following categories, knowledge of content and teaching, knowledge of content and students, and knowledge of content and curriculum (Ball et al., 2008)

Subject matter knowledge (SMK): a domain of mathematical knowledge for teaching (Ball et al., 2008) that is specific to content knowledge and refers to the amount of knowledge and the way it is organized within a teacher's schema (Schulman, 1986), which includes three distinct categories, common content knowledge, specialized content knowledge, and horizon content knowledge (Ball et al., 2008).

Limitations of the Study

The research portion of this study was conducted over three workshop-type sessions in an instructional setting where teachers are removed from the classroom. Therefore, how the content was transferred to students was beyond the scope of this study and limited whether this study could be described as useful for impacting students' thinking and understanding of fraction division.

In a paper reviewing nine professional development studies, Yoon, Duncan, Lee, Scarloss, and Shapley (2007) found that studies with more than 14 contact hours "showed a positive and significant effect on student achievement" (p. 19). Due to circumstances outside of the researcher's control, the professional development experiences were limited to the availability of the teachers as determined by their campus principals; therefore, workshop sessions were limited to a total of 10.5 hours of seat time.

Loucks-Horsley et al. (2010) contend that stand-alone workshops "may fall short of providing a well-rounded professional development experience" (p. 265) and urge that the stand-alone model should be combined with another, such as classroom observations or demonstration lessons, for a more effective learning experience. While the researcher continued to provide support for study participants and district officials throughout the school year, this work was beyond the scope of the current study and results from these additional experiences were not recorded in this report; however, participants self-reported how they anticipated their instructional practices would be impacted by session activities and discussions.

The sample group included seven teacher participants from the same school district who have similar demographics; therefore, the results may not be generalizable to a larger population of in-service teachers in Texas where participants are included from multiple districts and represent a wider range of demographics.

While Texas standards somewhat mirror those of the *Common Core State Standards for Math* (NGA Center & CCSSO, 2010), there are slight differences. Because the instruction delivered during the professional development series focused on Texas math standards, the results may not be transferable to teachers utilizing a different set of standards or those teaching in schools outside of Texas.

Additionally, because the researcher was also the professional development facilitator, discrete conversations and moments during the professional development series may have been missed. While sessions were recorded to capture critical discussions, the researcher recognizes that separate individuals may have been able to focus on different aspects and capture more of the workshop experience. In addition, biases held by the researcher may have impacted the design of the professional development experience.

Conclusion

Research involving both pre-service and in-service teachers reflects a need to provide additional support for the development of teachers' mathematical knowledge for teaching, specifically when it relates to transferring and communicating that knowledge to others. Fractions, an area most impacted by instruction that is detached from meaning, weaves its way through the K-8 curriculum standards. This topic is a critical component for understanding algebraic relationships. In order to support teachers' understanding of these essential concepts, it is important to consider how professional development may be used to support changes in teachers' understanding of the development of fraction computations. The literature lacks an abundance of studies that provide successful ways to do this. This study sought to explore how content-focused professional development sessions could be used to impact teacher knowledge in this area that would, in turn, influence teachers' work with students. Chapter Two provides a detailed review of the literature in the areas of teacher knowledge, division of fractions, and professional development.

CHAPTER TWO

Literature Review

The educational landscape of the 20th century was a tug-of-war dominated by the voices of traditionalists, behaviorists, like Edward Thorndike, and constructivists, like Jean Piaget. While the turmoil surrounding how to approach the education of students in America placed a heightened awareness on how learning occurred in schools, the groundwork for the way students would learn and teachers would teach in the 21st century began to emerge.

In the mid-19th century, the prevailing belief about the role of education was rooted in a traditionalist view. At this time, education was viewed as a method for transmitting the important ideas surrounding the cultural heritage and societal norms of the time. Student learning was orchestrated by a teacher, who, using a "lecture-recitation" (Posner, 2004, p. 45) type format, emphasized facts gleaned from reading a textbook. Viewing education through this lens led to a more "traditional teacher-dominated learning process" (Pines, 1982, p. 103).

In the 1950's, the emergence of and frenzied panic over the fall-out from World War II and the aeronautical advancements of other countries during the space race changed the focus of school curriculum and educational objectives. Teaching and learning in schools began to emphasize more science and mathematics through "disciplines of knowledge and the way scholars in those disciplines under[stood] their structure" (Posner, 2004, p. 53). Within this school of thought, "education at all levels was regarded as crucial to the achievement of national goals." Through the "structures of the disciplines" lens, a strong focus on math and science education led to a curriculum that emphasized inquiry-based learning.

After the mathematics and science-dominated 1950's and 1960's era, behavioral psychologists insisted that the focus of education should not be on the content that students are learning but on the behaviors that result from the learning. The focus of education then became how students acquired specific behaviors, or the conditions of learning. The result of this view of curriculum and instruction lead to standards that were chiseled down to a series of steps or observable behaviors that had to be mastered before a student could move forward. This school of thought perpetuated a procedural view of mathematics where curriculum was divided into objectives to be taught by the teacher.

In the second half of the 20th century, the work of Swiss psychologist Jean Piaget began to take center stage. Piaget theorized that "the mind both assimilates new ideas into an existing structure and also accommodates new ideas by reorganizing this structure" (Posner, 2004, p. 61). This idea became the basis for constructivism. Unlike the traditionalists, who viewed people as recorders of information, emphasizing rote learning and memorization, constructivists saw people as constructors of knowledge and emphasized thinking and sense-making.

Constructing Knowledge and the Role of the Teacher- Two Critical Lenses

Much of the work that is done in schools today is based on the basic idea of constructivism, that people are builders of knowledge. In fact, both the current *Common Core State Standards* (NGA Center & CCSSO, 2010) and the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012) include standards where students are expected to do more than just memorize facts and formulas. They are expected to construct meaning from the content which they are learning. In order to become successful learners of mathematics, students must have a teacher who is not only knowledgeable about mathematics content and skills but also knowledgeable enough about instruction in order to support students as they attempt to make sense of the learning. This issue of teacher effectiveness was highlighted by the work of two important figures- Lev Vygotsky and George Polya.

Lev Vygotsky's theory of cognitive development, where he theorized that children learn from a more knowledgeable person, i.e. the teacher, supports the notion that what the teacher knows and can do is essential to the success of the students under his or her charge (Vygotsky, 1978). In addition to understanding that learning occurs via a more knowledgeable person, Vygotsky also theorized that learning occurs when students are in the zone of proximal development, where the level of instruction supports the mental processes that are about to be developed. The role of the teacher, as viewed through Vygotsky's lens, reveals, as the more knowledgeable person, the teacher must have the tools to scaffold learning and support students in a way that will help them develop the skills necessary to become successful learners.

In 1945, George Polya highlighted the ideas of Vygotsky, when he advised that when students are solving problems, support that is provided in the form of questions at the student's level of development "challenges the curiosity of . . . students by setting them problems proportionate to their knowledge," "helps them to solve their problems with stimulating questions," and "give[s] them a taste for, and some means of, independent thinking" (Polya, 1945, p. v). Through the lens of George Polya, the teacher is not meant to be the dispenser of knowledge but more of a construction mentor helping the student understand what to do and how to use the learning tools he or she has acquired after years of learning.

Studies, such as the one conducted by Wright, Horn, and Sanders (1994), reveal that teacher effectiveness is the most important factor regarding student achievement. The work of both Lev Vygotsky and George Polya highlights the importance of an effective and well-prepared teacher who can help students construct knowledge. Furthermore, with this greater emphasis on the teacher as the facilitator of student construction of knowledge, a need to examine the knowledge teachers bring with them into the classroom begins to surface.

Understanding Teachers' Patterns of Behavior

Alba Thompson (1984) examined how the "teachers' conceptions (their beliefs, views, and preferences) about the subject matter and its teaching play an important role in affecting their effectiveness as the primary mediators between the subject and the learners" (p. 105). In Thompson's study, the cognitive and metacognitive processes of three high school math teachers were observed while teaching in the classroom and then subsequently interviewed about their instructional practices. The findings of this study showed that the relationship between the teacher's conceptions and their instructional behaviors was complex and that those conceptions "play[ed] a significant . . . role in shaping the teachers' characteristic patterns of instructional behavior" (p. 183). More specifically, study findings showed that the way in which material was presented aligned closely with the professed conceptions of the math teachers included in the study. Additionally, the teachers' instructional decisions were affected by aspects of teaching that were not related to mathematics knowledge and content, such as “the social and emotional make-up of their class” (p. 183). Moreover, this study revealed that no simple relationship between beliefs and teaching practices existed.

The New Paradigm

On the heels of the research conducted by Thompson (1984) and after conducting their own research in the area, Lee Schulman (1986) and his colleagues found a need for “a more coherent theoretical framework” to better understand “the complexities of teacher understanding and transmission of content knowledge” (p. 9). To address this need, Shulman developed an organized system divided into categories of knowledge for teaching. This system separated teacher knowledge into seven categories: general pedagogical knowledge, knowledge of learners’ characteristics, knowledge of educational context, knowledge of educational purposes and values, content knowledge, curriculum knowledge, and pedagogical content knowledge. Shulman’s work focused on the last three categories, labeled as “the missing paradigm in research on teaching,” as such, pedagogical content knowledge is considered the “most influential” (Petrou & Goulding, 2011, p. 11) of the three.

Shulman (1986) described pedagogical content knowledge as a blend of subject-matter knowledge and knowledge of pedagogy and as “the most useful forms of representation of those [content area] ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others” (Petrou & Goulding, 2011, p. 9). In addition, Schulman (1986) stated that pedagogical content knowledge includes a teacher’s insight into the factors that cause the learning of a specific topic or skill to be simple or problematic, understanding the knowledge and misconceptions that students bring with them into the classroom setting, and knowledge of techniques, strategies, and methods that will most help students' brains reorganize and restructure the content and skills in a way that sets them on the path to understanding.

Digging Deeper—Refining Mathematical Knowledge for Teaching

Since Shulman (1986) first introduced his organizational system of facets underpinning teacher knowledge, researchers struggled to understand this blended new form. In a study conducted by Hill et al. (2004), researchers examined the relationship between mathematical knowledge for teaching and teaching practice. The results of this work led to in several important findings. First, the mathematical knowledge needed for classroom teaching consists of more than the common knowledge held by "any well-educated adult" (p. 27). Second, the mathematical knowledge needed to teach a grade level extends beyond the content knowledge needed to teach the grade-level's content and skills. Third, "how an individual holds and uses . . . knowledge to generate representations, interpret student work, or analyze mistakes" (p. 27) rather than the amount of mathematical content knowledge they possess should be the focus of future research. Lastly, the specialized knowledge needed to teach should continue to be examined.

Two decades after Shulman introduced pedagogical content knowledge, Ball, Thames, and Phelps (2008) saw a need to subdivide Shulman's subject matter knowledge and pedagogical content knowledge in order to "identify mathematical knowledge that is demanded by the work teachers do" and "needed to perform the recurrent tasks of teaching mathematics to students" (Ball, et al., 2008, p. 399). Figure 2.1 illustrates this new organization.

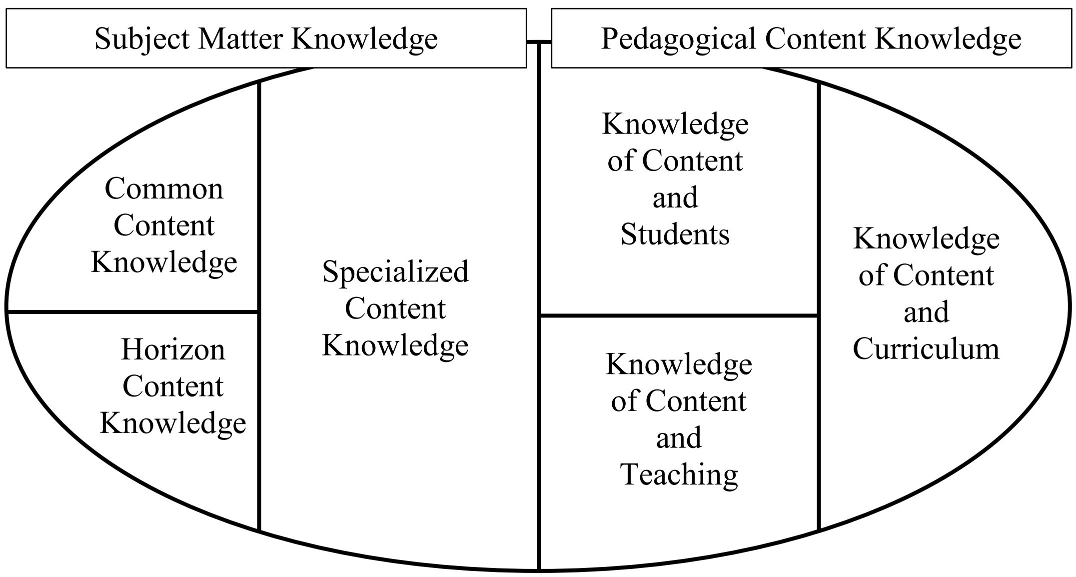


Figure 2.1. Mathematical knowledge for teaching as conceptualized by Ball et al., 2008, p. 403.

Subject Matter Knowledge

In this model, Ball et al. (2008), divided Shulman's (1986) subject matter knowledge into three categories: common content knowledge, horizon content knowledge, and specialized content knowledge. Common content knowledge is defined as the mathematical understanding held by those outside of the teaching field. Horizon knowledge is defined as the understanding of the mathematics that is on the horizon. In other words, teachers need to be aware of how the mathematics they teach is related to the mathematics students will learn in other grade levels in order to make appropriate instructional decisions. Ball et al. (2008) paid special attention to specialized content knowledge and defined it as the knowledge unique to the field of teaching. In addition, it is described as “an unpacking of [the] mathematics” needed to perform “the everyday tasks of teaching” (p. 400).

Pedagogical Content Knowledge

Shulman’s (1986) pedagogical content knowledge was divided into three separate categories as well to reflect the interplay between knowledge of pedagogy and content knowledge: knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. Ball et al. (2008) defined knowledge of content and students as a blend of the two areas where teachers can anticipate how students will respond to the way specific skills are presented, “how they are likely to think,” and “what they will find confusing” (p. 9).

Knowledge of content and teaching is a blend of knowing about teaching and knowing about the content of mathematics and emphasizes understanding “mathematical knowledge of the design of instruction” (Ball et al., 2008, p. 401). For example, knowledge of content and teaching involves how teachers sequence the components of instruction, choose initial examples and distinguish those that elicit deeper thinking, evaluate the effectiveness of different representations, and determine which methods and procedures are best (p. 401).

The last category, knowledge of content and curriculum, is based on Shulman’s (1986) initial organization of teacher knowledge and is consistent with his definition that defines this knowledge as understanding the “full-range of programs” (p. 10) available for teaching particular content, the instructional resources and materials available, and “the set of characteristics that serve as both indications and contradictions” (p. 10) present in available programs.

The Mathematical Knowledge of Teaching

This new system of organizing teachers' mathematical knowledge for teaching sheds light on the processes that teachers need to possess to be effective classroom teachers. Thames and Ball (2010) took an in-depth look at the mathematics teachers used when teaching. Through this work, they discovered that teachers needed more than just content knowledge to adequately address students' questions and that the "math understanding [displayed] involved . . . posing questions, interpreting students' answers, providing explanations, and using representations" (p. 222). In addition, their work found that teachers "require[d] an abundance of mathematical skill and of usable mathematical knowledge—mathematical knowledge in and of teaching" (p. 223). Table 2.1 shows the most frequently observed teacher tasks in the classroom (Ball & Thames, 2010, p. 233):

Table 2.1

Frequent tasks of teaching

▪ Posing mathematical questions	▪ Analyzing student errors
▪ Giving and appraising explanations	▪ Appraising students' unconventional ideas
▪ Choosing or designing tasks	▪ Mediating a discussion
▪ Using and choosing representations	▪ Attending to and using math language
▪ Recording mathematical work on the board	▪ Defining terms mathematically and accessibly
▪ Selecting and sequencing examples	▪ Choosing or using math notation

Thames and Ball (2010) stated, "teaching is not merely about doing mathematics oneself, but about helping students learn to do it" (p. 228). When considering the tasks in Table 2.1 above, the mathematical demands of a teachers' job are shown to be complex and multi-faceted. Being able to understand and identify these demands allowed researchers to identify and understand the "mathematical knowledge needed for teaching" (p. 228).

The research of Ball et al. (2005) identified several of the same tasks from Table 2.1 and stressed that "each of these common tasks of teaching involves mathematical reasoning as much as it does pedagogical thinking" (p. 21). In addition, they added that "the teacher has to think from the learner's perspective and to consider what it takes to understand a mathematical idea for someone seeing it for the first time" (p. 21).

Effects of Teacher Knowledge

Like Thompson (1984), other studies have labeled the relationship between teacher knowledge and teacher beliefs as a complex one. Charlambous (2015) studied how teacher knowledge and pedagogical beliefs relate to teaching practices. The findings of this study revealed that having a deficit in either teacher knowledge or pedagogical beliefs can interfere with the effect that the other facet had on teaching practice. In addition, the study revealed that when pedagogical beliefs were found to align closely with a standards-based teaching approach, one based on conceptual understanding and reasoning, teachers had the "propensity to work in ways that [were] conducive to structuring mathematically rich environments" (p. 441).

In an article which discusses the mathematical knowledge needed for teaching, Thames and Ball (2010) assert,

> To differentiate teachers' levels of mathematical knowledge, numerous studies have examined whether a teacher has a certification in math or a degree as well as the number of math courses taken. But analyses of the correlations between these indicators and students' achievement gains reveal no advantage at the grades K-8 level and only slight advantage at the secondary level" (p. 221).

However, studies (Ball et al., 2005; Hill et al., 2005) have shown a correlation between content knowledge and student achievement. In fact, the results of a study with 700 elementary school teachers and 3,000 students revealed a positive relationship between mathematical content knowledge, measured by answering items correlated to both common and specialized mathematical knowledge, and student achievement (Ball et al., 2005; Hill et al., 2005). In fact, their results showed what would be the equivalent of two to three weeks of extra instruction that would have been gained by the students of the above-average teacher when compared to the average teacher. Additionally, the researchers added that the difference between the average teacher and the above-average teacher "was comparable to the size of the effect of socioeconomic status on student gain scores" (Ball et al., 2005, p. 44).

Relating Subject Matter Knowledge to Pedagogical Content Knowledge

Ma (2010) stated "limited subject matter knowledge restricts a teacher's capacity to promote conceptual learning among students" (p. 36). She went on to say a "strong belief" in teaching for understanding "cannot remedy or supplement a teacher's disadvantage in subject matter knowledge" (p. 36). However, the work of Charlambous (2015) emphasized that "strong knowledge alone cannot ensure that teachers engage in work that lends itself to promoting mathematically rich environments" (p. 442).

A study conducted by Hill et al. (2008) found that teachers with high mathematical knowledge for teaching displayed many affordances, or possibilities, to positively affect the "classroom culture and instruction" (p. 433) and a low number of deficits, or errors, in instruction. Conversely, teachers with low mathematical knowledge for teaching displayed low affordances and a high number of deficits. The study findings also revealed that other factors, including beliefs, played a part in this relationship. Examining whether a relationship existed between a teacher's affordances and/or deficits and student achievement was beyond the scope of this study and was identified as a study limitation by the authors.

These studies support the notion that teaching is a complex job. While no simple relationship exists, there is indeed a connection between subject matter knowledge and pedagogical content knowledge. The studies also reveal a need to even the playing field and provide opportunities for all teachers to obtain the knowledge to successfully fulfill the role of a mathematics teacher.

Using Professional Development to Improve Teacher Knowledge

The study conducted by Ball et al. (2005) also included a professional development component. The goal was to determine whether mathematical knowledge for teaching could be gained by teachers in a more traditional professional development session via a summer workshop. These results showed that content knowledge for teaching did increase among the teachers after participation in the summer workshop. Researchers also reported seeing "greater performance gains on [the] measures . . . related to the length of the institutes and to curricula that focused on proof, analysis, exploration, communication, and representation" (p. 45). Based on these findings, researchers concluded that when "true measures of teacher and student learning (p.45)" are used rather than perceptual ones, a more universal understanding about how teachers learn mathematics can be constructed.

A study conducted by Hill and Ball (2004) utilized both a content-focused pre-test and post-test to anchor a series of professional development sessions. The focus of the study was centered on increasing the mathematical knowledge for teaching of in-service teachers in the number and operations strand of mathematics. The findings of this study revealed that "the opportunity to engage in mathematical analysis, reasoning, and communication can improve teacher knowledge" (p. 346).

In terms of the potential impact professional development workshops can have on teachers' instructional practices and decisions, a report based on reform efforts in the California school system showed that when in-service teachers participated in workshops that involved the mathematics curriculum used with students, there was an increase in classroom practices promoted by the training opportunities, such as extending investigations, initiating mathematical discussions about problems and solutions, and writing in mathematics (Cohen & Hill, 2004). In addition, the report also revealed a connection between the length of the workshop and the number of reform practices used in the classroom with students, longer workshops equated to a greater use of reform practices.

Components of Effective Professional Development

Research conducted by Borko (2004) offers additional support for the use of professional development to impact teacher knowledge. In a review of studies that utilized professional development experiences to improve teaching and learning, evidence exists which reveals the use of "high-quality professional development programs can help teachers deepen their knowledge and transform their teaching" (p. 5). After a study of individual teachers, Borko (2004) offers the following recommendations surrounding effective professional development:

- Opportunities for teachers to engage in activities as learners, such as solving mathematical problems, are effective.
- The inclusion of opportunities to observe student thinking increased teachers' awareness of the role of student thinking in the learning process and the importance of building on student understanding and misconceptions after careful listening.
- Teachers attempted to incorporate teaching practices learned in their professional development workshop into their classroom lessons and activities.

Ball, Lubienski, and Mewborn (2001) state that professional development for teachers, in its current state, "lacks consistency and coherence" (p. 437); however, the findings of Borko (2004) offer some sound recommendations for creating more effective professional development experiences for educators.

Using a Workshop Model of Professional Development

Loucks-Horsley et al., (2010) define workshops as "structured opportunities for educators to learn from facilitators or leaders with specialized expertise" (p. 260) where specific learning goals are selected for study. In addition, the components included in Table 2.2 have also been identified as effective features of workshops and provide professional development facilitators with a road map of critical elements to include in professional development opportunities for teachers.

Table 2.2 offers recommendations for *how* to deliver content to teachers during professional development workshops (Loucks-Horsely et al., 2010). In the paragraphs that follow, the researcher will address the *what* of the professional development series through an analysis of the current field of research.

Table 2.2

Effective features of workshops

▪ Establishes a clear vision and goals for a targeted group of people	▪ Participants develop products that reflect their learning
▪ Offers quality content that is valuable and addresses participants' learning goals and learning styles	▪ Provides opportunities for explorations and to address real-world problems
▪ Includes opportunities for participants to interact with each other	▪ Establishes a community of respect where all ideas are valued, and participants feel safe to try new things
▪ Uses each moment to the fullest	▪ Participants receive support and feedback from expert facilitators
▪ Session evaluations and participant input are used to design workshop content and adjust workshop content	▪ Connects new knowledge to participants' current knowledge and expertise
▪ Participants are provided with resources to take home with them	

Understanding Fractions

Meaning of a Fraction

Often, children see fractions as having two distinct whole-number parts (Siegler et al., 2010), rather than as a number. This is problematic because it does not allow children to see the relationship that exists between the numerator and denominator. "Fractions name the relationship between the collection of parts and the whole, not the size of the whole or its parts" (Smith, 2002, p. 8). Furthermore, a fraction refers to an "underlying rational number" (Lamon, 2012, p. 30), a single number, that illustrates "the relative amount conveyed by those symbols" (p. 30).

An Informal Understanding of Fractions

Research (NCTM, 2000; Van de Walle et al., 2019) in effective teaching practices suggests that fraction instruction should begin with connecting everyday situations with simple fractions that originate from authentic real-world problems, "starting with the common fractions expressed in the language [students] bring to the classroom" (NCTM, 2000, p. 82). For example, when you ask two young children how to share one cookie, they will most likely tell you to cut the cookie in half. While they may not understand the full meaning of a half, they can tell you to cut the cookie down the middle and give one half to each of them. This implies at least an informal understanding of fractions that all children bring with them into the classroom that they have developed from experiences in their world (Empson, 1995; Mack, 1990; Mack & Campbell, 1993; Smith, 2002).

After a five-week unit on fractions with a group of first graders, research conducted by Empson (1995) revealed that first graders were able to solve equal-sharing problems based on their intuitive understanding of fractions. Thus, they were building a conceptual understanding of fractions because they had no procedural understanding on which to rely. In fact, one of the first-grade teachers, who had taught fifth grade before teaching first grade, remarked that her students "understood fractions better than many children whose instruction is based on simply learning symbolic procedures" (p. 114). Studies such as those conducted by Mack (1990) and Empson (1995) provide support for the use of children's informal understanding of fractions to give meaning to procedures with symbols and algorithms.

Children's' Development and Understanding of Fractions: Four Considerations

Research suggests that fraction instruction needs to closely match the way in which children develop their understanding of fractions. Based on this research, there are four considerations that emerge involving children's development and understanding of fractions: 1.) Conceptual understanding of fractions should be developed before computations and algorithms are introduced (Burns, 2000; Cramer, Post, & delMas, 2002; Mack, 1990; NCTM, 2014; Sharp et al., 2002); 2.) Fraction instruction should include models and emphasize the unit and its equally partitioned parts (Cramer et al., 2002; Lamon, 2012; NCTM, 1989; Peck & Jencks, 1981); 3.) Instruction should present problems in the context of real-world situations instead of contrived scenarios (Burns, 2000; Mack, 1990; NCTM, 1989; Sharp et al., 2002; Van de Walle & Lovin, 2006); 4.) Instruction should emphasize the use of multiple representations, such as physical, concrete and pictorial models, numbers, and symbols (Burns, 2000; Cramer et al., 2002; NCTM, 1989; NCTM 2014; Sharp et al., 2002). The sections that follow discuss each consideration in more detail based on research in the field.

Conceptual Understanding of Fractions

In order to build a conceptual understanding, fraction instruction should include opportunities to build on students' informal knowledge of fractions and a variety of manipulative models, pictures, diagrams, and real-world objects. Procedural knowledge should be developed over time (NCTM, 2014) and students should not be rushed into the use of algorithms until a conceptual understanding has been ascertained (Sharp et al., 2002). Furthermore, as students develop procedural fluency, their ability to approach fraction operations will become more flexible (Van de Walle et al., 2019).

One way to help students build this conceptual knowledge base is to build on the informal knowledge students bring into the classroom. In a study conducted by Mack (1990), she used instructional sessions and interviews with eight sixth-grade students to determine how they were able to use their informal knowledge base to support their understanding of fraction symbols and procedures. Mack noted that students came to instruction “with a rich store of informal knowledge about fractions” that they were “able to build on to give meaning to formal symbols and procedures” (p. 29).

Mack (1990) also noted that students’ informal knowledge had limitations and was “disconnected from their knowledge of symbolic fractions and procedures” (p. 29). These results provide support for the use of instruction to fill the gap. Fraction instruction should begin with students’ informal knowledge and then their understanding should be deepened using concrete models, including manipulatives, and diagrams. In addition, Burns (2000) asserts students should “deal with fractions concretely” (p. 223) before they move on to representing them symbolically.

In a study of 66 fourth and fifth grade classrooms conducted by Cramer et al. (2002), the effect on student achievement was evaluated after students received instruction using either a commercial curriculum, like a traditional textbook, or lessons from the Rational Number Project, or RNP. Teachers were randomly assigned to one of the two groups and provided fraction instruction with the given curriculum for 28-30 days. While both study textbooks included some references to manipulative models and pictures, they “played only a cursory role in the development of fraction ideas” (Cramer et al., 2002, p. 120). The development of "student competence at the symbolic level" (Cramer et al., 2002, p. 120) was the primary goal. The Rational Number Project curriculum used various manipulatives, pictures, and real-life contexts to allow students to "explore mathematical ideas in multiple ways" (Cramer et al., 2002, p. 118) before they connected the representations to symbols. In addition, the lessons required students to connect fraction representations to multiple models, both continuous and discrete. After assessing students with written tests, the study revealed significant differences in the achievement level of students using the two different programs. The study showed significant differences in four areas: 1.) conceptual understanding, 2.) comparison of the relative sizes of fractions, 3.) ability to estimate sums and differences, and 4.) transference of knowledge to tasks that were not explicitly taught. In addition, the students who received the RNP curriculum showed an overwhelming difference in their use of conceptual models to make sense of the problems on the written assessment. The results of this study provide strong support for curriculum and instruction that develops students' understanding of fractions in a conceptual way.

Emphasis of the Unit

For students to understand the unit and its equal parts, fraction instruction should include a.) activities that require the student to determine the unit; b.) activities that provide opportunities for students to practice partitioning the unit into equal parts; c.) models that are continuous and discrete. NCTM (1989) emphasizes "the concept of a unit and its subdivision into equal parts is fundamental to understanding fractions" (p. 57). When presented with a fraction model, the first question students need to answer is "What is the unit?" (Lamon, 2012, p. 98). Students who are unable to recognize the relationship between the unit and its equally partitioned parts will struggle to make sense of fraction concepts. Lamon (2012) states that students must learn to work with units of different types and understand that the unit they choose does not affect the solution.

Cramer et al. (2002) conclude that instructional learning activities "should involve fractions that are easily modeled" and "the importance of the unit and its subdivision into equal parts should be emphasized" (p. 112). In a study conducted by Donald Peck and Stanley Jencks (1981), they noted that nearly all the children interviewed knew the denominator indicated how many parts into to which to subdivide the whole; however, less than half demonstrated that the parts had to be equal in size. These results reveal that fraction curriculum must provide a multitude of activities where students partition units into equal parts. These experiences should also include opportunities for students to identify pictures that are partitioned equally (Smith, 2002).

Lamon (2012) adds that students should be fluent working with various types of models, including: one or more continuous items, such as a circle or rectangle; continuous items that are prepartitioned, such as a candy bar or set of stamps; discrete objects, such as a set of playing cards or crayons; and composite units, such as a single package with multiple items inside, like cans of soda. It is essential that fraction instruction include a myriad of models and opportunities for students to identify the unit and divide it into equal parts.

Real-world Situations

In order to emphasize fractions in the real-world, fraction instruction should include age-appropriate real-world situations students may have previously experienced and real-world, or contextual, situations students have enough prior experience to understand (Burns, 2000; Mack, 1990; Sharp et al, 2002; Van de Walle & Lovin, 2006; Van de Walle et al., 2019). In the study conducted by Mack (1990), she concluded that while the students had “misconceptions related to their knowledge of fraction symbols and procedures” (p. 22), they had a “substantial store of informal knowledge about fractions” (p. 22) that afforded them the ability to respond to tasks when presented in the context of real-world situations. When problems are presented in the context of real-world situations, students can relate what they know and understand about the world to the problem in order to make sense of it (Sharp et al, 2002).

Burns (2000) advocates that fraction instruction build on the experiences students have encountered previously in their lives. This allows young children to develop an authentic use for fractions. For example, returning to the cookie-sharing scenario, children know a single cookie must be cut in order to share it with another child. They may have difficulty describing the fractional part of the cookie they will receive as one-half, but they know the pieces should be the same size for fair-sharing. Early grades teachers can build on this knowledge and expand the situation to sharing with three or four children and help students understand the partitioning of the unit before naming the quantities.

NCTM (1989) states “all fractions at the K-4 level should involve fractions that are useful in everyday life, that is, fractions that are easily modeled” (p. 57). This statement suggests that instruction at this level include situations children encounter in their everyday worlds. In addition, classroom teachers can provide opportunities for students to identify fractions in the classroom and surrounding school areas.

Multiple Representations

NCTM (2014) recommends the use of using multiple representations to help students foster a deeper understanding of mathematics and improve their problem-solving skills; therefore, students' knowledge of fractions should reflect a deep understanding of the "connections among symbols, models, pictures, and context" (Cramer et al., 2002, p. 112). Petit el al. (2016) and Siegler et al. (2010) assert using visual representations helps students make sense of fraction concepts and operations with fractions. In addition, for students to be able to transfer knowledge between representations, fraction instruction should include physical, concrete and pictorial models, numbers, and symbols, as well as, opportunities for students to demonstrate solutions using more than one representation.

Furthermore, as revealed in a study conducted by Gupta and Wilkerson (2015), when fraction instruction incorporated manipulatives, physical objects manipulated by hand, students' understanding of fraction concepts was positively impacted. These results provide strong support for the seamless use of various fraction models during instruction regarding fractions (Van de Walle et al., 2019).

Cramer et al. (2002) suggest the results of their study, comparing the achievement of fourth and fifth grade students who used the Rational Number Project curriculum with those who used the commercial curricula, indicate students should be exposed to multiple representations, including "multiple manipulative models" (p. 138), to better help them develop initial fraction understanding over an extended period of time. The use of multiple models, both commercially produced and teacher- or student-made, allows students flexibility in demonstrating their knowledge. In addition, the use of more than one model is beneficial so that students do not develop an incorrect affiliation with a particular-shaped object. For example, Burns (2000) suggests that if children's early fraction experiences only deal with manipulative models that are round, children may incorrectly generalize that fractions are in fact round.

Sharp et al. (2002) assert that students' "use of pictures enable them to understand and resolve situations" (p. 27) that they may have trouble understanding without a model. Student-invented pictures not only allow students to make sense of the problem in a way that permits them to determine the solution, they also allow students to demonstrate their understanding to someone else. The picture then becomes the record of their solution strategy.

NCTM states beginning fraction instruction "needs to emphasize oral language and connect . . . to the models" (NCTM, 1989, p. 57). They add that symbolic language of fractions such as one-fourth and six-fourths, "should be introduced only after children have developed the concepts of oral language necessary for symbols to be meaningful" (NCTM, 1989, p. 58) and then connected carefully to the model and language associated with the meaning of these fractions.

Teacher Knowledge and the Division of Fractions

As stated in Chapter One, multiple studies have indicated a need for American teachers, to have a deeper understanding of fractions (Ball, 1990, 1993; Borko et al., 1992; Koichu et al., 2013; Ma, 2010; Zhou et al., 2006) as studies in this area revealed several deficits and inconsistencies in fraction understanding, including the division of fractions, and the ability to communicate it accurately and effectively.

As part of Ma's (2010) research on what it means to learn and teach mathematics, she asked a group of American and Chinese teachers to solve the following problem:

$$1\frac{3}{4} \div \frac{1}{2}$$

Ma's (2010) research revealed that responses were widely varied. Of the 21 American teachers she interviewed, nine were able to correctly use the algorithm and complete the answer, which included simplifying the quotient to lowest terms. Two teachers correctly used the algorithm but were unable to simplify the answer to lowest terms. The remaining 10 teachers displayed a variety of errors and misunderstandings, including displaying an incomplete algorithm and having an incomplete answer, demonstrating a fragmented memory of the algorithm and producing no answer, or using the wrong strategy and producing no answer. All 72 of the Chinese teachers used a correct algorithm and produced a correct answer.

While 43% of the American teachers were able to successfully determine the answer to the division problem, none were able to produce a representation of the task. During the interviews, Ma (2010) noted the following misconceptions: confusing division by one-half with division by two and confusing division by one-half with multiplication by one-half. Ma's (2010) research also revealed another concern. As teachers created and solved their story problem representations and arrived at a different solution than the mathematical task yielded, they were unable to rectify the discrepancy. Based on these observations, Ma concluded that because the teachers did not have a conceptual understanding of division of fractions, "their computational knowledge was limited and flimsy" (p. 69).

