ABSTRACT

Gravitational Waves and Cosmology

J. R. Fier, Ph.D.

Advisor: Anzhong Wang, Ph.D.

The recent discovery of GW140915 and the confirmation of the existence of gravitational waves (GWs) has garnered the attention of many physicists as they seek to understand their behavior as they travel across the universe. In this dissertation, one will find the study singularities which may arise in plane GWs, and cosmological perturbations may affect GWs as they propagate through an expanding, inhomogeneous universe. It is found that in the BJR coordinates, singularities arise at the focused point $u = u_s$, except in the two cases: (i) $\alpha = 1/2, \forall\ \chi_n$, and (ii) $\alpha = 1, \chi_i = 0$, where χ_n are the coefficients in the expansion and α is a parameter. When observing GWs produced from remote astrophysical sources, one finds that there are three scales to consider, λ, L_c, and L which denote the typical wavelength of the GW, the scale of the cosmological perturbations, and the size of the observable universe, respectively. The Einstein equations were calculated for GWs on the cosmic scale, and the geometric optics approximation found the gravitational integrated Sachs-Wolfe effects created by both the cosmological scalar and tensor perturbations.

Gravitational Waves and Cosmology

by

J. R. Fier, B.S.

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Dwight Russell, Ph.D., Interim Chairperson

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Approved by the Dissertation Committee

Anzhong Wang, Ph.D.

Gerald Cleaver, Ph.D.

Lorin S. Matthews, Ph.D.

Dwight Russell, Ph.D.

Klaus Kirsten, Ph.D.

Accepted by the Graduate School
May 2021

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

"Deep in the human unconscious is a pervasive need for a logical universe that makes sense. But the real universe is always one step beyond logic." - Frank Herbert

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CHAPTER ONE

Introduction to General Relativity

1.1 Gravitational Theory

During the 1500's the foundation for a more modern gravitational theory began with Italian astronomer and physicist Galileo Galilei as he studied the motion of objects and celestial bodies. As an astronomer, he was able to devise a more precise telescope for watching the stars and rigorously track the movement of the celestial bodies. He was the first to discover the moons Io, Europa, Ganymede, and Callisto orbiting Jupiter. As a natural philosopher, he studied the motion of objects from pendulums to falling bodies. He proposed that all bodies fall with uniform acceleration, as the acceleration is independent from its mass. This idea was the basis for Newton's first law of motion. Galileo's work on falling bodies stood as the basis for Newton's theory of gravity.

Around the same time, astronomer Johannes Kepler was able to describe the movements of planets with his three laws of planetary motion developed in his books *Astronomia nova* and *Harmonices mundi* [2,3]. The findings stated: (*i*) every planet moves in an ellipse with the Sun at the center of one of the to foci, (*ii*) a planet moving relative to the Sun sweeps out equal areas in equal time intervals, and (*iii*) the square of a planet's orbital period is proportional to the cube of the semi-major axis of its orbit.

Only one year after Galileo passed away, Isaac Newton was born in 1643 in England. Having been inspired by the works of the natural philosophers like Galileo, Newton began devising a way to explain the effects of gravity. Newton developed Calculus as the mathematical language which he would describe the world. This was the beginning of the modern form of physics that we see today in which phenomena are described using mathematical equations. The culmination of his work led to his Universal Law of Gravitation, published in *Philosophiæ Naturalis Principia Mathematica* in 1687 [1]. In these writings, Newton stated his three Laws of Motion: (i) the law of inertia, (ii) the law of dynamics, and (iii) the law of action and reaction.

Newton's theory developed the idea that mass creates a gravitational potential which creates an attractive force between it and all other masses in the form of the gravitational force,

$$\vec{F}_g = -\frac{Gm_1m_2}{r^2}\hat{r} \tag{1.1}$$

The strength of the gravitational force is proportional to inverse square law, and G is the gravitational constant valued at $G = 6.67410^{-11}\ m^3kg^{-1}s^{-2}$. Newton's theory was highly successful as it is able to accurately describe the motions of the planets in the solar system, producing Kepler's results. The theory is also able to predict the equations of motions of objects, describing their trajectory and movement when subjected to the gravitational force. Even though Newton's theory has had great success, it is not a perfect theory and has some shortcomings. Newton's theory of gravitation holds that time is an absolute value, and that the gravitational forces between objects act instantaneously. Observational experiments have shown that light falls, and therefore is subject to the gravitational force even though light has no mass. The theory is also unable to account for the anomalous precession of the perihelion of Mercury's orbit around the Sun. Newton's theory also struggles when dealing with velocities that approach the speed of light.

In 1905, Einstein published a series of articles called *Annus Mirabilis* [4] where he proposed his theory of relativity that is based on two postulates: (i) All inertial observers are equivalent, and the laws of physics are invariant in an inertial frame, and (ii) The speed of light is constant, and the same for all observers. Einstein's ideas toss aside Newton's proposals of absolute space and time, stating that all events are relative. Space and time are no longer separate quantities, but are connected to each other, and measuring them depends on the frame of the observer. Einstein updated the Galilean transformation, which connects two inertial frames of reference, to account for time and relativistic moving bodies:

$$\begin{aligned} t' &= \beta\left(t - \frac{vx}{c^2}\right) \\ x' &= \beta(x - vt) \\ y' &= y \\ z' &= z. \end{aligned} \tag{1.2}$$

These are the Lorentz transformations for a test body moving in the x-direction, and where one can define the Lorentz factor:

$$\beta = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{1.3}$$

The Lorentz factor shows that an observer in one inertial frame of reference can measure length and time differently than another inertial observer in the manifestation of length contraction and time dilation. But if all inertial observers are equivalent, then there must be some invariant quantity that relates the results of the two observers.

Einstein stated that the square of the interval between two events must remain invariant under a Lorentz transformation, similar to the manner in which the square of the distance remains invariant under a Galilean transformation. If one looks at the infinitesimal interval between two events, there is

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{1.4}$$

This is the invariant four-dimensional spacetime called Minkowski spacetime, which is flat and provides the background geometry.

So far, this theory only applies to situations that are in inertial frames, making it a *special* case of relativity. In 1915, Einstein generalized his theory in which gravity worked into his formulation [5]. Including gravity allows for non-inertial reference frames, where he made the realization that a frame that is linearly accelerated relative to an inertial frame is identical to that of a frame that is at rest in a gravitational field, and an observer would not be able to tell the difference. The mathematical framework of general relativity relies on adopting an all-encompassing tensorial form. To start, the line element can be written in the form,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{1.5}$$

where the metric, $g_{\mu\nu}$, will be the building block for the Einstein equations. The metric is a rank-2 symmetric tensor that defines the distances and lengths of vectors in the space time. If the norm of the vector X^μ is zero, then X^μ is said to be a null vector and classifies things like light-ray geodesics. The metric also defines the orthonormality of vectors, where it is said that two vectors X^mu and Y^μ are orthogonal if

$$g_{\mu\nu}X^\mu Y^\nu = 0. \tag{1.6}$$

The metric can also be inverted by the definition

$$g_{\mu\lambda}g^{\lambda\nu} = \delta^{\nu}_{\mu}, \tag{1.7}$$

where δ^{ν}_{μ} is the identity tensor. Using the metric, one can also raise and lower indices on any type of tensor, with the exception of rank-0 tensors, which are scalar values and contain no indices, by

$$\begin{aligned} g_{\mu\lambda}X^{\mu\nu\ldots}_{\alpha\beta\ldots} &= X^{\nu\ldots}{}_{\lambda\alpha\beta\ldots}, \\ g^{\alpha\lambda}X^{\mu\nu\ldots}_{\alpha\beta\ldots} &= X^{\mu\nu\alpha\ldots}{}_{\beta\ldots}. \end{aligned} \tag{1.8}$$

Now that the metric is defined, one can now use it to start building up Einstein's field equations. First, there needs to be a new way to take a derivative using a covariant derivative. Since the space is no longer flat, there needs to be an extra term added to the derivative, called the Christoffel symbols, which describes how the metric deviates along a curved space. The Christoffel symbols are defined as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\big(g_{\mu\rho,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\big), \tag{1.9}$$

where partial derivatives are written as $\partial_\rho g_{\mu\nu} = g_{\mu\nu,\rho}$. With this termed defined, the covariant derivative of a generic mixed tensor can be written as

$$\nabla_\rho T^{\mu\ldots}_{\nu\ldots} = \partial_\rho T^{\mu\ldots}_{\nu\ldots} + \Gamma^{\mu}_{\rho\lambda}T^{\lambda\ldots}_{\nu\ldots} + \ldots - \Gamma^{\lambda}_{\rho\nu}T^{\mu\ldots}_{\lambda\ldots} - \ldots\ . \tag{1.10}$$

With the covariant derivatives defined, along with the Christoffel symbols we can now define the Riemann tensor, otherwise known as the curvature tensor which depends on the second derivatives of the metric and is defined as

$$R^{\alpha}_{\beta\mu\nu} = \partial_\mu\Gamma^{\alpha}_{\beta\nu} - \partial_\nu\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\lambda}_{\beta\nu}\Gamma^{\alpha}_{\lambda\mu} - \Gamma^{\lambda}_{\beta\mu}\Gamma^{\alpha}_{\lambda\nu}. \tag{1.11}$$

The Riemann tensor is an anti-symmetric tensor defined by

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = R_{\mu\nu\alpha\beta}. \tag{1.12}$$

The Ricci tensor is formed by contracting the upper index α, with the lower index μ,

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}. \tag{1.13}$$

Taking a trace of the Ricci tensor gives the Ricci scalar,

$$R = G^{\mu\nu} R_{\mu\nu}, \tag{1.14}$$

which gives everything needed to construct Einstein's field equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1.15}$$

where $T_{\mu\nu}$ is the energy-momentum tensor which accounts for things like matter, perfect fluids, or electromagnetic fields. The left-hand side of the equation describes the curvature of space, whether it is flat or curved. The right hand side describes matter makeup of the universe. One can conclude from this equation that matter creates curvature in space, and space describes how matter is distributed. A massive object, like the Sun, bends space around it, and the gravity generated by the Sun is a manifestation of this curvature.

In order for the general theory of relativity to be valid, then it must match previous predictions made by gravity, as well as provide some of the predictions made by general relativity.

1.2 Tests of General Relativity

Since his development of general relativity in 1915, Einstein proposed three tests that would determine the validity of his theory. In this next section the classical tests of general relativity will be discussed in the chronological order in which they were confirmed

1.2.1 The Precession of the Perihelion of Mercury

The first evidence of the validity of general relativity was the prediction of the precession of the perihelion of mercury. When Newton developed his theory of gravitation, he was able to accurately describe the motions of the planets that circle the Sun. There was one problem that was present in Newtonian theory, the precession of Mercury's perihelion. The planet's motion seemed to align perfectly with Newton's predictions until French astronomer Urbain Le Verrier began studying the data from the orbit of Mercury around the Sun. He noticed that the perihelion of Mercury had a slow precession about its perihelion, which was not accounted for by Newton's theory. Le Verrier proposed the existence of another planet, even closer to the Sun that he named Vulcan, to explain why Mercury's orbit was so strange. Since then, physicists attempted to calculate the period of this precession and were able to get so close as to only have a remainder of 43 arcseconds/century unaccounted for. From Newtonian gravity, the equations of motion of a particle in orbit, known as Binet's equation, are written as

$$\frac{d^2u}{d\phi^2} + u = \frac{Gm}{h^2}, \tag{1.16}$$

where $u = 1/r$, ϕ is the polar angle and $h = L/m$ is a constant where L is the angular momentum.

In 1915 Einstein was able to apply his equations to the problem. Using GR, Einstein developed his own equation of motion for Mercury using general relativity, and was able to provide the equation,

$$\frac{d^2u}{d\phi^2} + u = \frac{Gm}{h^2} + \frac{3Gm}{c^2}u^2. \tag{1.17}$$

We can see the differences in the two equations, where there is an extra term in the equation, which perfectly provides the missing remainder of 43 arcseconds/century that was needed to match observational results.

1.2.2 Deflection of Light

Another one of Einstein's predictions from his theory of relativity is that light will bend through curved spacetimes. The Einstein equivalence principle states that an observer cannot tell the difference between being in uniform acceleration or being under the influence of gravity. Since gravity is related to spacetime curvature, Einstein proposed that light would follow null geodesics that follow the curvature of space. This means the light rays moving across space would not necessarily follow straight line paths and that light moving near a massive object would have a deflection as it passed near the massive object as it was drawn in by its curvature. The experiment to prove this was to utilize a solar eclipse. If one was able to measure the stars at night in the exact position that the solar eclipse was to occur, then during totality when the moon has completely eclipsed the sun one can take pictures again of the stars around the sun and compare the two pictures. Einstein calculated the deflection of light rays "which just graze the Sun" to be a 1.75 arcsecond deflection.

The perfect time to measure this phenomenon was approaching with the solar eclipse of 1919 [8]. This task was taken up by English astronomer Sir Arthur Eddington, where he took two expeditions to different spots to view the total solar eclipse. One team went to Sobral, Brazil while the other team, along with Eddington, went to the island of Príncipe in Africa. The teams designed to take pictures of the Hyades

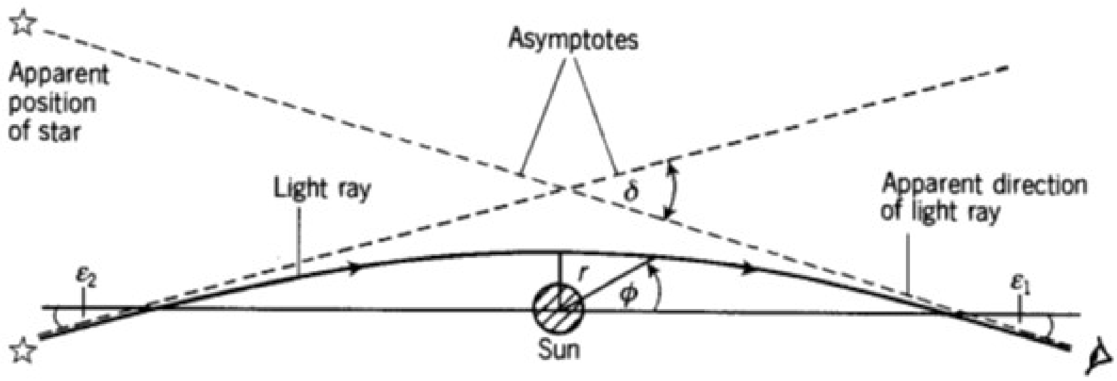

Figure 1.1: Light passing nearby a massive object with have its trajectory deflected. The result of such deflection cumulates in an apparent position of the light source that does not reflect its actual position in the sky [150].

star cluster, which would be directly behind the point where the solar eclipse was going to take place in the sky. Comparing the before and after pictures, Eddington and his team were able to conclude that the positions of the stars had changed between the two pictures, confirming the fact that massive bodies bend light rays.

1.2.3 Gravitational Redshift

The last of the classical tests put forth by Einstein to verify his theory of general relativity was the prediction of a gravitational redshift. Photons travelling out of a gravitational well should naturally lose some of their energy as they leave. The loss of energy will manifest in a change of frequency, shifting the light towards red. This prediction took a while to actually prove. In the 1959 two physicists, Robert Pound and Glen Rebka, developed an experiment to test it [9,10]. The setup was simple, to shoot photons from an iron (Fe) emitter upwards and measure the frequency shift in the spectral lines. The reason that this was difficult in the first place is that photons being emitted from the Fe emitter would exchange momentum with the nuclei which would lower a photon's overall energy, shifting it towards red.

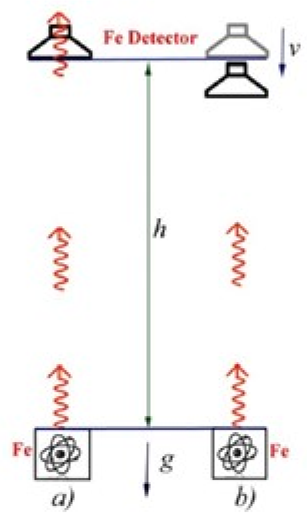

Figure 1.2: An illustration of the experiment carried out by Pound and Rebka. In a) one sees photons released by the Fe emitter being redshifted and passing through the detector at the other end. In b) we see the detector moving at speed v toward the Fe emitter where the photons are absorbed by the detector [11].

To truly test whether the redshift was due to gravitation or a Doppler effect, they made use of the Mössbauer effect, which allowed for recoil-free emissions of photons. This keeps a photon's energy the same as it leaves the lattice that the Fe emitter source was placed in, and ensures that any redshift would be due to a photon leaving the gravitational well.

In the first part of the experiment, Pound and Rebka emitted photons from the Fe emitter, which were gravitationally redshifted and passed straight through the stationary Fe detector. The experiment did not detect Fe spectral lines due to this gravitational red shift. In the second part of the experiment, the Fe detector was moving down at a constant velocity toward the emitter. Doing this creates a Doppler blueshift in the Fe spectral lines coming from the emitter. Pound and Rebka were able to calculate the speed at which the Fe detector should move in order to completely cancel out the effects of gravitational redshift. In this part of the experiment, the

Fe detectors were able to detect the spectral lines, indicating that the redshift was indeed from the photon moving through a gravitational well, confirming Einstein's prediction of gravitational red shift.

1.3 Predictions of General Relativity

There have been two predictions made by GR that have only been recently confirmed. Einstein's equations predicted the existence of black holes (BHs) and gravitational waves (GWs). A BH was theorized to be an object whose mass was so large that it creates an area of intense gravitational pull so strong that light cannot escape it. GWs are a direct result from linearizing the metric into background and perturbation terms, which produces wave equations. These GWs were found to propagate at the speed of light, and cause ripples in spacetime itself, bending and stretching space or any object that they pass through.

An important note before we move on, for the rest of the dissertation we shall adopt the convention of natural units setting $c = G = 1$ where c is the speed of light, and G is the gravitational constant. These units are useful in that we can treat all dimensions (t, x, y, z) with the same units of length which allows us to simplify equations.

1.3.1 Black Holes

The first person to find a solution to Einstein's field equations was Karl Schwarzschild, a German physicist, one year after the field equations were formulated in 1916. Schwarzschild developed the solution for a static mass, yielding the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2. \tag{1.18}$$

By examining this metric, we can notice that there are two areas for singularity found in it. The first being at $r = 0$, which would be the point at which the mass resides. The second singularity resides at the point $r = 2m$, which has become known as the Schwarzschild radius, r_s. This radius helps classify exactly what a black hole is, which is any object whose radius is smaller than its Schwarzschild radius. It was realized in 1958 that the Schwarzschild radius was the black hole's event horizon.

In 2020 physicists were first able to image the shadow of the M87 BH [64–69]. Imaging BHs is a difficult task because they are black and do not emit light themselves. That is not to say that the whole area around a BH is dark though; due to the high gravity in the area, BHs are able to capture light rays and put them into unstable circular orbits. The unstable orbits cause some of the photons to fall into the BH which can never be observed, and allows for some of the photons to escape just before they get to the event horizon. These pictures have confirmed the existence of BHs and have shown the validity of GR through its predictions of these objects.

1.3.2 Gravitational Waves

The last prediction of general relativity that we will discuss is the prediction of gravitational waves made in 1916, one year after GR was formulated. GWs are formed by linearly expanding the metric around a flat background metric,

$$g_{\mu\nu} = \gamma_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{1.19}$$

where $\gamma_{\mu\nu}$ is the flat background metric, and $h_{\mu\nu}$ is a perturbation in the space time. The smallness parameter ϵ corresponds to the ratio of the gravitational wavelength and the size of the observable universe, making $\epsilon \ll 1$. Expanding upon the Einstein field equations, one is able to produce a wave equation for the perturbation in the absence of matter, $T_{\mu\nu} = 0$,

$$\Box \bar{h}_{\mu\nu} = 0. \tag{1.20}$$

Here, the d'Alembert operator is defined as $\Box = -\partial_t^2 + \nabla^2$, and the trace-reversed perturbation is $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}h$, and h is the trace, defined as $h = \gamma^{\mu\nu}h_{\mu\nu}$. This equation shows that GWs, which are perturbations in spacetime, travel at the speed of light. These GWs have extremely small amplitudes and are difficult to detect without precise equipment.

Plane gravitational waves are parallel-propagated along null geodesics and oscillate in perpendicular to the direction of travel. The canonical form of the plane GW propagating along the z-direction takes the form

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{32} & -h_{22} \end{pmatrix}. \tag{1.21}$$

As the matrix shows, the GW relies only upon two functions, $h_{22}(t-z)$ and $h_{23}(t-z)$, which are functions of t and z, and propagates along the z-direction. We can infer from this that two polarization states arise from plane GWs in which the "+" polarization state is related to the h_{22} term, and the "×" polarization state is related to the h_{23} terms. These two polarization states create oscillation effects that can be measured by laser interferometers. If the GWs passed through a ring of test particles, then the "+" polarization and the "×" polarization have oscillations that looks like those in Figure 1.3.

In the 1960's, experiments were devised to create equipment that could actually detect these waves. The idea was that very precise laser interferometers would be

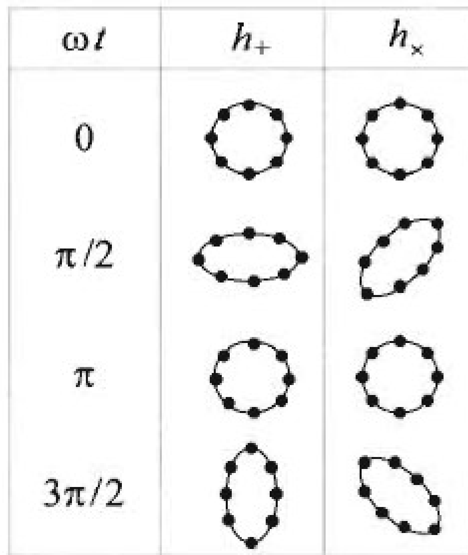

Figure 1.3: The effects of a gravitational wave passing through a ring of test particles. The passing wave displaces the particles depending on the + or × polarization [145].

able to detect the passing of a GW. Using laser interferometers set up perpendicular to each other, the lasers would be able to detect distortions in space as the GW passes. As the wave passes through the arms of the interferometer, one arm will stretch while the other shrinks. The variation in the arm length creates a phase difference between the two beams that can be measured. Using the laser interferometer, detectors were constructed around the 2000's including LIGO, Virgo, TAMA 300, and GEO 600. In 2016, LIGO was the first to confirm the existence of GWs with GW150914. LIGO, which uses interferometer arms 4 km long, was able to measure the GWs emitted in the merger of a binary black hole system, confirming the existence of gravitational waves [60–63]. The merger involved two black holes with masses, $m_1 = 29M_\odot$ and $m_2 = 36M_\odot$, where $M_\odot$ denotes the solar mass, $M_\odot \simeq 2 \times 10^{30} kg$. The coalescence of the two BHs consists of three phases. First is the orbital inspiral, which is the state that the two masses are in as they orbit around each other. When the two bodies

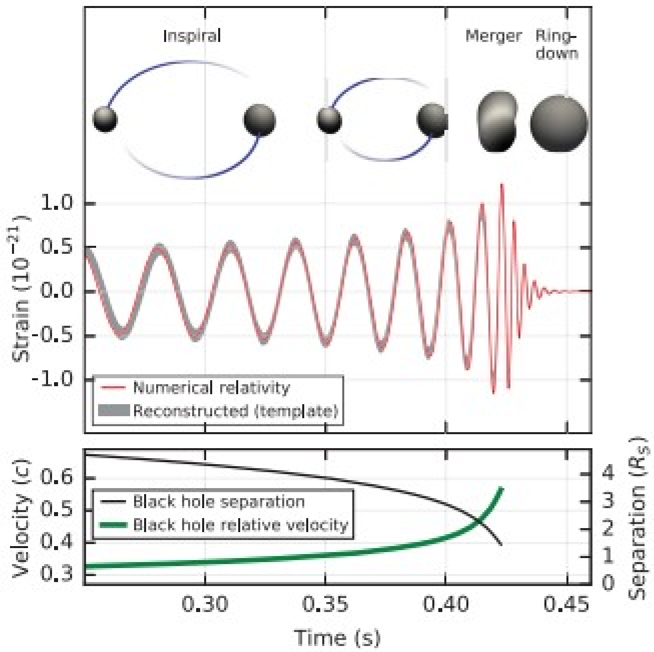

Figure 1.4: Data for the merger of GW150914. The frequency of the black holes GWs increases from 35Hz to 150Hz over 0.2s, where the amplitude reaches a maximum at their merger [17].

are sufficiently close to each other they go through the merger phase, where the two bodies are coalescing into one body. The last part is the ringdown phase where the new BH comes into equilibrium and the generation of GWs disappears.

1.4 Structure of the Dissertation

The rest of the dissertation will be organized as follows: Ch. 2 discusses the work found in [12]; the work was led by Tongzheng Wang under the supervision of Anzhong Wang, where Tongzheng and the rest of the authors completed the calculations and checked the work. Ch. 3 discusses the work found in [59]; the work was led by myself under the supervision of Anzhong Wang, with the other authors also contributing to the derivations and discussions of the research presented. Ch. 4 summarizes the results of the works presented as well as provide some final concluding remarks.

