ABSTRACT

Asymptotic Arc-Components in Inverse Limits of Dendrites

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We study asymptotic behavior arising in inverse limit spaces of dendrites. In particular, the inverse limit is constructed with a single unimodal bonding map, for which points have unique itineraries and the critical point is periodic. Using symbolic dynamics, sufficient conditions for two rays in the inverse limit space to have asymptotic parameterizations are given. Being a topological invariant, the classification of asymptotic parameterizations would be a useful tool when determining if two spaces are homeomorphic.

Asymptotic Arc-Components in Inverse Limits of Dendrites

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A Dissertation

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CHAPTER ONE

Introduction

1.1 Motivation

Inverse limits are a powerful tool and have been extensively studied; not just in research papers, but in many introductory textbooks as well. See, for example, [12], [13], [39],[22], [9], or [4]. While available in any category, they have become a staple in the subfield of Topology known as Continuum Theory. Their utility resides in the relative ease accorded in generating complicated continua. Being "complicated continua," it is often difficult to determine whether two inverse limit spaces are homeomorphic. Determining whether or not two spaces are homeomorphic is the raison d'être of a topologist, and it is the author's hope that this dissertation facilitates such work.

In [20], Henk Bruin considers inverse limit spaces on the interval, and develops a technique for embedding their arc-components in the plane ($\mathbb{R}^2$). It has long been known that inverse limit spaces on the interval are planar ([10]), but there are few results detailing what form the embeddings take. For more on this line of inquiry, see [18], [7], or [36]. Bruin's work from [20] is extended in [22], wherein sufficient conditions for two arc-components, arising from such spaces, to be asymptotic are given. As a topological invariant, the classification of asymptotic arc-components would be useful knowledge in determining whether two spaces are homeomorphic. A survey of Bruin's results from [20] and [22] is the subject of Chapter 4.

In [1], Stewart Baldwin provides a scheme, for a certain collection of dynamical systems, so that the act of identifying points with their respective itineraries is continuous. He then proceeds to give "admissibility" criterion, and the collection of all itineraries satisfying this criterion are shown to be a dendrite. This subsequently allows many concepts, previously relegated to studying dynamics of the interval, to be brought forth to study dynamical systems of dendrites. Of particular note, in [2], Baldwin proves a generalization of Ingram's Conjecture for k-stars. An outline of this work is the subject of Chapter 3.

The impetus of this dissertation, given in detail in Chapter 6, is to take Bruin's work on asymptotic arc-components, and modify it to be applicable to inverse limits of dendrites, as advanced by Baldwin in [1]. Many surprising differences arose in this undertaking. For one, Bruin gave no guarantee that the arc-components in his work necessarily existed; indeed, many do not. All exist in the dendrite setting. Moreover, in the dendrite case, arc-components contain branch points; something not present in the interval case. This ultimately results in a countably infinite collection of distinct asymptotic rays arising on two distinct arc-components in the dendrite's inverse limit; again, something not present in the interval case.

The remainder of this chapter is purposed with educating the casual reader on basic terminology, notation, and background results.

1.2 Inverse Limits of Continua

A *continuum* is a metric[1] space that is compact and connected. The interested reader is invited to consult [42] to see many of the varied examples of continua. The continua with which this dissertation is principally concerned are closed intervals, dendrites, and their respective inverse limit spaces. A *dendrite* is a continuum which is locally connected and uniquely arc-wise connected. If C is a continuum, a point $p \in C$ is said to be a *branch point* if $C - \{p\}$ consists of three or more components. A point $q \in C$ is said to be an *endpoint* of C provided that if A and B are any two subcontinua of C, each containing q, then we have $A \subseteq B$ or $B \subseteq A$.

We use the symbols $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}_-$ to denote, respectively, the positive integers, the integers, and the negative integers. The unit interval, $[0, 1]$ is denoted by I.

[1] Many authors replace "metric" with "Hausdorff"

Moreover, due to the annoyance that simply writing "$[a, b]$" implies (at least to the average mathematician) that $a < b$; we adopt the notation $\langle a; b\rangle$ to denote the closed interval with a and b as endpoints, irrespective of whether $a < b$ or $b < a$. Suppose X is a topological space, and $f : X \to X$ is a continous map. We define the *inverse limit* of X with *bonding map* f, denoted (X, f), as follows[2] :

$$(X, f) = \{(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) : x_i \in X \text{ and } f(x_{i-1}) = x_i \text{ for all integers i}\}$$

The following results are well-known. See, for example, [42].

Theorem 1.1. *The inverse limit of nonempty compact metric spaces is itself nonempty, compact and metric. Furthermore, if C is a continuum, then (C, f) is also a continuum.*

Proposition 1.1. *For each $i \in \mathbb{Z}_-$, let $\pi_i(\hat{x}) = x_i$, where $\hat{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)$ is an element of the inverse limit space (X, f). Then π_i is a continuous function from (X, f) to X.*

The function π_i in the previous proposition is known as the *i-th projection.* We will make frequent use of projection mappings in upcoming chapters. A function f is said to be *unimodal* if it is locally one-to-one, excepting a single point known as the *critical point.* In this dissertation, we will exclusively consider inverse limit spaces of continua where the bonding map is unimodal.

Example 1.1. For each $a \in (0, 2]$, let

$$T_a(x) = \begin{cases} ax \text{ if } 0 \leq x \leq 1/2 \\ a(1-x) \text{ if } 1/2 \leq x \leq 1 \end{cases}$$

The collection $\{T_a\}_{a\in(0,2]}$ is known as the *symmetric family of tent maps.* Each of these is unimodal on the unit interval, with critical point $c = 1/2$.

[2] In many texts, the inverse limit is defined much more generally. However, this definition will suit our purposes.

Example 1.2. For each $a \in (0, 4]$, let $L_a : I \to I$ be given by $L_a(x) = ax(1-x)$. The collection $\{L_a(x)\}_{a \in (0,4]}$ is referred to as the *logistic family*. As with the tent maps, each member of the logistic family is unimodal on the unit interval, and each has $1/2$ as its critical point.

Example 1.3. Of course, our definition of unimodal does not require the domain of a unimodal map to be an interval. Let T be a simple triod (i.e., a space homeomorphic to the letter "T"), and let p_1, p_2, and p_3 be the endpoints. Let p_0 be a point in the open arc with p_3 and the branch point as endpoints. Define $f : T \to T$ by first setting $f(p_i) = p_{(i+1)}$, where addition is done modulo 4. Extending linearly yields a unimodal map with critical point p_0.

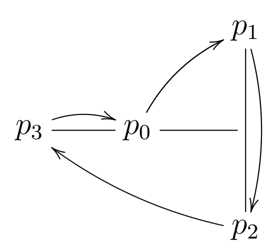

Figure 1.1. The unimodal map on a triod from Example 1.3

Example 1.4. The map $T_2 : I \to I$ is known as the *full tent map*. The corresponding space (I, T_2) is known to be homeomorphic to the B-J-K continuum (for Brouwer, Janiszewski, and Knaster). Its construction is given by the following procedure: Let C denote the Cantor Ternary set. Let C_0 be the collection of all semicircles in the upper half-plane which are symmetric about the line $x = 1/2$ and whose endpoints reside in C. For each $i \in \mathbb{N}$, let C_i be the collection of all semicircles in the lower half-plane which are symmetric about $x = 5/(2{*}3^i)$ and whose with endpoints reside in C. The B-J-K continuum is defined to be $\cup_{i=1}^{\infty} C_i$.

Locally, it is useful to think of this space as a Cantor set cross an arc; however, the point $(0, 0)$, being an endpoint, is an inhomogeneity. This endpoint belongs to a dense ray winding through the space, pictured in Figure 1.2.

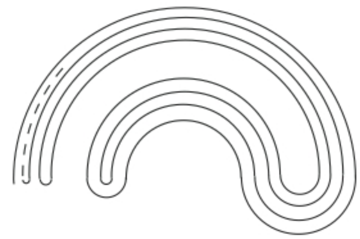

Figure 1.2. The dense ray of the B-J-K Continuum

In general, classifying the spaces (I, T_a) has been difficult, and much research has been relegated to the cases where the orbit of the critical point is finite. If $a \neq b$, it is natural to wonder whether the spaces (I, T_a) and (I, T_b) are homeomorphic. This proposition, known in the literature as *Ingram's Conjecture*, has garnered copious attention over the past 20 years, see, for example [5], [15], [21], [26], [11],[33], [37], [40], [45], [48], or [49]. In [37], it was shown that this conjecture is true when the critical points are periodic. This result was extended in [48] for the case when the orbit of the critical point is finite. In [45], this result was shown to be true when the critical point is non-recurrent. Recently, a complete proof of Ingram's Conjecture has surfaced in [4].

Naturally, developing a generalization of Ingram's Conjecture for a larger class of spaces would be a welcome development. As mentioned earlier, such a step was taken in [2], in the case of k-stars. It is the author's hope that this dissertation is a step in the right direction for dendrites.

CHAPTER TWO

The Kneading Map

In this chapter, we give a cursory overview of the kneading theory. This is a useful tool, allowing us to encode points in a topological space with a symbol set. Analyzing the pattern of these symbols then yields results pertaining to the dynamics of the point in question. For a more thorough exposition, see [17], [27], [24], [38], or [46].

2.1 Itineraries

Suppose $f : X \to X$ is a unimodal map with critical point c, and that $X - \{c\}$ consists of at least two components. We define a *pseudoleg* to be a union of any of the components of $X - \{c\}$, so that f is one-to-one on this union. We shall assume the orbit of c intersects with at most two pseudolegs. Let C_1 denote a pseudoleg containing $f(c)$, and let C_2 denote the other pseudoleg. We define the *itinerary* of a point $x \in I$, denoted $\imath(x) = \imath_0\imath_1\imath_2\ldots$ as follows:

$$\imath_n = \begin{cases} 1 \text{ if } f^n(x) \in C_1 \\ 2 \text{ if } f^n(x) \in C_2 \\ * \text{ if } f^n(x) = c \end{cases}$$

The *kneading sequence* is the itinerary of the critical point c. Our definition differs slightly from that in the prevailing literature, wherein the kneading sequence is taken to be the itinerary of $f(c)$. Both sequences are equally informative, so the difference is purely cosmetic. Additionally, we often write c_n in place of $f^n(c)$. If

$$\hat{x} = (\ldots, x_{-2}, x_{-1}, x_0, x_1 \ldots) \in (X, f),$$

we define the *backwards itinerary* of $\hat{x}$, denoted $e(\hat{x}) = \ldots e_{-3}e_{-2}e_{-1}$, similarly to the itinerary. Moreover, if $e = \ldots e_{-3}e_{-2}e_{-1}$ and $\widetilde{e} = \ldots \widetilde{e}_{-3}\widetilde{e}_{-2}\widetilde{e}_{-1}$ are two backwards itineraries, we define the *first discrepancy* between e and $\widetilde{e}$, denoted $\underline{d}(e, \widetilde{e})$, to be $\min\{i : e_{-i} \neq \widetilde{e}_{-i}\}$. We will primarily be concerned with backwards itineraries e so that $e_{-i} \neq *$ for all i. Otherwise, there would be only a single point with the backwards itinerary e.

2.2 It's "Cutting Time" Time

For the entirety of this chapter, we consider unimodal functions on the unit interval. For each n, a *branch* of f^n is any maximal interval on which f^n is monotone. Our principal concern will be regarding *central branches*, that is, one of the two branches whose boundary contains the critical point c. We rarely distinguish between the two central branches, and often speak of "the" central branch for f^n. We can get away with this primarily because we will concern ourselves with symmetric unimodal maps (e.g. maps from the logistic family or from the family of symmetric tent maps), whereupon the images of the central branches are equal[1] .

For each n, let J_n denote the central branch for f^n, and let $D_n = f^n(J_n)$. If $c \in D_n$, then we call n a *cutting time*. We denote the preimage of c in J_n by z_n. We call z_n a *closest precritical point*. If we need to make discernments, we assume z_n is the closest precritical point less than c, and $\hat{z}_n$ is the closest precritical point greater than c. Of course, some unimodal maps may not have any cutting times. Take, for example, any of the tent maps with slope less than 1. However, the dynamics of such examples are dreadfully boring, and we shall speak of them no more. Hence, all unimodal maps considered henceforth shall be assumed to have the critical point in their image. From this, it follows that the first cutting time, which we will denote by S_0, is equal to 1. We denote the sequence of cutting times as $\{S_k\}_{k\geq 0}$.

Proposition 2.1. *For each n, $J_n = [z_{S_{k-1}}, c]$, where $k = \max\{i : S_i \leq n\}$.*

[1] For theorems involving asymptotic behavior, we can weaken this restriction so that the images are *eventually* the same.

Proof. Note that $f^{S_{k-1}}(z_{S_{k-1}}) = c$. Hence, if $n > S_{k-1}$, f^n will not be one-to-one on $[a, c]$ if $a < z_{S_{k-1}}$. Hence, $J_n \subseteq [z_{S_{k-1}}, c]$.