Zhou et al.'s (2006) work echoed the results of Ma (2010) and revealed that when provided with an assessment to evaluate the subject matter knowledge of American teachers, fraction problems involving basic concepts, computations, and word problems "were difficult for the American teachers, but not for the Chinese teachers" (p. 446). When combined with an evaluation of teachers' pedagogical content knowledge, Zhou et al. (2006) concluded there were "deficits in U.S. teachers' knowledge of fractions (SMK) and their ability to communicate their fraction knowledge to students (PCK)" (p. 453). Based on this research, researchers concluded that the quality of the training in-service teachers receive should be improved. Zhou et al. (2006) also contend that teachers should be treated as learners and that instruction should focus of teaching teachers the mathematics needed to be successful in the classroom rather than focusing on how to teach the content.

Van de Wall et al. (2019) assert, "division of fractions remains one of the most mysterious algorithms in K-8 mathematics". They go on to say, "we want to avoid this mystery at all costs and help students really understand when and how to divide with situations involving fractions" (p. 394). Coupled with the results of studies like Ma (2010) and Zhou et al. (2006), the need to examine how teachers have come to know fractions and operations with them emerges. The following sections discuss the research on what it means to understand division of fractions, how children develop this understanding, and issues surrounding making sense of division with fractions.

The Meaning of Division with Fractions- Common Concepts

Based on her interviews with teachers, Ma's (2010) research revealed common concepts and skills the teachers used to justify and explain their thinking regarding the division of fractions task. These common concepts included: meaning of multiplication with fractions, meaning of multiplication with whole numbers, meaning of division with whole numbers, the conception of inverse operations, meaning of addition, concept of unit, and concept of fraction. This “knowledge package” (p. 77), and the relationships between the concepts, is illustrated in Figure 2.2.

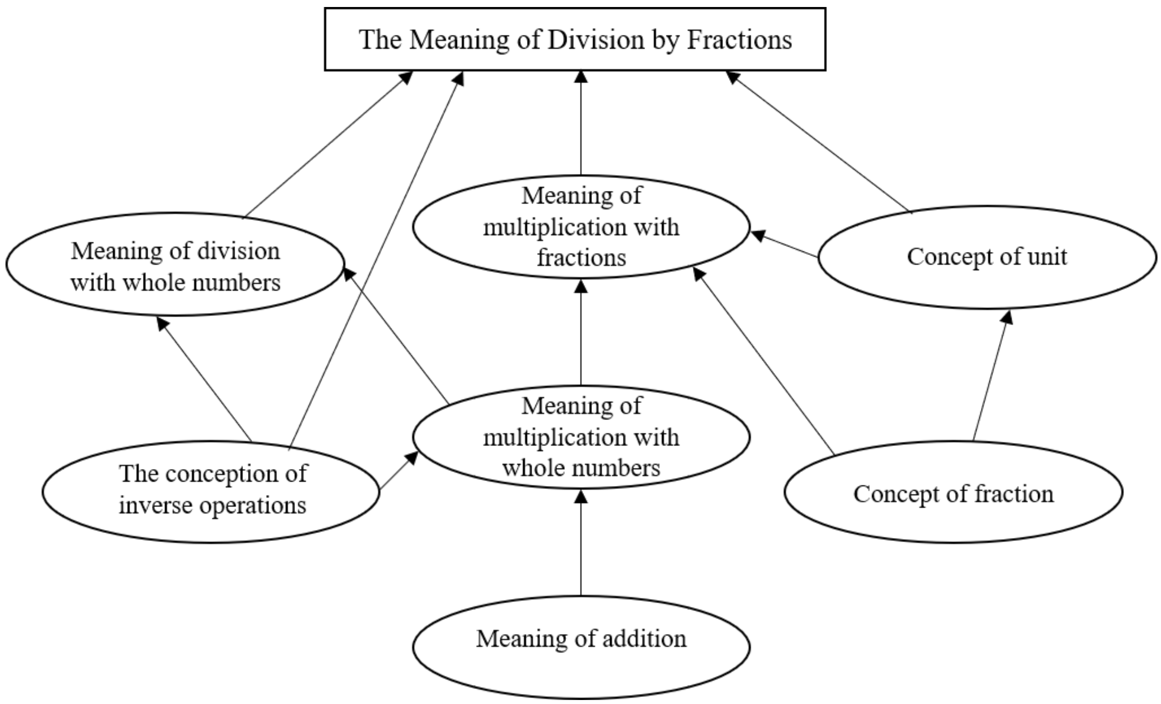


Figure 2.2. Division of fractions knowledge package (Ma, 2010, p. 77).

It is clear from Ma’s (2010) research and Figure 2.2. that some of the teachers attempted to make connections to multiplication with fractions to make better sense of the fraction task on which they were evaluated. The diagram illustrates that there is not a single path to understanding the division of fractions. Instead the diagram reveals that there are multiple, complex paths to understanding division with fractions and that many concepts are interrelated; however, Ma’s (2010) research suggests that when connections between concepts are made, our understanding of them is reinforced and deepened. While the Chinese teachers were able to make solid connections between the concepts, many of the American teachers fell short as they attempted to make connections by relying on incomplete or erroneous ideas and misconceptions which ultimately lead to failure to successfully represent the initial task in three ways, with an equation, model, and contextual situation.

Issues in Understanding Division with Fractions

Connecting Whole Number Division to Division of Fractions

In both the *Common Core State Standards* (NGA Center & CCSSO, 2010) and the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012), division of whole numbers first appears in grade three, usually proceeding multiplication. These initial understandings are related to representing division situations, using fact families, making connections to multiplication, and dividing within 100.

Understanding division involves two situations, partitive and quotative (Coughlin, 2011; Dixon & Tobias, 2013; Lamon, 2012; Petit et al., 2016). Partitive division, commonly called sharing, involves taking the total, or dividend, and dividing it into a specific number of equal sets (Van de Walle & Lovin, 2006). The second situation, commonly called measuring, involves taking the dividend and dividing it into groups of an equal size.

Studies have shown that when children first begin making sense of division with fractions, they make connections, albeit sometimes inappropriate, to division with whole numbers (Mack, 1995; NCTM, 2010, 2013; Sidney & Alibali, 2017). In a study conducted by Mack (1995), findings revealed that when third grade students were presented with fraction problems in a symbolic form, they incorrectly thought of the whole numbers as fractional parts, such as converting the whole number one from the expression 1 - $\frac{3}{4}$, into one-fourth. Conversely, when students were provided with a fractional computation embedded in a contextual situation, they were able to achieve the correct response.

In addition, the results of the same study (Mack, 1995) revealed that students use their prior knowledge of symbolic computations with whole numbers to make sense of symbolic computations with fractions. Although this research shows that students tend to overgeneralize their knowledge of computations with whole numbers to solve fraction computations, NCTM (2000), Van de Walle and Lovin (2006), and Van de Walle et al. (2019) recommend using whole number computations to help students make connections to the meaning of computations with fractions.

Modeling Fraction Division

Research suggests that creating models for fraction computations is a necessary skill for students to solidify the concept (Sharp et al., 2002; Van de Walle & Lovin, 2006). However, in the study conducted by Ma (2010), not one of the U.S. teachers was able to create a pedagogically correct representation for the fraction division task even though 43% of the teachers were able to arrive at the correct solution. Ma (2010) contends that an inadequate understanding of computational procedures for the division of fractions impeded teachers' understanding of the meaning of the operation. On the other hand, 65 of the 72 Chinese teachers were able to generate 80 representations for the fraction division task, including a mix of three different models.

While U.S. education emphasizes two division models, partitive and quotative, a review of middle grades textbooks reveals an abundance of quotative tasks and a limited number of partitive tasks for the division of fractions (Van de Walle & Lovin, 2006).

Although partition problems seem to be the more popular representation for students when working with whole number division or whole-number divisors in fraction division problems, Van de Walle and Lovin (2006) argue that “the sharing concept appears to break down when the divisor is a fraction” (p. 173). In quotative fraction division tasks, students are measuring a specific fractional quantity and counting the number of quantities obtained; however, the thinking behind the process is more complicated for partitive tasks.

Interviews with the Chinese teachers in the study conducted by Ma (2010) shed some light on the break down in understanding. In a whole number division partitive situation, the amount of several groups is known. The question then is how many in each group. When dividing by a fractional amount in a partitive situation, the amount for a fractional group, or part of a whole, is known, so the question becomes how much is one (Van de Walle & Lovin, 2006). Although the change in wording is slight, the change in meaning is difficult to conceptualize and connect to whole number division. This may contribute to the challenge of fully understanding division with fractions.

Concept of Unit— in Division Situations

Another issue that confounds understanding fractions is how to handle remainders. In whole number division, the remainder indicates a specific number of whole units that are remaining; however, in fraction division, the remainder refers to a fraction of a unit remaining. While students seem to have no trouble interpreting remainders in whole number division situations, interpreting remainders in fraction division problems, especially in quotative situations (Petit et al., 2016) seems to be a bit more challenging.

Philipp and Hawthorne (2015) state that one of the reasons students struggle with fraction division problems is because they do not understand the referent unit. Van de Walle et al. (2019) identify students recording “remainders based on the whole, rather than the share/serving” (p. 403) as a common misconception for dividing fractions. When working with remainders in fraction division tasks, students must be able to identify to what the remainder is referring, to part of the divisor or part of a whole. Consider the following task:

> A baker has $3\frac{1}{4}$ cups of sugar. She needs $\frac{3}{4}$ of a cup of sugar for each batch of cookies. How many batches of cookies will she be able to make?

While many students will arrive at a correct solution of 4 batches of cookies with one-fourth cup of sugar remaining, understanding how to interpret the one-fourth “highlights a common difficulty experienced by students” (Philipp & Hawthorne, 2015, p. 243)—does it refer to one-fourth cup of sugar or one-fourth of the amount of sugar needed for a recipe?

Philipp and Hawthorne (2015) argue “to make sense of fraction operations conceptually, one must attend explicitly to the units and to the fact that multiplication and division require one to conceptualize multiple units and view a quantity in two ways” (p. 243). Returning to the example above, because the question asks how many batches of cookies can be made, the remainder would be expressed as one-fourth cup of sugar, which is equal to one-third of the recipe.

To remedy this, Philipp and Hawthorne (2015) urge that when working with fraction division problems, it is important to be explicit about the unit, or whole. They add that this instructional implication should begin as soon as fraction instruction begins.

In other words, instead of just referring to a fractional amount, as a practice, teachers should include the unit in their reference to the fraction.

Invert and Multiply—the Traditional Algorithm

Research in the area of fraction computations indicates the importance of students developing number sense with fractions instead of beginning the unit of study with lessons on using common denominators and traditional rules for fractions (Neagoy, 2017; Van de Walle & Lovin, 2006). Like Ma's study (2010) revealed, a survey of American educators will reveal an overwhelming use of the traditional algorithm for the division of fractions. However, Van de Walle and Lovin (2006) state that "invert the divisor and multiply is probably one of the most mysterious rules in elementary mathematics" (p. 173).

In a study conducted by Borko et al. (1992), an observation of a lesson by a student teacher revealed that the teacher was unable to provide a concrete representation or justification for the traditional invert and multiply algorithm. This lack of understanding raised a myriad of concerns for researchers whose observation of the teacher led to a study of pre-service teachers' "knowledge, beliefs, thinking, and actions related to the teaching of mathematics" (p. 199). This classroom vignette revealed an alarming area of concern like that discovered by Ma (2010). While predominant in U.S. education, use of the traditional invert and multiply algorithm lacks connection to the meaning of division of fractions and often leads to an incomplete understanding of the skill.

Conclusion

Division with fractions is arguably the most difficult of the fraction computations. As discussed in Chapter One, research indicates a needed for teachers to have a deeper understanding of division with fractions (Ball, 1990; Borko et al., 1992; Koichu et al., 2013; Ma, 2010; Zhou et al., 2006). Understanding how teacher knowledge can be developed for in-service teachers is an area where much research is still needed to better understand and address the issue; however, several studies have shown success using professional development workshops to improve teacher knowledge. The purpose of this exploratory multiple case study was to explore how a workshop model of professional development can be used to improve teacher knowledge and impact instructional decisions in the area of division of fractions. Chapter Three provides an overview of the methodology for the current study.

CHAPTER THREE

Methodology

The purpose of this exploratory multiple case study was to explore how an understanding of the division of fractions could be developed through a series of focused professional development workshop-type sessions and used as an intervention to strengthen the pedagogical content knowledge and impact the instructional practices of fifth-grade math teachers utilizing the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012), or TEKS. As mentioned in Chapters One and Two, prior studies noted the need for more in-service professional development for teachers (Ma, 2010; Zhou et al., 2006); therefore, in this study, the researcher explored how teacher participants' understanding of the division of fractions changed over the course of three workshop sessions. All aspects of the study's research methodology are reported in this chapter. Through the next sections, the researcher describes the research design and rationale, the setting, the participants, data collection procedures and analysis, and reliability and validity considerations for the current study.

Research Questions

This study was designed to explore how a series of professional development workshops could be used to impact teacher participants' understanding of and instructional practices for the division of fractions. The research questions that focused this study were:

Primary research question:

1.) In what ways does the professional development series impact teacher participants' understanding of the division of fractions?

Secondary research questions:

2a.) How does the professional development series impact teacher participants' instructional practices?

2b.) What aspects of the workshop model of professional development are most effective for teacher learning?

Research Design and Rationale

The research design selected for this study was an exploratory multiple case study design. Similar studies (Charalambous, 2015; Hill et al., 2008; Thompson, 1984) designed to better understand teacher knowledge and beliefs utilized a case study research design to provide in-depth descriptions of teacher participants. This study was suited to a case study research design because the cases, the teacher participants, were "clearly identifiable" and the aim of the study was to "provide an in-depth understanding of the cases" (Creswell, 2018 p. 100).

According to Yin (2017), a case study method is preferred when the main question under investigation is a "how" question, the researcher is unable to control the behavioral events, and the focus of the study is a contemporary issue (p. 2). The purpose of the study was to explore how, in what ways, in-service classroom teachers' understanding of fractions and pedagogical content knowledge in the area of division of fractions could be developed through and potentially changed by a series of professional development workshop sessions; therefore, an exploratory multiple case study design was appropriate (Yin, 2017).

An exploratory multiple case study research design was used to highlight the experiences of each teacher participant during the professional development workshop series, as each teacher was a unit of analysis, or case (Yin, 2017). A cross-case analysis was also conducted to compare the experiences of each case and generalize about changes, or lack thereof, among the cohort.

Participants and Site

The major focus of this study was to explore how a deeper understanding of the division of fractions was developed and improved through a professional development workshop series. In order to best understand this phenomenon, criterion sampling techniques were used to select teacher participants for this study because cases were selected based on specific criteria. The school district selected for this study was chosen because they volunteered to participate, and district officials were willing and able to provide time for their teachers to attend the half-day workshop sessions and participate in study activities.

While the participants in the workshop series included general education teachers, as well as, Special Education teachers and an instructional coach, participants in the study included only the seven fifth-grade classroom teachers. Each of the fifth-grade teachers taught math in a departmentalized setting using the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012) and worked in a small Texas town. Teacher participants were from the same public school district, located in Central Texas, and were employed at either of the two intermediate schools that housed students in the fifth and sixth grade.

Due to the small size of the school district, there were only seven fifth grade teachers. All fifth-grade teachers participated in the study.

The selected district has just under 8,000 students with the following demographic breakdown: Economically Disadvantaged, 30.44%; Special Education, 7.63%; African American, 11.88%; Hispanic, 23.42%; White/Anglo, 57.01%; Asian, 4.65%. These demographics are somewhat reflective of the state data; however, state testing data, from the Texas Education Agency, for all students, just over 5,000,000, for the 2017-2018 school year reflects a Hispanic population of 52% and a White/Anglo population of 28.8%. The percentage of students that meet the criteria for economically disadvantaged in the state of Texas is at 58.5%.

Protection of Human Subjects

This research study used data gathered from assessments and performance tasks, observations, interviews, teacher reflections, and questionnaires. Prior to the collection of data, permission was obtained from the Institutional Review Board (IRB) at the researcher's university of study. A copy of the approval paperwork was placed in Appendix A. To protect the identity of the teacher participants and to conceal names of the campuses on which they work, pseudonyms have been used. While increasing student achievement is an anticipated effect and a goal of increasing teacher knowledge in the area of the division of fractions, no student data was collected or utilized for this study. A copy of the consent form is available in Appendix B.

Workshop Series

The purpose of this study was to explore how teacher participants' understanding of the division of fractions could be impacted and further developed through a series of focused professional development workshop-type sessions. Therefore, the workshop series was designed to provide the most effective learning experience for the participants.

Workshop sessions occurred during the fall at approximately one-month intervals. Each session included three and a half hours of seat time and a similar format of activities and discussion. In addition to increasing the pedagogical content knowledge of the participants, workshop sessions also aimed to:

- provide opportunities for participants to solve conceptual and contextual fraction division problems and engage in a variety of activities as learners, like the ones they may use with their students
- increase participants' awareness of the role of student thinking
- help teachers translate the work from the workshop sessions to their work with their students
- assist teachers in preparing for their upcoming fraction division unit

Rational for Workshop Format

Traditionally, mathematics instruction surrounding fraction operations has involved the memorization of procedures and processes. Specifically, the teaching of fraction division has included widespread use of the invert and multiply procedure, which means to invert the second fraction and then multiply both fractions together (Neagoy, 2017). As mentioned in Chapter One, this process is typically taught devoid of meaning and without connection to other fraction concepts.

The worldwide research of Jo Boaler (2018) and colleagues reveals that students who memorize facts and methods are among the lowest-achieving students. Consequently, her research also reveals that countries with high numbers of memorizing students were among the lower achieving nations. Interestingly, her work also indicates "students who combined self-monitoring and relational strategies outscored memorizing students by more than a year's worth of schooling" (Boaler, Munson, Williams, 2018, p. 4).

The study of mathematics is conceptual based; therefore, "it is important for students to be thinking slowly, deeply, and conceptually about mathematical ideas, not racing through methods that they try to memorize" (Boaler et al., 2018, p. 5). Boaler et al.'s (2018) research in understanding how the brain processes mathematics indicates that new information takes up a great deal of space in the brain as it processes where it fits within the existing schema and connects to other concepts and ideas; however, over time, as our understanding develops, the knowledge becomes compressed and takes up a smaller amount of space in the brain.

Many students, and adults alike, dislike mathematics because they are unable to understand mathematics the way mathematicians do, as "a subject of depth and connections" (Boaler, 2016, p. 32). Boaler et al. (2018) argues this is because the brain is unable to compress methods, only concepts. Therefore, if students are only exposed to methods and memorization, they will not develop the necessary components to become successful mathematical thinkers.

As teachers, it is important to provide students with opportunities to experience mathematics where they must "think deeply and conceptually" (Boaler et al., 2018, p. 5).

The main goal of this study was to positively influence teacher knowledge; therefore, workshop sessions included activities which could be used in the classroom in order to model the rationale for and importance of their use with students.

In her book *Mathematical Mindsets* (2016), Jo Boaler states,

> Teachers are the most important resource for students. They are the ones who can create exciting mathematical environments, give students the positive messages they need, and take any math task and make it one that piques students' curiosity and interest (p. 57).

Because teachers are a valuable resource for students and past studies have shown that beliefs play a role in the knowledge that teachers bring with them into the classroom (Charalambous, 2015; Thompson, 1984) it is important that they too receive positive messages about mathematics and participate in activities that will provide those messages for students. For this reason, workshop sessions included activities from Boaler et al.'s (2018) *Mindset Mathematics* series. A discussion surrounding the way these tasks are designed to address mindset is included later in this section.

Workshop Format

Each workshop session followed a similar format and included the following major components:

- Overview of learning objectives and essential questions
- Introduction of three big ideas surrounding the session topic
- Quick mini lesson to develop a shared understanding of the content
- Exploration and discussion regarding student thinking with a Who's Correct task
- Introduction of each session activity with an engaging contextual situation

- Time to explore and make sense of each session activity with another participant
- Discussion of each activity and how it can be modified for use in the classroom

Design of Workshop Tasks

Boaler (2016) suggests that in order to provide students with space for learning and help them develop mathematical mindsets, curricular resources must be redesigned and adapted to be more open. As teachers design and adapt tasks, Boaler (2016) recommends addressing the following aspects of the tasks to increase their power (pp. 77–90):

- encourage multiple methods, pathways, and representations
- make it an inquiry task
- assign a problem/task before teaching the methods needed to solve the problem
- add a visual component
- create a low floor and high ceiling task
- add a component where students must convince and reason

The activities used in the workshop series addressed the recommendations above in the following ways:

- encouraged participants to “think visually” and “connect visual and numerical representations” to promote engagement and support connections in the brain (Boaler et al., 2018, p. 13)
- provided opportunities for participants to utilize their varied experiences to make sense of the tasks

- allowed participants to play with the math and devise their own methods and procedures
- offered specific problems to complete but allowed participants to select their own problems to complete as well
- asked participants to explain responses, reason about solutions, and convince others in the group that their reasoning was correct

For continuity, each session was comprised of activities that included the components listed above. This process not only modeled how to create rich tasks that open the mathematical learning for participants but also provided opportunities for maximum discussion and potential to change teacher knowledge and mathematical mindsets.

Data Collection

Creswell (2018) states "a hallmark of a good qualitative case study is that it presents an in-depth understanding of the case" (p. 98). To accomplish this goal and search for consistency among the findings, this research study included several forms of data using a process called triangulation to create validating evidence (Creswell, 2018; Yin, 2017). This section will provide a detailed explanation of each type of data that was collected during the data collection process. For this study, data was collected from an initial fraction content knowledge test, initial and final workshop performance tasks, interviews, workshop observations, regular reflections from the participants, and exit questionnaires. Copies of the data collection tools are available in Appendices C, D, and E.

Fraction Content Knowledge Test

According to Hill, Sleep, Lewis, and Ball (2007), a widely used method for investigating teachers' mathematical knowledge for teaching is via the use of mathematical tasks and interviews; therefore, several assessments were administered to "understand the nature and extent of teachers' knowledge" (p. 127) at both the onset and conclusion of the workshop series. As a reference point to support the study's primary research question regarding the impact of the professional development series on teacher knowledge, teacher participants completed a fraction content knowledge test, which was piloted for length and comprehensibility with a different group of educators before being administered to the cohort.

The problems on this test were taken from *Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures* (Hill & Ball, 2004) and the *State of Texas Assessment of Academic Readiness*, or STAAR, (Texas Education Agency, 2019). Each item was selected because it reflected a big idea of fractions, as described in Chapter One, or it assessed teacher understanding of fraction division. Item one assessed whether participants could conceptualize the unit. Items three, four, and five assessed the participants' ability to partition equally and items two, six, seven, and eight assessed participants' ability to divide fractions. A copy of the fraction content knowledge test is available in Appendix D.

Pre-Workshop Performance Task

At the onset of the professional development series, a baseline for understanding the knowledge regarding teaching the division of fractions that each teacher participant brought to the experience was established using a pre-workshop performance task.

Follow-up interviews were used to gain a deeper understanding of participants' responses on the assessment items.

The pre-workshop performance task, which was piloted with a different group of educators for length and comprehensibility before being administered to the cohort, included mathematical tasks designed to assess the essential tasks of teaching which are exemplified within the domains of knowledge of content and teaching (KCT) and knowledge of content and student (KCS). As identified by Ball et al. (2008) and Hill et al. (2008), the tasks included sequencing content for instruction, selecting examples, evaluating the advantages and disadvantages of certain representations, determining the most effective examples to use, conceptions and misconceptions, including common errors, surrounding a topic, and understanding what students find easy and difficult about a topic.

While the performance task included items which would be appropriate for students to complete, the assessment went beyond grade-level expectations to "develop a deeper and more comprehensive view" of the division of fractions (Conference Board of the Mathematical Sciences, 2012, p. 23). The purpose of this assessment was to evaluate situations that may arise in the classroom setting during a division of fractions unit, such as analyzing student errors and answering questions that may be posed by students (Hill et al., 2007). A copy of the questionnaire is available in Appendix C and a detailed explanation of each assessment item follows.

Explanation of Performance Tasks

Question 1. The first question assessed the teacher participants' ability to sequence content for instruction and was adapted, to include specific combinations of whole numbers, mixed numbers, and fractions, from a similar task from the released items of *Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures* (Hill et al., 2004). Teachers were provided with a set of contextual situations and asked to sequence the tasks in the order they felt would be best to support student understanding of the division of fractions. The purpose of this task was for teacher participants to demonstrate an understanding of an instructional sequence appropriate for how students develop an understanding of the division of fractions. For example, beginning with a task which involves whole numbers and moving to a task with non-unit fractions would illustrate moving from a task that requires whole number thinking to more specific fractional reasoning for division.

While teacher participant responses were expected to differ, an appropriate sequencing of the tasks included task b, a, c, d. Task b, which illustrated division with whole numbers, was the best place to begin because elementary students are accustomed to partitive (sharing) problems and can intuitively solve these problems without having a full understanding of the procedural knowledge surrounding the division of fractions (Empson, 1995; Van de Walle & Lovin, 2006). Because task a required measuring quantities of one-fourth, a unit fraction, it is easier for students to conceptualize and they may be able to draw a picture with ease. Task c was like task b but used a non-unit fraction. Because it involved using three-fourths, students may be more easily able to make connections to the task that was completed using one-fourth. Task d was a partitive task and students could rely on their understanding of whole numbers to divide the total number of miles. However, students may struggle with what to do with the remaining mile and a half which requires them to divide the quantity into eights to share it equally among the four days.

Question 2. The second question assessed teacher participants' ability to evaluate the advantages and disadvantages of certain representations and demonstrate an understanding of what students find easy and difficult about the division of fractions. It was adapted, to include specific values, from a similar task from the released items of *Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures* (Hill et al., 2004) and modified to create an opportunity for participants to invent their own problem and justify their response. Teachers were provided with a contextual task that included a non-unit fraction and asked to create a simpler version of the task to prepare the students for solving the target task. The purpose of this task was to determine whether teacher participants could select a task which would be easier for students, i.e. a task with a unit fraction rather than a non-unit fraction. Knowledge of content and teaching also refers to the ability of teachers to articulate the advantages and disadvantages of using certain representations. This task was designed to help better understand a teacher's ability to do this for this topic of instruction.

While participant responses were expected to vary widely, the best way to prepare students to respond to the target task would be to use a problem where a whole number is divided by a unit fraction because students need only to attend to how many pieces they are able to create with each whole instead of separating them into single parts and then grouping them to create the divisor. It was also anticipated that teacher participants may indicate that dividing by a unit fraction is easier for students than dividing by a non-unit fraction. It is important to note that dividing by non-unit fractions is beyond the scope of both the *Common Core State Standards* (NGA Center and CCSSO, 2010) and *The Texas Essential Knowledge and Skills* (Texas Education Agency, 2012) for grade 5; however, teachers would be expected to apply the reasoning used to complete the unit fraction to the task with non-unit fractions. This task also assessed whether teachers could sequence tasks for level of difficulty, which was also assessed with the first task; therefore, the response to this task was compared to the response for the initial task.

Question 3. The third question assessed teacher participants' ability to identify the conceptions and misconceptions, including common errors, surrounding the division of fractions, and determine the most effective examples to use. This item was adapted from the released items of *Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures* (Hill et al., 2004) to create a similar formatted question where participants were asked to identify an error made after reviewing student work. Teacher participants were provided with a student response to a division of fractions problem and asked to identify the error the student was making. Participants were then asked to devise a follow-up task and representation to help the student mitigate the misconception. The purpose of this task was to determine whether teacher participants could diagnose student errors and select examples to use to alleviate the errors.

Ball et al., (2008) states that teachers who possess knowledge of content and students are familiar with common student errors surrounding a topic and proceed with appropriate instructional decisions to correct the misconception. When selecting a representation to use to follow-up on the misconception, “a teacher needs to be able to map each element of the computation to the representation” (Hill, Sleep, Lewis, & Ball, 2007). In this task, the student is simply dividing a dividend of 20 by a divisor of four, not one-fourth. An appropriate follow-up task may be to ask the student to count the number of halves in a quantity of 20 and then record the task with appropriate symbols so that the student can use the understanding gained when the divisor is one-half and then apply that thinking to a problem with a divisor of one-fourth.

Question 4. The fourth question assessed teacher participants’ ability to sequence content and select examples for instruction. This item was modeled after a similarly formatted task from the released items of *Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures* (Hill et al., 2004) which required selection of examples that could be used to develop a variety of strategies for comparing fractions. Participants were provided with several tasks to use with their students and asked to identify the task that would best help students understand how to divide fractions and lead to the development of an understanding of the traditional algorithm used for dividing fractions. After selecting a task to use to support students’ intuitive thinking regarding division, teacher participants were expected to create a contextual situation to illustrate the problem. The purpose of this task was to help students develop an intuitive method for the traditional fraction algorithm. Using both a dividend and a divisor with common denominators would allow students to divide more easily.

While it was anticipated that teacher participants may make a case for the use of any of the three, Task A included common denominators and could be used to help students develop a method for dividing with fractions because the division situation connected more to the thinking they use when they divide whole numbers. That is, make groups of the divisor using the dividend. The quotative, or measurement, model is used to help students connect the task to a contextual situation (Van de Walle & Lovin, 2006) and can be a steppingstone to move from pictorial models to a more algorithmic approach.

Question 5. The fifth question assessed teacher participants' ability to select examples for instruction and was a duplicate of the task used in Ma's (2010) study. Teacher participants were provided with a fraction division problem and then asked to create a contextual situation surrounding the problem. Participants were then asked to write an equation and create a model that could be used to represent the task. The purpose of this task was to determine whether teacher participants were able to create multiple representations for a division problem, including a corresponding contextual problem, model, and equation.

Ma (2010) states that for teachers to be able to have enough pedagogical knowledge to represent a situation, they must "be able to comprehensibly understand it" (p. 83), an area of concern that resulted from her research. For this task, responses indicated whether the teacher participant knew what to represent and whether the model matched the situation represented in the contextual problem, i.e. a partitive or quotative situation. The equation indicated whether the teacher participant was able to achieve the correct solution.

Post-Workshop Performance Task

Like the pre-workshop performance task, a post-workshop performance task was administered to compare participants' knowledge and thinking at the conclusion of the workshop series with their knowledge and thinking at the onset. The questions and tasks of this culminating assessment mimicked the structure and content of the initial task. This allowed for a more direct comparison of participant thinking and changes in level of knowledge. A copy of the performance task is available in Appendix C.

Interviews

Like the way researchers used follow-up interviews to address erroneous responses in a study conducted by Graeber, Tirosh, and Glover (1989), interviews with teacher participants were used to follow-up on incorrect answers or answers which seemed inconsistent with other responses recorded on the baseline assessments. During the interview of teacher participants, a series of questions was devised to obtain additional information surrounding a response to a mathematical task. To prepare for each interview, an analysis chart was created to document participant responses, comments regarding errors and inconsistences, and follow-up questions to ask during the interview. To address responses which were considered incorrect, like question 5, teacher participants were asked to explain their response and how the solution may be verified for accuracy and reasonableness (Graeber et al., 1998) using the other representations. For example, if there was inconsistency among a participants' equation, model, and contextual situation, the researcher might ask, "How can you use your model to check your solution?" or "Does your situation match your model?". This method allowed questions to be tailored to the responses given by each individual teacher participant.

In addition, interview questions were used to gain more information about a response. For example, if a teacher participant selected an order which included using a scenario with a unit fraction before a scenario with a non-unit fraction on question 1 but chose not to use a scenario with a unit fraction before introducing the task in question 2, follow-up questions were used to gain a better understanding of the teacher participants' intended instructional decisions.

Observations and Field Notes

To add additional support to and substantiate the evidence for the analysis of each case, participant observations were conducted during each workshop session. Participant observations were used because the observations were recorded by the professional development facilitator, who is also the researcher. Observations included insights of teacher participants during group discussions, as well as, actions observed during workshop activity sessions. These observations were then used to explain and support results from the fraction content knowledge test, performance tasks, interviews, and workshop reflections.

To collect observations, workshop sessions were video recorded using multiple pieces of audio-visual equipment that captured small group discussions, as well as, large group happenings. Videos were then analyzed and a modified transcription process was used. This process included organizing videos by session, reviewing videos, and recording participant responses and questions during small group work and large group activities and discussion. Specific pieces of content or participant comments and questions were recorded along with a timestamp indicating the session number, video number, and approximate time of the action or question/response. This data served as field notes for each session of the professional development series.

Written Teacher Reflections

Throughout the series, teacher participants completed regular written reflections and recorded responses to over-arching workshop goals. Written reflections included explanations regarding how, or even if, participants felt their schema about the division of fractions had changed and how the sessions impacted their teaching. Specifically, participants were asked to reflect on the goals of each session (illustrated in Table 3.1) and then demonstrate their understanding in pictures, words, and/or numbers.

Reflection questions remained consistent throughout the professional development series and included the following: 1.) Refer to the objectives for today's session. Choose one goal and explain how this session has added to your professional repertoire in pictures, numbers, or words. 2.) Describe one activity we did today that you felt was most impactful. How will you use this knowledge with your students in the classroom? 3.) What questions do you still have about today's objectives? What are you wondering now?

Exit Questionnaires

Two exit-questionnaires were administered to gather background information about the teacher participants' teaching experience and feedback on the impact of the workshop format. To provide an in-depth analysis of each teacher participant, the information gleaned from the exit questionnaires was used to add context to the data gathered from the pre- and post-workshop assessments, follow-up interviews, observations and field notes from the workshop sessions, and written reflections. A copy of the questionnaires is available Appendix E.

Table 3.1

Summary of workshop session objectives and essential questions

Workshop	Objectives	Essential Questions
Workshop #1	• Explore the relationship between part and whole by creating visual fraction models with pattern blocks • Understand the importance of identifying the unit and equal partitioning	• What is a fraction? • Why is it important to be able to identify the unit? How was the unit represented differently across the models? • Why is equal partitioning important? How can we emphasize this with our students?
Workshop #2	• Discriminate between partitive and quotative situations • Connect the big ideas of whole number division to the division of fractions • Explore the division of fractions through contextual situations and visual models	• What does it mean to divide? • What is the difference between partitive (sharing) and quotative (measuring) situations? • How does whole number division connect to the division of fractions?
Workshop #3	• Use invented procedures to make connections to traditional algorithms for the division of fractions • Investigate models for the division of fractions, including the common denominator model • Discover ways to transition from visual models to the traditional algorithm • Use the learning from the three workshop sessions to develop student lessons	• How do we translate our actions with models into more formal language? • Describe an alternative approach to the traditional model for dividing fractions. • How does the process of dividing fractions with models translate to a more algorithmic approach? • How do I translate the work I did here to my classroom?

First, a questionnaire was used to gain background knowledge about each teachers' demographics, teacher preparation path, experience, and beliefs about mathematics (what do you enjoy most/least about teaching math). Designed to be open-ended, the questionnaire allowed teacher participants the opportunity to provide details for each question and elaborate on areas of their choosing.

Second, an exit questionnaire was used to glean information about the workshop format and the impact the series had on the teachers' understanding of the content. This information was used to answer research question 2b and to offer implications for those wanting to replicate the workshop activities and use the results of this study to create similar training and professional development experiences for teachers with whom they work.

Data Analysis

Yin (2017) contends there are four principles to which a high-quality analysis attends: all evidence and possible rival interpretations addressed, most important study concept emphasized, and demonstration of an expert understanding of the subject under investigation. The researcher attended to each of these principles throughout the data analysis process. The paragraphs that follow provide a detailed description of how data was analyzed in the current study.