Ch. 2 gives a brief review over the singularities appearing in the BJR coordinates, and then the study of tidal forces and distortions felt by a typical class of observers, whose movements are confined within the $(u, v)-$plane, and show explicitly that tidal forces and distortions of these observers are finite in only two cases. Since a lot of studies of memory effects of GWs have been carried out in the Brinkmann coordinates, the singular behavior of the hypersurface $u = u_s$ in the Brinkmann coordinates will be considered and one will find that the singular behavior of the function $\mathcal{A}(u)$ is the only function to appear in the Brinkmann metric.

Ch. 3 begins by introducing the three different length scales, λ, L_c, and L and it is shown that for GWs to be detected by the current and foreseeable ground- and space-based detectors, such GWs can be well approximated as high frequency GWs. The Einstein field equations are then derived and one finds that, to make the backreaction of the GWs to the background negligible, as well as to have the linearized Einstein field equations for $h_{\mu\nu}$ to be valid, $|h_{\mu\nu}| \ll 1$ must hold. There is also a brief review on the cosmological background that consists of both the cosmological and tensor perturbations. The gauge conditions are next considered and it is shown that the three different gauge conditions, the Lorenz, traceless, and spatial gauges, can be imposed simultaneously, even when the background spacetime is not *in vacuo* as long as the high frequency approximations are valid. By imposing only the spatial gauge condition, one can write the Einstein field equations for the GWs. Next, we study the GWs with the geometrical optics approximation, and calculate the effects of the cosmological scalar and tensor perturbations on the amplitudes and phases of such GWs, and find the explicit expressions of the integrated Sachs-Wolfe effects due to both the cosmological scalar and tensor perturbations.

CHAPTER TWO

Singularities of Plane Gravitational Wave spacetimes in General Relativity

This chapter is published as [12]: T. Wang, J. R. Fier, B. Li, G. Lv, Z. Wang, Y. Wu, A. Wang," Singularities of plane gravitational waves and their memory effects," Gen. Relativity and Gravitation, 52 (2020) 21

2.1 Introduction

The memory effects of gravitational waves (GWs) have attracted a lot of attention (see, for example, [13–16] and references therein), especially after the recent observations of several GWs emitted from remote binary systems of either black holes [17–20] or neutron stars [21]. Such effects might possibly be detected by LISA [23] or even by the current generation of detectors, such as LIGO and aVIRGO [24]. Recently, such investigations gained new momentum due to the close relations between asymptotically symmetric theorems of soft gravitons and GW memory effects [25, 26]. The characteristic feature of these effects is the permanent displacement of a test particle after the burst of a GW passes [27–32]. In addition, the passage of the GW affects not only the position of the test particle, but also its velocity. In fact, the change in the velocity of the particle is also permanent [33–37]. When far from the sources, the emitted GWs can be well-approximated by plane GWs, a subject that has been extensively studied, including their nonlinear interactions [38–40]. The spacetimes for plane GWs can be cast in various forms, depending on the choice of the coordinates and gauge-fixing. One of these was originally due to Baldwin, Jeffery and Rosen (BJR) [41, 42]. Despite its several attractive features, the system of the BJR coordinates is often singular within a finite width of a wave, and when studying the asymptotic behavior of the spacetime, an extension beyond this singular surface is needed. In this, one can notice that there exist two kinds of singularities in plane gravitational wave spacetimes, one represents coordinate singularities, which can be removed by proper coordinate transformations, and the other represents spacetime singularities, and physical quantities, such as distortions of test particles, become infinitely large when such singularities are approaching. Therefore, in the latter case these singularities already represent the boundaries of the spacetimes and extensions beyond them are not only impossible, but also not needed. Since gravitational memory effects and soft graviton theorems are closely related to the asymptotic behaviors of plane GW spacetimes, in the latter the spacetimes cannot be used to study such properties.

In general relativity, there are powerful Hawking-Penrose theorems [43], from which one can see that spacetimes with quite "physically reasonable" conditions are singular. However, the theorems do not tell the nature of the singularities, and Ellis and Schmidt classified them into two different kinds, spacetime curvature singularities and coordinate singularities [44]. Spacetime curvature singularities are further divided into two sub-classes, scalar curvature singularities and non-scalar curvature singularities. If any of the fourteen independent scalars [48] constructed from the 4-dimensional Riemann tensor $R_{\mu\nu\lambda\sigma}$ and its derivatives is singular, then the spacetime is said to be singular, and the corresponding singularity is a scalar one. If none of these scalars is singular, spacetimes can be still singular. In particular, tidal forces and/or distortions (which are the double integrals of the tidal forces), experienced by an observer, may become infinitely large [49]. This kind of singularity is usually referred to as a non-scalar curvature singularity.

In the spacetimes of plane GWs, all the fourteen independent scalars vanish identically [38, 39], so in such spacetimes the singularities can be either non-scalar (but real spacetime) singularities or coordinate singularities. In this chapter, we shall clarify this important point, by studying tidal forces and distortions of freely falling observers. In particular, we find that the singularities can be in general characterized by

$$\chi(u) \equiv e^{-U(u)/2} = \left(u - u_s\right)^{\alpha} \hat{\chi}(u) \tag{2.1}$$

where the plane GWs are moving along the null direction of $u = Constant$, $\chi(u)$ is defined in Eq.(2.1), $\alpha > 0$, and $\hat{\chi}(u)$ is given by Eq.(2.25) with $\hat{\chi} \neq 0$. The Einstein vacuum field equations require $0 < \alpha \leq 1$. Then, one finds that the tidal forces and distortions are finite across the singular surface,

$$u = u_s, \tag{2.2}$$

only in two particular cases,

$$\begin{aligned} &(i)\ \alpha = \frac{1}{2},\ \forall\ \chi_n \\ &(ii)\ \alpha = 1,\ \chi_i = 0\ (i = 1, 2, 3), \end{aligned} \tag{2.3}$$

where χ_n are the coefficients appearing in the expansion in Eq.(2.25). Therefore, all the plane GW spacetimes are physically singular at the focused point $u = u_s$, the only exceptions being the ones with $\alpha = \frac{1}{2}$ or 1. As a result, all the plane GW spacetimes cannot be used to study memory effects and soft graviton theorems, except the ones with $\alpha = \frac{1}{2}$, 1, as only these spacetimes can possibly be extended to null infinity, whereby memory effects and soft graviton theorems can be studied.

It should be noted that, although the measurement of the physical states in the parameter space $\alpha \in (0, 1]$ is infinitesimal, there still exist an infinite number of

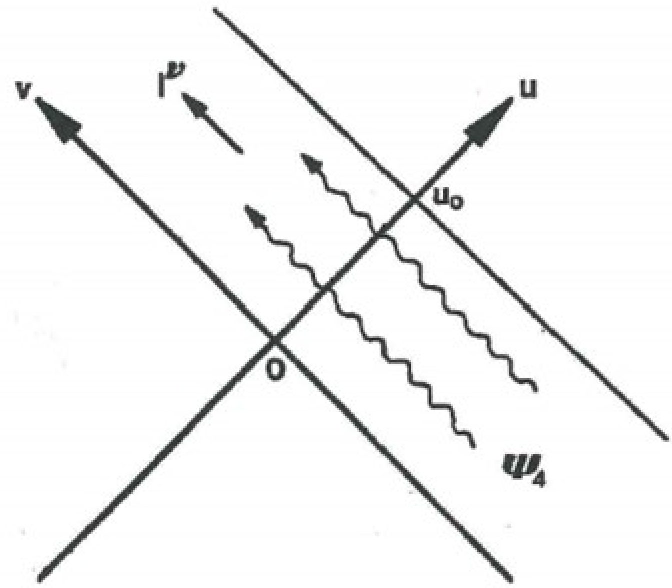

Figure 2.1: A plane gravitational wave moving along the null surfaces u = constant, with support only in the region $0 \leq u \leq u_0$, where Ψ_4 denotes the only non-vanishing component of the Weyl tensor, and is given by Eq.(2.6) in the linearly polarized case.

solutions of the Einstein field equations, which satisfy the asymptotic properties of Eq.(2.1) and (2.3) at the focusing point $u = u_s$. This can be seen clearly when one works in the Brinkmann coordinates [51], as to be shown below.

2.2 Singularities in spacetimes of Plane Gravitational Waves

The spacetimes for plane GWs in BJR coordinates can be cast in the form [38–40],

$$ds^2 = -2e^{-M}dudv + e^{-U}\left[e^{V}cosh(W)dy^2 - 2sinh(W)dydz + e^{-V}cosh(W)dz^2\right], \quad (2.4)$$

where M, U, V and W are functions of u only. The spacetime in general represents a plane GW moving along the null surfaces $u = Constant$ with two polarizations, one is along the y-axis, often referred to as the "+" polarization, and the other is along an axis which is at a 45° angle with respect to the y-axis, often referred to as the "×"

polarization. According to the Petrov classifications, the corresponding spacetimes belong to Petrov Type N [38, 39].

When the metric coefficients are functions of both u and v, an interesting phenomenon arises, the gravitational Faraday rotation, in this case, the medium is provided by the nonlinear interaction of the oppositely moving gravitational wave [40,52,152].

2.3 Linearly Polarized Plane Gravitational Waves

Note that by rescaling the null coordinate $u \to u' = \int e^{-M(u)} du$, without loss of the generality, one can always set,

$$M = 0, \tag{2.5}$$

a gauge that will be adopted in this chapter. In addition, for our current purpose, it is sufficient to consider only the linearly polarized case in which $W = 0$, so the metric takes the simple form

$$ds^2 = -2dudv + e^{-U(u)}\left(e^{V(u)}dy^2 + e^{-V(u)}dz^2\right). \tag{2.6}$$

It can be shown that the corresponding Riemann tensor has only two independent components, given, respectively, by

$$R_{uyuy} = \frac{1}{4}e^{-(U-V)}\left[2(U'' - V'') - (U' - V')^2\right] \tag{2.7}$$

$$R_{uzuz} = \frac{1}{4}e^{-(U+V)}\left[2(U'' + V'') - (U' + V')^2\right], \tag{2.8}$$

where $U' \equiv dU/du$, etc. All the fourteen independent scalars [48], made of the Riemann tensor and its derivatives, vanish identically [38, 39], so there are no scalar singularities in the spacetimes of plane GWs.

Decomposing the Riemann tensor into the Weyl and Ricci tensor [38,39], each has only one independent component. In particular, the independent component of the Ricci tensor is given by

$$R_{uu} = U'' - \frac{1}{2}\big(U'^2 + V'^2\big), \tag{2.9}$$

while the independent component of the Weyl tensor is given by

$$\Psi_4 \equiv -C_{\mu\nu\alpha\beta} n^\mu \bar{m}^\nu n^\alpha \bar{m}^\beta = -\frac{1}{2}A^2\big(V'' - U'V'\big) \tag{2.10}$$

which represents the plane GWs propagating along the hypersurfaces $u = Const.$, as illustrated in Figure 2.1, where

$$\begin{aligned} l^\mu \equiv& A^{-1}\delta^\mu_v, \\ n^\mu \equiv& A\delta^\mu_u, \\ m^\mu \equiv& \zeta^2\delta^\mu_2 + \zeta^3\delta^\mu_3, \\ \bar{m}^\mu \equiv& \bar{\zeta}^2\delta^\mu_2 + \bar{\zeta}^3\delta^\mu_3, \end{aligned} \tag{2.11}$$

form a null tetrad with A being an arbitrary function of u only, and

$$\zeta^2 \equiv \frac{e^{(U-V)/2}}{\sqrt{2}}, \;\; \zeta^3 \equiv i\frac{e^{(U+V)/2}}{\sqrt{2}}. \tag{2.12}$$

(an over bar denotes the complex conjugate). As noticed on various occasions, the BJR coordinates are not harmonic, typically not global, and contain coordinate singularities, see, for example, [14, 54, 55]. To overcome these problems, the Brinkmann coordinates $(\hat{u}, \hat{v}, \hat{y}, \hat{z})$ are often used, defined by

$$\begin{aligned} \hat{v} \equiv& v + \frac{1}{4}y^2 e^{V-U}\big(V' - U'\big) - \frac{1}{4}z^2 e^{-V-U}\big(V' + U'\big), \\ \hat{u} \equiv& u, \\ \hat{y} \equiv& e^{(V-U)/2}y, \\ \hat{z} \equiv& e^{-(V+U)/2}z, \end{aligned} \tag{2.13}$$

in terms of which, the metric (2.6) takes the form [51]

$$ds^2 = -2d\hat{u}d\hat{v} + d\hat{y}^2 + d\hat{z}^2 + \frac{1}{2}\mathcal{A}(\hat{u})\left(\hat{y}^2 - \hat{z}^2\right)d\hat{u}^2, \tag{2.14}$$

where

$$\mathcal{A}(\hat{u}) \equiv \frac{1}{2}\left[2\left(V'' - U''\right) + \left(V' - U'\right)^2\right]. \tag{2.15}$$

As mentioned previously, this chapter serves to point out that these singularities are not always due to the coordinates. In fact, all singularities are real spacetime singularities at the focused point $u = u_s$, except the ones that asymptotically behave as that given by Eqs.(2.1) and (2.3) at the neighborhood of the focused point. To verify this claim, one finds that it is easier to work in the BJR coordinates. Since the nature of singularities does not depend on the choice of coordinates, they must be singular in any coordinate system, including the Brinkmann system of coordinates.

2.4 *Spacetime Singularities*

In the vacuum case, the Einstein field equations $R_{\mu\nu} = 0$ have only one independent component, given by $R_{uu} = 0$, and from Eq.(2.9) we find that it can be written as

$$\chi'' + \omega^2\chi = 0, \tag{2.16}$$

where

$$\chi \equiv e^{-U/2}, \;\; \omega \equiv \frac{1}{2}V'. \tag{2.17}$$

Then, from Eq.(2.16) one can see that, for any given initial value, $\chi(u_s) > 0$, there always exists a moment, say, $u = u_s$ at which χ vanishes [14],

$$\chi(u_s) = 0, \;\; or \;\; U(u_s) = +\infty, \tag{2.18}$$

that is, a singularity of the metric (2.6) appears at $u = u_s$, which is surely not a scalar singularity, since, as mentioned above, all the fourteen independent scalars made of the Riemann tensor in such spacetimes vanish identically. Even from this, one cannot immediately tell if the singularity is a coordinate or physical singularity, as non-scalar spacetime singularities may be present [44]. In particular, distortions of a freely falling observer, which is the double integral of the tidal force with respect to the proper time of the observer, can diverge [49].

To calculate distortions of a freely falling observer, first consider the trajectory of the observer. In the present chapter, we just consider trajectories in the (u, v)-plane, that is, $(u, v, y, z) = (u(\lambda), v(\lambda), y_0, z_0)$, where λ denotes the proper time of the observer, and y_0 and z_0 are constants. Then, the time-like geodesics are simply given by

$$u = \gamma_0 \lambda, \;\; v = \frac{\lambda}{2\gamma_0}, \;\; , y = y_0, \;\; z = z_0, \tag{2.19}$$

where γ_0 is an integration constant. One will define $e^{\mu}_{(0)} \equiv dx_{\mu}/d\lambda$, one can construct a tetrad, $e^{\mu}_{(a)}$ $(a = 0, 1, 2, 3)$ by

$$\begin{aligned} e^{\mu}_{(0)} =& \gamma_0 \delta^{\mu}_{u} + \frac{1}{2\gamma_0}\delta^{\mu}_{v}, \\ e^{\mu}_{(1)} =& \gamma_0 \delta^{\mu}_{u} - \frac{1}{2\gamma_0}\delta^{\mu}_{v}, \\ e^{\mu}_{(2)} =& e^{(U-V)/2}\delta^{\mu}_{y}, \\ e^{\mu}_{(3)} =& e^{(U+V)/2}\delta^{\mu}_{z}, \end{aligned} \tag{2.20}$$

which satisfies the relations

$$e^{\mu}_{(\alpha)} e^{\nu}_{(\beta)} g_{\mu\nu} = \eta_{\alpha\beta}, \;\; e^{\mu}_{(\alpha);v} e^{v}_{(0)} = 0. \tag{2.21}$$

That is, they are unit orthogonal vectors and are parallelly transported along the time-like geodesics, so that they form a freely falling frame [49]. Then, the projection of the Riemann tensor onto this frame

$$R_{(a)(b)(c)(d)} \equiv R_{\mu\nu\lambda\rho} e^{\mu}_{(a)} e^{\nu}_{(b)} e^{\lambda}_{(c)} e^{\rho}_{(d)}, \tag{2.22}$$

yields two independent components:

$$\begin{aligned} R_{(0)(2)(0)(2)} &= \gamma_0^2 e^{U-V} R_{uyuy}, \\ R_{(0)(3)(0)(3)} &= \gamma_0^2 e^{U+V} R_{uzuz}, \end{aligned} \tag{2.23}$$

where $R_{\mu\nu\lambda\rho}$'s are given by Eqs.(2.7) and (2.8).

To study the nature of the singularities at $u = u_s$, one assumes that in the neighborhood of $u = u_s$, the function χ takes the form,

$$\chi(u) = (u - u_s)^{\alpha} \hat{\chi}(u), \tag{2.24}$$

where $\alpha > 0$ and $\hat{\chi}(u_s) \neq 0$. Thus, expanding it as

$$\hat{\chi}(u) = \sum_{n=0}^{\infty} \chi_n (u - u_s)^n, \tag{2.25}$$

one must assume that $\chi_0 \neq 0$, since $\hat{\chi}(u_s) \neq 0$. Then, using Eqs.(2.16) and (2.17) it is shown that,

$$\begin{aligned} V' =& \left(- \frac{4\chi''}{\chi} \right)^{1/2} \\ =& \frac{2}{u - u_s} \left[\alpha(1-\alpha) - 2\alpha(u - u_s)\frac{\hat{\chi}'}{\hat{\chi}} - (u - u_s)^2 \frac{\hat{\chi}''}{\hat{\chi}} \right]^{1/2}, \\ U =& - 2ln(\chi) = -2\alpha ln(u - u_s) - 2ln\big[\hat{\chi}(u)\big]. \end{aligned} \tag{2.26}$$

Note that in writing the above expression for V' we had chosen the plus sign, without loss of generality. To study the singular behavior of the solutions at the focused point further, it is found to be convenient to consider the cases with and without $\alpha = 1$ separately.

2.4.1 α=1

In this case, inserting Eq.(2.25) into Eq.(2.26), one obtains,

$$\begin{aligned}
V' =& \frac{2\sqrt{-\frac{2\chi_1}{\chi_0}}}{(u-u_s)^{1/2}}\sum_{n=0}^{\infty} v_n (u-u_s)^n,\\
V'' =& \frac{2\sqrt{-\frac{2\chi_1}{\chi_0}}}{(u-u_s)^{3/2}}\sum_{n=0}^{\infty} (n-\frac{1}{2}) v_n (u-u_s)^n,\\
U' =& -\frac{1}{u-u_s}\left[1+\frac{\chi_1}{\chi_0}(u-u_s)-\frac{\chi_1^2-2\chi_0\chi_2}{\chi_0^2}(u-u_s)^2+...\right],
\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}
v_0 =& 1,\\
v_1 =& -\frac{\chi_1^2-3\chi_0\chi_2}{2\chi_0\chi_1},\\
v_2 =& \frac{3\chi_1^4-10\chi_0\chi_1^2\chi_2-9\chi_0^2\chi_2^2+24\chi_0^2\chi_1\chi_3}{8\chi_0^2\chi_1^2},
\end{aligned} \tag{2.28}$$

and χ_n are coefficients appearing in Eq. (2.25). Hence, from Eqs.(2.7), (2.8) and (2.23) one finds,

$$\begin{aligned}
R_{(0)(2)(0)(2)} = -R_{(0)(3)(0)(3)} &= \frac{1}{2}\gamma_0^2(U'V'-V'')\\
&= \frac{3\sqrt{-\chi_1/(2\chi_0)}\gamma_0^2}{(u-u_s)^{3/2}} + \frac{3(\chi_1^2+5\chi_0\chi_2)\gamma_0^2}{2\chi_0^2\sqrt{-2\chi_1/\chi_0}(u-u_s)^{1/2}} + \mathcal{O}\Big((u-u_s)^{1/2}\Big),
\end{aligned} \tag{2.29}$$

and,

$$\int d\lambda \int d\lambda R_{(0)(2)(0)(2)}(\lambda) = 6\sqrt{\frac{-2\chi_1\gamma_0}{\chi_0}}(\lambda-\lambda_s)^{1/2} + \mathcal{O}\Big((\lambda-\lambda_s)^{3/2}\Big), \tag{2.30}$$

which is finite as $\lambda \to \lambda_s$, where $\lambda_s \equiv u_s/\gamma_0$.

2.4.2 $\alpha \neq 1$

In this case, one can write the functions U and V as,

$$\begin{aligned}
V' =& \frac{2}{u-u_s}\sum_{n=0}^{\infty} v_n (u-u_s)^n, \\
V'' =& \frac{2}{(u-u_s)^2}\sum_{n=0}^{\infty}(n-1) v_n (u-u_s)^n, \\
U' =& -\frac{2}{u-u_s}\left[\alpha + \frac{\chi_1}{\chi_0}(u-u_s) - \frac{\chi_1^2 - 2\chi_0\chi_2}{\chi_0^2}(u-u_s)^2 + ...\right],
\end{aligned} \tag{2.31}$$

but now with the coefficients defined as

$$\begin{aligned}
v_0 =& \sqrt{\alpha(1-\alpha)}, \\
v_1 =& -\frac{\chi_1 \alpha}{\chi_0\sqrt{\alpha(1-\alpha)}}, \\
v_2 =& -\frac{\alpha\left[\chi_1^2\alpha(2\alpha-1) + 2\chi_0\chi_2(1+\alpha-2\alpha^2)\right]}{2\chi_0^2\left[\alpha(1-\alpha)\right]^{3/2}}.
\end{aligned} \tag{2.32}$$

Clearly, to have the metric coefficient V be real, one must assume that $0 < \alpha < 1$. Then, one can find

$$\begin{aligned}
R_{(0)(2)(0)(2)} = -R_{(0)(3)(0)(3)} =& \frac{1}{2}\gamma_0^2(U'V' - V'') \\
=& \frac{\gamma_0^2(1-2\alpha)\sqrt{\alpha(1-\alpha)}}{(u-u_s)^2} \\
&+ \frac{2\chi_1\alpha(2\alpha-1)\gamma_0^2}{\chi_0\sqrt{\alpha(1-\alpha)}(u-u_s)} \\
&- \frac{\alpha\gamma_0^2}{2\chi_0^2\left[\alpha(1-\alpha)\right]^{3/2}}\Big\{\chi_1^2\alpha\big(-7+12\alpha-8\alpha^2\big) + 2\chi_0\chi_2\big(-1+\alpha-8\alpha^2+8\alpha^3\big)\Big\} \\
&+ \mathcal{O}(u-u_s).
\end{aligned} \tag{2.33}$$

Note that only the first term leads to a divergence in the distortions. In fact, the distortions are written as

$$\int d\lambda \int d\lambda R_{(0)(2)(0)(2)} = (2\alpha-1)\sqrt{\alpha(1-\alpha)}ln(\lambda-\lambda_s) + \mathcal{O}\Big[(\lambda-\lambda_s)ln(\lambda-\lambda_s)\Big], \tag{2.34}$$

for $0 < \alpha < 1$. Clearly, the tidal forces are always singular unless $\alpha = 1/2$. Combining the above with the case $\alpha = 1$, one concludes that unless

$$(i)\ \alpha = \frac{1}{2},\ or\ (ii)\ \alpha = 1, \tag{2.35}$$

the singularities located at the focused point $u = u_s$ are always real spacetime singularities.

2.5 Singularities in Brinkmann Coordinates

As mentioned previously, gravitational memory effects are frequently studied in the Brinkmann coordinates. Thus, it is of interest to see how the metric behaves in the neighborhood of $u = u_s$ in the Brinkmann coordinates. From Eqs. (2.17) and (2.24), it is found that

$$\begin{aligned} U =& -2\alpha ln(u-u_s) - 2ln\big(\hat{\chi}(u)\big),\\ V'^2 =& \frac{4\alpha(1-\alpha)}{(u-u_s)^2} - \frac{4}{\hat{\chi}}\left(\hat{\chi}'' + \frac{2\alpha\hat{\chi}'}{u-u_s}\right), \end{aligned} \tag{2.36}$$

where $\hat{\chi}(u)$ is expanded in the neighborhood of $u = u_s$ as given by Eq.(2.25).

In the vacuum, $R_{\mu\nu} = 0$, Eq.(2.16)) holds, from which it is written

$$2U'' - U'^2 = V'^2. \tag{2.37}$$

Then, Eq.(2.15) reduces to,

$$\mathcal{A}(u) = V'' - U'V'. \tag{2.38}$$

Note that in writing the above expression one uses the coordinate transformations (2.13), from which $u = \hat{u}$. Inserting Eqs.(2.36) and (2.25) into Eq.(2.38), the behavior of $\mathcal{A}(u)$ in the neighborhood of $u = u_s$ can be found.

It is interesting to note that tidal forces between two nearby null geodesics of $\hat{v}, \hat{y}, \hat{z} = Const.$ were studied in the Brinkmann coordinates (2.14) in [56], where it was determined that $R_{y\hat{u}y\hat{u}} = \mathcal{A}(\hat{u})/2$ describes diffeomorphism-invariant curvature information. Therefore, the divergence of $\mathcal{A}(\hat{u})$ at the focusing point $u = u_s$ implies the existence of a spacetime singularity. In the following it is shown that this is consistent with the conclusions obtained in the last section for the case $\alpha = 1/2$, and helps to understand the case $\alpha = 1$ in more detail. One finds that it is convenient to consider the cases, (i) $0 < \alpha < 1$, $\alpha \neq 1/2$; (ii) $\alpha = 1/2$; and (iii) $\alpha = 1$, separately.

