ABSTRACT

Existence and Uniqueness of Solutions of Boundary Value Problems by Matching Solutions

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In this dissertation, we investigate the existence and uniqueness of boundary value problems for the third and nth order differential equations by matching solutions. Essentially, we consider the interval $[a, c]$ of a BVP as the union of the two intervals $[a, b]$ and $[b, c]$, analyze the solutions of the BVP on each, and then match the proper ones to be the unique solution on the whole domain.

In the process of matching solutions, boundary value problems with different boundaries, especially at the matching point b, would be quite different for the requirements of conditions on the nonlinear term. We denote the missing derivatives in the boundary conditions at the matching point b by k_1 and k_2. We show how $y^{(k_2)}(b)$ varies with respect to $y^{(k_1)}(b)$, where y is a solution of the BVP on $[a, b]$ or $[b, c]$. Under certain conditions on the nonlinear term, we can get a monotone relation between $y^{(k_2)}(b)$ and $y^{(k_1)}(b)$, on $[a, b]$ and $[b, c]$, respectively. If the monotone relations are different on $[a, b]$ and $[b, c]$, then we can finally get a unique value for $y^{(k_1)}(b)$ where the k_2nd derivative of two solutions on $[a, b]$ and $[b, c]$ are equal and we can join the two solutions together to obtain the unique solution of our original BVP. If the relations are the same, then we will arrive at the situation that the k_2nd order derivatives of two solutions at b on $[a, b]$ and $[b, c]$ are decreasing with respect

to the k_1st derivatives at b at different rates, and by analyzing the relations more in detail, we can finally get a unique value for the k_1st derivative of solutions of BVP's on $[a, b]$ and $[b, c]$, which are matched to be a unique solution of the BVP on $[a, c]$. In our arguments, we use the Mean Value Theorem and the Rolle's Theorem many times.

As the simplest models, third order BVP's are considered first. Then, in the following chapters, nth order problems are studied. Lastly, we provide an example and some ideas for our future work.

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CHAPTER ONE

History and Introduction

In this dissertation, we are primarily concerned with matching solutions of certain nonlocal boundary value problems for the nth order ordinary differential equation $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$, which will be of different forms in different boundary value problems, for $n \geq 3$, on $[a, b]$ and on $[b, c]$ to obtain the existence and uniqueness of solutions of nonlocal boundary value problems for the same equation on $[a, c]$.

Matching solutions of boundary value problems is intimately involved with interface problems for which an intermediate boundary point corresponds to a point of interface [1, 25, 31, 39]. For such problems, smooth as possible interfacing is desired. Otherwise, leakage or impulses in transfer rates occur. Most matching results deal with smoothing one possible break in some order derivative. In this dissertation, we deal with smoothing by matching, when gaps in the derivatives at the interface point involve several successive derivatives, in which cases, there is great difficulty in transfer across the interface; and so the hypotheses for matching can be seemingly strong.

The solution-matching technique was first used by Bailey *et al.* [2]. They considered the solutions of two-point boundary value problems for the second order differential equation $y'' = f(x, y, y')$ by matching solutions of initial value problems. They assumed the existence and uniqueness of solutions on each interval and concluded the existence and uniqueness of solutions on the combined interval. Then, in 1973, Barr and Sherman [3] assumed monotonicity conditions on f and applied the solution-matching technique to third order equations and generalized to equations of arbitrary order. The boundedness of f was assumed. In 1978, Moorti [20] applied the monotonicity condition on f and solution matching method to the nth order boundary value problems. He also gave several general conditions on f and boundary conditions to get the existence and uniqueness of the third order boundary value problems. In 1981, Murthy *et al.* [21] and Rao *et al.* [29], in a certain sense, generalized the monotonicity of f of third order differential equations and introduced an auxiliary monotone function g. In 1983, Henderson [9] generalized to nth order BVP's and considered more general boundary conditions. In 1993, Taunton *et al.* [14] analyzed the properties of solutions of differential inequalities involved with the auxiliary monotone function g of the third order boundary value problems. In 2001, Henderson *et al.* [13] generated the solution method of nth order differential equations on time scales. Henderson *et al.* [15, 12, 11] used the solution matching method to obtain the existence and uniqueness of several new kinds of nonlocal boundary value problems of third order and nth order differential equations. Since then, a lot of work has been done on existence and uniqueness of certain BVP's for third order or higher order differential equations, differential systems or differential equations on time scales by matching solutions. We refer the readers to [4, 6, 17, 22, 27, 28, 29, 33], etc.

This dissertation considers three-point and nonlocal multi-point boundary value problems for third order and nth order differential equations. Several kinds of gaps in boundary conditions at the matching point are considered, in particular. In Chapter Two, a comprehensive analysis of the solution matching method applied to the third order differential equations is obtained when the gap in the boundary conditions is odd. In Chapter Three, we consider the case for the third order equations when the gap is even. Chapter Four is about the nth order problems when gaps in boundary conditions are odd. Chapter Five is for the case of nth order boundary value problems with even gap. Chapter Six provides an example. In Chapter Seven, some ideas for future research are discussed.

CHAPTER TWO

Nonlocal Boundary Value Problems with Odd Gaps in Boundary Conditions of Third Order Differential Equations

2.1 Introduction

There has been a lot of work on three-point boundary value problems for third order differential equations, see [7, 8, 10, 18, 19, 16, 23, 24, 26, 32, 34, 36, 38], etc. Solution-matching is one of the methods to explore the existence and uniqueness of solutions. As we stated in Chapter One, several papers [3, 4, 5, 11, 14, 21, 29] applied the solution-matching method to obtain existence and uniqueness of solutions to three-point boundary value problems associated with third order nonlinear differential equations. Barr *et al.* [3] and Rao *et al.* [30] used a solution matching technique and suitable 'Liapunov-like' function on third and nth order differential boundary value problems. In 2005, Henderson *et al.* [15] considered five-point boundary value problems of third order differential equations by using solution matching.

Here, we consider multi-point boundary value problems of third order differential equations. For different kinds of gaps in boundary conditions, we need different thoughts and conditions on the nonlinear term.

The following is our differential equation

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \qquad x \in [a, c], \tag{2.1}$$

with one of the multi-point boundary conditions

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y(b) = y_2, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{2.2}$$

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y'(b) = y_2, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{2.3}$$

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y''(b) = y_2, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{2.4}$$

where $a < \xi_1 < \xi_2 < \cdots < \xi_s < b < \eta_1 < \eta_2 < \eta \cdots < \eta_t < c$, $s, t \in \mathbb{N}$, $a_i, b_j > 0$ for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots t$, $\sum_{i=1}^{s} a_i = \sum_{j=1}^{t} b_j = 1$, and $y_1, y_2, y_3 \in \mathbb{R}$.

Among the boundary conditions at b, in the case of (2.2), the value of the first derivative and the second derivative of solutions are missing; in the case of (2.4), the function value and the value of the first derivative of solutions are missing. In both cases, the difference of the missing derivatives is 1, which is odd.

In Sections 2.2 and 2.3, we study the existence and uniqueness of solutions of (2.1), (2.2) and (2.1), (2.4), respectively, that is, the cases with odd gaps in boundary conditions at b. The case of (2.1), (2.3) with even gaps in boundary conditions at b will be studied in Chapter Three.

2.2 The Case of (2.1), (2.2)

In this section, we are concerned with the existence and uniqueness of solutions of (2.1), (2.2), restated as follows, over the interval $[a, c]$:

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \qquad x \in [a, c], \tag{2.1}$$

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y(b) = y_2, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3. \tag{2.2}$$

Throughout this section, it is assumed that $f : [a, c] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and that solutions of initial value problems (IVP's) for (2.1) are unique and exist on all of $[a, c]$.

Based on the idea of solution-matching, we break our BVP (2.1), (2.2) into four BVP's. Now consider the following four sets of boundary conditions,

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y(b) = y_2, \quad y'(b) = m, \tag{2.5}$$

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y(b) = y_2, \quad y''(b) = m, \tag{2.6}$$

$$y(b) = y_2, \quad y'(b) = m, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{2.7}$$

$$y(b) = y_2, \quad y''(b) = m, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{2.8}$$

where $m \in \mathbb{R}$, and we are going to obtain that (2.1), (2.3) has a unique solution by matching solutions of the BVP (2.1), (2.5) on $[a, b]$ with solutions of (2.1), (2.7) on $[b, c]$, or solutions of (2.1), (2.6) on $[a, b]$ with solutions of (2.1), (2.8) on $[b, c]$.

Conditions on f for the case of (2.1), (2.2) are as follows:

(A1) For any $v \in \mathbb{R}$, $f(x, v_0, v_1, v) - f(x, u_0, u_1, v) > 0$ when $x \in (a, b]$, $-v_0 \geq -u_0$, and $v_1 > u_1$; or when $x \in [b, c)$, $v_0 \geq u_0$, and $v_1 > u_1$.

With condition (A1), following the similar idea to [11], we can get the following two lemmas which are essentially important for the matching process.

The first lemma shows the relation between the change in the values of the first derivative and that of the second derivative at the matching point b of solutions of (2.1) on the interval $[a, b]$ satisfying $y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1$ and $y(b) = y_2$.

Lemma 2.1. *Suppose p and q are solutions of (2.1) satisfying $y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1,\ y(b) = y_2$ on $[a, b]$ and let $w = p - q$ so that w satisfies*

$$\begin{aligned} &w'''(x) = f(x, p(x), p'(x), p''(x)) - f(x, q(x), q'(x), q''(x)), \qquad x \in [a, b], \\ &w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0, \quad w(b) = 0. \end{aligned}$$

If the condition (A1) is satisfied, then $w'(b) = 0$ if and only if $w''(b) = 0$, and $w'(b) > 0$ if and only if $w''(b) > 0$.

Proof. ($\Rightarrow$) The necessity of equalities.

Suppose $w'(b) = 0$ and $w''(b) \neq 0$. Without loss of generality, we suppose $w''(b) > 0$. Since $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = \sum_{i=1}^{s} a_i (w(a) - w(\xi_i)) = 0$ and $a_i > 0$, there is some $x_0 \in (a, \xi_s)$ such that $w'(x_0) = 0$. By $w'(b) = 0$, there is some $x_1 \in (x_0, b)$ such that $w''(x_1) = 0$. Without loss of generality, we assume $w''(x) > 0$ for $x \in (x_1, b]$.

Hence,

$$w'''(x_1) = \lim_{x \to x_1^+} \frac{w''(x) - w''(x_1)}{x - x_1} \geq 0.$$

Since $w(b) = w'(b) = 0$, we have that $w(x) > 0$ and $w'(x) < 0$ for $x \in [x_1, b)$. Therefore by condition (A1), $w'''(x_1) < 0$, which is a contradiction.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w'(b) \neq 0$ and $w''(b) = 0$. Without loss of generality, we assume $w'(b) < 0$. By $w(b) = w''(b) = 0$ and condition (A1), we have that $w'''(b) < 0$. Then in a left neighborhood of b, $w''(x) > 0$. By $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0$, there is some $x_0 \in (a, \xi_s)$ such that $w'(x_0) = 0$. Then from $w'(x_0) = 0$ and $w'(b) < 0$, there is some $x_1 \in (x_0, b)$ such that $w''(x_1) < 0$. Then there is some $x_2 \in (x_1, b)$ such that $w''(x) > 0$ for $x \in (x_2, b)$ and $w''(x_2) = 0$. By $w'(b) < 0$ and $w(b) = 0$, we have that $w'(x) < 0$ and $w(x) > 0$ for $x \in [x_2, b)$. Then, from condition (A1), we have that $w'''(x_2) < 0$. However,

$$w'''(x_2) = \lim_{x \to x_2^+} \frac{w''(x) - w''(x_2)}{x - x_2} \geq 0.$$

This is a contradiction.

($\Rightarrow$) The necessity of inequalities.

Assume $w'(b) > 0$ and $w''(b) < 0$. By $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0$, there is some $x_0 \in (a, \xi_s)$ such that $w'(x_0) = 0$. Since $w'(b) > 0$, there is some $x_1 \in [x_0, b)$ such that $w'(x_1) = 0$ and $w'(x) > 0$ for $x \in (x_1, b]$, and from $w(b) = 0$ we have $w(x) < 0$ for $x \in [x_1, b)$. Then by the Mean Value Theorem and $w''(b) < 0$, there is some $x_2 \in (x_1, b)$ such that $w''(x_2) = 0$ and $w''(x) < 0$ for $x \in (x_2, b]$. Application of the condition (A1) gives us that $w'''(x_2) > 0$. However,

$$w'''(x_2) = \lim_{x \to x_2^+} \frac{w''(x) - w''(x_2)}{x - x_2} \leq 0,$$

which is a contradiction.

($\Leftarrow$) The sufficiency of inequalities.

By simply switching the sign of w in the proof of the necessity of inequality, we can get a contradiction too. Hence $w'(b) > 0$ if $w''(b) > 0$. $\square$

The next lemma shows the relation between the change in values of the first derivative and that of the second derivative at the matching point b of solutions of (2.1) on the interval $[b, c]$ satisfying $y(b) = y_2$, and $\sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3$.

Lemma 2.2. *Suppose p and q are solutions of (2.1) satisfying $y(b) = y_2$, and $\sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3$ on $[b, c]$ and let $w = p - q$ so that w satisfies*

$$
\begin{aligned}
&w'''(x) = f(x, p(x), p'(x), p''(x)) - f(x, q(x), q'(x), q''(x)), \qquad x \in [b, c],\\
&w(b) = 0, \ \sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0.
\end{aligned}
$$

If the condition (A1) is satisfied, then $w'(b) = 0$ if and only if $w''(b) = 0$, and $w'(b) > 0$ if and only if $w''(b) < 0$.

Proof. ($\Rightarrow$) The necessity of equalities.

Suppose $w'(b) = 0$ and $w''(b) \neq 0$. Without loss of generality, we suppose $w''(b) > 0$. Since $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = \sum_{j=1}^{t} b_j (w(\eta_j) - w(c)) = 0$ and $b_j > 0$ for $j = 1, 2, \ldots, t$, there is some $x_0 \in (\eta_1, c)$ such that $w'(x_0) = 0$. By $w'(b) = 0$, there is some $x_1 \in (b, x_0)$ such that $w''(x_1) = 0$. Without loss of generality, we assume $w''(x) > 0$ for $x \in [b, x_1)$. Hence,

$$w'''(x_1) = \lim_{x \to x_1^-} \frac{w''(x) - w''(x_1)}{x - x_1} \leq 0.$$

Since $w(b) = w'(b) = 0$ and $w''(x) > 0$ for $x \in [b, x_1)$, we have that $w(x) > 0$ and $w'(x) > 0$ for $x \in (b, x_1]$. Therefore by condition (A1), $w'''(x_1) > 0$, which yields a contradiction. Hence, $w''(b) = 0$ if $w'(b) = 0$.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w'(b) \neq 0$ and $w''(b) = 0$. Without loss of generality, we assume $w'(b) > 0$. By $w(b) = w''(b) = 0$ and condition (A1), we have that $w'''(b) > 0$. Then, in a right neighborhood of b, $w''(x) > 0$. By $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0$, there is some $x_0 \in (\eta_1, c)$ such that $w'(x_0) = 0$. Then from $w'(b) > 0$ and the Mean Value Theorem, there is some $x_1 \in (b, x_0)$ such that $w''(x_1) < 0$. Hence, there is some $x_2 \in (x_1, b)$ such that $w''(x) > 0$ for $x \in (b, x_2)$ and $w''(x_2) = 0$. By $w'(b) > 0$ and $w(b) = 0$, we have that $w'(x) > 0$ and $w(x) > 0$ for $x \in (b, x_2]$. Then, from condition (A1), it follows that $w'''(x_2) > 0$. However,

$$w'''(x_2) = \lim_{x \to x_2^-} \frac{w''(x) - w''(x_2)}{x - x_2} \leq 0.$$

This is a contradiction.

($\Rightarrow$) The necessity of inequalities.

Assume $w'(b) > 0$ and $w''(b) > 0$. By $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0$, there is some $x_0 \in (\eta_1, c)$ such that $w'(x_0) = 0$. Since $w'(b) > 0$, there is some $x_1 \in (b, x_0]$ such that $w'(x_1) = 0$ and $w'(x) > 0$ for $x \in [b, x_1)$. Then by the Mean Value Theorem and $w''(b) > 0$, there is some $x_2 \in (b, x_1)$ such that $w''(x_2) = 0$ and $w''(x) > 0$ for $x \in [b, x_2)$. We can see that $w(x) > 0$ and $w'(x) > 0$ for $x \in (b, x_2]$. Application of the condition (A1) gives us that $w'''(x_2) > 0$. However,

$$w'''(x_2) = \lim_{x \to x_2^-} \frac{w''(x) - w''(x_2)}{x - x_3} \leq 0,$$

which yields a contradiction.

($\Leftarrow$) The sufficiency of inequalities.

We can simply switch the sign of w in the proof of the necessity of inequality and then get a contradiction, too. Hence $w'(b) > 0$ if $w''(b) < 0$. $\square$

Lemma 2.3. *Let $y_1, y_2, y_3 \in \mathbb{R}$ be given and assume condition (A1) is satisfied. Then, given $m \in \mathbb{R}$, each of the BVP's for (2.1) satisfying any of conditions (2.5), (2.6), (2.7), or (2.8) has at most one solution.*

Proof. Here we prove the uniqueness of solutions of (2.1), (2.5) for any $m \in \mathbb{R}$. The other cases will be very similar based on Lemmas 2.1 or 2.2.

Suppose there are two solutions p and q of (2.1) satisfying (2.5). Let $w = p - q$. Then, we can see that w satisfies

$$w'''(x) = f(x, p(x), p'(x), p''(x)) - f(x, q(x), q'(x), q''(x)), \qquad x \in [a, b],$$
$$w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0, \quad w(b) = 0, \quad w'(b) = 0.$$

By Lemma 2.1, we have $w''(b) = 0$. From the assumption that solutions of IVP's for (2.1) are unique and exist on all of $[a, c]$, it follows that $p \equiv q$ on $[a, c]$. □

Lemma 2.4. *Let $y_1, y_2, y_3 \in \mathbb{R}$ be given. Assume (A1) is satisfied. Then, the BVP (2.1), (2.2) has at most one solution.*

Proof. Suppose p and q are two solutions of the BVP (2.1), (2.2). Let $w = p - q$. Then, w satisfies

$$w'''(x) = f(x, p(x), p'(x), p''(x)) - f(x, q(x), q'(x), q''(x)), \qquad x \in [a, c],$$
$$w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0, \quad w(b) = 0, \quad \sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0.$$

Since $p \neq q$, by Lemmas 2.1 and 2.2, we have $w'(b) \neq 0$ and $w''(b) \neq 0$. Without loss of generality, we suppose $w'(b) > 0$. By Lemma 2.1, we have $w''(b) > 0$, however, Lemma 2.2 gives us $w''(b) < 0$. A contradiction. Therefore, $p \equiv q$ over $[a, c]$. □

Given any $m \in \mathbb{R}$, let $\alpha_1(x, m)$, $u_1(x, m)$, $\beta_1(x, m)$, $v_1(x, m)$ denote the solutions, when they exist, of the boundary value problems of (2.1) satisfying (2.5), (2.6), (2.7), or (2.8), respectively.

Lemma 2.5. *Suppose that condition (A1) is satisfied and that for each $m \in \mathbb{R}$, there exist solutions of (2.1) satisfying each of the conditions (2.5), (2.6), (2.7) and (2.8), respectively. Then, $\alpha_1''(b, m)$ and $\beta_1''(b, m)$ are, respectively, strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$. Also, $u_1'(b, m)$ and $v_1'(b, m)$ are, respectively, strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$.*

Proof. By Lemmas 2.1 and 2.2, it is easy to see that $\alpha_1''(b,m)$ and $u_1'(b,m)$ are strictly increasing functions of m, and $\beta_1''(b,m)$ and $v_1'(b,m)$ are strictly decreasing functions of m.

We prove here that the range of $\alpha_1''(b,m)$, as a function of m, is all of $\mathbb{R}$. The proofs of other cases are very similar. To show $\{\alpha_1''(b,m)|m \in \mathbb{R}\} = \mathbb{R}$, let $l \in \mathbb{R}$. Consider the solution $u_1(x,l)$ of (2.1) satisfying (2.6) and the solution $\alpha_1(x, u_1'(b,l))$ of (2.1) satisfying (2.5). Then, both $u_1(x,l)$ and $\alpha_1(x, u_1'(b,l))$ satisfy (2.1) and the boundary conditions $y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1$, $y(b) = y_2$ and $\alpha_1'(b, u_1'(b,l)) = u_1'(b,l)$. By Lemma (2.3), $\alpha_1(x, u_1'(b,l)) \equiv u_1(x,l)$ for $x \in [a,b]$. Hence, $\alpha_1''(b, u_1'(b,l)) = u_1''(b,l) = l$. Therefore, $l \in \{\alpha_1''(b,m)|m \in \mathbb{R}\}$, that is, $\{\alpha_1''(b,m)|m \in \mathbb{R}\} = \mathbb{R}$. □

Theorem 2.1. *Assume f satisfies condition (A1) and that for each $m \in \mathbb{R}$, there exist solutions of (2.1) satisfying each of the conditions (2.5), (2.6), (2.7) and (2.8), respectively. Then, (2.1), (2.2) has a unique solution.*

Proof. We may prove the theorem by matching α_1 with β_1 or u_1 with v_1. Here we take advantage of α_1 and β_1. For any $m \in \mathbb{R}$, we have a solution $\alpha_1(x,m)$ of (2.1), (2.5) and a solution $\beta_1(x,m)$ of (2.1), (2.7). From Lemma 2.5, $\alpha_1''(b,m)$ and $\beta_1''(b,m)$ are, respectively, strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$. Then, there is a unique $m_0 \in \mathbb{R}$ such that $\alpha_1''(b,m_0) = \beta_1''(b,m_0)$. Therefore, the following piecewise defined function

$$y(x) = \begin{cases} \alpha_1(x,m_0), & x \in [a,b], \\ \beta_1(x,m_0), & x \in [b,c] \end{cases}$$

is the unique solution of (2.1), (2.2). □

2.3 The Case of (2.1), (2.4)

In this section, we are concerned with the existence and uniqueness of solutions of (2.1), (2.4) on the interval $[a,c]$, which is recalled as follows:

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \qquad x \in [a,c], \tag{2.1}$$

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y''(b) = y_2, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{2.4}$$

In this case, our conditions on f need to be independent of the last two terms. Hence, the differential equation we will consider here takes the form

$$y'''(x) = f(x, y(x)), \qquad x \in [a, c].$$

However, to be consistent, we still use the equation labeled (2.1) for this equation.

Throughout this section, it is assumed that $f : [a, c] \times \mathbb{R} \to \mathbb{R}$ is continuous and that solutions of IVP's for (2.1) are unique and exist on all of $[a, c]$.

Consider the following list of four boundary conditions,

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y''(b) = y_2, \quad y(b) = m, \tag{2.9}$$

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y''(b) = y_2, \quad y'(b) = m, \tag{2.10}$$

$$y''(b) = y_2, \quad y(b) = m, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{2.11}$$

$$y''(b) = y_2, \quad y'(b) = m, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{2.12}$$

where $m \in \mathbb{R}$, and we are going to obtain that (2.1), (2.4) has a unique solution by matching solutions of the BVP (2.1), (2.9) on $[a, b]$ with solutions of (2.1), (2.11) on $[b, c]$, or solutions of (2.1), (2.10) on $[a, b]$ with solutions of (2.1), (2.12) on $[b, c]$.