The data collection methods described earlier in the chapter have been designed to align with specific research questions and provide evidence for that question. Table 3.2 shows how each data source was used to address the study's research questions.

Yin (2017) states that the data analysis component of case study research involves "examining, categorizing, tabulating, testing, or otherwise recombining evidence, to produce empirically based findings" (p. 132). As a first step, Yin recommends that researchers "play" (p. 133) with the data to search for ideas and patterns that may lead to worthwhile findings. In addition, placing evidence into a matrix of categories and creating displays of data are also recommended ways to manipulate information (Yin, 2017).

Table 3.2

Summary of the data collection methods used to answer each research question

Research Question	Data Collection Method(s)
1.) In what ways does the professional development series impact teacher participants' understanding of the division of fractions?	• Fraction Content Knowledge Test • Pre- and Post-Performance Tasks • Interviews • Workshop Observations/Field Notes • Workshop Reflections • Demographic Questionnaire
2a.) How does the professional development series impact teacher participants' instructional practices?	• Workshop Reflections • Workshop Observations/Field Notes • Impact of Professional Development Series Questionnaire
2b.) What aspects of the workshop model of professional development are most effective for teacher learning?	• Impact of Professional Development Series Questionnaire • Workshop Reflections • Workshop Observations/Field Notes

After playing with the collected data, Yin (2017) recommends selecting an analytic strategy to use as a guide through the analysis process. One of the suggested strategies is to "follow the propositions" that directed the study. Yin (2017) contends that propositions help organize the analysis, as well as, indicate "contextual conditions to be described" (p. 136) and explanations to investigate. Specifically, the propositions that led to this study are the ideas that:

- teacher knowledge can be increased with professional development workshops
- a deeper understanding of fraction operations can be developed using tasks that emphasize contextual situations and visual models

Together with the analytic strategy described above, additional data analysis techniques were employed to complete an in-depth analysis of the collected data. Data was coded, and pattern matching was used to begin looking for categories and themes. A constant comparison method was used to analyze the cases for similarities and differences regarding the development of a deeper understanding of fraction division and overall changes in teacher knowledge from the pre-workshop to post-workshop assessments. The paragraphs that follow provide an in-depth description of how these techniques were used in the current study.

Pattern Matching

Pattern matching is considered the most advantageous technique used for case study analysis (Yin, 2017). As defined by Yin (2017), pattern matching is an "analysis of case study data by comparing or matching the pattern within the collected data with a pattern defined prior to data collection" (p. 240). For this study, pattern matching was utilized to determine whether specific changes, or lack thereof, observed during workshop sessions and indicated by participant responses on workshop assessments and reflections indicated a deeper understanding of the division of fractions. Pattern matching was employed by comparing evidence collected for each teacher to the propositions that were established and described earlier in the chapter. During the data analysis phase of this study, repeated comparisons were conducted to address reasonable threats to reliability (Yin, 2017).

Constant Comparison

Once data for each teacher participant was collected via the pre- and post-workshop assessments, interviews, workshop observations and field notes, and written reflections, it was coded. The codes were then used to identify key ideas and recurring themes. After identifying the key ideas, the codes were used to look for similarities and differences among them. Categories were then constructed and named with the similarly coded data. As additional data emerged, it was compared with existing categories and themes. During this phase of the data analysis process, new themes and key ideas emerged.

Cross-Case Analysis

Yin (2017) describes the cross-case synthesis technique used in a multiple case study design as one which will generate findings that are stronger and more "robust" than those that may result from a single case design (p. 216). The cross-case analysis was utilized in order to generalize about changes among the cohort to inform others how a similar workshop series may be used with teachers in other districts across the state and offer recommendations for changes that might improve the effectiveness of the intervention.

Validity and Reliability

Yin (2017) states there are four tests, construct validity, internal validity, external validity, and reliability, which have been used to "establish the quality of any empirical social research" (p. 45). According to Yin (2017), issues surrounding validity and reliability can be addressed using a variety of case study tactics. The paragraphs that follow detail the procedures and tactics that were used to address validity and reliability for the current study.

Table 3.3 details the researcher's timeline for conducting research activities and the actions that took place at each specific point on the research timeline.

Table 3.3

Summary of the research plan for the study

Time Frame	Activity	Actions
Mid-September 2018	Meeting #1	• Launch workshop series • Administer pre-workshop performance task and fraction content knowledge test • Conduct follow-up interviews with selected teacher participants
Late September 2018	Meeting #2/ Workshop Session #1	• Collect observations • Complete workshop reflections • Transcribe video data
Late October 2018	Meeting #3/ Workshop Session #2	• Collect observations • Complete workshop reflections • Transcribe video data
Late November 2018	Meeting #4/ Workshop Session #3	• Collect observations • Complete workshop reflections • Administer exit questionnaires (demographics and impact of professional development series) • Transcribe video data
Early December 2018	Meeting #5	• Celebrate learning • Administer post-workshop performance task
Late December 2018	Analyze Data	• Transcribe data • Code data
January 2018	Analyze Data	• Employ data analysis techniques

Construct Validity

Yin (2017) defines construct validity as "the accuracy with which a case study's measures reflect the concepts being studied" (p. 238). To address construct validity, Yin (2017) maintains that two areas must be addressed: clearly define the topic under investigation in terms of specific concepts and "identify operational measures that match the concepts" (p. 46). For this study, the topic under investigation is how teacher knowledge, specifically pedagogical content knowledge, in division of fractions can be improved and instructional decisions impacted via a workshop model of professional development. The measure used to determine whether teacher knowledge improved after the conclusion of the workshop series was an open-ended response performance task focused on teacher knowledge in the area of division of fractions. As mentioned in Chapter Two, several studies, like the one conducted by Hill and Ball (2004), have been conducted which utilized professional development methods, including college courses, summer seminars, and workshops, to improve teacher knowledge after assessing teachers with a content-focused assessment tool.

While multiple-choice assessments have been used widely in the field, Hill et al., (2007) warned against making sweeping deductions regarding a teacher's level of knowledge using a multiple-choice assessment and identified concerns surrounding the use of assessments where the tasks have a single correct response. First, multiple-choice or short answer responses limit what can be gleaned from the teacher participant and does not capture fully the mathematical knowledge and reasoning of the test taker. Second, because teaching requires making complex judgments based on multiple pieces of incoming data, evaluating teacher knowledge using items that yield an answer that is either right or wrong does not address the complexity of the judgment. Third, due to the wide-spread concerns and controversy surrounding the use of multiple-choice items to assess students, teachers may be resistant to this type of assessment as they may feel it is inherently unable to express the complexity of the knowledge they possess. Fourth, there are questions regarding the validity of using a single-response item format to evaluate teachers in this way. Finally, tests where items are presented in a multiple-choice or short-answer format are most appropriately analyzed using statistical analysis measures such as t-test or ANOVA.

To address these concerns and strengthen the validity of the performance task and the fraction content knowledge test, most of the assessments included open-response items that did not have a specific right or wrong response. In addition, teacher participants' responses were used to glean more information about each teacher's level of knowledge regarding the division of fractions and help develop a fuller picture of his or her abilities, rather than passing judgment on the participants' overall mathematical knowledge for teaching.

In addition to using open-response items to address confirmability, data from multiple sources of evidence was used to create a detailed picture of each teacher participant, including interviews, direct observations, and regular reflections. Data triangulation was used to corroborate findings and validate changes gleaned from the pre-workshop performance task and fraction content knowledge test to the post-workshop performance task. Specifically, after analyzing the initial performance task and follow-up interviews, the workshop observations, field notes, and reflections were used to gain a better understanding of how each participant's understanding of the division of fractions developed over the course of the experience. The final performance task was then used to confirm the collected data and compare to the pre-workshop assessments. In addition, a chain of evidence was established to link the case study report to the original research questions.

Internal Validity

Internal validity is defined as "the strength of the cause-effect link made by a case study" (Yin, 2017, p. 239). Yin (2017) further states that internal validity addresses whether a relationship truly exists between two factors or whether a third factor is in play. To address this, pattern matching was used to connect collected evidence to initial predicted outcomes (Yin, 2017). As discussed in Chapter One, the researcher anticipated that teacher participants' pedagogical content knowledge would be positively influenced by targeted professional development in the division of fractions. In addition, the researcher expected to observe an increase in the teacher participants' understanding of fractions. As discussed in Chapter One, both suppositions are supported by literature in the field. Rival explanations, or "plausible alternatives" (Yin, 2017, p. 240), for explaining case study findings were also addressed using pattern matching.

External Validity

External validity, or the transferability of a case study, relates to the "extent to which the findings from a case study can be analytically generalized to other situations that were not part of the original study" (Yin, 2017, p. 238). To address issues surrounding external validity, the researcher used replication logic in this exploratory multiple case study. When replication logic was applied, after the data collection protocol was employed for the first case, it was applied in the same way to all subsequent cases. A cross-case analysis was then applied to strengthen the study results and demonstrate the transferability of the results. As an additional measure of external validity, teacher participants were asked to review their interview data for errors or inconsistences.

Reliability

According to Yin (2017), reliability refers to "the consistency and repeatability of the research procedures used in a case study" (p. 240). To address reliability, Yin (2017) recommends making "as many steps as operational as possible" (p. 49). To do this, the researcher documented the case study protocol, illustrated in Table 3.3, and developed a case study database.

Conclusion

This chapter outlined the research questions, research design, participants, data collection methods, analysis procedures, and description of validity and reliability measures for this exploratory multiple case study. This research design allowed the researcher to highlight the experiences of each teacher participant during the professional development workshop series and used an in-depth analysis to describe changes or validations in their thinking and instructional practices. The remaining two chapters, Chapters Four and Five, will reveal the results and discuss the findings of the study.

CHAPTER FOUR

Results

Employing the use of an exploratory multiple case study design, this study explored how teacher participants' understanding of the division of fractions could be further developed through a series of focused professional development workshop-type sessions and used as an intervention to increase the pedagogical content knowledge of fifth-grade math teachers utilizing state curriculum standards, the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012), or TEKS, as a guide. Multiple assessments, including a fraction content knowledge test, pre- and post-performance task, follow-up interviews, workshop reflections, session observations and field notes, as well as, demographic and workshop feedback questionnaires were collected to aid the researcher in responding to the following research questions:

Primary research question:

1.) In what ways does the professional development series impact teacher participants' understanding of the division of fractions?

Secondary research questions:

2a.) How does the professional development series impact teacher participants' instructional practices?

2b.) What aspects of the workshop model of professional development are most effective for teacher learning?

This chapter begins with an overview of the professional development sessions, including the rationale for the design of the sessions and a summary of the observations for each session. Next, a brief overview of the participants, including their years of teaching experience, years teaching mathematics, and selection for this study is included. Then, following the standard of a multiple case study design, each participant is presented as a separate case. The narrative for each case includes, the results of their fraction content knowledge test, pre- and post-division of fractions performance task, insight gleaned from the follow-up interview which had to be conducted by phone due to a problem with the participants' district's network infrastructure, workshop observations/field notes, workshop reflections, and both exit questionnaires. Following the analysis of the individual cases, a cross-case analysis is presented to compare the cases and reveal emerging themes. In the conclusion of this chapter, the researcher synthesizes the results from all data collection sources to answer the primary and secondary research questions for this study.

Workshop Sessions

As stated in Chapter One, recommendations regarding the development of meaningful fraction algorithms using a problem-based approach by Sharp, Garofalo, and Adams (2002) and Van de Walle and Lovin (2006) formed the basis of the conceptual framework that was used to guide the workshop series for teacher participants. Recommendations included: 1.) helping students develop fraction concepts through personal contexts and contextual tasks (Sharp et al., 2002; Van de Walle & Lovin, 2006); 2.) encouraging students to use invented algorithms, procedures, and estimation to solve problems (Neagoy, 2017; Petit et al., 2015; Siegler et al., 2010; Sharp et al., 2002; Van de

Walle & Lovin, 2006, Van de Walle et al., 2019); and 3.) allowing students to explore the operations using their own language, pictures, and models (Roddick and Silvas-Centeno, 2007; Sharp et al., 2002; Van de Walle & Lovin, 2006).

The design of the workshop sessions centered on these recommendations. While the format of each session followed a similar pattern: discussion of big ideas, opening task, session activity, and culminating discussion, the focus of each session narrowed in scope. Detailed lesson plans for each session appear in Appendix F. Each session was recorded and then analyzed, using a modified transcription process, for supporting information. A summary of the observations for each session follows.

The initial meeting included an introduction to the study and two assessments, a fraction content knowledge test and initial division of fractions performance task. Interviews were then conducted following a preliminary review of the pre-assessments. The next two meetings included the first and second days of the professional development series. Due to a change in the schedule of the participating school district, it became necessary to combine meetings four and five. Therefore, the final division of fractions performance task was completed at the conclusion of the third and final workshop session. In addition, because they were found to be unmanageable due to the participants' demanding work schedules, follow-up interviews were not conducted to gain additional insight following the final performance task.

Workshop Session One

The major objectives for the first session included exploring the relationship between part and whole and understanding the importance of identifying the unit and equal partitioning. As anticipated, at the onset of the workshop session, participants sat close to another participant with whom they were comfortable or with whom they taught with on their campus; therefore, participants' seating was shuffled before the first activity to get them out of their comfort zones. This allowed them to be more engaged and rely less on familiar relationships to support them during the activities. For example, the professional development facilitator noted, during the first meeting, that Cindy and a Special Education teacher with whom she worked were very comfortable with each other as they sat together during the pre-assessments and talked to each other for a good portion of the time, frustrating other participants. For that reason, the two teachers were intentionally split up during the professional development experience to help them gain more from the session.

Throughout the activities, the participants worked well together; however, the researcher noticed that most groups settled into roles or where one teacher emerged as the dominant voice in the group often teaching the other person how to complete the activities. While music was used to lighten the mood during work times, it prevented some discussions from being heard.

Workshop Session Two

The major objectives for the second workshop session included distinguishing between partitive and quotative situations, connecting whole number and fraction division, and exploring the division of fractions through contextual situations. Right away, the professional development facilitator observed that the mood was much lighter as the participants laughed and joked together before the start of the session. Throughout this session, the researcher observed a lot of in-depth discussion. In particular, the teachers conversed about the difference between partitive and quotative tasks so intently that the professional development facilitator had to interrupt the group and ask them to stop discussing the task until everyone had a chance to work through it on their own.

Once again, the teachers really worked together to determine the responses for a card-making task where participants divided paper into cards of different fractional amounts. During this time, participants helped each other make sense of the specific problem and determined a plan of action. When one of the participants asked a question about how to address the remainder, a conversation ensued surrounding the importance of being able to interpret the remainder in terms of the correct unit. As participants worked through the second activity, where they partitioned fractional quantities into a specific number of groups, the researcher observed them struggling with the values needed to complete the task. This bit of struggle led to a discussion surrounding the teachers' own learning experiences as students.

Workshop Session Three

The major objectives for the third workshop session included using invented procedures to make connections to the traditional algorithm for the division of fractions, investigating models for the division of fractions, and exploring ways to transition from visual models to the traditional algorithm. Like the second session, the professional development facilitator observed that the mood at the onset of the session was light. The researcher also noted a comradery amongst the teachers as they often worked together as a cohort even when they had been asked to work alone.

The opening discussion surrounding creating overarching conjectures regarding fraction operations elicited a lot of conversation. This led to a lengthy discussion where the teachers voiced frustration regarding the knowledge students bring with them into the fifth-grade classroom and how lower grade-level teachers prepare them.

The Participants

As stated in Chapter Three of this report, participants in the study included seven fifth-grade teachers who teach the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012). Each of these seven participants attended all three of the professional development workshops. Table 4.1 provides an overview of the teaching experience of the participants, additional demographic information deemed important that was requested from participants after the final workshop session. A copy of this additional questionnaire is available in Appendix E.

Table 4.1

Summary of participants teaching experiences

Pseudonym	Teaching Certification	Years of Teaching Experience	Years Teaching K – 5 Math	Years Teaching 6 – 12 Math
Molly	Early Childhood – Grade 6	19	17	2
Cindy	Grades 1 – 8 all subjects	17.5	7	0
Erin	Grades 1 – 8 all subjects; Grades 1 – 8 Math	20	20	0
Gina	Early Childhood – Grade 6	23	23	0
Amber	Grades 1 – 8 all subjects	17	13	0
Tina	Early Childhood – Grade 6; Grades 4 – 8 General	6	5	1
Leah	Grades 1 – 8 all subjects; Grades 1 – 8 Math; Grades 4 – 8 Master Math Teacher	13	6	7

Individual Case Profiles

In the sections that follow, the researcher provides an in-depth description of each teacher participant, including the results of her fraction content knowledge test, pre- and post-division of fractions performance tasks, insight gleaned from the follow-up interview, workshop observations/field notes, workshop reflections, and demographic and workshop questionnaires.

The Case of Molly

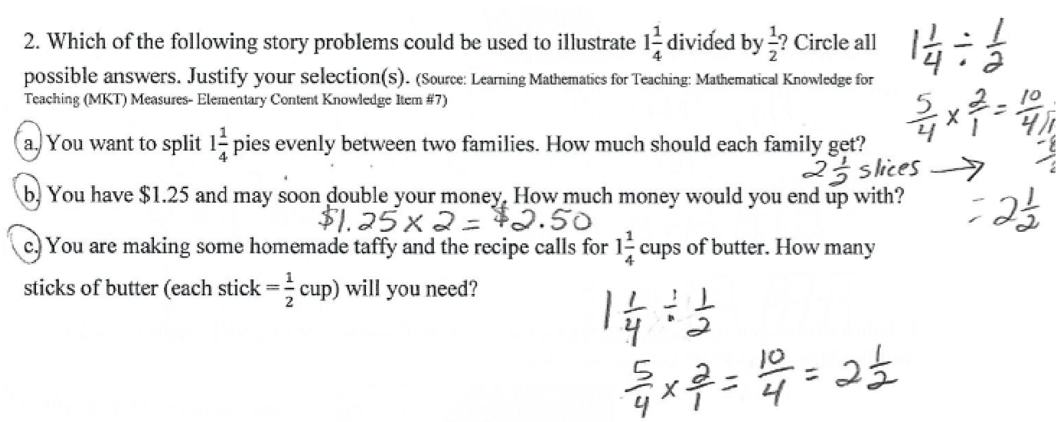
Molly's teaching experience reflected more of the traditional educator preparation program, one achieved via a university degree program. Molly completed her undergraduate work at a public university in Texas. She has 20 years of teaching experience, including teaching grade four and both fifth and sixth grade math.

On her demographics questionnaire, Molly stated that she loves teaching math and makes it a goal to "[instill] a love of math into [her] students." She added that she enjoys watching her students grasp the content and "then be able to teach others what they have learned." Conversely, Molly identified "lack of student effort" as the aspect of teaching mathematics she dislikes the most.

Throughout the workshop experience, Molly attended each session with a positive attitude and participated fully. Molly appeared to enjoy the opportunities to collaborate with her colleagues and noted that she would like additional opportunities to "talk about teaching math with [her] co-workers." During each of the workshop sessions, Molly was observed discussing session activities with her partner and participating in small group discussions; however, she remained very quiet and did not offer her thoughts or experiences in large group discussions.

On the fraction content knowledge test, Molly's work illustrated a mix of correct and incorrect responses; however, her responses to the tasks involving fraction operations, tasks two, six, seven, and eight, showed some areas of inaccurate thinking. In addition, a lack of justification on all four tasks provided limited insight into her reasoning. Three of the four tasks were aligned to fifth grade curriculum standards in the state and have appeared on previous state assessments.

Most notably, on the second task, Molly selected all three scenarios as illustrations of $1\frac{1}{4} \div \frac{1}{2}$ (as shown in Figure 4.1). As Molly's work reveals, she incorrectly connected the idea of dividing by half with multiplying by two in contextual situations. For each scenario, Molly recorded a number sentence which resulted in multiplying the initial quantity by two to obtain the solution. For example, for the first problem, Molly arrived at a solution of two and a half, a solution which is inconsistent with the context of the problem because if you share one and one-fourth pies with someone, it would not be possible to receive more than the total initial quantity.


2. Which of the following story problems could be used to illustrate $1\frac{1}{4}$ divided by $\frac{1}{2}$? Circle all possible answers. Justify your selection(s). (Source: Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures- Elementary Content Knowledge Item #7)

a. You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?

b. You have $1.25 and may soon double your money. How much money would you end up with?

c. You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?

$1\frac{1}{4} \div \frac{1}{2}$

$\frac{5}{4} \times \frac{2}{1} = \frac{10}{4}$

$2\frac{1}{2}$ slices →

$= 2\frac{1}{2}$

$1.25 x 2 = $2.50

$1\frac{1}{4} \div \frac{1}{2}$

$\frac{5}{4} \times \frac{2}{1} = \frac{10}{4} = 2\frac{1}{2}$

Figure 4.1. Molly's response to task two on the fraction content knowledge test.

On task six, Molly was able to illustrate a model that shows five and one-fourth; however, it is unclear how she divided the initial quantity into three equal-sized groups. In addition, her work indicated she used the invert and multiply procedure to arrive at a solution. Her solution is incorrect because she added the denominators together instead of multiplying them together in the invert and multiply step.

On task seven, Molly's use of the invert and multiply procedure allowed her to arrive at a correct quotient; however, her model illustrated six groups of one-eighth rather than one-eighth divided into six equal-sized groups. On the final task, Molly's work indicated a correct expression of $3 \div \frac{1}{3}$ and included a final solution matching that of the model (a number line labeled 0 – 3 with nine equally-spaced jumps between the endpoints) after the use of the invert and multiply procedure to arrive at a quotient of nine.

Most of Molly's responses on the initial performance task indicated some understanding of the tasks of teaching the division of fractions; however, she exhibited considerable difficulty with the final task where she was asked to create a contextual situation, model, and an equation for a fraction division problem.

Molly's response, as shown in Figure 4.2, illustrates a contextual situation where the dividend is divided by two, which is inconsistent with the stated expression of $1\frac{3}{4} \div \frac{1}{2}$. She then recorded an equation where she used the invert and multiply procedure to arrive at a solution of three and a half. Molly's model illustrated the initial quantity; however, it is unclear how it was used to help her make sense of the task.

During the follow-up interview, when asked to explain whether her situation and model represented the same expression, Molly stated that the dotted lines in the model represented dividing the pieces in half. She then added that each parent would receive seven brownies. After discussing the situation aloud for a few minutes, Molly was unable to connect the situation to her model.

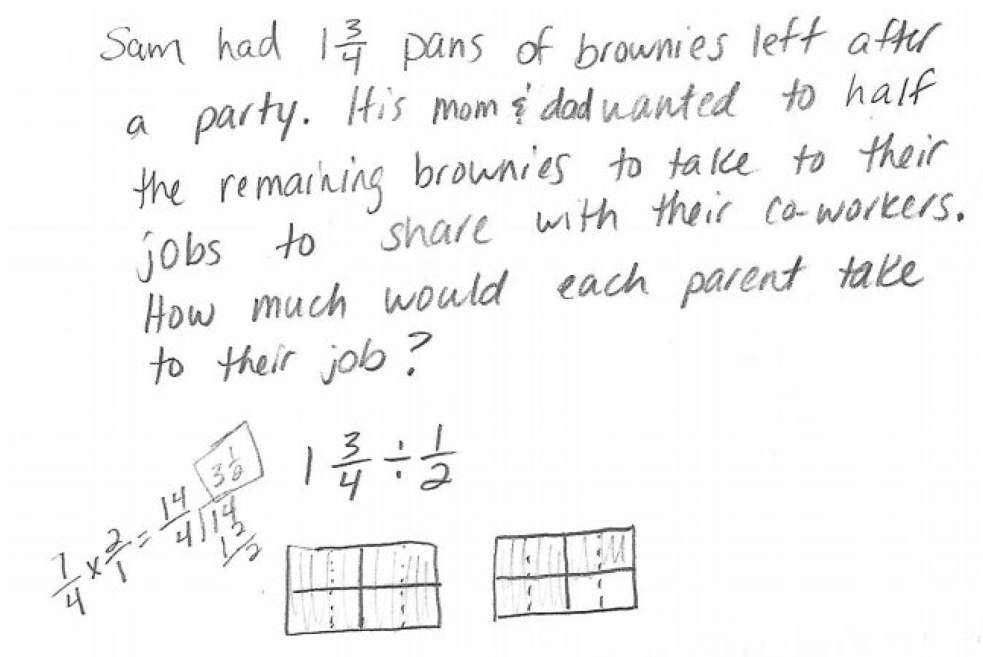


Figure 4.2. *Molly's response to task five on the initial performance task.*

During the professional development experience, Molly participated fully but was more vocal working with her partner or small group than she was in large group discussions. In fact, on her workshop feedback questionnaire, she stated that she enjoyed "being able to complete the workshop with [her] teaching team." She added that she "appreciate[d] anytime with [her] co-workers where [she was able to] learn and gather information to implement into [her] teaching."

Throughout the workshop series, Molly commented that she enjoyed the hands-on nature of the sessions. For example, in her reflections for session one, Molly stated the activity she felt was most impactful was "creating pictures and telling what fraction a shape represent[s] in the picture" during the pattern block pictures activity. Similarly, Molly stated in her reflections for session two that the card-making activity was most impactful because she was able to create a visual to represent what was happening in the contextual situation with her model. Then, in her reflections for session three, Molly stated she found the pizza-making activity to be most impactful because she "love[s] the connection you can see in making the models and filling out the recording sheet" to show the solution for each of the problems based on the initial contextual situation. In addition, Molly stated that she "appreciated how [the making pizza activity] ties modeling to the algorithm and allows students to see where the information comes from."

Molly's performance on the final performance task was like her performance on the initial task with one important difference—her response to task five. On the initial performance task, Molly was unable to connect her contextual situation and model to her equation; however, she was successfully able to connect her model and her equation on the final performance task (as shown in Figure 4.3). At first glance, Molly's contextual situation appears to match the other two representations but is a bit ambiguous because she wrote "put half of the brownies in different containers" and then asked for the number of containers needed to store all the brownies. Because Molly's model accurately depicted dividing the total quantity into half-pan portions, it seems as though Molly just neglected to include "half of a pan of brownies" in her contextual situation; however, the semantics of her response make it difficult to be sure of the intended meaning.

Even though Molly was quiet and reserved throughout the professional development experience, it is clear Molly valued the time she spent with her colleagues and desired to use the activities she experienced with her students. In fact, on her workshop feedback questionnaire, Molly commented that she would have liked for the workshop sessions to occur in the summer or earlier in the school year so that “more material could have been used this year.”

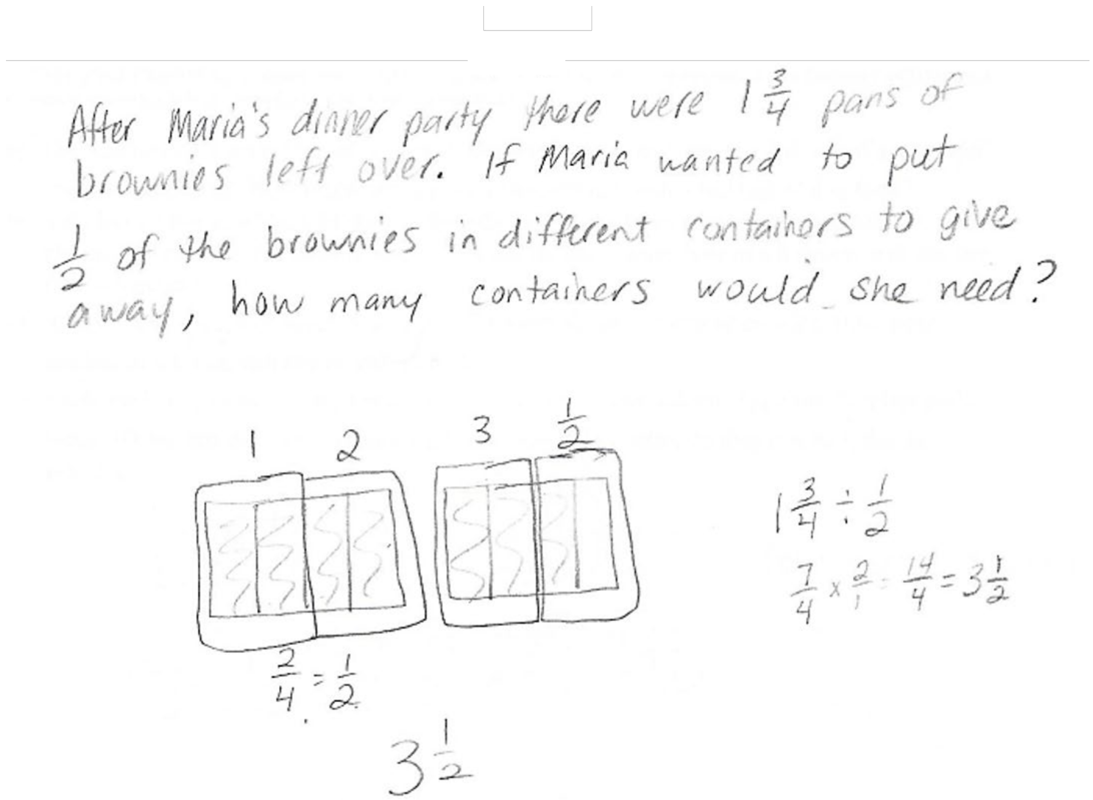


Figure 4.3. Molly’s response to task five on the final performance task.

The Case of Cindy

Cindy’s teaching experience reflected a lengthier version of a traditional teacher education path. She attended three post-secondary institutions, both community and university, in various parts of the U.S. before completing her education degree in her late twenties. She did complete one semester of student teaching and has been in the classroom for 17.5 years.

Cindy stated that she loves math, the "black and white part of it," and feels she has been successful at getting her students to pass, potentially without a deep understanding, but she feels her students leave without hating math anymore. When asked what she enjoys most about teaching math, she stated she likes that there is more than one answer. On the other hand, she stated she least likes when students come to her class with deficiencies because she believes it is harder for them to catch-up and is frustrated when she must continuously reteach old content.

Throughout the workshop series, Cindy attended each session with a smile on her face and a ready-to-work attitude. She participated in all discussions and was vocal about her understanding, or lack thereof, and, at times, exhibited a negative attitude about herself and her ability to teach mathematics. During the first session, Cindy stated that she had very few skills and that, because of her age, she struggles with using pictures and models because that is not how she was taught.

On the fraction content knowledge test, Cindy's responses were generally correct; however, in some cases, she did not provide a justification, or her justification was inconsistent with her response. For example, Cindy had difficulty selecting examples of dividing by one-half (task two), where she selected both the dividing by two problem (a) and multiplying by two problem (b), leaving the dividing by one-half task, problem (c) unselected.

Most of Cindy's responses on the initial performance task indicated some understanding of the tasks of teaching the division of fractions; however, she exhibited considerable difficulty with the final task where she was asked to create a contextual situation, model, and an equation for a fraction division problem.

Cindy’s response, as shown in Figure 4.4, illustrates a contextual situation where the dividend is divided by two, which is inconsistent with the stated expression of $1\frac{3}{4} \div \frac{1}{2}$. She then recorded an equation that is consistent with division by two and arrived at an incorrect response of seven-eighths.

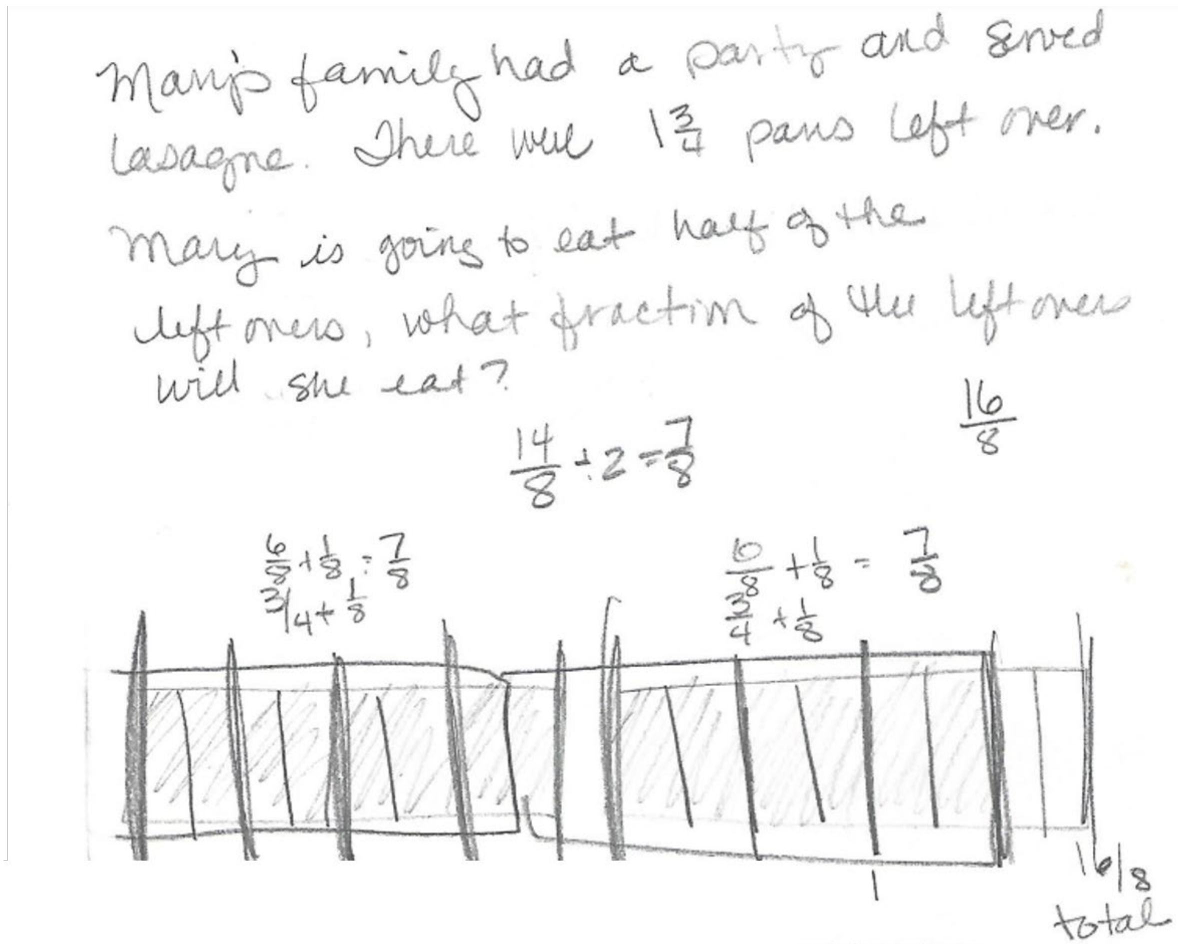


Figure 4.4. Cindy’s response to task five on the initial performance task.

During the follow-up interview, when asked to give a solution to $1\frac{3}{4} \div \frac{1}{2}$, Cindy responded with a correct quotient of $3\frac{1}{2}$ but stated that the solution didn’t seem right. When probed further, Cindy acknowledged the disconnect but was unable to rectify the difference in the solutions. Cindy requested additional time to rethink the task. She was diligent about trying and did not want to give up. While working, she stated "if I have $1\frac{3}{4} \div \frac{1}{2}$, I am essentially doubling it [the quantity]." Cindy eventually said, "I don't know" and the interview ended.