Suppose that $S_1 \leq n < S_2$, and recall that $S_0 = 1$. Then, if x and y reside in the interval $[z_1, c]$, we must have $f^n(x) \neq f^n(y)$. Otherwise, there would be a precritical point in between x and y, implying the existence of a closest precritical point $z_{S_j} \in [z_1, c]$ for some $S_j \leq n < S_2$. This is an absurdity. Hence, $J_n \supseteq [z_1, c] = [z_{S_0}, c]$. Proceeding inductively, suppose that $J_n \supseteq [z_{S_{k-1}}, c]$ whenever $S_k \leq n < S_{k+1}$, and now suppose that $S_{k+1} \leq n < S_{k+2}$. Assume that x and y are elements of $[z_{S_k}, c]$. If $f^n(x) = f^n(y)$, then there exists a precritical point z in $[z_{S_k}, c]$ for some iterate of f less than n. In turn, this implies the existence of a closest precritical point z_{S_j}, perhaps (but not necessarily) equal to z, in $[z, c)$. However, this contradicts the inductive hypothesis. Hence, the result follows. $\square$

The sequence of cutting times can be surmised from the kneading sequence as the following proposition demonstrates. First, however, we define the function σ, with domain $\{*, 1, 2\}^{\mathbb{N}}$ (or $\{*, 1, 2\}^{\mathbb{Z}}$), so that $\sigma(\hat{x}) = \hat{y}$, where $y_i = x_{i+1}$. The function σ is referred to as the *shift map*.

Proposition 2.2. *[22, Lemma 5] Let $n_0 = 1$, and for $k > 0$, let $n_k = n_{k-1} + \underline{d}(\sigma^{n_{k-1}}\nu, \nu)$. Then, for each k, $S_k = n_k$.*

If $\alpha \in \{1, 2\}$, we write α' to denote the unique element of $\{1, 2\} - \{\alpha\}$. Suppose $\nu = \nu_1\nu_2\nu_3\ldots$ is the kneading sequence of a unimodal map. We use V_n to abbreviate $\nu_1\nu_2\ldots\nu_{n-1}\nu_n$, and we use W_n to represent $\nu_1\nu_2\ldots\nu_{n-1}\nu_n'$. If $\alpha = \alpha_1\alpha_2\ldots$ is a sequence where each $\alpha_i \in \{*, 1, 2\}$, then we say α is *admissible* if there exists a point x with $\imath(x) = \alpha$. Moreover, if $\alpha = \alpha_1\ldots\alpha_n$ is a finite sequence, we say α is admissible if there exists a point x whose itinerary begins with α.

Proposition 2.3. *The word W_n is admissible if and only if n is a cutting time. In particular, $\underline{d}(\sigma^n\nu, \nu)$ is a cutting time for each n.*

Proof. If W_n is admissible, then there exists distinct points with itineraries beginning V_n and W_n, between which there must lie a preimage of c under f^n. This happens if and only if n is a cutting time. $\square$

From the previous two propositions, it follows that $S_k - S_{k-1}$ is a cutting time for each k. Hence, we define the *kneading map*, denoted $Q(k)$, via

$$S_{Q(k)} = \underline{d}(\sigma^{n_{k-1}}\nu, \nu), \text{ so that } S_{Q(k)} = S_k - S_{k-1}.$$

Additionally, we have the following decomposition of the kneading sequence:

$$\nu = 1W_{S_{Q(1)}}W_{S_{Q(2)}}W_{S_{Q(3)}}\dots$$

At this point, examples are likely in order. We hope the reader appreciates the excursion.

Example 2.1. Suppose $f(x)$ is the full tent map, ie,

$$f(x) = \begin{cases} 2x \text{ if } x \leq 1/2 \\ -2(1-x) \text{ if } x \geq 1/2 \end{cases}$$

This map has kneading sequence $\nu = 12^\infty$, and kneading map $Q(k) \equiv 0$. Hence, $S_{Q(k)} \equiv 1$, from which it follows that the sequence of cutting times is given by

$$\{S_k\}_{k\geq 0} = 1, 2, 3, 4, \dots$$

and we see that each natural number is a cutting time.

Example 2.2. Suppose $f(x)$ has kneading map $Q(k) = k - 1$. Then

$$\{S_{Q(k)}\}_{k\geq 1} = S_0, S_1, S_2, S_3, S_4, \dots$$

and the cutting times are the powers of 2. The function $f(x)$ is known in the literature as the *Feigenbaum map*. Additional monikers include a *Coullet Tresser* *map*, or, for reasons that should now be obvious, the 2^∞ *map*. In particular, it has a point of period 2^n for each $n \in \mathbb{N}$, and no other periodic points. Its kneading sequence is given by

$$\nu = 1\ 2\ 11\ 1212\ 12111211 \ldots$$

Surely, anything known by at least three names has been extensively studied. See, for example [8], [17], [25], [28], or [29].

Example 2.3. Suppose $f(x)$ has kneading map $Q(k) = \max\{0, k-2\}$. Then

$$\{S_{Q(k)}\}_{k \geq 1} = S_0, S_0, S_1, S_2, S_3, \ldots$$

and the cutting times are the Fibonacci numbers. For obvious reasons, this map is known as the *Fibonacci map*. The kneading sequence is

$$\nu = 1\ 2\ 2\ 11\ 121\ 12212\ 12211122 \ldots$$

For more on the combinatorics of this wonderful map, see [17], [23], [32], or [50].

In the case of the Fibonacci and Feigenbaum maps, we asserted that functions with the given kneading maps exists. The vigilant reader may have been disturbed by this bold assertion. As it happens, the Fibonacci map is realized as a symmetric tent map T_a with $a \approx 1.729211932$, and the Feigenbaum map is realized as a logistic map L_a with $a \approx 3.569945668$. Indeed, given an arbitrary kneading map $Q(k)$; it is not the case that there necessarily exists a unimodal map with $Q(k)$ as its kneading map. In [31], the following result was given.

$$\begin{tikzcd}
z_{Q(k)-1} \arrow[r, dash] \arrow[d] & c_{S_{k-1}} \arrow[r, dash] \arrow[d] & z_{Q(k)} \arrow[r, dash] \arrow[dr] & c & {} \arrow[d, bend left, "f^{S_{Q(k)}}"] \\
c_{S_{Q^2(k)}} \arrow[r, dash] & c_{S_k} \arrow[rr, dash] & & c & {}
\end{tikzcd}$$

Figure 2.1. The Geometry of $\{Q(k+j)\}_{j\geq 1} \leq \{Q(Q^2(k)+j)\}_{j \geq 1}$

Proposition 2.4. *There exists a unimodal map on the unit interval with kneading sequence* ν *if and only if* $\{Q(k+j)\}_{j\geq 1} \leq \{Q(Q^2(k)+j)\}_{j\geq 1}$ *for all nonnegative integers* k *(here, the sequences are ordered lexicographically).*

The proof of Proposition 2.4, particularly the "if" direction, is quite technical. However, the inequality $\{Q(k+j)\}_{j\geq 1} \leq \{Q(Q^2(k)+j)\}_{j\geq 1}$ is simply imposing a geometric condition, pictured in Figure 2.1.

Proposition 2.5. $f^{S_k}(J_{S_k}) = \langle c_{S_k}; c_{S_{Q(k)}} \rangle$

Proof. Recall that $J_{S_k} = [z_{S_{k-1}}, c]$, and $f^{S_k}(c) = c_{S_k}$, so we get one endpoint for free. The other one is also easily observed:

$$\begin{aligned} f^{S_k}(z_{S_{k-1}}) &= f^{S_{Q(k)}} \circ f^{S_{k-1}}(z_{S_{k-1}}) \\ &= f^{S_{Q(k)}}(c) \\ &= c_{S_{Q(k)}} \end{aligned}$$

□

As the name no doubt implies, the critical point is of import; its orbit is worth keeping track of. The following helps out nicely with this task.

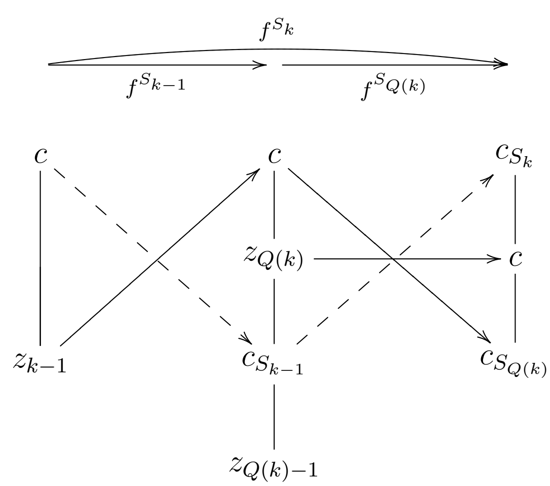

Figure 2.2. The Geometry of Proposition 2.6

Proposition 2.6. $c_{S_{k-1}} \in [z_{S_{Q(k)-1}}, z_{S_{Q(k)}}] \cup [\hat{z}_{S_{Q(k)}}, \hat{z}_{S_{Q(k)-1}}]$.

Proof. Recall that $f^{S_k} = f^{S_{k-1}} \circ f^{S_{Q(k)}}$, and that $J_{S_k} = [z_{S_{k-1}}, c]$ and $J_{S_{Q(k)}} = [z_{S_{Q(k)-1}}, c]$. Monotonicity of f^{S_k} on J_{S_k} implies that $c_{S_{k-1}} \in [z_{S_{Q(k)-1}}, \hat{z}_{S_{Q(k)-1}}]$. If $c_{S_{k-1}} \in (z_{S_{Q(k)}}, \hat{z}_{S_{Q(k)}})$, then $c \notin D_{S_k}$, contradicting that S_k is a cutting time. $\square$

We conclude this section with a well-known result pertaining to arc-components of (I, f) (see, for example, [16] or [17]). First, however, a definition is in order. A unimodal map $f : I \to I$ is said to be *longbranched* if its associated kneading map, $Q(k)$, is bounded.

Theorem 2.1. *[18, Corollary 2.10] Suppose $f : I \to I$ is longbranched. Let $\hat{x}$ and $\hat{y}$ be elements of (I, f), and let e and $\widetilde{e}$ denote their respective backwards itineraries. Then $\hat{x}$ and $\hat{y}$ are in the same arc-component of (I, f) if and only if there exists an n so that whenever $i > n$ we have $e_{-i}(x) = e_{-i}(y)$.*

Of particular note, if the critical point for f is periodic, then f is longbranched. Chapter 4 is primarily concerned with such examples.

2.3 Co-Cutting Times

Suppose $f : I \to I$ is unimodal with kneading sequence ν. We define the sequence of *co-cutting times*, denoted $\{T_k\}_{k\geq 0}$, as follows:

$$T_0 = \min\{i > 1 : \nu_i = 1\}, \text{ and } T_k = T_{k-1} + \underline{d}(\sigma^{T_{k-1}}\nu, \nu).$$

Given ν, the sequence of co-cutting may not be defined. Indeed, if $\nu = 12^{\infty}$, as is the case with the full tent map, T_0 is undefined. Most results concerning co-cutting times are beyond the scope of this dissertation; however, the following result will come into play in the following chapter.

Proposition 2.7. *[19, Lemma 2] The following properties hold:*

(i) The sequences of cutting and co-cutting times are disjoint.

(ii) Closest returns for the critical point appear either at cutting or at co-cutting times.

(iii) The difference between two consecutive co-cutting times is a cutting time.

Based on the previous proposition, we may define the *co-kneading map*; denoted $\widetilde{Q}(k)$, as follows:

$$T_{\widetilde{Q}(k)} = \underline{d}(\sigma^{T_{k-1}}\nu, \nu), \text{ so that } T_{\widetilde{Q}(k)} = T_k - T_{k-1}.$$

As with the cutting times, the co-cutting times provide a nice decomposition of the kneading sequence:

$$\nu = 12^{T_0-2}1W_{S_{\widetilde{Q}(1)}}W_{S_{\widetilde{Q}(2)}}W_{S_{\widetilde{Q}(3)}}\ldots.$$

However, even if an explicit formula for the kneading map, $Q(k)$, is known; determining a formula for the co-kneading map, $\widetilde{Q}(k)$, is often an extremely difficult combinatorics problem, as is the reverse.

Example 2.4. Let τ be the kneading sequence of the Fibonacci map, so that

$$\tau = 1\ 2\ 2\ 11\ 121\ 12212\ 12211122\ldots.$$

Then we have:

$$\begin{aligned}
T_0 = 4 &\text{ and } S_{\widetilde{Q}(1)} = 2\\
T_1 = 6 &\text{ and } S_{\widetilde{Q}(2)} = 1\\
T_2 = 7 &\text{ and } S_{\widetilde{Q}(3)} = 2\\
T_3 = 9 &\text{ and } S_{\widetilde{Q}(4)} = 1\\
T_4 = 10 &\text{ and } S_{\widetilde{Q}(5)} = 1\\
&\vdots
\end{aligned}$$

CHAPTER THREE

Inverse Limits of Dendrites

In this Chapter, we will leave the setting of the unit interval, and delve into dendrites, as advanced in [1] and [2]. The use of itineraries from the previous chapter will carry over, with one major difference: the points and the itineraries are one and the same!