2.5.1 $0 < \alpha < 1$, $\alpha \neq 1/2$

In this case, inserting Eqs.(2.36) and (2.25) into Eq.(2.15), one finds that,

$$\mathcal{A}(u) = \sum_{n=-2}^{\infty} \mathscr{A}_n (u - u_s)^n, \tag{2.39}$$

where the first three coefficients that show the singular behavior of $\mathcal{A}(u)$ are given by

$$\begin{aligned}
\mathscr{A}_{-2} &= -2(1-2\alpha)\sqrt{\alpha(1-\alpha)}, \\
\mathscr{A}_{-1} &= \frac{4\chi_1\alpha(1-2\alpha)}{\chi_0\sqrt{\alpha(1-\alpha)}}, \\
\mathscr{A}_0 &= -\frac{\alpha}{\left[\alpha(1-\alpha)\right]^{3/2}\chi_0^2}\Big(\chi_1^2\alpha(7-12\alpha+8\alpha^2) \\
&\quad + 2\chi_0\chi_2(1-\alpha+8\alpha^2-8\alpha^3)\Big).
\end{aligned} \tag{2.40}$$

Since $0 < \alpha < 1$ and $\alpha \neq 1/2$, along with $\mathscr{A}_{-2} \neq 0$, the leading divergent term now is $(u-u_s)^{-2}$, and $\mathcal{A}(u)$ behaves as

$$\mathcal{A}(u) = \frac{\mathscr{A}_{-2}(\alpha)}{(u-u_s)^2} + \frac{\mathscr{A}_{-1}(\alpha)}{(u-u_s)} + \mathscr{A}_0(\chi_0, \chi_1, \chi_2) + \mathcal{O}(u-u_s) \tag{2.41}$$

in the neighborhood of $u = u_s$, where $\mathscr{A}_{-1}(\alpha)$ is a function of α only, which is also non-zero for $0 < \alpha < 1$ and $\alpha \neq 1/2$, as it can be seen from Eq.(2.40). As mentioned in the last section, the spacetime now is singular, and no extension beyond this surface is possible, so $u = u_s$ represents a real boundary of the spacetime. This is consistent with the analysis in the Brinkmann coordinates given in [56].

From Eq.(2.36) it is shown that

$$\begin{aligned} U(u) =& -2\alpha ln(u-u_s) + \hat{U}(u), \\ V(u) =& 2\sqrt{\alpha(1-\alpha)} ln(u-u_s) + \hat{V}(u), \end{aligned} \tag{2.42}$$

where $\hat{U}$ and $\hat{V}$ are regular and finite functions of u across the hypersurface $u = u_s$. Note that in writing down the above expressions, the positive sign of $\hat{V}$ is used without loss of generality, as was previously done. In addition, $\hat{U}$ and $\hat{V}$ are not independent, as they must satisfy the field equation (2.37).

2.5.2 $\alpha = 1/2$

In this case, the singularity at $u = u_s$ is a coordinate singularity, which can be removed by the coordinate transformations of Eq.(2.13), and the resulted metric is the Brinkmann metric (2.14) with

$$\mathcal{A}(u) = \sum_{n=0}^{\infty} \mathscr{B}_n (u-u_s)^n, \tag{2.43}$$

where the first term $\mathscr{B}_0$ is given by

$$\mathscr{B}_0 = -\frac{6\left(\chi_1^2 + 2\chi_0\chi_2\right)}{\chi_0^2}. \tag{2.44}$$

Clearly, in this case $\mathcal{A}(u)$ is well-behaved in the neighborhood of $u = u_s$, and the Brinkmann metric (2.14) can be considered as its extension beyond the hypersurface $u = u_s$. If $\mathcal{A}(u)$ is obtained in such a manner, then the extension is unique. Again, this is consistent with the analysis in the Brinkmann coordinates presented in [56].

On the other hand, from Eq.(2.36) one finds that

$$
\begin{aligned}
U(u) &= -ln(u-u_s) + \hat{U}(u), \\
V(u) &= ln(u-u_s) + \hat{V}(u), \ (\alpha = 1/2),
\end{aligned}
\tag{2.45}
$$

where $\hat{U}$ and $\hat{V}$ are regular and finite functions of u across the hypersurface $u = u_s$, and are related each other through Eq.(2.37).

2.5.3 $\alpha = 1$

In this case, from Eq.(2.36), it is found that,

$$
\begin{aligned}
U =& -2ln(u-u_s) + \hat{U}(u), \\
V'^2 =& -\frac{\hat{\chi}'}{\hat{\chi}} \frac{8}{u-u_s} - 4\frac{\hat{\chi}''}{\hat{\chi}},
\end{aligned}
\tag{2.46}
$$

where $\hat{\chi}$ takes the form of Eq.(2.25) with $\chi_0 \neq 0$. Thus, depending on values of χ_1, χ_2 and χ_3, the function $\mathcal{A}(u)$ can have different singular behaviors. Therefore, in the following, they will be considered separately. In the case that $\chi_1 \neq 0$, one finds that

$$
\begin{aligned}
U =& -2ln(u-u_s) + \hat{U}(u), \\
V =& 4\sqrt{2}\mathcal{D}_1 (u-u_s)^{1/2} + \mathcal{O}\Big((u-u_s)^{3/2}\Big), \\
\mathcal{A} =& \frac{1}{(u-u_s)^{3/2}} \sum_{n=0}^{\infty} \mathscr{C}_n (u-u_s)^n,
\end{aligned}
\tag{2.47}
$$

where, $\mathcal{D}_1 \equiv \sqrt{-\chi_1/\chi_0}$, $\hat{U}$ is regular and finite functions of u across the hypersurface $u = u_s$, and the leading terms of $\mathscr{C}_n$ that clearly shows the singular behavior of $\mathcal{A}$ are given by

$$
\begin{aligned}
\mathscr{C}_0 =& 3\sqrt{2}\mathcal{D}_2, \\
\mathscr{C}_1 =& -\frac{3}{\sqrt{2}\chi_0\mathcal{D}_1} \big(\chi_1^2 + 5\chi_0\chi_2\big), \\
\mathscr{C}_2 =& -\frac{3}{4\sqrt{2}\chi_0^4\mathcal{D}_1^3} \big(9\chi_1^4 - 14\chi_0\chi_1^2\chi_2 + 21\chi_0^2\chi_2^2 - 56\chi_0^2\chi_1\chi_3\big).
\end{aligned}
\tag{2.48}
$$

It is interesting to note that in the current case the Brinkmann metric is still singular at $u = u_s$, although the distortions felt by the freely falling observers defined by Eq.(2.19) are all finite. Hence, now there are two possibilities: (i) Distortions felt by other freely falling observers diverge at $u = u_s$, so the singularity is a real spacetime singularity, and the spacetime cannot be extended beyond this surface. (ii) Distortions felt by any of freely falling observers are finite, and the singularity is a coordinate one. The results given in [56], show that the current case belongs to the first possibility, as the tidal forces for two nearby null geodesics of $\hat{v}$, $\hat{y}$, $\hat{z} = Constant$ become unbounded at $u = u_s$.

The second case to consider is when $\chi_1 = 0$ and $\chi_2 \neq 0$, from which one finds,

$$\begin{aligned}
U =& - 2ln(u - u_s) + \hat{U}(u),\\
V =& 2\sqrt{6}\mathcal{D}_2(u - u_s) + \mathcal{O}\Big((u - u_s)^2\Big),\\
\mathcal{A} =& \frac{1}{u - u_s}\sum_{n=0}^{\infty}\mathscr{D}_n(u - u_s)^n,
\end{aligned} \tag{2.49}$$

where $\mathscr{D}_2 \equiv \sqrt{-\chi_2/\chi_0}$, $\hat{U}$ is regular and finite functions of u across the hypersurface $u = u_s$, and the leading terms of $\mathscr{D}_n$ are given by

$$\begin{aligned}
\mathscr{D}_0 &= 4\sqrt{6}\mathcal{D}_2,\\
\mathscr{D}_1 &= \frac{6\sqrt{6}\mathcal{D}_2\chi_3}{\chi_2},\\
\mathscr{D}_2 &= -\frac{4\sqrt{2}\mathcal{D}_2^3}{\chi_2^3}\big(3\chi_2^3 - 3\chi_0\chi_3^2 + 10\chi_0\chi_2\chi_4\big).
\end{aligned} \tag{2.50}$$

Thus, the Brinkmann metric is now also singular near the hypersurface $u = u_s$, and the corresponding spacetimes are physically singular at $u = u_s$.

The next case is for $\chi_1 = \chi_2 = 0$ and $\chi_3 \neq 0$ for which one finds,

$$\begin{aligned}
U =& -2ln(u-u_s) + \hat{U}(u),\\
V =& \frac{8\sqrt{3}}{3}\mathcal{D}_3(u-u_s)^{3/2} + \mathcal{O}\Big((u-u_s)^{5/2}\Big),\\
\mathcal{A} =& \frac{1}{(u-u_s)^{1/2}}\sum_{n=0}^{\infty}\mathscr{E}_n(u-u_s)^n,
\end{aligned} \tag{2.51}$$

where $\mathcal{D}_3 \equiv \sqrt{-\chi_3/\chi_0}$ and

$$\begin{aligned}
\mathscr{E}_0 =& 10\sqrt{3}\mathcal{D}_3,\\
\mathscr{E}_1 =& \frac{35\mathcal{D}_3\chi_4}{\sqrt{3}\chi_3}.
\end{aligned} \tag{2.52}$$

Again, in this case the Brinkmann metric is also singular, and a spacetime curvature singularity is developed on the focusing hypersurface $u = u_s$.

The last case to look at is $\chi_1 = \chi_2 = \chi_3 = 0$ and $\chi_4 \neq 0$, where it is found that

$$\begin{aligned}
U =& -2ln(u-u_s) + \hat{U}(u),\\
V =& 2\sqrt{5}\mathcal{D}_4(u-u_s)^2 + \mathcal{O}\Big((u-u_s)^3\Big),\\
\mathcal{A} =& \sum_{n=0}^{\infty}\mathscr{G}_n(u-u_n)^n,
\end{aligned} \tag{2.53}$$

where $\mathcal{D}_4 \equiv \sqrt{\chi_4/\chi_0}$ and $\mathscr{G}_0 = 12\sqrt{5}\mathcal{D}_4$. In this case, it is clear that the Brinkmann metric becomes non-singular, and Eq.(2.13) represents an extension of the singular BJR metric (2.6) beyond the hypersurface $u = u_s$. So, in this case it is sure that the singularity encountered in the BJR metric is a coordinate one, and the Brinkmann metric (2.14) is one of its extensions. Note that the extension will be unique, if $\mathcal{A}(u)$ is analytical across $u = u_s$.

2.6 Examples of $\mathcal{A}(u)$

In the studies of gravitational wave memory effects, several interesting cases have been considered. For example, in [14, 57], the function $\mathcal{A}(u)$ was chosen as

$$\mathcal{A}(u) = \frac{1}{2}\frac{d^3 e^{-u^2}}{du^3} = 2u(3 - 2u^2)e^{-u^2}. \tag{2.54}$$

With this definition of $\mathcal{A}(u)$, Eqs.(2.15) and (2.37) become

$$\begin{aligned} 2(V'' - U'') + (V' - U')^2 &= 2\mathcal{A}(u), \\ 2U'' - U'^2 &= V'^2, \end{aligned} \tag{2.55}$$

which can be solved to find U and V. However, due to the non-linearity of these equations, usually it is difficult to find analytical solutions. In [14], it was numerically found that the singularity in the BJR coordinates occurs at $u_s \simeq 0.593342$. From Eq.(2.57) one can see that $\mathcal{A}(u)$ is finite and well behaved in the neighborhood of this point. So, it must belong to either the case with $\alpha = 1/2$, or the case with $\alpha = 1$ and $\chi_i = 0$ $(i = 1, 2, 3)$. Due to the high non-linearity between the BJR and Brinkmann coordinates, it is difficult to get a definite answer. Some modified versions of the above example were considered in [15, 37, 58].

Another example is the case with [16],

$$\mathcal{A}(u) = \frac{2}{\pi}\frac{\varepsilon^2}{(u^2 + \varepsilon^2)^2}, \tag{2.56}$$

where ε is a constant. When ε is very small, the above expression gives rise to an impulse of gravitational waves, recently studied in [58]. Clearly, in all of these models $\mathcal{A}(u)$ is always finite and well-behaved across the singularity located at $u = u_s$ in the BJR coordinates, so they all belong to the non-singular cases presented in the current chapter.

CHAPTER THREE

Gravitational Wave Cosmology in the High Frequency Approximation

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3.1 Introduction

The detection of the first gravitational wave (GW) from the coalescence of two massive black holes (BHs) by the advanced Laser Interferometer Gravitational-Wave Observatory (aLIGO) marked the beginning of a new era, *the GW astronomy* [60]. Following this observation, soon more than 50 GWs were detected by the LIGO/Virgo scientific collaboration [61–63]. The increased interest on GWs and BHs has further gained momentum after the detection of the shadow of the M87 BH [64–69].

One of the remarkable observational results is the discovery that the mass of an individual BH in these binary systems can be much larger than that which was previously expected, both theoretically and observationally [70–72], leading to the proposal and refinement of various formation scenarios (see, for example, [73–76], and references therein). A consequence of this discovery is that the early inspiral phase may also be detectable by space-based observatories, such as LISA [77], TianQin [78], Taiji [79], and DECIGO [80], for several years prior to their coalescence [81,82]. Multiple observations with different detectors which are sensitive to different frequencies in the signals from the same source can provide an excellent opportunity to study the evolution of a binary in detail. Since different detectors observe at disjoint frequency bands, together they cover different evolutionary stages of the same binary system.

Each stage of the evolution carries information about different physical aspects of the source. As a result, multi-band GW detections will provide an unprecedented opportunity to test different theories of gravity in the strong field regime [83].

Recently, some of the present authors generalized the post-Newtonian (PN) formalism to certain modified theories of gravity and applied it to the quasi-circular inspiral of compact binaries. In particular, we calculated in detail the waveforms, GW polarizations, response functions and energy losses due to gravitational radiation in Brans-Dicke (BD) theory [84], screened modified gravity (SMG) [85–87], and gravitational theories with parity violations [88–91] to the leading PN order, with which we then considered projected constraints from the third-generation detectors. Such studies have been further generalized to triple systems [92, 93] in Einstein-aether (æ-) theory [94–96]. When applying such formulas to the first relativistic triple system discovered in 2014 [97], a study of the radiation power found that quadrupole emission has almost the same amplitude as that in general relativity (GR), but the dipole emission can be as large as the quadrupole emission. This can provide a promising window to place severe constraints on æ-theory with multi-band GW observations [98, 99].

More recently, the problem of a binary system of non-spinning bodies in a quasi-circular inspiral within the framework of æ-theory [100–105], was examined and the results provided the explicit expressions for the time-domain and frequency-domain waveforms, GW polarizations, and response functions for both ground- and space-based detectors in the PN approximation [106]. In particular, when going beyond the leading order in the PN approximation, the non-Einsteinian polarization modes contain terms that depend on both the first and second harmonics of the orbital phase. With this in mind, an analytical calculation of the corresponding parameterized post-Einsteinian parameters was made, generalizing the existing framework to allow for different propagation speeds among scalar, vector and tensor modes, without assuming the magnitude of the coupling parameters, and meanwhile allowing the binary system to have relative motion with respect to the æther field. Such results will in particular allow for the easy construction of Einstein-æther templates that could be used in Bayesian tests of GR in the future.

It is remarkable to note that the space-based detectors mentioned above, together with the current and forthcoming ground-based ones, such as KAGRA [107], Voyager [108], the Einstein Telescope (ET) [109] and Cosmic Explorer (CE) [110], are able to detect GWs emitted from such systems with the redshift as great as $z \simeq 100$ [111], which will have profound scientific consequences. In particular, GWs propagating over such long cosmic distances will carry valuable information not only about their sources, but also about the detail of the cosmological expansion and inhomogeneities of the universe, whereby a completely new window to explore the universe by using GWs is opened. So far our understanding of the universe comes almost entirely from observations of electromagnetic waves (possibly with the important exceptions of cosmic rays and neutrinos) [112].

In this chapter, a generalization is made of the studies above to the cases in which the GWs are first generated by remote astrophysical sources and then propagate in the inhomogeneous universe through cosmic distances before arriving at either space- and/or ground-based detectors. It should be noted that such studies have recently attracted lots of attention (see, for example, [113] and references therein). In particular, using Isaacson's high frequency GW formulas [114, 115], Laguna *et* *al* studied the gravitational analogue of the electromagnetic integrated Sachs-Wolf (iSW) effects in cosmology, and found that the phase, frequency, and amplitude of the GWs experience iSW effects, in addition to the magnifications on the amplitude from gravitational lensing [116]. More recently, Bertacca *et al* connected the results of Laguna *et al* obtained in real space frame to the observed frame, by using the cosmic rulers formulas [117], whereby the corrections to the luminosity distance due to velocity, volume, lensing and gravitational potential effects were calculated [118].

On the other hand, Bonvin *et al* [119] studied the effects of the universe on the gravitational waveform, and found that the acceleration of the universe and the peculiar acceleration of a binary with respect to the observer distort the gravitational chirp signals from the simplest GR prediction, not only a mere time independent rescaling of the chirp mass, but also the intrinsic parameter estimations for binaries visible by LISA. In particular, the effect due to the peculiar acceleration can be much larger than the one due to the universe acceleration. Moreover, peculiar accelerations can introduce a bias in the estimation of parameters such as the time of coalescence and the individual masses of the binary. An error in the estimation of the time of coalescence made by LISA will have an impact on the prediction of the time at which the signal will be visible by ground based interferometers, for signals spanning both frequency bands. The correlations of such GWs with lensing fields from the cosmic microwave background and galaxies were studied [120], which led to a new window to explore our universe by gravitational weak lensing was proposed. Lately, GWs propagating in the curved universe has been further generalized to scalar-tensor theories [121], including Horndeski [122–124] and SMG [124] theories.

It should be noted that in all these studies, the cosmological tensor perturbations have been neglected. As observing primordial GWs (the tensor perturbations) is one of the main goals in the current and forthcoming cosmological observations [125], in this chapter we shall consider the cosmological background that consists of both the scalar and tensor perturbations, but restrict ourselves only to Einstein's theory, and leave the generalizations to other theories of gravity to other occasions. To describe the GWs propagating through the inhomogeneous universe from cosmic distances to observers properly, we first introduce three scales, λ, L_c and L, which denote, respectively, the typical wavelength of GWs, the scale of the cosmological perturbations, and the size of the observable universe. For GWs to be detected by the current and foreseeable detectors, we find that the condition

$$\lambda \ll L_c \ll L, \tag{3.1}$$

always holds. Such GWs can be approximated as *high-frequency GWs* and the signals can be well separated from the background $\gamma_{\mu\nu}$ by averaging the spacetime curvatures over a scale ℓ, where $\lambda \ll \ell \ll L_c$, and the total metric of the spacetime is given by

$$g_{\mu\nu} = \gamma_{\mu\nu} + \epsilon h_{\mu\nu}, \tag{3.2}$$

where $\epsilon \equiv \lambda/L$, and $\gamma_{\mu\nu}$ denotes the background, while $h_{\mu\nu}$ represents the GWs. In order for the backreaction of the GWs to the background spacetimes to be negligible, we must assume that $|h_{\mu\nu}| \ll 1$, in addition to the condition $\epsilon \ll 1$, which are also the conditions for the linearized Einstein field equations for $h_{\mu\nu}$ to be valid.

Such studies can be significantly simplified by properly imposing gauge conditions, such as *the spatial, traceless, and Lorenz gauges*, given, respectively, by

$$\chi_{0\mu} = 0, \tag{3.3}$$

$$\chi = 0, \tag{3.4}$$

$$\nabla^{\nu}\chi_{\mu\nu} = 0, \tag{3.5}$$

where

$$\chi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}h, \quad h \equiv \gamma^{\mu\nu}h_{\mu\nu}, \tag{3.6}$$

and ∇^{ν} denotes the covariant derivative with respect to $\gamma_{\mu\nu}$. We show that these three different gauge conditions can be imposed simultaneously, even when the background is not vacuum, as long as the high-frequency GW approximations are valid.

However, to develop the formulas that can be applicable to as many cases as possible, in this chapter the linearized Einstein field equations are written down explicitly for $\chi_{\mu\nu}$ by imposing only the spatial gauge. Applying these formulas together with the geometrical optic approximations to such GWs, one finds that they still move along null geodesics and the polarization bi-vector is parallel-transported, even when both the cosmological scalar and tensor perturbations are present. In addition, the gravitational integrated Sachs-Wolfe (iSW) effects was also calculated due to these two kinds of perturbations, whereby the dependencies of the amplitude, phase and luminosity distance of the GWs on these perturbations are read off explicitly.

Before proceeding to the next section, it should be noted that GWs produced by remote astrophysical sources and then propagating through the homogeneous and isotropic universe have been systematical studied by Ashtekar and his collaborators through a series of papers [126–132], and various subtle issues were clarified in the deSitter background [133–135]. In particular, in this chapter the signature of the metric is (-,+,+,+), while the Christoffel symbols, Riemann and Ricci tensors, as well as the Ricci scalar, are defined in Chapter 1. The covariant derivative ∇_α denotes the full covariant derivative with respect to the metric $g_{\mu\nu}$, and defines

$$\big(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\big)X^\alpha = R^\alpha{}_{\beta\mu\nu}X^\beta. \tag{3.7}$$

The Einstein field equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}, \tag{3.8}$$

where $\kappa \equiv 8\pi G$, with G denoting the Newtonian constant, and c the speed of light. In addition to ∇_α, the covariant derivative is introduced as $\bar{\nabla}_\alpha$ with respect to the homogeneous metric $\bar{\gamma}_{\mu\nu}$, where

$$\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} + \epsilon_c\hat{\gamma}_{\mu\nu}, \tag{3.9}$$

with $\epsilon_c \equiv L_c/L \ll 1$. The conventions, $A_{(\mu\nu)} \equiv \big(A_{\mu\nu} + A_{\nu\mu}\big)/2$ and $A_{[\mu\nu]} \equiv \big(A_{\mu\nu} - A_{\nu\mu}\big)/2$ are also adopted.

3.2 Gravitational Waves Propagating in an Inhomogeneous Universe

In this section, we shall consider GWs first produced by remote astrophysical sources and then propagating in cosmic distances through the inhomogeneous universe, before arriving at detectors. To study such GWs, first consider several characteristic lengths that are highly relevant to their generations and propagation and polarizations.

3.2.1 Characteristic Scales of the Background

In this section, the inhomogeneous universe is to be considered the background which includes two parts, the homogeneous and isotropic universe and its inhomogeneous perturbations, given by $\bar{\gamma}_{\mu\nu}$ and $\hat{\gamma}_{\mu\nu}$, respectively, so the background metric $\gamma_{\mu\nu}$ can be written as

$$\begin{aligned}\gamma_{\mu\nu} =& \bar{\gamma}_{\mu\nu} + \epsilon_c \hat{\gamma}_{\mu\nu} + \mathcal{O}\left(\epsilon_c^2\right), \\ \gamma^{\mu\nu} =& \bar{\gamma}^{\mu\nu} - \epsilon_c \hat{\gamma}^{\mu\nu} + \mathcal{O}\left(\epsilon_c^2\right),\end{aligned} \tag{3.10}$$

where $\epsilon_c,\ |\hat{\gamma}| \ll 1$, and

$$\begin{aligned}&\gamma^{\mu\lambda}\gamma_{\nu\lambda} = \delta^\mu_\nu + \mathcal{O}\left(\epsilon_c^2\right), \\ &\bar{\gamma}^{\mu\lambda}\bar{\gamma}_{\nu\lambda} = \delta^\mu_\nu + \mathcal{O}\left(\epsilon_c^2\right), \\ &\hat{\gamma}^\mu_\nu \equiv \bar{\gamma}^{\mu\alpha}\hat{\gamma}_{\alpha\nu}, \\ &\hat{\gamma}^{\mu\nu} \equiv \bar{\gamma}^{\mu\alpha}\bar{\gamma}^{\nu\beta}\hat{\gamma}_{\alpha\beta},\end{aligned} \tag{3.11}$$

and so on.