During the workshop series, Cindy participated fully and gave 100% effort during each task, despite her negative self-talk where she referred to herself as "not smart enough" and stated that "math majors" have an advantage over others when it comes to certain areas of mathematics. For example, even after acknowledging she was lost while working with her partner on an activity in session one which required participants to: 1.) build shapes with a variety of pattern blocks, 2.) identify a shape, and 3.) connect the shape to a specific unit fraction, she asked her partner for help and continued to try and make sense of the activity. Afterwards, she confessed that using the pattern blocks "opened up an avenue for [her] to use more models in [her] teaching of fractions."

Throughout the experience, Cindy consistently remarked that she enjoyed using the manipulatives, such as pattern blocks, sheets of paper, and Cuisenaire rods. She added that they made a "huge" difference in "what [she was] doing" because it helped her to "see fractions in division," recognizing that the Cuisenaire rod task helped her see why dividing a fraction is answered with a fraction.

Even when tasks required participants to work beyond what they taught in fifth grade; Cindy continued to persevere. During a discussion of how to address remainders in contextual situations, especially when they did not match the remainder obtained when using a calculator or the traditional algorithm, she actively engaged in the discussion surrounding the meaning of remainders while still grappling with her own understanding.

In fact, during the third session, she excitedly shared that she had challenged her higher-achieving students in the classroom with tasks where there was a remainder. She, herself, admitted to still struggling with the idea of the fractional remainder and the changing units but stated that she was able to initiate a great discussion with her students regarding it.

During the final session, Cindy offered great insight on how she solved a problem where she had to partition a whole number into fractional parts that were not unit fractions. Despite being partially shut down by the group, she continued to explain her "invented procedure" which was exactly right. In her reflections for the last session, Cindy remarked that she was "wowed by the thought of finding common denominators, then dividing the numerators," a concept the participants "discovered" during the final hands-on activity.

While many of Cindy's responses, which showed some understanding of the tasks of teaching the division of fractions, on the final performance task remained the same, she was able to provide much more detailed answers. For example, on task number three, regarding identifying and rectifying a student's misconception, she stated, "the student assumes you <u>always</u> get a quotient smaller than your dividend when you divide." She followed with, "that's not true for divisors smaller than one" which is a concept the participants grappled with during the final session.

In addition, Cindy referred to using common denominators twice in her explanations, including for task five (which was the same as the initial performance task). On the final performance task, Cindy correctly recorded an equation, represented the problem with a contextual situation, and illustrated a matching model for task five (refer to Figure 4.5). Cindy's response showed a better understanding of the remainder, something she struggled with throughout the workshop series, as she named it in three ways, one-half of a serving, one-fourth of the pan, or one-seventh of the original amount.

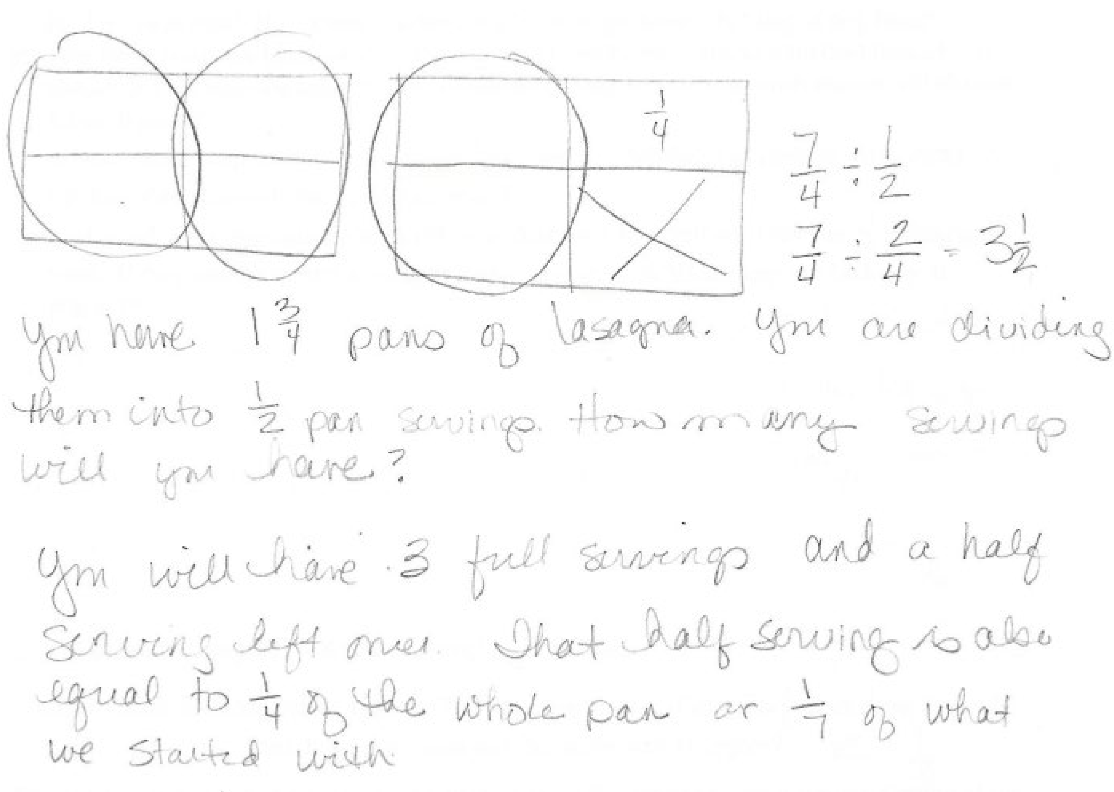


Figure 4.5. Cindy's response to task five on the final performance task.

Most notably, the biggest changes observed of Cindy include her own mindset. On the last day of the workshop series, she excitedly remarked that she wanted to go back to her initial performance task and "redo her lasagna problem," the last task, because of her new knowledge. In addition, Cindy stated she made several changes to her classroom instruction, including offering more challenging tasks for students who were ready for them and changing her traditional "keep, change, flip" mantra, a reference to how to remember the steps for using the traditional algorithm for the division of fractions, to multiplying by the reciprocal because it helped students better understand what they were doing—a change she was excited to tell the group about.

The Case of Erin

Erin's teaching experience reflected the more traditional educator preparation process. She attended a public university in Texas where she completed a two-year teacher preparation program consisting of three semesters of observations in a classroom setting, with exposure to lesson-planning and implementation, and a final semester of student teaching. Erin has been teaching for twenty years, which includes teaching in both self-contained and a specialized mathematics setting. Erin also has three years of experience as a math interventionist.

Erin stated that math has always been her favorite subject and the subject with which she is most comfortable teaching. When asked what she enjoys most about teaching math, she stated she sees math as a "never-ending puzzle," so she enjoys seeing the students "fitting the pieces together" and would love to see her mathematics instruction be more about critical thinking than standardized testing and deadlines. She added that it is often hard to get students' investment in the subject because of outside factors such as a parent who is not invested or who may be sending a student negative messages about math.

Throughout the workshop series, Erin remained reserved but often appeared more confident and comfortable with the math content than the other participants. While she was not as vocal as others during group discussions, when she did share her thinking, it was well thought-out. During each session, Erin was often observed supporting her partner or small group members in making sense of the tasks and processing the mathematical ideas at hand.

On the fraction content knowledge test, Erin's responses were mostly correct; however, some of her responses were incomplete. On the initial performance task, most of Erin's responses indicated a stronger understanding than most of the other participants. Most notably, on task five, Erin's model correctly matched her contextual situation; however, she incorrectly named the remainder in the model which was inconsistent with the remainder shown in her equation after use of the invert and multiply procedure (see Figure 4.6). The same process for naming the remainder yielded an incorrect response on the fraction content knowledge test as well.

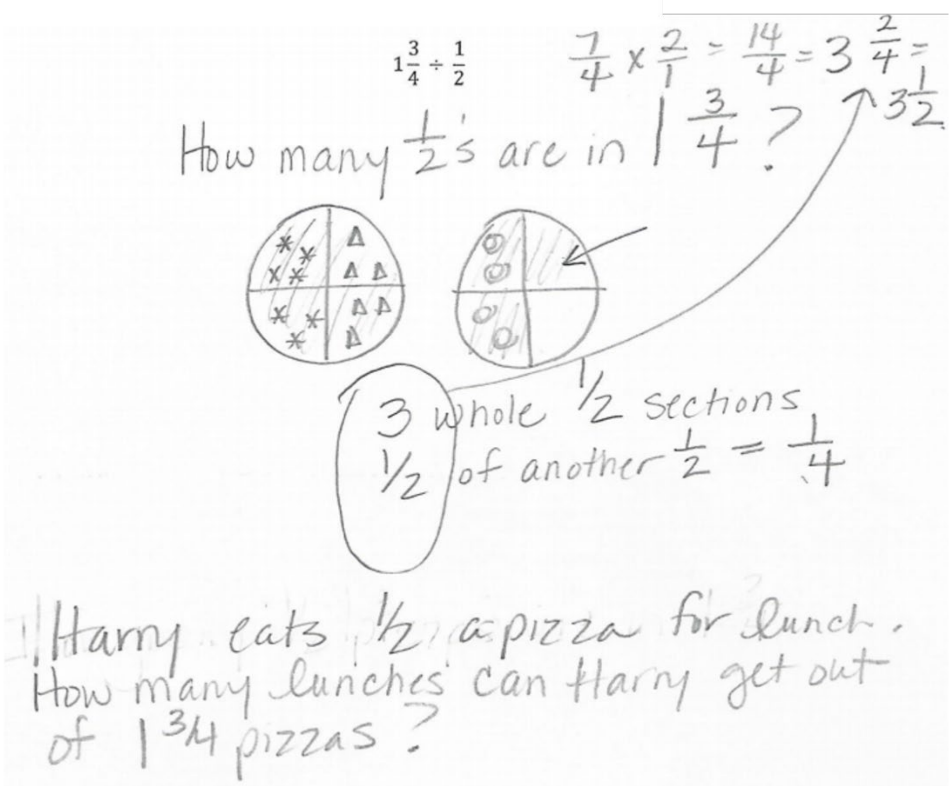


Figure 4.6. Erin's response to task five on the initial performance task.

During the follow-up interview, when asked if the quotient she obtained after using the algorithm matched the quotient illustrated in her model, Erin was able to correctly rectify the mismatch. After reviewing the problem, she stated there was half of a group remaining which yielded a remainder of one-half of a lunch serving of pizza. She added that one-fourth refers to the remaining one-fourth of a whole pizza.

During the workshop series, Erin participated fully and provided support and insight to others and the group as they grappled with making meaning from the activities. Erin's contributions during group discussions were mostly student-centered and offered strategies for helping students make sense of the division of fractions. For example, she talked about connecting whole number division to the division of fractions to help students make sense of dividing with fractions because fractions are abstract. She also stated that she feels teachers "step in too quickly" and stressed the importance of allowing students an opportunity to explore and make sense of the math on their own.

Throughout the workshop series, Erin consistently referred to having the opportunity to explore, discover, and make sense of the tasks as being the most impactful aspect of the professional development experience. Erin also commented that the use of models in the final hands-on activity, where participants where partitioning partial blocks of cheese into equal-sized fractional servings, led her to the common denominator model, something she had "never really thought about using" before but that helped her "contemplate what was really happening."

Erin's responses on the final performance task mirrored those of the initial task with some notable differences. For example, Erin selected the first problem (a) on task #1 as her initial task because it could be easily modeled. She selected the third problem (c) as the second task because it was easier to sketch. This sequence is the reverse of how she responded on the initial assessment. The sequencing of the second problem (b) and the fourth problem (d) remained the same, after the other problems.

On the second task, Erin mentioned that she would begin with a scenario which illustrated 12 divided by two and then move to 12 divided by one-half in order to show the connection to whole number division. While this response aligned with how Erin stated she teaches this content in the classroom, it is inconsistent with her response to task one where she stated she would use the second problem (b), which illustrates a whole number divided by a whole number, after introducing the first (a) and third (c) problems.

For the fourth task, Erin's response illustrated the use of common denominators with the third problem (c), which is not written with common denominators, as in the first problem (a). This may potentially be because the first problem, written with common denominators, will yield a remainder, whereas the third problem (c) is six-thirds partitioned into groups of two-sixths, essentially resulting in a problem where two wholes are partitioned into groups of two-sixths.

Additionally, Erin's response indicated the use of a procedure that is inconsistent with her model (see Figure 4.7). The fraction $\frac{36}{6}$ appears to be the quantity obtained if the invert and multiply procedure is used to solve the problem, as $\frac{6}{3}\ x\ \frac{6}{2} = \frac{36}{6}$. This is unclear based on the information provided in the response but is consistent with Erin's use of the traditional algorithm, on both the initial and final assessments, to address tasks four and five.

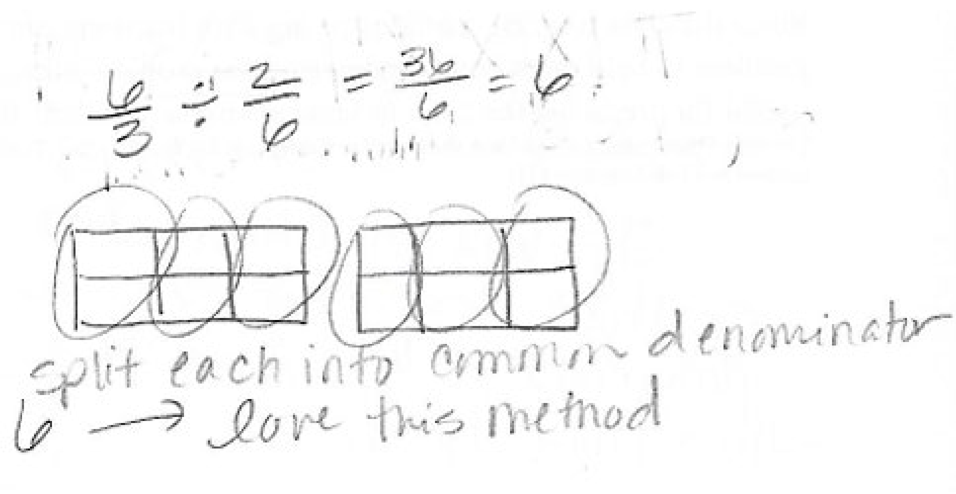


Figure 4.7. Erin's response to task four on the final performance task.

During the first session, Erin stated that fraction concepts were not her favorite things to teach but felt they were necessary. In her reflections for session three, Erin admitted that "fraction concepts are still a work in progress" for her but added that she would like to rework her lesson plans to incorporate the ideas, activities, and materials used in the professional development series.

The Case of Gina

Like Erin, Gina's teaching experience reflected a more traditional educator preparation process. She attended a public university in Texas where she completed field experiences once a week and participated in one semester of student teaching. Gina has been teaching for 23 years, which based on conversations during the workshop experience, seem to include teaching a diverse population of students.

Gina stated that she loves teaching math and "engaging with students as they discover there's more than one way to solve a problem." When asked what she enjoys most about teaching math, she stated she likes the "conceptual learning connection to

[the] procedural process" and that she would like to be able to include more time with hands-on materials, such as manipulatives, because too much time is spent completing paper and pencil tasks. Gina also echoed the sentiments of several other participants when asked what she likes least about teaching math. She responded, there is "so much content" to teach and "more time is needed for students to master content."

Throughout the professional development experience, Gina was personable and vocal as she regularly participated in group discussions and freely shared her teaching experiences with others. During the first session, when the group discussed norms for the workshop experience, Gina was overheard sharing that she wanted to open her mind to new ideas because she gets "stuck in a rut." She added that she likes teaching fractions but wishes students would "get it" and stated that she wanted to feel better equipped to teach fractions because she feels like students have a hard time with the concept as it's just a process they teach "without a real understanding."

On the fraction content knowledge test, Gina responded to all questions correctly and her justifications are consistent with the answers she recorded; however, at the beginning of the assessment, Gina exhibited difficulty with the first task, which involved determining a fraction to represent a whole of two circles, each divided into fourths, with five parts shaded, and asked for clarification.

Like Erin, Gina's responses on the initial performance assessment indicated a strong understanding surrounding teaching the division of fractions. In fact, she was one of two fifth-grade teachers who selected the pair of fractions with common denominators to explore the division of fractions before moving to the traditional algorithm (task four). In her explanation, she referenced that it is easier to divide when both fractions have a common denominator. Albeit somewhat confusing, Gina's model does illustrate finding the number of sets of three-sixths in ten-sixths. She attempts to create a corresponding equation but ultimately used the invert and multiply procedure to find the quotient.

On the final task of the initial performance assessment, Gina illustrated a correct contextual situation and correctly used a number line to model the action of the situation (see Figure 4.8). While Gina's model illustrated three and a half groups, her unit remains unclear as her work indicated that one slice equals half of a person. Then, below the number line model, Gina used the invert and multiply procedure to obtain the answer three and a half meals. Several attempts were made to schedule an interview time with Gina; however, due to scheduling conflicts, she was unable to complete the follow-up interview to address some of the inconsistencies in her work.

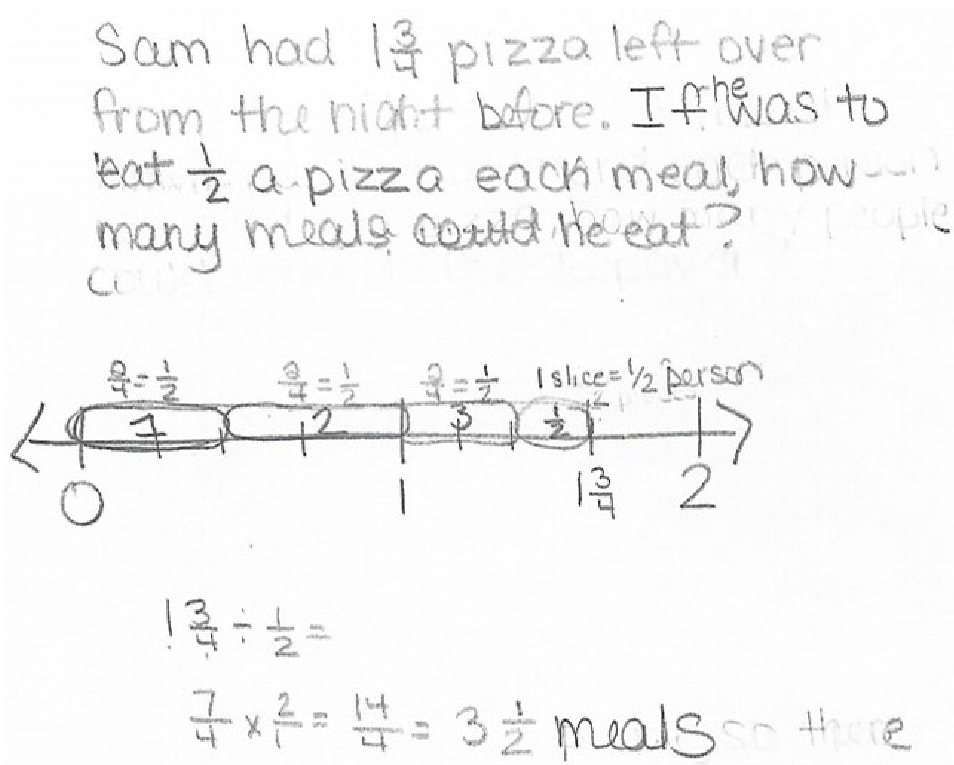


Figure 4.8. Gina's response to task five on the initial performance task.

During the workshop series, Gina participated fully and was vocal throughout all the group discussions. In fact, on her workshop feedback questionnaire, when asked what she enjoyed most about the workshop format, she wrote, “conversations about thought process in division of fractions.” She also wrote that while she felt the goals of the workshop series could be accomplished using a different professional development format, “the interaction and discussion made [the workshop sessions] more meaningful.” One notable response recorded on the workshop feedback questionnaire included a statement where Gina said she “loved that [they] were allowed to share with one another without fear of being wrong,” something that is surely important to students as well.

Gina approached each task with a positive attitude and was often observed sharing her teaching experiences and beliefs about the teaching of mathematics with her partner or small group. For example, during a small group discussion in the second workshop session, she discussed how she helps her students know how to begin analyzing a word problem. Then, during the large group discussion, she stated that she did not like her students coming to class with math tricks, because she is “not about tricks” and “not about shortcuts.”

During the professional development experience, Gina commented that the activities completed during the sessions were most impactful for her which aligned with her comment during the first session where she stated that she does better when she has a chance to practice. For example, during the first session she stated that she found “using shape manipulatives to show how one-third or another fraction is represented within other shapes” most impactful and would use this activity with her students. This is in reference to the pattern block activity where participants created a figure with the pattern blocks and then named a fractional part using one of the shapes represented in the figure. During the discussion surrounding this activity, Gina was overheard saying "with pictures [the students] feel like they can explain it."

In the second workshop session, Gina used the results of the card-making activity to make some sweeping generalizations about fraction division. For example, during the discussion, she stated "the larger the part, the less cards you could make." She echoed this reasoning in her reflections for the session when she described the activity as being most impactful and indicated she would use the understanding she gained from the activity to demonstrate for her students how the size of the cards connects to the number of cards that could be made, i.e. "the smaller the fraction, the more cards can be made" and vice versa. Using the activities in this way certainly aligned with her enjoyment of connecting conceptual learning to procedural processes and helping her students "connect to the algorithm."

During the final workshop session, when participants were given time to explore fraction division in a new contextual setting, Gina was the only participant who used repeated subtraction to complete the problems where participants divided a whole number into fractional parts that were not unit fractions and partitioned a fraction into fractional parts. In her reflections for this session, Gina remarked that discussing multiple ways to solve division of fractions problems, such as using common denominators and repeated subtraction, was most impactful for her.

On the final performance assessment, most of Gina's responses remained the same; however, on task five, she used repeated subtraction to solve the problem with no indication of the use of the invert and multiply process to solve or verify the solution. As seen in Figure 4.9, her work indicated three whole groups, which is correct. Her work also illustrated a remainder of one-fourth, which is technically incorrect because her question asks, "how many friends did she share her candy with."

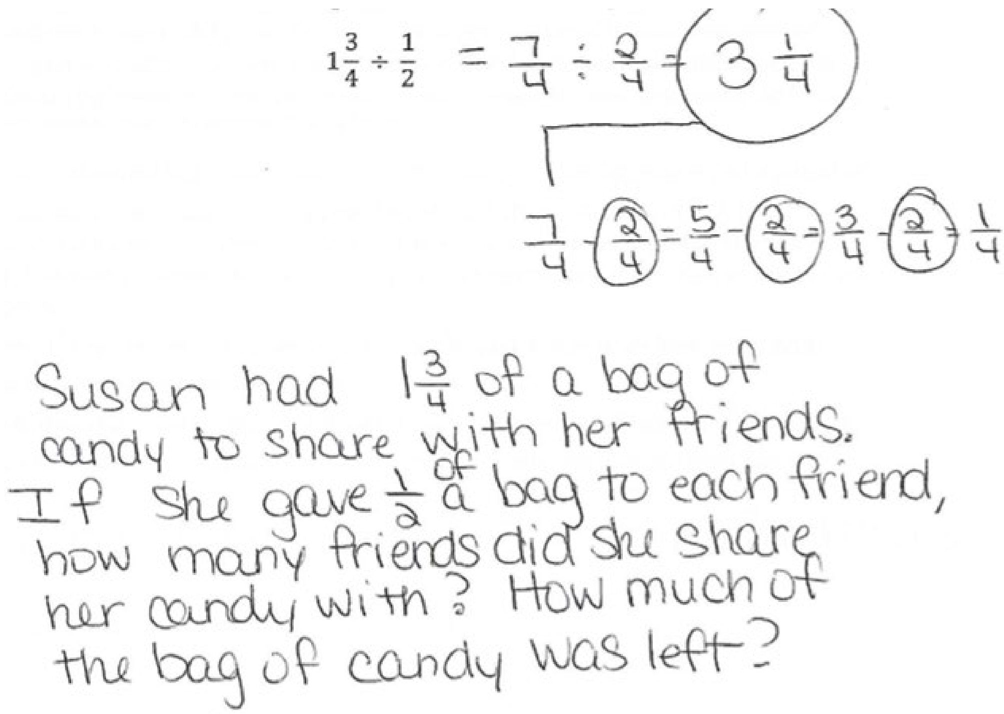


Figure 4.9. Gina's response to task five on the final performance task.

Like the initial task, Gina's work displayed some difficulty with naming the remainders. Her solution of three and one-fourth indicated that Susan could share the leftovers with a fourth of a friend which does not make sense; however, the one-fourth may be in response to her second question in the contextual situation where she asked, "how much of the bag of candy was left." The solution of one-fourth would accurately address this final question if the unit was a full bag of candy. Mathematically speaking though, the solution should be three and one-half where the one-half refers to the remaining one-fourth of the bag which is equal to one-half of a serving of candy. Like several of the other participants, it was clear she was still grappling with the idea of fractional remainders and how to accurately name them based on the context of the problem and the questions being asked.

Based on comments made throughout the workshop series and Gina's written responses, it is clear she finds time for students to explore with manipulatives an important aspect of mathematics instruction as she stated that this workshop format had more time to explore than other workshops she had previously attended. In fact, when asked what she would change about the workshop format, she requested "advice about how to use manipulatives in the classroom" which again aligned with her desire to begin with conceptual learning and then move to procedures and algorithms.

The Case of Amber

Amber's teaching experience reflected a traditional educator certification path. She attended a public university in Texas where she majored in Elementary Education and received a Bachelor of Science degree. She has been in the classroom for 17 years.

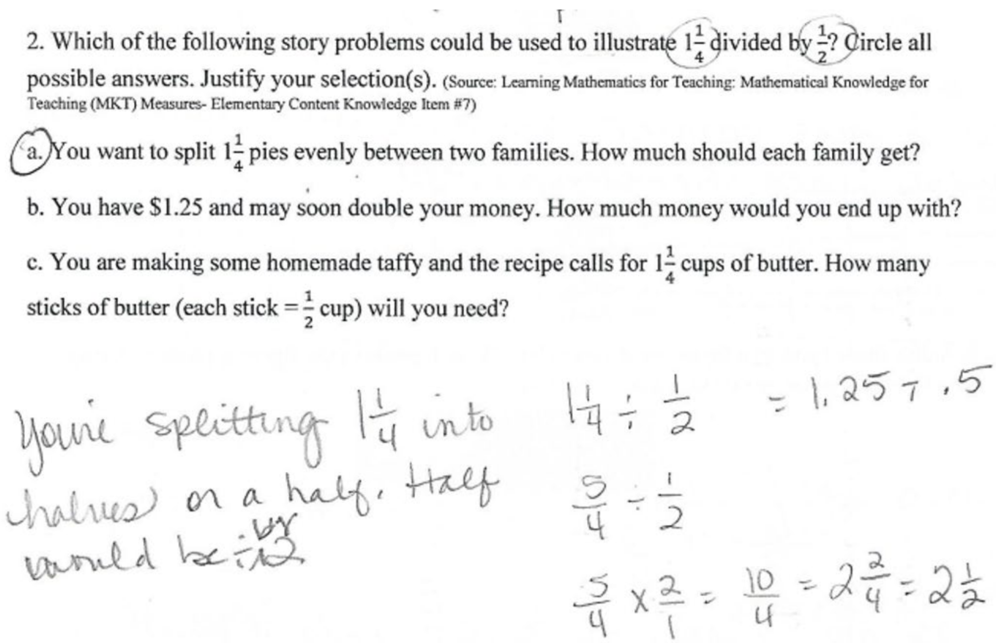
When asked what she enjoys most about teaching math, Amber stated, "that it changes daily" and "it's never boring." She added that she "love[s] the conversations and the ah ha moments." She listed time constraints, a topic at the forefront of several discussions during the professional development series, and teaching to prepare students for the state's standardized test as the aspects of teaching math she enjoys the least.

Amber described her experience as a mathematics teacher as evolving as she has learned more and more through the years. She stated that "workshops like this [the division of fractions workshop series] are the most meaningful and powerful because you can take something away from it." Similarly, in her response to what she would most like to change about her mathematics instruction, she responded that she would like "more lessons like these that [they] can really go back and use in the classroom for deeper understanding right off the bat."

Throughout the workshop series, Amber participated fully to get the most from each of the sessions. While she was often quiet and reserved during group discussions, when she did interject with her thoughts and opinions, she offered very intentional and thoughtful insight relevant to making instruction most impactful for her students. In addition, observations of her through the duration of the workshop series showed Amber deep in thought during activities and attempting to make sense of the mathematics with her partner or small group. Often, she was also observed quietly offering insight to her student teacher who attended the first and second sessions with her.

On the fraction content knowledge test, Amber's responses were mostly correct; however, her responses to the tasks involving fraction operations, tasks two, six, seven, and eight, showed some areas of inconsistency. Three of the four tasks are aligned to fifth grade curriculum standards in the state and have appeared on previous state assessments.

Most notably, on the second task, Amber selected the first problem as one that illustrated $1\frac{1}{4} \div \frac{1}{2}$ (see Figure 4.10). As Amber's work reveals, she equated splitting the pies evenly between two families with splitting something into half-size portions. She then added "half would be divide by two." She used the invert and multiply procedure, which was also used on tasks six and seven, to complete the task. She arrived at a solution of two and a half, a solution which is inconsistent with the context of the problem because if you share one and one-fourth pies with someone, it would not be possible to receive more than the total initial quantity.



2. Which of the following story problems could be used to illustrate $1\frac{1}{4}$ divided by $\frac{1}{2}$? Circle all possible answers. Justify your selection(s). (Source: Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures- Elementary Content Knowledge Item #7)

a. You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?

b. You have \$1.25 and may soon double your money. How much money would you end up with?

c. You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?

You're splitting $1\frac{1}{4}$ into halves or a half. Half would be $\div\frac{1}{2}$

$1\frac{1}{4} \div \frac{1}{2} = 1.25 \div .5$

$\frac{5}{4} \div \frac{1}{2}$

$\frac{5}{4} \times \frac{2}{1} = \frac{10}{4} = 2\frac{2}{4} = 2\frac{1}{2}$

Figure 4.10. Amber's response to task two on the fraction content knowledge test

On tasks six and seven, Amber was able to successfully illustrate and use a model to determine the solution and justify each task; however, it was sometimes difficult to follow her written solution paths. In addition, on task eight, where participants were asked to write an equation and a scenario to match the number line model (illustrating $3 \div \frac{1}{3}$), Amber correctly created a representation to match the model but created a scenario which indicated 3 wholes shared between three people, or 3 divided by 3.

Many of Amber's responses on the initial performance task, where she voiced frustration over some group members talking while she was trying to work, revealed a lack of understanding of the tasks of teaching fractions. For example, on the first task where she was asked to order a set of problems according to how she might sequence them in instruction, she recorded problem d (mixed number divided by a whole number) as the first task in the sequence and problem a (whole number divided by a unit fraction) as the second task. Problems b and c were not included in the sequence. As her justification, Amber recorded “IDK” because she did not know what to write. Amber was able to give more of an explanation during the interview, where she identified problem a as her initial task because it is easiest and can be modeled. She then sequenced the remaining problems as b, then d, and c. She noted the increasing difficulty of problems d and c and stated that “students tend to freak out more with fractions.”

For the second problem on the performance task, Amber’s initial response regarding an alternate problem situation to use during instruction indicated Amber identified the original task as a multiplication problem. The problem Amber selected to use instead of the original problem illustrated a multiplication situation which would result in students multiplying 12 by $\frac{5}{8}$. When asked whether the problem situation she recorded represented a multiplication or division situation during the follow-up interview, Amber responded that she “misunderstood the problem” and that the problem situation she wrote was indeed a multiplication situation. Amber recognized that the original problem situation required dividing 12 by $\frac{5}{8}$ and stated that “the question is asking how many groups of five-eighths are in 12.” Amber was able to reason that if the serving size was $\frac{5}{8}$ of an apple, “you must be able to get more than 12” servings. She then stated the result would be a little over 19 servings. She added that “division and multiplication are the same thing because you’re multiplying by the reciprocal” but was confused about how this similarity connected to the problem situations.

Like Cindy, Amber’s response to task five, as shown in Figure 4.11, also illustrated a contextual situation where the dividend is divided by two, which is inconsistent with the stated expression of $1\frac{3}{4} \div \frac{1}{2}$; however, Amber correctly used the invert and multiply procedure to arrive at a solution of $3\frac{1}{2}$. Interestingly, Amber concluded that "each person would get $3\frac{1}{2}$ slices because your whole would now be four slices" which seems to be verified by her model. While it appeared to make sense at the onset, the change in units, from pizza to slices, made the solution more challenging to understand. Based on the logic Amber used on task two of the performance task where she was able to label the quotient correctly, her response of $3\frac{1}{2}$ would reflect the number of pizzas, not the number of slices. It is possible Amber's solution may also reflect $3\frac{1}{2}$ fourths which would be consistent with the solution for $1\frac{3}{4} \div 2$; however, it is difficult to discern the meaning based on the information in Amber's written response.

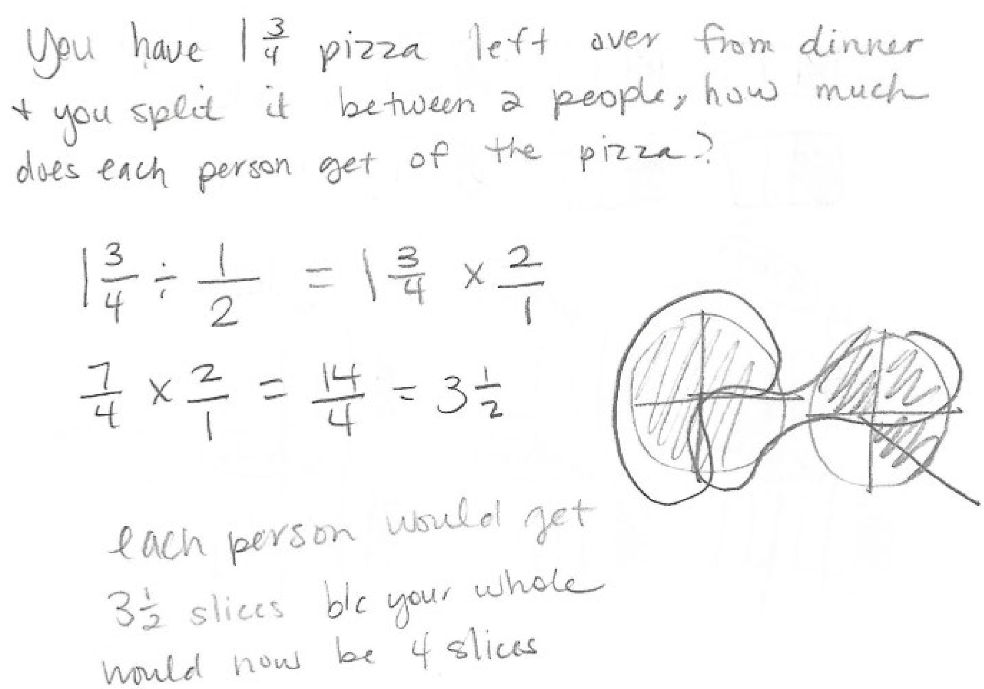


Figure 4.11. Amber's response to task five on the initial performance task.

In response to questions about task five, during the follow-up interview, Amber stated that $1\frac{3}{4} \div \frac{1}{2}$ is the same as $1\frac{3}{4} \times 2$. Amber was diligent and tried to talk her way through the task during the follow-up interview commenting that she should have used rectangles and talked about creating eighths to better represent the task. Ultimately though, Amber was unable to rectify the inconsistencies in her response.