3.1 A Symbolic Representation

We begin by topologizing the symbol space $\{*, 1, 2\}$ with the basis $\{\{1\}, \{2\}, \{*, 1, 2\}\}$; extending this topology to the product spaces $\{*, 1, 2\}^{\mathbb{N}}$ and $\{*, 1, 2\}^{\mathbb{Z}}$ in the usual way. If a and b are elements of $\{*, 1, 2\}$, $a \approx b$ means either $a = b$ or at least one of a or b is $*$. We extend the definition of $\approx$ to the product spaces in the obvious way. An element $\tau = \tau_0\tau_1\tau_2\ldots$ of $\{*, 1, 2\}^{\mathbb{N}}$ is said to be *acceptable* if $\tau_0 = *$ and $\sigma^n(\tau) \approx \tau$ implies $\sigma^n(\tau) = \tau$ for each $n \in \mathbb{N}$. Given an acceptable sequence τ, an element $\alpha \in \{*, 1, 2\}^{\mathbb{N}}$ is said to be τ-*admissible* $\sigma^n(\alpha) \approx \tau$ implies $\sigma^n(\alpha) = \tau$ for each non-negative integer n. The space D_τ is defined to be the set of all τ-admissible sequences, and was shown in [1] to be a dendrite.

Suppose X is a continuum, and $f : X \to X$ is unimodal. Then f is said to be *tentish* if, for any elements x and y of X, we have $\imath(x) \neq \imath(y)$ whenever $x \neq y$. In this case, we also say that f has the *unique itinerary property.* This is important, because our topology on $\{*, 1, 2\}^{\mathbb{N}}$ is non-Hausdorff, and we wish to concern ourselves with a proper subspace which satisfies the Hausdorff property.

Another important concept in the study of dynamical systems is conjugation. Let $f : X \to X$ and $g : Y \to Y$ be continuous functions. We say that f and g are *topologically conjugate* provided there is a homeomorphism $h : X \to Y$ so that $h \circ f = g \circ h$. The homeomorphism h is referred to as a *conjugacy.* The import of conjugacies is that they preserve many, but certainly not all, dynamical behaviors of the system consisting of X and f. As a rule of thumb, conjugacies preserve topological properties, but need not preserve metric or analytic properties. Of particular note, asymptotic arc-components, once we've defined this concept in the following chapter, will easily be seen to be preserved by conjugacies (since the composition of continuous functions is in turn continuous). For a more precise overview, please consult [14], [34], or [35].

Proposition 3.1. *[1, 2.17] Let $f : D \to D$ be a unimodal dendrite map, with the property that any two distinct elements of D have distinct itineraries. Then the itinerary map $\imath : D \to D_\tau$ is a homeomorphism onto its range, and $\imath$ is a conjugacy between f and $\sigma|_{\imath(D)}$.*

Let $\hat{x} = \ldots x_{-2}x_{-1}.x_0x_1 \ldots \in \{*, 1, 2\}^{\mathbb{Z}}$. For each $n \in \mathbb{Z}$, define $\pi_n(\hat{x}) = x_nx_{n+1}x_{n+2}\ldots$. Let $\hat{D}_\tau = \{\hat{x} \in \{*, 1, 2\}^{\mathbb{Z}} : \pi_n(\hat{x}) \in D_\tau \text{ for all } n \in \mathbb{Z}\}$, and let $\hat{\sigma}$ denote the shift map on $\hat{D}_\tau$.

Proposition 3.2. *[2, 2.4] Define $h : \hat{D}_\tau \to (D_\tau, \sigma)$ by setting*

$$h(\hat{x}) = (\ldots, \pi_{-1}(\hat{x}), \pi_0(\hat{x}), \pi_1(\hat{x}), \ldots).$$

Then h is a conjugacy between $\hat{\sigma}$ and the corresponding shift map for (D_τ, σ).

Proposition 3.3. *[1, 2.25] Let A be an arc in $\hat{D}_\tau$ with endpoints $\hat{x}$ and $\hat{y}$, and suppose $k = \min\{i | x_i \not\approx y_i\}$ is finite. Then if $\hat{z} \in A$ and $i < k$, we have $x_i \approx z_i \approx y_i$.*

If $S \subseteq D_\tau$ is a finite set of points, we let $[S]$ denote the smallest subcontinuum of D_τ containing S. We may use $[x, y]$, in place of $[\{x, y\}]$, to denote the unique arc in D_τ having x and y as endpoints, and $(x, y) = [x, y] - \{x, y\}$. Given two points x and y of D_τ, it is often useful to find a point in (x, y). The following technique, dubbed the "μ-process," was developed in [1], and was useful in proving many results (e.g. that D_τ is connected). If $x = x_0x_1\ldots$ and $y = y_0y_1\ldots$ are distinct elements of D_τ, then there exists a minimal n so that $x_n \not\approx y_n$. Define $\mu'(x,y) = \mu'_1\mu'_2\ldots$ as follows. If $i < n$, let $\mu'_i \in \{x_i, y_i\}$ be chosen so that $\mu'_i \neq *$. If $i = n$, set $\mu'_i = *$, and for $i > n$, $\mu'_i = \tau_{i-n}$. Then there exists a unique $\mu \in D_\tau$ with $\mu \approx \mu'(x,y)$. Moreover, $\mu \in (x,y)$.

Based on Ingram's Conjecture, we state the following:

Conjecture 1. *If $\tau \neq \nu$, then D_τ is not homeomorphic to D_ν.*

While the upcoming sections do not completely address the above conjecture, it is the author's hope that they are a step in the right direction.

We conclude this section with a characterization of arc-components in $\hat{D}_\tau$ in the case where τ is periodic. If $\hat{x} = \ldots x_{-2}x_{-1}.x_0x_1\ldots$ is an element of $\hat{D}_\tau$, we define the *backwards itinerary* of $\hat{x}$, denoted $e(\hat{x})$, to be the reverse sequence $\ldots x_{-3}x_{-2}x_{-1}$. We define the equivalence class e_* via $e_* = \{e(\hat{x}) : \hat{x} \in \hat{D}_\tau$ and $e(\hat{x})_i = e_i$ whenever $i \leq M$ for some $M \in \mathbb{Z}_-\}$. Given two backwards itineraries $e = e(\hat{x})$ and $\widetilde{e} = e(\hat{y})$, we define the *sequence of discrepancies* as follows. Let $k_1 = \min\{i | e_{-i} \neq \widetilde{e}_{-i}\}$ and inductively define $k_{i+1} = \min\{i > k_i | e_{-i} \neq \widetilde{e}_{-i}\}$. We call k_1 the *first discrepancy* between e and $\widetilde{e}$. If k_{i+1} does not exist, we leave it undefined and say the sequence of discrepancies is finite. In this case, the sequences e and $\widetilde{e}$ have the same tails, and we have $e_* = \widetilde{e}_*$.

Proposition 3.4. *[2, 2.7] Let $\hat{x}$ and $\hat{y}$ be points in $\hat{D}_\tau$, where τ is of period N. Let $\{k_i\}$ denote the sequence of discrepancies between $e = e(\hat{x})$ and $\widetilde{e} = e(\hat{y})$. Then $\hat{x}$ and $\hat{y}$ are in the same arc-component if and only if $\{k_i\}$ is finite or if there exists a natural number M so that if k_i,$k_j \geq M$, we have $k_i \equiv k_j \mod N$ and, for each i, $x_{-k_{i+1}}\ldots x_{-k_i+1} \approx (\tau_0\ldots\tau_{N-1})^{n_i} \approx y_{-k_{i+1}}\ldots y_{-k_i+1}$, where $n_i = (k_{i+1} - k_i)/N$.*

3.2 Pre-Critical Points in D_τ

In this section, we present some preliminary results concerning closest pre-critical points in the space D_τ. To this end, we begin by generalizing many of the definitions given in Chapter 2 so as to be applicable to the spaces D_τ. For each k, we define a *branch* for D_τ to be a maximal subset of a pseudoleg on which σ^k is one-to-one. A *central branch* is a branch whose boundary contains the critical point τ. We define a point $z \in D_\tau$ to be a *closest precritical point* provided that, for some n, $\sigma^n(z) = \tau$ and $(z, \tau) \cap \sigma^{-m}(\tau) = \emptyset$ for all $0 \leq m \leq n$.

Proposition 3.5. *Let $z \in D_\tau$ with $\sigma^n(z) = \tau$ for some $n \in \mathbb{N}$. Then z is a closest precritical point if and only if $\underline{d}(z, \tau) > n$.*

Proof. Suppose z is a closest precritical point, with $\sigma^n(z) = \tau$. Suppose $\underline{d}(z, \tau) \leq n$. Then, by application of the μ-process to z and τ, there exists a point y in (z, τ) with $\sigma^k(y) = \tau$ for some $k < m$. This contradicts z being a closest precritical point.

Next, suppose $\sigma^n(z) = \tau$ and $\underline{d}(z, \tau) > n$. Then σ^n is one-to-one on $[z, \tau]$, which in turn implies there are no precritical points in $[z, \tau]$ for any iterates of σ less than n. Hence, z is a closest precritical point. $\square$

A distinct difference between unimodal maps on D_τ and unimodal maps on I, is that closest precritical points exist for each natural number in D_τ. This is not the case for the invterval, wherein there is a closest precritical point for f^n if and only if $n = S_k$ for some k.

Proposition 3.6. *Suppose τ is non-periodic. Then, for each $n \in \mathbb{N}$ there exists a closest precritical point in $\sigma^{-n}(\tau)$. Moreover, such a precritical point resides in each pseudoleg of D_τ*

Proof. For each $n \in \mathbb{N}$, let $z_n = 1\tau_1 \ldots \tau_{n-1}\tau$. The admissiblility of z_n follows from τ being non-periodic. That z_n is a closest precritical point follows from the previous proposition. Replacing the 0-th coordinate of z_n with a 2 yields a closest precritical point in the remaining pseudoleg. $\square$

Table 3.1 comes from the result of applying the μ-process to z_n and τ. Based upon the observations residing therein, we have the following two propositions.

Table 3.1. Closest precritical points when τ has Fibonacci combinatorics

i	z_i	Interval Point?	$\underline{d}(z_i,\tau)$	$\mu(z_i,\tau)$
1	1τ	Yes	2	z_2
2	11τ	Yes	3	z_3
3	112τ	Yes	5	z_5
4	1122τ	No	6	z_6
5	11221τ	Yes	8	z_8
6	11221τ	No	7	z_7
7	112211τ	No	9	z_9
8	1122111τ	Yes	13	z_{13}
9	11221112τ	No	12	z_{12}
10	112211121τ	No	11	z_{11}
11	1122111211τ	No	14	z_{14}
12	11221112112τ	No	13	z_{13}
13	112211121122τ	Yes	21	z_{21}

Proposition 3.7. *Suppose τ is non-periodic, and is admissible as a unimodal map on the interval, and the sequences of cutting and co-cutting times are unbounded. Then, for each $m \in \mathbb{N}$, $\mu(z_m,\tau) = z_n$ for some $n > m$. Moreover, $\mu(z_{S_k},\tau) = z_{S_{k+1}}$ and $\mu(z_{T_k},\tau) = z_{T_{k+1}}$.*

Proof. It is easily seen that $\mu(z_{S_k},\tau) = z_{S_{k+1}}$ and $\mu(z_{T_k},\tau) = z_{T_{k+1}}$. The result follows.

□

Proposition 3.8. *Suppose τ is non-periodic, is admissible as a unimodal map on the interval, and the sequences of cutting and co-cutting times are unbounded. Then $D_\tau - \{\tau\}$ consists of at least four components.*

Proof. By Proposition 3.1, each z_{S_k} necessarily resides on a separate component of $D_\tau - \{c\}$ from each z_{T_k}. And, by Proposition 2.7, these sequences are disjoint. Hence, the result follows. □

Apparently, each application of the μ-process to z_n and τ (eventually) results in a z_{S_k} or a z_{T_k}. This leads to the following conjectue.

Conjecture 2. *Suppose τ is non-periodic, is admissible as a unimodal map on the interval, and the sequences of cutting and co-cutting times are unbounded. Then $D_\tau - \{\tau\}$ consists of exactly four components.*

The truth (or falsehood) of this conjecture relies on whether we may define a sequence; $\{U_k\}$, by setting

$$U_0 = n, \text{ where } n \in \mathbb{N}, \text{ and } U_k = U_{k-1} + \underline{d}(\sigma^{U_{k-1}}\tau, \tau)$$

so that $\{U_k\}$ is disjoint from both the cutting and the co-cutting times of τ (regarded as a kneading sequence for an interval map).

CHAPTER FOUR

Asymptotic Behavior in Inverse Limit Spaces of the Unit Interval

As mentioned in the Introduction, this chapter summarizes the results from [22] and [20] which are relevant to our studies of dendrites. All the results and definitions in this chapter will be repurposed in Chapter 5; hence, the hurried reader may skip this chapter entirely. However, it is worth a perusal, insofar as it illustrates many differences between inverse limits of the interval, and those of dendrites.

4.1 Folding Patterns

Suppose τ is a periodic kneading sequence of period N, so that

$$\tau = (\tau_1 \tau_2 \dots \tau_{n-1} *)^\infty.$$

For the remainder of this chapter, we will redefine $\tau_N \in \{1, 2\}$ so that $\tau_1 \tau_2 \dots \tau_N$ contains an even number of 1's. The purpose of this redefinition is for technical reasons relating to admissibility, which are detailed in [27].

Proposition 4.1. *[18] Suppose f has a periodic critical point, and let $\hat{x}$ and $\hat{y}$ be points in (I, f). Then $\hat{x}$ and $\hat{y}$ are in the same arc-component if and only if there exists a natural number M so that $e_{-i}(\hat{x}) = e_{-i}(\hat{y})$ for all $i \geq M$.*

Let $\hat{x} = (\dots, x_{-2}, x_{-1}, x_0, x_1, \dots) \in (I, f)$. Let

$$T(\hat{x}) := \{\hat{y} \in (I, f) : y_{-i} = x_{-i} \text{ for all } i \in \mathbb{N}\}.$$

If $e = e(\hat{x})$ is the backwards itinerary of $\hat{x}$, we may write $T(e)$ in place of $T(\hat{x})$.