Conditions on f for the case of (2.1), (2.4) are stated as the following (A2):

(A2) The function f is of the form $f(x, v)$ and $f(x, v) - f(x, u) > 0$, when $x \in (a, b)$, $v < u$; or when $x \in (b, c)$, $v > u$.

Now with condition (A2) and similar ideas to Section 2.2, we present a whole set of results for (2.1), (2.4). The first are our lemmas showing relations between the change in function values and in values of the first order derivatives of solutions of (2.1) at b satisfying $y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1$, $y''(b) = y_2$, or $y''(b) = y_2$, $\sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3$, respectively.

Lemma 2.6. *Suppose p and q are solutions of (2.1) satisfying $y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1$ and $y''(b) = y_2$ on $[a, b]$, and let $w = p - q$ so that w satisfies*

$$w'''(x) = f(x, p(x)) - f(x, q(x)), \qquad x \in [a, b],$$
$$w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0, \ w''(b) = 0.$$

If the condition (A2) is satisfied, then $w(b) = 0$ if and only if $w'(b) = 0$, and $w(b) > 0$ if and only if $w'(b) > 0$.

Proof. ($\Rightarrow$) The necessity of equalities.

Suppose $w(b) = 0$ and $w'(b) \neq 0$. Without loss of generality, we suppose $w'(b) > 0$. By $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = \sum_{i=1}^{s} a_i(w(a) - w(\xi_i)) = 0$ and $a_i > 0$ for $i = 1, 2, \ldots, s$, there is some $x_0 \in (a, \xi_s)$ such that $w'(x_0) = 0$. Since $w'(b) > 0$, there is some $x_1 \in [x_0, b)$ such that $w'(x) > 0$ for $x \in (x_1, b]$ and $w'(x_1) = 0$. From $w(b) = 0$, we see that $w(x) < 0$ for $x \in [x_1, b)$. Since $w'(x_1) = 0$ and $w'(b) > 0$ and the Mean Value Theorem, we have some $x_2 \in (x_1, b)$ such that $w''(x_2) > 0$. By $w''(b) = 0$ and the Mean Value Theorem, there is some $x_3 \in (x_2, b)$ such that $w'''(x_3) < 0$. However, from condition (A2), we have $w'''(x) > 0$ for $x \in [x_1, b)$. This is a contradiction.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w'(b) = 0$ and $w(b) \neq 0$. Without loss of generality, we assume $w(b) < 0$. Since w is continuous on $[a, b]$, so w is negative on a left neighborhood of b. By condition (A2), we know that $w'''(x) > 0$ on that deleted left neighborhood. From $w'(b) = 0$, $w(b) < 0$, and $w''(b) = 0$, we have that on that deleted left neighborhood of b, $w''(x) < 0$, $w'(x) > 0$, $w(x) < 0$.

By $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0$, there is some $x_0 \in (a, \xi_s)$ such that $w'(x_0) = 0$. Hence, there is some $x_1 \in [x_0, b)$ such that $w'(x_1) = 0$ and $w'(x) > 0$ for $x \in (x_1, b]$. So, $w(x) < 0$ for $x \in [x_1, b]$, which implies $w'''(x) > 0$ for $x \in [x_1, b)$ and $w''(x) < 0$ for $x \in [x_1, b)$. However, by $w'(x_1) = 0$ and $w'(b) = 0$, there is some $x_2 \in (x_1, b)$ such that $w''(x_2) = 0$, which leads to a contradiction. Therefore, $w(b) = 0$.

($\Rightarrow$) The necessity of inequalities.

Assume $w(b) > 0$ and $w'(b) < 0$. Similarly, by $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0$, there is some $x_0 \in (a, \xi_s)$ such that $w'(x_0) = 0$. Since $w'(b) < 0$, there is some $x_1 \in [x_0, b)$ such that $w'(x_1) = 0$ and $w'(x) < 0$ for $x \in (x_1, b]$. Hence $w(x) > 0$ for $x \in [x_1, b]$. By condition (A2), $w'''(x) < 0$ for $x \in [x_1, b)$. By $w''(b) = 0$, we have that $w''(x) > 0$ for $x \in [x_1, b)$. However, by $w'(x_1) = 0$ and $w'(b) < 0$ and the Mean Value Theorem, there is some $x_2 \in (x_1, b)$ such that $w''(x_2) < 0$, which is a contradiction.

($\Leftarrow$) The sufficiency of inequalities.

We assume that $w(b) < 0$ and $w'(b) > 0$. Then, we get the same situation as the proof of necessity by replacing w with $-w$, which also implies a contradiction. Hence $w(b) > 0$ if $w'(b) > 0$. $\square$

Lemma 2.7. *Suppose p and q are solutions of (2.1) satisfying $y''(b) = y_2$ and $\sum_{j=1}^{t} b_j y(\eta_j)$ $-y(c) = y_3$ on $[b, c]$ and let $w = p - q$ so that w satisfies*

$$w'''(x) = f(x, p(x)) - f(x, q(x)), \qquad x \in [b, c],$$
$$w''(b) = 0, \quad \sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0.$$

If f satisfies the condition (A2), then $w(b) = 0$ if and only if $w'(b) = 0$, and $w(b) > 0$ if and only if $w'(b) < 0$.

Proof. ($\Rightarrow$) The necessity of equalities.

Suppose $w(b) = 0$ and $w'(b) \neq 0$. Without loss of generality, we suppose $w'(b) > 0$. By $w(b) = 0$, we know on a deleted right neighborhood of b, w is positive. By (A2), we know that $w''' > 0$ on that deleted right neighborhood of b, which from $w''(b) = 0$ implies that w stays positive and increasing on (b, c), which contradicts $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0$. Therefore, $w'(b) = 0$.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w'(b) = 0$ and $w(b) \neq 0$. Without loss of generality, we suppose $w(b) > 0$. Since w is continuous on $[a, b]$, so w is positive on a right neighborhood of

b. From condition (A2), we have that $w''' > 0$ on that deleted neighborhood. Since $w''(b) = 0$ and $w'(b) = 0$, we have that w is increasing and positive on (b, c), which is contrary to $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0$. Hence, $w(b) = 0$.

($\Rightarrow$) The necessity of inequalities.

Assume $w(b) > 0$ and $w'(b) > 0$. By condition (A2), we will get a similar contradiction to that in the proofs of equivalence of equalities. Hence $w(b) > 0$ and $w'(b) < 0$.

($\Leftarrow$) The sufficiency of inequalities.

We assume that $w(b) < 0$ and $w'(b) < 0$. Then, we get the same situation as the proof of necessity with opposite signs of $w(b)$ and $w'(b)$, which also implies a contradiction. Hence $w(b) > 0$ and $w'(b) < 0$. $\square$

With these two lemmas, we are ready to discuss the uniqueness and existence of solutions of (2.1), (2.4). We first consider the uniqueness of solutions of each of the BVP's for (2.1) satisfying any of (2.9), (2.10), (2.11), or (2.12).

Lemma 2.8. *Let $y_1, y_2, y_3 \in \mathbb{R}$ be given and assume condition (A2) is satisfied. Then, given $m \in \mathbb{R}$, each of the BVP's for (2.1) satisfying any of conditions (2.9), (2.10), (2.11), or (2.12) has at most one solution.*

Proof. By using Lemmas 2.6 and 2.7, the proofs are based on the same idea as that of Lemma 2.3. We omit them here. $\square$

Lemma 2.9. *Let $y_1, y_2, y_3 \in \mathbb{R}$ be given. Assume the condition (A2) is satisfied. Then, the BVP (2.1), (2.4) has at most one solution.*

Proof. Suppose for some $y_1, y_2, y_3 \in \mathbb{R}$, there exist distinct solutions p and q of (2.1), (2.4). Let $w = p - q$. Then, from Lemmas 2.6 and 2.7, we get $w(b) \neq 0$, $w'(b) \neq 0$.

Without loss of generality, we suppose $w(b) > 0$. Then, by Lemma 2.6, $w'(b) > 0$. But by Lemma 2.7, $w'(b) < 0$. This is a contradiction. Hence, $p \equiv q$ on $[a, c]$. $\square$

Now we show that certain derivatives of solutions of (2.1) satisfying each of (2.9), (2.10), (2.11), or (2.12), respectively, are monotone functions of m at b. For notation purposes, given any $m \in \mathbb{R}$, let $\alpha_2(x,m)$, $u_2(x,m)$, $\beta_2(x,m)$, $v_2(x,m)$ denote the solutions, when they exist, of the boundary value problems of (2.1) satisfying (2.9), (2.10), (2.11), or (2.12), respectively.

Lemma 2.10. *Suppose that the condition (A2) is satisfied and that for each $m \in \mathbb{R}$, there exist solutions of (2.1) satisfying each of the conditions (2.9), (2.10), (2.11), or (2.12), respectively. Then, $\alpha_2'(b,m)$ and $\beta_2'(b,m)$ are, respectively, strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$. Also, $u_2(b,m)$ and $v_2(b,m)$ are, respectively, strictly increasing and strictly decreasing functions of m with ranges all of $\mathbb{R}$.*

Proof. By using Lemmas 2.6 and 2.7, the proof is very similar to that of Lemma 2.5. □

Finally, we arrive at our existence result for (2.1), (2.2), which is obtained by solution matching.

Theorem 2.2. *Assume the condition (A2) is satisfied and that for each $m \in \mathbb{R}$, there exist solutions of (2.1) satisfying each of the conditions (2.9), (2.10), (2.11) and (2.12), respectively. Then, (2.1), (2.4) has a unique solution.*

Proof. By using Lemma 2.10, the proof is very similar to that of Theorem 2.1. □

CHAPTER THREE

Nonlocal Boundary Value Problems with Even Gaps in Boundary Conditions for Third Order Differential Equations

3.1 Introduction

In this chapter, we again use solution matching to study the uniqueness and existence of solutions for the nonlocal boundary value problem for third order differential equations (2.1), (2.3). For convenience, we recall

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \qquad x \in [a, c], \tag{2.1}$$

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y'(b) = y_2, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{2.3}$$

where $a < \xi_1 < \xi_2 < \cdots < \xi_s < b < \eta_1 < \eta_2 < \eta \cdots < \eta_t < c$, $s, t \in \mathbb{N}$, $a_i, b_j > 0$ for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots t$, $\sum_{i=1}^{s} a_i = \sum_{j=1}^{t} b_j = 1$, and $y_1, y_2, y_3 \in \mathbb{R}$.

Examining the boundary conditions at b, we can see that the function value and the value of the second order derivative of solutions are missing. The difference of their order of derivatives (or gaps) is two, which is even. At the time of this work, no previous work has been done on the existence and uniqueness of solutions of this BVP with even gaps at b based on the techniques of solution-matching.

3.2 The Case of (2.1), (2.3)

In this case, our conditions require that f is independent of the last two terms. Hence, the differential equation we will consider here for this case is

$$y'''(x) = f(x, y(x)), \qquad x \in [a, c].$$

However, to be consistent, we still use the equation label (2.1) for this equation.

Consider the following list of four boundary conditions,

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y'(b) = y_2, \quad y(b) = m, \tag{3.1}$$

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y'(b) = y_2, \quad y''(b) = m, \tag{3.2}$$

$$y'(b) = y_2, \quad y(b) = m, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{3.3}$$

$$y'(b) = y_2, \quad y''(b) = m, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_3, \tag{3.4}$$

where $m \in \mathbb{R}$, and we are going to obtain that (2.1), (2.3) has a unique solution on $[a, c]$ by matching solutions of the BVP's (2.1), (3.1) on $[a, b]$ with solutions of (2.1), (3.3) on $[b, c]$, or matching solutions of (2.1), (3.2) on $[a, b]$ with solutions of (2.1), (3.4) on $[b, c]$.

Throughout this section, it is assumed that $f : [a, c] \times \mathbb{R} \to \mathbb{R}$ is continuous and that solutions of IVP's for (2.1) are unique and exist on all of $[a, c]$. A monotone condition on f for the case of (2.1), (2.3) is the same as the condition (A2) and we recall it as follows:

(A2) Function f is of the form $f(x, v)$ and $f(x, v) - f(x, u) > 0$, when $x \in (a, b)$, $v < u$; or when $x \in (b, c)$, $v > u$.

The next two lemmas are our elementary results for our main theorem, which describe the monotonicity relations between the changes in the function values and changes in the values of the second derivative of solutions of (2.1), (2.3) at b.

Lemma 3.1. *Suppose p and q are solutions of (2.1) satisfying $y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1$, $y'(b) = y_2$ on $[a, b]$, and let $w = p - q$ so that w satisfies*

$$w'''(x) = f(x, p(x)) - f(x, q(x)), \qquad x \in [a, b],$$
$$w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0, \ w'(b) = 0.$$

If the condition (A2) is satisfied, then $w(b) = 0$ if and only if $w''(b) = 0$, and $w(b) > 0$ if and only if $w''(b) < 0$.

Proof. ($\Rightarrow$) The necessity of equalities.

Suppose $w(b) = 0$ and $w''(b) \neq 0$. Without loss of generality, we assume $w''(b) > 0$. By $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = \sum_{i=1}^{s} a_i(w(a) - w(\xi_i) = 0$ and $a_i > 0$ for $i = 1, 2, \ldots, s$, there is some $x_0 \in (a, \xi_s)$ such that $w'(x_0) = 0$. Since $w'(b) = 0$, there is some $x_1 \in (x_0, b)$ such that $w''(x_1) = 0$. From $w''(b) > 0$, there is some $x_2 \in [x_1, b)$ such that $w''(x) > 0$ for $x \in (x_2, b]$ and $w''(x_2) = 0$. By the Mean Value Theorem, there is some $x_3 \in (x_2, b)$ such that $w'''(x_3) > 0$. However, from $w(b) = w'(b) = 0$, we have $w(x) > 0$ and $w'(x) < 0$ for $x \in [x_2, b)$, which together with the condition (A2) imply $w'''(x) < 0$ for $x \in [x_2, b)$. This is a contradiction.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w''(b) = 0$ and $w(b) \neq 0$. Without loss of generality, we assume $w(b) > 0$. With the same condition (A2), the proof is the same as that of the sufficiency of equalities of Lemma 2.6.

($\Rightarrow$) The necessity of inequalities.

Assume $w(b) > 0$ and $w''(b) > 0$. Similarly, by $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0$, there is some $x_0 \in (a, \xi_s)$ such that $w'(x_0) = 0$. Since $w'(b) = 0$, there is some $x_1 \in (x_0, b)$ such that $w''(x_1) = 0$. Since $w''(b) > 0$, there is $x_2 \in [x_1, b)$ such that $w''(x) > 0$ for $x \in (x_2, b]$ and $w''(x_2) = 0$. From $w(b) > 0$ and $w'(b) = 0$, it follows that $w(x) > 0$ and $w'(b) < 0$ for $x \in [x_2, b)$. By the condition (A2), $w'''(x) < 0$ for $x \in [x_2, b)$. However, from $w''(x_2) = 0$ and $w''(b) > 0$ and the Mean Value Theorem, there is some $x_3 \in (x_2, b)$ such that $w'''(x_3) > 0$. This is a contradiction. Hence if $w(b) > 0$, then $w''(b) < 0$.

($\Leftarrow$) The sufficiency of inequalities.

We assume that $w(b) < 0$ and $w''(b) < 0$. By replacing w by $-w$, we are in the same situation as in the proof of the necessity of inequalities and we will arrive at a contradiction. Hence $w(b) < 0$, if $w'(b) < 0$. $\square$

Lemma 3.2. *Suppose p and q are solutions of (2.1) satisfying $y'(b) = y_2$ and $\sum_{j=1}^{t} b_j y(\eta_j)$*

$-y(c) = y_3$ on $[b,c]$ and let $w = p - q$ so that w satisfies

$$w'''(x) = f(x,p(x)) - f(x,q(x)), \qquad x \in [b,c],$$
$$w'(b) = 0, \ \sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0.$$

If the condition (A2) is satisfied, then $w(b) = 0$ if and only if $w''(b) = 0$, and $w(b) > 0$ if and only if $w''(b) < 0$.

Proof. ($\Rightarrow$) The necessity of equalities.

Suppose $w(b) = 0$ and $w''(b) \neq 0$. Without loss of generality, we suppose $w''(b) > 0$. From $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = \sum_{j=1}^{t} b_j (w(\eta_j) - w(c)) = 0$ and $b_j > 0$ for $j = 1, 2, \ldots, t$, we have that there is some $x_0 \in (\eta_1, c)$ such that $w'(x_0) = 0$. The Mean Value Theorem and $w'(b) = w'(x_0) = 0$ imply that there is some $x_1 \in (b, x_0)$ such that $w''(x_1) = 0$. By $w''(b) > 0$, there is some $x_2 \in (b, x_1]$ such that $w''(x_2) = 0$ and $w''(x) > 0$ for $x \in [b, x_2)$. $w(b) = 0$ gives us that $w(x) > 0$ for $x \in (b, x_2]$, which together with condition (A2) implies that $w'''(x) > 0$ for $x \in (b, x_2]$. However, $w''(b) > 0$ and $w''(x_2) = 0$ and the Mean Value Theorem, there is some $x_3 \in (b, x_2)$ such that $w'''(x_3) < 0$, which is again a contradiction.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w''(b) = 0$ and $w(b) \neq 0$. Without loss of generality, we suppose $w(b) > 0$. Under the condition (A2), the proof will be the same as that of the sufficiency of equalities of Lemma 2.7.

($\Rightarrow$) The necessity of inequalities.

Assume $w(b) > 0$ and $w''(b) > 0$. By condition (A2), the proof is similar to that of the necessity of equalities. A contradiction yields that $w''(b) < 0$, if $w(b) > 0$.

($\Leftarrow$) The sufficiency of inequalities.

We assume that $w(b) < 0$ and $w''(b) < 0$. Then, we have the same situation as the proof of necessity with opposite sign of w, which leads to a contradiction. Hence $w(b) > 0$, if $w''(b) < 0$. $\square$

With the above two fundamental lemmas, we are in a position to show some more lemmas leading to our matching ideas.

Lemma 3.3. *Let $y_1, y_2, y_3 \in \mathbb{R}$ be given and assume condition (A2) is satisfied. Then, given $m \in \mathbb{R}$, each of the BVP's for (2.1) satisfying any of conditions (3.1), (3.2), (3.3), or (3.4) has at most one solution.*

Proof. By using Lemmas 3.1 and 3.2, the proofs are based on the same idea as that of Lemma 2.3. □

Now we show that solutions of (2.1) satisfying each of (3.1), (3.2), (3.3), or (3.4), respectively, are monotone functions of m at b. For notation purposes, given any $m \in \mathbb{R}$, let $\alpha_3(x,m)$, $u_3(x,m)$, $\beta_3(x,m)$, $v_3(x,m)$ denote the solutions, when they exist, of the boundary value problems of (2.1) satisfying (3.1), (3.2), (3.3), or (3.4), respectively.

Lemma 3.4. *Suppose that condition (A2) is satisfied and that for each $m \in \mathbb{R}$, there exist solutions of (2.1) satisfying each of the conditions (3.1), (3.2), (3.3), (3.4), respectively. Then, all of $\alpha_3''(b,m)$, $\beta_3''(b,m)$, $u_3(b,m)$ and $v_3(b,m)$ are strictly decreasing functions of m with ranges all of $\mathbb{R}$.*

Proof. First, by using Lemmas 3.1 and 3.2, $\alpha_3''(b,m)$, $\beta_3''(b,m)$, $u_3(b,m)$ and $v_3(b,m)$ are all strictly decreasing functions of m.

Then, we prove the range of $\alpha_3''(b,m)$ as a function of m is all of $\mathbb{R}$. The proofs of other cases are very similar. It suffices to show $\{\alpha_3''(b,m) | m \in \mathbb{R}\} = \mathbb{R}$. Let $l \in \mathbb{R}$. Consider the solution $u_3(x,l)$ of (2.1) satisfying (3.2) and the solution $\alpha_3(x, u_3''(b,l))$ of (2.1) satisfying (3.1). Then, both $u_3(x,l)$ and $\alpha_3(x, u_3(b,l))$ satisfy (2.1) and the boundary conditions $y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1$, $y'(b) = y_2$ and $\alpha_3(b, u_3(b,l)) = u_3(b,l)$. By Lemma 3.1, $\alpha_3(x, u_3(b,l)) \equiv u_3(x,l)$ for $x \in [a,b]$. Hence, $\alpha_3''(b, u_3(b,l)) = u_3''(b,l) = l$. Therefore, $l \in \{\alpha_3''(b,m) | m \in \mathbb{R}\}$, that is, $\{\alpha_3''(b,m) | m \in \mathbb{R}\} = \mathbb{R}$.

□

When applying the solution-matching technique to get our main results in this chapter, we consider matching solutions of BVP (2.1), (3.1) with solutions of BVP (2.1), (3.3). Next, under certain Lipschitz conditions of f, we obtain some bounds to the rate of change of the second order derivative of solutions of (2.1) at b with respect of $m \in \mathbb{R}$.

Lemma 3.5. *Suppose f satisfies the condition (A2), and suppose there is some $M_1 > 0$, such that*

$$f(x,v) - f(x,u) \geq -M_1(v-u), \ \forall\, x \in (a,b), \ \forall\, v \geq u \in \mathbb{R}. \tag{3.5}$$

Assume for each $m \in \mathbb{R}$, there exists a solution $\alpha_3(x,m)$ of (2.1) satisfying (3.1). Let $m_1, m_2 \in \mathbb{R}$ with $m_1 < m_2$. Then,

$$\alpha_3''(b,m_2) - \alpha_3''(b,m_1) > -M_1(b-a)(m_2-m_1). \tag{3.6}$$

Proof. Let $m_1, m_2 \in \mathbb{R}$ with $m_1 < m_2$ be fixed. We denote $\Phi(x) = \frac{\alpha_3(x,m_2)-\alpha_3(x,m_1)}{m_2-m_1}$. Then, $\Phi(x)$ satisfies

$$\begin{aligned}
&\Phi'''(x) = \frac{f(x,\alpha_3(x,m_2)) - f(x,\alpha_3(x,m_1))}{m_2-m_1}, \qquad x \in [a,b],\\
&\Phi(b) = 1, \quad \Phi'(b) = 0, \quad \Phi(a) - \sum_{i=1}^{s} a_i \Phi(\xi_i) = 0,
\end{aligned}$$

and by Lemma 3.4, $\Phi''(b) < 0$. It suffices to show that $\Phi''(b) > -M_1(b-a)$.

Since $\Phi(a) - \sum\limits_{i=1}^{s} a_i\Phi(\xi_i) = \sum\limits_{i=1}^{s} a_i(\Phi(a) - \Phi(\xi_i)) = 0$ and $a_i > 0$, for $i = 1, 2, \ldots, s$, there is some $x_0 \in (a, \xi_s)$ such that $\Phi'(x_0) = 0$. By $\Phi'(b) = 0$, there is some $x_1 \in (x_0, b)$ such that $\Phi''(x_1) = 0$. Since $\Phi''(b) < 0$, there is some $x_2 \in [x_1, b)$ such that $\Phi''(x_2) = 0$ and $\Phi''(x) < 0$ for $x \in (x_2, b]$. From $\Phi'(b) = 0$, we can see that $\Phi'(x) > 0$ for $x \in [x_2, b)$. Since $\Phi'(x_0) = 0$, there is some $x_3 \in [x_0, x_2)$ such that $\Phi'(x_3) = 0$ and $\Phi'(x) > 0$ for $x \in (x_3, b)$.