The size of the observational universe is about $L \simeq 8.8 \times 10^{26}\ m$. On the other hand, in the momentum space of the cosmological perturbations, $L_c \simeq 1/k$, where k denotes the typical wavenumber of the perturbations, and L_c the length over which the change of the cosmological perturbations becomes appreciable. When the modes are outside the Hubble horizon, it can be shown that $L_c/L \simeq 10^{-5}$. But, once they re-enter the horizon these modes decay suddenly and then oscillate rapidly about a minimum [146]. In addition, the density perturbation $\delta\rho$ is the order of $\delta\rho/\bar{\rho} \simeq 10^{-5}$. So, it is quite reasonable to assume that,

$$\epsilon_c \simeq \frac{L_c}{L} \ll 1. \tag{3.12}$$

3.2.2 Typical Gravitational Wavelengths

For the second generation of the ground-based detectors, such as LIGO, Virgo, and KAGRA, the wavelength of the detected GWs are $\lambda \simeq 10^5 \sim 10^7\ m$, while the wavelength of GWs to be detected by the space-based detectors, such as LISA, TianQin and Taiji, are $\lambda \simeq 10^8 \sim 10^{12}\ m$. Therefore, the ground-based detectors have $\epsilon \simeq \lambda/L \in (10^{-22},\ 10^{-20})$, while the space-based detectors have $\epsilon \in (10^{-19},\ 10^{-15})$. Thus, for the GWs to be detected by the current and foreseeable detectors, the following is always true:

$$\frac{\lambda}{L_c} = \frac{\epsilon}{\epsilon_c} \ll 1. \tag{3.13}$$

Therefore, *all such GWs can be well approximated as high frequency GWs with respect to the distance over which the inhomogeneities of the Universe change significantly.*

3.2.3 Einstein Field Equations

Following the above analyses, one finds that λ, L_c and L denote, respectively, the characteristic length over which $h_{\mu\nu}$, $\hat{\gamma}_{\mu\nu}$ or $\bar{\gamma}_{\mu\nu}$ changes significantly. Thus, their derivatives are typically of the orders

$$\begin{aligned}
&\partial\bar{\gamma} \sim \frac{\bar{\gamma}}{L}, \quad \partial^2\bar{\gamma} \sim \frac{\bar{\gamma}}{L^2}, \\
&\partial\hat{\gamma} \sim \frac{\hat{\gamma}}{L_c}, \quad \partial^2\hat{\gamma} \sim \frac{\hat{\gamma}}{L_c^2}, \\
&\partial h \sim \frac{h}{\lambda}, \quad \partial^2 h \sim \frac{h}{\lambda^2}.
\end{aligned} \tag{3.14}$$

To estimate orders of terms, following Isaacson [114], we regard L as order of unity, and say that the metric (3.2) contains a high-frequency GW, if and only if there exists a family of coordinate systems (related by infinitesimal coordinate transformations), in which one finds

$$\epsilon \ll \epsilon_c \ll 1, \tag{3.15}$$

and

$$\begin{aligned}
&\bar{\gamma}_{\mu\nu},\ \bar{\gamma}_{\mu\nu,\alpha},\ \bar{\gamma}_{\mu\nu,\alpha\beta} \simeq \mathcal{O}(1),\\
&\hat{\gamma}_{\mu\nu} \simeq \mathcal{O}\left(\hat{\gamma}\right), \quad \hat{\gamma}_{\mu\nu,\alpha} \simeq \mathcal{O}\left(\hat{\gamma}/\epsilon_c\right),\\
&\hat{\gamma}_{\mu\nu,\alpha\beta} \simeq \mathcal{O}\left(\hat{\gamma}/\epsilon_c^2\right),\\
&h_{\mu\nu} \simeq \mathcal{O}\left(h\right), \quad h_{\mu\nu,\alpha} \simeq \mathcal{O}\left(h/\epsilon\right),\\
&h_{\mu\nu,\alpha\beta} \simeq \mathcal{O}\left(h/\epsilon^2\right),
\end{aligned} \tag{3.16}$$

where $\gamma_{\mu\nu,\alpha} \equiv \partial\gamma_{\mu\nu}/\partial x^{\alpha}$, etc. Note that, in contrast to [114], here one does not assume $h_{\mu\nu} \simeq \mathcal{O}(1)$, in order to neglect the backreaction of the GWs to the background spacetime $\gamma_{\mu\nu}$, as to be shown below.

Expanding the Riemann and Ricci tensors $R_{\mu\nu\alpha\beta}\left(g_{\mu\nu}\right)$ and $R_{\mu\nu}\left(g_{\mu\nu}\right)$ in terms of ϵ, one finds [114, 145],

$$\begin{aligned}
R_{\alpha\beta\gamma\delta}\left(g_{\mu\nu}\right) &= {R_{\alpha\beta\gamma\delta}}^{(0)} + \epsilon {R_{\alpha\beta\gamma\delta}}^{(1)} + \epsilon^2 {R_{\alpha\beta\gamma\delta}}^{(2)} + \mathcal{O}\left(\epsilon^3\right),\\
R_{\alpha\beta}\left(g_{\mu\nu}\right) &= {R_{\alpha\beta}}^{(0)} + \epsilon {R_{\alpha\beta}}^{(1)} + \epsilon^2 {R_{\alpha\beta}}^{(2)} + \mathcal{O}\left(\epsilon^3\right),
\end{aligned} \tag{3.17}$$

where

$$
\begin{aligned}
{R_{\alpha\beta\gamma\delta}}^{(0)} &= R_{\alpha\beta\gamma\delta}\left(\gamma_{\mu\nu}\right), \\
{R_{\alpha\beta\gamma\delta}}^{(1)} &= \frac{1}{2}\Big[h_{\beta\gamma;\alpha\delta} + h_{\alpha\delta;\beta\gamma} - h_{\alpha\gamma;\beta\delta} - h_{\beta\delta;\alpha\gamma} \\
&\qquad + {R_{\alpha\sigma\gamma\delta}}^{(0)} h^{\sigma}_{\beta} - {R_{\beta\sigma\gamma\delta}}^{(0)} h^{\sigma}_{\alpha}\Big], \qquad (3.18) \\
{R_{\alpha\beta}}^{(0)} &= R_{\alpha\beta}\left(\gamma_{\mu\nu}\right), \\
{R_{\alpha\beta}}^{(1)} &= \frac{1}{2}\gamma^{\rho\tau}\Big(h_{\tau\alpha;\beta\rho} + h_{\tau\beta;\alpha\rho} \\
&\qquad - h_{\rho\tau;\alpha\beta} - h_{\alpha\beta;\rho\tau}\Big), \qquad (3.19) \\
{R_{\alpha\beta}}^{(2)} &= \frac{1}{4}\Big\{{h^{\rho\tau}}_{;\beta} h_{\rho\tau;\alpha} + 2h^{\rho\tau}\big(h_{\tau\rho;\alpha\beta} + h_{\alpha\beta;\tau\rho} \\
&\qquad - h_{\tau\alpha;\beta\rho} - h_{\tau\beta;\alpha\rho}\big) + 2h^{\tau;\rho}_{\beta}\big(h_{\tau\alpha;\rho} - h_{\rho\alpha;\tau}\big) \\
&\qquad - \left(2{h^{\rho\tau}}_{;\rho} - h^{;\tau}\right)\left(h_{\tau\alpha;\beta} + h_{\tau\beta;\alpha} - h_{\alpha\beta;\tau}\right)\Big\}. \qquad (3.20)
\end{aligned}
$$

Here the semi-colon ";" denotes the covariant derivative with respect to the background metric $\gamma_{\mu\nu}$. For the sake of convenience, we shall also use ∇_λ to denote the covariant derivative with respect to $\gamma_{\mu\nu}$, so one finds $h_{\mu\nu;\lambda} \equiv \nabla_\lambda h_{\mu\nu}$, etc. The background metric $\gamma_{\mu\nu}$ ($\gamma^{\mu\nu}$) is also used to lower (raise) the indices of $h_{\mu\nu}$, such as

$$
\begin{aligned}
h^{\mu}_{\nu} &\equiv \gamma^{\mu\alpha} h_{\alpha\nu} = \gamma_{\nu\alpha} h^{\mu\alpha}, \\
h &\equiv h^{\lambda}_{\lambda} = \gamma^{\alpha\beta} h_{\alpha\beta}, \qquad (3.21)
\end{aligned}
$$

and so on.

The background curvatures ${R_{\alpha\beta\gamma\delta}}^{(0)}(\gamma)$ and ${R_{\alpha\beta}}^{(0)}(\gamma)$ can be further expanded in terms of ϵ_c, as

$$
\begin{aligned}
{R_{\alpha\beta\gamma\delta}}^{(0)}(\gamma) &= \bar{R}_{\alpha\beta\gamma\delta}(\bar{\gamma}) + \epsilon_c \hat{R}_{\alpha\beta\gamma\delta}(\hat{\gamma}) + \epsilon_c^2 \hat{R}^{(2)}_{\alpha\beta\gamma\delta}(\hat{\gamma}) + \mathcal{O}\left(\epsilon_c^3\right), \\
{R_{\alpha\beta}}^{(0)}(\gamma) &= \bar{R}_{\alpha\beta}(\bar{\gamma}) + \epsilon_c \hat{R}_{\alpha\beta}(\hat{\gamma}) + \epsilon_c^2 \hat{R}^{(2)}_{\alpha\beta}(\hat{\gamma}) + \mathcal{O}\left(\epsilon_c^3\right), \qquad (3.22)
\end{aligned}
$$

where

$$\hat{R}_{\alpha\beta\gamma\delta}(\hat{\gamma}) = \frac{1}{2}\Big[\hat{\gamma}_{\beta\gamma|\alpha\delta} + \hat{\gamma}_{\alpha\delta|\beta\gamma} - \hat{\gamma}_{\alpha\gamma|\beta\delta} - \hat{\gamma}_{\beta\delta|\alpha\gamma} + \bar{R}_{\alpha\sigma\gamma\delta}\hat{\gamma}^{\sigma}_{\beta} - \bar{R}_{\beta\sigma\gamma\delta}\hat{\gamma}^{\sigma}_{\alpha}\Big], \tag{3.23}$$

$$\hat{R}_{\alpha\beta}(\hat{\gamma}) = \frac{1}{2}\bar{\gamma}^{\rho\tau}\Big(\hat{\gamma}_{\tau\alpha|\beta\rho} + \hat{\gamma}_{\tau\beta|\alpha\rho} - \hat{\gamma}_{\rho\tau|\alpha\beta} - \hat{\gamma}_{\alpha\beta|\rho\tau}\Big), \tag{3.24}$$

and $\hat{R}^{(2)}_{\alpha\beta}(\hat{\gamma})$ is given by Eq.(3.20) with the replacement $(h_{\alpha\beta}, \nabla_\mu) \to \big(\hat{\gamma}_{\alpha\beta}, \bar{\nabla}_\mu\big)$. Here the vertical bar "|" denotes the covariant derivative with respect to $\bar{\gamma}_{\mu\nu}$, which is also denoted by $\bar{\nabla}_\lambda$, so that $\hat{\gamma}_{\rho\tau|\alpha} \equiv \bar{\nabla}_\alpha\hat{\gamma}_{\rho\tau}$, etc. Taking $L \simeq \mathcal{O}(1)$ and considering Eq.(3.16) one finds

$$\bar{R}^{\alpha}{}_{\beta\gamma\delta},\ \bar{R}_{\alpha\beta} \simeq \mathcal{O}(1), \tag{3.25}$$

$$\epsilon_c\hat{R}^{\alpha}{}_{\beta\gamma\delta},\ \epsilon_c\hat{R}_{\alpha\beta} \sim \mathcal{O}\left(\hat{\gamma}/\epsilon_c\right),$$

$$\epsilon_c^2\hat{R}^{(2)}_{\alpha\beta\gamma\delta},\ \epsilon_c^2\hat{R}^{(2)}_{\alpha\beta} \simeq \mathcal{O}\left(\hat{\gamma}^2\right), \tag{3.26}$$

$$\epsilon R_{\alpha\beta\gamma\delta}{}^{(1)},\ \epsilon R_{\alpha\beta}{}^{(1)} \simeq \mathcal{O}\left(h/\epsilon\right),$$

$$\epsilon^2 R_{\alpha\beta\gamma\delta}{}^{(2)},\ \epsilon^2 R_{\alpha\beta}{}^{(2)} \simeq \mathcal{O}\left(h^2\right). \tag{3.27}$$

To write down the Einstein field equations, first note that

$$\left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right)\chi_{\gamma\delta} = -R^{\sigma}{}_{\gamma\alpha\beta}{}^{(0)}\chi_{\sigma\delta} - R^{\sigma}{}_{\delta\alpha\beta}{}^{(0)}\chi_{\gamma\sigma}. \tag{3.28}$$

Then, one finds that in terms of $\chi_{\mu\nu}$, $R_{\alpha\beta}{}^{(1)}$ is given by

$$R_{\alpha\beta}{}^{(1)} = \frac{1}{2}\Big(2R_{\gamma\alpha\beta\sigma}{}^{(0)}\chi^{\gamma\sigma} + R^{\sigma(0)}_{\alpha}\chi_{\beta\sigma} + R^{\sigma(0)}_{\beta}\chi_{\alpha\sigma} + \nabla_\alpha\nabla^\delta\chi_{\beta\delta} + \nabla_\beta\nabla^\delta\chi_{\alpha\delta}\Big) - \frac{1}{2}\Box\chi_{\alpha\beta} + \frac{1}{4}\gamma_{\alpha\beta}\Box\chi, \tag{3.29}$$

where $\Box\chi_{\alpha\beta} \equiv \gamma^{\mu\nu}\chi_{\alpha\beta;\mu\nu}$, and

$$\chi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}h,$$
$$\chi \equiv \gamma^{\mu\nu}\chi_{\mu\nu} = -h. \tag{3.30}$$

It should be noted that in [114] Isaacson considered the vacuum case, for which ${R_{\alpha\beta}}^{(1)} = 0$, that is

$$\Box\chi_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}\Box\chi - \nabla_\alpha\nabla^\delta\chi_{\beta\delta} - \nabla_\beta\nabla^\delta\chi_{\alpha\delta}$$
$$+ 2{R_{\alpha\gamma\beta\sigma}}^{(0)}\chi^{\gamma\sigma} - R_\alpha^{\sigma(0)}\chi_{\beta\sigma} - R_\beta^{\sigma(0)}\chi_{\alpha\sigma} = 0, \tag{3.31}$$

which is precisely Eq.(5.7) of [114], after the differences between the conventions used here and the ones used in [114] are taken into account.

In the present chapter we consider the propagation of GWs through the inhomogeneous universe, which has non-zero Riemann and Ricci tensors. So, we expect that the corresponding Einstein field equations for $h_{\mu\nu}$ are different from Eq.(3.31). To see this, first note that

$$g^{\mu\nu} = \gamma^{\mu\nu} - \epsilon h^{\mu\nu} + \epsilon^2 h^\mu_\alpha h^{\alpha\nu} + O\left(\epsilon^3\right),$$
$$R \equiv g^{\mu\nu}R_{\mu\nu} = R^{(0)} + \epsilon R^{(1)} + \epsilon^2 R^{(2)} + O\left(\epsilon^3\right), \tag{3.32}$$

where

$$R^{(0)} \equiv \gamma^{\mu\nu}R^{(0)}_{\mu\nu},$$
$$R^{(1)} \equiv \gamma^{\mu\nu}R^{(1)}_{\mu\nu} - h^{\mu\nu}R^{(0)}_{\mu\nu}$$
$$= \nabla^\alpha\nabla^\beta\chi_{\alpha\beta} - \chi^{\alpha\beta}R^{(0)}_{\alpha\beta} + \frac{1}{2}\left(\Box + R^{(0)}\right)\chi,$$
$$R^{(2)} \equiv \gamma^{\mu\nu}R^{(2)}_{\mu\nu} - h^{\mu\nu}R^{(1)}_{\mu\nu} + h^\mu_\alpha h^{\alpha\nu}R^{(0)}_{\mu\nu}. \tag{3.33}$$

Inserting Eqs.(3.17) and (3.32) into the Einstein field equations, one finds that

$$\begin{aligned}
R^{(0)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R^{(0)} &+ \epsilon\left[R^{(1)}_{\mu\nu} - \frac{1}{2}\left(\gamma_{\mu\nu}R^{(1)} + h_{\mu\nu}R^{(0)}\right)\right] \\
&+ \epsilon^2\left[R^{(2)}_{\mu\nu} - \frac{1}{2}\left(\gamma_{\mu\nu}R^{(2)} + h_{\mu\nu}R^{(1)}\right)\right] + O\left(\epsilon^3\right) \\
&= \kappa\left(T^{(0)}_{\mu\nu} + \epsilon\mathcal{T}_{\mu\nu}\right),
\end{aligned} \tag{3.34}$$

where $T^{(0)}_{\mu\nu}$ denote the energy-momentum tensor that produces the background, while $\mathcal{T}_{\mu\nu}$ denotes the astrophysical source that produces the GWs.

3.2.4 Separation of GWs from the Background

To separate GWs produced by astrophysical sources from the inhomogeneous background, one can average the field equations over a length scale ℓ, which is much larger than the typical wavelength of the GWs but much smaller than L_c,

$$\lambda \ll \ell \ll L_c. \tag{3.35}$$

Then, this process will extract the slowly varying background from GWs, as the latter will vanish when averaging over such a scale. In particular,

$$\begin{aligned}
&\langle\gamma_{\mu\nu}\rangle = \gamma_{\mu\nu}, \ \left\langle {R_{\mu\nu\alpha\beta}}^{(0)}\right\rangle = {R_{\mu\nu\alpha\beta}}^{(0)}, \\
&\left\langle {R_{\mu\nu}}^{(0)}\right\rangle = {R_{\mu\nu}}^{(0)}, \ \left\langle {T_{\mu\nu}}^{(0)}\right\rangle = {T_{\mu\nu}}^{(0)},
\end{aligned} \tag{3.36}$$

$$\langle h_{\mu\nu}\rangle = \left\langle {R_{\mu\nu}}^{(1)}\right\rangle = \left\langle R^{(1)}\right\rangle = 0, \tag{3.37}$$

$$\begin{aligned}
&\left\langle {R_{\mu\nu}}^{(2)}\right\rangle = \left\langle {R_{\mu\nu}}^{(2)}\right\rangle_\ell, \ \left\langle R^{(2)}\right\rangle = \left\langle R^{(2)}\right\rangle_\ell, \\
&\left\langle h_{\mu\nu}R^{(1)}\right\rangle = \left\langle h_{\mu\nu}R^{(1)}\right\rangle_\ell, \ \left\langle\mathcal{T}_{\mu\nu}\right\rangle = \left\langle\mathcal{T}_{\mu\nu}\right\rangle_\ell.
\end{aligned} \tag{3.38}$$

Note that quadratic terms of $h_{\mu\nu}$ may survive such an averaging process, if two modes are almost equal but with different signs, although each of them represents a high frequency mode. For example, for $h_{\mu\nu} \propto e^{i\omega_1 x}$ and $h_{\alpha\beta} \propto e^{-i\omega_2 x}$, one finds

$h_{\mu\nu}h_{\alpha\beta} \propto e^{i\omega_{12}x}$, where $\omega_{12} \equiv \omega_1 - \omega_2$. Thus, although $\omega_1,\ \omega_2 \gg 1$, one can have $\omega_{12} \ll 1$, if $\omega_1 \simeq \omega_2$. Therefore, due to the nonlinear interactions among different modes, low frequency modes can be produced, which will survive with such averaging processes. If one is only interested in the linearized Einstein field equations of $h_{\mu\nu}$, such modes must be taken of care properly. With this in mind, taking the average of Eq.(3.34) the equations are written as

$$R^{(0)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R^{(0)} + \epsilon^2 \left\langle {G_{\mu\nu}}^{(2)} \right\rangle_\ell = \kappa \left(T^{(0)}_{\mu\nu} + \epsilon \left\langle \mathcal{T}_{\mu\nu} \right\rangle_\ell \right), \tag{3.39}$$

where

$${G_{\mu\nu}}^{(2)} \equiv R^{(2)}_{\mu\nu} - \frac{1}{2}\left(\gamma_{\mu\nu}R^{(2)} + h_{\mu\nu}R^{(1)}\right), \tag{3.40}$$

which is a quadratic function of $h_{\mu\nu}$. Then, substituting Eqs.(3.39) and (3.40) back to Eq.(3.34), the high-frequency part takes the form

$$\begin{aligned} R^{(1)}_{\mu\nu} - &\frac{1}{2}\left(\gamma_{\mu\nu}R^{(1)} + h_{\mu\nu}R^{(0)}\right) \\ &+\epsilon \left\langle G^{(2)}_{\mu\nu} \right\rangle^{\text{high}} = \kappa \left\langle \mathcal{T}_{\mu\nu} \right\rangle^{\text{high}}, \end{aligned} \tag{3.41}$$

where

$$\begin{aligned} \left\langle G^{(2)}_{\mu\nu} \right\rangle^{\text{high}} &\equiv G^{(2)}_{\mu\nu} - \left\langle {G_{\mu\nu}}^{(2)} \right\rangle_\ell, \\ \left\langle \mathcal{T}_{\mu\nu} \right\rangle^{\text{high}} &\equiv \mathcal{T}_{\mu\nu} - \left\langle \mathcal{T}_{\mu\nu} \right\rangle_\ell. \end{aligned} \tag{3.42}$$

On the other hand, from Eqs.(3.25)-(3.27) one finds that

$$\begin{aligned} G^{(0)}_{\mu\nu} \equiv R^{(0)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R^{(0)} \simeq \mathcal{O}\left(\hat{\gamma}/\epsilon_c\right), \\ \left\langle {G_{\mu\nu}}^{(2)} \right\rangle_\ell \simeq \mathcal{O}\left(h^2/\epsilon^2\right), \quad T^{(0)}_{\mu\nu} \simeq \mathcal{O}\left(\epsilon_c^{-1}\right). \end{aligned} \tag{3.43}$$

Note that, after introducing the cosmological perturbation scale L_c, the leading order of $G^{(0)}_{\mu\nu}$ becomes $G^{(0)}_{\mu\nu} \simeq \epsilon_c \hat{R}_{\mu\nu} \simeq \mathcal{O}\left(\hat{\gamma}/\epsilon_c\right)$, instead of L^{-2} [121]. The same is true for

$T^{(0)}_{\mu\nu}$, as it can be seen from Appendix A. Then, from Eq.(3.39) one finds that each term has the following order

$$\mathcal{O}\left(\hat{\gamma}/\epsilon_c\right) + \mathcal{O}\left(h^2\right) = \mathcal{O}\left(\hat{\gamma}/\epsilon_c\right) + \epsilon\mathcal{O}\left(\langle\mathcal{T}_{\mu\nu}\rangle_\ell\right). \tag{3.44}$$

Therefore, to have the backreaction of the GWs to the background be negligible, so that the background spacetime $\gamma_{\mu\nu}$ is uniquely determined by $T^{(0)}_{\mu\nu}$, i.e.,

$$R_{\mu\nu}{}^{(0)} - \frac{1}{2}\gamma_{\mu\nu}R^{(0)} = \kappa T_{\mu\nu}{}^{(0)}, \tag{3.45}$$

one must assume that

$$h^2 \ll \frac{\hat{\gamma}}{\epsilon_c}, \tag{3.46}$$

$$\epsilon \cdot \left|\langle\mathcal{T}_{\mu\nu}\rangle_\ell\right| \ll \frac{\hat{\gamma}}{\epsilon_c}. \tag{3.47}$$

In addition, from Eq.(3.41) one finds that

$$\epsilon\left\langle G^{(2)}_{\mu\nu}\right\rangle^{\text{high}} \simeq \mathcal{O}\left(h^2/\epsilon\right). \tag{3.48}$$

Therefore, in order for the quadratic terms from $G^{(2)}_{\mu\nu}$ not to affect the linear terms of the leading orders h/ϵ^2 and h/ϵ^1 in Eq.(3.41), one must assume that

$$|h| \ll 1. \tag{3.49}$$

With the above conditions, one finds that Eq.(3.41) can be written as

$$\begin{aligned}\Box\chi_{\alpha\beta}+&\gamma_{\alpha\beta}\nabla^\gamma\nabla^\delta\chi_{\gamma\delta} - \nabla_\alpha\nabla^\delta\chi_{\beta\delta} - \nabla_\beta\nabla^\delta\chi_{\alpha\delta}\\ &+2R_{\alpha\gamma\beta\sigma}{}^{(0)}\chi^{\gamma\sigma} = \kappa\Big(\mathcal{F}_{\alpha\beta} - 2\left\langle\mathcal{T}_{\alpha\beta}\right\rangle^{\text{high}}\Big).\end{aligned} \tag{3.50}$$

where

$$
\begin{aligned}
\mathcal{F}_{\alpha\beta} \equiv & \frac{1}{\kappa}\Big\{R^{\sigma}{}_{\alpha}{}^{(0)}\chi_{\beta\sigma} + R^{\sigma}{}_{\beta}{}^{(0)}\chi_{\alpha\sigma} - \chi_{\alpha\beta}R^{(0)} \\
& + \gamma_{\alpha\beta}\chi^{\gamma\delta}R_{\gamma\delta}{}^{(0)}\Big\} \\
= & \chi_{\beta\delta}T^{\delta}{}_{\alpha}{}^{(0)} + \chi_{\alpha\delta}T^{\delta}{}_{\beta}{}^{(0)} + \gamma_{\alpha\beta}\chi^{\gamma\delta}T_{\gamma\delta}{}^{(0)} \\
& - \frac{1}{2}\gamma_{\alpha\beta}\chi T^{(0)}.
\end{aligned} \tag{3.51}
$$

From the above derivations, one can see that the linearized Einstein field equations (3.50) are valid only to the two leading orders, ϵ^{-2} and ϵ^{-1}. For orders higher than this, these equations are not applicable. This is particularly true for the zeroth-order of ϵ. In addition, since $\epsilon_c^{-1} \ll \epsilon^{-1}$, one finds that in Eq.(3.50) the term

$$
\mathcal{F}_{\alpha\beta},\ 2R_{\alpha\gamma\beta\sigma}{}^{(0)}\chi^{\gamma\sigma} \simeq \mathcal{O}(\hat{\gamma}h/\epsilon_c) \ll \mathcal{O}(h/\epsilon), \tag{3.52}
$$

can also be neglected in comparing the terms that are of orders ϵ^{-2} or ϵ^{-1}. However, in order to compare these results with the ones obtained in [114–116], we shall keep it and drop the corresponding terms only at the end of our calculations.