Proposition 4.2. *[20, Lemma 1] $\pi_0 : T(\hat{x}) \to I$ is a homeomorphism onto its image. In particular, the closure of $T(\hat{x})$ is homeomorphic to either a point (if $e_{-i}(\hat{x}) = *$ for some i) or a closed interval (if $e_{-i}(\hat{x}) \neq *$ for all i).*

Given a kneading sequence $\tau = \tau_1\tau_2\ldots$, we define the *parity function* as follows:

$$P(n) = \begin{cases} \text{even, if } \tau_1\tau_2\ldots\tau_n \text{ contains an even number of 1's} \\ \text{odd, if } \tau_1\tau_2\ldots\tau_n \text{ contains an odd number of 1's} \end{cases}$$

Let $e = e(\hat{x})$ be a backwards itinerary. We define

$$\alpha_{-1}(e) = \sup\{n \geq 1 : e_{-n+1}\ldots e_{-1} = \tau_1\ldots\tau_{n-1} \text{ and } P(n) = \text{odd}\}$$

$$\alpha_1(e) = \sup\{n \geq 1 : e_{-n+1}\ldots e_{-1} = \tau_1\ldots\tau_{n-1} \text{ and } P(n) = \text{even}\}$$

If the backwards itinerary in question is understood, we may simply write α_i in place of $\alpha_i(e)$. We also note the possibility that $\alpha_i(e) = \infty$.

Proposition 4.3. *[20, Lemma 2] Let $\hat{x} \in (I, f)$ and let $e = e(\hat{x})$. Suppose $\alpha_{-1}(e)$ and $\alpha_1(e)$ are both finite, and $e_{-i} \neq *$ for all $i \in \mathbb{N}$. Then $\pi_0(T(\hat{x})) = \langle c_{\alpha_{-1}}; c_{\alpha_1}\rangle$. Moreover, if $\hat{y} \in (I, f)$ is such that $e_{-i}(y) = e_{-i}(x)$, except for $i = \alpha_1(e(\hat{x}))$ (or $i = \alpha_{-1}(e(\hat{x}))$), then $T(\hat{x})$ and $T(\hat{y})$ share a common boundary point.*

The previous proposition allows us to track itinerial changes, as we move along an arc-component; beginning in $T(\hat{x})$. To this end, we need a few more definitions. Given a backwards itinerary $e = \ldots e_{-2}e_{-1}$, we define the backwards itineraries $Re = \ldots\psi_{-2}\psi_{-1}$ and $R^{-1}e = \ldots\xi_{-2}\xi_{-1}$ via the following:

$$\psi_{-i} = \begin{cases} e_{-i}, \text{ if } i \neq \alpha_1(e) \\ e'_{-i}, \text{ if } i = \alpha_1(e) \end{cases}$$

$$\xi_{-i} = \begin{cases} e_{-i}, \text{ if } i \neq \alpha_{-1}(e) \\ e'_{-i}, \text{ if } i = \alpha_{-1}(e) \end{cases}$$

We then continue inductively. If $n > 1$, let

$$R^n e = \begin{cases} R^{-1}(R^{n-1}e), \text{ if } n \text{ is even} \\ R(R^{n-1}e), \text{ if } n \text{ is odd} \end{cases}$$

Table 4.1. Folding pattern for $e = 1^\infty$ and $\tau = (121)^\infty$

i	$\alpha_i(e)$	$R^i e$	$\pi_0(T(R^i e))$
0	undefined	1^∞	$\langle c_2; c_1\rangle$
1	1	$1^\infty 2$	$\langle c_3; c_1\rangle$
2	3	$1^\infty 212$	$\langle c_3; c_1\rangle$
3	1	$1^\infty 211$	$\langle c_5; c_1\rangle$
4	5	$1^\infty 21211$	$\langle c_5; c_1\rangle$
5	1	$1^\infty 21212$	$\langle c_3; c_1\rangle$
6	3	$1^\infty 21112$	$\langle c_3; c_1\rangle$

and if $n < 1$, let

$$R^n e = \begin{cases} R^{-1}(R^{n+1}e), \text{ if } n \text{ is odd} \\ R(R^{n+1}e), \text{ if } n \text{ is even} \end{cases}$$

The reader is cautioned that $R^{n+m}e \neq R^n(R^m e)$ if m is odd.

For each $i \in \mathbb{Z}$, let $\alpha_i(e) = \underline{d}(R^{n+1}e, R^n e)$. We leave α_0 undefined. The sequence $\{\alpha_i(e)\}$ is said to be the *folding pattern* for the backwards itinerary e. For the most part, we shall only concern ourselves with the α_i's where $i \geq 1$. Table 4.1 gives the first few iterates of the folding pattern, and respective projections, when $\tau = (121)^\infty$ and $e = 1^\infty$.

4.2 Asymptotic Arc-Components in (I, f)

Suppose C and $\widetilde{C}$ are two arc-components in (I, f), and that ρ is a metric compatible with the topology of (I, f). Then we say C and $\widetilde{C}$ are *asymptotic* provided that there exists parameterizations $\phi : [0, \infty) \to C$ and $\widetilde{\phi} : [0, \infty) \to \widetilde{C}$ so that $\rho(\phi(t), \widetilde{\phi}(t)) \to 0$ as $t \to \infty$.

The study of asymptotic arc-components was first undertaken in the context of substitution tiling spaces (see [30] or [43]). These spaces are beyond our scope. Suffice it to say, they appear as orientable coverings of the spaces (I, f) which we have been considering. Hence, many results from the study of substitution tiling spaces may be adapted to service us. Of particular note, in [22], the author adopted such a result, which we now state.

Theorem 4.1. *[6, Proposition 4] Suppose the kneading sequence for the unimodal map f has period N. Then (I, f) has at most $2N-4$ asymptotic arc-components (discounting the N arc-components with endpoints).*

It is known that the bound provided by the previous proposition is sharp. In [22], the author found exactly $2N-4$ asymptotic arc-components for many examples (e.g. when the kneading sequence for f is $(12111)^\infty$, 8 asymptotic arc-components have been found). The author utilized a characterization of asymptotic arc-components based on backwards itineraries, which we come to presently.

Proposition 4.4. *[22, Proposition 1] Two arc-components C and $\widetilde{C}$ in (I, f) are aymptotic if and only if they contain points with respective backwards itineraries e and $\widetilde{e}$ so that the following hold:*

(1) $\underline{d}(R^n e, R^n \widetilde{e}) \to \infty$ *as* $n \to \infty$.

(2) $|c_{\alpha_n} - c_{\widetilde{\alpha}_n}| \to 0$ *as* $n \to \infty$*, where* $\{\alpha_i\}$ *and* $\{\tilde{\alpha}_i\}$ *are the respective folding patterns of* e *and* $\widetilde{e}$.

Of particular note, when the kneading sequence is of period N, the condition $|c_{\alpha_n} - c_{\widetilde{\alpha}_n}| \to 0$ can be replaced with $\alpha_n \equiv \widetilde{\alpha}_n \mod N$ for all n sufficiently large. We now state the primary theorems from [22], which we will adapt to the service of inverse limits of dendrites in the coming chapters. Each of the following three theorems assumes that f is not renormalizable, and its corresponding kneading sequence, τ, is of period N.

Theorem 4.2. *Suppose* $\tau = (12^{N-2}1)^\infty$ *and* $l \in \{1, 2, \dots, N-2\}$. *Then the backwards itineraries*

$$e = \overline{12^{N-3}1}12^{N-2}1 \text{ and}$$
$$\widetilde{e} = \overline{12^{N-3}1}12^{N-2}12^l,$$

correspond to asymptotic arc-components.

Theorem 4.3. *Suppose that* $k < N$ *is such that* $\underline{d}(\sigma^k\tau, \tau) \geq N - k$ *and* $P(k) =$ *odd. Then the backwards itineraries*

$$e = \overline{\nu_1 \dots \nu_k'}\nu_1 \dots \nu_N \text{ and}$$
$$\widetilde{e} = \overline{\nu_1 \dots \nu_{N-k}'\nu_{N-k+1} \dots \nu_k}\nu_1 \dots \nu_{N-k}'\nu_{N-k+1} \dots \nu_N',$$

correspond to asypmtotic arc-components, provided that e *and* $\widetilde{e}$ *are admissible.*

Theorem 4.4. *Suppose that* $k < N$ *is such that* $\underline{d}(\sigma^k\tau, \tau) \geq N - k$ *and* $P(k) =$ *even. Then the backwards itineraries*

$$e = \overline{\nu_{N-k+1} \dots \nu_k'\nu_1 \dots \nu_N} \text{ and}$$
$$\widetilde{e} = \overline{\nu_{N-k+1} \dots \nu_k'\nu_1 \dots \nu_N}\nu_{N-k+1} \dots \nu_N',$$

correspond to asymptotic arc-components, provided that e *and* $\widetilde{e}$ *are admissible.*

We wish to emphasize that the backwards itineraries in the previous theorems need not be admissible. In fact, most are not. Additionally, the proofs of these theorems are highly technical, and rely heavily on the theory of cutting times. For a taste of just how technical these sorts of results can be, the reader is invited to peruse Chapter 7. In Chapter 6, we state and prove analogues of these theorems for $\hat{D}_\tau$, for which the proofs are greatly simplified.

CHAPTER FIVE

Preliminary Results for $\hat{D}_\tau$

The purpsose of this chapter is to repurpose many of the lemmas and propositions from Chapter 4 to suit the purposes of inverse limits of dendrites, as covered in Chapter 3. Many significant differences arise; which should reward the careful reader.

Suppose $\hat{x} \in \hat{D}_\tau$. For each $N \in \mathbb{Z}$, let $T_N(\hat{x}) = \{\hat{y} \in \hat{D}_\tau : y_i \approx x_i \text{ for all } i \leq N\}$, and let $T(\hat{x}) = T_{-1}(\hat{x})$. If $e = e(\hat{x})$, we may write $T_N(e)$ in place of $T_N(\hat{x})$. It is easily verified that $T_N(\hat{x})$ is a closed subset of $\hat{D}_\tau$, and is homeomorphic to $\pi_N(T_N(\hat{x}))$.

Lemma 5.1. *[1, 2.14]Suppose $y \in D_\tau$ and e is a backwards itinerary for some point in $\hat{D}_\tau$. Then there exists a unique $\hat{x} \in \hat{D}_\tau$ such that $\pi_0(\hat{x}) = y$ and $\hat{x} \in T(e)$.*

Lemma 5.2. *$T_N(e)$ is uniquely arc-wise connected.*

Proof. If $e_i = *$ for some $i \leq N$, then $T_N(e)$ consists of a single point. Suppose $e_i \neq *$ for all $i \leq N$. Let $\hat{x}$ and $\hat{y}$ be distinct elements of $T_N(e)$. Since D_τ is arc-wise connected, there exists an arc A in D_τ having $\pi_N(\hat{x})$ and $\pi_N(\hat{y})$ as endpoints. Let $k = \min\{i > N : x_i \neq y_i\}$ and let $z \in A$. By admissibility of $\pi_N(\hat{x})$ and $\pi_N(\hat{y})$, k is finite. By Proposition 3.3, if $N \leq i < k$, we have $x_i \approx z_{i-N+1} \approx y_i$. For $i \geq N$, let $p'_i = z_{i-N+1}$, and for $i < N$, let $p'_i = e_i$. By the previous lemma, there exists a unique $\hat{p} \in \hat{D}_\tau$ with $\hat{p} \approx p'$. Moreover, $\pi_N(\hat{p}) = z$. Hence, we have $\pi_N(T(e)) \supseteq A$. Since π_N is a homeomorphism from $T_N(e)$ onto its image, $\pi_N^{-1}(A)$ is a subarc of $T_N(e)$, with endpoints $\hat{x}$ and $\hat{y}$. Furthermore, since D_τ is uniquely arc-wise connected, $T_N(e)$ is as well. $\square$

Proposition 5.1. *$T_N(e)$ is a subcontinuum of $\hat{D}_\tau$.*

Proof. Since $T_N(e)$ is a closed subset of the compact space $\hat{D}_\tau$, we have that $T_N(e)$ is compact. This, coupled with the previous proposition, yields the desired result. $\square$

If e and $\widetilde{e}$ are backwards itineraries occurring in $\hat{D}_\tau$, we seek a way to determine when $T(e)$ and $T(\widetilde{e})$ are share a common boundary point. To that end, we make the following definition: for each $0 \leq i < N$, let $\beta^i(e) = \max\{k : e_{-k}e_{-(k-1)} \dots e_{-1} \approx \tau_0\tau_1 \dots \tau_{k-1} \text{ and } k \equiv i \mod N\}$. If no such match exists, we will leave $\beta^i(e)$ undefined.