Next, we show $\Phi(x) > 0$ for $x \in [x_2, b]$. Suppose this is not true. Then, from $\Phi'(x_3) = 0$ and $\Phi'(x) > 0$ for $x \in (x_3, b)$ and $x_3 \in [x_0, x_2)$, we have that $\Phi(x_2) \leq 0$

and $\Phi(x) < 0$ for $x \in [x_3, x_2)$. From condition (A2), $\Phi'''(x) > 0$ for $x \in [x_3, x_2)$. However, from $\Phi'(x_3) = 0$ and $\Phi'(x_2) > 0$, there is some $x_4 \in (x_3, x_2)$ such that $\Phi''(x_4) > 0$. Also from $\Phi''(x_2) = 0$, there is some $x_5 \in (x_4, x_2) \subset (x_3, x_2)$ such that $\Phi'''(x_5) < 0$. This is a contradiction to $\Phi'''(x) > 0$ for $x \in [x_3, x_2)$. Therefore, $\Phi(x) > 0$ for $x \in [x_2, b]$.

Now from $\Phi(x) > 0$ for $x \in [x_2, b]$ and $\Phi'(x) > 0$ for $x \in [x_2, b)$, it is easy to see that $0 < \Phi(x) < 1$ for $x \in [x_2, b)$. Then, by (3.5), for $x \in [x_2, b)$

$$\begin{aligned}
\Phi'''(x) &= \frac{f(x, \alpha_3(x, m_2)) - f(x, \alpha_3(x, m_1))}{m_2 - m_1} \\
&\geq \frac{-M_1(\alpha_3(x, m_2) - \alpha_3(x, m_1))}{m_2 - m_1} \\
&= -M_1 \Phi(x) \\
&> -M_1.
\end{aligned}$$

Next we show that $\Phi''(b) > -M_1(b-a)$. Suppose this is not true. Then, $\Phi''(b) \leq -M_1(b-a)$. By $\Phi''(x_2) = 0$, there is some $x_5 \in (x_2, b)$ such that

$$\Phi'''(x_5) = \frac{\Phi''(b) - \Phi''(x_2)}{b - x_2} \leq \frac{-M_1(b-a)}{b - x_2} = -M_1 \frac{(b-a)}{b - x_2} < -M_1,$$

which is a contradiction to $\Phi'''(x) > -M_1$ for $x \in [x_2, b)$. Therefore, $\Phi''(b) > -M_1(b-a)$. $\square$

Lemma 3.6. *Suppose f satisfies the condition (A2) and there is a continuous function $M_1(x)$ for $x \in [a, b]$, such that*

$$f(x, v) - f(x, u) \leq -M_1(x)(v - u), \ \forall\, x \in (a, b), \ \forall\, v \geq u \in \mathbb{R}, \tag{3.7}$$

where $M_1(x) > 0$, for $x \in [a, b)$, and

$$\frac{\sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \int_r^b M_1(e) de dr dl}{\sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \left(1 + \int_r^b M_1(e) \frac{(b-e)^2}{2} de\right) dr dl} \geq \frac{2}{(b-a)^2}. \tag{3.8}$$

Assume for each $m \in \mathbb{R}$, there exists a solution $\alpha_3(x, m)$ of (2.1) satisfying (3.1). Let $m_1 < m_2 \in \mathbb{R}$. Then,

$$\alpha_3''(b, m_2) - \alpha_3''(b, m_1) < -\frac{2(m_2 - m_1)}{(b-a)^2}. \tag{3.9}$$

Proof. Let $m_1 < m_2 \in \mathbb{R}$ be fixed. We denote

$$\Phi(x) = \frac{\alpha_3(x, m_2) - \alpha_3(x, m_1)}{m_2 - m_1}.$$

Then $\Phi(x)$ satisfies

$$\begin{aligned}
&\Phi'''(x) = \frac{f(x, \alpha_3(x, m_2)) - f(x, \alpha_3(x, m_1))}{m_2 - m_1}, \qquad x \in [a, b], \\
&\Phi(b) = 1, \quad \Phi'(b) = 0, \quad \Phi(a) - \sum_{i=1}^{s} a_i \Phi(\xi_i) = 0,
\end{aligned}$$

and by Lemma 3.4, $\Phi''(b) < 0$. It suffices to show that $\Phi''(b) < -\frac{2}{(b-a)^2}$. Suppose this is not true. Then, $\Phi''(b) \geq -\frac{2}{(b-a)^2}$.

By $\Phi(b) = 1$, $\Phi'(b) = 0$, and $\Phi''(b) \geq -\frac{2}{(b-a)^2}$, we have that

$$\begin{aligned}
\Phi(x) &= \Phi(b) - \int_x^b \Phi'(l)dl = \Phi(b) + \int_x^b \int_l^b \Phi''(r)drdl \\
&= \Phi(b) + \int_x^b \int_l^b \left(\Phi''(b) - \int_r^b \Phi'''(e)de\right) drdl \\
&= 1 + \Phi''(b) \cdot \frac{(b-x)^2}{2} - \int_x^b \int_l^b \int_r^b \Phi'''(e)dedrdl.
\end{aligned}$$

Next, we show $\Phi(x) > 0$ for $x \in [a, b]$. Assume this is not true. Let $x_0 \in [a, b)$ such that $\Phi(x_0) = 0$ and $\Phi(x) > 0$ for $x \in (x_0, b]$. Then, by (3.7),

$$\Phi'''(x) = \frac{f(x, \alpha_3(x, m_2)) - f(x, \alpha_3(x, m_1))}{m_2 - m_1} \leq -M_1(x)\Phi(x), \quad \forall x \in (x_0, b].$$

Hence, by $\Phi''(b) \geq -\frac{2}{(b-a)^2}$,

$$\begin{aligned}
\Phi(x_0) &= 1 + \Phi''(b) \cdot \frac{(b-x_0)^2}{2} - \int_{x_0}^b \int_l^b \int_r^b \Phi'''(e)dedrdl \\
&\geq 1 + \Phi''(b) \cdot \frac{(b-x_0)^2}{2} + \int_{x_0}^b \int_l^b \int_r^b M_1(e)\Phi(e)dedrdl
\end{aligned}$$

$$
\begin{aligned}
&> 1+\Phi''(b)\cdot\frac{(b-x_0)^2}{2} \\
&\geq 1-\frac{(b-x_0)^2}{(b-a)^2} \\
&\geq 0,
\end{aligned}
$$

which is a contradiction to $\Phi(x_0)=0$.

From $\Phi(x)>0$ for $x\in[a,b]$, we have that $\Phi'''(x)\leq 0$ for $x\in[a,b]$. Hence, by (3.7), $\Phi'''(x)\leq -M_1(x)\Phi(x)$ for $x\in[a,b]$. Therefore,

$$
\begin{aligned}
\Phi(x) &= 1+\Phi''(b)\cdot\frac{(b-x)^2}{2}-\int_x^b\int_l^b\int_r^b \Phi'''(e)dedrdl \\
&\geq 1+\Phi''(b)\cdot\frac{(b-x)^2}{2}+\int_x^b\int_l^b\int_r^b M_1(e)\Phi(e)dedrdl \\
&> 1+\Phi''(b)\cdot\frac{(b-x)^2}{2}.
\end{aligned}
$$

Now, we use the expression

$$\Phi(x)=1+\Phi''(b)\cdot\frac{(b-x)^2}{2}-\int_x^b\int_l^b\int_e^b \Phi'''(e)dedrdl.$$

From $\Phi(a)-\sum\limits_{i=1}^{s}a_i\Phi(\xi_i)=0$, we have that

$$
\begin{aligned}
&\Phi''(b)\cdot\frac{(b-a)^2}{2}-\int_a^b\int_l^b\int_r^b \Phi'''(e)dedrdl \\
=\ &\sum_{i=1}^{s}a_i\left(\Phi''(b)\cdot\frac{(b-\xi_i)^2}{2}-\int_{\xi_i}^b\int_l^b\int_r^b \Phi'''(e)dedrdl\right),
\end{aligned}
$$

that is,

$$\Phi''(b)\cdot\sum_{i=1}^{s}a_i\left(\frac{(b-\xi_i)^2-(b-a)^2}{2}\right) = -\sum_{i=1}^{s}a_i\int_a^{\xi_i}\int_l^b\int_r^b \Phi'''(e)dedrdl.$$

By

$$\sum_{i=1}^{s}a_i\left(\frac{(b-\xi_i)^2-(b-a)^2}{2}\right)=-\sum_{i=1}^{s}a_i\int_a^{\xi_i}\int_l^b drdl,$$

$\Phi'''(x)\leq -M_1(x)\Phi(x)$ for $x\in[a,b]$, and $\Phi(x)>1+\Phi''(b)\cdot\frac{(b-x)^2}{2}$, we have that

$$-\Phi''(b)\sum_{i=1}^{s}a_i\int_a^{\xi_i}\int_l^b drdl$$

$$
\begin{aligned}
&= -\sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \int_r^b \Phi'''(e) dedrdl \\
&\geq \sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \int_r^b M_1(e)\Phi(e) dedrdl \\
&> \sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \int_r^b M_1(e) \left(1 + \Phi''(b) \cdot \frac{(b-e)^2}{2}\right) dedrdl \\
&= \sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \int_r^b M_1(e) dedrdl + \Phi''(b) \sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \int_r^b M_1(e) \frac{(b-e)^2}{2} dedrdl,
\end{aligned}
$$

which give that

$$
\begin{aligned}
& -\Phi''(b) \sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \left(1 + \int_r^b M_1(e) \frac{(b-e)^2}{2} de\right) drdl \\
> & \sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \int_r^b M_1(e) dedrdl,
\end{aligned}
$$

that is,

$$
-\Phi''(b) > \frac{\sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \int_r^b M_1(e) dedrdl}{\sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \left(1 + \int_r^b M_1(e) \frac{(b-e)^2}{2} de\right) drdl}
$$

By (3.8), we have

$$
-\Phi''(b) > \frac{2}{(b-a)^2},
$$

which is a contradiction to the assumption $-\Phi''(b) \leq \frac{2}{(b-a)^2}$. Therefore, $\Phi''(b) < -\frac{2}{(b-a)^2}$. □

Lemma 3.7. *Suppose f satisfies the condition (A2) and there is some $M_2 > 0$, such that*

$$
f(x, v) - f(x, u) \leq M_2(v - u), \ \forall\, x \in (b, c), \ \forall\, v \geq u \in \mathbb{R}. \tag{3.10}
$$

Assume for each $m \in \mathbb{R}$, there exists a solution $\beta_3(x, m)$ of (2.1) satisfying (3.3). Let $m_1 < m_2 \in \mathbb{R}$. Then,

$$
\beta_3''(b, m_2) - \beta_3''(b, m_1) > -M_2(c-b)(m_2 - m_1). \tag{3.11}
$$

Proof. Let $m_1 < m_2 \in \mathbb{R}$ be fixed. We denote $\Psi(x) = \frac{\beta_3(x,m_2)-\beta_3(x,m_1)}{m_2-m_1}$. Then $\Psi(x)$ satisfies

$$\Psi'''(x) = \frac{f(x,\beta_3(x,m_2)) - f(x,\beta_3(x,m_1))}{m_2-m_1}, \qquad x \in [b,c],$$
$$\Psi(b) = 1, \quad \Psi'(b) = 0, \quad \sum_{j=1}^{t} b_j\Psi(\eta_j) - \Psi(c) = 0,$$

and by Lemma 3.4, $\Psi''(b) < 0$. We need to show that $\Psi''(b) > -M_2(c-b)$.

By $\sum_{j=1}^{t} b_j\Psi(\eta_j) - \Psi(c) = \sum_{j=1}^{t} b_j(\Psi(\eta_j) - \Psi(c)) = 0$ and $b_j > 0$ for $j = 1, 2, \ldots, t$, there is some $x_0 \in (\eta_1, c)$ such that $\Psi'(x_0) = 0$. By $\Psi'(b) = 0$, there is some $x_1 \in (b, x_0)$ such that $\Psi''(x_1) = 0$ and $\Psi''(x) < 0$ for $x \in [b, x_1)$. It follows that $\Psi'(x) < 0$ for $x \in (b, x_1]$. Then there is some $x_2 \in (x_1, x_0]$ such that $\Psi'(x) < 0$ for $x \in (b, x_2)$ and $\Psi'(x_2) = 0$.

Now we want to show $\Psi(x) > 0$ for $x \in [b, x_1]$. Otherwise, by $\Psi'(x) < 0$ for $x \in (b, x_1) \subset (b, x_2)$ we suppose $\Psi(x_1) \leq 0$. Then for $x \in (x_1, x_2]$, $\Psi(x) < 0$. By condition (A2), $\Psi'''(x) \leq 0$ for $x \in (x_1, x_2]$. Since $\Psi''(x_1) = 0$, $\Psi''(x) \leq 0$ for $x \in (x_1, x_2]$. However, $\Psi'(x_1) < 0$ and $\Psi'(x_2) = 0$ and the Mean Value Theorem yield that there is some $x_3 \in (x_1, x_2)$ such that $\Psi''(x_3) > 0$. This is a contradiction. Therefore, $\Psi(x) > 0$ for $x \in [b, x_1]$ and so $0 \leq \Psi(x) < 1$ for $x \in (b, x_1)$.

Then, by (3.10), for $x \in (b, x_1)$,

$$\Psi'''(x) = \frac{f(x,\beta_3(x,m_2)) - f(x,\beta_3(x,m_1))}{m_2-m_1} \leq M_2\Psi(x) < M_2.$$

Since $\Psi''(x_1) = 0$, we claim that $\Psi''(b) > -M_2(c-b)$. Suppose this is not true, i.e., $\Psi''(b) \leq -M_2(c-b)$. Then there is some $x_4 \in (b, x_1)$ such that

$$\Psi'''(x_4) = \frac{\Phi''(x_1) - \Phi''(b)}{x_1 - b} \geq M_2\frac{c-b}{x_1-b} > M_2.$$

This is a contradiction. Therefore $|\Psi''(b)| < M_2(c-b)$ or $\Psi''(b) > -M_2(c-b)$. □

Lemma 3.8. *Suppose f satisfies the condition (A2) and there is a continuous function $M_2(x)$ for $x \in [b,c]$, such that*

$$f(x,v) - f(x,u) \geq M_2(x)(v-u), \ \forall\, x \in (b,c), \ \forall\, v \geq u \in \mathbb{R}, \tag{3.12}$$

where $M_2(x) > 0$, for $x \in (b, c]$, and

$$\frac{\sum_{j=1}^{t} b_j \int_{\eta_j}^{c} \int_b^l \int_b^r M_2(e)dedrdl}{\sum_{j=1}^{t} b_j \int_{\eta_j}^{c} \int_b^l \left(1 + \int_b^r M_2(e)\frac{(e-b)^2}{2}de\right) drdl} \geq \frac{2}{(c-b)^2}. \tag{3.13}$$

Assume for each $m \in \mathbb{R}$, there exist solutions $\beta_3(x, m)$ of (2.1) satisfying (3.3). Let $m_1 < m_2 \in \mathbb{R}$. Then,

$$\beta_3''(b, m_2) - \beta_3''(b, m_1) < -\frac{2(m_2 - m_1)}{(c-b)^2}. \tag{3.14}$$

Proof. Let $m_1 < m_2 \in \mathbb{R}$ be fixed. We denote $\Psi(x) = \frac{\beta_3(x,m_2)-\beta_3(x,m_1)}{m_2-m_1}$. Then $\Psi(x)$ satisfies

$$\Psi'''(x) = \frac{f(x, \beta_3(x, m_2)) - f(x, \beta_3(x, m_1))}{m_2 - m_1}, \qquad x \in [b, c],$$
$$\Psi(b) = 1, \quad \Psi'(b) = 0, \quad \sum_{j=1}^{t} b_j \Psi(\eta_j) - \Psi(c) = 0,$$

and by Lemma 3.4, $\Psi''(b) < 0$. Then, we only need to show that $\Psi''(b) < -\frac{2}{(c-b)^2}$. Suppose this is not true. Then, $\Psi''(b) \geq -\frac{2}{(c-b)^2}$.

Similarly as in the proof of Lemma 3.6, by $\Psi(b) = 1$, $\Psi'(b) = 0$, and $\Psi''(b) \geq -\frac{2}{(c-b)^2}$, we have that

$$\begin{aligned}\Psi(x) &= \Psi(b) + \int_b^x \int_b^l \left(\Psi''(b) + \int_b^r \Psi'''(e)de\right) drdl \\ &= 1 + \Psi''(b) \cdot \int_b^x \int_b^l drdl + \int_b^x \int_b^l \int_b^r \Psi'''(e)dedrdl.\end{aligned}$$

Next, we show $\Psi(x) > 0$ for $x \in [b, c]$. Assume it is not true. Let $x_0 \in (b, c]$ such that $\Psi(x_0) = 0$ and $\Psi(x) > 0$ for $x \in [b, x_0)$. Then, by (3.12) and $\Psi''(b) \geq -\frac{2}{(c-b)^2}$,

$$\begin{aligned}\Psi(x_0) &= 1 + \Psi''(b) \cdot \frac{(x_0 - b)^2}{2} + \int_b^{x_0} \int_b^l \int_b^r \Psi'''(e)dedrdl \\ &\geq 1 + \Psi''(b) \cdot \frac{(x_0 - b)^2}{2} + \int_b^{x_0} \int_b^l \int_b^r M_2(e)\Psi(e)dedrdl \\ &> 1 + \Psi''(b) \cdot \frac{(x_0 - b)^2}{2} \geq 1 - \frac{(x_0 - b)^2}{(c-b)^2}\end{aligned}$$

$$\geq \quad 0,$$

which is a contradiction to $\Psi(x_0) = 0$. Hence, $\Psi(x) > 0$ for $x \in [b, c]$, and so for $x \in (b, c]$,

$$\Psi(x) > 1 + \Psi''(b) \cdot \int_b^x \int_b^l drdl.$$

Notice

$$\Psi(x) = 1 + \Psi''(b) \cdot \int_b^x \int_b^l drdl + \int_b^x \int_b^l \int_b^r \Psi'''(e)dedrdl$$

and $\sum_{j=1}^{t} b_j \Psi(\eta_j) = \Psi(c)$, so we have that

$$\begin{aligned}
& -\Psi''(b) \cdot \sum_{j=1}^{t} b_j \int_{\eta_j}^{c} \int_b^l drdl \\
= \quad & \sum_{j=1}^{t} b_j \int_{\eta_j}^{c} \int_b^l \int_b^r \Psi'''(e)dedrdl \\
\geq \quad & \sum_{j=1}^{t} b_j \int_{\eta_j}^{c} \int_b^l \int_b^r M_2(e)\Psi(e)dedrdl \\
> \quad & \sum_{j=1}^{t} b_j \int_{\eta_j}^{c} \int_b^l \int_b^r M_2(e) \left(1 + \Psi''(b) \cdot \int_b^e \int_b^u dvdu\right) dedrdl,
\end{aligned}$$

that is, by (3.13),

$$\begin{aligned}
-\Psi''(b) \quad > \quad & \frac{\sum_{j=1}^{t} b_j \int_{\eta_j}^{c} \int_b^l \int_b^r M_2(e)dedrdl}{\sum_{j=1}^{t} b_j \int_{\eta_j}^{c} \int_b^l \left(1 + \int_b^r M_2(e)\frac{(e-b)^2}{2}de\right) drdl} \\
\geq \quad & \frac{2}{(c-b)^2}.
\end{aligned}$$

This is a contradiction to the assumption $\Psi''(b) \geq -\frac{2}{(c-b)^2}$. Therefore, $\Psi''(b) < -\frac{2}{(c-b)^2}$. $\square$

The next lemma is about the existence and uniqueness of an intersection point of two continuous and strictly decreasing functions with ranges all of $\mathbb{R}$. The proof is based on some calculus analysis.

Lemma 3.9. *Assume $\mu(x), \omega(x) \in C(\mathbb{R})$ and both are strictly decreasing functions and range all of $\mathbb{R}$. Suppose $\exists\, \sigma_1 < \sigma_2 < 0$ such that*

$$\mu(x_2) - \mu(x_1) \leq \sigma_1(x_2 - x_1), \qquad \omega(x_2) - \omega(x_1) \geq \sigma_2(x_2 - x_1), \qquad \forall\, x_1 < x_2.$$

Then, there exist a unique $x_0 \in \mathbb{R}$ such that $\mu(x_0) = \omega(x_0)$.

Proof. First, we prove the existence. Suppose $\mu(x) \neq \omega(x)$ for any $x \in \mathbb{R}$. Then, either $\mu(x) < \omega(x)$ or $\mu(x) > \omega(x)$, $\forall\, x \in \mathbb{R}$.

Case 1: $\omega(x) < \mu(x)$, $\forall\, x \in \mathbb{R}$.

Let some $\bar{x} \in \mathbb{R}$ be fixed. We consider two lines $l_1(x) = \mu(\bar{x}) + \sigma_1(x - \bar{x})$ and $l_2(x) = \omega(\bar{x}) + \sigma_2(x - \bar{x})$ for $x \in \mathbb{R}$.

Since $l_1(\bar{x}) = \mu(\bar{x})$ and $\mu(x) - \mu(\bar{x}) \leq \sigma_1(x - \bar{x})$ for $x > \bar{x}$, we have that $\mu(x) \leq l_1(x)$ for $x > \bar{x}$. From $l_2(\bar{x}) = \omega(\bar{x})$ and $\omega(x) - \omega(\bar{x}) \geq \sigma_2(x - \bar{x})$ for $x > \bar{x}$, it follows that $\omega(x) \geq l_2(x)$ for $x > \bar{x}$.

Notice $l(\tilde{x}) = l_2(\tilde{x})$ for $\tilde{x} = \frac{\mu(\bar{x}) - \omega(\bar{x})}{\sigma_2 - \sigma_1} + \bar{x} > \bar{x}$. Hence, $\mu(\tilde{x}) \leq l_1(\tilde{x}) = l_2(\tilde{x}) \leq \omega(\tilde{x})$, which is a contradiction to $\omega(x) < \mu(x)$, $\forall\, x \in \mathbb{R}$.

Case 2: $\omega(x) > \mu(x)$, $\forall\, x \in \mathbb{R}$.

Let some $\bar{x} \in \mathbb{R}$ be fixed. We still consider the two lines $l_1(x) = \mu(\bar{x}) + \sigma_1(x - \bar{x})$ and $l_2(x) = \omega(\bar{x}) + \sigma_2(x - \bar{x})$ for $x \in \mathbb{R}$.

By $l_1(\bar{x}) = \mu(\bar{x})$ and $\mu(\bar{x}) - \mu(x) \leq \sigma_1(\bar{x} - x)$ for $x < \bar{x}$, we have that $\mu(x) \geq l_1(x)$ for $x < \bar{x}$. From $l_2(\bar{x}) = \omega(\bar{x})$ and $\omega(\bar{x}) - \omega(x) \geq \sigma_2(\bar{x} - x)$ for $x < \bar{x}$, it follows that $\omega(x) \leq l_2(x)$ for $x < \bar{x}$.