Like Molly, during the professional development experience, Amber participated fully but was more vocal working with her partner or small group than she was in large group discussions. In fact, on her workshop feedback questionnaire, she stated that she "liked the small group structure" of the workshop sessions. She added that the participants "all had input" and were able to have "deep conversations." Amber believes structuring the sessions in this way "led to more discussion, partner work, and pondering."

Throughout the workshop sessions, Amber commented that she appreciated the opportunity to experience the activities. For example, during the first session, after reflecting on the pattern block pictures activity, Amber stated, "using the manipulatives reminded [her] how important it is for kids to be able to manipulate wholes and parts, so they become familiar with trading them out." She also stated that "the drawing of the shapes was most impactful because [she] kept [her pictures] simple but seeing the more complex shapes [made by other participants] made [her] think a bit deeper to challenge [herself]."

During the second session, Amber commented that after the opening discussion about the different types of division situations, she "can better explain the different types of division and what the quotient represents." She also remarked that, using the *Creating*

Cards activity sheet, she was able to relate the models she created for each of the problems to the standard algorithm. Then, during session three, Amber stated that she enjoyed reading the article which explained the reason why the invert and multiply procedure works. Specifically, she stated she liked the explanation of the invert and multiply procedure resulting from getting a common denominator and then dividing the numerators. Amber added that she tells her students the reason, but the article showed the connection in more detail.

Amber's performance on the final performance task was markedly different than on the first assessment. For example, when Amber ordered the problems on the first task, Amber identified problem b, with whole numbers, as the initial task to use to connect to what students already know about division. She then selected problems a, d, and c and justified the sequence by noting that the progression from whole number and unit fraction to mixed number and whole number to mixed number and fraction as moving from more concrete and easier to illustrate to needing a deeper understanding to solve the problems successfully. Amber repeated this line of thought on the second task, stating she would change the problem to model $12 \div 2$, including the whole number initially so that students can "feel successful and realize that they know how to do this." She then stated that she would then replace the whole number with a fraction because students would then have the foundation to better understand.

On task five of the final performance task, Amber correctly recorded a situation that illustrated the number of halves in one and three-fourths; however, the contextual situation seems incomplete (see Figure 4.12) as she stated the cake will be split in halves but then asked how many days the cake will last. To complete the situation, it seems

Amber needed to specify that Caron will eat one-half each day. Amber was also able to correctly model the action of the situation and label the remainder correctly. Like the initial performance task, Amber still used the invert and multiply procedure to complete the equation.

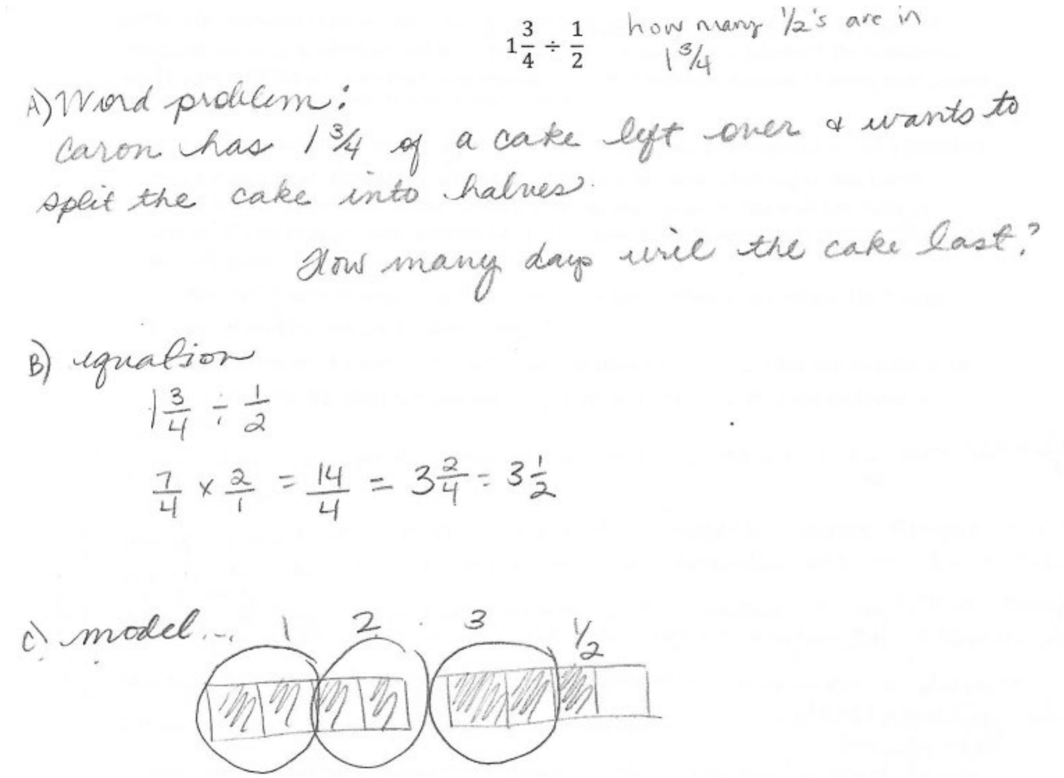


Figure 4.12. Amber's response to task five on the final performance task

Based on Amber's workshop reflections and questionnaire responses, it was clear she valued possessing a deeper understanding of mathematics, her own, as well as, her students. During the workshop experience, Amber noted that having students share their thinking and reasoning with the whole class can help them learn from each other. In fact, Amber remarked that she used the pattern block pictures activity to introduce common denominators and her students were able to make the connections. Amber also noted that she would like to challenge her gifted and talented students with problems involving non-unit fractions to help them develop a deeper understanding. Amber's desire to help her students think more deeply was evident in her contributions to group discussions too. During the last session, when the group discussed the practicality of providing learning experiences that took more time than others but helped students develop a deeper understanding, she chimed in by saying, "when she starts the year making [students] think, they are amazing by the end."

Amber's responses also revealed a desire to better understand fractions herself. She noted she felt she needed more practice using the Cuisenaire rods to model how to partition a fraction into equal-sized units. She also commented that she feels like she needs "to keep digging deeper for more and more understanding to be the best [she] can be." In addition, on the workshop feedback questionnaire, Amber stated she would attend workshops formatted in a similar way and requested more opportunities to work with fractions.

The Case of Tina

Tina's teaching experience reflected a non-traditional educator preparation path, one achieved via a state-approved program for university degree holders that leads to educator certification. Tina attended a public university in Texas and graduated with a bachelor's degree in University Studies but received her teacher certification through an alternative certification program from one of the regional educational service centers in Texas. Tina has 12 years of experience in education, including six years of teaching as a certified teacher and six years, before certification, as an intervention aide.

When asked how she would describe her experience as a mathematics teacher, Tina stated that she “absolutely love[s] the content;” however, “recently, the type of learners [in her classroom] has made it very challenging. Tina also stated that she likes seeing “students struggle productively and have ah ha moments,” but dislikes “all of the other tasks involved with teaching in general” and feels there is a “lack of time to teach [to] the rigor of the TEKS.” If given the opportunity, Tina would like to change the format of her instruction to more of a math workshop model or guided math, like guided reading, format.

Throughout the workshop series, Tina was focused and participated fully during each session. Tina approached each activity with a willingness to learn and was observed trying to connect the new knowledge gained with her existing schema. Tina participated in all group discussions, often being vocal about trying to rectify the disequilibrium created by the new learning and asking questions to try and make sense of the mathematics.

On the fraction content knowledge test, Tina’s responses were mostly correct; however, there were two notable areas of misunderstanding. First, on the second task, where participants were asked to select a problem that could be represented by $1\frac{1}{4} \div \frac{1}{2}$, Tina stated “none of the choices . . . because you would be multiplying $1\frac{1}{4}$ by 2,” meaning none of the situations illustrated $1\frac{1}{4}$ multiplied by two. Second, on the last task, Tina recorded two contextual situations for the division of fractions number line task. As illustrated in Figure 4.13, Tina recorded a multiplication situation leading to an equation equivalent to 3 x 3 = 9 which is consistent with her thinking from the second task where she stated that division is multiplication. The second situation recorded by Tina does illustrate the model.

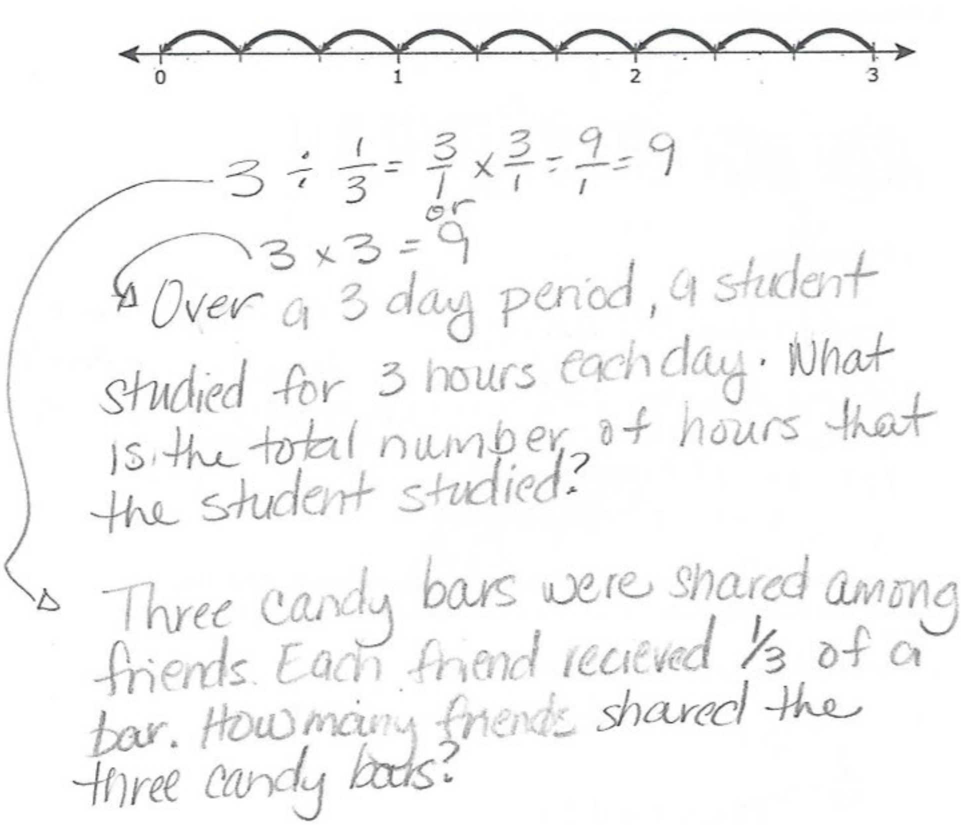


Figure 4.13. Tina’s response to task eight on the fraction content test.

Tina’s responses on the initial performance task indicated a stronger understanding of the tasks of teaching the division of fractions. Most notably, like on the fraction content knowledge test, Tina likened the division of fractions to multiplication throughout her responses. For example, on the first task, where participants were asked to sequence the problems for instruction, Tina selected problem a as her initial task stating that “each lb. is divided in four equal parts showing that division with fractions cause[s] a multiplicative outcome.” Similarly, on task three, when describing the error made by the student, Tina stated “the student is not realizing that [when] dividing by less than one whole, you are creating more parts; therefore, multiplication is taking place." Additionally, during the follow-up interview when asked to explain her model on task five, Tina stated that "splitting it down the middle" is the equivalent of multiplying by two. She added that this results in doubling the number of parts which helped her explain the result of $\frac{14}{4}$ after using the invert and multiply procedure to perform $\frac{7}{4}\ x\ \frac{2}{1}$.

During the workshop series, Tina participated fully and was not afraid to share her thinking, even before her thoughts were fully developed, and ask questions as she attempted to make sense of the tasks. For example, during the third session, Tina tried to make a conjecture, to help the group generalize about fraction division. She stated, "When your divisor is less than one whole, your quotient will be greater than your dividend." The group then helped provide examples to confirm or deny the conjecture.

Tina's comments and questions were also student-centered and revolved around how she could use the activities to help her students understand fractions more deeply. For example, during the second session, Tina initiated a discussion surrounding how to address struggling students who may be confused by using models.

In session one, Tina stated that fractions were fun to teach, but she found it hard for her students to understand the basics of them. After the pattern block activity where the participants built wholes and identified a specific partitioning of the units, Tina stated that "us[ing] a variety of whole representations can stretch a students' mind to reconsider the ideas that they may already have about fractions." She added that she would use a variation of the pattern block activity to give students an opportunity to "explore fractions and wholes with a variety of representations."

During the second session, Tina admitted that she tends to overlook division situations when fractions are involved but added that she would “use the understanding of whole number division to introduce the understanding of fraction division and determine solutions in partitive and quotative situations.” In addition, Tina initiated a discussion about looking at fractions as division problems in and of themselves and how they would connect to algebra in the later years.

Tina was also the first participant to bring a discrepancy to the group’s attention. She and her partner found that some of the remainders they were recording in the card-making activity were not the same as the remainders obtained by performing the invert and multiply procedure or by using a calculator to obtain the quotients. This gave the entire group a chance to grapple with the idea of having a different remainder based on the unit used—something several of the participants were still grappling with at the end of the workshop experience. Again, in session three, Tina attempted to make sense of a remainder in the pizza-making task. The group convinced her that her thinking was incorrect and helped her arrive at the correct reminder.

Throughout the workshop series, Tina consistently referred to having the opportunity to use hands-on learning, such as creating pictures from pattern blocks, dividing paper to make different-sized cards, and drawing models for pizza, to make sense of the tasks as being the most impactful aspect of the professional development experience. Tina added “it helps to work through problems and discuss with peers” as she felt the discussions helped lead her to a deeper understanding.

Tina’s responses on the final performance task were like those of the initial task with one notable difference. On task five, Tina noted the remainder to the problem using two different units—cups and batches. This was a topic of discussion throughout the professional development experience as it is one of the aspects of fraction division that makes it difficult to understand.

Throughout the workshop series, Tina was observed not only making sense of the mathematics for herself but also trying to determine how the learning would be most useful for her students. She also talked about teaching multiplication and division simultaneously, as she felt they were intricately connected, using pictures and conjectures to help her students remember the type of quotient they should expect to get in specific situations.

The Case of Leah

Leah's teaching experience reflected a traditional educator preparation path. She attended a university in Texas, majored in Elementary Education, and minored in mathematics. Leah holds three different teaching certificates: 1-8 General, 1-8 Mathematics, and 4-8 Master Math Teacher. Leah has been in the classroom for 13 years, including seven years in 7th grade and 4th and 5th grade math.

Leah stated that she "choose[s] to teach math because it is what she loves to do." When asked what she enjoys most about teaching math, Leah responded that she likes "listening to kids talk about math and figure things out on their own." On the other hand, she does not enjoy assigning homework or forcing her students to put forth effort in her class. Leah also stated that if she could change something about her math instruction, she would provide more time to teach the fifth-grade math curriculum. Given the timing of the mandated state assessment, fifth-grade teachers must finish teaching content in March so that students are ready to take the standardized math test in early April.

Throughout the professional development experience, Leah participated in each session with confidence, a positive attitude, and ready to talk about math. Leah participated in all discussions, was vocal, sometimes even talking over others, and freely shared her opinions and experiences with the group. From the onset of the experience, it was clear Leah possessed a stronger understanding of mathematics than many others in the room. In fact, several participants commented that they go to Leah when they need help with the content or with planning lessons. During discussions, Leah was often the first one to speak up and it appeared that others waited for her to respond first within the large group. Leah was not afraid to voice her opinion and disagreed with the professional development facilitator on several occasions.

On the fraction content knowledge test, Leah responded to every question correctly and gave enough explanations to justify her thinking. Leah's performance on the initial performance task indicated a strong understanding of the tasks of teaching the division of fractions with one notable exception— task five where she was asked to create a contextual situation, model, and an equation for a fraction division problem.

Like Cindy, Leah's response, as shown in Figure 4.14, illustrates a contextual situation where the dividend is divided by two, which is inconsistent with the stated expression of $1\frac{3}{4} \div \frac{1}{2}$. Leah then recorded an equation that is consistent with this thinking and arrived at an incorrect response of seven-eighths.

During the follow-up interview, when asked to talk through her approach to the task, Leah responded that giving half of the candy to someone was not consistent with division by one-half because it illustrates division by two. Leah then readjusted her contextual situation to illustrate the number of halves in one and three-fourths, a scenario which was consistent with the stated expression. After creating a new scenario, Leah offered a new model which illustrated three full groups of one-half and a remaining piece. When asked how to deal with the remainder, Leah renamed the remainder as one-half but stated that naming the remainder is troublesome when using the model.

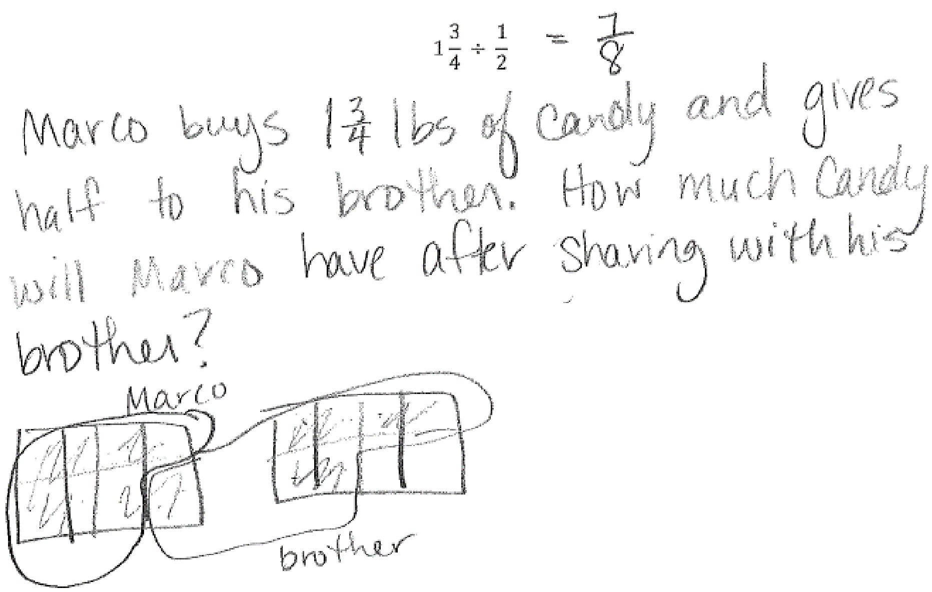


Figure 4.14. Leah's response to task five on the initial performance task.

Leah stated she was excited about teaching fractions and would rather do problems with fractions and decimals than others. During the workshop series, Leah was often observed supporting her colleagues and exhibited confidence when attacking each task. For example, Leah helped Cindy understand the directions during the first activity of session one and shared her processes and thinking throughout the activities.

During the first session, after the participants spent a considerable amount of time grappling with the creation and labeling of their pattern block pictures, Leah provided the group with a strategy, which made sense to her, for determining the fraction to represent each of the pattern block shapes in the pattern block picture. Her strategy included visualizing the parts and decomposing the pictures into green triangles in order to find the common denominator as all the shapes, except the orange square, can be decomposed into the green triangles in this way. She then added that in some of the pictures, she used the smallest shape that could be represented in the picture.

Despite the fact that Leah talked about learning mathematics from a textbook, in a very traditional way, as a student and not being exposed to the methods emphasized in the workshop series during her college years, Leah consistently remarked that she enjoyed exploring with manipulatives, such as pattern blocks, models for making cards, and models for division of fractions. She even commented that, “once she could see it, she could better understand it.” Although Leah struggled with some of the activities, such as the *Cuisenaire Trains* task, she was observed constantly making connections and trying to make sense of the mathematics using the manipulatives.

Leah’s responses on the final performance task were like her initial responses with two notable differences. For example, on the fourth task of the initial performance assessment, like several of the other participants, Leah indicated she would use problem c, which required finding the number of sets of two-sixths that are in six-thirds, or two wholes, because six-thirds can be divided by two-sixths without leaving a remainder. On the final performance assessment, Leah referenced using problem a, which requires finding the number of sets of three-sixths that are in ten-sixths, because the common denominators would make the model very simple to create and follow.

On the final performance task, Leah correctly recorded an equation, represented the problem with a contextual situation, and illustrated a matching model for task five (refer to Figure 4.15).

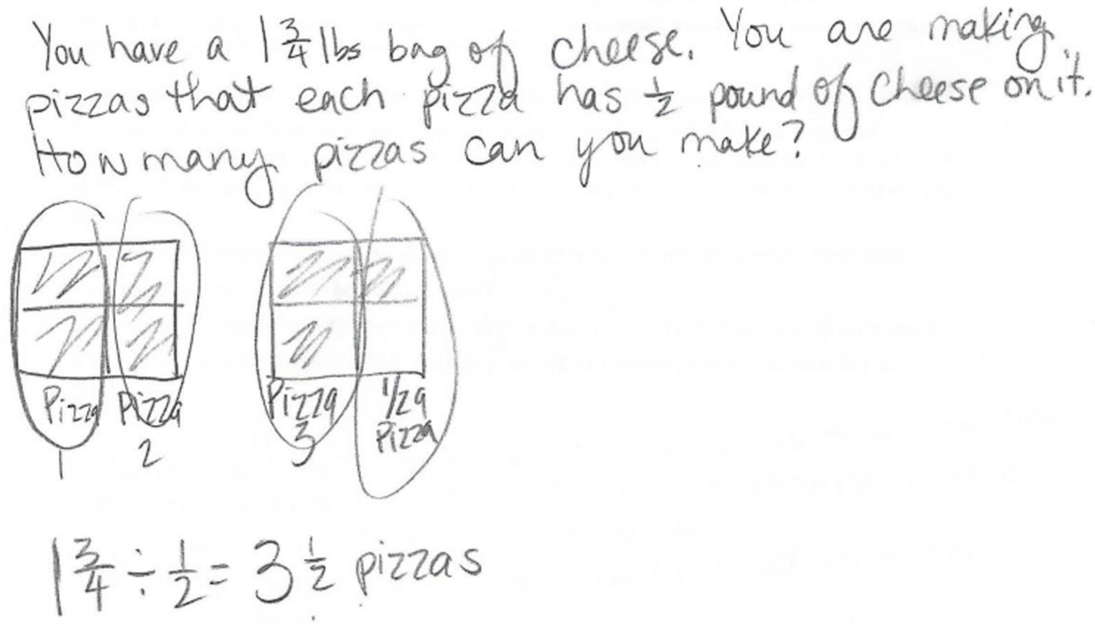


Figure 4.15. Leah's response to task five on the final performance task.

It was evident throughout the professional development series that many of the participants predominently used the invert and multiply procedure to teach the division of fractions after some initial experiences with models. Therefeore, it is worth mentioning that Leah felt the most impactful activity during session three was the reading and discussion of an article surrounding the reason why the invert and multiply procedure is used. In fact, in her reflections for session three, Leah stated, "I will never teach reciprocal the same again. I want my students to know why we use the reciprocal and where it comes from."

Cross-Case Analysis

To conclude the data analysis process, all seven cases in this study were compared to create a cross-case analysis. During the data analysis process, the constant comparison technique was used to search for common themes and attributes of the cases. The results of this analysis were first organized by examining and comparing the results of the fraction content knowledge test and initial performance task. Then individual participant results and case narratives were reviewed for similarities and differences. Next, a comparison of the emerging themes from the individual cases was completed. Finally, each research question was addressed by comparing the data collected through the pre- and post-assessments, follow-up interviews, participants' workshop reflections, session observations and field notes, and both the demographic and workshop feedback questionnaires.

The cross-case analysis includes topics related to common areas of difficulty on the fraction content knowledge test, as well as, an in-depth analysis of question five on the performance task. This discussion addresses challenges surrounding understanding the meaning of division of fractions, confusing the division of one-half with dividing by two, rectifying discrepancies among participants' equations, models, and contextual situations, and the effects of misunderstanding the traditional algorithm. This section ends with a discussion of each of the research questions considering the collected data.

Fraction Content Knowledge

As illustrated in Table 4.2, most of the participants came to the professional development experience with the ability to respond successfully to most of the questions regarding fraction content and skills the participants were expected to teach and reinforce in fifth grade.

The most notable area of difficulty was the second task which required participants to select a contextual situation that illustrated $1\frac{1}{4} \div \frac{1}{2}$. Participant responses revealed confusion between dividing by two, multiplying by two, and dividing by one-half. Specifically, throughout her work, Tina mentioned the division of fractions was essentially multiplication and treated it as such. While no other participants included this notion in their written justifications, it was evident in their work. For example, five of the seven participants used the invert and multiply procedure to solve at least one fraction division problem on the fraction content knowledge test or the initial performance task.

Table 4.2

Summary of results for the fraction content knowledge test

Participant	Q. 1	Q. 2	Q. 3	Q. 4	Q. 5	Q. 6	Q. 7	Q. 8
Molly	0	0	1	1	1*	0*	1*	1*
Cindy	0	0*	1*	2	2	2	1	2
Erin	0	2	2	2	2	2	1	2
Gina	2	2	2	2	2	1	2	2
Amber	2	0	2	2	2	2	2	1
Tina	2	0	2	2	2	2	2	2
Leah	2	2	2	2	2	2	2	2

Key: 0 = no part of the task was completed correctly; 1 = part of the task was completed correctly, i.e. solution, equation, model, contextual situation, or justification; 2 = all parts of the task were completed correctly; * = response missing a component

In addition, it is also worth noting that three of the participants were unable to successfully name the fraction illustrated in the first task in which the picture illustrated two circles where each one was divided into four equal parts. Five of the eight parts were shaded. Participants were given information indicating that the two circles equated to one whole. The five equally shaded parts would then represent five-eighths; however, three of the participants recorded one and three-fourths as the fraction that named the picture.

Understanding Division of Fractions Through Task Five

Meaning of division of fractions. As illustrated in Table 4.3, 71% of the participants were able to successfully calculate the quotient, as indicated by their work on the initial performance task, using the invert and multiply procedure, for $1\frac{3}{4} \div \frac{1}{2}$. The two remaining teachers incorrectly calculated $1\frac{3}{4} \div \frac{1}{2}$ as $1\frac{3}{4} \div 2$. In addition, of the seven participants, 43% were able to create a conceptually accurate model and contextual situation to represent $1\frac{3}{4} \div \frac{1}{2}$. Those participants who were unable to produce an accurate model and/or contextual situation provided a representation which demonstrated confusion of division by one-half and division by two.

Table 4.3

Summary of results for task five of the initial performance task

Participant	Equation	Model	Contextual Situation
Molly	-	N	N
Cindy	N	N	N
Erin	-	Y	Y
Gina	-	Y	Y
Amber	-	N	N
Tina	-	Y	Y
Leah	N	N	N

Key: Y = correct; N = incorrect; a (-) indicates a correct equation that resulted from the use of the invert and multiply procedure

Confusing division by ½ with division by two. Four of the seven participants recorded a contextual situation or model that represented the dividend divided by two. This misconception was represented by dividing the initial quantity between two people or into two equal parts. All four of these participants used food to represent the situation. Three of the four participants included situations where the food was shared between two people. Leah's contextual situation below is representative of the four participants' thinking surrounding the task.

> Marco buys 1¾ lbs. of candy and gives half to his brother. How much candy will Marco have after sharing with his brother?

Like Leah, two additional participants used the term "half" in their contextual situation to indicate splitting the quantity into two equal parts. Amber's situation explicitly included the phrase "split it between 2 people," indicating dividing the initial quantity into two equal parts. Except for Molly, whose model was incomplete, each of the four participant's models matched their contextual situation and showed the quantity being divided into two equal parts.

Rectifying discrepancies. During the follow-up interviews, each of the four participants was asked to rethink the problem included in task five. Of the four participants who responded to task five with a contextual situation and/or model that represented division by two instead of division by one-half, only Leah was able to rectify the discrepancy and correctly adjusted her contextual situation and model to determine the number of halves in the initial quantity.

Molly, whose equation indicated a correct quotient, recognized the disconnect but was unable to rectify the misconception. Cindy, whose work did not include a correct equation, was able to provide a correct quotient but was unable to rectify the difference in the quotient of three and one-half, which she stated, "didn't seem right," and her contextual situation and model. Amber, whose work did indicate a correct quotient, was unable to rectify the difference between her solution and model because her model showed that each person would receive three and a half slices of pizza (which were illustrated as fourths), a detail that was most likely missed by Amber because the solution matched her initial quotient.

Misunderstanding the procedure produces inaccurate representations. During their follow-up interviews, Cindy, Amber, and Tina referred to partitioning into halves as doubling. When Cindy tried to make sense of the task after calculating a different quotient, she stated, "I am essentially doubling it" in reference to the division problem $1\frac{3}{4} \div \frac{1}{2}$. And, during her follow-up interview, Amber stated, "$1\frac{3}{4} \div \frac{1}{2}$ is the same as $1\frac{3}{4}$ x 2."

Use of the invert and multiply procedure led some participants, like Cindy, Amber, and Tina, to believe that multiplication yields the same results as division when working with fractions in a conceptual way. The way the invert and multiply procedure was used by each of these participants illustrated the doubling of the initial quantity; however, neither Cindy nor Amber was able to connect this thinking to her conceptual model or use this line of thinking to develop an accurate contextual situation. Interestingly, while Tina referred to "splitting [something] down the middle" as the "equivalent of multiplying by two," both her model and her contextual situation were accurate and consistent with division by one-half.

Responding to the Research Questions

Research Question 1: In what ways does the professional development series impact teacher participants' understanding of the division of fractions?

The greatest indicator of the impact of the professional development series on the participants' understanding of the division of fractions was the comparison of their initial responses to task five of the performance assessment to their final responses. As noted earlier, while 71% of the participants were able to correctly determine a solution for $1\frac{3}{4} \div \frac{1}{2}$, only 43% of the participants were able to represent the situation accurately using a model and contextual situation; however, on the final performance task, all seven of the participants were able to obtain a correct solution and devise a model and contextual situation which represented the problem.

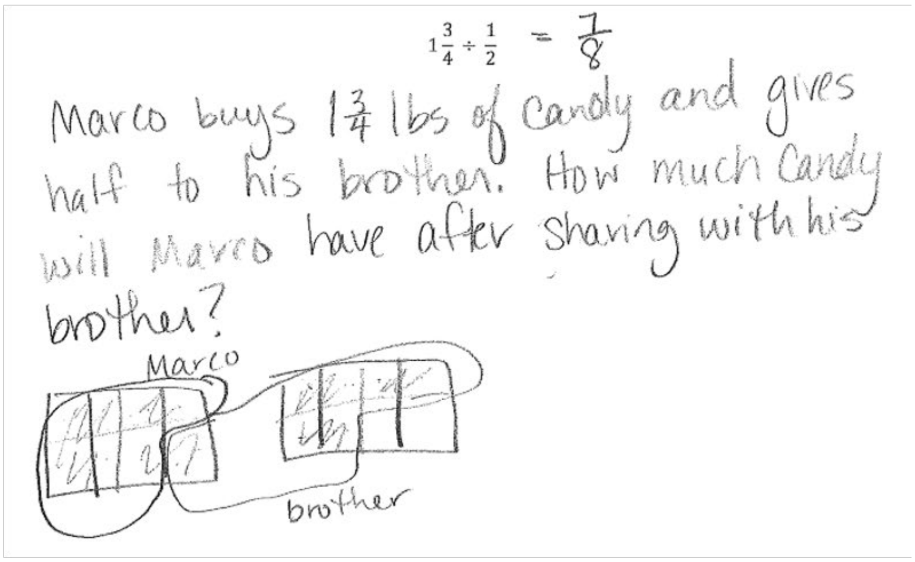
Leah's initial and final responses to task five are illustrated in Figure 4.16. The initial response provided by Leah was indicative of the initial thinking illustrated by most of the participants where they illustrated a situation where they were expected to partition a dividend into equal-sized halves as a situation where they divided the dividend into two equal parts.

In addition to the changes in the participants' responses from the initial performance task to the final performance task, there were several moments throughout the professional development series where participants exhibited changes in their thinking. The paragraphs that follow describe these changes.

Partitive vs. quotative situations. During the second professional development session, teachers participated in a discussion surrounding the difference between partitive and quotative division situations. Several participants commented that they were unfamiliar with the terms. Others were able to connect the terms with the work they had done with whole numbers, such as "how many in each group" or "how many groups." During the discussion, teachers worked together to connect the term 'partitive' to "fair sharing" and the term 'quotative' to "measurement division" (Neagoy, 2017, p. 181).

Several teachers contributed to the discussion trying to rectify the difference between the two terms and connect them to the work they do with their students.

INITIAL



FINAL

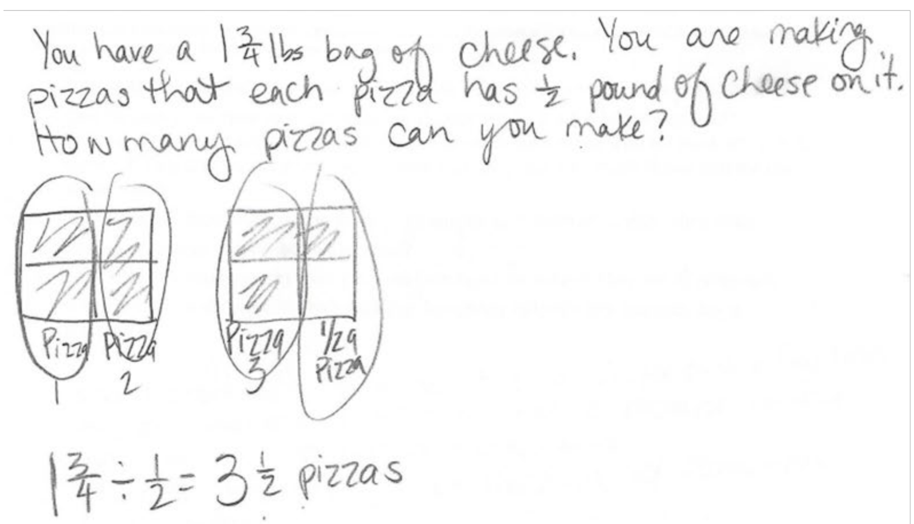


Figure 4.16. Leah's initial and final responses to task five of the performance assessment.

For example, during the discussion, Leah was very vocal about her understanding of partitive and quotative situations. She believed a quotative situation involved the division of a whole number by a fraction. Leah also believed a partitive situation involved the division of a fraction by a whole number. This line of thinking is consistent with a statement she made during the follow-up interview after the fraction content knowledge test and initial performance task.

When the discussion then turned to the “per unit” interpretation of division and a task illustrated a whole number divided by a fraction, Leah grappled with the task along with the other participants and admitted, in her workshop reflections, that she was “still wondering/thinking about partitive if your divisor is a fraction,” indicating the session activities were impacting her thinking about the division of fractions.

Dealing with remainders. The first division-focused activity with which the participants had the opportunity to take part in was one which required teachers to take six sheets of paper and divide them into cards of different fractional sizes, such as two-thirds, one-eighth, and three-eighths. While the teachers were able to easily respond to tasks where they divided the paper into cards with fractional sizes that were unit fractions, like one-half, one fourth, and one-sixth, they exhibited considerable difficulty when faced with a fraction such as seven-eighths.

When six wholes are divided into groups of seven-eighths, there will be a portion of paper leftover that will not result in another opportunity to create a complete card as each whole will have one-eight remaining. The six-eights that remain will not make a new card. The participants really struggled with how to name this leftover portion. Should it be recorded as six-eighths of a sheet of paper or six-sevenths of a card?