3.2.5 The Inhomogeneous Universe

In this subsection, we shall give a very brief introduction over the flat FRW universe with its linear scalar and tensor perturbations, described by the metric (A.11). In terms of the conformal coordinates $x^{\mu} = (\eta, x^i)$, $(i = 1, 2, 3)$, the metrics are defined as

$$
\begin{aligned}
\bar{\gamma}_{\mu\nu} &= a^2(\eta)\eta_{\mu\nu}, \\
\bar{\gamma}^{\mu\nu} &= a^{-2}(\eta)\eta^{\mu\nu},
\end{aligned} \tag{3.53}
$$

with $\eta_{\mu\nu} = \text{diag}\,(-1, +1, +1, +1)$. The coordinate η is related to the cosmic time via the relation, $\eta = \int \frac{dt}{a(t)}$.

Following the standard process, one decomposes the linear perturbations $\hat{\gamma}_{\mu\nu}$ into scalar, vector and tensor modes,

$$\hat{\gamma}_{\mu\nu} = a^2(\eta) \left(\begin{array}{c|c} -2\phi & \partial_i B - S_i \\ \hline sym & -2\psi\delta_{ij} + 2\partial_{ij}E + 2\partial_{(i}F_{j)} + H_{ij} \end{array} \right), \tag{3.54}$$

where,

$$\partial^i S_i = \partial^i F_i = 0, \quad \partial^i H_{ij} = 0 = H^i_i, \tag{3.55}$$

with $\partial^i \equiv \delta^{ij}\partial_j$ and $H^i_j \equiv \delta^{ik}H_{kj}$. However, the vector mode will decay quickly with the expansion of the universe, and can be safely neglected [147,148]. Then, using the gauge transformations, as shown explicitly in Appendix A, one can always set

$$B = E = 0, \tag{3.56}$$

in which the gauge is completely fixed. This is often referred to as the Newtonian gauge, under which the gauge-invariant quantities defined in Eq.(A.11) become,

$$\Phi = \phi, \quad \Psi = \psi, \ (B = E = 0), \tag{3.57}$$

that is, in the Newtonian gauge, the potentials ϕ and ψ are equal to the gauge-invariant ones, Φ and Ψ. Therefore, with this gauge and ignoring the vector part, one finds,

$$\begin{aligned} \hat{\gamma}_{\mu\nu} &= a^2(\eta) \left(\begin{array}{c|c} -2\phi & 0 \\ \hline 0 & H_{ij} - 2\psi\delta_{ij} \end{array} \right), \\ \hat{\gamma}^{\mu\nu} &= a^{-2}(\eta) \left(\begin{array}{c|c} -2\phi & 0 \\ \hline 0 & H^{ij} - 2\psi\delta^{ij} \end{array} \right). \end{aligned} \tag{3.58}$$

The rest of this chapter will be restricted to this gauge, and the values of the Christoffel symbols, Riemann tensors, Ricci tensors, and Ricci scalars can be found in Appendix B.

3.3 Linearized Field Equations for GWs in an Inhomogeneous Universe

In this section, we shall consider the field equations for $\chi_{\mu\nu}$ given by Eq.(3.50) in the inhomogeneous cosmological background of Eq.(3.9) with the Newtonian gauge (3.56), by neglecting the vector perturbations, for which $\hat{\gamma}_{\mu\nu}$ and $\hat{\gamma}^{\mu\nu}$ are given by Eq.(3.58).

3.3.1 Gauge Fixings fow GWs

Before writing down these linearized field equations explicitly, let us first consider the gauge freedom for $\chi_{\mu\nu}$. At the end of the last section, we had considered the gauge transformations for the cosmological perturbations, and had already used the gauge freedom

$$\tilde{x}^{\mu} = x^{\mu} + \epsilon_c \zeta^{\mu}, \tag{3.59}$$

to set $B = E = 0$ [cf. Eq.(3.56)], the so-called Newtonian gauge, as shown explicitly in Appendix A. These choices completely fix the gauge freedom for the cosmological perturbations. In this subsection, we shall consider another kind of gauge transformations for the GWs, given by

$$\breve{x}^{\alpha} = x^{\alpha} + \epsilon \xi^{\alpha}, \tag{3.60}$$

where

$$\xi_{\alpha} \simeq \mathcal{O}\left(\epsilon h\right), \quad \xi_{\alpha;\beta} \simeq \mathcal{O}\left(h\right), \quad \xi_{\alpha;\beta;\gamma} \simeq \mathcal{O}\left(h/\epsilon\right). \tag{3.61}$$

Since $\epsilon_c \gg \epsilon$, one can see that to the first order of ϵ_c, the background metric $\gamma_{\mu\nu}$ does not change under the coordinate transformations (3.60), that is,

$$\breve{\gamma}_{\mu\nu} = \gamma_{\mu\nu} + \mathcal{O}\left(\epsilon_c^2\right), \tag{3.62}$$

a property that is required for the transformations (3.60) to be the gauge transformations only for the GWs. On the other hand, under the coordinate transformations (3.60), the metric transforms as

$$\begin{aligned}\check{g}_{\mu\nu} \equiv& \check{\gamma}_{\mu\nu} + \epsilon \check{h}_{\mu\nu} + \mathcal{O}\left(\epsilon^2\right) \\ =& \gamma_{\mu\nu} + \epsilon\left(h_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu}\right) + \mathcal{O}\left(\epsilon^2\right),\end{aligned} \tag{3.63}$$

that is,

$$\check{h}_{\mu\nu} = h_{\mu\nu} - 2\xi_{(\mu;\nu)}. \tag{3.64}$$

Hence, one finds,

$$\begin{aligned}\check{R}_{\alpha\beta\gamma\delta}{}^{(1)} - R_{\alpha\beta\gamma\delta}{}^{(1)} &= -\mathcal{L}_\xi R_{\alpha\beta\gamma\delta}{}^{(0)} = \mathcal{O}\left(h\hat{\gamma}/\epsilon_c\right), \\ \check{R}_{\alpha\beta}{}^{(1)} - R_{\alpha\beta}{}^{(1)} &= -\mathcal{L}_\xi R_{\alpha\beta}{}^{(0)} = \mathcal{O}\left(h\hat{\gamma}/\epsilon_c\right),\end{aligned} \tag{3.65}$$

as can be seen from Eqs.(3.25)-(3.27), and (3.61), where $\mathcal{L}_\xi$ denotes the Lie derivative. Therefore, Eq.(3.50) is gauge-invariant only up to $\mathcal{O}\left(h\hat{\gamma}/\epsilon_c\right)$. However, since $\epsilon_c^{-1} \ll \epsilon^{-1}$, terms that are order of ϵ^{-2} and ϵ^{-1} are still gauge-invariant, while the ones of order of ϵ^0 are not. This is because in the scale λ the spacetime appears locally flat, and the curvature is locally gauge-invariant. Thus, provided that the following conditions hold,

$$|h|,\ |\hat{\gamma}| \ll 1, \quad \epsilon \ll \epsilon_c \ll 1, \tag{3.66}$$

the GW produced by an astrophysical source can be considered as a high-frequency GW, and their low-frequency components are negligible, so that the local-flatness behavior carries over to the case in which the background is even curved.

On the other hand, from the field equations (3.50) one can see that they will be considerably simplified if the Lorenz gauge is chosen,

$$\nabla^{\nu}\check{\chi}_{\mu\nu} = 0, \tag{3.67}$$

where

$$\begin{aligned}\check{\chi}_{\mu\nu} \equiv& \check{h}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\check{h} \\ =& \chi_{\mu\nu} - 2\nabla_{(\mu}\xi_{\nu)} + \gamma_{\mu\nu}\nabla_{\lambda}\xi^{\lambda},\end{aligned} \tag{3.68}$$

as it can be seen from Eq.(3.64), where $\xi_{\mu} \equiv \gamma_{\mu\nu}\xi^{\nu}$. Then, one finds that the Lorenz gauge (3.67) requires

$$\Box\xi_{\mu} + R^{(0)\nu}{}_{\mu}\xi_{\nu} = \nabla^{\nu}\chi_{\mu\nu}. \tag{3.69}$$

Note that $R^{(0)\nu}{}_{\mu}\xi_{\nu} \simeq \mathcal{O}(h\hat{\gamma}\epsilon/\epsilon_c) \ll \mathcal{O}(h/\epsilon)$, so it can be neglected to the order of ϵ^{-1}. Clearly, for any given $\chi_{\mu\nu}$ (with some proper continuous conditions [149], which are normally assumed always to exist), the above equation in general has non-trivial solutions [114].

In addition, Eq.(3.69) does not completely fix the gauge. In fact, the gauge residual,

$$\check{\check{x}}^{\alpha} = \check{x}^{\alpha} + \epsilon\varsigma^{\alpha}, \tag{3.70}$$

exists, for which the Lorenz gauge (3.67) still holds,

$$\nabla^{\nu}\check{\check{\chi}}_{\mu\nu} = 0, \tag{3.71}$$

as long as ς^{α} satisfies the conditions

$$\Box\varsigma_{\mu} + R^{(0)\nu}{}_{\mu}\varsigma_{\nu} = 0. \tag{3.72}$$

Again, in this equation the term ${R^{(0)}}^{\nu}_{\mu}\varsigma_\nu \simeq \mathcal{O}(h\epsilon\hat{\gamma}/\epsilon_c)$ is negligible compared with $\Box\varsigma_\mu \simeq \mathcal{O}(h/\epsilon)$.

An interesting question is whether one can use this gauge residual further to set

$$\check{\check{\chi}}_{0\mu} = 0. \tag{3.73}$$

To answer this question, it must first be noted that if this is the case, ς_μ must satisfy the additional conditions

$$\nabla_0\varsigma_\nu + \nabla_\nu\varsigma_0 - \gamma_{0\nu}\nabla_\alpha\varsigma^\alpha = \check{\chi}_{0\nu}. \tag{3.74}$$

Clearly, for any given $\gamma_{\mu\nu}$ and $\check{\chi}_{\mu\nu}$ (again with certain regular conditions [149]), in general the above equation has solutions. However, ς_ν also needs to satisfy Eq.(3.72). To see if these conditions are consistent, one will take the covariant derivative ∇^μ in both sides of Eq.(3.74), which results in

$$\begin{aligned}\nabla_\nu\nabla_0\varsigma^\nu + &\Box\varsigma_0 - \nabla_0\nabla_\nu\varsigma^\nu \\ = &\Box\varsigma_0 + {R^{(0)}}_{0\alpha}\varsigma^\alpha = 0 = \nabla^\nu\check{\chi}_{0\nu}.\end{aligned} \tag{3.75}$$

Therefore, we conclude that *it is consistent to impose the Lorenz and spatial gauges simultaneously, even when the background is curved* [114].

Finally, we note that the traceless condition

$$\chi = 0, \tag{3.76}$$

was also introduced in [114]. In fact, provided that the Lorenz gauge $\nabla^\nu\chi_{\mu\nu} = 0$ holds, from the field equations (3.50) one finds

$$\Box\chi + 2{R_{\alpha\beta}}^{(0)}\chi^{\alpha\beta} = \kappa\gamma^{\alpha\beta}\left(\mathcal{F}_{\alpha\beta} - 2\left\langle\mathcal{T}_{\alpha\beta}\right\rangle^{\text{high}}\right). \tag{3.77}$$

Note that the two terms $\mathcal{F}_{\alpha\beta}$ and $2R_{\gamma\sigma}{}^{(0)}\chi^{\gamma\sigma}$ are of order $h\hat{\gamma}/\epsilon_c$, as shown above and can be dropped in comparing them with terms of order h/ϵ. Therefore, far from the source ($\mathcal{T}_{\alpha\beta} = 0$), if the Lorenz gauge holds, one can also consistently impose the traceless gauge. Together with the Lorenz and spatial gauges, it leads to the well-known traceless-transverse (TT) gauge, frequently used when the background is Minkowski [145, 150, 151].

It should be noted that in curved backgrounds the above three different gauge conditions can be imposed simultaneously only for high frequency GWs, and are valid only up to the order of ϵ^{-1} [114]. In other situations, when imposing them, one must take great caution, as these constraints in general represent many more degrees than the four degrees of the gauge freedom that the general covariance normally allows.

3.3.2 Field Equations for GWs

To write down explicitly the field equations (3.50) for $\chi_{\mu\nu}$, and to make the expressions as applicable as possible, in Appendix A, we only impose the spatial gauge

$$\chi_{0\mu} = 0, \; (\mu = 0, 1, 2, 3), \tag{3.78}$$

and then calculate each term appearing in Eq.(3.50), before putting them together to finally obtain the explicit expressions for each component of the field equations. In particular, the non-vanishing components of $\mathcal{F}_{\alpha\beta}$ and $2R_{\gamma\alpha\sigma\beta}{}^{(0)}\chi^{\gamma\sigma}$ are given by Eqs. (B.2) and (B.10), while the ones of $\Box\chi_{\alpha\beta}$ are given by Eqs.(B.11) and (B.13). The term $\gamma_{\alpha\beta}\nabla^{\gamma}\nabla^{\delta}\chi_{\gamma\delta}$ is given by Eqs.(B.15) and (B.16), while $\nabla_{\alpha}\nabla^{\delta}\chi_{\beta\delta}$ is given by

Eq.(B). Setting

$$\mathcal{G}_{\alpha\beta} \equiv \Box \chi_{\alpha\beta} + \gamma_{\alpha\beta} \nabla^{\gamma} \nabla^{\delta} \chi_{\gamma\delta} - \nabla_{\alpha} \nabla^{\delta} \chi_{\beta\delta} - \nabla_{\beta} \nabla^{\delta} \chi_{\alpha\delta}$$
$$+ 2R_{\alpha\gamma\beta\sigma}{}^{(0)} \chi^{\gamma\sigma}, \tag{3.79}$$

one finds that the field equations (3.50) take the form

$$\mathcal{G}_{\alpha\beta} = \kappa \left(\mathcal{F}_{\alpha\beta} - 2 \left\langle \mathcal{T}_{\alpha\beta} \right\rangle^{\text{high}} \right), \tag{3.80}$$

where the non-vanishing components of $\mathcal{G}_{\alpha\beta}$ are given by Eqs.(B.20) - (B.22).

3.4 The Geometrical Optics Approximation

To study the propagation of GWs in our inhomogeneous universe, first note that, when far away from the source that produces the GWs, $\mathcal{T}_{\mu\nu} = 0$. Then, Eq.(3.80) reduces to

$$\mathcal{G}_{\alpha\beta} = \kappa \mathcal{F}_{\alpha\beta}, \; (\mathcal{T}_{\mu\nu} = 0) \,. \tag{3.81}$$

Following Isaacson [114] and Laguna et al [116], one considers the geometrical optics approximation, defined as

$$\chi_{\alpha\beta} = Re \left(A_{\alpha\beta} e^{i\varphi/\epsilon} \right) = Re \left(e_{\alpha\beta} \mathcal{A} e^{i\varphi/\epsilon} \right), \tag{3.82}$$

where $e_{\alpha\beta}$ denotes the polarization tensor with

$$e^{\alpha\beta} e^{*}_{\alpha\beta} = 1, \tag{3.83}$$

and $\mathcal{A}$ and φ are real and characterize, respectively, the amplitude and phase of the GWs with $e^{\alpha\beta} \equiv \gamma^{\alpha\mu}\gamma^{\beta\nu} e_{\mu\nu}$. Note that in writing the above expression the change $\varphi_I \to \varphi/\epsilon$ is made by following Laguna et al [116], where φ_I is the quantity used by Isaacson [114]. With this in mind, both the amplitude $\mathcal{A}$ and the phase φ are slowly changing functions [114],

$$\partial_\alpha \varphi \simeq \mathcal{O}(1), \quad A^{\alpha\beta}{}_{;\gamma} \simeq \mathcal{O}(1). \tag{3.84}$$

With the gauge (3.78) the approximation may follow

$$A_{0\beta} = 0 = e_{0\beta}. \tag{3.85}$$

Moreover, as shown in the last section, in addition to the spatial gauge, one can consistently impose the Lorenz and traceless gauges

$$\nabla^\nu \chi_{\mu\nu} = 0,$$
$$\chi = 0. \tag{3.86}$$

Then, from Eqs.(3.81) and (3.85) one finds that the Lorenz gauge yields

$$\nabla^\nu A_{\mu\nu} + \frac{i}{\epsilon} k^\nu A_{\mu\nu} = 0, \tag{3.87}$$

where $k_\alpha \equiv \nabla_\alpha \varphi$ and $k^\alpha \equiv \gamma^{\alpha\beta} k_\beta$. Considering Eq.(3.84) one finds that, to the leading order (ϵ^{-1})

$$k^\nu A_{\mu\nu} = 0 \quad \Rightarrow \quad k^\nu e_{\mu\nu} = 0. \tag{3.88}$$

Therefore, *the propagation direction of the GW is orthogonal to its polarization plane spanned by the bivector* $e_{\mu\nu}$. Note that the first term in Eq.(3.4) is of order ϵ^0, and should be discarded. Otherwise, it will lead to inconsistent results, as mentioned above. Therefore, in the rest of this chapter we shall ignore such terms without further notifications. See [114, 116, 121] for more details.

In addition, the traceless condition requires

$$\gamma^{\alpha\beta} e_{\alpha\beta} = 0. \tag{3.89}$$

Substituting Eq.(3.81) into Eq.(3.80) and considering Eq.(3.84) and the Lorenz gauge (3.86), the field equations to the orders of ϵ^{-2} and ϵ^{-1} are given, respectively, by

$$\epsilon^{-2}: \quad k^{\mu}k_{\mu}A_{\alpha\beta} = 0, \tag{3.90}$$

$$\epsilon^{-1}: \quad k^{\mu}\nabla_{\mu}e_{\alpha\beta} + \left(k^{\mu}\nabla_{\mu}\ln\mathcal{A} + \frac{1}{2}\nabla_{\mu}k^{\mu}\right)e_{\alpha\beta} = 0. \tag{3.91}$$

Since $A_{\mu\nu} \neq 0$, from Eq.(3.90) one finds

$$k^{\lambda}k_{\lambda} = 0. \tag{3.92}$$

Then, for such a null vector k^{μ}, one may always define a curve $x^{\mu} = x^{\mu}(\lambda)$ by setting

$$\frac{dx^{\mu}(\lambda)}{d\lambda} \equiv k^{\mu}, \tag{3.93}$$

where λ denotes the affine parameter along the curve. It is clear that such a defined curve is a null geodesic

$$k^{\lambda}\nabla_{\mu}k_{\lambda} = k^{\lambda}\nabla_{\lambda}k_{\mu} = 0, \tag{3.94}$$

where $\nabla_{\mu}k_{\lambda} = \nabla_{\mu}\nabla_{\lambda}\varphi = \nabla_{\lambda}\nabla_{\mu}\varphi = \nabla_{\lambda}k_{\mu}$, that is, *GWs are always propagating along null geodesics in our inhomogeneous universe, even when both the cosmological scalar and tensor perturbations are all present, as long as the geometrical optics approximation are valid.*

On the other hand, multiplying $e^{\alpha\beta}$ on both sides of Eq.(3.91) and taking Eq.(3.83) into account, one finds that

$$\frac{d}{d\lambda}\ln\mathcal{A} + \frac{1}{2}\nabla_{\mu}k^{\mu} = 0, \tag{3.95}$$

where $d/d\lambda \equiv k^{\nu}\nabla_{\nu}$. Introducing the current $J^{\mu} \equiv \mathcal{A}^2k^{\mu}$ of the gravitons moving along the null geodesics, the above equation can be written in the form

$$\nabla_{\mu}J^{\mu} = 0. \tag{3.96}$$

Therefore, *the current of the gravitons moving along the null geodesics defined by* k^μ *is conserved, even when the primordial GWs (or cosmological tensor perturbations) are present (*$H_{ij} \neq 0$*).* Inserting Eq.(3.95) into Eq.(3.91), one finds that

$$k^\mu \nabla_\mu e_{\alpha\beta} = 0. \tag{3.97}$$

Thus, *the polarization bivector* $e_{\alpha\beta}$ *is still parallel-transported along the null geodesics, even when the primordial GWs are present.*

It is interesting to note that *Eqs.(3.4)-(3.97) hold not only for the inhomogeneous universe, but also for any curved background, as long as the geometrical optics approximation are applicable to the high frequency GWs.*

To study the GWs further, one can expand $\chi_{\mu\nu}$ in terms of ϵ_c as,

$$\hat{\chi}_{\mu\nu} = \chi^{(0)}_{\mu\nu} + \epsilon_c \chi^{(1)}_{\mu\nu} + \mathcal{O}\left(\epsilon_c^2\right), \tag{3.98}$$

and then consider them order by order.

3.4.1 GWs Propagating in a Homogeneous and Isotropic Background

To the zeroth-order of ϵ_c, the background metric is $\gamma_{\mu\nu} \simeq \bar{\gamma}_{\mu\nu} = a^2\eta_{\mu\nu}$, and

$$\chi_{\mu\nu} \simeq \chi^{(0)}_{\mu\nu} + \mathcal{O}\left(\epsilon_c\right), \tag{3.99}$$

where the geometric optics approximation is set as

$$\chi^{(0)}_{\mu\nu} \equiv A^{(0)}_{\mu\nu} e^{i\varphi^{(0)}/\epsilon} = e^{(0)}_{\mu\nu} \mathcal{A}^{(0)} e^{i\varphi^{(0)}/\epsilon}. \tag{3.100}$$

Then, from Eqs.(3.96) and (3.97) one immediately obtain

$$\bar{\nabla}_\nu \left(\mathcal{A}^{(0)^2} k^{(0)\nu}\right) = 0, \tag{3.101}$$

$$\frac{d}{d\lambda} e^{(0)}_{ij} = 0, \tag{3.102}$$

where $k^{(0)}_\mu \equiv \bar{\nabla}_\mu \varphi^{(0)} = \left(\varphi^{(0)}{}_{,\eta}, \varphi^{(0)}{}_{,i}\right)$, and $k^{(0)\mu} \equiv \bar{\gamma}^{\mu\nu} k^{(0)}_\nu$.

3.4.2 *Gravitational iSW Effects*

The derivation of the iSW effect in cosmology is based on the fact that electromagnetic radiation propagates along null geodesics in the inhomogeneous universe. Laguna et al [116] took advantage of the fact that GWs are also propagating along null geodesics and derived the gravitational iSW effect for GWs when only the cosmological scalar perturbations are present ($H_{ij} = 0$). In this subsection, we shall generalize their studies further to the case where both the cosmological scalar and tensor perturbations are present. As shown by Eq.(3.92), even when both of them are present, the GWs produced by astrophysical sources are still propagating along the null geodesics. Therefore, such a generalization is straightforward.

In particular, one can first introduce the conformal metric $\tilde{\gamma}_{\mu\nu}$ by

$$\begin{aligned} d\tilde{s}^2 =& \tilde{\gamma}_{\mu\nu} dx^\mu dx^\nu \equiv a^{-2} \gamma_{\mu\nu} dx^\mu dx^\nu \\ =& -\left(1 + 2\epsilon_c \phi\right) d\eta^2 + \Big[\left(1 - 2\epsilon_c \psi\right) \delta_{ij} + H_{ij}\Big] dx^i dx^j. \end{aligned} \tag{3.103}$$

Since $\gamma_{\mu\nu}$ and $\tilde{\gamma}_{\mu\nu}$ are related to each other by a conformal transformation, the null geodesics $x^\mu(\lambda)$ in the $\gamma_{\mu\nu}$ spacetime are the same as $\tilde{x}^\mu(\tilde{\lambda})$ in the $\tilde{\gamma}_{\mu\nu}$ spacetime, where

$$d\lambda = a d\tilde{\lambda}, \quad k^\mu = \frac{1}{a^2} \tilde{k}^\mu, \tag{3.104}$$

and $\tilde{\lambda}$ is the affine parameter of the null geodesics $\tilde{x}^\mu$ in the spacetime of $\tilde{\gamma}_{\mu\nu}$.

The advantage of working with the metric $\tilde{\gamma}_{\mu\nu}$ is that the zeroth-order spacetime now becomes the Minkowski spacetime, and the corresponding null geodesics are the straight lines given by

$$\frac{d\tilde{x}^{(0)\mu}(\tilde{\lambda})}{d\tilde{\lambda}} \equiv \tilde{k}^{(0)\mu}. \tag{3.105}$$

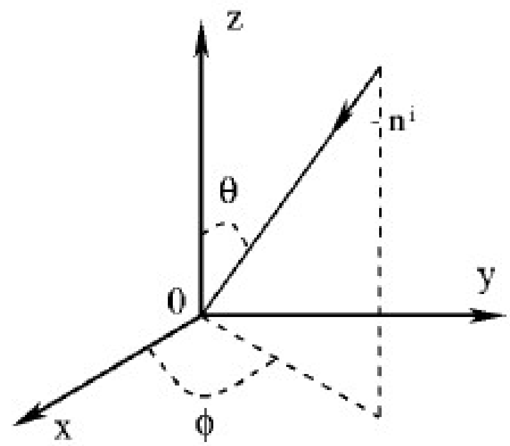

Figure 3.1: A gravitational wave is propagating along the spatial direction $\tilde{k}^{(0)i} \equiv -n^i$ to the observer located at the origin.