At $\tilde{x} = \frac{\omega(\bar{x}) - \mu(\bar{x})}{\sigma_1 - \sigma_2} + \bar{x} < \bar{x}$, $l(\tilde{x}) = l_2(\tilde{x})$. Then, $\omega(\tilde{x}) \leq l_2(\tilde{x}) = l_1(\tilde{x}) \leq \mu(\tilde{x})$, which is a contradiction to $\omega(x) > \mu(x)$, $\forall\, x \in \mathbb{R}$.

Next, we show the uniqueness. Suppose there two distinct numbers $x_1, x_2 \in \mathbb{R}$ such that $\mu(x_i) = \omega(x_i)$ for $i = 1, 2$. Without loss of generality, suppose $x_1 < x_2$. Then,

$$\sigma_2(x_2 - x_1) \leq \omega(x_2) - \omega(x_1) = \mu(x_2) - \mu(x_1) \leq \sigma(x_2 - x_1),$$

which is a contradiction to $\sigma_1 < \sigma_2$. Therefore, there is a unique x_0 such that $\mu(x_0) = \omega(x_0)$. □

Now, we are in the position to show our main result.

Theorem 3.1. *Suppose that f satisfies condition (A2) and that for each $m \in \mathbb{R}$, there exist solutions $\alpha_3(x,m)$, $u_3(x,m)$, $\beta_3(x,m)$, $v_3(x,m)$ of (2.1) satisfying each of the conditions (3.1), (3.2), (3.3), (3.4), respectively. Suppose f satisfies one of the following:*

(H1): there is some $M_1 > 0$ and a continuous function $M_2(x)$ for $x \in [b,c]$ with $M_2(x) > 0$ for $x \in (b,c]$, such that

$$0 > f(x,v) - f(x,u) \geq -M_1(v-u),\ \forall\, x \in (a,b),\ \forall\, v > u \in \mathbb{R},$$

$$f(x,v) - f(x,u) \geq M_2(x)(v-u),\ \forall\, x \in (b,c),\ \forall\, v > u \in \mathbb{R},$$

where

$$M_1(b-a) < \frac{2}{(c-b)^2},$$

and

$$\frac{\sum_{j=1}^{t} b_j \int_{\eta_j}^{c} \int_b^l \int_b^r M_2(e)\,de\,dr\,dl}{\sum_{j=1}^{t} b_j \int_{\eta_j}^{c} \int_b^l \left(1 + \int_b^r M_2(e)\frac{(e-b)^2}{2}de\right) dr\,dl} \geq \frac{2}{(c-b)^2};$$

or

(H2): there is some $M_2 > 0$ and a continuous function $M_1(x)$ for $x \in [a,b]$ with $M_1(x) > 0$ for $x \in [a,b)$, such that

$$f(x,v) - f(x,u) \leq -M_1(x)(v-u),\ \forall\, x \in (a,b),\ \forall\, v > u \in \mathbb{R},$$

$$0 < f(x,v) - f(x,u) \leq M_2(v-u),\ \forall\, x \in (b,c),\ \forall\, v > u \in \mathbb{R},$$

where

$$\frac{\sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \int_r^b M_1(e)\,de\,dr\,dl}{\sum_{i=1}^{s} a_i \int_a^{\xi_i} \int_l^b \left(1 + \int_r^b M_1(e)\frac{(b-e)^2}{2}de\right) dr\,dl} \geq \frac{2}{(b-a)^2},$$

and

$$\frac{2}{(b-a)^2} > M_2(c-b).$$

Then the BVP (2.1), (2.3) has a unique solution.

Proof. We show the proof for the case that f satisfies (H1). The proof for the other case is very similar and omitted here.

First, we prove the existence of solutions of the BVP (2.1), (2.3). Since for any $m \in \mathbb{R}$, there exist solutions $\alpha_3(x,m)$, $u_3(x,m)$, $\beta_3(x,m)$, $v_3(x,m)$ of (2.1) satisfying each of the conditions (3.1), we consider $\alpha_3''(b,m)$, $u_3(b,m)$, $\beta_3''(b,m)$, $v_3(b,m)$ as functions of m. By Lemma 3.4, they are all strictly decreasing continuous functions.

For any $m_1 < m_2 \in \mathbb{R}$, from Lemma 3.5, we have $\alpha_3''(b,m_2) - \alpha_3''(b,m_1) > -M_1(b-a)(m_2-m_1)$; and from Lemma 3.8, we have $\beta_3''(b,m_2) - \beta_3''(b,m_1) < -\frac{2(m_2-m_1)}{(c-b)^2}$. Notice, $-M_1(b-a) > -\frac{2}{(c-b)^2}$. By Lemma 3.9, there is a unique $m_0 \in \mathbb{R}$ such that $\alpha_3''(b,m_0) = \beta_3''(b,m_0)$. Then the piecewise defined function

$$y(x) = \begin{cases} \alpha_3(x,m_0), & x \in [a,b], \\ \beta_3(x,m_0), & x \in [b,c], \end{cases}$$

is a solution of (2.1), (2.3).

Second, we prove the uniqueness. Suppose there are two solutions $y_1(x)$ and $y_2(x)$ of (2.1), (2.3). Then, we have some $m_1 = y_1(b)$ and $m_2 = y_2(b)$ such that $\alpha_3(x,m_1) = y_1(x)$ for $x \in [a,b]$, $\beta_3(x,m_1) = y_1(x)$ for $x \in [b,c]$, $\alpha_3(x,m_2) = y_2(x)$ for $x \in [a,b]$, and $\beta_3(x,m_2) = y_2(x)$ for $x \in [b,c]$. By Lemma 3.3, $m_1 \neq m_2$. Without loss of generality, we suppose $m_2 > m_1$. Then by (H1) and Lemmas 3.5 and 3.8, we have that $\beta_3''(b,m_2) - \beta_3''(b,m_1) < -\frac{2(m_2-m_1)}{(c-b)^2}$, and $\alpha_3''(b,m_2) - \alpha_3''(b,m_1) > -M_1(b-a)(m_2-m_1)$, that is,

$$-M_1(b-a)(m_2-m_1) < \alpha_3''(b,m_2) - \alpha_3''(b,m_1) = \beta_3''(b,m_2) - \beta_3''(b,m_1) < -\frac{2(m_2-m_1)}{(c-b)^2},$$

which is a contradiction to $-M_1(b-a) > -\frac{2}{(c-b)^2}$.

Therefore, $y(x)$ above is the unique solution of (2.1), (2.3). $\square$

CHAPTER FOUR

Nonlocal Boundary Value Problems of nth Order Differential Equations with $k_2 - k_1$ Being Odd

4.1 Introduction

In this chapter, we extend our results in Chapter Two for third order problems to more general conclusions for nth order BVP's based on the solution-matching technique. Basically, we are concerned with the existence and uniqueness of solutions of BVP's on an interval $[a, c]$ for the nth order ordinary differential equation,

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x)), \quad n \geq 3, \quad x \in [a, c], \tag{4.1}$$

satisfying the boundary conditions,

$$\begin{aligned} & y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+2}, \ 0 \leq i \leq k_1 - 1, \\ & y^{(i)}(b) = y_{i+1}, \ k_1 + 1 \leq i \leq k_2 - 1, \\ & y^{(i)}(b) = y_i, \ k_2 + 1 \leq i \leq n - 1, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_n, \end{aligned} \tag{4.2}$$

where $a < \xi_1 < \xi_2 < \cdots < \xi_s < b < \eta_1 < \eta_2 < \cdots < \eta_t < c$, and $y_1, y_2, \ldots, y_n \in \mathbb{R}$, $k_1, \ k_2 \in \mathbb{Z}$ such that $0 \leq k_1 < k_2 \leq n - 1$ and $k_2 - k_1$ is odd.

It is assumed throughout this chapter that $f : [a, c] \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that solutions of IVP's for (4.1) are unique and exist on the entire interval $[a, c]$. Moreover, k_1 and k_2 are fixed.

Given the following set of boundary conditions,

$$\begin{aligned} & y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+2}, \ 0 \leq i \leq k_1 - 1, \ y^{(k_1)}(b) = m, \\ & y^{(i)}(b) = y_{i+1}, \ k_1 + 1 \leq i \leq k_2 - 1, \quad y^{(i)}(b) = y_i, \ k_2 + 1 \leq i \leq n - 1, \end{aligned} \tag{4.3}$$

$$\begin{aligned} & y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+2}, \ 0 \leq i \leq k_1 - 1, \ y^{(k_2)}(b) = m, \\ & y^{(i)}(b) = y_{i+1}, \ k_1 + 1 \leq i \leq k_2 - 1, \quad y^{(i)}(b) = y_i, \ k_2 + 1 \leq i \leq n - 1, \end{aligned} \tag{4.4}$$

$$y^{(i)}(b) = y_{i+2},\ 0 \le i \le k_1 - 1,\ \ y^{(i)}(b) = y_{i+1},\ k_1 + 1 \le i \le k_2 - 1,$$
$$y^{(k_1)}(b) = m,\ \ y^{(i)}(b) = y_i,\ k_2 + 1 \le i \le n - 1,\ \ \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_n, \tag{4.5}$$

$$y^{(i)}(b) = y_{i+2},\ 0 \le i \le k_1 - 1,\ \ y^{(i)}(b) = y_{i+1},\ k_1 + 1 \le i \le k_2 - 1,$$
$$y^{(k_2)}(b) = m,\ \ y^{(i)}(b) = y_i,\ k_2 + 1 \le i \le n - 1,\ \ \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_n, \tag{4.6}$$

where $m \in \mathbb{R}$, we will match solutions of the BVP's (4.1), (4.3) on $[a, b]$ with solutions of (4.1), (4.5) on $[b, c]$, or solutions of (4.1), (4.4) on $[a, b]$ with solutions of (4.1), (4.6) on $[b, c]$, to obtain a desired unique solution of (4.1), (4.2). The condition that $k_2 - k_1$ is odd is key here.

Concerning three-point BVP's for nth order differential equations (4.1), the special cases of $k_2 = n - 1$ and $k_1 = n - 2$ were discussed in [20, 9]. Here, we explore more general cases, that is, we consider nonlocal multi-point BVP's and $k_2 - k_1$ is only required to be odd.

Monotonicity conditions on f will guarantee that the postulation of the value of the k_1st or k_2nd order derivative of a solution of (4.1) at b presupposes a knowledge of the values of all derivatives at b. The parity of the order n of the differential equation also plays a role since the odd or even property of $n - k_1$ will evoke different monotonicity conditions on f. In Section 4.2 and Section 4.3, we will separately consider the case of $n - k_1$ being even and the case of $n - k_1$ being odd, to give some basic lemmas on the relation between the change in values of the k_1st order derivative and the change in values of the k_2nd order derivative of two solutions of (4.1) at b that satisfy the boundary conditions (4.2), respectively, on the interval $[a, b]$ and the interval $[b, c]$. Different monotonicity conditions will be imposed on f with respect to $[a, b]$ and $[b, c]$ for distinct cases that arise. In Section 4.4, based on our results in Sections 4.2 and 4.3, the existence and uniqueness of solutions of (4.1), (4.2) are obtained.

4.2 Preliminaries for the Case That $n - k_1$ Is Even

In this section, we impose some monotonicity conditions of f, which depend on whether $k_2 = n-1$ or where k_2 lies between 1 and $n-1$. We choose from the following list of conditions.

(**B1**): *If $k_2 = n-1$, then for any $v_{n-1} = u_{n-1}$,*

$$f(x, v_0, v_1, \ldots, v_{n-1}) - f(x, u_0, u_1, \ldots, u_{n-1}) > 0$$

when $x \in (a, b]$ for $(-1)^{n-i} v_i \geq (-1)^{n-i} u_i,\ 0 \leq i \leq k_2 - 1,\ i \neq k_1,\ v_{k_1} > u_{k_1}$; when $x \in [b, c)$ for $v_i \geq u_i,\ 0 \leq i \leq k_2 - 1,\ i \neq k_1,\ v_{k_1} > u_{k_1}$;

(**B2**): *The function f is of the form $f(x, u_0, u_1, u_2, \ldots, u_{k_2})$, and*

$$f(x, v_0, v_1, \ldots, v_{k_2}) - f(x, u_0, u_1, \ldots, u_{k_2}) > 0$$

when $x \in (a, b]$ for $(-1)^{n-i} v_i \geq (-1)^{n-i} u_i,\ 0 \leq i \leq k_2,\ i \neq k_1,\ v_{k_1} > u_{k_1}$; when $x \in [b, c)$ for $v_i \geq u_i,\ 0 \leq i \leq k_2,\ i \neq k_1,\ v_{k_1} > u_{k_1}$.

When $k_2 = n - 1$, our next two lemmas are true under each of Conditions (B1) and (B2). Condition (B1) is the same as the Condition in [9]. With (B1), $f(x, u_0, u_1, \ldots, u_{n-1})$ is strictly monotone in u_{k_1} and f does not have to be monotone with respect to the last variable. Condition (B2) is stronger than (B1) since f is required to be monotone in all u_i, for $0 \leq i \leq k_2 = n - 1$ including u_{n-1}, but Condition (B2) contains cases when $k_2 < n - 1$.

In the following two lemmas, we show the relations between the change in values of the k_1st order derivative and the change in values of the k_2nd order derivative of two solutions of (4.1) at b that satisfy the boundary conditions (4.2), respectively, on the interval $[a, b]$ and on the interval $[b, c]$. These two lemmas are important for producing our main results in Section 4.3. All conclusions in the two lemmas are proved by contradiction.

Lemma 4.1. *Assume f satisfies one of conditions (B1) and (B2) if $k_2 = n - 1$ and f satisfies conditions (B2) if $k_2 < n - 1$. Suppose p and q are solutions of (4.1) on*

$[a, b]$ so that $w = p - q$ satisfies the following boundary conditions:

$$w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0, \quad w^{(i)}(b) = 0,\ 0 \le i \le n-1,\ i \ne k_1,\ k_2.$$

Then, $w^{(k_1)}(b) = 0$ if and only if $w^{(k_2)}(b) = 0$. Also, $w^{(k_1)}(b) > 0$ if and only if $w^{(k_2)}(b) > 0$.

Proof. The proof for the case that f satisfies (B1) is omitted here, see [9] for reference.

($\Rightarrow$) The necessity of the equalities.

Suppose $w^{(k_1)}(b) = 0$ and $w^{(k_2)}(b) \ne 0$. Without loss of generality, we assume $w^{(k_2)}(b) > 0$.

Since $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = \sum_{i=1}^{s} a_i (w(a) - w(\xi_i)) = 0$ and $a_i > 0$ for $i = 1, 2, \ldots, s$, $w^{(i)}(b) = 0$, $0 \le i \le k_2 - 1$ (note $1 \le k_2 \le n-1$), and $w^{(k_2)}(b) > 0$, by repeated applications of Rolle's Theorem, there exists $x_1 \in (a, b)$ such that $w^{(k_2)}(x_1) = 0$, $w^{(k_2)}(x) > 0$, for $x \in (x_1, b]$, and $(-1)^{k_2 - i} w^{(i)}(x) > 0$, i.e., $(-1)^{n-i} w^{(i)}(x) < 0$, for $0 \le i \le k_2 - 1$ and $x \in [x_1, b)$. In particular, $w^{(k_1)}(x) < 0$, for $x \in [x_1, b)$.

If f satisfies Condition (B2) on $[x_1, b)$, then we have that

$$w^{(n)}(x) \ = \ f(x, p, p', \ldots, p^{(n-1)}) - f(x, q, q', \ldots, q^{(n-1)}) < 0, \quad \text{for} \ \ x \in [x_1, b).$$

By $w^{(k_2)}(x_1) = 0$, $w^{(k_2)}(b) > 0$, $w^{(i)}(b) = 0$ for $k_2 + 1 \le i \le n-1$, the face that $n - k_2$ is odd and repeated applications of Rolle's Theorem, there is some $x_2 \in (x_1, b)$ such that $w^{(n)}(x_2) > 0$, which is a contradiction to $w^{(n)} < 0$ for $x \in [x_1, b)$.

Therefore, $w^{(k_2)}(b) = 0$, if $w^{(k_1)}(b) = 0$.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w^{(k_2)}(b) = 0$ and $w^{(k_1)}(b) \ne 0$. Without loss of generality, we assume $w^{(k_1)}(b) > 0$. By (B2), $w^{(n)}(b) > 0$. Hence by $w^{(i)}(b) = 0$, for $k_2 + 1 \le i \le n-1$ and odd $n - k_2$, in a left neighborhood of b, $w^{(k_2)}(x) < 0$.

Case 1: $k_1 = 0$.

By $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = \sum_{i=1}^{s} a_i(w(a) - w(\xi_i)) = 0$ and $a_i > 0$ for $i = 1, 2, \dots, s$, $w^{(i)}(b) = 0$, for $k_1 + 1 \le i \le k_2 - 1$, and repeated applications of Rolle's Theorem, there is some $x_1 \in (a, b)$ such that $w^{(k_2)}(x_1) = 0$ and $w^{(k_2)}(x) < 0$ for $x \in (x_1, b)$. Then by $w^{(i)}(b) = 0$, for $k_1 + 1 \le i \le k_2 - 1$, we have $(-1)^{(k_2 - i)} w^{(i)}(x) < 0$ for $x \in [x_1, b)$ and $k_1 + 1 \le i \le k_2 - 1$. Hence, $w^{(k_1+1)}(x) < 0$ for $x \in [x_1, b)$ and $w^{(k_1)}(b) = w(b) > 0$ imply $w^{(k_1)}(x) = w(x) > 0$ for $x \in [x_1, b]$.

Case 2: $k_1 > 0$.

By $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = \sum_{i=1}^{s} a_i(w(a) - w(\xi_i)) = 0$ and $a_i > 0$ for $i = 1, 2, \dots, s$, $w^{(i)}(b) = 0$, for $k_1 + 1 \le i \le k_2 - 1$, $k_1 \ge 0$, and repeated applications of Rolle's Theorem, there is some $x_0 \in (a, b)$ such that $w^{(k_1)}(x_0) = 0$, $w^{(k_1)}(x) > 0$ for $x \in (x_0, b]$. By $w^{(i)}(b) = 0$ for $0 \le i \le k_1 - 1$, we have $(-1)^{k_1 - i} w^{(i)}(x) > 0$ for $x \in [x_0, b)$ and $0 \le i \le k_1 - 1$.

From $w^{(k_1)}(x_0) = 0$, $w^{(k_1)}(x) > 0$ for $x \in (x_0, b]$, $w^{(i)}(b) = 0$ for $0 \le i \le k_1 - 1$, the odd $k_2 - k_1$, and repeated applications of Rolle's Theorem, there is some $x_0' \in (x_0, b)$ such that $w^{(k_2)}(x_0') > 0$. Since in a left neighborhood of b, $w^{(k_2)}(x) < 0$. Hence, there is some $x_1 \in [x_0', b)$ such that $w^{(k_2)}(x_1) = 0$ and $w^{(k_2)}(x) < 0$ for $x \in (x_1, b)$. By $w^{(i)}(b) = 0$ for $k_1 + 1 \le i \le k_2 - 1$ and $x \in [x_1, b)$, we have $(-1)^{n-i} w^{(i)}(x) > 0$ for $x \in [x_1, b)$ and $0 \le i \le k_2 - 1$.

In either case, we can find some $x_1 \in (a, b)$ such that $w^{(k_2)}(x_1) = 0$ and $w^{(k_2)}(x) < 0$ for $x \in (x_1, b)$, and $(-1)^{n-i} w^{(i)}(x) > 0$ for $x \in [x_1, b)$ and $0 \le i \le k_2 - 1$.

Suppose f satisfies (B2). Then, $w^{(n)}(x) > 0$ for $x \in [x_1, b]$. By $w^{(k_2)}(x_1) = 0$, $w^{(k_2)}(b) = 0$, $w^{(i)}(b) = 0$, for $k_2 + 1 \le i \le n - 1$, the odd $n - k_2$, and repeated applications of Rolle's Theorem, there exist some $x_2 \in (x_1, b)$ such that $w^{(n)}(x_2) < 0$. This is a contradiction.

In summary, $w^{(k_1)}(b) = 0$, when $w^{(k_2)}(b) = 0$.

($\Rightarrow$) The necessity of inequalities.

Suppose $w^{(k_1)}(b) > 0$ and $w^{(k_2)}(b) < 0$. The proof is pretty similar to that of the sufficiency of equalities, since we have both $w^{(k_1)}(b) > 0$ and $w^{(k_2)}(x) < 0$ in a left neighborhood of b. Hence, we can also get a contradiction and so $w^{(k_2)}(b) > 0$, if $w^{(k_1)}(b) > 0$.

($\Leftarrow$) The sufficiency of inequalities.

We assume that $w^{(k_2)}(b) > 0$ and $w^{(k_1)}(b) < 0$. Then, we are in the same situation as in the proof of necessity of inequalities by replacing w with $-w$, which also yields a contradiction. Hence, the sufficiency is true. □

Lemma 4.2. *Assume f satisfies one of conditions (B1) and (B2) if $k_2 = n-1$ and f satisfies conditions (B2) if $k_2 < n-1$. Suppose p and q are solutions of (4.1) on $[b,c]$ so that $w = p - q$ satisfies the following boundary conditions:*

$$w^{(i)}(b) = 0,\ 0 \le i \le n-1,\ i \ne k_1, k_2, \quad \sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0.$$

Then, $w^{(k_1)}(b) = 0$ if and only if $w^{(k_2)}(b) = 0$. Also, $w^{(k_1)}(b) > 0$ if and only if $w^{(k_2)}(b) < 0$.

Proof. The proof for the case that f satisfies (B1) is referred to [9] for reference.

($\Rightarrow$) The necessity of equalities.

Assume $w^{(k_1)}(b) = 0$ and $w^{(k_2)}(b) \ne 0$. Without loss of generality, we suppose $w^{(k_2)}(b) > 0$. By $w^{(i)}(b) = 0$, for $0 \le i \le k_2 - 1$ (note $1 \le k_2 \le n-1$), and $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = \sum_{j=1}^{t} b_j (w(\eta_j) - w(c)) = 0$ and $b_j > 0$ for $1 \le j \le t$, and repeated applications of Rolle's Theorem, we have an $x_1 \in (b,c)$ such that $w^{(k_2)}(x) > 0$ on $[b, x_1)$, and $w^{(k_2)}(x_1) = 0$. It follows from $w^{(i)}(b) = 0$, for $0 \le i \le k_2 - 1$, that $w^{(i)}(x) > 0$ on $(b, x_1]$, for $0 \le i \le k_2 - 1$. In particular, $w^{(k_1)}(x) > 0$ on $(b, x_1]$.

Suppose f satisfies (B2). Then, $w^{(n)}(x) > 0$ for $x \in [x_1, b)$. By $w^{(k_2)}(b) > 0$, $w^{(k_2)}(x_1) = 0$, $w^{(i)}(b) = 0$, for $k_2 + 1 \le i \le n-1$, and repeated applications of

Rolle's Theorem, we have some $x_2 \in (b, x_1)$ such that $w^{(n)}(x_2) < 0$. A contradiction. Therefore, if $w^{(k_1)}(b) = 0$, then $w^{(k_2)}(b) = 0$.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w^{(k_1)}(b) \neq 0$ and $w^{(k_2)}(b) = 0$. Without loss of generality, $w^{(k_1)}(b) > 0$. By (B2), we have $w^{(n)}(b) > 0$. So, in a right neighborhood of b, $w^{(k_2)}(x) > 0$.