Proposition 5.2. *Suppose τ is of period N, and e is a backwards itinerary with $e_{-k} = *$ for some positive integer k. Then, there exists a unique $0 \leq i < N$ such that $\beta^i(e)$ is defined. In particular, $i \equiv k \mod N$.*

Proof. This follows easily from the definition of admissiblity. $\square$

Proposition 5.3. *Suppose e is a backwards itinerary, with $e_i \neq *$ for all i and τ is of period N. Suppose $\beta^k(e)$ is defined. Define $\widetilde{e}$ by setting $\widetilde{e}_i$ to be either 1 or 2 when $\beta^k \leq i < 0$ and $i \equiv k \mod N$, and $\widetilde{e}_i = e_i$ otherwise. If $e \neq \widetilde{e}$, then $T(e) \cap T(\widetilde{e})$ consists of a single point.*

Proof. Suppose $\beta = \beta^k(e)$ is finite. Define $\hat{p}$ by setting

$$\hat{p}_i = \begin{cases} e_i \text{ if } i < -\beta \\ \tau_{k+i} \text{ if } i \geq -\beta \end{cases}$$

Note that $\pi_{-\beta}(\hat{p}) = \tau$. Hence, if $i \geq -\beta$, $\pi_i(\hat{p}) \in D_\tau$. Moreover, if $i < -\beta$, then $\pi_i(\hat{p}) \not\approx \tau$, since k was chosen maximally modulo N. Thus, $\hat{p} \in \hat{D}_\tau$.

Now, we show $\hat{p} \in T(e)$. Let $O = \dots O_{-1} \times O_0 \times O_1 \dots$ be a basic open set containing $\hat{p}$. Pick $n \in \mathbb{N}$ so that whenever $|i| > n$ we have $O_i = \{*, 1, 2\}$. Define $\hat{x}$ by setting

$$\hat{x}_i = \begin{cases} e_i \text{ if } i < 0 \\ p_i \text{ if } 0 \leq i < n \text{ and } p_i \neq * \\ 2 \text{ otherwise} \end{cases}$$

Then we clearly have $\hat{x} \in O \cap T(e)$. Since $T(e)$ is closed, it follows that $\hat{p} \in T(e)$. That $\hat{p} \in T(\widetilde{e})$ follows similarly.

It remains to show that $T(e) \cap T(\widetilde{e}) \subseteq \{\hat{p}\}$. Suppose $\hat{q} \in T(e) \cap T(\widetilde{e})$. Pick j minimally so that $e_j \neq \widetilde{e}_j$, and note that $j \equiv \beta \mod N$. If $\hat{q}_j \neq *$, we may construct an open set containing $\hat{q}$, and missing one of either $T(e)$ or $T(\widetilde{e})$. Hence, $\hat{q}_j = *$. If $\hat{q}_{-\beta} \neq *$, then $\pi_{-\beta}(\hat{q}) \approx \tau$, but $\pi_{-\beta}(\hat{q}) \neq \tau$, contradicting admissibility. Hence, $\hat{q}_{-\beta} = *$, which implies $\hat{q} = \hat{p}$.

The proof for when $\beta = \infty$ is similar. □

Corollary 5.1. *Suppose τ is of period N, and let e be a backwards itinerary occurring in $\hat{D}_\tau$. Then $\pi_0 T(e) \supseteq [\{\sigma^i(\tau) | \beta^i(e)$ is defined$\}]$.*

Proof. By Lemma 5.3, whenever $\beta^i(e)$ is defined, it corresponds to a boundary point of $T(e)$ which projects to $\sigma^i(\tau)$. $\pi_0 T(e)$ is connected, and $[\{\sigma^i(\tau) | \beta^i$ is defined$\}]$ is the smallest connected subset of D_τ containing the $\sigma^i(\tau)$'s. □

Example 5.1. Let $\tau = \overline{*112}$ and $e = 1^\infty$. Then the point $\hat{p}_1$ with backwards itinerary $1^\infty *$ is a boundary point for $T(e)$, and is adjacent to the continuum $T(e_1)$, where $e_1 = 1^\infty 2$. Moreover, $T(e)$ also shares boundaries with $T(e_2)$ and $T(e_3)$, where $e_2 = 1^\infty 21$ and $e_3 = 1^\infty 211$. Hence, it follows that $T(e)$ contains a branch point (i.e., a point $\hat{p}$ so that $T(e) - \{\hat{p}\}$ consists of more than two components). This corresponds to the *central branching point* of D: 1^∞(cf [1], Theorem 1.22 and Definition 1.23).

In general, a boundary point of $T(e)$ may also be a branch point. For example, again let $\tau = \overline{*112}$, and define $\hat{p}$ to be the point with backwards itinerary $1^\infty * 112 * 112$. Then $\hat{p} \in T(e_1) \cap T(e_2) \cap T(e_3)$, where $e_1 = 1^\infty 21112$, $e_2 = 1^\infty 21121112$, and $e_3 = 1^\infty 22112$. And, if $\beta^k(e) = \infty$ for some k, there exists an infinite collection of backwards itineraries, whose corresponding continua share a common boundary point. For example, for each n, let $e^n = (1112)^\infty(2112)^n$. Then $(*112)^\infty.(*112)^\infty \in \bigcap_{n=1}^\infty T(e^n)$.

Lemma 5.3. *Suppose $A = [\hat{x}, \hat{y}]$ is an arc in $\hat{D}_\tau$, where τ is of period N. If $\{\hat{z}^i\}$ is a sequence of points of A converging to $\hat{y}$, then there exists an integer M so that if $i, j \geq M$, we have $e(\hat{z}^i) = e(\hat{z}^j)$.*

Proof. Suppose $e(\hat{z}^i) \neq e(\hat{z}^j)$ infinitely often. Then, between each such pairing, there exists a $\hat{p}^i$ with $\hat{p}^i_{-k(i)} = *$ for some $k(i) \in \mathbb{N}$. By passing to a subsequence if necessary, we may assume each $k(i)$ is congruent modulo N. It follows that $\hat{y}$ is a shift of $(\tau_0\tau_1 \ldots \tau_{N-1})^\infty.(\tau_0\tau_1 \ldots \tau_{N-1})^\infty$, as otherwise we may construct an open set containing $\hat{y}$ and at most finitely many of the $\hat{p}^i$'s. By Proposition 3.4, $e_*(\hat{p}^i) \approx e_*(\hat{y})$ for each i. Fix i_0, and pick M so that whenever $j > M$, we have $\hat{p}^{i_0}_{-j} \approx \hat{y}_{-j}$. By Proposition 3.3, whenever $j > M$ and $i \geq i_0$, we have $\hat{p}^i_{-j} \approx \hat{y}_{-j}$. This leaves only finitely many options for $e(\hat{p}^i)$ when $i \geq i_0$. Hence, there exists an i_1 so that whenever $i, j \geq i_1$, we have $e(\hat{p}^i) = e(\hat{p}^j)$. By admissibility, this implies $\hat{p}^i = \hat{p}^j$ whenever $i, j \geq i_1$, providing a contradiction. □

Proposition 5.4. *Suppose $A = [\hat{x}, \hat{y}]$ is an arc in $\hat{D}_\tau$, where τ is of period N. Then there exists finitely many backwards itineraries occuring on A.*

Proof. If $\hat{z} \in [\hat{x}, \hat{y}]$ with $\hat{z}_{-i} \neq *$ for all $i \in \mathbb{N}$, then the set $\{\hat{p} \in [\hat{x}, \hat{y}] : e(\hat{p}) = e(\hat{z})\}$ is open in the subspace $[\hat{x}, \hat{y}]$. If $\hat{z} \in [\hat{x}, \hat{y}]$ with $\hat{z}_{-i} = *$ for some i, then, by Lemma 5.3 we may find an open set (in the topology of $[\hat{x}, \hat{y}]$) containing $\hat{z}$ and at most three backwards itineraries. This gives us an open cover of the compact space $[\hat{x}, \hat{y}]$. Taking a finite subcover concludes the proof. □

Suppose $\phi : [0, \infty) \to \hat{D}_\tau$ is a continous bijection. We call the image of ϕ a *ray*, and ϕ a *parameterization*. Suppose $\phi : [0, \infty) \to \hat{D}_\tau$ parameterizes a ray Φ. Let

$e = e(\phi(0))$, and suppose that $e_{-i} \neq *$ for all $i \leq -1$. As s increases, the backwards itineraries $e(\phi(s))$ may also change. Pick s_1 minimal so that $e(\phi(s_1)) \neq e$. Then, by Proposition 5.4, $\phi(s_1)$ is a boundary point between Te and $Te(\phi(s_1+\varepsilon_1))$ for sufficiently small ε_1. Let $Re = e(\phi(s_1+\varepsilon_1))$. Continue inductively, picking $s_n > s_{n-1}$ minimally with $e(\phi(s_n)) \neq R^{n-1}e$. Then $\phi(s_n)$ is a boundary point between $TR^{n-1}e$ and $Te(\phi(s_n+\varepsilon_n))$ for sufficiently small ε_n, and let $R^n e = e(\phi(t_n+\varepsilon_n))$. We define the *folding pattern*, $\{\alpha_n(\Phi)\}$, or simply $\{\alpha_i\}$ when the ray Φ is understood, by letting α_n be the sequence of discrepancies between $R^{n-1}e$ and $R^n e$. If Φ' is a ray originating in $T(\widetilde{e})$ with folding pattern $\{\tilde{\alpha}_i\}$ we let $d_n(\Phi, \Phi')$ denote the first discrepancy between $R^n e$ and $R^n\widetilde{e}$. Note that each α_n is a (potentially finite) sequence, each element of which is congruent modulo N. We let $C(\alpha_n)$ denote the least nonnegative element of this congruence class. Let ϕ' parameterize Φ', and let d be a metric compatible with the topology of $\hat{D}_\tau$. We say the rays Φ and Φ' are *asymptotic* provided that $d(\phi(s), \phi'(s)) \to 0$.

There are notable differences in our use of $R^n e$, as compared with the development in [22, 20], for the space (I, f). In particular, in (I, f), $R^n e$ is a function of the backwards itinerary e. Whereas in our treatement, given a backwards itinerary e, there are multiple valid choices for Re, depending on which of the $\beta^i e$'s are defined. Indeed, if $\beta^i e = \infty$ for some i, there are infinitely many options for Re. Additionally, given a backwards itinerary e, once choices have been assigned to $R^n e$ for each $n \in \mathbb{N}$, this defines a unique ray in $\hat{D}_\tau$. We state this more formally with the following proposition.

Proposition 5.5. *Suppose $e = e(\hat{x})$ for some $\hat{x} \in \hat{D}_\tau$. Let e^1 be a backwards itinerary, distinct from e, so that $T(e^1)$ shares a common boundary point with $T(e)$. Let γ_1 denote the sequence of discrepancies between e and e^1. Continue inductively, letting $e^n \notin \{e^i : i < N\}$ be chosen so that $T(e^n)$ and $T(e^{n-1})$ share a common boundary* *point, and let γ_n denote the sequence of discrepancies between e^{n-1} and e^n. Then there exists a unique ray Φ in $\hat{D}_\tau$, so that $\alpha_n(\Phi) = \gamma_n$ for all $n \in \mathbb{N}$.*

Proof. This follows from Lemmas 5.2 and 5.3. $\square$

Example 5.2. Suppose $\tau = \overline{*112}$, and let $e = \overline{1112}$, $e^1 = \overline{2112}$, $e^2 = \overline{2112}2111$, and $e^3 = \overline{2112}2211$. Then, by Proposition 5.3, there exists a ray Φ beginning in Te, and travelling through Te^3 by way of Te^1 and Te^2. For such a ray, we have $e^1 = Re$, $e^2 = R^2e$, $e^3 = R^3e$, and the folding pattern for Φ begins with $\alpha_1 = \{4n\}_{n=1}^{\infty}$, $\alpha_2 = \{1\}$, and $\alpha_3 = \{3\}$.

Proposition 5.6. $\pi_0(R^n e \cap \Phi) = [\sigma^{C(\alpha_{n-1})}(\tau), \sigma^{C(\alpha_n)}(\tau)]$

Proof. This follows from Proposition 5.1. $\square$

The following can be stated more generally (cf. [22], Proposition 1); but the following will suit our purposes.

Proposition 5.7. *Suppose ρ is a metric compatible with the topology of D_τ, and τ is of period N. Let Φ and Φ' be rays in $\hat{D}_\tau$ with respective folding patterns $\{\alpha_i\}$ and $\{\tilde{\alpha}_i\}$. If $\rho(\sigma^{C(\alpha_n)}\tau, \sigma^{C(\tilde{\alpha}_n)}\tau) \to 0$ and $d_n(\Phi, \Phi') \to \infty$, then Φ and Φ' are asymptotic.*

Proof. Without loss of generality, suppose $\rho(\sigma^{C(\alpha_n)}\tau, \sigma^{C(\tilde{\alpha}_n)}\tau) = 0$ for all n. Let $e = e(\phi(0))$ and $e' = e(\phi'(0))$. Then $\pi_0(R^n e \cap \Phi) = \pi_0(R^n e' \cap \Phi')$. For each n, let $\phi_n : [n, n+1] \to (R^n e \cap \Phi)$ be a parameterization of $(R^n e \cap \Phi)$. Similarly, let $\phi'_n : [n, n+1] \to (R^n e' \cap \Phi')$ parameterize $(R^n e' \cap \Phi')$ so that $\pi_0(\phi(t)) = \pi_0(\phi'(t))$, and expand these in the obvious way to get the parameterizations ϕ and ϕ'. The condition $d_n(\Phi, \Phi') \to \infty$ implies that $d(\phi(t), \phi'(t)) \to 0$, where d is a metric for $\hat{D}_\tau$. $\square$

CHAPTER SIX

Main Results

In this chapter, we show that the sufficeint conditions for arc-components residing in (I, f) carry over to D_τ. The proofs relating to $\hat{D}_\tau$ are far less technical than their counterparts for (I, f).