Thus, to simplify our calculations, $\tilde{\gamma}_{\mu\nu}$ will be used as the background metric. In particular, to the zeroth-order of ϵ_c, the wavevector is

$$\tilde{k}^{(0)\mu} = \left(1, -n^i\right), \tag{3.106}$$

where $\tilde{k}^{(0)i} \equiv -n^i$ represents the spatial direction of the GWs from the source propagating to the observer [cf. Fig. 3.1]. Then, from Eq.(3.101),

$$\frac{d}{d\tilde{\lambda}} \ln \left(a\mathcal{A}^{(0)}\right) = -\frac{1}{2}\tilde{k}^{(0)\nu}{}_{,\nu} = 0, \tag{3.107}$$

which implies that the quantity defined by

$$\mathcal{Q} \equiv \mathcal{R}\mathcal{A}^{(0)}, \tag{3.108}$$

is constant along the GW path, and will be determined by the local wave-zone source solution, where $\mathcal{R} \equiv ar$ denotes the physical distance between the observer and the source, while r denotes the comoving distance, given by $r \equiv \sqrt{(x_e - x_r)^2 + (y_e - y_r)^2 + (z_e - z_r)^2}$, where $x_e^i \equiv (x_e, y_e, z_e)$ and $x_r^i \equiv (x_r, y_r, z_r)$ are the spatial locations of the source and observer, respectively.

In the following, we shall set up the coordinates as follows [116]: The observer is located at the origin with its proper time denoted by τ and world line $x^\mu(\tau)$. Denoting the time to receive the GW by τ_r, this event will be recorded as $x^\mu(\tau) = (\tau_r, \vec{o})$. The emission time of the GW by an astrophysical source corresponds to the proper time τ_e of the observer with $x^\mu(\tau) = (\tau_e, \vec{o})$. Then, the GW will move along the null geodesics, described by $\tilde{x}^\mu(\tilde{\lambda}) = \tilde{x}^{(0)\mu}(\tilde{\lambda}) + \epsilon_c \tilde{x}^{(1)\mu}(\tilde{\lambda})$, which corresponds to the wave vector $\tilde{k}^\mu(\tilde{\lambda}) = \tilde{k}^{(0)\mu}(\tilde{\lambda}) + \epsilon_c \tilde{k}^{(1)\mu}(\tilde{\lambda})$, where $\tilde{x}^{(0)\mu}(\tilde{\lambda}) = \left(\tilde{\lambda}, (\tilde{\lambda}_r - \tilde{\lambda}) n^i\right)$, and $\tilde{\lambda}_r$ is the moment when the GW arrives at the origin with $\tau(\tilde{\lambda}_r) = \tau_r$.

The effects of the scalar and tensor perturbations are manifested from the perturbations of the null geodesics. Considering the fact $\tilde{\Gamma}^{(0)\mu}_{\nu\lambda} = 0$ in the $\tilde{\gamma}_{\mu\nu}$ spacetime, one finds that, to the first-order of ϵ_c, $\tilde{k}^{(1)\mu}(\tilde{\lambda})$ is given by

$$\frac{d\tilde{k}^{(1)\mu}}{d\tilde{\lambda}} + \tilde{\Gamma}^{(1)\mu}_{\alpha\beta} \tilde{k}^{(0)\alpha} \tilde{k}^{(0)\beta} = 0, \tag{3.109}$$

where $\tilde{\Gamma}^{(1)\mu}_{\alpha\beta}$ denotes the Christoffel symbols of the first-order of ϵ_c. As mentioned previously, for the scalar perturbations, one shall not assume that $\psi = \phi$, that is, the trace of the anisotropic stress of the universe does not necessarily vanish, as shown by Eq.(A.15) in Appendix A. Then, for $\mu = 0$ Eq.(3.109) is written as

$$\frac{d}{d\lambda} \tilde{k}^{(1)0} = \partial_\tau (\phi + \psi) - 2\frac{d\phi}{d\lambda} - \frac{1}{2} n^k n^l \partial_\tau H_{kl}, \tag{3.110}$$

where

$$\frac{d\Phi}{d\lambda} \equiv \left(\partial_\tau - n^i \partial_i\right) \Phi. \tag{3.111}$$

Thus, integrating Eq.(3.110) the wavevector is written as

$$\begin{aligned} \tilde{k}^{(1)0} = & - \left.(\phi + \psi)\right|_{\lambda_e} + \frac{1}{2} n^k n^l H_{kl}|_{\lambda_e} \\ & - 2 \left.\phi\right|^{\lambda}_{\lambda_e} + I^{(s)}_{iSW} - \frac{1}{2} I^{(t)}_{iSW}, \end{aligned} \tag{3.112}$$

where $I^{(s)}_{iSW}$ represents the gravitational iSW effect due to the cosmological scalar perturbations, and was first calculated in [116]. The new term $I^{(t)}_{iSW}$ is the gravitational integrated effect due to the cosmological tensor perturbations. They are given, respectively, by

$$I^{(s)}_{iSW} \equiv \int_{\lambda_e}^{\lambda} \partial_\tau(\phi+\psi)d\lambda', \tag{3.113}$$

$$I^{(t)}_{iSW} \equiv n^k n^l \int_{\lambda_e}^{\lambda} \partial_\tau H_{kl} d\lambda'. \tag{3.114}$$

On the other hand, the spatial components of the wave-vector are given by

$$\begin{aligned}\frac{d}{d\lambda}\tilde{k}^{(1)i}_{\parallel} =& -n^i\Bigg[\partial_\tau(\phi+\psi)+\frac{d}{d\lambda}(\phi-\psi)\\ &-\frac{1}{2}n^k n^l\left(\frac{dH_{kl}}{d\lambda}+\partial_\tau H_{kl}\right)\Bigg],\end{aligned} \tag{3.115}$$

$$\begin{aligned}\frac{d}{d\lambda}\tilde{k}^{(1)i}_{\perp} =& - \perp^{ij}\Bigg[\partial_j(\phi+\psi)-n^k\frac{dH_{jk}}{d\lambda}\\ &-\frac{1}{2}n^k n^l \partial_j H_{kl}\Bigg],\end{aligned} \tag{3.116}$$

where the wavevector is split into parallel and perpendicular modes $\tilde{k}^{(1)i} = \tilde{k}^{(1)i}_{\parallel} + \tilde{k}^{(1)i}_{\perp}$, with the parallel component of the spatial wave-vector being defined by $\tilde{k}^{(1)i}_{\parallel} = n^i n_j \tilde{k}^{(1)j}$, and the perpendicular component by $\tilde{k}^{(1)i}_{\perp} = \perp^i_j \tilde{k}^{(1)j}$. The projection operator $\perp^i_j$ is defined by $\perp^i_j = \delta^i_j - n^i n_j$, with $n_i \equiv \delta_{ik} n^k$. After integrating the above two equations yield

$$\tilde{k}_{\parallel}^{(1)i} = -\, n^i \left[(\psi - \phi)|_{\lambda_e}^{\lambda} - \frac{1}{2} n^k n^l H_{kl}|_{\lambda_e}^{\lambda} \right.$$
$$\left. + I_{iSW}^{(s)} - \frac{1}{2} I_{iSW}^{(t)} \right], \tag{3.117}$$
$$\tilde{k}_{\perp}^{(1)i} = - \perp^{ij} \left[\int_{\lambda_e}^{\lambda} \partial_j (\phi + \psi) d\lambda' - n^k H_{jk}|_{\lambda_e}^{\lambda} \right.$$
$$\left. - \frac{1}{2} n^k n^l \int_{\lambda_e}^{\lambda} \partial_j H_{kl} d\lambda' \right]. \tag{3.118}$$

The GW phase is then given by

$$\frac{d\varphi}{d\lambda} = \phi + \psi - \frac{1}{2} n^k n^l \int_{\lambda_e}^{\lambda} H_{kl} d\lambda', \tag{3.119}$$

which leads to

$$\delta\varphi = \varphi - \varphi_e = \int_{\lambda_e}^{\lambda} (\phi + \psi) d\lambda'$$
$$- \frac{1}{2} n^k n^l \int_{\lambda_e}^{\lambda} H_{kl} d\lambda'. \tag{3.120}$$

The frequency of the GW is defined as $\omega = -u^{\mu} k_{\mu}$, where u^{μ} is the 4-velocity of the fluid of the universe, from which one finds that the ratio of receiving and emitting frequencies is given by

$$\frac{\omega_r}{\omega_e} = \frac{1 - \Upsilon}{1 + z}, \tag{3.121}$$

where $1 + z \equiv a_r / a_e$, and

$$\Upsilon \equiv \phi|_{\lambda_e}^{\lambda_r} + v^i n_i + \frac{1}{2} n^k n^l H_{kl}|_{\lambda_e}^{\lambda_r}$$
$$- I_{iSW}^{(s)}|_{\lambda_r} + \frac{1}{2} I_{iSW}^{(t)}|_{\lambda_r}. \tag{3.122}$$

In addition, setting $\mathcal{A} = \mathcal{A}^{(0)}(1 + \xi)$, from Eq.(3.101) one gets

$$-2 \frac{d\xi}{d\lambda} = \partial_{\tau} \tilde{k}^{(1)0} + \partial_i \tilde{k}_{\parallel}^{(1)i} + \partial_i \tilde{k}_{\perp}^{(1)i} + \tilde{\Gamma}_{\mu\nu}^{(1)\mu} \tilde{k}^{(0)\nu}, \tag{3.123}$$

where

$$
\begin{aligned}
\partial_\tau \tilde{k}^{(1)0} =& \partial_\tau \left(-2\phi + I^{(s)}_{iSW} - \frac{1}{2} I^{(t)}_{iSW} \right), \\
\partial_i \tilde{k}^{(1)i}_{\parallel} =& \frac{d}{d\lambda} \left(\psi - \phi + I^{(s)}_{iSW} \right) \\
& - \partial_\tau \left(\psi - \phi + I^{(s)}_{iSW} \right) \\
& - \frac{1}{2} \frac{d}{d\lambda} \left(n^k n^l H_{kl} + I^{(t)}_{iSW} \right) \\
& + \frac{1}{2} \partial_\tau \left(n^k n^l H_{kl} + I^{(t)}_{iSW} \right), \\
\partial_i \tilde{k}^{(1)i}_{\perp} =& - \perp^{ij} \left[\int_{\lambda_e}^{\lambda} \partial_i \partial_j (\phi + \psi) d\lambda' \right. \\
& \left. - n^k \partial_i H_{kj} - \frac{1}{2} n^k n^l \int_{\lambda_e}^{\lambda} \partial_i \partial_j H_{kl} d\lambda' \right], \\
\tilde{\Gamma}^{(1)\mu}_{\mu\nu} \tilde{k}^{(0)\nu} =& \frac{d}{d\lambda} (\phi - 3\psi).
\end{aligned}
\tag{3.124}
$$

Notice that in the last term there are no contributions from the tensor perturbations.

Collecting all of this together, Eq.(3.124) yields

$$
\begin{aligned}
-2\frac{d\xi}{d\lambda} =& - \partial_\tau (\phi + \psi) + \frac{d}{d\lambda} \left(-2\psi + I^{(s)}_{iSW} \right) \\
& - \perp^{ij} \int_{\lambda_e}^{\lambda} \partial_i \partial_j (\phi + \psi) d\lambda' \\
& + \frac{1}{2} n^k n^l \partial_\tau H_{kl} - \frac{1}{2} \frac{d}{d\lambda} \left(n^k n^l H_{kl} + I^{(t)}_{iSW} \right) \\
& + \perp^{ij} n^k \partial_i H_{jk} \\
& + \frac{1}{2} \perp^{ij} n^k n^l \int_{\lambda_e}^{\lambda} \partial_i \partial_j H_{kl} d\lambda',
\end{aligned}
\tag{3.125}
$$

which has the general solution

$$
\begin{aligned}
\xi =& - \psi|^{\lambda}_{\lambda_e} + \frac{1}{2} \perp^{ij} \int_{\lambda_e}^{\lambda} \int_{\lambda_e}^{\lambda'} \partial_i \partial_j (\phi + \psi) d\lambda' d\lambda'' \\
& - \frac{1}{2} n^k \left[- \frac{1}{2} n^l H_{kl}|^{\lambda}_{\lambda_e} + \perp^{ij} \int_{\lambda_e}^{\lambda} \partial_i H_{jk} d\lambda' \right. \\
& \left. + \frac{1}{2} \perp^{ij} n^l \int_{\lambda_e}^{\lambda} \int_{\lambda_e}^{\lambda'} \partial_i \partial_j H_{kl} d\lambda' d\lambda'' \right].
\end{aligned}
\tag{3.126}
$$

In terms of the gravitational tensorial iSW effect defined by Eq.(3.114), the above expression can be written in the form

$$
\begin{aligned}
\xi = &\left(\psi - \frac{1}{4}n^k n^l H_{kl}\right)\Bigg|_{\lambda_e}^{\lambda} + \frac{1}{2}I_{iSW}^{(t)} \\
&- \frac{1}{4}\perp^{ij}\int_{\lambda_e}^{\lambda}\int_{\lambda_e}^{\lambda'}\partial_i\partial_j\left[n^k n^l H_{kl} - 2\left(\phi+\psi\right)\right]d\lambda'' d\lambda' \\
&- \frac{1}{2}n^k\int_{\lambda_e}^{\lambda}\partial^l H_{kl} d\lambda'.
\end{aligned}
\tag{3.127}
$$

Combining all of our results together, one can construct the gravitational waveform through Eq.(3.81), from which it is written as

$$
\begin{aligned}
h_{\mu\nu} &= \chi_{\mu\nu} - \frac{1}{2}\chi\gamma_{\mu\nu} = e_{\mu\nu}\tilde{h}, \\
\tilde{h} \equiv \mathcal{A}e^{i\varphi} &= \frac{(1+z)\mathcal{Q}}{d_L}(1+\xi)e^{i(\varphi_e+\delta\varphi)},
\end{aligned}
\tag{3.128}
$$

where $\delta\varphi$ and ξ are given, respectively, by Eqs.(3.120) and (3.127), and $d_L \equiv (1+z)\mathcal{R}$ is the luminosity distance. Note that in writing the expression for the response function $\tilde{h}$ the smallness parameter is set to $\epsilon = 1$.

For a binary system, the values $\mathcal{Q}$ and φ_e are written as [116, 151]

$$
\begin{aligned}
\mathcal{Q} =&\mathcal{M}_e\left(\pi f_e \mathcal{M}_e\right)^{2/3}, \\
\varphi_e =&\varphi_c - \left(\pi f_e \mathcal{M}_e\right)^{-5/3},
\end{aligned}
\tag{3.129}
$$

where $\mathcal{M}_e$ and f_e denote, respectively, the intrinsic chirp mass and frequency of the binary, and ϕ_c is the value of the phase at the merge, at which $f = \infty$. Therefore, the function $\tilde{h}$ for a binary system can be cast in the form

$$
\tilde{h} = \frac{\mathcal{M}_r}{D_L}(\pi f_r \mathcal{M}_r)^{2/3} e^{i(\varphi_e+\delta\varphi)},
\tag{3.130}
$$

where the modified luminosity distance D_L and the chirp mass $\mathcal{M}_r$ measured by the observer are given, respectively, by

$$D_L \equiv \frac{d_L}{1 - \Upsilon - \xi}, \quad \mathcal{M}_r \equiv \left(\frac{1+z}{1-\Upsilon}\right)\mathcal{M}_e, \tag{3.131}$$

where Υ is given by Eq.(3.122).

CHAPTER FOUR

Conclusions and Closing Remarks

4.1 Gravitational Theory

In this dissertation, we discussed the nature of GWs as they propagate through the universe. We looked at the effects of singularities and tidal forces produced by GWs, and how early universe scalar and tensor perturbations affect GWs as they propagate in an inhomogeneous universe.

The memory effects of gravitational waves are tightly related to the asymptotical properties of the spacetime at the future null infinity (see Ref. [16, 23, 37] and references therein), and so are the soft gravitons and black holes [25, 26]. However, it is well-known that in the BJR coordinates Eq.(2.6), the metric coefficients often become singular, and extensions beyond the singularities are needed before studying these important issues.

In Chapter 2, we have first pointed out that such extensions are not always possible, as some of these singularities are physically real singularities. In particular, distortions experienced by freely falling observers in the (u, v)-plane can be divergent, and any objects trying across the singular surface will be destroyed by these distortions. As a result, in these cases the singularities actually represent the boundaries of the spacetimes. In particular, if the metric coefficient e^{-U} vanishes at the singularity $u = u_s$ as,

$$\chi^{-U/2} = (u = u_s)^{\alpha}\hat{\chi}(u),$$

where $\alpha \in (0, 1]$, which is required for the metric coefficients to be real, and $\hat{\chi}(u_s) \simeq 0$, we found that distortions experienced by such freely falling observers always diverge, unless $\alpha = 1/2$ or $\alpha = 1$. Therefore, only in the cases where $\alpha = 1/2$ or 1, the spacetimes at $u = u_s$ are possibly non-singular, and extensions of the spacetimes beyond this surface is needed, whereby we are able to study the memory effects of gravitational waves and soft gravitons and black holes.

Coordinate transformations from the BJR coordinates to the Brinkmann coordinates are carried out by Eq.(2.13). It is interesting to note that in the Brinkmann coordinates there is only one unknown function $\mathcal{A}$, while in the BJR coordinates there are two, U and V. However, the vacuum Einstein field equation (3.2) relates U to V, so finally there is only one independent component in that system too. In fact, for any given V, from Eq.(2.37) one can find U, and then the function $\mathcal{A}$ is uniquely determined by Eq.(2.38). It is also interesting to note that the inverse is not unique, that is, for any given $\mathcal{A}(u)$, Eq.(2.56) will have a family of solutions of the form, $U(u, u_1, u_2)$ and $V(u, v_1, v_2)$, where u_i's and v_i's are the integration constants. With the above in mind, we find that $\mathcal{A}$ is finite and well-behaved across $u = u_s$ for $\alpha = 1/2$ for any given χ_n, where χ_n are the expansion coefficients of $\hat{\chi}(u)$, given in Eq.(2.25). However, in the case $\alpha = 1$, we found that $\mathcal{A}$ is finite and well-behaved across $u = u_s$ only when $\chi_1 = \chi_2 = \chi_3 = 0$. If any of these three coefficients is not zero, $\mathcal{A}(u)$ will be singular across $u = u_s$, although the distortions of the freely falling observers considered in this chapter are finite. There are two possibilities for these cases: (i) The corresponding spacetimes are indeed singular, and distortions become unbounded across $u = u_s$ for other kinds of observers. (ii) The corresponding singularities are coordinate ones, but the proper coordinate transformations are not given by Eq.(2.13), and instead they are given by something else. The results on the studies of the tidal forces between two nearby null geodesics in the Brinkmann coordinates presented in [56] show that the possibility (i) is the right answer, as the tidal forces will diverge when $\mathcal{A}(u)$ does. Therefore, it is concluded that only in the two cases given by Eq.(2.1) are the spacetimes not singular at the focusing surface $u = u_s$, and extensions beyond the hypersurface are needed, in order to obtain maximal spacetimes.

Finally we note that our results are expected to be valid when both of the two polarizations exist, that is, $W \simeq 0$ in Eq.(2.4), although in the this chapter we only considered the case $W = 0$.

In Chapter 3, we have systematically studied GWs, which are first produced by some remote compact astrophysical sources, and then propagate in our inhomogeneous universe through cosmic distances before arriving at the detectors. Such GWs will carry valuable information of both their source and the cosmological expansion and inhomogeneities of the universe, whereby a completely new window to explore our universe by using GWs is opened. As the third generation (3G) detectors, such as the space-based ones, LISA [77], TianQin [78], Taiji [79], DECIGO [80], and the ground-based ones, ET [109] and CE [110], are able to detect GWs emitted from such sources as far as at the redshift $z \simeq 100$ [111], it is very important and timely to carry out such studies systematically. Such studies were already initiated some years ago [116, 118, 119] in the framework of Einstein's theory, and more recently in scalar-tensor theories [121–124, 124].

In order to effectively characterize such systems, we first introduced three scales, λ, L_c and L, which represent, respectively, the typical wavelength of the

GWs, the scale of the cosmological perturbations, and the size of our observable universe. For GWs to be detected by the current and foreseeable (both ground- and space-based) detectors, in Chapter 3.2 we showed that the relation

$$\lambda \ll L_c \ll L,$$

is always true, that is, such GWs can be well approximated as *high frequency GWs*, for which the general formulas were already developed by Isaacson more than half century ago [114, 115].

However, Isaacson considered only the case where the background is vacuum, while in [116, 118, 119] only the cosmological scalar perturbations were considered. In this chapter, we considered the most general case in which the background also includes the cosmological tensor perturbations. The inclusion of the latter is important, as now one of the main goals of cosmological observations is the primordial GWs (the tensor perturbations) [125]. In the non-vacuum case, (in Ch.3.2) we showed explicitly that the conditions

$$|h_{\mu\nu}| \ll 1, \quad \epsilon \ll \epsilon_c \ll 1,$$

must hold, in order for the backreaction of the GWs to the background to be neglected, and the linearized Einstein field equations given by Eq.(3.50) to hold, where the total metric of the spacetime is expanded as $g_{\mu\nu} = \gamma_{\mu\nu} + \epsilon h_{\mu\nu}$, with $\gamma_{\mu\nu} (\equiv \bar{\gamma}_{\mu\nu} + \epsilon_c \hat{\gamma}_{\mu\nu})$ representing the background.

In Ch.3.3, we considered the gauge choices, and found that the three different gauge conditions, *spatial, traceless, and Lorenz*, given respectively by Eqs.(3.3) - (3.5), can be still imposed *simultaneously*, even when both the cosmological scalar and tensor perturbations are present, as long as the GWs can be approximated as the high-frequency GWs. However, by imposing only the spatial gauge (3.3), the linearized Einstein field equations (3.50) are explicitly given in Appendix B. If $\chi_{\mu\nu}$ is decomposed into two parts,

$$\chi_{\mu\nu} = \chi_{\mu\nu}^{(0)} + \epsilon_c \chi_{\mu\nu}^{(1)} + \mathcal{O}\left(\epsilon_c^2\right),$$

the field equations for $\chi_{\mu\nu}^{(1)}$ are given explicitly in Appendix C.

As an application of the general formulas developed in Ch.3.2 and 3.3, in Ch. 3.4 we studied the GWs by using the geometrical optics approximation,

$$\chi_{\alpha\beta} = e_{\alpha\beta}\mathcal{A}e^{i\varphi/\epsilon},$$

where $e_{\alpha\beta}$ represents the polarization tensor, $\mathcal{A}$ and φ denote, respectively, the amplitude and phase of the GWs. We showed explicitly that even when both the cosmological scalar and tensor perturbations are present, *such GWs are still propagating along null geodesics, and the current of gravitons moving along the null geodesics is conserved, and the polarization tensor is parallel-transported*, i.e.,

$$k^\lambda \nabla_\lambda k^\mu = 0, \quad k^\lambda \nabla_\lambda e_{\alpha\beta} = 0, \quad \nabla^\lambda J_\lambda = 0,$$

where $k_\mu \equiv \nabla_\mu \varphi$, $J_\mu \equiv \mathcal{A}^2 k_\mu$. In fact, *these are true for any curved background, provided that: (a) the GWs can be considered as high-frequency GWs; and (b) the geometrical optics approximation are valid.*

With these remarkable features, we calculated the effects of the cosmological scalar and tensor perturbations on the amplitude $\mathcal{A}$ and phase φ, given by Eqs.(3.120), (3.127) and (3.128). Restricting to GWs produced by a binary system, the effects of the cosmological perturbations, both scalar and tensor, on the luminosity distance and the chirp mass are given explicitly by Eq.(3.131), which represent a natural generalization of the results obtained in [116,118,119] to the case in which the cosmological tensor perturbations are also present.

The applications of our general formulas developed in this chapter to other studies are immediate, including the gravitational analogue of the electromagnetic Faraday rotations [143,144,152,153], and their detections by the space- and ground-based detectors. We wish to return to these important issues in the near future. It would be also very important to extend such studies to include the relations between the GWs and their sources, high-order corrections to the geometrical optics approximations, and more interestingly the non-high frequency GWs.

APPENDICES

APPENDIX A

Decomposition of Cosmological Perturbations and Gauge Choices

Following [147, 148], the linear perturbations $\hat{\gamma}_{\mu\nu}$ can be decomposed into scalar, vector and tensor modes, and given explicitly by Eq.(3.54).