Case 1: $k_1 = 0$.

By $w^{(i)}(b) = 0$, $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0$, for $k_1 + 1 \leq i \leq k_2 - 1$, and repeated applications of Rolle's Theorem, there is some $x_1 \in (b, c)$ such that $w^{(k_2)}(x_1) = 0$ and $w^{(k_2)}(x) > 0$ for $x \in (b, x_1)$. Then by $w^{(i)}(b) = 0$, for $k_1 + 1 \leq i \leq k_2 - 1$, we have $w^{(i)}(x) > 0$ for $x \in (b, x_1]$ and $k_1 + 1 \leq i \leq k_2 - 1$. Hence, $w^{(k_1+1)}(x) > 0$ for $x \in (b, x_1]$ and $w^{(k_1)}(b) = w(b) > 0$ imply $w^{(k_1)}(x) = w(x) > 0$ for $x \in [b, x_1]$.

Case 2: $k_1 > 0$.

By $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0$, $w^{(i)}(b) = 0$, for $0 \leq i \leq k_1 - 1$, $k_1 \geq 0$, and repeated applications of Rolle's Theorem, there is some $x_0 \in (b, c)$ such that $w^{(k_1)}(x_0) = 0$, $w^{(k_1)}(x) > 0$ for $x \in [b, x_0)$. By $w^{(i)}(b) = 0$ for $0 \leq i \leq k_1 - 1$, we have $w^{(i)}(x) > 0$ for $x \in (b, x_0]$ and $0 \leq i \leq k_1 - 1$.

From $w^{(k_1)}(x_0) = 0$, $w^{(k_1)}(x) > 0$ for $x \in (x_0, b]$, $w^{(i)}(b) = 0$ for $k_1 + 1 \leq i \leq k_2 - 1$, and repeated applications of the Mean Value Theorem, there is some $x_0' \in (x_0, b)$ such that $w^{(k_2)}(x_0') < 0$. Since in a right neighborhood of b, $w^{(k_2)}(x) > 0$. Hence, there is some $x_1 \in (b, x_0')$ such that $w^{(k_2)}(x_1) = 0$ and $w^{(k_2)}(x) > 0$ for $x \in (b, x_1)$. By $w^{(i)}(b) = 0$ for $k_1 + 1 \leq i \leq k_2 - 1$, we have $w^{(i)}(x) > 0$ for $x \in (b, x_1)$ and $k_1 + 1 \leq i \leq k_2 - 1$.

In either case, we can find some $x_1 \in (b, c)$ such that $w^{(k_2)}(x_1) = 0$ and $w^{(k_2)}(x) > 0$ for $x \in (b, x_1)$, and $w^{(i)}(x) > 0$ for $x \in (b, x_1]$ and $0 \leq i \leq k_2 - 1$.

Suppose f satisfies (B2). Then, $w^{(n)}(x) > 0$ for $x \in [b, x_1]$. By $w^{(k_2)}(x_1) = 0$, $w^{(k_2)}(b) = 0$, $w^{(i)}(b) = 0$, for $k_2 + 1 \leq i \leq n - 1$, the odd $n - k_2$, and repeated applications of the Mean Value Theorem, we have that there exist some $x_2 \in (b, x_1)$ such that $w^{(n)}(x_2) < 0$. This is a contradiction.

In summary, $w^{(k_1)}(b) = 0$, when $w^{(k_2)}(b) = 0$.

($\Rightarrow$) The necessity of inequalities.

Suppose $w^{(k_1)}(b) > 0$ and $w^{(k_2)}(b) > 0$. Then by a similar proof to that of the necessity of equalities of this lemma since $w^{(k_1)}(x) > 0$ and $w^{(k_2)}(x) > 0$ in a right neighborhood of b, we can arrive at a contradiction, too. Therefore, if $w^{(k_1)}(b) > 0$, then $w^{(k_2)}(b) < 0$.

($\Leftarrow$) The sufficiency of inequalities.

Suppose $w^{(k_2)}(b) < 0$. To prove $w^{(k_1)}(b) > 0$, we suppose $w^{(k_1)}(b) < 0$. By replacing w with $-w$ and using the results from the necessity of inequalities of this lemma, we also get a contradiction. Hence, the sufficiency is true. $\square$

4.3 Preliminaries for the Case That $n - k_1$ Is Odd

In this section, we use the monotonicity condition (B3) on f for our next two lemmas which are similar to Lemmas 2.1 and 2.2 but under the situation that $n - k_1$ is odd.

(**B3**): *The function f is of the form $f(x, u_0, u_1, u_2, \ldots, u_{k_1})$, and*

$$f(x, v_0, v_1, \ldots, v_{k_1}) - f(x, u_0, u_1, \ldots, u_{k_1}) \geq 0$$

when $x \in (a, b]$ for $(-1)^{n-i} v_i \geq (-1)^{n-i} u_i$, $0 \leq i \leq k_1$; when $x \in [b, c)$ for $v_i \geq u_i$, $0 \leq i \leq k_1$. Also, if $k_1 = 0$, $f(x, v) - f(x, u) > 0$ when $x \in (a, b)$ for $v < u$; when $x \in (b, c)$ for $v > u$.

Lemma 4.3. *Assume f satisfies condition (B3). Suppose p and q are solutions of*

(4.1) on $[a,b]$ so that $w = p - q$ satisfies the following boundary conditions:

$$w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = w^{(i)}(b) = 0,\ 0 \le i \le n-1,\ i \ne k_1, k_2.$$

Then, $w^{(k_1)}(b) = 0$ if and only if $w^{(k_2)}(b) = 0$. Also, $w^{(k_1)}(b) > 0$ if and only if $w^{(k_2)}(b) > 0$.

Proof. ($\Rightarrow$) The necessity of inequalities.

Suppose $w^{(k_1)}(b) = 0$ and $w^{(k_2)}(b) \ne 0$. Without loss of generality, we assume $w^{(k_2)}(b) > 0$.

Since $w(a) - \sum\limits_{i=1}^{s} a_i w(\xi_i) = 0$, $w^{(i)}(b) = 0$, $0 \le i \le k_2 - 1$ (note $1 \le k_2 \le n-1$), and $w^{(k_2)}(b) > 0$, by repeated applications of Rolle's Theorem, there exists $x_1 \in (a,b)$ such that $w^{(k_2)}(x_1) = 0$, $w^{(k_2)}(x) > 0$, for $x \in (x_1, b]$, and $(-1)^{n-i} w^{(i)}(x) = (-1)^{k_2 - i} w^{(i)}(x) > 0$, for $0 \le i \le k_2 - 1$, on $[x_1, b)$. By (B3) regardless of $k_1 = 0$ or $k_1 \ge 1$, we have $w^{(n)}(x) \ge 0$ for $x \in [x_1, b)$. However, from $w^{(k_2)}(x_1) = 0$, $w^{(k_2)}(b) > 0$, $w^{(i)}(b) = 0$, $k_2 + 1 \le i \le n-1$, the even $n - k_2$, and repeated applications of the Mean Value Theorem, there is some $x_2 \in (x_1, b)$ such that $w^{(n)}(x_2) < 0$, which is a contradiction.

So, if $w^{(k_1)}(b) = 0$, then $w^{(k_2)}(b) = 0$.

($\Leftarrow$) The sufficiency of equalities.

Suppose $w^{(k_2)}(b) = 0$ and $w^{(k_1)}(b) \ne 0$. Without loss of generality, we suppose $w^{(k_1)}(b) > 0$.

Case 1: $k_1 \ge 1$.

From $w(a) - \sum\limits_{i=1}^{s} a_i w(\xi_i) = 0$, $w^{(i)}(b) = 0$, $0 \le i \le k_1 - 1$, $w^{(k_1)}(b) > 0$, and repeated applications of Rolle's Theorem, there is some $x_1 \in (a,b)$ such that $w^{(k_1)}(x_1) = 0$, $w^{(k_1)}(x) > 0$ and $(-1)^{k_1 - i} w^{(i)}(x) < 0$, for $x \in [x_1, b)$ and $0 \le i \le k_1 - 1$. By (B3), we have that $w^{(n)}(x) \le 0$ for $x \in [x_1, b]$. However, $w^{(k_1)}(x_1) = 0$, $w^{(k_1)}(b) > 0$, $w^{(i)}(b) = 0$, $k_1 + 1 \le i \le n-1$, the odd $n - k_1$, and repeated applications of the Mean Value Theorem, there is some $x_2 \in (x_1, b)$ such that $w^{(n)}(x_2) > 0$, which is a contradiction.

Case 2: $k_1 = 0$.

When $k_1 = 0$, we have either there is some $x_1 \in (a, b)$ such that $w^{(k_1)}(x_1) = 0$, $w^{(k_1)}(x) > 0$ for $x \in [x_1, b)$, or $w^{(k_1)}(x) > 0$ for $x \in [a, b]$. The proof of the former situation is the same as Case 1. We omit it here. Now we prove for the second situation, i.e., $w^{(k_1)}(x) > 0$ for $x \in [a, b]$. By (B3), we have $w^{(n)}(x) < 0$ for $x \in [a, b)$. However, from $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0$, $w^{(i)}(b) = 0$, $1 \leq i \leq n-1$, the odd $n - k_1$, and repeated application of Rolle's Theorem, there is some $x_2 \in (a, b)$ such that $w^{(n)}(x_2) = 0$. This is a contradiction.

($\Rightarrow$) The necessity of inequalities.

Suppose $w^{(k_2)}(b) > 0$ and $w^{(k_1)}(b) < 0$.

Case 1: $k_1 \geq 1$.

From $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0$, $w^{(i)}(b) = 0$, $0 \leq i \leq k_1 - 1$, $w^{(k_1)}(b) > 0$, and repeated applications of Rolle's Theorem, there is some $x_1 \in (a, b)$ such that $w^{(k_1)}(x_1) = 0$, $w^{(k_1)}(x) > 0$ and $(-1)^{k_1 - i} w^{(i)}(x) < 0$, for $x \in [x_1, b)$ and $0 \leq i \leq k_1 - 1$. By (B3), we have that $w^{(n)}(x) \leq 0$ for $x \in [x_1, b]$. However, $w^{(k_1)}(x_1) = 0$, $w^{(k_1)}(b) > 0$, $w^{(i)}(b) = 0$, $k_1 + 1 \leq i \leq k_2 - 1$, the odd $k_2 - k_1$, and repeated applications of the Mean Value Theorem, there is some $x_2 \in (s_1, b)$ such that $w^{(k_2)}(x_2) > 0$. By $w^{(k_2)}(x_2) > 0$, $w^{(k_2)}(b) < 0$, $w^{(i)}(b) = 0$, $k_2 - 1 \leq i \leq n-1$, the even $n - k_2$, and repeated application of the Mean Value Theorem, there is some $x_3 \in (x_2, b)$ such that $w^{(n)}(x_3) > 0$, which is a contradiction.

Case 2: $k_1 = 0$.

When $k_1 = 0$, we have either there is some $x_1 \in (a, b)$ such that $w^{(k_1)}(x_1) = 0$, $w^{(k_1)}(x) > 0$ for $x \in [x_1, b)$, or $w^{(k_1)}(x) > 0$ for $x \in [a, b]$. Similarly, the proof of the former situation is the same as Case 1. We omit it here. Now we prove for the second situation, i.e., $w^{(k_1)}(x) > 0$ for $x \in [a, b]$. By (B3), we have $w^{(n)}(x) < 0$ for $x \in (a, b)$. However, from $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0$, $w^{(i)}(b) = 0$, $1 \le i \le k_2 - 1$, the even $n - k_2$, and repeated application of Rolle's Theorem, there is some $x_2 \in (a, b)$ such that $w^{(n)}(x_2) > 0$. This is a contradiction.

($\Leftarrow$) The sufficiency of inequalities.

Negate the sign of w. Then, from the proof for necessity of inequalities above in this lemma, we can conclude the sufficiency is also true. $\square$

Lemma 4.4. *Assume f satisfies Condition (B3). Suppose p and q are solutions of (4.1) on $[b, c]$ and $w = p - q$ satisfies the following boundary conditions:*

$$w^{(i)}(b) = 0,\ 0 \le i \le n - 1,\ i \ne k_1, k_2, \quad \sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0.$$

Then, $w^{(k_1)}(b) = 0$ if and only if $w^{(k_2)}(b) = 0$. Also, $w^{(k_1)}(b) > 0$ if and only if $w^{(k_2)}(b) < 0$.

Proof. The proof of each direction can be established by similar ideas to that of the corresponding direction of Lemma 4.3. We omit them here.

$\square$

4.4 Existence and Uniqueness of Solutions of (4.1), (4.2)

In this section, we mainly discuss the uniqueness and existence of solutions of (4.1), (4.2) by using Lemmas 4.1–4.4. We first consider the uniqueness of solutions of each of the BVP's for (4.1) satisfying any of (4.3), (4.4), (4.5), or (4.6), respectively.

(**H**): We assume f satisfies one of conditions (B1) and (B2) if $n - k_1$ is even and $k_2 = n - 1$; f satisfies condition (B2) if $n - k_1$ is even and $k_2 < n - 1$; f satisfies condition (B3) if $n - k_1$ is odd.

Lemma 4.5. *Let $y_1, y_2, \ldots, y_n \in \mathbb{R}$ be given and assume (H) is satisfied. Then, given $m \in \mathbb{R}$, each of the BVP's for (4.1) satisfying any of conditions (4.3), (4.4), (4.5), or (4.6) has at most one solution.*

Proof. By Lemma 4.1, Lemma 4.2 or Lemma 4.3, Lemma 4.4, we use similar ideas to the proof of Lemma 2.3 and get the uniqueness of solutions of (4.1) satisfying any of conditions (4.3), (4.4), (4.5), or (4.6). □

Lemma 4.6. *Let $y_1, y_2, \ldots, y_n \in \mathbb{R}$ be given and assume (H) is satisfied. Then, the BVP (4.1), (4.2) has at most one solution.*

Proof. Base on Lemma 4.1, Lemma 4.2 or Lemma 4.3, Lemma 4.4, the proof is similar to that of Lemma 2.4. □

For notation purposes, given any $m \in \mathbb{R}$, let $\alpha_4(x, m)$, $u_4(x, m)$, $\beta_4(x, m)$, $v_4(x, m)$ denote the solutions, when they exist, of the BVP's of (4.1) satisfying (4.3), (4.4), (4.5), or (4.6), respectively. Next, we show that $\alpha_4^{(k_2)}(b, m)$, $u_4^{(k_1)}(b, m)$, $\beta_4^{(k_2)}(b, m)$, $v_4^{(k_1)}(b, m)$, respectively, are strictly monotone functions of m.

Lemma 4.7. *Assume (H) is satisfied and that for each $m \in \mathbb{R}$, there exist solutions of (4.1) satisfying each of the conditions (4.3), (4.4), (4.5), (4.6), respectively. Then, $\alpha_4^{(k_2)}(b, m)$ and $u_4^{(k_1)}(b, m)$ are strictly decreasing functions of m with ranges all of $\mathbb{R}$; $\beta_4^{(k_2)}(b, m)$ and $v_4^{(k_1)}(b, m)$ are strictly increasing functions of m with ranges all of $\mathbb{R}$.*

Proof. With Lemma 4.1, Lemma 4.2 or Lemma 4.3, Lemma 4.4, the proofs are very similar to those of Lemma 2.5. □

Now, we are in the position to show our existence result for (4.1), (4.2) by matching solutions.

Theorem 4.1. *Assume (H) is satisfied and that for each $m \in \mathbb{R}$, there exist solutions of (4.1) satisfying each of the conditions (4.3), (4.4), (4.5), (4.6), respectively. Then, (4.1), (4.2) has a unique solution.*

Proof. We prove the existence from Lemma 4.7. We could match α_4 with β_4 or u_4 with v_4. Here we use α_4 and β_4. Since $\alpha_4^{(k_2)}(b,m)$ and $\beta_4^{(k_2)}(b,m)$ are, respectively, strictly decreasing and strictly increasing functions of m with ranges all of $\mathbb{R}$, there exists a unique $m_0 \in \mathbb{R}$ such that $\alpha_4^{(k_2)}(b,m_0) = \beta_4^{(k_2)}(b,m_0)$. Then,

$$y(x) = \begin{cases} \alpha_4(x,m_0), & a \le x \le b, \\ \beta_4(x,m_0), & b \le x \le c, \end{cases}$$

is a solution of (4.1), (4.2) and by Lemma 4.6, $y(x)$ is the unique solution. $\square$

CHAPTER FIVE

Nonlocal Boundary Value Problems of nth Order Differential Equations with $k_2 - k_1$ Being Even, $k_1 = 0$ And $k_2 = n - 1$

5.1 Introduction

In this chapter, we are going to study the situation that the difference of k_1 and k_2 is even by applying the solution-matching technique to nonlocal multi-point nth order BVP's for $y^{(n)}(x) = f(x, y^{(k_1)})$ with $k_1 = 0$ and $k_2 = n - 1$, that is,

$$y^{(n)}(x) = f(x, y(x)), \quad n \geq 3, \quad x \in [a, c], \tag{5.1}$$

satisfying the boundary conditions,

$$\begin{aligned} & y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+1}, \; 1 \leq i \leq n-2, \\ & \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_n, \end{aligned} \tag{5.2}$$

where $a < \xi_1 < \xi_2 < \cdots < \xi_s < b < \eta_1 < \eta_2 < \cdots < \eta_t < c$, $y_1, y_2, \ldots, y_n \in \mathbb{R}$ and $k_2 - k_1 = (n-1) - 0 = n - 1$ is even.

Our ideas here are based on the same as those in Chapter Three for third order problems and the results in this chapter are generalizations for nth order problems of those in Chpater 3. More details for the nth order problems need to be taken care of.

Throughout this chapter we assume the following assumptions: $f : [a, c] \times \mathbb{R} \to \mathbb{R}$ is continuous and that solutions of IVP's for (5.1) are unique and exist on the entire interval $[a, c]$.

Given the following set of boundary conditions,

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+1}, \; 1 \leq i \leq n-2, \quad y(b) = m, \tag{5.3}$$

$$y(a) - \sum_{i=1}^{s} a_i y(\xi_i) = y_1, \quad y^{(i)}(b) = y_{i+1},\ 1 \leq i \leq n-2, \quad y^{(n-1)}(b) = m, \tag{5.4}$$

$$y^{(i)}(b) = y_{i+1},\ 1 \leq i \leq n-2, \qquad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_n, \qquad y(b) = m, \tag{5.5}$$

$$y^{(i)}(b) = y_{i+1},\ 1 \leq i \leq n-2, \quad \sum_{j=1}^{t} b_j y(\eta_j) - y(c) = y_n, \quad y^{(n-1)}(b) = m, \tag{5.6}$$

where $m \in \mathbb{R}$, we will match solutions of the BVP's (5.1), (5.3) on $[a,b]$ with solutions of the BVP's (5.1) ,(5.5) on $[b,c]$, or solutions of (5.1), (5.4) on $[a,b]$ with solutions of (5.1), (5.6) on $[b,c]$, to obtain a desired unique solution of (5.1), (5.2).

The monotonicity conditions on f that lead to our results of this chapter are as follows:

(**C**): $f(x,v) - f(x,u) > 0$ when $x \in (a,b)$ and $v - u < 0$; when $x \in (b,c)$ and $v - u > 0$.

5.2 Preliminary Lemmas

In our following preliminary lemmas, we present some monotonicity relations between the change in the function values and the change in the values of the k_2nd, i.e. $n-1$st, order derivative of solutions at b of (5.1), (5.2).

Lemma 5.1. *Assume f satisfies the condition (C). Suppose p and q are solutions of (5.1) on $[a,b]$ so that $w = p - q$ satisfies the following boundary conditions:*

$$w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0, \quad w^{(i)}(b) = 0,\ 1 \leq i \leq n-2,$$

Then, $w(b) = 0$ if and only if $w^{(n-1)}(b) = 0$; $w(b) > 0$ if and only if $w^{(n-1)}(b) < 0$.

Proof. ($\Rightarrow$) The necessity of the equalities.

Suppose $w(b) = 0$ and $w^{(n-1)}(b) \neq 0$. Without loss of generality, we assume that $w^{(n-1)}(b) > 0$.

By $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = 0$, we know there is some $x_0 \in (a, \xi_s)$ such that $w'(x_0) = 0$. Then, from $w^{(i)}(b) = 0$ for $1 \leq i \leq n-2$ and Rolle's Theorem, it follows that there is some $x_1 \in (x_0, b)$ such that $w^{(k_2)}(x_1) = 0$ and $w^{(k_2)}(x) > 0$ for $x \in (x_1, b]$, which implies that $w(x) > 0$ for $x \in [x_1, b)$ since $k_2 - k_1$ is even and $w^{(i)}(b) = 0$ for $0 \le i \le k_2 - 1$. By the condition (C), $w^{(n)}(x) < 0$ for $x \in [x_1, b)$. However, since $w^{(k_2)}(x_1) = 0$, $w^{(k_2)}(x) > 0$ for $x \in (x_1, b]$ and $k_2 = n-1$, by the Mean Value Theorem, we have there is some $x_2 \in (x_1, b)$ such that $w^{(n)}(x_2) < 0$, which is a contradiction.

($\Leftarrow$) The sufficiency of the equalities.

Suppose $w(b) > 0$, $w^{(n-1)}(b) = 0$. Then, $w(x) > 0$ in a left neighborhood of b. By condition (C), we have $w^{(n)}(x) < 0$ in a deleted left neighborhood of b.

By $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = \sum_{i=1}^{s} a_i (w(a) - w(\xi_i)) = 0$, $w^{(i)}(b) = 0$ for $1 \le i \le n-1$, and repeated applications of the Rolle's Theorem, there is some $x_1 \in (a, b)$ such that $w^{(n)}(x_1) = 0$ and $w^{(n)}(x) < 0$ for $x \in (x_1, b)$, which implies that $w(x) > 0$ for $x \in [x_1, b]$ since n is odd and $w(b) > 0$. By condition (C), $w^{(n)}(x) < 0$ for $x \in [x_1, b)$. A contradiction to $w^{(n)}(x_1) = 0$.

($\Rightarrow$) The necessity of the inequalities.

Suppose $w(b) > 0$ and $w^{(n-1)}(b) > 0$. By condition (C), $w^{(n)}(x) < 0$ in a deleted left neighborhood of b.

Similarly as before, by $w(a) - \sum_{i=1}^{s} a_i w(\xi_i) = \sum_{i=1}^{s} a_i (w(a) - w(\xi_i)) = 0$, $w^{(i)}(b) = 0$ for $1 \le i \le n-2$ and Rolle's Theorem, we have some $x_1 \in (a, b)$ such that $w^{(n-1)}(x_1) = 0$. Since $w^{(k_2)}(b) > 0$, there is some $x_2 \in [x_1, b)$ such that $w^{(n-1)}(x_2) = 0$ and $w^{(n-1)}(x) > 0$ for $x \in (x_2, b]$. From $w(b) > 0$ and $w^{(i)}(b) = 0$ for $1 \le i \le n-2$ and the fact that n is odd, it follows that $w(x) > 0$ for $x \in [x_2, b]$, which implies by condition (C) that $w^{(n)}(x) < 0$ for $x \in [x_2, b)$. However, by $w^{(n-1)}(x_2) = 0$, $w^{(n-1)}(b) > 0$ and the Mean Value Theorem, we have that there is some $x_3 \in (x_2, b)$ such that $w^{(n)}(x_3) > 0$, which is a contradiction.