6.1 A Last Minute Lemma

Lemma 6.1. *Suppose e and $\widetilde{e}$ are backwards itineraries, and let $j \leq \underline{d}$, where $\underline{d}$ is the first discrepancy between e and $\widetilde{e}$. Let $w_1 w_2 \ldots w_j \in \{1, 2\}^j$,*

$$\psi = \ldots e_{-(j+2)} e_{-(j+1)} w_1 w_2 \ldots w_j, \text{ and}$$

$$\widetilde{\psi} = \ldots \widetilde{e}_{-(j+2)} \widetilde{e}_{-(j+1)} w_1 w_2 \ldots w_j.$$

Then there exists an n so that $\psi = R^n e$ and $\widetilde{\psi} = R^n \widetilde{e}$.

Proof. By Proposition 3.4, there exists rays R and $\widetilde{R}$, respectively originating in $T(e)$ and $T(\widetilde{e})$ and peregrinating through $T(\psi)$ and $T(\widetilde{\psi})$. Hence, there exists integers n_1 and n_2 so that $R^{n_1} e = \psi$ and $R^{n_2} \widetilde{e} = \widetilde{\psi}$. Hence, we need only show $n_1 = n_2$.

Suppose $n_1 = 1$. Let $\{k_i\}_{i=1}^m$ be the sequence of discrepancies between e and ψ. Then $e_{-k_m} \ldots e_{-1} \approx \tau_0 \ldots \tau_{k_m - 1}$. Since $k_m \leq j$, we also have $\widetilde{e}_{-k_m} \ldots \widetilde{e}_{-1} \approx \tau_0 \ldots \tau_{k_m - 1}$. Moreover, by Proposition 5.3, all the k_i's are congruent modulo the period of the kneading sequence. Hence, $T(\widetilde{e})$ and $T(\widetilde{\psi})$ share a common boundary point, and $n_2 = 1$. Now, suppose $n_1 > 1$ and we proceed by induction. By Proposition 3.3, the first discrepancy between $R^{n_1 - 1} e$ and e is at most k. By the inductive hypothesis, the last discrepancy between $R^{n_1 - 1} e$ and $R^{n_1 - 1} \widetilde{e}$ is at most k. Apply the same argument as used in the base case to conclude the proof. $\square$

Analogues of the following theorems were given in [22], and stated in Chapter 4, for inverse limits of unimodal maps of the interval. Before proceeding, we will introduce some notation. If $x = x_0x_1x_2\dots$ and $y = y_0y_1y_2\dots$ are elements of D_τ, let $\underline{d}(x,y) = \min\{i : x_i \not\approx y_i\}$. Additionally, if $a \in \{1,2\}$, we use a' to denote the unique element of $\{1,2\} - \{a\}$.

6.2 Sufficient Conditions for Asymptotic Arc-components in $\hat{D}_\tau$

Theorem 6.1. *Suppose $\tau = \overline{*12^{N-2}}$, where $\tau_0 = *$. Let $l \in \{1, 2, \dots, N-2\}$. If*

$$e = \overline{12^{N-3}1}12^{N-2}1 \text{ and}$$

$$\widetilde{e} = \overline{12^{N-3}1}12^{N-2}12^{l},$$

then there exists asymptotic rays originating in $T(e)$ and $T(\widetilde{e})$.

Proof. Let

$$Re = \overline{12^{N-3}1}12^{N-2}12^{N-2}1$$

$$R\widetilde{e} = \overline{12^{N-3}1}12^{N-2}12^{l-1}1$$

By Lemma 6.1, there exists an integer n so that we may define

$$R^n e = \overline{12^{N-3}1}12^{N-2}12^{N-l-1}12^{l-1}$$

$$R^n\widetilde{e} = \overline{12^{N-3}1}12^{N-2}112^{l-1}$$

Next, take $\alpha_{n+1}(e) = l+1$ and $\alpha_{n+1}(\widetilde{e}) = N+l+1$, and we have

$$R^{n+1}e = \overline{12^{N-3}1}12^{N-2}12^{N-l-2}112^{l-1}$$

$$= e2^{N-l-2}112^{l-1}$$

$$R^{n+1}\widetilde{e} = \overline{12^{N-3}1}12^{N-2}12^{N-2}112^{l-1}$$

$$= \widetilde{e}2^{N-l-2}112^{l-1}$$

Hence, each of $R^{n+1}e$ and $R^{n+1}\widetilde{e}$ are left-shifts of the original itineraries e and $\widetilde{e}$, and all folds thus far have been congruent modulo N. We now proceed by induction. Suppose n_{i-1} is defined so that

$$R^{n_{i-1}}e = eA \text{ and}$$

$$R^{n_{i-1}}\widetilde{e} = \widetilde{e}A,$$

and all folds thus far have been congruent modulo N. Pick m so that

$$R^m e = \overline{12^{N-3}1}12^{N-2}1V_{|A|}$$

$$R^m \widetilde{e} = \overline{12^{N-3}1}12^{N-2}12^l V_{|A|}$$

where, V_n is a {1,2}-block with $V_n \approx \tau_1 \dots \tau_n$. Next, take $\alpha_{m+1}(e) = |A| + N + 1$ and $\alpha_{m+1}(\widetilde{e}) = |A| + 1$ and obtain

$$R^{m+1}e = \overline{12^{N-3}1}12^{N-2}12^{N-2}1V_{|A|}$$

$$R^{m+1}\widetilde{e} = \overline{12^{N-3}1}12^{N-2}12^{l-1}1V_{|A|}$$

Define $n_i - 1$ so that

$$R^{n_i-1}e = \overline{12^{N-3}1}12^{N-2}12^{N-l-1}V_{|A|+l}$$

$$R^{n_i-1}\widetilde{e} = \overline{12^{N-3}1}12^{N-2}1V_{|A|+l}$$

Take $\alpha_{n_i}(e) = |A| + l + 1$ and $\alpha_{n_i}(\widetilde{e}) = |A| + l + N + 1$, and we have

$$R^{n_i}e = \overline{12^{N-3}1}12^{N-2}12^{N-l-2}1V_{|A|+l}$$

$$= e2^{N-l-2}1V_{|A|+l}$$

$$R^{n_i}\widetilde{e} = \overline{12^{N-3}1}12^{N-2}12^{N-2}1V_{|A|+l}$$

$$= \widetilde{e}2^{N-l-2}1V_{|A|+l}$$

and all folds have been congruent modulo N. Hence, we have constructed rays Φ and Φ' for which $d_n(\Phi, \Phi') \to \infty$, and $\pi_0(R^n e \cap \Phi) = \pi_0(R^n \widetilde{e} \cap \Phi')$ for all n. Proposition 5.7 implies the constructed rays are asymptotic. □

Theorem 6.2. *Suppose there exists a* $N/2 < k < N$ *such that* $\underline{d}(\sigma^k\tau, \tau) \geq N - k$, *and let* $\nu_1 \ldots \nu_N \approx \tau_1 \ldots \tau_N$, *where* $\nu_N \in \{1, 2\}$ *is chosen so that* $\underline{d}(\sigma^k\nu, \tau) > N - k$. *If*

$$\begin{aligned} e &= \overline{\nu_1 \ldots \nu_k'}\nu_1 \ldots \nu_N \text{ and} \\ \widetilde{e} &= \overline{\nu_1 \ldots \nu_{N-k}'\nu_{N-k+1} \ldots \nu_k}\nu_1 \ldots \nu_{N-k}'\nu_{N-k+1} \ldots \nu_N', \end{aligned}$$

then there exists asymptotic rays originating in $T(e)$ *and* $T(\widetilde{e})$.

Proof. We begin by taking $\alpha_1(e) = N + 1$ and $\alpha_1(\widetilde{e}) = 1$, and obtain

$$\begin{aligned} Re &= \overline{\nu_1 \ldots \nu_k'}\nu_1 \ldots \nu_k\nu_1 \ldots \nu_N \\ R\widetilde{e} &= \overline{\nu_1 \ldots \nu_{N-k}'\nu_{N-k+1} \ldots \nu_k}\nu_1 \ldots \nu_{N-k}'\nu_{N-k+1} \ldots \nu_N \end{aligned}$$

The hypothetical condition $\underline{d}(\sigma^k\nu, \tau) > N - k$ implies $\nu_1 \ldots \nu_k\nu_1 \ldots \nu_{N-k} = \nu_1 \ldots \nu_N$. Additionally, since the first discrepancy between e and $\widetilde{e}$ is not less than k, we may apply Proposition 3.3 and pick n so that

$$\begin{aligned} R^n e &= \overline{\nu_1 \ldots \nu_k'}\nu_1 \ldots \nu_k\nu_1 \ldots \nu_{N-k}\nu_1 \ldots \nu_k \\ &= \overline{\nu_1 \ldots \nu_k'}\nu_1 \ldots \nu_N\nu_1 \ldots \nu_k \\ R^n\widetilde{e} &= \overline{\nu_1 \ldots \nu_{N-k}'\nu_{N-k+1} \ldots \nu_k}\nu_1 \ldots \nu_{N-k}'\nu_1 \ldots \nu_k \end{aligned}$$

Next, take $\alpha_{n+1}(e) = N + k + 1$ and $\alpha_{n+1}(\widetilde{e}) = k + 1$, and we have

$$\begin{aligned} R^{n+1}e &= \overline{\nu_1 \ldots \nu_k'}\nu_1 \ldots \nu_k\nu_1 \ldots \nu_N\nu_1 \ldots \nu_k \\ &= \overline{\nu_1 \ldots \nu_k'}\nu_1 \ldots \nu_N\nu_{N-k+1} \ldots \nu_N\nu_1 \ldots \nu_k \\ &= e\nu_{N-k+1} \ldots \nu_N\nu_1 \ldots \nu_k \\ R^{n+1}\widetilde{e} &= \overline{\nu_1 \ldots \nu_{N-k}'\nu_{N-k+1} \ldots \nu_k}\nu_1 \ldots \nu_{N-k}\nu_1 \ldots \nu_k \\ &= \overline{\nu_1 \ldots \nu_{N-k}'\nu_{N-k+1} \ldots \nu_k}\nu_1 \ldots \nu_{N-k}'\nu_{N-k+1} \ldots \nu_N\nu_1 \ldots \nu_k \\ &= \widetilde{e}\nu_{N-k+1} \ldots \nu_N\nu_1 \ldots \nu_k \end{aligned}$$

Table 6.1. The itineraries from Theorems 6.2 and 6.3 for τ up to period 5

τ	$\nu_1 \ldots \nu_N$	k	e	$\widetilde{e}$	Theorem
$(*12)^\infty$	121	2	$1^\infty 21$	2^∞	6.2
	122	2	$(1122)^\infty$	$(1122)^\infty 21$	6.3
$(*122)^\infty$	1221	3	$(121)^\infty 1221$	2^∞	6.2
	1222	3	$(211222)^\infty$	$(211222)^\infty 221$	6.3
$(*112)^\infty$	1121	3	$(112)^\infty 1112$	$(212)^\infty 2122$	6.2
	1122	3	$(111122)^\infty$	$(111122)^\infty 121$	6.3
$(*1222)^\infty$	12221	4	$(1221)^\infty 12221$	2^∞	6.2
	12222	4	$(22112222)^\infty$	$(22112222)^\infty 2221$	6.3
$(*1221)^\infty$	12211	4	$(1222)^\infty 12211$	$(2221)^\infty 2$	6.2
	12212	3	$(121)^\infty 12212$	$(112)^\infty 11$	6.2
	12211	3	$(112211)^\infty$	$(112211)^\infty 212$	6.3
	12212	4	$(22212212)^\infty$	$(22212212)^\infty 2211$	6.3
$(*1211)^\infty$	12111	4	$(12)^\infty 111$	$(2211)^\infty 2$	6.2
	12112	3	$(122)^\infty 12112$	1^∞	6.2
	12111	3	$(212111)^\infty$	$(212111)^\infty 112$	6.3
	12112	4	$(21212112)^\infty$	$(21212112)^\infty 2111$	6.3
$(*1122)^\infty$	11221	4	$(12111221)^\infty$	$(12111221)^\infty 1222$	6.3
	11222	4	$(12111222)^\infty$	$(12111222)^\infty 1221$	6.3
$(*1121)^\infty$	11211	3	$1^\infty 211$	$(122)^\infty 12$	6.2
	11211	4	$(1122)^\infty 11211$	$(212)^\infty 12$	6.2
	11212	3	$(111212)^\infty$	$(111212)^\infty 211$	6.3
	11212	4	$(12211212)^\infty$	$(12211212)^\infty 1211$	6.3
$(*1112)^\infty$	11121	4	$1^\infty 21$	$(2112)^\infty 2$	6.2
	11122	4	$(11111122)^\infty$	$(11111122)^\infty 1121$	6.3

Hence, we have shifted copies of the original backwards itineraries, and all folds have been congruent modulo N. As before, proceed inductively to conclude the proof. □

Theorem 6.3. *Suppose there exists a $N/2 < k < N$ such that $\underline{d}(\sigma^k\tau, \tau) \geq N-k$, and let $\nu_1 \ldots \nu_N \approx \tau_1 \ldots \tau_N$, where $\nu_N \in \{1,2\}$ is chosen so that $\underline{d}(\sigma^k\nu, \tau) = N-k$. If*

$$e = \overline{\nu_{N-k+1} \ldots \nu'_k \nu_1 \ldots \nu_N} \text{ and}$$
$$\widetilde{e} = \overline{\nu_{N-k+1} \ldots \nu'_k \nu_1 \ldots \nu_N} \nu_{N-k+1} \ldots \nu'_N,$$

then there exists asymptotic rays originating in $T(e)$ and $T(\widetilde{e})$.