Participants really wrestled with the idea of naming the unused portion of paper. This was especially true for those teachers who then used the traditional algorithm, invert the second fraction and multiply, to determine the solution. This process yielded a remainder of six-sevenths. A lengthy discussion ensued with the group trying to rectify the difference. It took some time, but the conversation eventually circled back to a discussion from the first workshop session where participants discussed the importance of conceptualizing the unit. Participants then realized that the solution six-eighths referred to the amount of paper that remained—six-eighths of a whole piece of paper. The solution of six-sevenths, achieved when using the standard algorithm, referred to the amount of an additional card that could be created.

This discussion continued through the second and third professional development sessions as participants continued to wrestle with the idea of the remainder. In fact, at least one participant, Cindy, used the activity with her high-performing students to challenge them. She admitted during a conversation while on a break in the third session that she was still working to understand how to name the remainder, but was willingly to challenge herself, and her students, with content that extended beyond the work fifth graders do in Texas, which only includes division of a whole number and a unit fraction—leaving no remainders.

In the reflections for session two, six of the seven participants stated they found the *Creating Cards* activity to be most impactful for them noting making models, drawing pictures, and discussing how to address the remainder as the most effective aspects of the activity. In addition, like Cindy, Leah noted that the discussion surrounding remainders motivated her to try using fractions for the *Creating Cards* activity which result in leftover paper.

Traditional algorithm. When the professional development series began, it was evident through initial conversations and pre-workshop assessments, that the traditional method was the primary mode used to compute when dividing with fractions despite the participants' standards emphasizing the use of models.

As the results of participants' performance on task five of the initial performance assessment (shown in Table 4.3) and task five of the final performance task (shown in Table 4.4) reveal, 71% of the teachers used the invert and multiply procedure to respond to the item on the initial performance task. Forty-three percent of the participants used the invert and multiply procedure on the final assessment.

Table 4.3

Summary of results for task five of the initial performance task

Participant	Equation	Model	Contextual Situation
Molly	-	N	N
Cindy	N	N	N
Erin	-	Y	Y
Gina	-	Y	Y
Amber	-	N	N
Tina	-	Y	Y
Leah	N	N	N

Key: Y = correct; N = incorrect; a (-) indicates a correct equation that resulted from the use of the invert and multiply procedure

Throughout the professional development experience, teachers used the traditional procedure to complete problem situations and to "check" their responses during some of the hands-on activities where it was difficult to determine the remainder. In addition, during the *Making Pizzas* activity where participants were asked to divide blocks of cheese into fractional portions, there was lots of discussion surrounding using the traditional algorithm to solve the problems which included non-unit fractions and whole numbers, as well as, tasks where participants were asked to divide a non-unit fraction by a non-unit fraction; however, many of the participants tried to find an alternative method.

Table 4.4

Summary of results for task five of the final performance task

Participant	Equation	Model	Contextual Situation
Molly	-	Y	Y
Cindy	Y	Y	Y
Erin	-	Y	Y
Gina	Y	Y	Y
Amber	-	Y	Y
Tina	Y	Y	Y
Leah	Y	Y	Y

Key: Y = correct; N = incorrect; a (-) indicates a correct equation that resulted from the use of the invert and multiply procedure

During the final professional development session, teachers participated in several activities related to understanding the origins of the traditional method for dividing fractions, including watching a video, tackling a problem-solving task, reading an article, and contributing to a culminating discussion. After completing the activities, there was a great deal of discussion surrounding the article and what it meant to invert and multiply. In fact, several participants reflected that the article was impactful for them and that they enjoyed reading it.

In addition, several of the teachers mentioned the *Reason Why We Invert and Multiply* (Cardone, 2015) article reading and discussion experience in their session reflections. For example, Leah stated she found the article most impactful and that she would never teach reciprocal the same way again. She added that she "want[s] [her] students to know why we use the reciprocal and where it comes from." In Amber's reflections, she remarked that she "liked the article and how it explained [the traditional] algorithm as finding common denominators." She added that she tells her students this every year but appreciated the detailed explanation in the article. In addition, Molly commented that she felt the activity emphasizing invented procedures tied to modeling and would help students see where the traditional algorithm comes from.

Research Question 2: How does the professional development series impact teacher participants' instructional practices?

To help students develop a mathematical mindset, Boaler (2016) recommends designing tasks that add a visual component to the work. Therefore, the tasks for this workshop series were selected to meet this goal. In their workshop reflections, participants were asked to describe one activity they found most impactful and then describe how they would transfer this knowledge to their students. The prevailing response to this question was the use of manipulatives, models, and other visuals to make sense of the concepts. The paragraphs that follow expand on this finding.

Use visuals to explore fraction concepts. All seven participants commented on the use of manipulatives, models, or visuals to teach fraction concepts. In workshop session one, participants were asked to complete a task where they explored unit fractions and the relationship between part and whole by creating visual models using pattern blocks. In her workshop reflections for session one, Cindy stated she "enjoyed using the colored [pattern blocks] to represent different fractions." She added that because most of the fraction manipulatives they use are labeled, using pattern blocks, which are unlabeled, "gives [students] the opportunity to explore pieces and wholes."

In her reflections, Erin commented that "the [pattern block] activity really gave a ton of jumping off places for students to explore fractions," including creating a picture and having students analyze the fractional parts or having students create and represent their own models. In addition, both Leah and Tina mentioned using pattern block pictures to provide additional ways to represent and review early fraction concepts before teaching operations. Likewise, Gina remarked, "using [pattern block] manipulatives to show how one-third or another fraction is represented within other shapes" is one way she planned to use the activity with her students.

During the second workshop session, participants used Cusienaire rods to create models for dividing a fraction by a whole number. When Cindy reflected on this activity, she stated, "using the rods to illustrate dividing fractions was awesome" because students "can see why dividing a fraction is answered with a fraction."

Use models to divide fractions. Regarding the division of fractions, five of the seven participants noted multiple times throughout their reflections that they would change their instruction to allow students the opportunity to explore division of fractions with models and/or visuals. Both the *Creating Cards* activity during session two and the *Making Pizzas* activity during session three emphasized the exploration of fraction division with visuals and models.

In their session reflections, Molly and Erin both commented that creating a visual model of each situation helped them better understand the action of division and connect the traditional algorithm to the model. In addition, Leah stated, "using pictures to really explain what's happening in division" added to her professional knowledge. Tina also remarked that "dividing the cards is a great way to introduce/explore the conceptual idea of dividing whole numbers by fractions." Tina added that she "will use visuals to show students that division does not always produce a smaller result"— a big idea of fraction division.

Draw pictures to record solutions. Another component of the workshop tasks included participants drawing models and using recording sheets to help them find solutions to the tasks. In addition to using manipulatives and creating models, the drawing aspect of recording solutions to the tasks was one of the most widely mentioned strategies in terms of the aspect of the professional development series participants found most impactful.

For example, in her workshop reflections for session two, Cindy commented that "drawing pictures made a huge difference" during the *Creating Cards* activity. Additionally, Leah noted that she "really liked drawing the models and coloring" to determine the number of cards that could be made from the six pieces of paper.

In addition, when Erin reflected on session three's *Making Pizzas* task, she stated, "when [she] started sketching out the models it hit [her] what was happening." She added, this aspect "is powerful for students to see as well." Tina also noted that she found "drawing models for pizza" most impactful and that she planned to use "repetitive modeling with various fractions," like the *Making Pizzas* task, "as students explore division of fractions."

Research Question 3: What aspects of the workshop model of professional development are most effective for teacher learning?

One of the features of the design for the professional development series included asking participants to explain responses, reason about solutions, and convince others in the group that their reasoning was correct. Overwhelmingly, the participants identified discussion as both an aspect of the workshop format that contributed to their ability to learn, as well as, the feature they enjoyed the most.

During both the second and third workshops, participants were eager to share their thoughts and ideas surrounding the tasks. For example, during the second workshop session, there was a lot of discussion regarding the difference between partitive and quotative situations. In fact, there was so much discussion between participants about strategies to differentiate between them, that the professional development facilitator had to interrupt the group and ask the participants to stop discussing the task until everyone had a chance to complete the work on their own.

After the teacher participants were given the opportunity to talk with one another, both Erin and Gina discussed how they helped their students breakdown a word problem and connect fraction division problems to whole number division. In addition, Leah was also very vocal during this discussion as she helped the other participants connect the terms ‘partitive’ and ‘quotative’ to the language “how many in each group” and “how many groups.” This in-depth conversation led to a discussion about the preparedness of elementary teachers, who are not mathematics specialists or possess a mathematics certification, to understand these subtle nuances of division.

Later in the session, Tina initiated a lengthy discussion about the difference between the contextual solutions she was discovering and the mathematical solutions that could be obtained by using either a calculator or the traditional algorithm. This discussion was so rich that it spanned both the second and third sessions and connected back to the work the participants completed during the first session where they carried out a task which exemplified why conceptualizing the unit is key and essential to understanding how to compute with fractions.

During the third session, when participants felt even more comfortable with each other and the professional development facilitator, a great deal of time was spent in discussion mode. Early on, teachers voiced frustration surrounding the knowledge, or lack thereof, and misconceptions that their students bring with them into fifth grade. There was also a great deal of discussion about the *Making Pizzas* task regarding how to make sense of the solutions, again leading to more discussion surrounding identifying the unit and corresponding remainder. Many of the participants wanted to go straight to the algorithm as the *Making Pizzas* tasks were harder to conceptualize; however, the participants really tried to devise their own invented methods of completing each of the tasks, which eventually lead to a discussion about the common denominator method of dividing fractions and what it means to "invert and multiply."

Throughout the series, discussion became an important mode for learning as participants grappled with content that stretched their professional beliefs about teaching mathematics and fraction concepts which went beyond their grade level standards. In fact, discussion was mentioned nine times in the workshop feedback questionnaires. Some of the feedback included the following comments:

- Gina stated she "loved that [they] were allowed to share with one another without the fear of being wrong." When asked whether she felt the goals of the workshop could have been accomplished using a different format, Gina replied, "Yes, but the interaction and discussion made it more meaningful."

- When asked what she felt contributed to her ability to learn when compared to other workshops she has attended, Amber responded that she "liked the small group structure." She added that they were all able to have "input and deep conversations" which led to "more discussion, partner work, and pondering."
- Tina, when asked what she enjoyed most about the workshop format, commented "discussions that lead to deeper understanding."
- In response to what she enjoyed most about the workshop format, Leah expressed that she enjoyed "talking about math" and "hearing others' thinking."

While not mentioned as frequently as discussion, both time to explore and work through the activities and the hands-on nature of the workshop series were mentioned throughout the workshop feedback questionnaires. For example, Cindy stated, "I loved all the activities. When I worked through the activities, it helped me see and understand where [students'] issues might be." In addition, Erin commented, "I really enjoyed getting to work through the activities. Most presenters just hand over activities and don't provide the time to do them. These are usually filed away and never thought about again."

Conclusion

In this chapter, results from the fraction content knowledge test, pre- and post-workshop performance tasks, follow-up interviews, participants' workshop reflections, session observations and field notes, and both the demographic and workshop feedback questionnaires were introduced, analyzed, and reported. Like the typical multiple case design, each teacher participant was analyzed and reported as an individual case.

Collectively, the cases were then analyzed using pattern matching and the constant comparison technique. A cross-case analysis was then conducted to search for similarities and differences among the seven cases with respect to themes that emerged from data analysis, as well as, the primary and secondary research questions. Additionally, the cross-case analysis incorporated "generalizations that people can learn from the case" (Creswell, 2018, p. 200) regarding the participants initial thinking about fraction division. A discussion of the implications of these research findings follows in Chapter Five.

CHAPTER FIVE

Discussion and Implications

The most important factor in student achievement is educator effectiveness and research suggests there is a critical need for American teachers to possess a deeper understanding of fractions (Ball, 1990; Borko et al., 1992; Koichu et al., 2013; Ma, 2010; Zhou et al., 2006). These studies revealed teacher deficits and inconsistencies in fraction understanding and the ability to communicate fraction knowledge accurately and effectively. To affect teacher knowledge surrounding fraction concepts, researchers have called for increasing opportunities for teachers to receive professional development while in the classroom so that they will be more effective with their students (Hill et al., 2005; Ma, 2010; Zhou et al., 2006)

In this exploratory multiple case study, the researcher explored how teacher participants' understanding of the division of fractions and their instructional practices could be further developed through a series of focused professional development workshop-type sessions and used as an intervention to increase the pedagogical content knowledge of fifth-grade math teachers utilizing the state curriculum standards, the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012), or TEKS, as a guide.

The major goal of the workshop series was to deliver lessons and learning experiences which provided participants with the opportunity to work with fractions in meaningful contexts, invent personal procedures and algorithms, and express thinking through informal language, pictures, and models. Workshop participants included seven fifth-grade teachers teaching the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012).

In this study, each of the teacher participants was treated as an individual case. The results of each teacher's data were then presented and analyzed. The cases were later compared with one another in a cross-case analysis. Results from each of the cases, as well as, the cross-case analysis were detailed in Chapter Four. The remaining sections of Chapter Five will present a discussion of significant findings and connect them to relevant literature including implications and recommendations, based on the results of the study, for state agencies, in-service professional developers, district mathematics curriculum coordinators, and teachers.

Discussion of Significant Findings Related to the Research Questions and Connections to the Literature

The purpose of this exploratory multiple case study was to explore how a workshop model of professional development can be used to improve teacher knowledge and impact instructional decisions in the area of division of fractions. Three research questions were designed to guide this study.

Primary research question:

1.) In what ways does the professional development series impact teacher participants' understanding of the division of fractions?

Secondary research questions:

2a.) How does the professional development series impact teacher participants' instructional practices?

2b.) What aspects of the workshop model of professional development are most effective for teacher learning?

Through the collection and analysis of multiple data sources, including pre- and post-assessments, follow-up interviews, written reflections, workshop observations and field notes, and participant questionnaires, three significant findings related to developing teachers' understanding of the division of fractions emerged. These findings include:

1. Connecting the action of whole number division with the division of fractions provides a solid foundation for understanding fraction division.
2. Developing lessons and learning experiences centered on real-world situations helps connect the abstract nature of fractions to a more conceptual understanding.
    - Beginning instruction with tasks which are rooted in familiar context help students better understand remainders in fraction division situations.
3. Possessing an incomplete understanding of the traditional invert and multiply procedure limits understanding of the division of fractions.

Finding One: Connecting Whole Number Division with the Division of Fractions

During the second professional development workshop, participants discussed the big ideas of division, including defining division as separating a quantity into equal groups or parts and understanding the different interpretations of division. As mentioned in Chapter Two, the two dominant interpretations for division include the partitive interpretation and the quotative interpretation (Neagoy, 2017). Partitive division, or fair sharing, involves dividing into a specified number of equal sets and quotative division, or measuring, involves dividing into equal-sized groups.

The activities for session two were designed to help teacher participants connect whole number division to fraction division. While both Erin and Leah remarked that they used the division interpretations to make connections to fraction division, Tina commented that she seemed to disconnect these foundational ideas about division where the division of fractions was involved. In fact, during an activity where participants were asked to create a partitive and quotative situation with fractions, several participants had a significant amount of difficulty creating examples of partitive and quotative situations with fractions and asked to use whole numbers instead.

Furthermore, when the professional development facilitator proposed replacing fractions with whole numbers in problem situations as a strategy for making sense of a fraction division task, participants were both curious and puzzled. This led to further discussion about connecting whole number division and the division of fractions and how students could use this strategy to support their thinking.

As Tina stated, sometimes operations with fractions are treated as a new set of procedures and division tends to lose its meaning when fractions are involved (NCTM, 2010); however, the action behind the operations mirrors that of whole numbers. In fact, NCTM (2010) describes operations with fractions and decimals as an underlying essential understanding of the big ideas of rational numbers and state, “the interpretation of the operations on rational numbers are essentially the same as those on whole numbers, but some interpretations require adaptation, and the algorithms are different” (NCTM, 2010, p. 8).

In a study conducted by Sidney and Alibali (2017), they found that "activating children's most relevant, structurally similar whole number concept (i.e., whole number division) immediately before asking them to model an analogous fraction concept (i.e., fraction division) increased children's likelihood of demonstrating the fraction concept correctly" (p. 51). NCTM refers to these as "parallel situations" that can be used to connect whole number division to the division of fractions because "rational numbers are a natural extension of the way that we use numbers" (NCTM, 2010, p. 7).

Using parallel situations became an important learning tool to support teacher understanding of fraction division throughout the professional development series as the participants grappled with more complex tasks each session. This finding provides support for use of parallel tasks in the classroom to help students understand the connection between the action of dividing with whole numbers and the action of fraction division. Studies have shown that students come to the classroom with informal familiarities developed from their own lived experiences to help them deal with tasks involving fractions (Empson, 1995; Mack, 1990; Mack & Campbell, 1993; Smith, 2002); therefore, connecting whole number operations with those of fractions provides an additional scaffold for students to help them develop a deeper understanding of fractions.

Finding Two: Using Real-world Situations Supports Conceptual Understanding

One of the goals of the professional development learning experience was to provide opportunities for participants to solve conceptual fraction division problems and engage in a variety of activities as learners, like the ones they may use with their students. Therefore, all the activities used during the professional development series centered on the context of familiar situations. This aspect of the experience aligns with the recommendations of researchers which indicated that building fraction tasks within the context of real-world situations with which students are familiar provides support for making sense of fractions (Mack, 1990; Burns, 2000; Sharp et al., 2002).

Developing learning experiences with which the teacher participants were familiar afforded them the opportunity to draw pictures and use strategies and methods which made sense for the specific context. In fact, all the teacher participants' reflections indicated that the context of each task allowed them to use visuals, i.e. pictures and models, to conceptualize the tasks and make sense of the learning. Six of the seven participants commented they found having a recording sheet with a place to draw and model each solution impactful. Five of these six participants then stated that providing a tool for students to model the situation is a change they planned to make in their own instructional practices surrounding teaching fraction division.

Familiar contexts help students identify the unit and understand remainders. In Chapter Two, confusion surrounding understanding the unit in a fraction division situation was identified as a barrier to understanding the division of fractions. In an article written by Philipp and Hawthorne (2015), they state "one must attend explicitly to the units and to the fact that multiplication and division require one to conceptualize multiple units and view a quantity in multiple ways" (p. 243). In addition, Petit et al. (2016) state that a quotative situation where the number of groups is unknown poses a greater challenge for students because "students sometimes have a difficult time identifying the unit in quotative division situations" (Petit et al., 2016, p. 190).

Difficulty interpreting the remainder in a quotative situation was a limitation for teacher participants during the professional development experience as well. Like Petit et al. (2016) stated, teachers began to struggle with the concept of a remainder during the *Creating Cards* activity where they divided six sheets of paper into a variety of fractional quantities—a quotative situation. The professional development facilitator created a recording sheet which included a column for the number of cards made and for leftover paper. Due to the ambiguity of the phrase "leftover paper," discussion ensued surrounding what was meant by the terminology—was it referring to the portion of a card that was left over or the portion of a single piece of paper that was left over.

Initially, participants showed no struggle with the activity and both the mood and conversation were light-hearted as teachers created pictures to model the tasks and discussed them with their partners. However, after Tina and Amber completed the first few problems where the solutions were whole numbers and began to confront the tasks that did not have whole number quotients, Tina initiated a discussion with the facilitator. All participants joined in as they began arriving at the tasks where use of a reminder was imminent. To record each of the different situations in this task, most participants drew six rectangles to represent the six pieces of paper and divided them into the number of parts indicated by the fractional size of the card, e.g. six divided by seven-eighths. Teachers then used colored pencils or another shading technique to differentiate between the whole cards. Using this process, a remainder of six fractional parts is easily recognizable; however, what is the unit? Does the remainder refer to six-eighths of a piece of paper or six-sevenths of a new card?

After the professional development facilitator referred back to the "Who's Correct" task used during the first session and reminded teacher participants that both responses are correct depending on the label used to record the solution, bewilderment set in. Tina then stated that when she used her calculator, the remainder did not seem to refer to the amount of paper that was left. For example, when a calculator is used to find the solution for six divided by seven-eighths, the result is 6.8571428. The fractional amount of 0.8571428 corresponds to the fraction six-sevenths, not six-eighths.

This discovery led to a new discussion—that of comparing an answer in context with that of the mathematical answer, like the one achieved with a calculator or using the traditional algorithm because both the calculator and the traditional algorithm assume the divisor to be the unit. Therefore, the remainder is named in terms of the divisor (Lamon, 2012). Still, participants wanted to know which response was the "right" one and had difficulty understanding that if the question does not specify the referent unit, that either answer could be considered correct if it was labeled appropriately.

Discussion around how to interpret remainders became a major factor in both the discussion of the *Creating Cards* and *Making Pizzas* activity as participants continued to wrestle with the concept and considered how they might bridge the discussion with their students. During the *Creating Cards* debrief, Cindy reiterated the importance of students possessing the ability to explain their answer with the remainder. In fact, Cindy returned to the discussion during a break in the third session to say that she had given the *Creating Cards* task to her students and had struggled to help them understand the remainders as she was still grappling with the concept herself.

The rich discussions that proceeded both the *Creating Cards* and *Making Pizzas* activities supported a recommendation by Petit et al. (2016) emphasizing "the need for students to interact with a variety of situations and contexts that include both partitive and quotative division requiring different interpretations of remainders" (p. 192). In addition, Philipp and Hawthorne (2015) suggest, as an instructional implication, “to make the unit, or whole, explicit whenever talking about fractions” (p. 246), such as stating “one-half of one cupcake” instead of just “one-half” even in the primary grades, beginning with Kindergarten.

Finding Three: An Incomplete Understanding of the Traditional Invert and Multiply Procedure

Chinnappan and Forrester (2013) state, “procedurally driven fraction knowledge has limited value and, indeed, could impede the development of the specialized content knowledge and pedagogical content knowledge necessary for quality mathematics teaching” (p. 894); however, from the onset of the professional development experience, beginning with the initial performance task where teacher participants were given an opportunity to demonstrate their pedagogical content knowledge surrounding the division of fractions, it was clear that the invert and multiply procedure was used most frequently to solve fraction division problems. As mentioned in Chapter Four, five of the seven participants arrived at a correct solution for task five; however, their work indicated the invert-and-multiply procedure was used. Of these five teachers, only three of them were then able to create a correct conceptual model and matching contextual situation.

During the follow-up interviews, it was evident that while some participants were able to successfully solve a fraction division problem using the invert and multiply procedure, they did not possess a full understanding of it. Ball (1993) found that many “teachers’ own mathematical experiences and understandings have not emphasized meaning and concepts” (p. 188); therefore, they tended to rely on the use of traditional rules and algorithms which may have limited their understanding of the meaning behind dividing fractions and prevented them from being able to "unpack the conceptual underpinnings of the content" (p. 188). For example, during Cindy's follow-up interview, when given a second opportunity to determine the quotient for $1\frac{3}{4} \div \frac{1}{2}$, she correctly responded 3½, albeit unsure of her accuracy. When the researcher probed further and asked whether this solution matched her model, she recognized the disconnect and tried to rectify it. During this think time, she stated, "I am essentially doubling it," referring to the quantity of 1¾. Molly's model also indicated that she attempted to divide each fourth in half to double the quantity. Molly was also unable to rectify the disconnect.

Even though Tina was able to create a correct conceptual model and contextual situation, her use of the invert and multiply procedure seemed to confuse her ability to connect the three representations (see Figure 5.1). For example, Tina's work showed $\frac{7}{4} \times \frac{2}{1} = \frac{14}{4}$ which she then simplified to three and a half wholes. When asked to explain her process in the follow-up interview, Tina responded that she initially made two wholes and shaded them to illustrate $1\frac{3}{4}$. She then stated that she divided the fourths into two, or halved them, to see if that made sense. This action resulted in her model showing $1\frac{6}{8}$ or $\frac{14}{8}$. When she was asked why she chose to divide her model into eighths, she was unsure of why. In her interview, Tina went on to say "splitting it down the middle" is the equivalent of multiplying by two which doubled the number of parts. She then connected this to $\frac{14}{4}$ and the need to make groups of $\frac{4}{4}$ to illustrate that one cake recipe needed four equal parts of flour. While this line of logic may have made some sense to Tina at the time, it does not demonstrate an efficient procedure for dividing with fractions using a model.

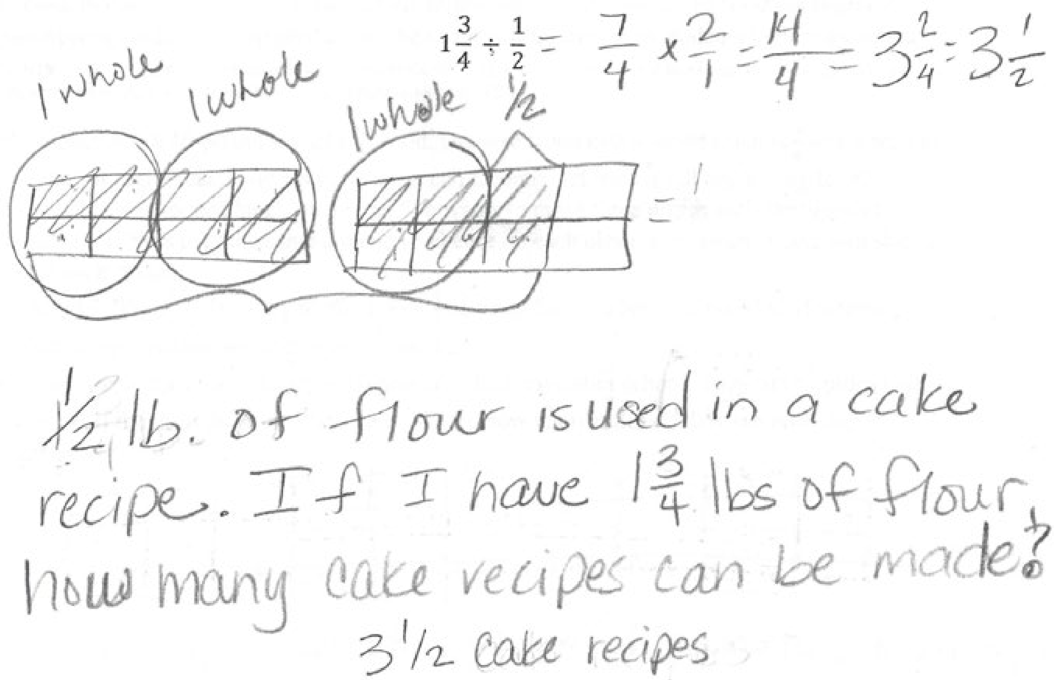


Figure 5.1. Tina's response to task five on the initial performance task.

NCTM (2010) states "multiplying by 2 is the inverse operation of dividing by 2" which supports the notion that "every division problem can be reformulated as a multiplication problem" (p. 15). While it is evident that all the teacher participants, either on their initial performance assessment or in their follow-up interview, were able to calculate the quotient for $1\frac{3}{4} \div \frac{1}{2}$ correctly, several teachers unsuccessfully attempted to connect the inverse relationship to their conceptual models. While it is difficult to say with 100% certainty that Molly's and Cindy's thinking were going in this direction, Tina saw $\frac{14}{4}$ in her model and made groups of four-fourths to arrive at three and one-half. In actuality, the $\frac{14}{4}$ that Tina saw is $\frac{14}{8}$ which is equivalent to $1\frac{6}{8}$ or $1\frac{3}{4}$. It is more likely that $\frac{14}{4}$ refers to 14 ÷ 4, as a division computation, and does not mathematically connect to the model. This incomplete understanding of the traditional invert and multiply procedure limited teacher participants' conceptual understanding of the division of fractions.

Van de Walle and Lovin (2006) assert that the invert the divisor and multiply method "may be one of the most poorly understood procedures in the K-8 curriculum" (p. 177) and, as can be seen in the participants' work, overreliance on the traditional algorithm contributes to incorrectly connecting a conceptual understanding of fraction division with multiplication. Both Van de Walle and Lovin (2006) and Neagoy (2017) urge that teachers should allow algorithms to develop naturally and advocate that students be provided the opportunity to develop invented methods for computing before exposure to the more traditional algorithms.

Additionally, Van de Walle and Lovin (2006) state "the understanding that children gain from working with invented strategies will make it much easier for [teachers] to teach the traditional methods" (p. 105). In fact, both Molly and Cindy commented on how modeling the fraction division situations helped them invent procedures that connected to the math behind the algorithm—this was especially true when participants were presented with division situations that moved beyond the work they do with their fifth graders in the classroom. Teacher participants were also intrigued during the debrief of the *Creating Cards* activity during session two where the professional development facilitator modeled how developing a line of thinking while reviewing the solutions to the tasks helped students "invent" a fraction division procedure that would lead to understanding the traditional algorithm.

Summary of Significant Findings Related to Research Questions

Three major themes emerged through the analysis of the teacher participants' pre- and post-assessments, follow-up interviews, workshop reflections, and exit questionnaires, as well as, through the researcher's observations and field notes:

1. Connecting whole number division to fraction division is essential to support understanding of the concept of the division of fractions.
2. Using familiar, real-world contexts supports a conceptual understanding of fraction division, specifically where remainders are involved.
3. Possessing an incomplete understanding of the traditional algorithm prevents one from acquiring a complete understanding of dividing with fractions.

During the professional development series, making connections to whole number operations provided participants with a foothold with which they could better understand fraction division situations. In addition, throughout the professional development series, teachers commented on how the accessible context of the tasks made them easier to conceptualize and model, both visually and with tools. Finally, diving into deep conversation surrounding the reason the invert and multiply procedure is used and how it connects to the action of fraction division provided a platform for teachers to better understand the procedure for themselves and more effectively use the algorithm with their students. All three of these findings connect to the findings of other researchers in both understanding fractions and the division of fractions.

Implications and Recommendations for Developing Teachers' Understanding of Fraction Division

Findings and results from this study offer several implications and recommendations for specific groups to help teachers develop both their mathematical content knowledge and pedagogical content knowledge surrounding the division of fractions. The results and findings of this study offer implications for classroom teachers as well. In addition to discussing the implications for each group, the researcher also provides recommendations to support the development of the two main areas of mathematical knowledge for teaching. These agencies and recommendations include:

- State Agencies
    - Bridge whole number division with fraction division in the required curriculum.
    - Create professional development for teachers illustrating appropriate teaching practices.
- School District Curriculum Leaders and Professional Development Personnel
    - Provide professional development in the area of fractions for teachers in the younger grades.
    - Design similar professional development workshops.
    - Determine next steps to further develop teachers' content knowledge in the area of mathematics.
- Teachers
    - Use contextual situations to teach fraction division.
    - Use visuals and models to teach fraction content.

The implications and recommendations for each of these groups are detailed in the sections that follow.

State Agencies

Each public school's curriculum is dictated by a state agency. Whether the curriculum is the *Common Core State Standards for Math* (NGA Center & CCSSO, 2010) or a specific set of state standards, like the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012), each state determines which standards teachers teach at each grade level. This means the standards must provide appropriate roadmaps that will lead to a complete understanding of fraction content and skills. This understanding should begin in the primary grades, i.e. Kindergarten, as well as, grades one and two, with a focus on foundational skills and essential vocabulary students will need to be successful with future fraction content in grades four and five.

During the professional development experience, a discussion surrounding the division of whole numbers led to a conversation about how fractions were introduced and whether students were familiar with recording a remainder as a fraction. Teachers revealed that this skill was not explicitly stated in their fifth-grade standards. In fact, it is not listed in the fourth-grade standards either (Texas Education Agency, 2012). It is, however, included in the *Common Core State Standards for Math* (NGA Center & CCSSO, 2010) within the curriculum standards for grade five.

In a book designed to decode the *Common Core State Standards for Math* (NGA Center & CCSSO, 2010), Gojak and Miles (2016) assert that when students begin working to interpret a fraction as division, they "may initially think that you cannot divide a smaller number by a bigger number"; however, providing "good problems to solve" and "many opportunities to explore with models" will help students develop a conceptual understanding of a topic that is often misunderstood because students are frequently told that division makes things smaller. Neagoy (2017) urges this false belief can be debunked by using patterns to illustrate how the quotient changes when the same dividend is divided by whole numbers and fractions.

Interpreting the remainder evolved into a major topic of conversation during the professional development experience as teachers grappled with the idea of fractional remainders and whether the remainder that resulted from a mathematical approach or a conceptual approach was accurate. Interpreting a fraction as a division situation can help bridge the gap between understanding whole number division and fraction division. Gojak and Miles (2016) state that "modeling what part of a full group is left over will help [students] understand the meaning of the remainder when it is expressed as a fraction" (p. 158).

Interestingly, on the initial performance task, when asked to sequence a set of problems involving fraction division, only three of the participants selected to start with task b which required students to divide 16 by three to determine the amount of cheese to use on each of three pizzas from a 16-ounce block of cheese. Dividing 16 by three yields a result of five and one-third. On the final performance task, two teachers selected this as a starting place. Cindy was the only teacher to select it as a starting place on both the initial and final performance task stating that she would "start with whole numbers." She added that it was important for students to understand what part of the division problem is first, the dividend, because after that, it didn't matter what numbers were used.

The teacher participants' performance on the sequencing instruction task shed some light on the importance of creating an explicit progression of learning, or learning trajectory, (Confrey, Maloney, & Corley, 2014) within the curriculum standards. Confrey, Maloney, and Corley (2014) define learning trajectories as "conjectures, based on empirical study of student learning and of how student ideas develop from naïve conceptions to learned ideas" over time (p. 721). Understanding fractions as division helps bridge students' understanding of division with whole numbers and fraction division as "students develop new strategies that combine elements of dealing and equipartitioning," a big idea of understanding fractions (Confrey, 2012). This provides students with additional opportunities to model contextual situations, deconstruct misconceptions, and make sense of remainders.

In conjunction with making the progression of understanding explicit and including a reference to interpreting fractions as division situations, state agencies should invest resources to provide quality professional development and learning opportunities for teachers. This way, they can ensure that each teacher is provided with professional development opportunities to successfully teach grade level content and skills.

School District Curriculum Leaders and Professional Development Personnel

Teacher educator and certification program developers play an important role in preparing teachers to teach mathematics to their students; however, learning does not stop once pre-service teachers enter the field. All teacher preparation programs are different; therefore, pre-service teachers bring different amounts of knowledge with them. Once teachers enter the classroom, school district curriculum leaders and professional development personnel should continue to support and enhance teachers' mathematical knowledge for teaching. Recommendations for curriculum leaders and professional development personnel include providing professional development in the younger grades, determining next steps to further develop teachers' content knowledge, and designing similar professional development workshops.

Provide professional development for teachers of Kindergarten through fourth grade. One of the chief concerns of the teacher participants during the professional development experience was the preparation of teachers in the grades preceding fifth grade as they felt these teachers were inadequately prepared to teach fraction content that would support the work they do with students. Reeder and Utley (2017) urge "there is a need for continued robust experiences with fractions" and for "elementary and middle level teachers to deepen their understanding of fractions and improve their pedagogical practices related to teaching fraction concepts" (p. 314). This becomes an essential recommendation for school districts. If teachers in the early grades do not possess a solid understanding of fraction content, they will be unable to prepare students with the thinking necessary to be successful in the middle grades.

Often, teachers receive professional development for a specific standard or set of related standards, such as fact fluency or computational strategies; however, these sessions are generally geared for a specific grade level or even two grade levels when content overlaps. While this method does provide targeted instructional strategies for grade-level teachers, it does not ensure systemic change across grade levels. In order to improve students' understanding of fraction concepts, teacher knowledge surrounding them must be improved as well.