($\Leftarrow$) The sufficiency of the inequalities.

Suppose $w(b) < 0$ and $w^{(n-1)}(b) < 0$. By assigning the opposite sign to w, we can also get a contradiction by the necessity of the inequalities. Hence, $w(b) > 0$, if $w^{(n-1)}(b) < 0$. $\square$

Lemma 5.2. *Assume f satisfies the condition (C). Suppose p and q are solutions of (5.1) on $[b, c]$ and $w = p - q$ satisfies the following boundary conditions:*

$$w^{(i)}(b) = 0,\ 1 \le i \le n-1, \quad \sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0.$$

Then, $w(b) = 0$ if and only if $w^{(n-1)}(b) = 0$; $w(b) > 0$ if and only if $w^{(n-1)}(b) < 0$.

Proof. ($\Rightarrow$) The necessity of the equalities.

Suppose $w(b) = 0$ and $w^{(n-1)}(b) \neq 0$. Without loss of generality, we assume that $w^{(n-1)}(b) > 0$.

By $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0$, we know there is some $x_0 \in (\eta_1, c)$ such that $w'(x_0) = 0$. Then, from $w^{(i)}(b) = 0$ for $1 \le i \le n-2$ and Rolle's Theorem, it follows that there is an $x_1 \in (b, x_0)$ such that $w^{(n-1)}(x_1) = 0$ and $w^{(n-1)}(x) > 0$ for $x \in [b, x_1)$, which implies by $w^{(i)}(b) = 0$ for $0 \le i \le n-2$ that $w(x) > 0$ for $x \in (b, x_1]$. By the condition (C), $w^{(n)}(x) > 0$ for $x \in (b, x_1]$. However, since $w^{(n-1)}(x_1) = 0$, $w^{(n-1)}(b) > 0$ and the Mean Value Theorem, we have there is some $x_2 \in (b, x_1)$ such that $w^{(n)}(x_2) < 0$, which is a contradiction.

($\Leftarrow$) The sufficiency of the equalities.

Without loss of generality, we suppose $w(b) > 0$ and $w^{(n-1)}(b) = 0$. By condition (C), we have $w^{(n)}(x) > 0$ in a deleted right neighborhood of b.

By $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0$, $w^{(i)}(b) = 0$ for $1 \le i \le n-1$, and repeated applications of Rolle's Theorem, there is some $x_1 \in (b, c)$ such that $w^{(n)}(x_1) = 0$ and $w^{(n)}(x) > 0$ for $x \in (b, x_1)$, which implies that $w(x) > 0$ for $x \in [b, x_1]$ by $w(b) > 0$ and $w^{(i)}(b) = 0$ for $1 \le i \le n-1$. By condition (C), $w^{(n)}(x) > 0$ for $x \in (b, x_1]$, which is a contradiction to $w(x_1) > 0$.

($\Rightarrow$) The necessity of the inequalities.

Suppose $w(b) > 0$ and $w^{(n-1)}(b) > 0$. By condition (C), $w^{(n)}(x) > 0$ in a deleted right neighborhood of b.

Similarly as before, by $\sum_{j=1}^{t} b_j w(\eta_j) - w(c) = 0$, $w^{(i)}(b) = 0$ for $1 \le i \le n-2$ and Rolle's Theorem, we have some $x_0 \in (b, c)$ such that $w^{(n-1)}(x_0) = 0$. Since $w^{(n-1)}(b) > 0$, there is some $x_1 \in (b, x_0]$ such that $w^{(n-1)}(x_1) = 0$ and $w^{(n-1)}(x) > 0$ for $x \in [b, x_1)$. By Mean Value Theorem, there is some $x_2 \in (b, x_1)$ such that $w^{(n)}(x_2) < 0$ and $x_3 \in (b, x_2)$ such that $w^{(n)}(x_3) = 0$ and $w^{(n)}(x) > 0$ for $x \in (b, x_3)$.

From $w(b) > 0$ and $w^{n-1)}(b) > 0$, $w^{(i)}(b) = 0$ for $1 \le i \le n-2$, and $w^{(n)}(x) > 0$ for $x \in (b, x_3)$, it follows that $w(x) > 0$ for $x \in [b, x_3]$, which by condition (C) implies that $w^{(n)}(x) > 0$ for $x \in (b, x_3]$. This is a contradiction.

($\Leftarrow$) The sufficiency of the inequalities.

Suppose $w(b) < 0$ and $w^{(n-1)}(b) < 0$. By assigning the opposite sign to w, we can also get a contradiction by the necessity of the inequalities. Hence, $w(b) > 0$ if $w^{(n-1)}(b) < 0$. □

5.3 *Existence and Uniqueness of Solutions of (5.1), (5.2)*

Before we show our main results, we need to establish some more lemmas.

Lemma 5.3. *Let $y_1, y_2, \ldots, y_n \in \mathbb{R}$ be given and assume condition (C) is satisfied. Then, given $m \in \mathbb{R}$, each of the BVP's for (5.1) satisfying any of conditions (5.3), (5.4), (5.5), or (5.6) has at most one solution.*

Proof. By using Lemmas 5.1 and 5.2, the proofs are based on the same idea as that of Lemma 2.3. □

For notation purposes, given any $m \in \mathbb{R}$, let $\alpha_5(x, m)$, $u_5(x, m)$, $\beta_5(x, m)$, $v_5(x, m)$ denote the solutions, when they exist, of the boundary value problems of (5.1) satisfying (5.3), (5.4), (5.5) or (5.6), respectively. Next we show that they are monotone functions of m at b.

Lemma 5.4. *Suppose that condition (C) is satisfied and that for each $m \in \mathbb{R}$, there exist solutions of (5.1) satisfying each of the conditions (5.3), (5.4), (5.5) and (5.6), respectively. Then, all of $\alpha_5^{(n-1)}(b,m)$, $\beta_5^{(n-1)}(b,m)$, $u_5(b,m)$ and $v_5(b,m)$ are strictly decreasing functions of m with ranges all of $\mathbb{R}$.*

Proof. By using Lemmas 5.1 and 5.2, the proof is very similar to that of Lemma 2.5. $\square$

The following four lemmas provide some bounds to the rate of change of the k_2nd order derivative of solutions of (5.1) at b, $\alpha_5^{(n-1)}(b,m)$ and $\beta_5^{(n-1)}(b,m)$, with respect of $m \in \mathbb{R}$. These lemmas extend Lemmas 3.5, 3.6, 3.7 and 3.8 from the third order problems to the nth order problems. The proofs are based on the similar reasoning and our results in this chapter are more general.

Lemma 5.5. *Suppose f satisfies the condition (C) and there is some $M_1 > 0$, such that*

$$f(x,v) - f(x,u) \geq -M_1(v-u), \ \forall\, x \in (a,b), \ \forall\, v \geq u \in \mathbb{R}. \tag{5.7}$$

Assume for each $m \in \mathbb{R}$, there exist solutions $\alpha_5(x,m)$ of (5.1) satisfying (5.3). Let $m_1, m_2 \in \mathbb{R}$ with $m_1 < m_2$. Then,

$$\alpha_5^{(n-1)}(b,m_2) - \alpha_5^{(n-1)}(b,m_1) > -M_1(b-a)(m_2-m_1), \tag{5.8}$$

Proof. Let $m_1, m_2 \in \mathbb{R}$ with $m_1 < m_2$ be fixed. We denote $\Phi(x) = \frac{\alpha_5(x,m_2)-\alpha_5(x,m_1)}{m_2-m_1}$. Then, $\Phi(x)$ satisfies

$$\Phi^{(n)}(x) = \frac{f(x,\alpha_5(x,m_2)) - f(x,\alpha_5(x,m_1))}{m_2-m_1}, \qquad x \in [a,b],$$

$$\Phi(b) = 1, \quad \Phi^{(i)}(b) = 0, \ 1 \leq i \leq n-2, \quad \Phi(a) - \sum_{i=1}^{s} a_i \Phi(\xi_i) = 0,$$

and by Lemma 5.1, $\Phi^{(n-1)}(b) < 0$. It suffices to show that $\Phi^{(n-1)}(b) > -M_1(b-a)$.

Since $\Phi(a) - \sum\limits_{i=1}^{s} a_i \Phi(\xi_i) = \sum\limits_{i=1}^{s} a_i(\Phi(a) - \Phi(\xi_i)) = 0$ and $a_i > 0$ for $i = 1, 2, \ldots, s$, there is some $x_0 \in (a, \xi_s)$ such that $\Phi'(x_0) = 0$. By $\Phi^{(i)}(b) = 0$ for $1 \leq i \leq n-2$

and repeated applications of Rolle's Theorem, there is some $x_1 \in (x_0, b)$ such that $\Phi^{(n-1)}(x_1) = 0$ and $\Phi^{(n-1)}(x) < 0$ for $x \in (x_1, b]$.

By $\Phi'(x_0) = 0$, $\Phi^{(n-1)}(x) < 0$ for $x \in (x_1, b]$, $\Phi^{(i)}(b) = 0$ for $1 \le i \le n-2$ and n is odd, there is some $x_2 \in (x_0, x_1)$ such that $\Phi'(x_2) = 0$, $\Phi'(x) > 0$ for $x \in (x_2, b)$.

Next, we show $\Phi(x) > 0$ for $x \in (x_1, b]$. Suppose it is not true, then from $\Phi'(x_2) = 0$, $\Phi'(x) > 0$ for $x \in (x_2, b)$ and $x_1 \in (x_2, b)$, we have that $\Phi(x_1) < 0$, $\Phi(x_3) = 0$ for some $x_3 \in (x_1, b)$, $\Phi(x) < 0$ for $x \in (x_1, x_3)$ and $\Phi(x) > 0$ for $x \in (x_3, b)$. From condition (C), $\Phi^{(n)}(x) < 0$ for $x \in [x_1, x_3)$. However, $\Phi^{(n-1)}(x_1) = 0$ and $\Phi^{(n-1)}(x) < 0$ for $x \in (x_1, b]$ imply that $\Phi^{(n)}(x) > 0$ for $x \in (x_1, x_3)$. A contradiction.

Now from $\Phi(x) > 0$ for $x \in (x_1, b]$ and $\Phi'(x) > 0$ for $x \in [x_1, b)$, it is easy to see that $0 < \Phi(x) < 1$ for $x \in (x_1, b)$. Then, by (5.7), for $x \in (x_1, b)$

$$\begin{aligned}
\Phi^{(n)}(x) &= \frac{f(x, \alpha_5(x, m_2)) - f(x, \alpha_5(x, m_1))}{m_2 - m_1} \\
&\ge \frac{-M_1(\alpha_5(x, m_2) - \alpha_5(x, m_1))}{m_2 - m_1} \\
&= -M_1 \Phi(x) \\
&> -M_1.
\end{aligned}$$

Next we want to show that $\Phi^{(n-1)}(b) > -M_1(b-a)$. Suppose it is not true. Then, $\Phi^{(n-1)}(b) \le -M_1(b-a)$. By $\Phi^{(n-1)}(x_1) = 0$, there is some $x_4 \in (x_1, b)$ such that

$$\Phi^{(n)}(x_4) = \frac{\Phi^{(n-1)}(b) - \Phi^{(n-1)}(x_1)}{b - x_1} \le \frac{-M_1(b-a)}{b - x_1} = -M_1 \frac{(b-a)}{b - x_1} < -M_1,$$

which is a contradiction to $\Phi^{(n)}(x) > -M_1$ for $x \in [x_1, b)$. Therefore, $\Phi^{(n-1)}(b) > -M_1(b-a)$. □

Lemma 5.6. *Suppose f satisfies the condition (C) and there is a continuous function $M_1(x)$ on $[a, b]$, such that*

$$f(x, v) - f(x, u) \le -M_1(x)(v - u), \ \forall\, x \in (a, b), \ \forall\, v \ge u \in \mathbb{R}, \tag{5.9}$$

where $M_1(x) > 0$, for $x \in [a,b)$, and

$$\begin{aligned}
& \frac{\sum_{i=1}^{s} a_i \overbrace{\int_a^{\xi_i} \int_{r_1}^{b} \cdots \int_{r_{n-1}}^{b}}^{n} M_1(r_n) \overbrace{dr_n \cdots dr_1}^{n}}{\sum_{i=1}^{s} a_i \overbrace{\int_a^{\xi_i} \int_{r_1}^{b} \cdots \int_{r_{n-2}}^{b}}^{n-1} \left(1 + \int_{r_{n-1}}^{b} M_1(r_n) \frac{(b-r_n)^{n-1}}{(n-1)!} dr_n\right) \overbrace{dr_{n-1} \cdots dr_1}^{n-1}} \\
\geq \; & \frac{(n-1)!}{(b-a)^{n-1}} \qquad (5.10)
\end{aligned}$$

Assume for each $m \in \mathbb{R}$, there exist solutions $\alpha_5(x,m)$ of (5.1) satisfying (5.3). Let $m_1 < m_2 \in \mathbb{R}$. Then,

$$\alpha_5^{(n-1)}(b, m_2) - \alpha_5^{(n-1)}(b, m_1) < -\frac{(n-1)!(m_2 - m_1)}{(b-a)^{n-1}}. \qquad (5.11)$$

Proof. Let $m_1 < m_2 \in \mathbb{R}$ be fixed. We denote $\Phi(x) = \frac{\alpha_5(x,m_2) - \alpha_5(x,m_1)}{m_2 - m_1}$. Then $\Phi(x)$ satisfies

$$\begin{aligned}
& \Phi^{(n)}(x) = \frac{f(x, \alpha_5(x,m_2)) - f(x, \alpha_5(x,m_1))}{m_2 - m_1}, \qquad x \in [a,b], \\
& \Phi(b) = 1, \quad \Phi^{(i)}(b) = 0, \ 1 \leq i \leq n-2, \quad \Phi(a) - \sum_{i=1}^{s} a_i \Phi(\xi_i) = 0,
\end{aligned}$$

and by Lemma 5.1, $\Phi^{(n-1)}(b) < 0$. Then, it suffices to show that $\Phi^{(n-1)}(b) < -\frac{(n-1)!}{(b-a)^{n-1}}$. Suppose this is not true. Then, $\Phi^{(n-1)}(b) \geq -\frac{(n-1)!}{(b-a)^{n-1}}$.

By $\Phi(b) = 1$, $\Phi^{(i)}(b) = 0$ for $1 \leq i \leq n-2$ and $\Phi^{(n-1)}(b) \geq -\frac{(n-1)!}{(b-a)^{n-1}}$, we have that

$$\begin{aligned}
\Phi(x) &= \Phi(b) + \overbrace{\int_x^{b} \int_{r_1}^{b} \cdots \int_{r_{n-2}}^{b}}^{n-1} \Phi^{(n-1)}(r_{n-1}) \overbrace{dr_{n-1} \cdots dr_1}^{n-1} \\
&= 1 + \overbrace{\int_x^{b} \int_{r_1}^{b} \cdots \int_{r_{n-2}}^{b}}^{n-1} \left(\Phi^{(n-1)}(b) - \int_{r_{n-1}}^{b} \Phi^{(n)}(r_n) dr_n\right) \overbrace{dr_{n-1} \cdots dr_1}^{n-1} \\
&= 1 + \Phi^{(n-1)}(b) \cdot \frac{(b-x)^{n-1}}{(n-1)!} - \overbrace{\int_x^{b} \int_{r_1}^{b} \cdots \int_{r_{n-1}}^{b}}^{n} \Phi^{(n)}(r_n) \overbrace{dr_n \cdots dr_1}^{n}.
\end{aligned}$$

Next, we show $\Phi(x) > 0$ for $x \in [a,b]$. Assume this is not true. Let $x_0 \in [a,b)$ such that $\Phi(x_0) = 0$ and $\Phi(x) > 0$ for $x \in (x_0, b]$. Then, by (5.9),

$$\Phi^{(n)}(x) = \frac{f(x,\alpha_5(x,m_2)) - f(x,\alpha_5(x,m_1))}{m_2 - m_1} \le -M_1(x)\Phi(x), \quad \forall x \in (x_0, b].$$

Hence, by $\Phi^{(n-1)}(b) \ge -\frac{(n-1)!}{(b-a)^{n-1}}$,

$$\begin{aligned}
\Phi(x_0) &= 1 + \Phi^{(n-1)}(b) \cdot \frac{(b-x_0)^{n-1}}{(n-1)!} - \overbrace{\int_{x_0}^{b}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n} \Phi^{(n)}(r_n)\,\overbrace{dr_n\cdots dr_1}^{n} \\
&\ge 1 + \Phi^{(n-1)}(b) \cdot \frac{(b-x_0)^{n-1}}{(n-1)!} + \overbrace{\int_{x_0}^{b}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n} M_1(r_n)\Phi(r_n)\,\overbrace{dr_n\cdots dr_1}^{n} \\
&> 1 + \Phi^{(n-1)}(b) \cdot \frac{(b-x_0)^{n-1}}{(n-1)!} \\
&\ge 1 - \frac{(b-x_0)^{n-1}}{(b-a)^{n-1}} \ge 0,
\end{aligned}$$

which is a contradiction to $\Phi(x_0) = 0$.

From $\Phi(x) > 0$ for $x \in [a,b]$, we have that $\Phi^{(n)}(x) \le 0$ for $x \in [a,b]$ by condition (C). Hence, from (5.9), we have $\Phi^{(n)}(x) \le -M_1(x)\Phi(x)$ for $x \in [a,b]$. Therefore, for $x \in [a,b]$,

$$\begin{aligned}
\Phi(x) &= 1 + \Phi^{(n-1)}(b) \cdot \frac{(b-x)^{n-1}}{(n-1)!} - \overbrace{\int_{x}^{b}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n} \Phi^{(n)}(r_n)\,\overbrace{dr_n\cdots dr_1}^{n} \\
&\ge 1 + \Phi^{(n-1)}(b) \cdot \frac{(b-x)^{n-1}}{(n-1)!} + \overbrace{\int_{x}^{b}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n} M_1(r_n)\Phi(r_n)\,\overbrace{dr_n\cdots dr_1}^{n} \\
&> 1 + \Phi^{(n-1)}(b) \cdot \frac{(b-x)^{n-1}}{(n-1)!}.
\end{aligned}$$

From the boundary conditions $\Phi(a) - \sum\limits_{i=1}^{s} a_i\Phi(\xi_i) = 0$, we have that

$$\begin{aligned}
&\Phi^{(n-1)}(b) \cdot \frac{(b-a)^{n-1}}{(n-1)!} - \overbrace{\int_{a}^{b}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n} \Phi^{(n)}(r_n)\,\overbrace{dr_n\cdots dr_1}^{n} \\
= &\sum_{i=1}^{s} a_i \left(\Phi^{(n-1)}(b) \cdot \frac{(b-\xi_i)^{n-1}}{(n-1)!} - \overbrace{\int_{\xi_i}^{b}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n} \Phi^{(n)}(r_n)\,\overbrace{dr_n\cdots dr_1}^{n} \right),
\end{aligned}$$

that is,

$$
\begin{aligned}
&-\Phi^{(n-1)}(b)\sum_{i=1}^{s}a_i\left(\frac{(b-\xi_i)^{n-1}-(b-a)^{n-1}}{(n-1)!}\right)\\
&= \ -\sum_{i=1}^{s}a_i\overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n}\Phi^{(n)}(r_n)\overbrace{dr_n\cdots dr_1}^{n}.
\end{aligned}
$$

By $\frac{(b-\xi_i)^{n-1}-(b-a)^{n-1}}{(n-1)!} = -\overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-2}}^{b}}^{n-1}1\,\overbrace{dr_{n-1}\cdots dr_1}^{n-1}$, $\Phi^{(n)}(x) \le -M_1(x)\Phi(x)$ for $x\in[a,b]$, and $\Phi(x) > 1+\Phi^{(n-1)}(b)\cdot\frac{(b-x)^{n-1}}{(n-1)!}$, we have that

$$
\begin{aligned}
&-\Phi^{(n-1)}(b)\sum_{i=1}^{s}a_i\overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-2}}^{b}}^{n-1}1\,\overbrace{dr_{n-1}\cdots dr_1}^{n-1}\\
= \ & -\sum_{i=1}^{s}a_i\overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n}\Phi^{(n)}(r_n)\overbrace{dr_n\cdots dr_1}^{n}\\
\ge \ & \sum_{i=1}^{s}a_i\overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n}M_1(r_n)\Phi(r_n)\overbrace{dr_n\cdots dr_1}^{n}\\
> \ & \sum_{i=1}^{s}a_i\overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n}M_1(r_n)\left(1+\Phi^{(n-1)}(b)\cdot\frac{(b-r_n)^{n-1}}{(n-1)!}\right)\overbrace{dr_n\cdots dr_1}^{n}\\
= \ & \sum_{i=1}^{s}a_i\overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n}M_1(r_n)\overbrace{dr_n\cdots dr_1}^{n}\\
&+\Phi^{(n-1)}(b)\sum_{i=1}^{s}a_i\overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n}M_1(r_n)\frac{(b-r_n)^{n-1}}{(n-1)!}\overbrace{dr_n\cdots dr_1}^{n},
\end{aligned}
$$

which give that

$$
\begin{aligned}
&-\Phi^{(n-1)}(b)\sum_{i=1}^{s}a_i\overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-2}}^{b}}^{n-1}\left(1+\int_{r_{n-1}}^{b}M_1(r_n)\frac{(b-r_n)^{n-1}}{(n-1)!}dr_n\right)\overbrace{dr_{n-1}\cdots dr_1}^{n-1}\\
&>\sum_{i=1}^{s}a_i\overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n}M_1(r_n)\overbrace{dr_n\cdots dr_1}^{n},
\end{aligned}
$$

that is, by (5.10),

$$
\begin{aligned}
-\Phi^{(n-1)}(b) &> \frac{\sum_{i=1}^{s} a_i \overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-1}}^{b}}^{n} M_1(r_n)\,\overbrace{dr_n\cdots dr_1}^{n}}{\sum_{i=1}^{s} a_i \overbrace{\int_a^{\xi_i}\int_{r_1}^{b}\cdots\int_{r_{n-2}}^{b}}^{n-1}\left(1+\int_{r_{n-1}}^{b} M_1(r_n)\frac{(b-r_n)^{n-1}}{(n-1)!}dr_n\right)\overbrace{dr_{n-1}\cdots dr_1}^{n-1}} \\
&\geq \frac{(n-1)!}{(b-a)^{n-1}},
\end{aligned}
$$

which is a contradiction to the assumption $-\Phi^{(n-1)}(b) \leq \frac{(n-1)!}{(b-a)^{n-1}}$. Therefore, our assumption is not true. Hence $\Phi^{(n-1)}(b) < -\frac{(n-1)!}{(b-a)^{n-1}}$. □

Lemma 5.7. *Suppose f satisfies the condition (C) and there is some $M_2 > 0$, such that*

$$f(x,v) - f(x,u) \leq M_2(v-u),\ \forall\, x \in (b,c),\ \forall\, v \geq u \in \mathbb{R}, \tag{5.12}$$

Assume for each $m \in \mathbb{R}$, there exist solutions $\beta_5(x,m)$ of (2.1) satisfying (3.3). Let $m_1 < m_2 \in \mathbb{R}$. Then,

$$\beta_5^{(n-1)}(b,m_2) - \beta_5^{(n-1)}(b,m_1) > -M_2(c-b)(m_2-m_1). \tag{5.13}$$

Proof. Let $m_1 < m_2 \in \mathbb{R}$ be fixed. We denote $\Psi(x) = \frac{\beta_5(x,m_2)-\beta_5(x,m_1)}{m_2-m_1}$. Then $\Psi(x)$ satisfies

$$
\begin{aligned}
&\Psi^{(n)}(x) = \frac{f(x,\beta_5(x,m_2)) - f(x,\beta_5(x,m_1))}{m_2-m_1}, \qquad x \in [b,c], \\
&\Psi(b) = 1,\ \ \Psi^{(i)}(b) = 0,\ 1 \leq i \leq n-1,\ \ \sum_{j=1}^{t} b_j\Psi(\eta_j) - \Psi(c) = 0,
\end{aligned}
$$

and by Lemma 5.2, $\Psi^{(n-1)}(b) < 0$. We need to show $\Psi^{(n-1)}(b) > -M_2(c-b)$.