The energy-momentum tensor ${T_{\mu\nu}}^{(0)}$ of a fluid takes the form [147],

$$T^{\mu(0)}_{\nu} = (\rho + p)\, u^{\mu} u_{\nu} + p\delta^{\mu}_{\nu} + \pi^{\mu}_{\nu}, \tag{A.1}$$

where u^{μ} is the 4-velocity of the fluid, ρ and p are its energy density and isotropic pressure, respectively, and π^{μ}_{ν} is the anisotropic stress tensor, which has only spatial components, i.e., $\pi^{\mu}_{0} = 0$. Setting

$$\begin{aligned} &\rho = \bar{\rho} + \epsilon_c \delta\rho, \quad p = \bar{p} + \epsilon_c \delta p, \\ &u^{\mu} = \bar{u}^{\mu} + \epsilon_c \delta u^{\mu}, \end{aligned} \tag{A.2}$$

where $\bar{u}^{\mu} = a^{-1}\delta^{\mu}_{\eta}$ is the 4-velocity of the fluid of the homogeneous and isotropic universe, and $\bar{\rho}$ and $\bar{p}$ are its energy density and isotropic pressure, respectively, we find that δu^{μ} can be decomposed as

$$\delta u^{\mu} = \frac{1}{a}\left(-\phi, \partial^i v + v^i\right), \tag{A.3}$$

where $\partial_i v^i = 0$. Then, from $u_{\mu} \equiv \gamma_{\mu\nu} u^{\nu} = \bar{u}_{\mu} + \epsilon_c \delta u_{\mu}$, we find that,

$$\delta u_{\mu} = a\left(-\phi, \partial_i v + \partial_i B + v_i - S_i\right), \tag{A.4}$$

which leads to $u^{\mu} u_{\mu} = -1 + \mathcal{O}\left(\epsilon_c^2\right)$, as expected.

On the other hand, setting $\pi_i^j = \epsilon_c \hat{\pi}_i^j$, similar to $\hat{\gamma}_{\mu\nu}$, the anisotropic stress tensor $\hat{\pi}_i^j$ can be decomposed into scalar, vector and tensor modes,

$$\hat{\pi}_i^j = \left(\partial^j \partial_i - \frac{1}{3}\delta_i^j \partial^2\right)\Pi + \frac{1}{2}\left(\partial_i \Pi^j + \partial^j \Pi_i\right) + \Pi^j{}_i, \tag{A.5}$$

where $\partial_i \Pi^i = 0 = \Pi^i{}_i$, $\partial_j \Pi^j{}_i = 0$, $\Pi^i \equiv \delta^{ik}\Pi_k$, $\Pi^i{}_j \equiv \delta^{ik}\Pi_{kj}$, $\partial^2 \equiv \partial^i \partial_i$, etc. Then, we find that

$$\begin{aligned}
T^0{}_0{}^{(0)} &= -\bar{\rho} - \epsilon_c \delta\rho, \\
T^0{}_i{}^{(0)} &= \epsilon_c \left(\bar{\rho} + \bar{p}\right)\left[\partial_i (v + B) + v_i - S_i\right], \\
T^i{}_0{}^{(0)} &= -\epsilon_c \left(\bar{\rho} + \bar{p}\right)\left(\partial^i v + v^i\right), \\
T^i{}_j{}^{(0)} &= \bar{p}\delta_j^i + \epsilon_c \left(\delta p \delta_j^i + \hat{\pi}_j^i\right).
\end{aligned} \tag{A.6}$$

Considering the gauge transformations,

$$\tilde{\eta} = \eta + \epsilon_c \zeta^0, \quad \tilde{x}^i = x^i + \epsilon_c \left(\partial^i \zeta + \zeta^i\right), \tag{A.7}$$

where $\partial_i \zeta^i = 0$, we find that,

$$\begin{aligned}
&\tilde{\phi} = \phi - \mathcal{H}\zeta^0 - \zeta^{0\prime}, \ \ \tilde{\psi} = \psi + \mathcal{H}\zeta^0, \\
&\tilde{B} = B + \zeta^0 - \zeta^\prime, \ \ \tilde{E} = E - \zeta, \\
&\tilde{\delta\rho} = \delta\rho - \zeta^0 \bar{\rho}^\prime, \ \ \tilde{\delta p} = \delta p - \zeta^0 \bar{p}^\prime, \\
&\tilde{v} = v + \zeta^\prime,
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
&\tilde{F}_i = F_i - \zeta_i, \ \ \tilde{S}_i = S_i + \zeta_i^\prime, \\
&\tilde{v}^i = v^i + \zeta^{i\prime},
\end{aligned} \tag{A.9}$$

$$\tilde{H}_{ij} = H_{ij}, \quad \tilde{\pi}_j^i = \pi_j^i, \tag{A.10}$$

where $\mathcal{H} \equiv a'/a$ with $a' \equiv da/d\eta$. From the above gauge transformations we can see that the following quantities are gauge-invariant,

$$\begin{aligned}
&\Phi \equiv \phi + \mathcal{H}\left(B - E'\right) + \left(B - E'\right)', \\
&\Psi \equiv \psi - \mathcal{H}\left(B - E'\right), \\
&\Phi_i \equiv S_i + F_i'.
\end{aligned} \tag{A.11}$$

On the other hand, if we choose $\zeta = E$, $\zeta^0 = E' - B$ and $\zeta_i = F_i$, we have

$$\tilde{B} = \tilde{E} = 0, \quad \tilde{F}_i = 0, \tag{A.12}$$

in which the gauge is completely fixed. This is often referred to as the Newtonian gauge. Then, we are left with six scalars, $(\phi, \psi, v, \delta\rho, \delta p, \Pi)$, two vectors, (S_i, v_i), and two tensors, (H_{ij}, Π_{ij}). However, the vector part decreases rapidly with the expansion of the universe, so we can safely set them to zero [147, 148],

$$S_i = F_i = v_i = \Pi_i = 0. \tag{A.13}$$

Then, for the scalar perturbations, there are six-independent equations, given, respectively, by [147],

$$\begin{aligned}
&\psi'' + 2\mathcal{H}\psi' + \mathcal{H}\phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\phi \\
&\qquad\qquad = 4\pi G a^2 \left(\delta p + \frac{2}{3}\nabla^2\Pi\right),
\end{aligned} \tag{A.14}$$

$$\psi - \phi = 8\pi G a^2 \Pi, \tag{A.15}$$

$$3\mathcal{H}\left(\psi' + \mathcal{H}\phi\right) - \nabla^2\psi = -4\pi G a^2 \delta\rho, \tag{A.16}$$

$$\psi' + \mathcal{H}\phi = -4\pi G a^2 \left(\bar{\rho} + \bar{p}\right) v, \tag{A.17}$$

$$\delta\rho' + 3\mathcal{H}\left(\delta\rho + \delta p\right) = \left(\bar{\rho} + \bar{p}\right)\left(3\psi' - \nabla^2 v\right), \tag{A.18}$$

$$\left[\left(\bar{\rho} + \bar{p}\right) v\right]' + \delta p + \frac{2}{3}\nabla^2\Pi - \left(\bar{\rho} + \bar{p}\right)\left(\phi + 4\mathcal{H}v\right). \tag{A.19}$$

Note that Eqs.(A.14) and (A.15) are obtained from the linearized (i, j)-components of the Einstein field equations, and Eqs.(A.16) and (A.17) are the energy and momentum constraints, while Eqs.(A.18) and (A.19) are obtained from the conservation of the energy-momentum tensor.

For the tensor perturbations, we have

$$H''_{ij} + 2\mathcal{H}H'_{ij} - \nabla^2 H_{ij} = 16\pi G a^2 \Pi_{ij}, \tag{A.20}$$

which is obtained from the equations $\delta {G^{(0)}}^i_j = \kappa \delta {T^{(0)}}^i_j$.

It must be noted that in writing the linearized field equations, (A.14) - (A.19), we had implicitly assumed that the quadratic terms $\epsilon_c^2 \hat{R}^{(2)}_{\mu\nu}(\hat{\gamma}) \simeq \mathcal{O}(\hat{\gamma}^2) \ll 1$, which is equivalent to

$$\hat{\gamma} \ll 1, \tag{A.21}$$

where $\hat{R}^{(2)}_{\mu\nu}(\hat{\gamma})$ is given by Eq.(3.20) with the replacement $(h_{\mu\nu}, \nabla_\alpha) \rightarrow \left(\hat{\gamma}_{\mu\nu}, \bar{\nabla}_\alpha\right)$. Otherwise, these quadratic terms cannot be neglected from the Einstein field equations for the background spacetimes,

$$\bar{G}_{\mu\nu}(\bar{\gamma}) + \epsilon_c \hat{G}_{\mu\nu}(\hat{\gamma}) + \epsilon_c^2 \hat{G}^{(2)}_{\mu\nu}(\hat{\gamma}) = \kappa T^{(0)}_{\mu\nu}, \tag{A.22}$$

where,

$$\begin{aligned} &\bar{G}_{\mu\nu}(\bar{\gamma}) \simeq \mathcal{O}(1), \quad \epsilon_c \hat{G}_{\mu\nu}(\hat{\gamma}) \simeq \mathcal{O}(\hat{\gamma}/\epsilon_c), \\ &\epsilon_c^2 \hat{G}^{(2)}_{\mu\nu}(\hat{\gamma}) \simeq \mathcal{O}(\hat{\gamma}^2), \end{aligned} \tag{A.23}$$

as can be seen from Eq.(3.26).

APPENDIX B

Decomposition of Cosmological Perturbations and Gauge Choices

In this Appendix, we shall calculate all the components of the quantities appearing in the field equations (3.80) for $\chi_{\alpha\beta}$, by imposing only the spatial gauge,

$$\chi_{0\mu} = 0.$$

In particular, to calculate the non-vanishing components of the tensor $\mathcal{G}_{\alpha\beta}$, we first note that,

$$\begin{aligned}
&\chi^{ij} \equiv \gamma^{i\mu}\gamma^{j\nu}\chi_{\mu\nu} = \gamma^{ik}\gamma^{jl}\chi_{kl} = \frac{1}{a^2}\Big\{\delta^{ik}\delta^{jl} + \epsilon_c\left[4\psi\delta^{ik}\delta^{jl} - \left(\delta^{ik}H^{jl} + \delta^{jl}H^{ik}\right)\right]\Big\}\hat{\chi}_{kl},\\
&\gamma_{ij}\chi^{ij} = \hat{\chi} + \epsilon_c\left(2\psi\hat{\chi} - H^{kl}\hat{\chi}_{kl}\right), \quad \chi \equiv \gamma^{\mu\nu}\chi_{\mu\nu} = \gamma^{ij}\chi_{ij} = \gamma_{ij}\chi^{ij},\\
&\gamma_{ij}\chi^{ik}\hat{\pi}^{j}_{k} = \Big[\hat{\pi}^{kl} + \epsilon_c\left(2\psi\hat{\pi}^{kl} - \hat{\pi}^{k}_{m}H^{ml}\right)\Big]\hat{\chi}_{kl},\\
&\chi^{\gamma\sigma}{T_{\gamma\sigma}}^{(0)} - \frac{1}{2}\chi T^{(0)} = \frac{1}{2}\left(\bar{\rho} - \bar{p}\right)\hat{\chi} + \frac{1}{2}\epsilon_c\Big[\left(\bar{\rho} - \bar{p}\right)\left(2\psi\hat{\chi} - H^{kl}\hat{\chi}_{kl}\right)\\
&\qquad\qquad + \left(\delta\rho - \delta p\right)\hat{\chi} + 2\hat{\pi}^{kl}\hat{\chi}_{kl}\Big],
\end{aligned} \tag{B.1}$$

where $\hat{\chi} \equiv \delta^{ij}\hat{\chi}_{ij}$, $\chi_{ij} \equiv a^2\hat{\chi}_{ij}$, $\hat{\pi}_{ij} \equiv \delta_{ik}\hat{\pi}^k_j$, etc. We find that the terms $\mathcal{F}_{\mu\nu}$ take the form,

$$
\begin{aligned}
\mathcal{F}_{00} =& - a^2\frac{1}{2}\Big\{(\bar{\rho}-\bar{p})\hat{\chi} \\
&+ \epsilon_c\big[(\bar{\rho}-\bar{p})\big(2(\psi+\phi)\hat{\chi} - H^{kl}\hat{\chi}_{kl}\big) + (\delta\rho-\delta p)\hat{\chi} + 2\hat{\pi}^{kl}\hat{\chi}_{kl}\big]\Big\}, \\
\mathcal{F}_{0i} =& - a^2\epsilon\big(\bar{\rho}+\bar{p}\big)\hat{\chi}_{ik}\partial^k v, \\
\mathcal{F}_{ij} =& a^2\frac{1}{2}\big[4\bar{p}\hat{\chi}_{ij} + (\delta\rho-\delta p)\hat{\chi}\delta_{ij}\big] \\
&+ a^2\frac{1}{2}\epsilon_c\Big\{\big[4\delta\rho\hat{\chi}_{ij} + (\delta\rho-\delta p)\hat{\chi}\delta_{ij}\big] + (\bar{\rho}-\bar{p})\big(\hat{\chi}H_{ij} - H^{kl}\hat{\chi}_{kl}\delta_{ij}\big) \\
&+ 2\big(\hat{\pi}^k_i\hat{\chi}_{jk} + \hat{\pi}^k_j\hat{\chi}_{ik} + \hat{\pi}^{kl}\hat{\chi}_{kl}\delta_{ij}\big)\big]\Big\}.
\end{aligned}
\tag{B.2}
$$

The Christoffel symboles are defined as,

$$
\Gamma^\mu_{\nu\lambda}\left(\gamma_{\alpha\beta}\right) = \bar{\Gamma}^\mu_{\nu\lambda} + \epsilon_c\mathcal{D}^\mu_{\nu\lambda}, \tag{B.3}
$$

Where $\bar{\Gamma}^\mu_{\nu\lambda}$ is our background Christoffel symbol composed of only of the zeroth-order background metric, $\bar{\gamma}_{\mu\nu}$, and $\mathcal{D}^\mu_{\nu\lambda}$ is defined as,

$$
\mathcal{D}^\mu_{\nu\lambda} \equiv \frac{1}{2}\bar{\gamma}^{\mu\sigma}\left(\hat{\gamma}_{\sigma\lambda,\nu} + \hat{\gamma}_{\nu\sigma,\lambda} - \hat{\gamma}_{\nu\lambda,\sigma}\right) - \bar{\Gamma}^\sigma_{\nu\lambda}\hat{\gamma}^\mu_\sigma. \tag{B.4}
$$

The non-vanishing terms of the Christoffel symbols are then,

$$
\begin{aligned}
\bar{\Gamma}^0_{00} =& \mathcal{H}, \quad \bar{\Gamma}^0_{ij} = \mathcal{H}\delta_{ij}, \quad \bar{\Gamma}^i_{0j} = \mathcal{H}\delta^i_j, \\
\mathcal{D}^0_{00} =& \phi', \quad \mathcal{D}^0_{0i} = \phi_{,i}, \\
\mathcal{D}^0_{ij} =& \mathcal{H}\left[H_{ij} - 2\left(\psi+\phi\right)\delta_{ij}\right] + \frac{1}{2}\left(H'_{ij} - 2\psi'\delta_{ij}\right), \\
\mathcal{D}^i_{00} =& \partial^i\phi, \quad \mathcal{D}^i_{0j} = \frac{1}{2}\left(H^{i\prime}_j - 2\psi'\delta^i_j\right), \\
\mathcal{D}^k_{ij} =& H^k_{(i,j)} - 2\psi_{(,i}\delta^k_{j)} - \frac{1}{2}\partial^k\left(H_{ij} - 2\psi\delta_{ij}\right).
\end{aligned}
\tag{B.5}
$$

The covariant derivative is then defined as,

$$
\nabla_\alpha\chi_{\mu\nu} = \partial_\alpha\chi_{\mu\nu} - \left(\bar{\Gamma}^\sigma_{\alpha\mu}\chi_{\sigma\nu} + \bar{\Gamma}^\sigma_{\alpha\nu}\chi_{\mu\sigma}\right) - \epsilon_c\big(\mathcal{D}^\sigma_{\alpha\mu}\chi_{\sigma\nu} + \mathcal{D}^\sigma_{\alpha\nu}\chi_{\mu\sigma}\big). \tag{B.6}
$$

Then, we find that the non-vanishing (independent) components of the Riemann tensor,

$$R_{\mu\nu\alpha\beta}{}^{(0)} = \bar{R}_{\mu\nu\alpha\beta} + \epsilon_c \hat{R}_{\mu\nu\alpha\beta}, \tag{B.7}$$

are given, respectively, by

$$\begin{aligned}
\bar{R}_{0i0j} &= a^2 \left(\mathcal{H}^2 - \frac{a''}{a} \right) \delta_{ij}, \\
\bar{R}_{minj} &= a^2 \mathcal{H}^2 \left(\delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj} \right),
\end{aligned} \tag{B.8}$$

and

$$\begin{aligned}
\hat{R}_{0i0j} =& a^2 \Bigg\{ \phi_{,ij} + \mathcal{H}\phi'\delta_{ij} + \left[(\psi'' + \mathcal{H}\psi') + 2\left(\frac{a''}{a} - \mathcal{H}^2 \right) \psi \right] \delta_{ij} \\
& - \frac{1}{2} \left[(H_{ij}'' + \mathcal{H} H_{ij}') + 2 \left(\frac{a''}{a} - \mathcal{H}^2 \right) H_{ij} \right] \Bigg\}, \\
\hat{R}_{0ijk} =& a^2 \Big[\mathcal{H} \left(\phi_{,j}\delta_{ik} - \phi_{,k}\delta_{ij} \right) + \left(\psi'_{,j}\delta_{ik} - \psi'_{,k}\delta_{ij} \right) + \frac{1}{2} \left(H'_{ij,k} - H'_{ik,j} \right) \Big], \\
\hat{R}_{ijkl} =& - 2a^2\mathcal{H}^2\phi \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \right) \\
& - a^2 \Big[\left(\delta_{jk}\psi_{,il} + \delta_{il}\psi_{,jk} - \delta_{ik}\psi_{,jl} - \delta_{jl}\psi_{,ik} \right) + 2\mathcal{H} \left(\psi' + 2\mathcal{H}\psi \right) \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \right) \Big] \\
& + \frac{1}{2} a^2 \Bigg\{ \left(H_{jk,il} + H_{il,jk} - H_{ik,jl} - H_{jl,ik} \right) - \mathcal{H} \Big[\delta_{il} \left(H'_{jk} + 2\mathcal{H} H_{jk} \right) \\
& + \delta_{jk} \left(H'_{il} + 2\mathcal{H} H_{il} \right) - \delta_{jl} \left(H'_{ik} + 2\mathcal{H} H_{ik} \right) - \delta_{ik} \left(H'_{jl} + 2\mathcal{H} H_{jl} \right) \Big] \Bigg\}.
\end{aligned} \tag{B.9}$$

Hence, we find that,

$$
\begin{aligned}
2R_{0i0j}{}^{(0)}\chi^{ij} =& 2\left(\frac{a''}{a}-\mathcal{H}^2\right)\hat{\chi}+\epsilon_c\Bigg\{2\left(\partial^i\partial^j\phi\right)\hat{\chi}_{ij}+2\mathcal{H}\phi'\hat{\chi}\\
&+2\big(\psi''+\mathcal{H}\psi'\big)\hat{\chi}-4\left(\frac{a''}{a}-\mathcal{H}^2\right)\psi\hat{\chi}\\
&-\big(H^{ij''}+\mathcal{H}H^{ij'}\big)\hat{\chi}_{ij}+2\left(\frac{a''}{a}-\mathcal{H}^2\right)H^{ij}\hat{\chi}_{ij}\Bigg\},\\
2R_{0jik}{}^{(0)}\chi^{jk} =& 2\epsilon_c\Bigg\{\mathcal{H}\left[\left(\partial_i\phi\right)\hat{\chi}-\left(\partial^k\phi\right)\hat{\chi}_{ik}\right]+\left(\partial_i\psi'\right)\hat{\chi}-\left(\partial^k\psi'\right)\hat{\chi}_{ik}\\
&+\frac{1}{2}\left[\left(\partial^k H_i^{j'}\right)-\left(\partial_i H^{jk'}\right)\right]\hat{\chi}_{jk}\Bigg\},\\
2R_{ikjl}{}^{(0)}\chi^{kl} =& 2\mathcal{H}^2\big(\delta_{ij}\hat{\chi}-\hat{\chi}_{ij}\big)+\epsilon_c\Bigg\{4\mathcal{H}^2\phi\left(\hat{\chi}_{ij}-\hat{\chi}\delta_{ij}\right)+4\mathcal{H}\psi'\left(\hat{\chi}_{ij}-\hat{\chi}\delta_{ij}\right)\\
&+2\Big[\left(\partial_i\partial_j\psi\right)\hat{\chi}+\left(\partial^k\partial^l\psi\right)\hat{\chi}_{kl}\delta_{ij}-\left(\partial^k\partial_i\psi\right)\hat{\chi}_{jk}-\left(\partial^k\partial_j\psi\right)\hat{\chi}_{ik}\Big]\\
&-2\mathcal{H}^2H^{kl}\hat{\chi}_{kl}\delta_{ij}+2\mathcal{H}^2H_{ij}\hat{\chi}-2\mathcal{H}H_{(i}^{k\,\prime}\hat{\chi}_{j)k}\\
&+\mathcal{H}H_{ij}{}'\hat{\chi}+\mathcal{H}H^{kl'}\hat{\chi}_{kl}\delta_{ij}+\left(2\partial_{(i}\partial^l H_{j)}^k-\partial^k\partial^l H_{ij}-\partial_i\partial_j H^{kl}\right)\hat{\chi}_{kl}\Bigg\}.
\end{aligned}
\tag{B.10}
$$

On the other hand, similar to the above expression, writing $\Box\chi_{\alpha\beta}$ in the form,

$$
\Box\chi_{\alpha\beta}\equiv\ \bar{\Box}\chi_{\alpha\beta}+\epsilon_c\hat{\Box}\chi_{\alpha\beta}, \tag{B.11}
$$

we find they are given, respectively, by

$$
\begin{aligned}
\bar{\Box}\chi_{00} &= 2\mathcal{H}^2\hat{\chi},\\
\bar{\Box}\chi_{0i} &= -2\mathcal{H}\partial^j\hat{\chi}_{ij},\\
\bar{\Box}\chi_{ij} &= -\hat{\chi}''_{ij}-2\mathcal{H}\hat{\chi}'_{ij}+\partial^2\hat{\chi}_{ij}+2\mathcal{H}^2\hat{\chi}_{ij},
\end{aligned}
\tag{B.12}
$$

and,

$$
\begin{aligned}
\hat{\Box}\chi_{00} =& -2\mathcal{H}\Big[2\left(\psi' - \mathcal{H}\psi\right)\hat{\chi} - \left(H^{ij\prime} - \mathcal{H}H^{ij}\right)\hat{\chi}_{ij}\Big], \\
\hat{\Box}\chi_{0i} =& \left(\partial^j \phi'\right)\hat{\chi}_{ij} + 2\left(\partial^j \phi\right)\left(\hat{\chi}'_{ij} + \mathcal{H}\hat{\chi}_{ij}\right) + \left(\partial^j \psi'\right)\hat{\chi}_{ij} + 2\left(\psi' - 2\mathcal{H}\psi\right)\partial^j \hat{\chi}_{ij} \\
& + 2\mathcal{H}\left[\left(\partial^j \psi\right)\hat{\chi}_{ij} - \left(\partial_i \psi\right)\hat{\chi}\right] - \left(H^{jk\prime} - 2\mathcal{H}H^{jk}\right)\partial_k \hat{\chi}_{ij} + \mathcal{H}\left(\partial_i H^{jk}\right)\hat{\chi}_{jk}, \\
\hat{\Box}\chi_{ij} =& 2\phi\hat{\chi}''_{ij} + \left(\phi' + 4\mathcal{H}\phi\right)\hat{\chi}'_{ij} + \left(\partial^k \phi\right)\partial_k \hat{\chi}_{ij} - 4\mathcal{H}^2\phi\hat{\chi}_{ij} \\
& + 2\psi\partial^2\hat{\chi}_{ij} + 4\partial_{(i}\psi\partial^k\hat{\chi}_{j)k} + 3\left(\partial^k\psi\right)\partial_k\hat{\chi}_{ij} - 4\left(\partial^k\psi\right)\partial_{(i}\hat{\chi}_{j)k} \\
& + 2\partial^k\partial_{(i}\psi\hat{\chi}_{j)k} + 2\left(\partial^2\psi\right)\hat{\chi}_{ij} - 2\partial_{(i}\partial^k\psi\hat{\chi}_{j)k} - \psi'\hat{\chi}'_{ij} - 2\left(\psi'' + 4\mathcal{H}\psi'\right)\hat{\chi}_{ij} \\
& - H^{kl}\partial_k\partial_l\hat{\chi}_{ij} - 2\partial^l H^k_{(i}\partial_l\hat{\chi}_{j)k} - 2\partial_{(i}H^{kl}\partial_l\hat{\chi}_{j)k} \\
& + 2\partial^k H^l_{(i}\partial_l\hat{\chi}_{j)k} + 2H^{k\prime}_{(i}\hat{\chi}'_{j)k} + H^{k\prime\prime}_{(i}\hat{\chi}_{j)k} + 4\mathcal{H}H^{k\prime}_{(i}\hat{\chi}_{j)k} - \partial^2 H^k_{(i}\hat{\chi}_{j)k},
\end{aligned}
\tag{B.13}
$$

where $2\partial^k\partial_{(i}\psi\hat{\chi}_{j)k} \equiv \left(\partial^k\partial_i\psi\right)\hat{\chi}_{jk} + \left(\partial^k\partial_j\psi\right)\hat{\chi}_{ik}$, that is, the partial derivative acts only to the first function. The same is true for other terms, for example, $2\partial^l H^k_{(i}\partial_l\hat{\chi}_{j)k} \equiv \left(\partial^l H^k_i\right)\partial_l\hat{\chi}_{jk} + \left(\partial^l H^k_j\right)\partial_l\hat{\chi}_{ik}$.