Design similar professional development workshops. The workshop feedback questionnaire provided insight into what teachers liked most about the format of the workshop series. As discussed previously, teacher participants most enjoyed the discussion aspect of the workshop. While not unique to the workshop format, providing teachers an opportunity to tackle a task, as their students would, and then debrief the task with other teachers afterwards offered a chance for them to listen to the way others addressed the task, reason about the solutions, and discuss the efficiency and effectiveness of the approaches. This method was repeated throughout the professional development experience and contributed to the success of this workshop format. In fact, when asked whether they would attend a similar workshop format hosted by their school district, all seven participants replied yes.

Time is an important factor for classroom teachers as they are responsible for a large amount of content but only have a limited amount of time to teach it. Time was referenced on five of the seven teachers' workshop feedback questionnaires. Several teachers commented that missing time with their students to attend the workshop was stressful for them as they already felt pressed for time. Leah mentioned that she would have preferred the sessions to be closer together. While the selection of workshop dates was beyond the control of the professional development facilitator, the time between sessions was longer than preferred. Furthermore, changes to the teachers' schedules pushed back the date of the final session one week later than planned which caused it to occur near the end of the teachers' fraction division unit. With these ideas in mind, a multi-day workshop held at a different time of the year, possibly in the summer, may be preferable for teachers.

Determine next steps to further develop teachers' content knowledge. The professional development experience provided a springboard for strengthening teachers' understanding of the division of fractions; however, the conversation needs to continue. Teachers must continue to strengthen their understanding of essential concepts and skills because "it is not enough for teachers to rely on their past experiences as learners of mathematics" (Conference Board of the Mathematical Sciences, 2012, p. 23).

Specifically, more work needs to be done regarding procedures and algorithms, such as the common denominator approach, for dividing fractions, including an in-depth study on identifying when to use a specific strategy and helping students match the strategy to a fraction division situation. In addition, providing learning that goes beyond that of a teacher's current grade level helps support their understanding of fraction division on a larger scale that will in turn help them be better prepared to support their students (Hill et al., 2004).

Teachers

As mentioned in Chapter One, research shows that teacher effectiveness has the greatest impact on student achievement; as such, teachers provide instruction and interact with students daily. The following recommendations are suggested to transfer the most valuable aspects of the professional development experience, based on teacher participants' reflections and workshop feedback, to the classroom.

Use contextual situations to teach fraction division. One of the most impactful aspects of the design of the workshop series were the tasks that were used. As a goal for the experience, solving problems in real-world, contextual situations was a major success.

Using situations that were familiar to the participants allowed them to create models and visual representations that helped them break down the tasks and make sense of the remainder, a struggle for students and adults alike.

Additionally, presenting fraction division within the context of a real-world situation provides a comfortable platform from which students can work. Smith (2002) asserts that students' understanding of fractions originates in "children's social activities with physical objects" (p. 5). An example of this is fair sharing and halving, something with which most children have some experience. Empson's (1995) research demonstrated that even young children can tackle fraction division problems, sharing situations in particular, when in a familiar context because this knowledge is intuitive. Math teachers can use this intuitive understanding to build up to more challenging and abstract areas of fractions.

Use visuals to teach fraction content. Another impactful aspect of the professional development experience was the role of visuals. In each of the activities included in the workshop series, drawing and/or modeling was an essential component. In fact, it was the most mentioned aspect of instruction teacher participants stated they would change regarding their instructional practices surrounding fractions. Use of visuals, manipulatives, and models also led participants to develop their own algorithms and invented procedures, a teaching practice research has identified as essential for the development of meaningful fraction algorithms (Bezuk & Cramer, 1989; Sharp et al., 2002; Van de Walle & Lovin, 2006).

Recommendations for Future Research

Several ideas emerged from this study that future studies could use as a foundation on which to build. The following ideas could benefit from further investigation:

- Teacher Knowledge and Division of Fractions
  - Connection between teacher knowledge and student achievement
  - Role of the traditional algorithm in early grades fraction instruction
  - Alternate fraction division strategies
- Expanding the Current Study
  - Broader range of teachers
  - Connection to the *Common Core State Standards for Math*
  - Extend the professional development experience

Future Research on Teacher Knowledge and the Division of Fractions

Connecting teacher knowledge to student achievement. While this study did not explore how, or even if, student achievement was impacted by strengthening teachers' mathematical knowledge through the professional development experience, it is worthwhile research. Research conducted by Ball et al., (2005) and Hill et al., (2005) illustrated a positive relationship between mathematical content knowledge, measured by answering items correlated to both common and specialized mathematical knowledge, and student achievement. Investigating how professional development experiences, like the one used in this study, where the focus is improving teacher knowledge surrounding specific topics could even the playing field for students of average teachers when compared to above-average teachers.

The traditional algorithm. Throughout the professional development experience, use of the traditional algorithm played a major role in the way participants approached the tasks as it was always attached to the way they, individually, made sense of the division of fractions. In a study conducted by Chinnappan and Forrester (2014), results supported the argument that "procedurally driven fraction knowledge has limited value and, indeed, could impede the development of specialized content knowledge and pedagogical content knowledge necessary for quality mathematics teaching" (p. 894). In addition, Ma's (2010) work revealed that an inadequate understanding of computational procedures for the division of fractions impeded teachers' understanding of the meaning of the operation; however, it is used relentlessly in American education.

During the third and final session of the workshop series, teacher participants had an opportunity to discover the reason behind the invert and multiply procedure and discuss why it works; however, this brief introduction just scratched the surface in helping teachers make sense of its connection to fraction division and its inverse relationship with multiplication. More work in this area may better help teachers feel equipped to tackle teaching the division of fractions without the use of this technique until students are ready and able to articulate the meaning behind it and how and why it is used.

Selecting a fraction division strategy. When people think about how to solve a fraction division problem, typically, the first strategy that comes to mind is the traditional algorithm, or invert the second fraction and multiply; however, this is not the only method. As a matter of fact, during the professional development series, several teachers "discovered" the common denominator method where, like when adding and subtracting fractions, a common denominator is obtained before completing the operation (Yeping, 2008).

Additionally, McNamara (2015) offers the "divide-across" method, where students divide across the numerator and denominator of both fractions to obtain the quotient, as a suitable alternative in some fraction division situations. McNamara (2015) asserts that when students select a strategy based on the type of numbers in the problem, "this type of decision making on the part of the students, as opposed to blindly using an algorithm with little understanding, is a great way to support students to make sense of mathematical situations" (p. 125). Further research on other methods and algorithms that can be used instead of the traditional algorithm could be advantageous to teachers as they begin helping students make sense of fraction division.

Expanding the Current Study

Broader range of teachers. The current research study was designed to support fifth-grade teachers teaching the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012) to determine how, and if, their pedagogical content knowledge could be strengthened with a series of professional development workshops. While the results of this study show a great deal of promise for the process, future studies could examine whether these results could be achieved with a more diverse group of teachers, including those with varying levels of subject-matter knowledge and educational backgrounds, or a group of teachers from different grade levels.

Connection to The Common Core State Standards. The workshop series used in this study was designed specifically for Texas teachers teaching a specific set of state standards. While not an exact match, there are many similarities between the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012) and the *Common Core State Standards for Math* (NGA Center & CCSSO, 2010). The activities used in this study were not applicable to just Texas educators and could be used with a wider group of teachers. Future studies could explore whether the workshop series could be replicated with Common Core teachers teaching a similar set of standards at grade five.

Extending the professional development experience. In Chapter One, the length of the workshop was listed as a limitation of the study due to the limited availability of the teacher participants. Both the work of Yoon et al. (2007) and Loucks-Horsley et al. (2010) offer recommendations for providing more effective professional development experiences, including longer seat time and combining the experience with classroom observations or demonstration lessons. To expand on the work done in this study, future studies could be designed to include more seat time and an additional opportunity to support teachers outside of the workshop setting.

Reflections from the Researcher

The purpose of this study was to explore how teacher participants' understanding of the division of fractions and their instructional practices could be strengthened through a series of focused professional development workshop sessions and used as an intervention to increase the pedagogical content knowledge of fifth-grade math teachers utilizing the state curriculum standards, the *Texas Essential Knowledge and Skills* (Texas Education Agency, 2012), or TEKS, as a guide. The researcher anticipated the workshop series would positively influence the participants' teaching, instructional practices, and pedagogical content knowledge surrounding the division of fractions.

The field of research offered promise for the use of professional development to further develop teachers' mathematical knowledge for teaching, but little work had been done with in-service teachers in the area of division of fractions. Dissatisfied with the amount of literature available to recommend how to help in-service teachers strengthen their understanding of the mathematics they teach, the researcher sought to develop a professional development workshop series that addressed best practices in the teaching and learning of fractions, as well as, provided a well-crafted venue for learning. The aim of the researcher was to highlight aspects of the professional development series that worked well and identify areas that needed more time or could be improved to inform her own understanding of best practices for designing professional development workshops for educators.

Reflecting on the results of the study, the researcher noted several ideas that emerged from the work regarding teacher knowledge and the division of fractions. Use of the traditional algorithm dominates the teaching of the division of fractions but at high cost. Overuse of this strategy prevents teachers from fully understanding what it means to divide with fractions causing a disconnect between the instruction their students need to be successful with fraction division and the instruction received. The current field of research urges specific practices, such as real-world contexts that are familiar to students and visual models, be used when teaching fraction computations. Much of this research also urges traditional algorithms be left until later in a student's school career and favors students being afforded the opportunity to invent their own algorithms and procedures that they develop as their understanding evolves.

Results of this study also suggest that professional development can be successfully achieved via a series of professional development experiences, as opposed to a one-day session, where teachers have an opportunity to "play" with the math and tackle the tasks as their students might. In fact, because of this, teacher participants reflected that they planned to use the activities with their own students, even allowing students to work beyond what their mathematics standards require—a huge feat considering the discomfort that some of the teachers, like Cindy, had while working through the content themselves.

In addition, having the opportunity to debrief the activities and discuss how to transfer the learning to the classroom proved to be a successful aspect of the workshop series and was the aspect of the workshop format that participants stated most contributed to their learning and what they enjoyed the most. The workshop design, as outlined in Chapter Three, was shown to be a successful method of learning and approach to strengthening the subject-matter knowledge and pedagogical content knowledge of in-service teachers. Furthermore, the support teachers provided to each other during the experience reminded the researcher of the importance of collaboration as a component of professional development.

Yin (2017) states the case study is used "to contribute to our knowledge of individual, group, organizational, social, political, and related phenomena" (p. 4). The case study design of this study allowed the researcher to become immersed in the details of each teacher's story and invested in their professional growth. Building relationships with and getting to know each of the participants afforded the researcher the opportunity to work more closely with the group, an aspect of working with teachers that is often hard to obtain and difficult to succeed without. This may not have been possible if the researcher did not also serve as the professional development facilitator.

The researcher began this study with her own biases. As a lover of mathematics and as a former middle grades mathematics teacher, it is difficult to understand how well elementary school teachers are equipped to teach complex topics like the division of fractions; however, the results of this study proved that no matter what level of mathematics content knowledge one possesses or the number of years they have taught mathematics, both subject-matter knowledge and pedagogical content knowledge can be improved with the use of carefully-crafted and responsive professional development experiences. The recommendations and implications from this study will be beneficial for the researcher was she continues her work in educator development. Using hands-on activities, rooted in contextual situations, as well as, incorporating discussion will become staple components of her professional development work.

Conclusion

This study was predicated on the idea that the teacher is the "most expert knower of mathematics in the classroom" (Lampert, 1990, p. 41). Lampert (1990) also stated,

> Given my goal of teaching students a new way of knowing mathematics, I needed to demonstrate what it would look like for someone more expert than they to know mathematics in the way I wanted them to know it. The role I took in classroom discourse, therefore, was to follow and engage in mathematical arguments with students; this meant that I needed to know more than the answer

> or the rule for how to find it, and I needed to do something other than explain to them why the rules worked. I needed to know how to prove it to them, in the mathematical sense.

Designing a professional development workshop experience to support teachers in strengthening their own understanding of the division of fractions and developing instructional practices to help their students make sense of fraction division provided valuable insight into what is not only needed to support teachers in this area but others as well. Incorporating real-world contexts, visual models, and opportunities for discussion as critical components for learning are universal and can be applied to all professional development sessions in mathematics. In addition, while not a cohesive framework, the three recommended practices for the teaching and learning of fractions as introduced in Chapter One, through the use of contextual tasks, invented procedures, and contrived models, proved to help strengthen the teacher participants' understanding of the division of fractions and helped them make sense of the operation.

Much of the professional development received by in-service teachers is through the workshop model; however, little attention has been paid to the design of the workshop by professional development personnel. Using a workshop as more than just a method to dispense materials and share curricular changes is an essential change that must be made by our school districts if they hope to help teachers become expert knowers of mathematics. Tasks developed to highlight the idiosyncrasies of fraction computations, as well as, a carefully-designed professional development experience will help close the gap between effective and ineffective teachers, help teachers make sense of fraction operations, and increase student achievement—a fundamental goal of education.

APPENDICES

APPENDIX A

Baylor University IRB Approval

BAYLOR
UNIVERSITY

INSTITUTIONAL REVIEW BOARD – PROTECTION OF HUMAN SUBJECTS IN RESEARCH

NOTICE OF DETERMINATION OF NON-HUMAN SUBJECT RESEARCH

Principal Investigator:	Shametria Banks
Study Title:	Increasing Pedagogical Content Knowledge in Division of Fractions through a Three-Part Professional Development Series: A Multiple Case Study
IRB Reference #:	1315178
Date of Determination:	09/04/2018

The above referenced research project has been determined to not meet the definition of human subject research under the purview of the IRB according to federal regulations at 45 CFR 46.102(d) & (f). Specifically, this is not generalizable.

The following documents were reviewed:

- IRB Application submitted on 08/29/2018
- Protocol, dated 08/18/2018

This determination is based on the protocol and/or materials submitted. If the research is modified, you must contact this office to determine whether your modified research meets the definition of human subject research.

If you have any questions, please contact Deborah Holland at (254) 710-1438 or Deborah_L_Holland@baylor.edu.

Sincerely,

Deborah L Holland

Deborah L. Holland, JD, MPH
Assistant Vice Provost of Research
Director of Compliance

OFFICE OF THE VICE PROVOST FOR RESEARCH

One Bear Place #97310 • Waco, TX 76798-7310 • (254) 710-3708 • FAX (254) 710-7309 • http://www.baylor.edu/research/irb/

APPENDIX B

Consent Form

Baylor University

School of Education: Department of Curriculum & Instruction

<u>Consent Form for Research</u>

PROTOCOL TITLE:	Increasing Pedagogical Content Knowledge in Division of Fractions through a Three-Part Professional Development Series: A Multiple Case Study
PRINCIPAL INVESTIGATOR:	Shametria Routt Banks
SUPPORTED BY:	Baylor University

Introduction

Please read this form carefully. The purpose of this form is to provide you with important information about taking part in a research study. If any of the statements or words in this form are unclear, please let me know. I will be happy to answer any questions. You have the right to discuss this study with another person who is not part of the research team before making your decision whether to be in the study.

Taking part in this research study is up to you. If you decide to take part in this research study, I will ask you to sign this form. I will give you a copy of the signed form.

The person in charge of this study is Shametria Routt Banks (Faculty Advisor: Dr. Sandra Cooper). I will refer to this person as the "researcher" throughout this form.

Why is this study being done?

The purpose of this study is to explore how a deeper understanding of fraction division can be developed through a grade-level focused professional development workshop series.

The researcher is asking you to take part in this study because you are a fifth-grade math teacher who teaches the Texas Essential Knowledge and Skills (TEKS).

The research study will include about 7 participants.

How long will I take part in this research study?

The researcher expects that you will be in this research study for the duration of five workshop sessions conducted over a four-month period (September – December). During this time, the researcher will ask you to attend 5 workshop sessions at the location designated by [redacted].

What will happen if I take part in this research study?

If you agree to take part in this study, the researcher will ask you to sign the consent form before conducting any study activities.

Study Specifics:

- We will meet five times during the months of September, October, November, and December.

    - The first September meeting and the December meeting will be conducted after school. The main purpose of these meetings is to complete a performance task so that the researcher can compare September responses to December responses. The September meeting will also include a fraction-focused content test that will be used to better help the researcher understand your content knowledge surrounding fractions.
    - The second September meeting, as well as, the October and November meetings will be held during the second half of the school day. The district's secondary math coordinator will coordinate substitutes for each of the workshop participants for this time out of the classroom.
- The half-day workshop sessions will focus on teaching practices that will help develop a deeper understanding of fractions by presenting activities that build on students' informal knowledge, create meaningful contexts, and construct an understanding of fraction division based on whole number operations. In addition, participants will engage in a variety of learning experiences, discuss and analyze student work, and prepare lessons and activities to use in the classroom.
- During the workshop sessions, the following data will be collected.
    - Recorded observations to capture group discussions during activities
    - Anecdotal notes used in conjunction with recorded observations
    - Written reflections completed by participants to reflect on the goals of each meeting
    - Demographics/Background questionnaire (completed during the last workshop session)
    - Workshop feedback questionnaire (completed during the last workshop session)

If at any time, you feel uncomfortable with study activities or do not wish to continue with the study, please communicate your desire to leave the study in writing. Please note: Even if you decide to the leave the study, you may still choose to participate in the professional development sessions.

Audio/Video Recording

The researcher would like to make a video recording of participant groups during workshop sessions. If you are recorded it will be possible to identify you on the recording. The researcher will store these recordings in a locked cabinet and only approved study staff will be able to access them. The researcher will label these recordings with a session date and group number. The group number connects group members to the recording. The researcher will keep the key to the group numbers on a password-protected computer. Recordings will be stored for three years.

Video recording is required for this study. If you do not want to be recorded, you should not participate in this study.

What are the risks of taking part in this research study?

To the best of the researcher's knowledge, taking part in this study will not hurt you.

Are there any benefits from being in this research study?

You may or may not benefit from taking part in this study. Possible benefits include increased understanding of the division of fractions. This study has the potential to increase student achievement as well.

Others may benefit in the future from the information that is learned in this study, including curriculum designers and university teacher certification program designers.

What alternatives are available?

You may choose not to take part in this research study.

How Will You Keep My Study Records Confidential?

The researcher will keep the records of this study confidential by using a pseudonym to conceal each participants' name. A code key will be kept on a password-protected computer. The researcher will make every effort to keep your records confidential.

However, there are times when federal or state law requires the disclosure of your records.

The following people or groups may review your study records for purposes such as quality control or safety:

- The researcher and any member of her research team
- Authorized members of Baylor University who may need to see your information, such as administrative staff members from the Office of the Vice Provost for Research and members of the Institutional Review Board (a committee which is responsible for the ethical oversight of the study)

The study data will be stored in a locked file cabinet or on a password-protected computer.

The results of this study may also be used for teaching, publications, or presentations at professional meetings. If your individual results are discussed, your identity will be protected by using a code number or pseudonym rather than your name or other identifying information.

Study Participation and Early Withdrawal

Taking part in this study is your choice. You are free not to take part or to withdraw at any time for any reason. No matter what you decide, there will be no penalty or loss of benefit to which you are entitled. If you decide to withdraw from this study, the information that you have already provided will be kept confidential. You cannot withdraw information collected prior to your withdrawal.

The researcher may take you out of this study without your permission. This may happen because:

- The researcher thinks it is in your best interest
- You are unable to attend the required meetings
- Other administrative reasons

What will it cost me to take part in this research study?

There are no costs to you for taking part in this research study.

You can call the researcher with any concerns or questions about the research. Contact information is listed below:

- Principal Investigator: Shametria Banks
    - Phone: (512) 470-1908
    - The researcher can be reached via phone between the hours of 5 p.m. and 9 p.m.
- Faculty Advisor: Dr. Sandra Cooper
    - Phone: (254) 710-3246
    - Dr. Cooper can be reached during the hours of 8 a.m. to 5 p.m.

If you want to speak with someone **not** directly involved in this research study, you may contact the Baylor University IRB through the Office of the Vice Provost for Research at 254-710-1438. You can talk to them about:

- Your rights as a research subject
- Your concerns about the research
- A complaint about the research

Indicate your decision below for participation in the video recording as discussed earlier in this form:

Do you agree to let us make a video recording of you during this study?

______YES ______NO _______INITIALS

* Video recording is required for this study. If you do not want to be recorded, you should not participate in this study.

Statement of Consent

I have read the information in this consent form including risks and possible benefits. I have been given the chance to ask questions. My questions have been answered to my satisfaction, and I agree to participate in the study.

______________________________________ ______________________

Signature of Subject Date

Signature of Person Obtaining Consent:

I have explained the research to the subject and answered all his/her questions. I will give a copy of the signed consent form to the subject.

______________________________________ ______________________

Signature of Person Obtaining Consent Date

APPENDIX C

Division of Fractions Performance Task

1. To introduce the idea of division with fractions, a teacher decides to create a real-world problem for students to solve. Which of the following scenarios would be most appropriate? Determine an order to illustrate the way the tasks may be used with students in the classroom. Justify your reasoning. (Adapted from Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures- Elementary and Middle School Knowledge of Content and Teaching Item #23)

a) Deontre has a 10-pound bag of dog food. Deontre's dog eats a serving that is $\frac{1}{4}$ of a pound of food for each meal. How many servings can Deontre get from a full bag of dog food?
b) Tess has a 16-ounce block of cheese. She wants to make three pizzas with the block of cheese. If Tess uses the same amount of cheese on each pizza, how much cheese will she use for each pizza?
c) A baker has $3\frac{1}{4}$ cups of sugar. She needs $\frac{3}{4}$ of a cup for each batch of cookies. How many batches of cookies will she be able to make?
d) Each week, the cross-country team practices four days after school. They run $5\frac{1}{2}$ miles each week. If they run the same distance each day, how many miles do they run each day at practice?

2. A teacher plans to give the following problem to her class:

> Jonah's mom is making apple tarts for Thanksgiving dinner. If she uses $\frac{5}{8}$ of an apple for each tart, how many tarts can she make with 12 apples?

Since the class has just started working with fractions, she decides to use a simpler version of the problem to help them better understand the problem above. Write a problem that would be most useful for preparing the class to understand the problem above? Justify your response. (Adapted from Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures- Elementary and Middle School Knowledge of Content and Teaching Item #23)

3. A student's response to a division with fractions task is shown below. The student justified his answer because he believes that when you divide, a smaller quantity is yielded. (Adapted from Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures- Knowledge of Content and Students Item #11)

a.) What error is the student making?

b.) Explain how you help the student rectify this misconception. Include in your explanation, a plan to follow-up to this task with a representation to help the student understand his misunderstanding.

$$20 \div \frac{1}{4} = 5$$

4. Mr. Hinkle's class is learning to divide fractions. Before working toward developing the more traditional algorithm, he would like to select a task that would better help his students understand the process of dividing with fractions.

Which of the following problems would you choose for exploring the division of fractions process with students and better help them develop an algorithm? Circle ONE answer. Justify your selection and use a contextual situation with a model to show how the problem can be used to help students develop the skill. (Adapted from Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures- Elementary and Middle School Knowledge of Content and Teaching Item #21)

a. How many sets of $\frac{3}{6}$ are in $\frac{10}{6}$?

b. How many sets of $\frac{2}{3}$ are in $\frac{5}{4}$?

c. How many sets of $\frac{2}{6}$ are in $\frac{6}{3}$?

5. Intuitively, students have different ways of approaching finding solutions to problems involving the division of fractions. Imagine you are preparing for a unit on the division of fractions. Use the problem below to create a meaningful situation in a real-world context. Then represent the problem with a model and an equation. (Source: Ma, Liping; 2010; p. 55)

$$1\frac{3}{4} \div \frac{1}{2}$$

References:

Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures Items

Hill, H. C., and Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*, 330 – 351.

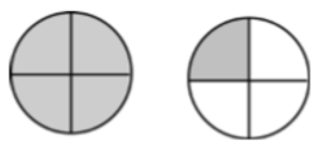
Knowing and Teaching Elementary Mathematics Item

Ma, L. (2010). Knowing and teaching elementary mathematics. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

APPENDIX D

Fraction Content Knowledge Test

1. Mrs. Johnson thinks it is important to vary the whole when she teaches fractions. For example, she might use five dollars to be the whole, or ten students, or a single rectangle. On one particular day, she uses a picture of two pizzas as the whole. What fraction of the whole is shown below? Justify your thinking. (Source: Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures- Elementary Content Knowledge Item #5)



2. Which of the following story problems could be used to illustrate $1\frac{1}{4}$ divided by $\frac{1}{2}$?

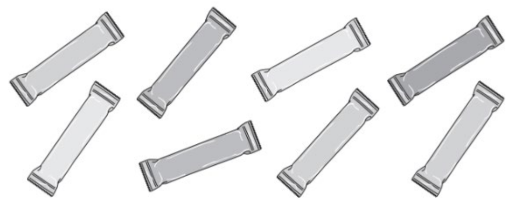
Circle all possible answers. Justify your selection(s). (Source: Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures- Elementary Content Knowledge Item #7)

a. You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?

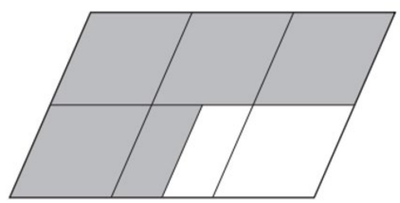
b. You have \$1.25 and may soon double your money. How much money would you end up with?

c. You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?

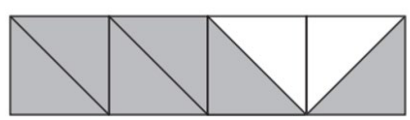
3. Lonny opened a new box of granola bars. Lonny and three of his friends equally shared the granola bars shown in the picture. What fraction of the granola bars did each of them get? Justify your thinking. (Source: 2016 STAAR Math Grade 3- Item #32)



4. Issac divided a figure into 7 pieces and shaded part of the figure as shown below. Issac said that $\frac{5}{7}$ of the figure is shaded. Is he correct? Explain. (Adapted from 2014 STAAR Math Grade 3- Item #33)



5. Indira shaded part of a figure, as shown below. What fraction of the figure is shaded? Justify your thinking. (Source: 2013 STAAR Math Grade 3- Item #14)



6. Mr. Atkinson has $5\frac{1}{4}$ lbs. of dry fish food. He will put an equal amount of food into 3 containers. How much fish food will be in each container? (Source: 2013 STAAR Math Grade 7- Item #25)

a. Write an equation and use a model to represent the situation.

b. Justify your thinking.

7. Cyril put a total of $\frac{1}{8}$ lb. of gravel into 6 fish tanks. He put the same amount of gravel into each tank. How many pounds of gravel did Cyril put into each fish tank? (Source: 2016 STAAR Math Grade 5- Item #13)

a. Write an equation and use a model to represent the situation.

b. Justify your thinking.

8. The model below represents a fraction division situation. Write an equation and a scenario that could be used to represent the model. (Adapted from 2016 STAAR Math Grade 5- Item #21)

References:

Learning Mathematics for Teaching: Mathematical Knowledge for Teaching (MKT) Measures Items
Hill, H. C., and Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*, 330 – 351.

STAAR Math Assessment Items
Texas Education Agency. (2019). *STAAR released test questions*. Retrieved from: https://tea.texas.gov/student.assessment/STAAR_Released_Test_Questions/

APPENDIX E

Teacher Participant Demographics Questionnaire

Respond to the questions below. Please provide as much detail as possible.

1. How long have you been a classroom teacher?

2. Describe your teacher preparation program? Please include where you completed your teaching certification credentials?

3. How would you describe your experience as a mathematics teacher?

4. What do you like most about teaching math?

5. What do you like least about teaching math?

6. If you could change one thing about your mathematics instruction, what would it be? Explain.

Teacher Participant Workshop Feedback Questionnaire

Respond to the questions below. Please provide as much detail as possible.

1. How does the format of this workshop compare with other workshops you have attended? Do you feel the workshop format contributed to your ability to learn? Explain.

2. What did you enjoy most about the workshop format?

3. What would you change about the workshop format?

4. Do you feel the goals of the workshop could have been accomplished using a different format? Explain.

5. If your district offered a similar workshop format for a topic of interest to you, would you attend? What topic would you be most interested in learning more about?

Additional Demographic Questions

Please respond to these additional demographic questions which will help deepen the case study.

1. Full Name
2. What type of teaching certificate do you hold, such as EC-6, Math 4 – 8?
3. How many years have you taught math (even in a self-contained setting) as a classroom teacher (K – 12)?
4. How many years have you taught elementary math (even in a self-contained setting) as a classroom teacher (Kindergarten – Grade 5)?
5. How many years have you taught secondary math (even in a self-contained setting) as a classroom teacher (Grades 6 -12)?
6. How did you feel about teaching the division of fractions before the professional development series?
7. How did you feel about teaching the division of fractions after the professional development series?

APPENDIX F

Workshop Lesson Plans

Workshop Session #1 Agenda and Lesson Plan Overview

Time	What	Why
Before 12:30	Fraction Teaching Challenges Entry Ticket	To describe the challenges of teaching fractions
12:30	Opening ▪ Welcome ▪ Introductions ▪ Super Teacher Name Plates ▪ That's Like Me	To get to know each other, understand what we'll do together today, and discuss how to bring our best selves to this learning experience
12:50	Review Agenda	To review our objectives, learning goals, and essential questions for the day and discuss Session Notes Catcher
1:00	Norms for the Afternoon	To frame how we will collaborate and complete our work together today
1:15	Building a Shared Knowledge: What are the big ideas of fractions?	To develop a shared understanding of the three big ideas of fractions
1:35	Teaching Fractions is Like . . .	To discuss the challenges of teaching fractions
1:45	Activity: Seeing Parts and Wholes	To explore unit fractions and the relationship between part and whole by creating visual models using pattern blocks
2:25	Break	
2:35	Activity: Seeing Parts and Wholes	To explore unit fractions and the relationship between part and whole by creating visual models using pattern blocks
3:15	Exit Ticket	To demonstrate understanding of the afternoon's learning objectives
3:25	Closing ▪ Appreciations ▪ Door Prizes ▪ Workshop Reflections	To remind ourselves about our learning experiences from the day, to appreciate ourselves and each other, and to reflect on the workshop session overall
3:45	End of Workshop	

Lesson Objectives

- Explore the relationship between part and whole by creating visual fraction models with pattern blocks
- Understand the importance of identifying the unit and of equal partitioning

Essential Questions

1. What is a fraction?
2. Why is it important to be able to identify the unit? How was the unit represented differently across the models?
3. Why is equal partitioning important? How can we emphasize this with our students?

Lesson Plan Overview

I. Building a Shared Knowledge: What are the big ideas of fractions?

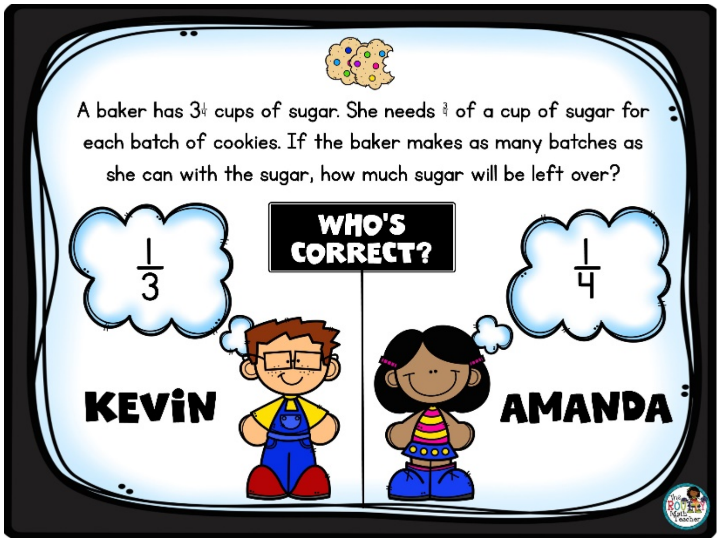
The purpose of this segment of the lesson plan is to build a shared understanding of the three big ideas of fractions: 1.) a fraction is a relationship between a part and a whole; 2.) conceptualizing the unit is key to understanding and operating with fractions; 3.) equal partitioning of the whole is essential to understanding the concept of fractions.

A. Big Ideas of Fractions: Discussion will include the following misconceptions for each big idea:

- Big Idea #1:
    - traditional use of the definition of a fraction as "part of a whole" limits student thinking to fractions that are less than one

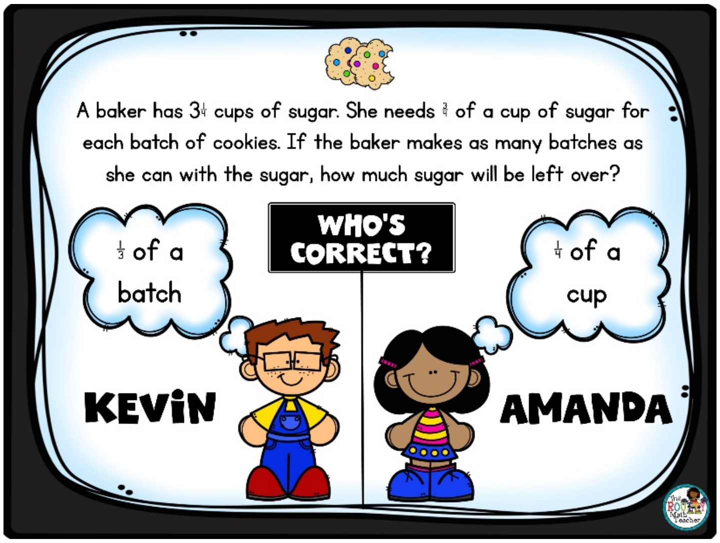
- Big Idea #2:
  - the whole is made of one piece which limits student understanding to area models
  - a fraction is smaller than the whole, the unit, or one
  - difficulty conceiving of or writing fractions greater than one
- Big Idea #3:
  - parts must be equal which limits student thinking to believing that fractional parts must be the same size and shape
  - parts need to be clearly delineated which prevents students from comparing the part to the whole, such as considering a rhombus to be two-thirds of a trapezoid (in a set of pattern bocks)
  - parts must have the same shape
  - shaded regions must be grouped into one part

B. Who's Correct Task: The purpose of this task is to challenge participants to conceptualize the unit and develop a deeper understanding of how different responses can be obtained when the unit is not clearly identified, a challenge in fraction division situations.



Who's Correct Task for Workshop Session 1

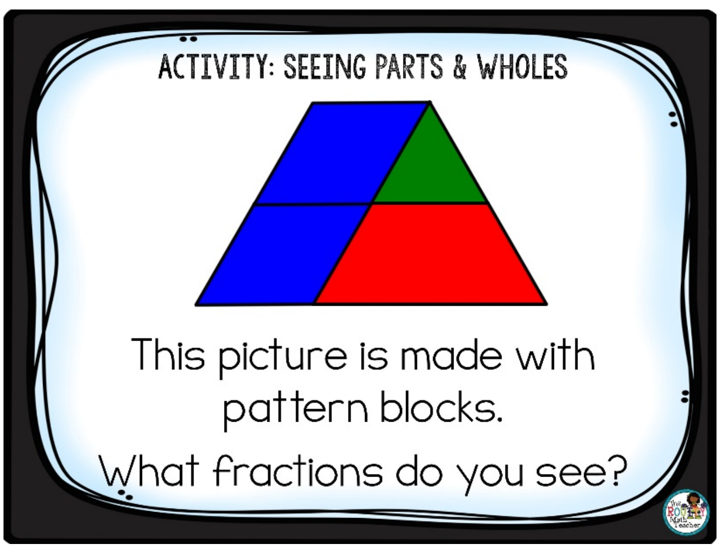
Participants will explore the question and determine the answer on their own before reviewing the students' responses. Once participants have created their own solution, they will compare their response with the students' responses and determine which student is correct. Afterwards, participants will discuss their reasoning with their table groups and come to a consensus. Once the final slide is revealed, participants will discuss how each student labeled their fraction and what they referred to as the whole. The professional development facilitator will then review why understanding the whole is important.