By $\sum_{j=1}^{t} b_j\Psi(\eta_j) - \Psi(c) = \sum_{j=1}^{t} b_j(\Psi(\eta_j) - \Psi(c)) = 0$ and $b_j > 0$ for $j = 1, 2, \ldots, t$, there is some $x_0 \in (\eta_1, c)$ such that $\Psi'(x_0) = 0$. By $\Psi^{(i)}(b) = 0$ for $1 \leq i \leq n-2$ and repeated applications of Rolle's Theorem, there is some $x_1 \in (b, x_0)$ such that $\Psi^{(n-1)}(x_1) = 0$ and $\Psi^{(n-1)}(x) < 0$ for $x \in [b, x_1)$. It follows that $\Psi'(x) < 0$ for

$x \in (b, x_1]$. Then there is some $x_2 \in (x_1, x_0]$ such that $\Psi'(x) < 0$ for $x \in (b, x_2)$ and $\Psi'(x_2) = 0$.

Similarly as in the proof of Lemma 5.5, we want to show $\Psi(x) > 0$ for $x \in [b, x_1)]$. Otherwise, by $\Psi'(x) < 0$ for $x \in (b, x_2)$ and $\Psi'(x_2) = 0$ we suppose there is some $x_3 \in (b, x_1)$ such that $\Psi(x_3) = 0$, $\Psi(x) > 0$ for $x \in [b, x_3)$ and $\Psi(x) < 0$ for $x \in (x_3, x_1)$. Then by condition (C), $\Psi^{(n)}(x) > 0$ for $x \in (b, x_3)$ and $\Psi^{(n)}(x) < 0$ for $x \in (x_3, x_1)$. However, since $\Psi^{(n-1)}(x_1) = 0$ and $\Psi^{(n-1)}(x) < 0$ for $x \in [b, x_1)$, by the Mean Value Theorem, there is some $x_4 \in (x_3, x_1)$ such that $\Psi^{(n)}(x_4) < 0$. A contradiction. Therefore, $\Psi(x) > 0$ for $x \in [b, x_1)$ and so $0 < \Psi(x) < 1$ for $x \in (b, x_1)$.

Then, by (5.12), for $x \in (b, x_1)$,

$$\Psi^{(n)}(x) = \frac{f(x, \beta_5(x, m_2)) - f(x, \beta_5(x, m_1))}{m_2 - m_1} \leq M_2 \Psi(x) < M_2.$$

Suppose $\Psi^{(n-1)}(b) > -M_2(c-b)$ is not true, i.e., $\Psi^{(n-1)}(b) \leq -M_2(c-b)$. Then there is some $x_5 \in (b, x_1)$ such that

$$\Psi^{(n)}(x_5) = \frac{\Phi^{(n-1)}(x_1) - \Phi^{(n-1)}(b)}{x_1 - b} \geq M_2 \frac{c-b}{x_1 - b} > M_2.$$

A contradiction. Therefore $\Psi^{(n-1)}(b) > -M_2(c-b)$. □

Lemma 5.8. *Suppose f satisfies the condition (C) and there is a continuous function $M_2(x)$ on $[b, c]$, such that*

$$f(x, v) - f(x, u) \geq M_2(x)(v - u), \ \forall\, x \in (b, c), \ \forall\, v \geq u \in \mathbb{R}, \tag{5.14}$$

where $M_2(x) > 0$, for $x \in (b, c]$, and

$$\frac{\displaystyle\sum_{j=1}^{t} b_j \overbrace{\int_{\eta_j}^{c} \int_{b}^{r_1} \cdots \int_{b}^{r_{n-1}}}^{n} M_2(r_n) \overbrace{dr_n \cdots dr_1}^{n}}{\displaystyle\sum_{j=1}^{t} b_j \overbrace{\int_{\eta_j}^{c} \int_{b}^{r_1} \cdots \int_{b}^{r_{n-2}}}^{n-1} \left(1 + \int_b^{r_{n-1}} M_2(r_n) \cdot \frac{(r_n - b)^{n-1}}{(n-1)!} dr_n\right) \overbrace{dr_{n-1} \cdots dr_1}^{n-1}} \geq \frac{(n-1)!}{(c-b)^{n-1}}. \tag{5.15}$$

Assume for each $m \in \mathbb{R}$, there exist solutions $\beta_5(x,m)$ of (5.1) satisfying (5.3). Let $m_1 < m_2 \in \mathbb{R}$. Then,

$$\beta_5^{(n-1)}(b,m_2) - \beta_5^{(n-1)}(b,m_1) < -\frac{(n-1)!(m_2-m_1)}{(c-b)^{n-1}}. \tag{5.16}$$

Proof. Let $m_1 < m_2 \in \mathbb{R}$ be fixed. We denote $\Psi(x) = \frac{\beta_5(x,m_2)-\beta_5(x,m_1)}{m_2-m_1}$. Then $\Psi(x)$ satisfies

$$\Psi^{(n)}(x) = \frac{f(x,\beta_5(x,m_2)) - f(x,\beta_5(x,m_1))}{m_2-m_1}, \qquad x \in [b,c],$$
$$\Psi(b) = 1, \quad \Psi^{(i)}(b) = 0, \ 1 \le i \le n-1, \quad \sum_{j=1}^{t} b_j \Psi(\eta_j) - \Psi(c) = 0.$$

By Lemma 5.2, $\Psi^{(n-1)}(b) < 0$. Then, it suffices to show that $\Psi^{(n-1)}(b) < -\frac{(n-1)!}{(c-b)^{n-1}}$. Suppose this is not true. Then, $\Psi^{(n-1)}(b) \ge -\frac{(n-1)!}{(c-b)^{n-1}}$.

Similarly as the proof of Lemma 5.6, by $\Psi(b) = 1$ and $\Psi^{(i)}(b) = 0$ for $1 \le i \le n-1$, we have that

$$\begin{aligned}
\Psi(x) &= \Psi(b) + \overbrace{\int_b^x \int_b^{r_1} \cdots \int_b^{r_{n-2}}}^{n-1} \left(\Psi^{(n-1)}(b) + \int_b^{r_{n-1}} \Psi^{(n)}(r_n)dr_n\right) \overbrace{dr_{n-1}\cdots dr_1}^{n-1} \\
&= 1 + \Psi^{(n-1)}(b) \cdot \frac{(x-b)^{n-1}}{(n-1)!} + \overbrace{\int_b^x \int_b^{r_1} \cdots \int_b^{r_{n-1}}}^{n} \Psi^{(n)}(r_n) \overbrace{dr_n \cdots dr_1}^{n}.
\end{aligned}$$

Next, we show $\Psi(x) > 0$ for $x \in [b,c]$. Assume it is not true. Let $x_0 \in (b,c]$ such that $\Psi(x_0) = 0$ and $\Psi(x) > 0$ for $x \in [b,x_0)$. Then, by (5.14) and $\Psi^{(n-1)}(b) \ge -\frac{(n-1)!}{(c-b)^{n-1}}$,

$$\begin{aligned}
\Psi(x_0) &= 1 + \Psi^{(n-1)}(b) \cdot \frac{(x-b)^{n-1}}{(n-1)!} + \overbrace{\int_b^x \int_b^{r_1} \cdots \int_b^{r_{n-1}}}^{n} \Psi^{(n)}(r_n) \overbrace{dr_n \cdots dr_1}^{n} \\
&\ge 1 + \Psi^{(n-1)}(b) \cdot \frac{(x-b)^{n-1}}{(n-1)!} + \overbrace{\int_b^x \int_b^{r_1} \cdots \int_b^{r_{n-1}}}^{n} M_2(r_n)\Psi(r_n) \overbrace{dr_n \cdots dr_1}^{n} \\
&> 1 + \Psi^{(n-1)}(b) \cdot \frac{(x-b)^{n-1}}{(n-1)!}
\end{aligned}$$

$$
\begin{aligned}
&\geq 1 - \frac{(x_0-b)^{n-1}}{(c-b)^{n-1}} \\
&\geq 0,
\end{aligned}
$$

which is a contradiction to $\Psi(x_0) = 0$. Hence, $\Psi(x) > 0$ for $x \in [b, c]$, and so $\Psi(x) > 1 + \Psi^{(n-1)}(b) \cdot \frac{(x-b)^{n-1}}{(n-1)!}$ for $x \in (b, c]$.

By $\sum_{j=1}^{t} b_j \Psi(\eta_j) = \Psi(c)$, we have that

$$
\begin{aligned}
& -\Psi^{(n-1)}(b) \cdot \sum_{j=1}^{t} b_j \frac{(c-b)^{n-1} - (\eta_j - b)^{n-1}}{(n-1)!} \\
= & \sum_{j=1}^{t} b_j \overbrace{\int_{\eta_j}^{c} \int_{b}^{r_1} \cdots \int_{b}^{r_{n-1}}}^{n} \Psi^{(n)}(r_n) \overbrace{dr_n \cdots dr_1}^{n} \\
\geq & \sum_{j=1}^{t} b_j \overbrace{\int_{\eta_j}^{c} \int_{b}^{r_1} \cdots \int_{b}^{r_{n-1}}}^{n} M_2(r_n) \Psi(r_n) \overbrace{dr_n \cdots dr_1}^{n} \\
> & \sum_{j=1}^{t} b_j \overbrace{\int_{\eta_j}^{c} \int_{b}^{r_1} \cdots \int_{b}^{r_{n-1}}}^{n} M_2(r_n) \left(1 + \Psi^{(n-1)}(b) \cdot \frac{(r_n - b)^{n-1}}{(n-1)!}\right) \overbrace{dr_n \cdots dr_1}^{n},
\end{aligned}
$$

which gives, by (5.15),

$$
\begin{aligned}
& -\Psi^{(n-1)}(b) \\
> & \frac{\sum_{j=1}^{t} b_j \overbrace{\int_{\eta_j}^{c} \int_{b}^{r_1} \cdots \int_{b}^{r_{n-1}}}^{n} M_2(r_n) \overbrace{dr_n \cdots dr_1}^{n}}{\sum_{j=1}^{t} b_j \overbrace{\int_{\eta_j}^{c} \int_{b}^{r_1} \cdots \int_{b}^{r_{n-2}}}^{n-1} \left(1 + \int_{b}^{r_{n-1}} M_2(r_n) \cdot \frac{(r_n-b)^{n-1}}{(n-1)!} dr_n\right) \overbrace{dr_{n-1} \cdots dr_1}^{n-1}} \\
\geq & \frac{(n-1)!}{(c-b)^{n-1}}.
\end{aligned}
$$

This is a contradiction to the assumption $\Psi^{(n-1)}(b) \geq -\frac{(n-1)!}{(c-b)^{n-1}}$. Therefore, $\Psi^{(n-1)}(b) < -\frac{(n-1)!}{(c-b)^{n-1}}$. □

Now, we are in the position to show our main results.

Theorem 5.1. *Suppose that f satisfies condition (C) and that for each $m \in \mathbb{R}$, there exist solutions $\alpha_5(x, m)$, $u_5(x, m)$, $\beta_5(x, m)$, $v_5(x, m)$ of (5.1) satisfying each of the* *conditions (5.3), (5.4), (5.5), (5.6), respectively. Suppose f satisfies one of the following (K1) or (K2):*

(K1): there are some $M_1 > 0$ and a continuous function $M_2(x)$ on $[b, c]$, such that

$$0 > f(x, v) - f(x, u) \geq -M_1(v - u), \ \forall\, x \in (a, b), \ \forall\, v > u \in \mathbb{R},$$

$$f(x, v) - f(x, u) \geq M_2(x)(v - u), \ \forall\, x \in (b, c), \ \forall\, v > u \in \mathbb{R},$$

where $M_1(b-a) < \frac{(n-1)!}{(c-b)^{n-1}}$, $M_2(x) > 0$ for $x \in (b, c]$, and

$$\frac{\sum_{j=1}^{t} b_j \overbrace{\int_{\eta_j}^{c} \int_{b}^{r_1} \cdots \int_{b}^{r_{n-1}}}^{n} M_2(r_n) \overbrace{dr_n \cdots dr_1}^{n}}{\sum_{j=1}^{t} b_j \overbrace{\int_{\eta_j}^{c} \int_{b}^{r_1} \cdots \int_{b}^{r_{n-2}}}^{n-1} \left(1 + \int_{b}^{r_{n-1}} M_2(r_n) \cdot \frac{(r_n-b)^{n-1}}{(n-1)!} dr_n\right) \overbrace{dr_{n-1} \cdots dr_1}^{n-1}}$$
$$\geq \frac{(n-1)!}{(c-b)^{n-1}}.$$

or

(K2): there are some $M_2 > 0$ and a continuous function $M_1(x)$ on $[a, b]$, such that

$$f(x, v) - f(x, u) \leq -M_1(x)(v - u), \ \forall\, x \in (a, b), \ \forall\, v > u \in \mathbb{R},$$

$$0 < f(x, v) - f(x, u) \leq M_2(v - u), \ \forall\, x \in (b, c), \ \forall\, v > u \in \mathbb{R},$$

where $\frac{(n-1)!}{(b-a)^{n-1}} > M_2(c-b)$, $M_1(x) > 0$ for $x \in [a, b)$, and

$$\frac{\sum_{i=1}^{s} a_i \overbrace{\int_{a}^{\xi_i} \int_{r_1}^{b} \cdots \int_{r_{n-1}}^{b}}^{n} M_1(r_n) \overbrace{dr_n \cdots dr_1}^{n}}{\sum_{i=1}^{s} a_i \overbrace{\int_{a}^{\xi_i} \int_{r_1}^{b} \cdots \int_{r_{n-2}}^{b}}^{n-1} \left(1 + \int_{r_{n-1}}^{b} M_1(r_n) \frac{(b-r_n)^{n-1}}{(n-1)!} dr_n\right) \overbrace{dr_{n-1} \cdots dr_1}^{n-1}}$$
$$\geq \frac{(n-1)!}{(b-a)^{n-1}}.$$

Then the BVP (5.1), (5.2) has a unique solution.

Proof. By using Lemmas 5.1–5.8 and Lemma 3.9, the proof is based on the same idea as that of Theorem 3.1. □

CHAPTER SIX

An Example for the Case That $k_2 - k_1$ Is Odd

In this chapter, we consider a special example of the BVP's (2.1), (2.2):

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \qquad x \in [a, c], \tag{6.1}$$

$$y(a) - y(\xi) = y_1, \quad y(b) = y_2, \quad y(\eta) - y(c) = y_3, \tag{6.2}$$

that is, it is the case with $s = t = 1$ in the boundary conditions (2.2). Using the idea of matching solutions, we consider the following sets of four boundary conditions, which are the special cases of (2.5), (2.6), (2.7), (2.8), respectively:

$$y(a) - y(\xi) = y_1, \quad y(b) = y_2, \quad y'(b) = m, \tag{6.3}$$

$$y(a) - y(\xi) = y_1, \quad y(b) = y_2, \quad y''(b) = m, \tag{6.4}$$

$$y(b) = y_2, \quad y'(b) = m, \quad y(\eta) - y(c) = y_3, \tag{6.5}$$

$$y(b) = y_2, \quad y''(b) = m, \quad y(\eta) - y(c) = y_3, \tag{6.6}$$

The homogeneous BVP's corresponding to the above four boundary value problems are as follows:

$$y^{(3)}(x) = 0 \tag{6.7}$$

satisfying each of

$$y(a) - y(\xi) = 0, \quad y(b) = 0, \quad y'(b) = 0, \tag{6.8}$$

$$y(a) - y(\xi) = 0, \quad y(b) = 0, \quad y''(b) = 0, \tag{6.9}$$

$$y(b) = 0, \quad y'(b) = 0, \quad y(\eta) - y(c) = 0, \tag{6.10}$$

$$y(b) = 0, \quad y''(b) = 0, \quad y(\eta) - y(c) = 0. \tag{6.11}$$

First, in Section 6.1, we are concerned about Green's functions and their properties. Then, in Section 6.2, we use the Contraction Mapping Principle to finally get the existence and uniqueness of solutions of (6.1), (6.2).

6.1 Green's Functions

The next four lemmas give us the Green's functions and some properties we need later in the following section.

Lemma 6.1. *(i) The Green's function of the homogeneous BVP (6.7), (6.8) is given by*

$$G_1(x,s) = \begin{cases} \frac{(b-x)^2(s-a)^2}{2(\xi-a)(2b-a-\xi)} - \frac{1}{2}(s-x)^2, & a \le x \le s \le b, \quad s < \xi, \\ \frac{(b-x)^2(s-a)^2}{2(\xi-a)(2b-a-\xi)}, & a \le s \le x \le b, \quad s < \xi, \\ \frac{(b-x)^2(2s-a-\xi)}{2(2b-a-\xi)} - \frac{1}{2}(s-x)^2, & a \le x \le s \le b, \quad s \ge \xi, \\ \frac{(b-x)^2(2s-a-\xi)}{2(2b-a-\xi)}, & a \le s \le x \le b, \quad s \ge \xi. \end{cases}$$

(ii) Then, solutions of (6.1), (6.3) can be expressed as

$$\begin{aligned} y(x) = \quad & y_2 + m(x-b) + \frac{(y_1 + m(\xi-a))(x-b)^2}{(\xi-a)(2b-a-\xi)} \\ & + \int_a^b G_1(x,s) f(s, y(s), y'(s), y''(s)) ds. \end{aligned} \tag{6.12}$$

(iii)

$$\begin{aligned} A_1 &:= \max_{x\in[a,b]} \int_a^b |G_1(x,s)| ds < \frac{1}{12}(b-a)^3, \\ B_1 &:= \max_{x\in[a,b]} \int_a^b \left| \frac{\partial G_1(x,s)}{\partial x} \right| ds < \frac{1}{6}(b-a)^2, \\ C_1 &:= \max_{x\in[a,b]} \int_a^b \left| \frac{\partial^2 G_1(x,s)}{\partial x^2} \right| ds < \frac{2}{3}(b-a). \end{aligned}$$

Proof. It is easy to check (i) and (ii). We here mainly prove (iii).

Notice for any $a \le x \le b$,

$$\int_a^b |G_1(x,s)| ds = \frac{(b-x)^2(\xi-a)^2}{6(2b-a-\xi)} + \frac{(b-x)^2(b-a)(b-\xi)}{2(2b-a-\xi)} - \frac{(b-x)^3}{6}.$$

Take the first derivative of the above expression with respect to x, and we have

$$\begin{aligned}
&\frac{d\left(\int_a^b |G_1(x,s)|ds\right)}{dx} \\
= \quad & \frac{(x-b)}{6(2b-a-\xi)}\left[2(\xi-a)^2+3(x-b)(2b-a-\xi)+6(b-a)(b-\xi)\right].
\end{aligned}$$

Hence, when $b-x=\frac{2(\xi-a)^2+6(b-a)(b-\xi)}{3(2b-a-\xi)}$, $\int_a^b |G_1(x,s)|ds$ attains its maximum value:

$$\begin{aligned}
A_1 : \quad = & \quad \max_{x\in[a,b]} \int_a^b |G_1(x,s)|ds = \frac{2\left[(\xi-a)^2+3(b-a)(b-\xi)\right]^3}{81(2b-a-\xi)^3} \\
= & \quad \frac{2(b-a)^3}{81}\cdot\left[\frac{(\xi-a)^2-3(\xi-a)+3}{2-(\xi-a)}\right]^3 \\
< & \quad \frac{(b-a)^3}{12},
\end{aligned}$$

where the supremum is at $\xi=a$.

Now we show $B_1<\frac{1}{6}(b-a)^2$. Note

$$\frac{\partial G_1(x,s)}{\partial x} = \begin{cases}
-\frac{(b-x)(s-a)^2}{(\xi-a)(2b-a-\xi)}-x+s, & a\le x\le s\le b, \quad s<\xi, \\
-\frac{(b-x)(s-a)^2}{(\xi-a)(2b-a-\xi)}, & a\le s\le x\le b, \quad s<\xi, \\
-\frac{(b-x)(2s-a-\xi)}{(2b-a-\xi)}-x+s, & a\le x\le s\le b, \quad s\ge\xi, \\
-\frac{(b-x)(2s-a-\xi)}{(2b-a-\xi)}, & a\le s\le x\le b, \quad s\ge\xi.
\end{cases}$$

Hence, for any $\frac{\xi+a}{2}\le x\le b$,

$$\begin{aligned}
&\int_a^b \left|\frac{\partial G_1(x,s)}{\partial x}\right| ds \\
= \quad & \frac{b-x}{6(2b-a-\xi)}\cdot\left[2(\xi-a)^2+6(b-a)(b-\xi)-3(b-x)(2b-a-\xi)\right].
\end{aligned}$$

Take the first derivative of the above expression with respect to x, and we have

$$\begin{aligned}
&\frac{d\left(\int_a^b \left|\frac{\partial G_1(x,s)}{\partial x}\right| ds\right)}{dx} \\
= \quad & \frac{1}{6(2b-a-\xi)}\left[-2(\xi-a)^2-6(b-a)(b-\xi)+6(b-x)(2b-a-\xi)\right],
\end{aligned}$$

which implies when $b-x=\frac{(\xi-a)^2+3(b-a)(b-\xi)}{3(2b-a-\xi)}$, $\int_a^b \left|\frac{\partial G_1(x,s)}{\partial x}\right| ds$ gets its maximum value over the interval $[\frac{\xi+a}{2},b]$:

$$\max_{x\in[\frac{\xi+a}{2},b]} \int_a^b \left|\frac{\partial G_1(x,s)}{\partial x}\right| ds = \frac{[(\xi-a)^2+3(b-a)(b-\xi)]^2}{18(2b-a-\xi)^2}$$

$$
\begin{aligned}
&= \frac{(b-a)^2}{18} \cdot \left[\frac{(\xi-a)^2 - 3(\xi-a) + 3}{2-(\xi-a)} \right]^2 \\
&< \frac{(b-a)^2}{8},
\end{aligned}
$$

where the supremum is at $\xi = a$.