On the other hand, defining

$$
\mathcal{G}^{(1)}_{\alpha\beta} \equiv \gamma_{\alpha\beta}\nabla^\gamma\nabla^\delta\chi_{\gamma\delta}, \tag{B.14}
$$

we find that

$$
\begin{aligned}
\mathcal{G}^{(1)}_{00} =& -\mathcal{G}^{(1)}_0 - \epsilon_c\left(2\phi\mathcal{G}^{(1)}_0 + \mathcal{G}^{(1)}_1\right), \quad \mathcal{G}^{(1)}_{0i} = \mathcal{G}^1_{i0} = 0, \\
\mathcal{G}^{(1)}_{ij} =& \delta_{ij}\mathcal{G}^{(1)}_0 + \epsilon_c\left[\delta_{ij}\left(\mathcal{G}^{(1)}_1 - 2\psi\mathcal{G}^{(1)}_0\right) + H_{ij}\mathcal{G}^{(1)}_0\right],
\end{aligned}
\tag{B.15}
$$

where,

$$
\begin{aligned}
\mathcal{G}_0^{(1)} \equiv& \mathcal{H}\hat{\chi}' + \left(\frac{a''}{a} + \mathcal{H}^2\right)\hat{\chi} + \partial^i\partial^j\hat{\chi}_{ij}, \\
\mathcal{G}_1^{(1)} \equiv& - 2\mathcal{H}\phi\hat{\chi}' - \left[2\left(\frac{a''}{a} + \mathcal{H}^2\right)\phi + \mathcal{H}\phi'\right]\hat{\chi} + 2\left(\partial^i\phi\right)\left(\partial^j\hat{\chi}_{ij}\right) + \left(\partial^i\partial^j\phi\right)\hat{\chi}_{ij} \\
& - \left(\psi' - 2\mathcal{H}\psi\right)\hat{\chi}' - \left[\psi'' + 3\mathcal{H}\psi' - \partial^2\psi - 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\psi\right]\hat{\chi} \\
& + (\partial^i\psi)\partial_i\hat{\chi} + 4\psi\partial^i\partial^j\hat{\chi}_{ij} - \left(\partial^i\partial^j\psi\right)\hat{\chi}_{ij} \\
& + \frac{1}{2}\left\{\left(H^{ij\prime} - 2\mathcal{H}H^{ij}\right)\hat{\chi}'_{ij} + \left[H^{ij\prime\prime} - 2\left(\frac{a''}{a} + \mathcal{H}^2\right)H^{ij} - \partial^2 H^{ij}\right]\hat{\chi}_{ij}\right. \\
& \left. - 4H^{ik}\partial_k\partial^j\hat{\chi}_{ij} - \left(\partial_k H^{ij}\right)\left(\partial^k\hat{\chi}_{ij}\right) - 2\left(\partial^i H^{jk}\right)\left(\partial_k\hat{\chi}_{ij}\right)\right\}. \qquad \text{(B.16)}
\end{aligned}
$$

On the other hand, defining,

$$
\mathcal{G}_{\alpha\beta}^{(2)} \equiv \nabla_\alpha\nabla^\delta\chi_{\beta\delta}, \qquad \text{(B.17)}
$$

we find that it has the following non-vanishing components,

$$
\begin{aligned}
\mathcal{G}^{(2)}_{00} =& -\frac{a'}{a}\hat{\chi}' - \left(\frac{a''}{a} - 2\mathcal{H}^2\right)\hat{\chi} \\
&+ \epsilon_c \Bigg[\mathcal{H}\phi'\hat{\chi} - \left(\partial^i\phi\right)\partial^k\hat{\chi}_{ik} + \left(\psi' - 2\mathcal{H}\psi\right)\hat{\chi}' + \left(\psi'' - 3\mathcal{H}\psi'\right)\hat{\chi} - 2\left(\frac{a''}{a} - 2\mathcal{H}^2\right)\psi\hat{\chi} \\
&- \frac{1}{2}\left(H^{ij\prime} - 2\mathcal{H}H^{ij}\right)\hat{\chi}'_{ij} - \frac{1}{2}\left(H^{ij\prime\prime} - 3\mathcal{H}H^{ij\prime}\right)\hat{\chi}_{ij} + \left(\frac{a''}{a} - 2\mathcal{H}^2\right)H^{ij}\hat{\chi}_{ij} \Bigg], \\
\mathcal{G}^{(2)}_{0i} =& \partial^k\hat{\chi}'_{ik} - \mathcal{H}\partial^k\hat{\chi}_{ik} + \epsilon_c \Bigg[\left(\partial^j\phi\right)\hat{\chi}'_{ij} + \left(\partial^j\phi' - \mathcal{H}\partial^j\phi\right)\hat{\chi}_{ij} + \mathcal{H}\left(\partial_i\phi\right)\hat{\chi} \\
&+ 2\psi\partial^k\hat{\chi}'_{ik} - \left(\partial^k\psi\right)\hat{\chi}'_{ik} + \left(3\psi' - 2\mathcal{H}\psi\right)\partial^k\hat{\chi}_{ik} + \left(\partial_i\psi\right)\hat{\chi}' \\
&- \left(\partial^k\psi' - \mathcal{H}\partial^k\psi\right)\hat{\chi}_{ik} + \left(\partial_i\psi' - \mathcal{H}\partial_i\psi\right)\hat{\chi} \\
&- H^{jk}\partial_k\hat{\chi}'_{ij} - \left(H^{jk\prime} - \mathcal{H}H^{jk}\right)\partial_k\hat{\chi}_{ij} - \frac{1}{2}H^{j\prime}_{i}\partial^k\hat{\chi}_{jk} \\
&- \frac{1}{2}H^{jk}_{,i}\hat{\chi}'_{jk} - \frac{1}{2}\left(H^{jk\prime} - \mathcal{H}H^{jk}\right)_{,i}\hat{\chi}_{jk} \Bigg], \\
\mathcal{G}^{(2)}_{i0} =& -\mathcal{H}\left(\partial^k\hat{\chi}_{ik} + \partial_i\hat{\chi}\right) + \epsilon_c \Bigg[\mathcal{H}\left(\partial_i\phi\right)\hat{\chi} - \mathcal{H}\left(\partial^k\phi\right)\hat{\chi}_{ik} \\
&+ \left(\psi' - 2\mathcal{H}\psi\right)\left(\partial^k\hat{\chi}_{ik} + \partial_i\hat{\chi}\right) + \mathcal{H}\left(\partial^k\psi\right)\hat{\chi}_{ik} + \left(\psi' - 3\mathcal{H}\psi\right)_{,i}\hat{\chi} \\
&- \frac{1}{2}\left(H^{jk\prime} - 2\mathcal{H}H^{jk}\right)\partial_i\hat{\chi}_{jk} - \frac{1}{2}H^{j\prime}_{i}\partial^k\hat{\chi}_{jk} \\
&+ \mathcal{H}H^{jk}\partial_k\hat{\chi}_{ij} - \frac{1}{2}\left(H^{jk\prime} - 3\mathcal{H}H^{jk}\right)_{,i}\hat{\chi}_{jk} \Bigg], \\
\mathcal{G}^{(2)}_{ij} =& \partial_i\partial^k\hat{\chi}_{jk} + \mathcal{H}^2\hat{\chi}\delta_{ij} + \epsilon_c \Bigg[\left(\partial^k\phi\right)\partial_i\hat{\chi}_{jk} + \left(\partial_i\partial^k\phi\right)\hat{\chi}_{jk} - 2\mathcal{H}^2\phi\hat{\chi}\delta_{ij} \\
&- \partial_i\left[\left(\partial^k\psi\right)\hat{\chi}_{jk}\right] + \partial_i\left[\left(\partial_j\psi\right)\hat{\chi}\right] + 2\partial_i(\psi\partial^k\hat{\chi}_{jk}) + \left(\partial_i\psi\right)\partial^k\hat{\chi}_{jk} \\
&+ \left(\partial_j\psi\right)\partial^k\hat{\chi}_{ik} - \left(\partial^k\psi\right)\partial^l\hat{\chi}_{kl}\delta_{ij} - 2\mathcal{H}\psi'\hat{\chi}\delta_{ij} \\
&- \partial_i(H^{kl}\partial_l\hat{\chi}_{jk}) - \frac{1}{2}\partial_i\left[\left(\partial_j H^{kl}\right)\hat{\chi}_{kl}\right] + \frac{1}{2}\left(\partial^k H_{ij} - H^k_{i,j} - H^k_{j,i}\right)\partial^l\hat{\chi}_{kl} \\
&+ \frac{1}{2}\mathcal{H}\left(H^{kl\prime} - 2\mathcal{H}H^{kl}\right)\hat{\chi}_{kl}\delta_{ij} + \frac{1}{2}\mathcal{H}\left(H'_{ij} + 2\mathcal{H}H_{ij}\right)\hat{\chi} \Bigg].
\end{aligned} \tag{B.18}
$$

Note that $\mathcal{G}^{(2)}_{\alpha\beta}$ is not symmetric, $\mathcal{G}^{(2)}_{\alpha\beta} \neq \mathcal{G}^{(2)}_{\beta\alpha}$, as can be seen from its definition given by Eq.(B.17).

Finally, defining $\mathcal{G}_{\alpha\beta}$ as

$$\mathcal{G}_{\alpha\beta} \equiv \ \Box\chi_{\alpha\beta} + \gamma_{\alpha\beta}\nabla^{\gamma}\nabla^{\delta}\chi_{\gamma\delta} - \nabla_{\alpha}\nabla^{\delta}\chi_{\beta\delta} - \nabla_{\beta}\nabla^{\delta}\chi_{\alpha\delta} + 2R_{\alpha\gamma\beta\sigma}{}^{(0)}\chi^{\gamma\sigma}, \tag{B.19}$$

we find that its non-vanishing components are given by,

$$\begin{aligned}
\mathcal{G}_{00} =& \mathcal{H}\hat{\chi}' - \left(\frac{a''}{a} + \mathcal{H}^2\right)\hat{\chi} - \partial^i\partial^j\hat{\chi}_{ij} \\
& + \epsilon_c\Bigg\{\mathcal{H}\phi'\hat{\chi} + \left(\partial^i\partial^j\phi\right)\hat{\chi}_{ij} - 2\phi\left(\partial^i\partial^j\hat{\chi}_{ij}\right) + \left[\psi'' + 7\mathcal{H}\psi' - 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\psi - \partial^2\psi\right]\hat{\chi} \\
& - \left(\psi' - 2\mathcal{H}\psi\right)\hat{\chi}' + \left(\partial^i\partial^j\psi\right)\hat{\chi}_{ij} - 4\psi\left(\partial^i\partial^j\hat{\chi}_{ij}\right) - \left(\partial^k\psi\right)\partial_k\hat{\chi} \\
& - \frac{1}{2}\left[\left(H^{ij\prime\prime} - \partial^2 H^{ij}\right) + 4\mathcal{H}H^{ij\prime} - 2\left(\frac{a''}{a} + \mathcal{H}^2\right)H^{ij}\right]\hat{\chi}_{ij} \\
& + \frac{1}{2}\left(H^{ij\prime} - 2\mathcal{H}H^{ij}\right)\hat{\chi}'_{ij} + 2H^i_k(\partial^k\partial^j\hat{\chi}_{ij}) + \frac{1}{2}(\partial^k H^{ij})\partial_k\hat{\chi}_{ij} + (\partial^i H^{jk})\partial^k\hat{\chi}_{ij}\Bigg\},
\end{aligned} \tag{B.20}$$

$$\begin{aligned}
\mathcal{G}_{0i} =& \mathcal{H}\partial_i\hat{\chi} - \partial^j\hat{\chi}'_{ij} \\
& + \epsilon_c\Bigg\{\left(\partial^j\phi\right)\hat{\chi}'_{ij} + 2\mathcal{H}\left(\partial^j\phi\right)\hat{\chi}_{ij} - 2\psi'(\partial^j\hat{\chi}_{ij}) + 2\mathcal{H}(\partial^i\psi)\hat{\chi} - 2\psi(\partial^j\hat{\chi}'_{ij}) \\
& + (\partial^j\psi)\hat{\chi}'_{ij} - (\partial_i\psi)\hat{\chi}' - (\psi' - 2\mathcal{H}\psi)\partial_i\hat{\chi} - \mathcal{H}(\partial_i H^{jk})\hat{\chi}_{jk} + \left(\partial^k H^{j\prime}_i\right)\hat{\chi}_{jk} \\
& + H^{j\prime}_i\left(\partial^k\hat{\chi}_{jk}\right) + H^{jk}\left(\partial_k\hat{\chi}'_{ij}\right) + \frac{1}{2}(\partial_i H^{jk})\hat{\chi}'_{jk} + \frac{1}{2}H^{jk\prime}(\partial_i\hat{\chi}_{jk}) - \mathcal{H}H^{jk}(\partial_i\hat{\chi}_{jk})\Bigg\},
\end{aligned} \tag{B.21}$$

$$
\begin{aligned}
\mathcal{G}_{ij} =& -\hat{\chi}''_{ij} + \partial^2\hat{\chi}_{ij} - 2\mathcal{H}\hat{\chi}'_{ij} + \mathcal{H}\delta_{ij}\hat{\chi}' + \delta_{ij}\left(\frac{a''}{a} + \mathcal{H}^2\right)\hat{\chi} + \partial^k\partial^l\hat{\chi}_{kl}\delta_{ij} - \partial_i\partial^k\hat{\chi}_{jk} - \partial_j\partial^k\hat{\chi}_{ik} \\
& + \epsilon_c\Bigg\{2\phi\hat{\chi}''_{ij} + \big(\phi' + 4\mathcal{H}\phi\big)\hat{\chi}'_{ij} - \left[\mathcal{H}\phi' + 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\phi\right]\delta_{ij}\hat{\chi} - 2\mathcal{H}\phi\delta_{ij}\hat{\chi}' \\
& - (\partial_j\partial^k\phi)\hat{\chi}_{ik} - (\partial_i\partial^k\phi)\hat{\chi}_{jk} + (\partial^k\phi)\partial_k\hat{\chi}_{ij} + 2(\partial^k\phi)\partial^l\hat{\chi}_{kl}\delta_{ij} - (\partial^k\phi)\partial_i\hat{\chi}_{kj} \\
& - (\partial^k\phi)\partial_j\hat{\chi}_{ik} + (\partial^k\partial^l\phi)\hat{\chi}_{kl}\delta_{ij} \\
& + 2\big(\partial^2\psi - \psi'' - 2\mathcal{H}\psi'\big)\hat{\chi}_{ij} + \big(\partial^2\psi - \psi'' - 3\mathcal{H}\psi'\big)\hat{\chi}\delta_{ij} - \psi'\big(\hat{\chi}'_{ij} + \hat{\chi}'\delta_{ij}\big) + 2\psi\partial^2\hat{\chi}_{ij} \\
& + \partial^k\psi\big(3\partial_k\hat{\chi}_{ij} + \partial_k\hat{\chi}\delta_{ij} - \partial_i\hat{\chi}_{jk} - \partial_j\hat{\chi}_{ik} + 2\partial^l\hat{\chi}_{kl}\delta_{ij}\big) + 2\psi\partial^k\partial^l\hat{\chi}_{kl}\delta_{ij} + (\partial^k\partial^l\psi)\hat{\chi}_{kl}\delta_{ij} \\
& - 4\partial_{(i}\psi\partial^k\hat{\chi}_{j)k} - 2\partial_{(i}\partial^k\psi\hat{\chi}_{j)k} - 4\psi\partial_{(i}\partial^k\hat{\chi}_{j)k} - 2\partial_{(i}\psi\partial_{j)}\hat{\chi} \\
& + \frac{1}{2}\big(H^{kl\prime} - 2\mathcal{H}H^{kl}\big)\hat{\chi}'_{kl}\delta_{ij} + \left[\frac{1}{2}H^{kl\prime\prime} - \left(\frac{a''}{a} + \mathcal{H}^2\right)H^{kl}\right]\hat{\chi}_{kl}\delta_{ij} + H^{k\prime\prime}_{(i}\hat{\chi}_{j)k} + 2\mathcal{H}H^{k\prime}_{(i}\hat{\chi}_{j)k} \\
& + 2H^{k\prime}_{(i}\hat{\chi}'_{j)k} + \left[\mathcal{H}\hat{\chi}' + \left(\frac{a''}{a} + \mathcal{H}^2\right)\hat{\chi} + \partial^k\partial^l\hat{\chi}_{kl}\right]H_{ij} - 2H^{jk}\partial_k\partial^m\hat{\chi}_{lm}\delta_{ij} \\
& - \frac{1}{2}\partial^2 H^{kl}\hat{\chi}_{kl}\delta_{ij} - \frac{1}{2}\partial_m H^{kl}\partial^m\hat{\chi}_{kl}\delta_{ij} - 2\partial^l H^k_{(i}\partial_l\hat{\chi}_{j)k} + 2\partial^k H^l_{(i}\partial_l\hat{\chi}_{j)k} \\
& - \partial^2 H^k_{(i}\hat{\chi}_{j)k} + \partial_{(i}H^{kl}\partial_{j)}\hat{\chi}_{kl} + \partial_{(i}H^k_{j)}\partial^l\hat{\chi}_{kl} + 2\partial_{(i}\partial^l H^k_{j)}\hat{\chi}_{kl} - \partial^k\partial^l H_{ij}\hat{\chi}_{kl} \\
& - \partial^k H_{ij}\partial^l\hat{\chi}_{kl} + 2H^{kl}\partial_{(i}\partial_l\hat{\chi}_{j)k} - \partial^k H^{ml}\partial_m\hat{\chi}_{kl}\delta_{ij} - H^{kl}\partial_k\partial_l\hat{\chi}_{ij}\Bigg\}. \qquad \text{(B.22)}
\end{aligned}
$$

APPENDIX C

Decomposition of Cosmological Perturbations and Gauge Choices

Following Eq.(3.98), we write $\hat{\chi}_{\alpha\beta}$ in the form,

$$\hat{\chi}_{\alpha\beta} \simeq \hat{\chi}^{(0)}_{\alpha\beta} + \epsilon_c \hat{\chi}^{(1)}_{\alpha\beta} + \mathcal{O}\left(\epsilon_c^2\right), \tag{C.1}$$

where to the zeroth-order, the TT gauge,

$$\hat{\chi}^{(0)}_{0\beta} = 0, \quad \hat{\chi}^{(0)} = 0, \quad \partial^i \hat{\chi}^{(0)}_{ij} = 0, \tag{C.2}$$

will be chosen. But, to the first order, we shall not impose the traceless and Lorentz gauge conditions. The only gauge that now we choose is,

$$\hat{\chi}^{(1)}_{0\beta} = 0. \tag{C.3}$$

With this gauge choice, to the first-order of ϵ_c, the non-vanishing components of the tensor $\mathcal{G}_{\alpha\beta}$ given by Eqs.(B.20)-(B.22) yield,

$$\mathcal{G}^{(1)}_{00} = \mathcal{H}\hat{\chi}'^{(1)} - \left(\frac{a''}{a} + \mathcal{H}^2\right)\hat{\chi}^{(1)} - \partial^i\partial^j\hat{\chi}^{(1)}_{ij} + \left[\partial^i\partial^j\left(\phi+\psi\right)\right]\hat{\chi}^{(1)}_{ij} + \hat{\mathcal{G}}^{(1)}_{00}, \tag{C.4}$$

$$\mathcal{G}^{(1)}_{0i} = \mathcal{H}\partial_i\hat{\chi}^{(1)} - \partial^j\hat{\chi}'^{(1)}_{ij} + \hat{\mathcal{G}}^{(1)}_{0i}, \tag{C.5}$$

$$\begin{aligned}\mathcal{G}^{(1)}_{ij} =& -\hat{\chi}''^{(1)}_{ij} + \partial^2\hat{\chi}^{(1)}_{ij} - 2\mathcal{H}\hat{\chi}'^{(1)}_{ij} + \mathcal{H}\delta_{ij}\hat{\chi}'^{(1)} + \delta_{ij}\left(\frac{a''}{a} + \mathcal{H}^2\right)\hat{\chi}^{(1)} \\ &+ \partial^k\partial^l\hat{\chi}^{(1)}_{kl}\delta_{ij} - \partial_i\partial^k\hat{\chi}^{(1)}_{jk} - \partial_j\partial^k\hat{\chi}^{(1)}_{ik} + \hat{\mathcal{G}}^{(1)}_{ij},\end{aligned} \tag{C.6}$$

where,

$$
\begin{aligned}
\hat{\mathcal{G}}^{(1)}_{00} = &- \frac{1}{2}\left[\left(H^{ij\prime\prime} - \partial^2 H^{ij}\right) + 4\mathcal{H}H^{ij\prime} - 2\left(\frac{a''}{a} + \mathcal{H}^2\right)H^{ij}\right]\hat{\chi}^{(0)}_{ij} \\
&+ \frac{1}{2}\left(H^{ij\prime} - 2\mathcal{H}H^{ij}\right)\hat{\chi}'^{(0)}_{ij} + \frac{1}{2}\left(\partial^k H^{ij}\right)\partial_k\hat{\chi}^{(0)}_{ij} + \left(\partial^i H^{jk}\right)\partial^k\hat{\chi}^{(0)}_{ij}, \qquad \text{(C.7)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathcal{G}}^{(1)}_{0i} = &+ \left(\partial^j\phi\right)\hat{\chi}'^{(0)}_{ij} + 2\mathcal{H}\left(\partial^j\phi\right)\hat{\chi}^{(0)}_{ij} + \left(\partial^j\psi\right)\hat{\chi}'^{(0)}_{ij} \\
&- \mathcal{H}\left(\partial_i H^{jk}\right)\hat{\chi}^{(0)}_{jk} + \left(\partial^k H'^j_i\right)\hat{\chi}^{(0)}_{jk} + H^{jk}\left(\partial_k\hat{\chi}'^{(0)}_{ij}\right) \\
&+ \frac{1}{2}H'^{jk}\left(\partial_i\hat{\chi}^{(0)}_{jk}\right) - \mathcal{H}H^{jk}\left(\partial_i\hat{\chi}^{(0)}_{jk}\right), \qquad \text{(C.8)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathcal{G}}^{(1)}_{ij} = &+ 2\phi\hat{\chi}''^{(0)}_{ij} + \left(\phi' + 4\mathcal{H}\phi\right)\hat{\chi}'^{(0)}_{ij} - \left(\partial_j\partial^k\phi\right)\hat{\chi}^{(0)}_{ik} - \left(\partial_i\partial^k\phi\right)\hat{\chi}^{(0)}_{jk} \qquad \text{(C.9)} \\
&+ \left(\partial^k\phi\right)\partial_k\hat{\chi}^{(0)}_{ij} - \left(\partial^k\phi\right)\partial_i\hat{\chi}^{(0)}_{kj} - \left(\partial^k\phi\right)\partial_j\hat{\chi}^{(0)}_{ik} + \left(\partial^k\partial^l\phi\right)\hat{\chi}^{(0)}_{kl}\delta_{ij} \\
&+ 2\left(\partial^2\psi - \psi'' - 2\mathcal{H}\psi'\right)\hat{\chi}^{(0)}_{ij} - \psi'\hat{\chi}'^{(0)}_{ij} + 2\psi\partial^2\hat{\chi}^{(0)}_{ij} \\
&+ \partial^k\psi\left(3\partial_k\hat{\chi}^{(0)}_{ij} - \partial_i\hat{\chi}^{(0)}_{jk} - \partial_j\hat{\chi}^{(0)}_{ik}\right) + \left(\partial^k\partial^l\psi\right)\hat{\chi}^{(0)}_{kl}\delta_{ij} \\
&+ \frac{1}{2}\left(H^{kl\prime} - 2\mathcal{H}H^{kl}\right)\hat{\chi}'^{(0)}_{kl}\delta_{ij} + \left[\frac{1}{2}H^{kl\prime\prime} - \left(\frac{a''}{a} + \mathcal{H}^2\right)H^{kl}\right]\hat{\chi}^{(0)}_{kl}\delta_{ij} \\
&+ H^{k\prime\prime}_{(i}\hat{\chi}^{(0)}_{j)k} + 2\mathcal{H}H^{k\prime}_{(i}\hat{\chi}^{(0)}_{j)k} + 2H^{k\prime}_{(i}\hat{\chi}'^{(0)}_{j)k} - \frac{1}{2}\partial^2 H^{kl}\hat{\chi}^{(0)}_{kl}\delta_{ij} - \frac{1}{2}\partial_m H^{kl}\partial^m\hat{\chi}^{(0)}_{kl}\delta_{ij} \\
&- 2\partial^l H^k_{(i}\partial_l\hat{\chi}^{(0)}_{j)k} + 2\partial^k H^l_{(i}\partial_l\hat{\chi}^{(0)}_{j)k} - \partial^2 H^k_{(i}\hat{\chi}^{(0)}_{j)k} + \partial_{(i}H^{kl}\partial_{j)}\hat{\chi}^{(0)}_{kl} + 2\partial_{(i}\partial^l H^k_{j)}\hat{\chi}^{(0)}_{kl} \\
&- \partial^k\partial^l H_{ij}\hat{\chi}^{(0)}_{kl} + 2H^{kl}\partial_{(i}\partial^l\hat{\chi}^{(0)}_{j)k} - \partial^k H^{ml}\partial_m\hat{\chi}^{(0)}_{kl}\delta_{ij}. \qquad \text{(C.10)}
\end{aligned}
$$

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