Proof. We begin by taking $\alpha_1(e) = N+1$ and $\alpha_1(\widetilde{e}) = 1$, and we obtain

$$\begin{aligned} Re &= \overline{\nu_{N-k+1}\ldots\nu'_k\nu_1\ldots\nu_N}\nu_{N-k+1}\ldots\nu_k\nu_1\ldots\nu_N \\ &= \overline{\nu_{N-k+1}\ldots\nu'_k\nu_1\ldots\nu_N}\nu_{N-k+1}\ldots\nu_k\nu_{k+1}\ldots\nu_N\nu_{N-k+1}\ldots\nu_N \\ R\widetilde{e} &= \overline{\nu_{N-k+1}\ldots\nu'_k\nu_1\ldots\nu_N}\nu_{N-k+1}\ldots\nu_N \end{aligned}$$

We then take the path to

$$\begin{aligned} R^{n-1}e &= \overline{\nu_{N-k+1}\ldots\nu'_k\nu_1\ldots\nu_N}\nu_{N-k+1}\ldots\nu_k\nu_{k+1}\ldots\nu'_N\nu_1\ldots\nu_k \\ R^{n-1}\widetilde{e} &= \overline{\nu_{N-k+1}\ldots\nu'_k\nu_1\ldots\nu_N}\nu_1\ldots\nu_k \end{aligned}$$

Next, we may take $\alpha_n(e) = k+1$ and $\alpha_n(\widetilde{e}) = N+k+1$, and we have

$$\begin{aligned} R^n e &= \overline{\nu_{N-k+1}\ldots\nu'_k\nu_1\ldots\nu_N}\nu_{N-k+1}\ldots\nu_k\nu_1\ldots\nu'_{N-k}\nu_1\ldots\nu_k \\ &= e\nu_{N-k+1}\ldots\nu_N\nu_1\ldots\nu_k \\ R^n\widetilde{e} &= \overline{\nu_{N-k+1}\ldots\nu'_k\nu_1\ldots\nu_N}\nu_{N-k+1}\ldots\nu_k\nu_1\ldots\nu_N\nu_1\ldots\nu_k \\ &= \overline{\nu_{N-k+1}\ldots\nu'_k\nu_1\ldots\nu_N}\nu_{N-k+1}\ldots\nu'_N\nu_{N-k+1}\ldots\nu_N\nu_1\ldots\nu_k \\ &= \widetilde{e}\nu_{N-k+1}\ldots\nu_N\nu_1\ldots\nu_k \end{aligned}$$

Hence, we have left-shifted copies of the original itineraries, and all folds have been congruent modulo N. As before, proceeding by induction concludes the proof.

□

Table 6.2 lists all asymptotic arc-components guaranteed by Theorems 6.1, 6.2, and 6.3 for all admissible τ up to period 5. There are some noticable differences between our results, and the comparable ones for (I, f) presented in [22] and Chapter 4. For example, for (I, f), the backwards itineraries may not be admissible. Moreover, the situations described in Theorems 6.2 and 6.3 are mutually exclusive in (I, f). As shown in Figure 6.1, these may happen concurrently in $\hat{D}_\tau$. However, in [22], the author was able to give more information on the asymptotic structure of the asymptotic arc-components in (I, f) (e.g., whether they form "fans," "cycles," or even combinations thereof). That $\hat{D}_\tau$ contains branch points makes analogous results difficult. The following corollaries address this line of inquiry.

Corollary 6.1. *The backwards itineraries in Theorems 6.1, 6.2, and 6.3 give rise to a countably infinite collection of asymptotic rays.*

Proof. In the inductive step, we may alter our choice for $V_{|A|}$. □

Corollary 6.2. *The asymptotic arc-components in Theorem 6.1 form a k-fan (i.e., the k rays are pairwise asymptotic).*

Proof. For each $j \in \{1, 2, \ldots N-2\}$, let $\widetilde{e}^j = \overline{12^{N-2}1}12^{N-1}12^j$. As shown in Theorem 6.1, each of the $\widetilde{e}^j$'s give rise to a ray $\widetilde{\phi}$ which is asymptotic to some ray Φ_j emanating from $T(e)$, and we need only show that each Φ_j is the same ray. The proof is similar to that of Theorem 6.1.

□

Proposition 6.1. *The backwards itineraries e and $\widetilde{e}$ in each of Theorems 6.1, 6.2 and 6.3 reside on distinct arc-components of D_τ*

Proof. With Proposition 3.4 in mind, it is easily seen that the backwards itineraries from Theorem 6.1 are on different arc-components. For Theorem 6.2, observe that

$$e = \overline{\nu_1 \ldots \nu_{N-k}\nu_{N-k+1} \ldots \nu'_k}\nu_1 \ldots \nu_{N-k}\nu_{N-k+1} \ldots \nu_N$$
$$\widetilde{e} = \overline{\nu_1 \ldots \nu'_{N-k}\nu_{N-k+1} \ldots \nu_k}\nu_1 \ldots \nu'_{N-k}\nu_{N-k+1} \ldots \nu'_N$$

It is easily seen that the respective tails for e and $\widetilde{e}$ "line up," having infinitely many discrepancies, and neither being equivalent to τ.

For Theorem 6.3, after rewriting e so that the tails "line up," we have:

$$e = \overline{\nu_{N-k+1} \ldots \nu_k \ldots \nu_N\nu_{N-k+1} \ldots \nu'_k}\nu_1 \ldots \nu_{N-k}\nu_{N-k+1} \ldots \nu_N$$
$$\widetilde{e} = \overline{\nu_{N-k+1} \ldots \nu'_k\nu_1 \ldots \nu_N}\nu_{N-k+1} \ldots \nu'_N$$

Again, it is easily noticed that the tails are discrepant infinitely often, with neither being equivalent to τ. Hence, the backwards itineraries e and $\widetilde{e}$ correspond to distinct arc-components. $\square$

CHAPTER SEVEN

Miscellaneous Results

The focus of this chapter is to present some results originating from early attempts to extend the results of Chapter 4 to the non-periodic case. Most results concern unimodal maps on the interval that are *Fibonacci-like*; that is, their kneading map is given by

$$Q(k) = \max\{0, k-d\} \text{ for some } d \in \mathbb{N}.$$

In Section 6.1, we concern ourselves with some properties of the folding patterns as they relate to Fibonacci-like maps. In 6.2, some limitations on folding patterns are imposed. In particular, it is shown that if $Q(k)$ is bounded, $|\alpha_{i+1} - \alpha_i|$ is bounded for all i, where $\{\alpha_i\}$ is any folding pattern arising from (I, f). This is relevant, in that it hints at the structure the folding patterns of asymptotic arc-components must take. For this chapter, so as to avoid an abundance of subscripts, we slightly alter our notation from previous chapters: We write z_n to denote the closest precritcal point with respect to f^{S_n}, rather than with respect to f^n. Before proceeding, the reader may wish to review Chapter 2.

7.1 Some Results Concerning Fibonacci Combinatorics

Proposition 7.1. *Suppose $Q(k) = \max\{0, k-d\}$ for some natural number d. Then, for each $n > d$, we have c_{S_n} and $c_{S_{n-d}}$ are on opposite sides of c*

Proof. S_n is a cutting time, and $f^{S_n}([z_{n-1}, c]) = \langle c_{S_n}; c_{S_{n-d}} \rangle$ □

Proposition 7.2. *Suppose $Q(k) = \max\{0, k-d\}$ and $n > d$. Then $f^{S_n}(z_{n+1}) \in \{z_{n-d+1}, \hat{z}_{n-d+1}\}$.*

Proof. First, note that

$$c = f^{S_{n+1}}(z_{n+1}) = f^{S_{n-d+1}}(f^{S_n}(z_{n+1})).$$

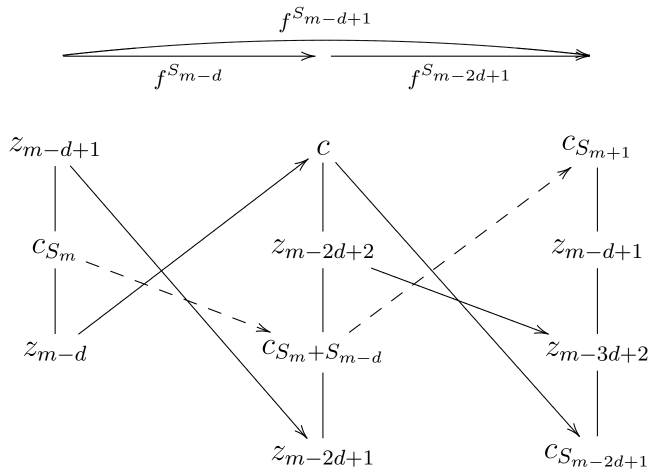

Figure 7.1. The Geometry of Corollary 7.1

Hence, $f^{S_n}(z_{n+1}) \in f^{-S_{n-d+1}}(c)$. We must verify that $f^{S_n}(z_{n+1})$ is a closest precritical point, ie, that $f^{S_n}(z_{n+1}) \in [z_{n-1}, \hat{z}_{n-1}]$. This follows, since

$$c_{S_n} \in [z_{n-d}, z_{n-d+1}] \cup [\hat{z}_{n-d+1}, \hat{z}_{n-d}]$$

and $f^{S_{n+1}}$ is one-to-one on each of $[z_n, c]$ and $[c, \hat{z}_n]$. $\square$

Lemma 7.1. *Suppose $Q(k)$ is non-decreasing. Let m and n be integers such that $S_m < n < S_{m+1}$ and $n - S_m \neq S_{Q(m+1)-1}$. Then $D_n \cap [z_{Q(m+1)}, \hat{z}_{Q(m+1)}] = \emptyset$.*

Proof. Suppose the claim is false. Pick n minimally with $S_m < n < S_{m+1}$ so that $D_n \cap [z_{Q(m+1)}, \hat{z}_{Q(m+1)}] \neq \emptyset$. Because n is not a cutting time, we have $c \notin D_n$. It immediately follows that either $D_n \cap [z_{Q(m+1)}, c] = \emptyset$ or $D_n \cap [c, \hat{z}_{Q(m+1)}] = \emptyset$. Without loss of generality, assume the latter.

Recall that

$$\begin{aligned} D_n &= \langle c_n; c_{n-S_m} \rangle \\ &\subseteq D_{n-S_m} \\ &= \langle c_{n-S_m}; c_{n-S_m-S_k} \rangle, \end{aligned}$$

where $k = \max\{j : S_j \leq n - S_m\}$. Suppose $n - S_m$ is not a cutting time. Then, by minimality of n, we know that D_{n-S_m} and $[z_{Q(m+1)}, c]$ have empty intersection. Since $n < S_{m+1}$, we have $n - S_m < S_{m+1} - S_m = S_{Q(m+1)}$. Hence, $k \leq Q(m+1)$. Furthermore, since $Q(j) < j$ for all j, and Q is non-decreasing, we have $z_{Q(k+1)} \leq z_{Q(Q(m+1)+1)} \leq z_{Q(m+1)}$. Hence, $D_{n-S_m} \cap [z_{Q(m+1)}, c] \neq \emptyset$, contradicting the minimality of n.

If $n - S_m$ is a cutting time, say S_j, then $D_n = \langle c_n; c_{S_j} \rangle$. We immediately obtain $c_{S_j} \notin [z_{Q(m+1)}, \hat{z}_{Q(m+1)}]$, since $c_{S_j} \in \langle z_{Q(j+1)-1}, z_{Q(j+1)} \rangle$, $j < Q(m+1) \leq m$ and Q is non-decreasing. Now, it suffices to show $c_n \notin [z_{Q(m+1)}, \hat{z}_{Q(m+1)}]$. Note that $c_n = f^{S_j}(c_{S_m})$ and $c_{S_m} \in \langle z_{Q(m+1)-1}; z_{Q(m+1)} \rangle$. As $f^{S_{Q(m+1)}}(z_{Q(m+1)}) = c$, and $S_j < S_{Q(m+1)}$, we have that $f^{S_j}(z_{Q(m+1)}) \in f^{-(S_{Q(m+1)-1} - S_j)}(c)$. Hence, $f^{S_j}(z_{Q(m+1)})$ is a (not necessarily closest) precritical point. Therefore, $f^{S_j}(z_{Q(m+1)}) \notin [z_{Q(m+1)}, \hat{z}_{Q(m+1)}]$. The condition $n - S_m \neq S_{Q(m+1)-1}$ implies the same for $f^{S_j}(z_{Q(m+1)-1})$. Since f^{S_j} is monotone on $\langle z_{Q(m+1)-1}; \hat{z}_{Q(m+1)-1} \rangle$, it follows that $c_n \notin [z_{Q(m+1)}, \hat{z}_{Q(m+1)}]$, as desired. □

Corollary 7.1. *Suppose $Q(k) = \max\{0, k-d\}$. Then, for each m and n with $S_m < n < S_{m+1}$, we have $D_n \cap [z_{m-d+1}, \hat{z}_{m-d+1}] = \emptyset$.*

Proof. We need only verify the case when $n - S_m = S_{m-d}$, wherein

$$D_n = \langle c_{(S_m + S_{m-d})}; c_{S_{m-d}} \rangle.$$

First, we note that $c_{S_{m-d}} \in [z_{m-2d}, z_{m-2d+1}] \cup [\hat{z}_{m-2d+1}, \hat{z}_{m-2d}]$. Hence, it suffices to show that $c_{S_m + S_{m-d})} \notin [z_{m-d+1}, \hat{z}_{m-d+1}]$.