Who's Correct Task for Workshop Session 1
(Solution Slide)

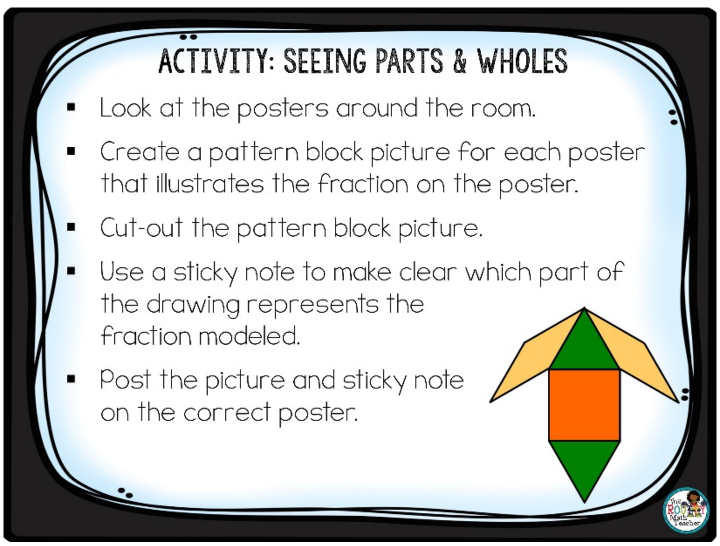
II. Activity: Seeing Parts and Wholes (Boaler, Munson, & Williams, 2018)

The purpose of this activity is to explore the relationship between the part and the whole in a fraction by creating models using pattern blocks.



Seeing Parts & Wholes Activity Launch for
Workshop Session 1

A. Launch: The purpose of the launch task is to introduce the thinking that will be needed to complete the explore activity. For this task, participants will look at a pattern block picture (as shown in the image above), choose a part, and name the fractional part. Participants will have an opportunity to think about a response and then share their responses with the group. As each response is shared, participants will need to agree on the way the part should be named. This activity will then lead participants into the next activity.



Seeing Parts & Wholes Activity for Workshop
Session 1

B. Explore: Participants will then move into the explore task where they will create a variety of pattern block pictures and identify a part that matches one of the following fraction posters: one-half, one-third, one-fourth, one-fifth, one-sixth, one-eighth, two-thirds, and three-fourths. There will also be a poster for other fractions participants discover as they create their pattern block pictures. Note: Participants' pictures will show a variety of fractions. Therefore, participants will need to select a specific part to identify and name on a sticky note to include with the pattern block picture on the specified poster.

After all groups have had enough time to complete the task, participants will complete a gallery walk of the posters and make a mental note of the pictures with which they disagree for later discussion.

After the gallery walk, participants will return to their seats and discuss the following questions as a large group:

- Are there any disagreements or challenges?
- What do the representations on each poster have in common?
- How do you know that each picture shows the fraction on the poster?
- What is needed to create the fraction on the poster?

Finally, participants will complete a review of the activity with a partner using the "StandUp, HandUp, PairUp" cooperative learning structure (Kagan & Kagan, 2009). For this activity, participants will walk around the room with their hand up while music plays. When the music stops, participants will high-five the nearest person. This will be their partner for the review activity. There will be three rotations. During each rotation, participants will answer one of the following questions:

- What is a fraction? How has your thinking about fractions changed?
- Do the number of pieces matter? Does the size of the pieces matter? How do we decide which to pay attention to?
- Describe the strategy you used to decompose the figures to determine what fraction you could make.

C. Wrap Up: As a culminating activity, participants will complete an exit task. The directions are as follows:

1. On an index card, create a shape where the blue rhombus is one-third. Be sure to label or explain how the rhombus represents one-third.
2. One the back of the card, explain how you could change the shape so that the rhombus represents a different fraction.
3. Be sure to write your name on the card.

Participants will then share their exit ticket with another participant using the "StandUp, HandUp, PairUp" cooperative learning structure (Kagan & Kagan, 2009).

Workshop Session #2 Agenda and Lesson Plan Overview

Time	What	Why
Before 12:30	Introducing Fractions Entry Ticket	To describe how participants introduce fraction division to their students
12:30	Review Agenda and Essential Questions	To review our objectives, learning goals, and essential questions for the day, and discuss Session Notes Catcher
12:45	Review Essential Questions from Session One	To review the essential questions from the previous session and connect them to today's learning
12:55	Building a Shared Knowledge: What are the big ideas of division? How do they connect to the division of fractions?	To develop a shared understanding of the three big ideas of division and explore how they connect with fraction division
1:35	Introducing Fractions . . .	To discuss how fraction division is introduced
1:40	Activity: *Creating Cards*	To explore fractions by making cards and cutting paper to develop the concept of fraction division
2:15	Break	
2:25	Activity: *Cuisenaire Trains*	To explore the patterns that emerge when partitioning a fraction into equal-sized pieces
3:00	Exit Ticket	To demonstrate understanding of the afternoon's learning objectives
3:20	Next Steps	To review the afternoon's activities and discuss how they can be applied in the classroom
3:25	Closing ▪ Appreciations ▪ Door Prizes ▪ Workshop Reflections	To remind ourselves about our learning experiences from the day, to appreciate ourselves and each other, and to reflect on the workshop session overall
3:45	End of Workshop	

Lesson Objectives

- Distinguish between the two interpretations of division, partitive and quotative situations
- Connect the big ideas of whole number division to the division of fractions

- Explore the division of fractions through contextual situations and visual models

Essential Questions

1. What does it mean to divide?
2. What is the difference between partitive (sharing) and quotative (measuring) situations?
3. How does whole number division connect to the division of fractions?

Lesson Plan Overview

I. Building a Shared Knowledge: What are the big ideas of division?

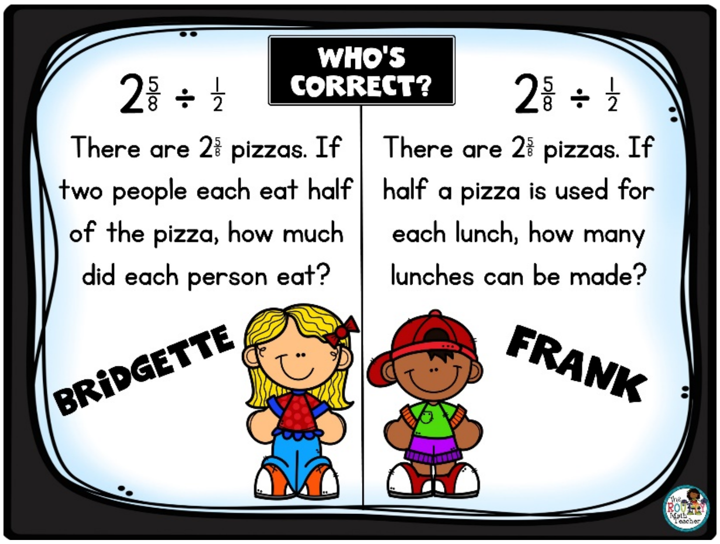
The purpose of this segment of the lesson plan is to build a shared understanding of the three big ideas of division: 1.) division means splitting into equal groups or parts; 2.) there are two different interpretations of division; 3.) the interpretations of division with rational numbers are essentially the same as with whole numbers but require adaptation (NCTM, 2010).

A. Big Ideas of Division: Discussion will include the following insight for each big idea:

- Big Idea #1:
    - understanding the action of division is critical to helping students understand fraction division
- Big Idea #2:
    - including both types of problems in instruction is important to helping students develop a deeper understanding of division and the division of fractions

- Big Idea #3:
    - explore "per unit" situations as an interpretation of the partitive situation when the divisor is a fraction

B. Who's Correct Task: The purpose of this task is to challenge participants to understand how fraction situations differ from whole number situations.

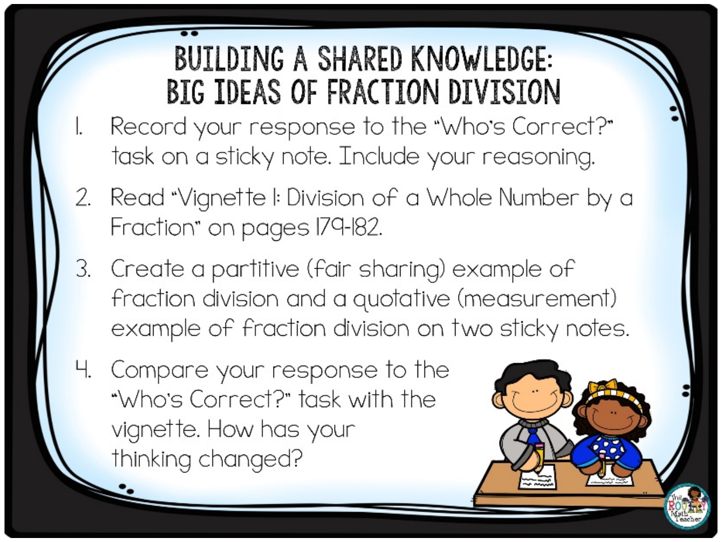


Who's Correct Task for Workshop Session 2

Participants will explore which situation models the fraction division problem on their own before discussing with the large group. Participants will record their response on a sticky note and include their reasoning. The sticky notes will then be collected for a quick review by the professional development facilitator.

C. Interpreting Division Situations: The purpose of this portion of the session will help participants interpret division situations, specifically regarding dividing something in half, dividing by two, or partitioning something into half-size portions, dividing by one-half. (This was revealed as an area of difficulty on the initial performance task.)

Participants will then complete the following set of instructions, beginning with number two.



Big Ideas Tasks for Workshop Session 2

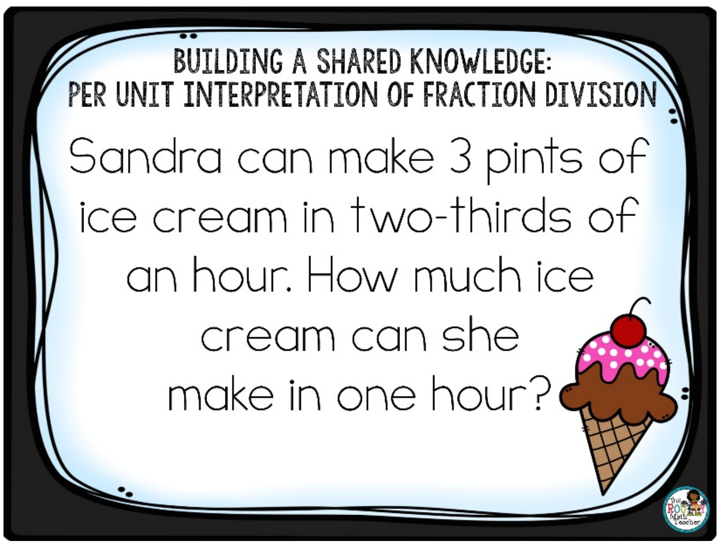
Note: The reading noted in the image above refers to the book *Unpacking Fractions: Classroom-tested Strategies to Build Students' Mathematical Understanding* (Neagoy, 2017).

After all participants have had an opportunity to complete the tasks, a group discussion will be held to review the following questions:

- What does this vignette reveal about our understanding of fraction division?
- What is quotative division? Give an example.
- What is partitive division? Give an example.
- Can you create an example of a partitive situation where the divisor is not a whole number?
- What are you wondering now?

Note: The professional development facilitator will return to the Who's Correct task to discuss which student correctly represented the fraction division problem. A discussion will follow.

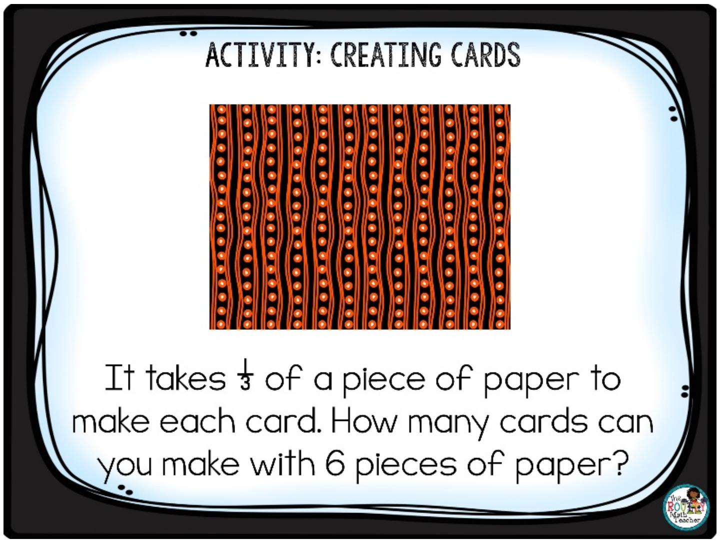
C. Partitive Situations- Dividing by a Fraction: The purpose of this activity is to make sense of sharing situations where the divisor is a fraction. Participants will complete the task below using their dry erase materials. The participants will then debrief the task as a large group.



Division Situations Task for Workshop Session 2

II. Activity: ***Creating Cards*** (Boaler, Munson, & Williams, 2018)

The purpose of this activity is to explore fraction division by cutting paper and making cards to help develop a deeper understanding of what it means to divide with fractions.

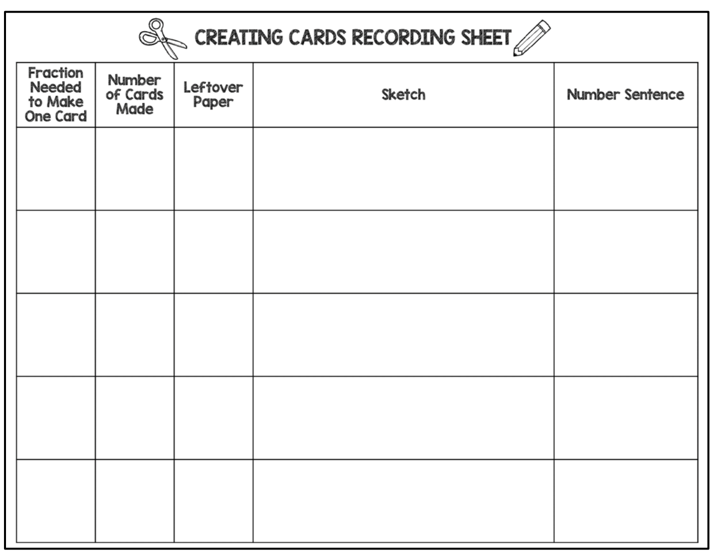


Creating Cards Activity Launch for Workshop
Session 2

A. Launch: The purpose of the launch task is to introduce the thinking that will be needed to complete the explore activity. For this task, participants will determine how many cards can be made from six pieces of paper if each card is made with one-third of a piece of paper. Participants will have an opportunity to think about a response individually and then share their response with their partner. Participants will need to justify how they determined the number of cards that could be made. This activity will then lead participants into the next activity.

Creating Cards Activity for Workshop Session 2

B. Explore: Participants will then move into the explore task (see the image above) where they will determine the number of cards that could be made using the following card sizes: two-thirds, three-fourths, five-sixths, three-eighths, seven-eighths, three-halves, and five-fourths. Participants will then be asked to add three additional card sizes of their choosing. A recording sheet will be used to record responses. (See the image below.)



CREATING CARDS RECORDING SHEET

Fraction Needed to Make One Card	Number of Cards Made	Leftover Paper	Sketch	Number Sentence

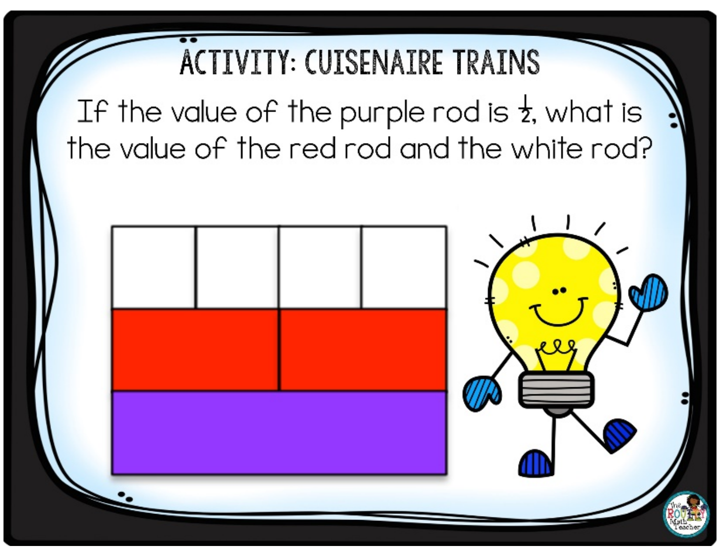
Creating Cards Recording Sheet for Workshop Session 2

After all groups have had enough time to complete the task, participants will share their responses as the instructor adds them to a large class chart. Then, participants will discuss the following questions as a large group:

- What fractions did you use? How many cards could you make?
- What strategies did you use to find the solutions?
- How did the number of cards you could make change when you changed the fraction of paper needed to make a card? Why did it change?
- Did any of the card sizes create leftover paper? What was leftover? How much? How is this quantity named?

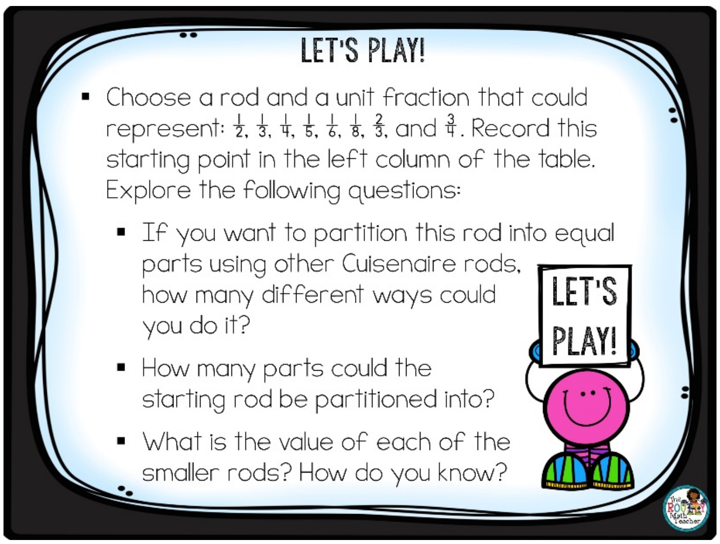
III. Activity: ***Cuisenaire Trains*** (Boaler, Munson, & Williams, 2018)

The purpose of this activity is to explore what it means to partition a unit fraction into equal-sized pieces (a whole number).



Cuisenaire Trains Activity Launch for Workshop Session 2

A. Launch: The purpose of the launch task is to introduce the thinking that will be needed to complete the explore activity. For this task, participants will determine the value of both the red and white rods if the value of the purple rod is one-half. Participants will have an opportunity to think about a response individually and then share their response with their partner. Participants will need to justify how they determined the value of each rod. This activity will then lead participants into the next activity.



Cuisenaire Trains Activity for Workshop Session 2

B. Explore: Participants will then move into the explore task (see the image above) where they will partition a starting rod into equal-sized units, using a rod of a different length, and then name the value of each partitioned unit. Participants will begin by selecting a starting rod and giving it a value (one-half, one-third, one-fourth, one-fifth, one-sixth, and one-eighth). Then they will determine ways to partition the starting rod into other equal-sized units represented by the other rods. For example, when a yellow rod represents one-fifth and is partitioned into five equal-sized white rods, each white rod will have a value of one-twenty-fifth. Two additional starting values are included for challenge: two-thirds and three-fourths. A recording sheet will be used to record responses.

After all groups have had enough time to complete the task, participants will share their responses as the instructor adds them to a large class chart. Then, participants will discuss the following questions as a large group:

- What interesting solutions did you find?
- How did you determine the value of each rod? What strategies did you develop?
- What patterns did you notice when partitioning the rods?
- What patterns do you notice in your table?
- Which rods had the most solutions? The fewest? Why do you think that is?
- Which fractions were the easiest to start with? The hardest? Why?

IV. Reviewing What We've Learned: For this portion of the session, participants will complete a review of both activities with a partner using the "StandUp, HandUp, PairUp" cooperative learning structure (Kagan & Kagan, 2009). For this activity, participants will walk around the room with their hand up while music plays. When the music stops, participants will high-five the nearest person. This will be their partner for the review activity. There will be four rotations. During each rotation, participants will answer one of the following questions:

- How did your picture or model help you show what was happening in each situation?
- How do you know that your solutions make sense?
- As each card got bigger, did it make sense to get more or less cards?

- What was happening when we took a fraction and divided it by a whole number?

V. Wrap Up: As a culminating activity, participants will complete an exit task. The directions are as follows:

Using an index card,

1. Determine how many cards you could make with eight sheets of paper if each card uses four-fifths of a sheet of paper. Record an equation to represent the situation.
2. Select a train to equal five-eighths. Partition it in two different ways. Record an equation for each partition.

Participants will then share their exit ticket with another participant using the "StandUp, HandUp, PairUp" cooperative learning structure (Kagan & Kagan, 2009).

Workshop Session #3 Agenda and Lesson Plan Overview

Time	What	Why
Before 12:30	Reflections Entry Ticket	To reflect on the learning from the past two sessions
12:30	Review Agenda and Essential Questions	To review our objectives, learning goals, and essential questions for the day, and discuss Session Notes Catcher
12:45	Review Essential Questions from Session Two	To review the essential questions from the previous session and connect them to today's learning
12:55	Building a Shared Knowledge: What are the big ideas of fraction division?	To develop a shared understanding of the three big ideas of fraction division
1:15	Reflecting on Fractions . . .	To reflect on the understanding and learning thus far
1:20	Activity: *Making Pizzas*	To explore dividing fractions by dividing cheese blocks into equal-sized fractional portions
1:50	Break	
2:00	Activity: Reason Why When You Invert and Multiply	To explore the reason why we invert and multiply
2:30	Next Steps	To review the afternoon's activities and discuss how they can be applied in the classroom
2:35	Lesson Planning	To translate our learning into actions by creating student learning experiences
3:25	Closing ▪ Shaping Your Reflections ▪ Appreciations ▪ Door Prizes ▪ Workshop Reflections	To review the afternoon's objectives, to appreciate ourselves and each other, and to reflect on the workshop session overall
3:45	End of Workshop	

Lesson Objectives

- Use invented procedures to make connections to traditional algorithms for the division of fractions

- Investigate models for the division of fractions, including the common denominator model
- Discover ways to transform from visual models to the traditional algorithm
- Use the learning from the three workshop sessions to develop student lessons

Essential Questions

1. How do our actions with models translate to more formal language?
2. What is an alternate approach to the traditional model for dividing fractions?
3. How does whole number division connect to the division of fractions?
4. How does the work we did together translate to the classroom?

Lesson Plan Overview

I. Building a Shared Knowledge: What are the big ideas of fraction division?

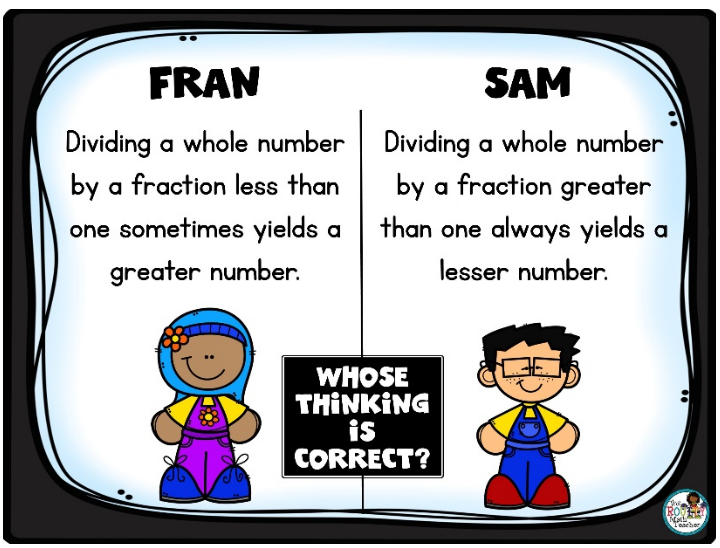
The purpose of the segment of the lesson plan is to build a shared understanding of the three big ideas of fraction division: 1.) division is the inverse of equal-groups multiplication; 2.) division does not always make thinks smaller; 3.) the units in division problems are often different.

A. Big Ideas of Fraction Division: Discussion will include the following insight for each big idea:

- Big Idea #1:
    - understanding the connection between multiplication and division and the idea of them being "inverse" operations is essential to understanding the traditional algorithm

- Big Idea #2:
  - thinking that division always makes things smaller offers an incomplete understanding of division and creates a roadblock in student thinking
- Big Idea #3:
  - because there are often multiple units in fraction division problems, it is important to understand which unit the solution refers to; a mathematical response, such as the one obtained on a calculator or when using an algorithm, to a situation can be different than a response obtained from solving a problem in context

B. Who's Correct Task: The purpose of this task is to challenge participants to develop a deeper understanding of fraction division.



Whose Thinking is Correct for Workshop Session 3

Participants will explore the statements on their own and determine whose thinking is correct. After discussing the statements, the instructor will engage the group in discussion about generating some conjectures, or rules, that apply to fraction division situations.

After all participants have had an opportunity to complete the task, a group discussion will be held to review the following questions:

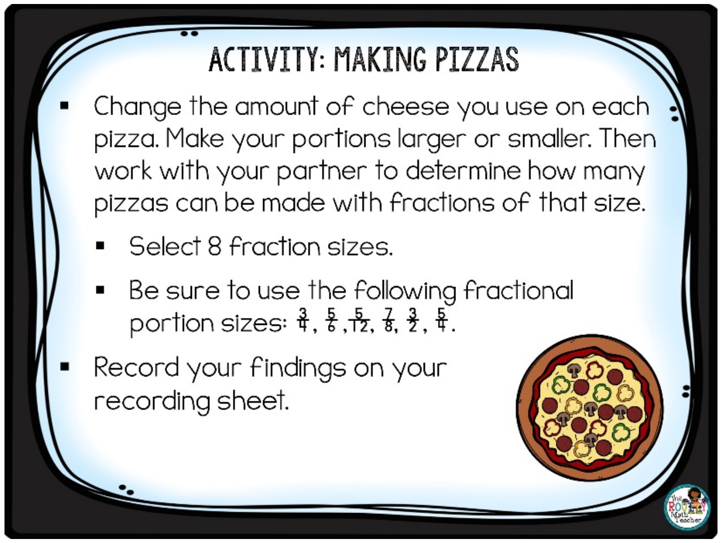
- What does it mean for division to be the inverse of multiplication?
- Students often develop the thinking that division makes things smaller. How can we help students refine their thinking to understand how fractions fit into the mix?
- The units in division problems are often different. How does this change the way we think about fraction division?

II. Activity: *Making Pizzas* (Note: This activity was adapted from the *Creating Cards* activity from Session Two written by Boaler, Munson, & Williams (2018) to create a similar task for developing a better understanding of dividing a whole number or a non-unit fraction by a non-unit fraction.)

The purpose of this activity is to explore fraction division by cutting blocks of cheese into equal-sized portions to make pizzas in order to help develop a deeper understanding of what it means to divide with fractions.

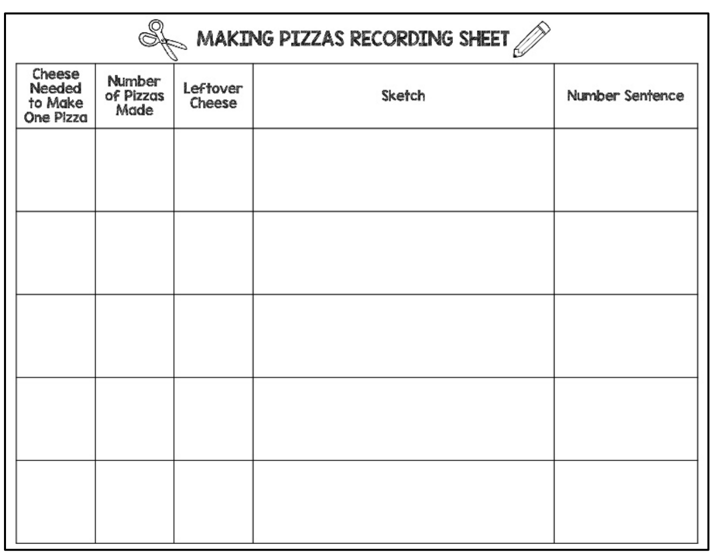
Making Pizzas Launch Activity for Workshop
Session 3

A. Launch: The purpose of the launch task is to introduce the thinking that will be needed to complete the explore activity. For this task, participants will determine how many pizzas can be made with four blocks of cheese. Participants will have an opportunity to think about a response individually and then share their response with their partner. Participants will need to justify how they determined the number of pizzas that could be made. This activity will then lead participants into the next activity.



Making Pizzas Activity for Workshop Session 3

B. Explore: Participants will then move into the explore task (see the image above) where they will determine the number of pizzas that could be made using the following equal-sized portions of cheese: three-fourths, five-sixths, five-twelfths, seven-eighths, three-halves, and five-fourths. Participants will then be asked to add two additional portion sizes of their choosing. A recording sheet will be used to record responses. (See the image below.)



MAKING PIZZAS RECORDING SHEET

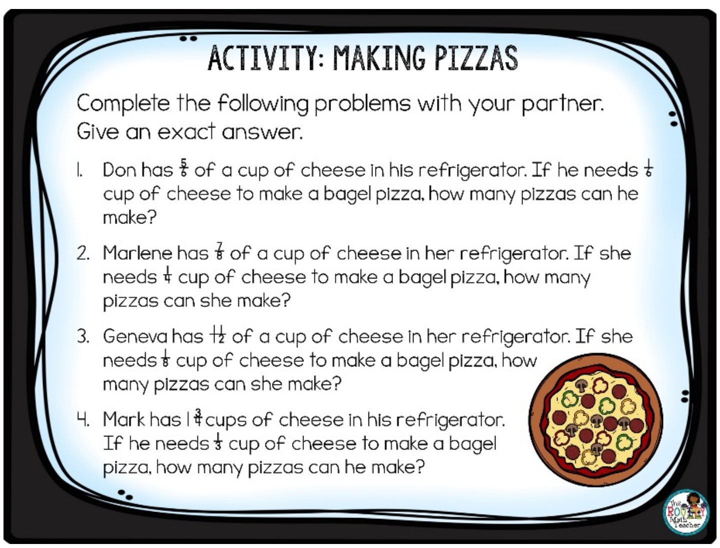
Cheese Needed to Make One Pizza	Number of Pizzas Made	Leftover Cheese	Sketch	Number Sentence

Making Pizzas Recording Sheet for Workshop Session 3

After all groups have had enough time to complete the task, participants will share their responses as the instructor adds them to a large class chart. Then, participants will discuss the following questions as a large group:

- What fractions did you use? How many pizzas could you make?
- What strategies did you use to find the solutions?
- How does the number of pizzas you could make change when you changed the size of the portion needed to make a pizza? How did it change?
- Did any of the portion sizes leave cheese remaining? What unit do we use to refer to the leftover amount?

C. Extend: After the initial experience, participants will return to the *Making Pizzas* task to simulate dividing a fraction by a fraction. The problems are displayed in the image below.



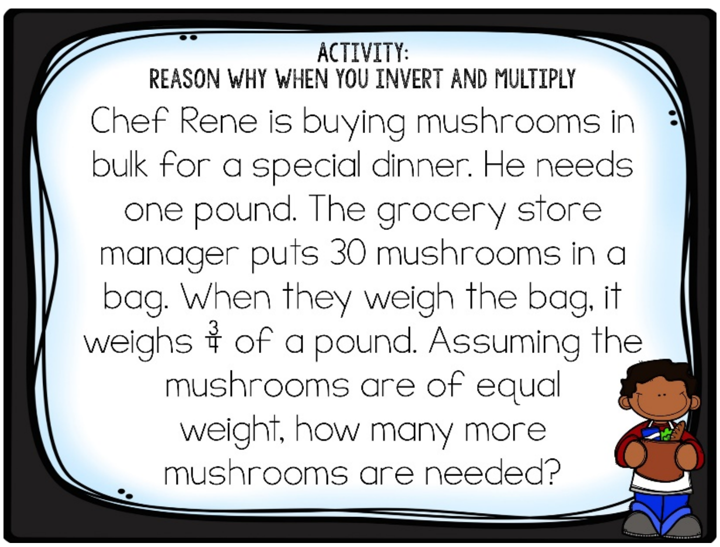
Making Pizzas Activity for Workshop Session 3

Before moving to the next task, participants will review the learning from both tasks with the following discussion questions:

- How did you model each situation?
- How do you know that your solutions make sense?
- What happened when we took a fraction and divided it by another fraction?
- What patterns did you notice?
- Make a conjecture. What kind of value do you get when you divide a fraction by a fraction?

III. Activity: Reason Why When We Invert and Multiply

The purpose of this activity is to explore the reason why we invert and multiply.



Reason Why When We Invert and Multiply Launch
Activity for Workshop Session 3

A. Launch: The purpose of the launch task is to introduce an additional contextual situation and allow participants to grapple with the task before it is introduced in the video later in the activity. For this task, participants will determine the additional quantity of mushrooms needed to complete the pound. Participants will have an opportunity to think about a response individually and then share their response with their partner. This activity will then lead participants into the next activity.

After participants have completed the task, the professional development facilitator will share a video (The Math Forum, 2015) of a teacher using this task with his students. During the video, the instructor discusses the importance of allowing students the opportunity to use stories to help them connect to concepts, use the Notice and Wonder strategy to develop a deeper understanding of the task, devise their own strategies and methods to solve the problem, and connect student-invented solution strategies to efficient procedures. Throughout the video, the instructor also shares some of the most common responses to the task, giving participants an opportunity to compare their strategies to the students' strategies and make connections to the traditional algorithm.

B. Article Review and Discussion: Participants read the "Reason Why When You Invert and Multiply" article (Cardone, 2015) and complete the following set of instructions, beginning with number two.

Reason Why When We Invert and Multiply Activity
for Workshop Session 3

After all participants have had an opportunity to complete the tasks, they will share their understanding of what it means to invert and multiply and an example of multiplying by the reciprocal. Then, a group discussion will be held to review the following questions:

- Multiplication and division are inverse operations. What does this mean?
- How can we help students connect whole number division with the action of dividing with fractions?
- Why does multiplying by the reciprocal work? Give an example.
- How has your thinking changed after reading the article?

IV. Reviewing What We've Learned: For this portion of the session, participants will complete a review of both activities with a partner using the "StandUp, HandUp, PairUp" cooperative learning structure (Kagan & Kagan, 2009). For this activity, participants will walk around the room with their hand up while music plays. When the music stops, participants will high-five the nearest person. This will be their partner for the review activity. There will be four rotations. During each rotation, participants will answer one of the following questions:

- How do today's activities impact your thinking about fraction division?
- What changes might you make to your instruction based on today's activities?
- How can we help students develop the concept of fraction division?
- What are you wondering now?

V. Lesson Planning Time: The purpose of this time is to allow participants an opportunity to create student-centered lesson plans based on the new learning gleaned from the workshop series. During this time, participants will complete the following tasks:

- Determine a launch for the unit. Consider using a contextual situation or a picture book, such as The Multiplying Menace Divides by Pam Calvert.
- Determine the activities you will use to develop the concept of division of fractions. Decide what changes need to be made, if any, before using the activity.
- Determine the formative assessments that will be used.

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