For $a \leq x \leq \frac{\xi+a}{2}$, it is easy to show that $\left|\frac{\partial G_1(x,s)}{\partial x}\right| \leq \left|\frac{\partial G_1(a,s)}{\partial x}\right|$ for all $a \leq s \leq b$. Therefore,

$$
\begin{aligned}
&\max_{x\in[a,\frac{\xi+a}{2}]} \int_a^b \left|\frac{\partial G_1(x,s)}{\partial x}\right| ds \leq \int_a^b \left|\frac{\partial G_1(a,s)}{\partial x}\right| ds \\
&= \frac{(\xi-a)(b-a)(3b-2a-\xi)}{6(2b-a-\xi)} \\
&= \frac{(b-a)^2}{6} \cdot \frac{(\xi-a)(3-2(\xi-a))}{2-(\xi-a)} \\
&< \frac{(b-a)^2}{6},
\end{aligned}
$$

where the supremum is at $\xi = b$. Hence,

$$
B_1 := \max_{x\in[a,b]} \int_a^b \left|\frac{\partial G_1(x,s)}{\partial x}\right| ds < \frac{1}{6}(b-a)^2.
$$

Last, we show $C_1 < \frac{2}{3}(b-a)$. Note

$$
\frac{\partial^2 G_1(x,s)}{\partial x^2} = \begin{cases}
\frac{(s-a)^2}{(\xi-a)(2b-a-\xi)} - 1, & a \leq x \leq s \leq b, \quad s < \xi, \\
\frac{(s-a)^2}{(\xi-a)(2b-a-\xi)}, & a \leq s \leq x \leq b, \quad s < \xi, \\
\frac{(2s-a-\xi)}{(2b-a-\xi)} - 1, & a \leq x \leq s \leq b, \quad s \geq \xi, \\
\frac{(2s-a-\xi)}{(2b-a-\xi)}, & a \leq s \leq x \leq b, \quad s \geq \xi,
\end{cases}
$$

and

$$
\frac{\partial^2 G_1(x,s)}{\partial x^2} \leq 0 \quad \text{for} \quad x \leq s \qquad \text{and} \qquad \frac{\partial^2 G_1(x,s)}{\partial x^2} \geq 0 \quad \text{for} \quad s \leq x.
$$

For $a \leq \xi$,

$$
\begin{aligned}
&\int_a^b \left|\frac{\partial^2 G_1(x,s)}{\partial x^2}\right| ds \\
&= \frac{2(x-a)^3 - (\xi-a)^3 + 3(b-x)(\xi-a)(2b-a-\xi) - 3(\xi-a)(b-a)(b-\xi)}{3(\xi-a)(2b-a-\xi)},
\end{aligned}
$$

whose maximum value is either at $x = a$ or $x = \xi$. Comparing the two values, we can see

$$\max_{x\in[a,\xi]} \int_a^b \left| \frac{\partial^2 G_1(x,s)}{\partial x^2} \right| ds = \int_a^b \left| \frac{\partial^2 G_1(a,s)}{\partial x^2} \right| ds = \frac{3(b-a)^3 - (\xi-a)^2}{3(2b-a-\xi)} < \frac{2}{3}(b-a),$$

where the supremum is at $\xi = b$.

For $a \le \xi$,

$$\int_a^b \left| \frac{\partial^2 G_1(x,s)}{\partial x^2} \right| ds = \frac{(\xi-a)^2 + 3(2b-a-\xi)x - 3b^2 + 3a\xi}{3(2b-a-\xi)}$$

which abtains its maximum value $\frac{(\xi-a)^2+3(b-a)(b-\xi)}{3(2b-a-\xi)}$at $x = b$. Therefore,

$$\max_{x\in[\xi,b]} \int_a^b \left| \frac{\partial^2 G_1(x,s)}{\partial x^2} \right| ds = \frac{(\xi-a)^2 + 3(b-a)(b-\xi)}{3(2b-a-\xi)} < \frac{1}{2}(b-a),$$

where the supremum is at $\xi = a$.

To sum up, we have

$$C_1 := \max_{x\in[a,b]} \int_a^b \left| \frac{\partial^2 G_1(x,s)}{\partial x^2} \right| ds < \frac{2}{3}(b-a).$$

$\square$

Lemma 6.2. *(i) The Green's function for the homogeneous BVP (6.7), (6.9) is as follows*

$$G_2(x,s) = \begin{cases} \frac{(b-x)(s-a)^2}{2(\xi-a)} - \frac{1}{2}(s-x)^2, & a \le x \le s \le b, \quad s < \xi, \\ \frac{(b-x)(s-a)^2}{2(\xi-a)}, & a \le s \le x \le b, \quad s < \xi, \\ \frac{(b-x)(2s-a-\xi)}{2} - \frac{1}{2}(s-x)^2, & a \le x \le s \le b, \quad s \ge \xi, \\ \frac{(b-x)(2s-a-\xi)}{2}, & a \le s \le x \le b, \quad s \ge \xi. \end{cases}$$

(ii) Then, the solution of (6.1), (6.4) can be expressed as

$$\begin{aligned} y(x) = \quad & y_2 + \frac{m(\xi-a)(2b-a-\xi) - 2y_1}{2(\xi-a)}(x-b) + \frac{m}{2}(x-b)^2 \\ & + \int_a^b G_2(x,s) f(s, y(s), y'(s), y''(s)) ds. \end{aligned} \tag{6.13}$$

(iii)

$$\begin{aligned} A_2 &:= \max_{x\in[a,b]} \int_a^b |G_2(x,s)|ds < \frac{1}{3}(b-a)^3, \\ B_2 &:= \max_{x\in[a,b]} \int_a^b \left|\frac{\partial G_2(x,s)}{\partial x}\right| ds < \frac{1}{2}(b-a)^2, \\ C_2 &:= \max_{x\in[a,b]} \int_a^b \left|\frac{\partial^2 G_2(x,s)}{\partial x^2}\right| ds < b-a. \end{aligned}$$

Proof. Parts (i) and (ii) are easy to check and are omitted. We next prove (iii).

Notice that by doing some calulus, for any $a \le x \le b$,

$$\int_a^b |G_2(x,s)|ds = -\frac{1}{6}(b-x)\left[(b-x)^2 - (\xi-a)^2 - 3(b-\xi)(b-a)\right].$$

Take the first derivative of the above expression with respect to x, and we have

$$\frac{d\left(\int_a^b |G_2(x,s)|ds\right)}{dx} = \frac{1}{2}\left[(b-x)^2 - \frac{1}{3}(\xi-a)^2 - (b-a)(b-\xi)\right].$$

Since $(b-\xi)^2 \le \frac{1}{3}(\xi-a)^2 + (b-a)(b-\xi) \le (b-a)^2$, when $(b-x)^2 = \frac{1}{3}(\xi-a)^2 + (b-a)(b-\xi)$, $\int_a^b |G_2(x,s)|ds$ attains its maximum value:

$$\begin{aligned} A_2 : &= \max_{x\in[a,b]} \int_a^b |G_2(x,s)|ds = \frac{1}{3}\left[\frac{1}{3}(\xi-a)^2 + (b-a)(b-\xi)\right]^{\frac{3}{2}} \\ &< \frac{(b-a)^3}{3}, \end{aligned}$$

where the supremum is at $\xi = a$.

Now we consider B_2. Note

$$\frac{\partial G_2(x,s)}{\partial x} = \begin{cases} -\frac{(s-a)^2}{2(\xi-a)} - x + s, & a \le x \le s \le b, \quad s < \xi, \\ -\frac{(s-a)^2}{2(\xi-a)}, & a \le s \le x \le b, \quad s < \xi, \\ -\frac{(2s-a-\xi)}{2} - x + s, & a \le x \le s \le b, \quad s \ge \xi, \\ -\frac{(2s-a-\xi)}{2}, & a \le s \le x \le b, \quad s \ge \xi. \end{cases}$$

Hence, for any $\frac{\xi+a}{2} \le x \le b$, every expression of $\frac{\partial G_2(x,s)}{\partial x}$ is negative and so

$$\int_a^b \left|\frac{\partial G_2(x,s)}{\partial x}\right| ds = \frac{1}{6}(\xi-a)^2 - \frac{1}{2}(b-x)^2 + \frac{1}{2}(b-a)(b-\xi),$$

which attains its maximum at $x = b$:

$$\max_{x\in[\frac{\xi+a}{2},b]} \int_a^b \left|\frac{\partial G_2(x,s)}{\partial x}\right| ds = \frac{1}{6}(\xi-a)^2 + \frac{1}{2}(b-a)(b-\xi) < \frac{1}{2}(b-a)^2, \quad \forall \xi \in (a,b).$$

For $a \le x \le \frac{\xi+a}{2}$, it is easy to check that $\left|\frac{\partial G_2(x,s)}{\partial x}\right| \le \left|\frac{\partial G_2(a,s)}{\partial x}\right|$ for any $a \le s \le b$. Therefore

$$\begin{aligned} & \max_{x\in[a,\frac{\xi+a}{2}]} \int_a^b \left|\frac{\partial G_2(x,s)}{\partial x}\right| ds \\ \le\ & \int_a^b \left|\frac{\partial G_2(a,s)}{\partial x}\right| ds = \int_a^b \frac{\partial G_2(a,s)}{\partial x} ds \\ =\ & \frac{1}{3}(\xi-a)^2 + \frac{1}{2}(\xi-a)(b-\xi) \\ <\ & \frac{1}{3}(b-a)^2, \end{aligned}$$

where the supremum is at $\xi = b$. Hence,

$$B_2 := \max_{x\in[a,b]} \int_a^b \left|\frac{\partial G_2(x,s)}{\partial x}\right| ds < \frac{1}{2}(b-a)^2.$$

From

$$\frac{\partial^2 G_2(x,s)}{\partial x^2} = \begin{cases} -1, & a \le x \le s \le b, \quad s < \xi, \\ 0, & a \le s \le x \le b, \quad s < \xi, \\ -1, & a \le x \le s \le b, \quad s \ge \xi, \\ 0, & a \le s \le x \le b, \quad s \ge \xi, \end{cases}$$

it follows that

$$C_2 := \max_{x\in[a,b]} \int_a^b \left|\frac{\partial^2 G_2(x,s)}{\partial x^2}\right| ds < b - a.$$

□

Based on the symmetry and similar reasoning to the proof of Lemmas 6.1 and 6.2, we can get the following two lemmas on $[b,c]$.

Lemma 6.3. *(i) The Green's function of the homogeneous BVP (6.7), (6.10) is as follows*

$$G_3(x,s) = \begin{cases} -\frac{(b-x)^2(2s-c-\eta)}{2(2b-c-\eta)}, & b \le x \le s \le c, \quad s < \eta, \\ -\frac{(b-x)^2(2s-c-\eta)}{2(2b-c-\eta)} + \frac{1}{2}(s-x)^2, & b \le s \le x \le c, \quad s < \eta, \\ \frac{(b-x)^2(s-c)^2}{2(c-\eta)(2b-\eta-c)}, & b \le x \le s \le c, \quad s \ge \eta, \\ \frac{(b-x)^2(s-c)^2}{2(c-\eta)(2b-\eta-c)} + \frac{1}{2}(s-x)^2, & b \le s \le x \le c, \quad s \ge \eta. \end{cases}$$

(ii) Then, the solution of (6.1), (6.5) can be expressed as

$$\begin{aligned} y(x) = \quad & y_2 + m(x-b) + \frac{y_3 + m(c-\eta)}{(c-\eta)(2b-\eta-c)}(x-b)^2 \\ & + \int_b^c G_3(x,s) f(s, y(s), y'(s), y''(s)) ds. \end{aligned} \tag{6.14}$$

(iii)

$$\begin{aligned} A_3 &:= \max_{x\in[b,c]} \int_b^c |G_3(x,s)| ds < \frac{1}{12}(c-b)^3, \\ B_3 &:= \max_{x\in[b,c]} \int_b^c \left| \frac{\partial G_3(x,s)}{\partial x} \right| ds < \frac{1}{6}(c-b)^2, \\ C_3 &:= \max_{x\in[b,c]} \int_b^c \left| \frac{\partial^2 G_3(x,s)}{\partial x^2} \right| ds < \frac{2}{3}(c-b). \end{aligned}$$

Lemma 6.4. *(i) The Green's function of the homogeneous BVP (6.7), (6.11) is as follows*

$$G_4(x,s) = \begin{cases} \frac{(2s-c-\eta)(x-b)}{2}, & b \le x \le s \le c, \quad s < \eta, \\ \frac{(2s-c-\eta)(x-b)}{2} + \frac{1}{2}(s-x)^2, & b \le s \le x \le c, \quad s < \eta, \\ -\frac{(x-b)(s-c)^2}{2(c-\eta)}, & b \le x \le s \le c, \quad s \ge \eta, \\ -\frac{(x-b)(s-c)^2}{2(c-\eta)} + \frac{1}{2}(s-x)^2, & b \le s \le x \le c, \quad s \ge \eta. \end{cases}$$

(ii) Then, the solution of (6.1), (6.6) can be expressed as

$$\begin{aligned} y(x) = \quad & y_2 + \frac{m(c-\eta)(2b-c-\eta) - 2y_3}{2(c-\eta)}(x-b) + \frac{m}{2}(x-b)^2 \\ & + \int_b^c G_4(x,s) f(s, y(s), y'(s), y''(s)) ds. \end{aligned} \tag{6.15}$$

(iii)

$$A_4 := \max_{x\in[b,c]} \int_b^c |G_4(x,s)|ds < \frac{1}{3}(c-b)^3,$$
$$B_4 := \max_{x\in[b,c]} \int_b^c \left|\frac{\partial G_4(x,s)}{\partial x}\right| ds < \frac{1}{2}(c-b)^2,$$
$$C_4 := \max_{x\in[b,c]} \int_b^c \left|\frac{\partial^2 G_4(x,s)}{\partial x^2}\right| ds < c-b.$$

6.2 Main Results

The Contraction Mapping Principle is stated as follows:

Lemma 6.5. *Let (X,d) be a non-empty complete metric space. Let $T: X \to X$ be a contraction mapping on X, i.e., there is a nonnegative real number $\alpha < 1$ such that $d(T(x),T(y)) \le \alpha d(x,y)$ for all $x, y \in X$. Then, the map T admits one and only one fixed point x^* in X (this means $T(x^*) = x^*$).*

Theorem 6.1. *For $f : [a,c] \times \mathbb{R}^3 \to \mathbb{R}$, suppose there are real numbers L_i, M_i, N_i such that for all $x \in [a,c]$ and $z_j, w_j \in \mathbb{R}$ for $i = 1,2,3,4,\ j = 1,2,3$,*

$$|f(x,z_1,z_2,z_3) - f(x,w_1,w_2,w_3)| \le L_i|z_1-w_1| + M_i|z_2-w_2| + N_i|z_3-w_3|,$$

where

$$\frac{L_1(b-a)^3}{12} + \frac{M_1(b-a)^2}{6} + \frac{2N_1(b-a)}{3} < 1, \tag{6.16}$$
$$\frac{L_2(b-a)^3}{3} + \frac{M_2(b-a)^2}{2} + N_2(b-a) < 1, \tag{6.17}$$
$$\frac{L_3(c-d)^3}{12} + \frac{M_3(c-d)^2}{6} + \frac{2N_3(c-d)}{3} < 1, \tag{6.18}$$
$$\frac{L_4(c-d)^3}{3} + \frac{M_4(c-d)^2}{2} + N_4(c-d) < 1. \tag{6.19}$$

Then, the BVP's (6.1) satisfying each of (6.3), (6.4), (6.5) and (6.6), have a unique solution for any $y_i \in \mathbb{R},\ i = 1,2,3,\ \xi \in (a,b)$, and $\eta \in (b,c)$, respectively. If condition (A1) on f is also satisfied, then (6.1), (6.2) has a unique solution for any $y_i \in \mathbb{R}$, $i = 1,2,3,\ \xi \in (a,b)$, and $\eta \in (b,c)$.

Proof. We use the Contraction Mapping Principle, Lemma 6.5, to prove our conclusions.

For $i = 1, 2$, consider the complete metric spaces $X_i = C^{(2)}([a,b],\mathbb{R})$ with $d_i(y,z) = L_i|y-z|_\infty + M_i|y'-z'|_\infty + N_i|y''-z''|_\infty$ for $y, z \in X$, with the maximum norm $|\cdot|_\infty$ in $C([a,b],\mathbb{R})$, and the complete metric spaces $E_j = C^{(2)}([b,c],\mathbb{R})$ with $d(y,z) = L_j|y-z|_\infty + M_j|y'-z'|_\infty + N_j|y''-z''|_\infty$ for $y, z \in E_j$, with the maximum norm $|\cdot|$ in $C([b,c],\mathbb{R})$, for $j = 1, 2$. According to parts (ii) of the Lemmas 6.1, 6.2, 6.3 and 6.4, we define the following four operators on X_1, X_2, E_1 and E_2, respectively:

$$\begin{aligned}
(T_1y)(x) &= y_2 + m(x-b) + \frac{(y_1 + m(\xi - a))(x-b)^2}{(\xi - a)(2b - a - \xi)} \\
&\quad + \int_a^b G_1(x,s) f(s, y(s), y'(s), y''(s))ds, \qquad \text{for } \ y \in X, \\
(T_2y)(x) &= y_2 + \frac{m(\xi - a)(2b - a - \xi) - 2y_1}{2(\xi - a)}(x-b) + \frac{m}{2}(x-b)^2 \\
&\quad + \int_a^b G_2(x,s) f(s, y(s), y'(s), y''(s))ds, \qquad \text{for } \ y \in X, \\
(T_3y)(x) &= y_2 + m(x-b) + \frac{y_3 + m(c-\eta)}{(c-\eta)(2b - \eta - c)}(x-b)^2 \\
&\quad + \int_b^c G_3(x,s) f(s, y(s), y'(s), y''(s))ds, \qquad \text{for } \ y \in E, \\
(T_4y)(x) &= y_2 + \frac{m(c-\eta)(2b - c - \eta) - 2y_3}{2(c-\eta)}(x-b) + \frac{m}{2}(x-b)^2 \\
&\quad + \int_b^c G_4(x,s) f(s, y(s), y'(s), y''(s))ds, \qquad \text{for } \ y \in E.
\end{aligned}$$

Now we show T_i's for $i = 1, 2, 3, 4$ are contraction mappings. Take T_1 as an example. Others can be proved similarly. For any $y, z \in X$, by Lemma 6.1, we have

$$\begin{aligned}
& |(T_1y)(x) - (T_1z)(x)| \\
= & \left| \int_a^b G_1(x,s)(f(s,y(s),y'(s),y''(s)) - f(s,z(s),z'(s),z''(s)))ds \right| \\
\le & \int_a^b |G_1(x,s)| \ |f(s,y(s),y'(s),y''(s)) - f(s,z(s),z'(s),z''(s))|ds \\
\le & \int_a^b |G_1(x,s)| \ \left[L_1|y(s)-z(s)| + M_1|y'(s)-z'(s)| + N_1|y''(s)-z''(s)|\right] ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_a^b |G_1(x,s)| \ [L_1|y-z|_\infty + M_1|y'-z'|_\infty + N_1|y''-z''|_\infty]\, ds \\
&\leq A_1 d(y,z) \leq \frac{(b-a)^3}{12} d(y,z).
\end{aligned}
$$

Similarly, we have

$$
|(T_1y)'(x) - (T_1z)'(x)| \leq \int_a^b \left| \frac{\partial G_1(x,s)}{\partial x} \right| ds \cdot d(y,z) \leq B_1 d(y,z) \leq \frac{(b-a)^2}{6} d(y,z)
$$

and

$$
|(T_1y)''(x) - (T_1z)''(x)| \leq \int_a^b \left| \frac{\partial^2 G_1(x,s)}{\partial x^2} \right| ds \cdot d(y,z) \leq C_1 d(y,z) \leq \frac{2(b-a)}{3} d(y,z).
$$

Therefore,

$$
\begin{aligned}
& d(T_1(y) - T_1(z)) \\
=\ & L_1|T_1(y) - T_1(z)|_\infty + M_1|(T_1y)' - (T_1z)'|_\infty + N_1|(T_1y)'' - (T_1z)''|_\infty \\
\leq\ & \left[\frac{L_1(b-a)^3}{12} + \frac{M_1(b-a)^2}{6} + \frac{2N_1(b-a)}{3} \right] d(y,z).
\end{aligned}
$$

By (6.16), T_1 is a contraction mapping, thus T_1 has one and only one fixed point $y \in C^{(2)}([a,b],\mathbb{R})$. In particular, the BVP (6.1) and (6.3) has a unique solution $\alpha(x)$.

Similary, by using (6.17), (6.18) and (6.19) and showing T_i's for $i = 2,3,4$ are contraction mappings, we get that the BVP's (6.1) satisfying each of (6.4), (6.5) and (6.6) have unique solutions $u(x)$, $\beta(x)$, $v(x)$, respectively. If f also satisfies the condition (A1), then by Theorem 2.1, the BVP (6.1) and (6.2) has a unique solution for any $y_i \in \mathbb{R}$, $i = 1,2,3$, $\xi \in (a,b)$, and $\eta \in (b,c)$. $\square$

CHAPTER SEVEN

Conclusion and Future Work

The solution matching techniques have been developed over 30 years and extended to many types of boundary value problems, but only a few deal with nonlocal boundary value problems, such as five-point BVP's, see [15, 24], etc. The problems discussed in this dissertation are involved with more general nonlocal boundary conditions. In addition, by our results on several different types of boundary conditions at the matching point b, the solution-matching technique has been extended further.

If we look forward to what future work this dissertation may lead to, we can consider several directions.

First, the most natural question would be whether these results could be extended to other types of boundary value problems for differential equations, difference equations, or dynamic systems on time scales. Different insights could be developed for different problems.

Second, Liapunov-like functions could be used to substitute the monotone conditions. Basically, the monotone properties of the nonlinear term f are essentially important for matching solutions, and Liapunov theory [37] is a good tool to weaken these monotonicity conditions. Some papers have already applied Liapunov theory to solution-matching methods on certain BVP's, see [24, 35]. We could also resort to Liapunov functions to generalize the conditions (A)'s, (B)'s and (C) in this dissertation. The cases with odd gaps should be easily dealt with. There might be some technical difficulties in the cases with even gaps since we have to consider numerical and monotone conditions at the same time. It is worthwhile to work on it.

Another idea could be to further generalize the cases with even gaps. Based on the analysis used in this dissertation, we only considered special situations with

$k_1 = 0$ and $k_2 = n - 1$ in Chapter Five. The natural question is, "how about other cases?" New methods or tools for analyzing, or stronger conditions, may be needed to cope with the more general problems.

Last, though continuous differentiability of f is a strong condition, yet we may consider using it to study boundary data smoothness for solutions, such as the k_2nd order derivative of $\Phi(b, m)$ or $\Psi(b, m)$ as defined in Chapter Three or Chapter Five with respect to the boundary value m of the k_1st order derivative of a solution at the matching point b, that is, $\Phi^{(k_2)}(b, m)$ or $\Psi^{(k_2)}(b, m)$ with respect to m, and then combine with the solution-matching technique to get the existence and uniqueness of solutions of certain BVP's. In this case, it would seem very plausible that the monotonicity of f may be not required.

The above are some questions and thoughts for further research that arise during the writing of this dissertation. We hope these are great directions in the future development of the solution matching technique.

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