Note the following three facts:

(1) $c_{S_m} \in [z_{m-d}, z_{m-d+1}] \cup [\hat{z}_{m-d+1}, \hat{z}_{m-d}]$

(2) $c_{S_{m+1}} \in [z_{m-d+1}, z_{m-d+2}] \cup [\hat{z}_{m-d+2}, \hat{z}_{m-d+1}]$

(3) $c_{S_{m+1}} = f^{S_{m-d+1}}(c_{S_m})$

By Proposition 7.2, we have $f^{S_{m-d}}([z_{m-d}, z_{m-d+1}) = \langle z_{m-2d+1}; c\rangle$, which implies $c_{S_m+S_{m-d}} \in \langle z_{m-2d+1}; c\rangle$. Without loss of generality, assume $c_{S_{m+1}} < c$. Since $c_{S_{m+1}} \notin \langle c_{S_{m-2d+1}}; z_{m-3d+2}\rangle = f^{S_{m-2d+1}}([z_{m-2d+2}; c])$, we have $c_{(S_m+S_{m-d})} \notin [z_{m-2d+2}, \hat{z}_{m-2d+2}]$. Hence, $c_{(S_m+S_{m-d})} \notin [z_{m-d+1}, \hat{z}_{m-d+1}]$, as desired. □

Proposition 7.3. *Suppose* $Q(k) = \max\{0, k-d\}$, *and let*

$$e = \ldots V_{S_{j_4}} V_{S_{j_3}} V_{S_{j_2}} V_{S_{j_1}},$$

where, for each i, *we have* $Q(j_i) = j_{i-2}$ *(i.e.,* $j_{i-2} = j_i - d$*). Then* e *is admissible.*

Proof. The interval(s) of points whose images have itineraries beginning $V_{S_{j_i}} V_{S_{j_{i-1}}}$ is $\langle z_{j_{i-1}}; z_{j_i}\rangle$. And,

$$\begin{aligned} f^{S_{j_i}}(\langle z_{j_{i-1}}; z_{j_i}\rangle) &= \langle c_{S_{j_i-d}}; c\rangle \\ &\supseteq \langle z_{j_i-2d+1}; c\rangle \\ &\supseteq \langle z_{j_i-d}, z_{j_{i-1}}\rangle \\ &= \langle z_{j_{i-2}}; z_{j_{i-1}}\rangle, \end{aligned}$$

which is the set of points with itineraries beginning $V_{S_{j_{i-1}}} V_{S_{j_{i-2}}}$. □

Proposition 7.4. *Suppose* $Q(k) = \max\{0, k-2\}$, *and* n *is an odd integer. Then*

$$V_{S_n-2} = V_{S_{n-2}} W_{S_{n-3}} V_{S_{n-4}} \ldots V_5 W_3 V_2 W_1.$$

Proof. Before proceeding, recall the following identity for the Fibonacci numbers, $\{F_n\} : \sum_{k=0}^{n} F_k = F_{n+2} - 1$. Hence, $\sum_{k=0}^{n} S_k = S_{n+2} - 2$, and the words in the statement of the proposition are of appropriate length. Simple observation yields that $V_{S_3-2} = V_3 = V_2 W_1$. Assume the result holds for $n-2$. For all n, we have $V_{S_n} = V_{S_{n-2}} = V_{S_{n-2}} W_{S_{n-3}} W_{S_{n-2}}$. Hence, $V_{S_n-2} = V_{S_{n-2}} W_{S_{n-3}} V_{S_{n-2}-2}$. Applying the inductive hypothesis to $V_{S_{n-2}-2}$ yields the desired result. □

Example 7.1. Let $Q(k) = \max\{0, k-2\}$. Then the backwards itinerary

$$e = \ldots V_{S_3} V_{S_2} V_{S_1} V_{S_0}$$

is admissible by Proposition 7.3. Let $\{\alpha_i\}$ denote the folding pattern of e. With the previous proposition in mind, a cursory inspection hints that $\alpha_i = S_{2i-1} - 1$ for all i. This is indeed the case, and we will now give a proof of this claim.

Suppose $\alpha_i \neq S_{2i-1} - 1$, where i is chosen minimally. Then

$$R^{i-1}e = \ldots V_{S_{2i-2}} V_{S_{2i-3}} V_{S_{2i-4}} \ldots W_3 V_2 W_1.$$

By the previous proposition and our assumption that $\alpha_i \neq S_{2i-1} - 1$ together imply that $\alpha_i > S_{2i-1} - 1$.

Case 1: $S_{2i-2} + S_{2i-1} - 1 < \alpha_i \leq \infty$. Pick n maximally so that $\sum_{i=0}^{n} S_i = S_{n+2} - 2 < \alpha_i$ (if $\alpha_i = \infty$, redefine α_i to be a large match for the kneading sequence). Let $M = S_{n+2} - 1$, and let $\hat{x} = (x_i)_{i \in \mathbb{Z}} \in T(R^{i-1}e)$. Then $x_{-(M+1)}$ has itinerary beginning $V_{S_n} V_{S_{n+1}}$. As such, $x_{-(M+1)} \in \langle z_n; z_{n+1} \rangle$. Similarly, since $x_{-(\alpha_i - 1)}$ has itinerary beginning V_{α_i} and $\alpha_i \geq S_{n+2} - 1$, we have $x_{-\alpha_i} \in [z_{n+2}, \hat{z}_{n+2}]$.

Pick m maximailly so that $S_m \leq \alpha_i - M$, and note that $S_m \leq S_{n+1}$. Hence, $\pi_{-\alpha_i} T(R^{i-1}e) \subseteq \langle z_{n+2}; c \rangle \subseteq J_{S_m}$. Therefore, $\pi_{-(\alpha_i - S_m)} T(R^{i-1}e) \subseteq D_{S_m} = \langle c_{S_m}; c_{S_{m-2}} \rangle$. If $S_m < \alpha_i - M$, Corollary 7.1 yields $D_{\alpha_i - M} \cap [z_{m-1}, \hat{z}_{m-1}] = \emptyset$. And, since $\pi_{-M} T(R^{i-1}e) \subseteq D_{\alpha_i - M}$, we have $\pi_{-M} T(R^{i-1}e) \cap [z_{m-1}, \hat{z}_{m-1}] = \emptyset$. As $m < n$, this is a contradiction to $x_{-m} \in \langle z_n; z_{n+1} \rangle$.

Case 2: $\alpha_i = S_{2i-2} + S_{2i-1} - 1$. This case cannot happen, since it would imply the kneading sequence ν begins $V_{S_{2i-2}} V_{S_{2i-3}}$.

Case 3: $S_{2i-1} - 1 < \alpha_i < S_{2i-2} + S_{2i-1} - 1$. Let $k = \alpha_i - S_{2i-1} - 1$. Then the itinerary of $x_{-(\alpha_i - 1)}$ begins $V_k V_{S_{2i-1} - 1}$. As $P(k)$ =even, k is not a cutting time. Pick m so that $S_m < k < S_{m+1}$, noting that $m < 2i - 2$. By Corollary 7.1, we have $D_k \cap [z_{m-1}, \hat{z}_{m-1}] = \emptyset$. Hence, $D_k \cap [z_{2i-2}, \hat{z}_{2i-2}] = \emptyset$, contradicting $x_{-(S_{2i-2} - 2)} \in [z_{2i-2}, \hat{z}_{2i-2}]$.

Hence, $\alpha_i = S_{2i-1} - 1$ for all i. As we follow the path along $R^i(e)$ as $i \to \infty$, we are approaching a point with backwards itinerary

$$\dots V_{S_{n-2}} W_{S_{n-3}} V_{S_{n-4}} \dots V_5 W_3 V_2 W_1.$$

7.2 A Bit on the Structure of Folding Patterns

Proposition 7.5. *Suppose f is unimodal with recurrent critical point c, $S_{Q(k)} < M$ for all k, and $\sigma^n \nu$ begins V_M. Then n is not a cutting time.*

Proof. Suppose $\sigma^n \nu$ begins V_N, where $N > S_M$ and that n is a cutting time, say S_k. Recall that $S_k = \sum_{j=0}^{k} S_{Q(k)}$. Hence, $W_{S_{Q(k)}}$ begins ν_{n+1} in the decomposition

$$\nu = 1 W_{S_{Q(1)}} W_{S_{Q(2)}} W_{S_{Q(3)}} \dots W_{S_{Q(k)}} \dots .$$

Recall that ν begins $10\dots$. Thus, if $S_{Q(k)} = 1$, we have $\nu_{n+1} = 0$ and if $S_{Q(k)} = 2$ we have $\nu_{n+1}\nu_{n+2} = 11$. Therefore, $S_{Q(k)} > 2$. Hence, $W_{S_{Q(k)}} = \nu_{n+1} \dots \nu_{n+S_{Q(k)}} = \nu_1 \dots \nu'_{S_{Q(k)}}$, contradicting that $\sigma^n \nu$ begins

$$V_N = \nu_1 \dots \nu_{S_{Q(k)}} \dots \nu_{S_M} \dots \nu_N.$$

Hence, n is not a cutting time. □

Proposition 7.6. *Suppose e is a backwards itinerary with folding pattern $\{\alpha_i\}$. Then, for each i, $c \in \pi_{-\alpha_i} T(R^i e) \cap \pi_{-\alpha_{i+1}} T(R^i e)$.*

Proof. Recall that $\pi_0 T(R^i e) = \langle c_{\alpha_i}; c_{\alpha_{i+1}} \rangle$. Hence, there exists a point $\hat{x} \in T(R^i e)$ for which $x_0 = c_{\alpha_i}$ and $x_{-\alpha_i} \in f^{-\alpha_i}(c_{\alpha_i})$. Furthermore, since

$$\pi_{-\alpha_i} T(R^i e) \subseteq [z_{\beta(\alpha_i - 1)}, \hat{z}_{\beta(\alpha_i - 1)}],$$

we conclude $x_{-\alpha_i} = c$. That proof that $c \in \pi_{-\alpha_{i+1}} T(R^i e)$ is similar. □

Proposition 7.7. *Suppose e is a backwards itinerary with folding pattern $\{\alpha_i\}$. Let i and j be integers satisfying $0 < i < j$ and $\alpha_j > \alpha_k$ for all $i < k < j$. Then $\pi_{-(\alpha_{j+1})} T(R^j e) \cap \pi_{-(\alpha_{j+1})} T(R^i e) = \emptyset$.*

Proof. Let $\hat{x}$ and $\hat{y}$ be points in (I, f) with respective backwards itineraries

$$R^j e = \dots e_{-\alpha_{j+1}} V_{\alpha_{j+1}-1}, \text{ and}$$
$$R^i e = \dots e_{-\alpha_{j+1}} V_{\alpha_{j+1}-\alpha_{i+1}-1} e_{-\alpha_{i+1}} V_{\alpha_{i+1}-1}$$

Hence, we see that $[z_{\beta(\alpha_{j+1}-1)}, \hat{z}_{\beta(\alpha_{j+1}-1)}]$ contains $x_{-\alpha_{j+1}}$; but not $y_{-\alpha_{j+1}}$, else $\hat{y}$ would have backwards itinerary $R^j e$. □

Proposition 7.8. *Suppose e is a backwards itinerary with folding pattern $\{\alpha_i\}$. Then, for each i, $|\alpha_{i+1} - \alpha_i|$ is a cutting time.*

Proof. Suppose $\alpha_{i+1} > \alpha_i$. Then $\pi_{-\alpha_{i+1}} T(R^{i+1}e) \subseteq [z_{\beta(\alpha_{i+1})}, \hat{z}_{\beta(\alpha_{i+1})}]$ and, by Proposition 7.6, we have $c \in \pi_{-\alpha_i} T(R^i e)$. Let $\hat{x} \in T(R^i e)$ with $x_{-\alpha_i} = c$. Then $x_{-\alpha_{i+1}} \in [z_{\beta(\alpha_{i+1})}, \hat{z}_{\beta(\alpha_{i+1})}]$, since $R^i e = \dots e_{-\alpha_{i+1}} V_{(\alpha_{i+1}-\alpha_i-1)} V_{\alpha_i - 1}$. Hence, we conclude that

$$[z_{\beta(\alpha_{i+1}-\alpha_i-1)}, \hat{z}_{\beta(\alpha_{i+1}-\alpha_i-1)}] \cap f^{-(\alpha_{i+1}-\alpha_i)}(c) \neq \emptyset.$$

If $\alpha_{i+1} - \alpha_i$ is not a cutting time, then there can be no precritical point under $f^{\alpha_{i+1}-\alpha_i}$ in $[z_{\beta(\alpha_{i+1}-\alpha_i-1)}, \hat{z}_{\beta(\alpha_{i+1}-\alpha_i-1)}]$. Hence, we have the desired result. The case where $\alpha_{i+1} < \alpha_i$ is similar. □

Proposition 7.9. *Suppose f is unimodal with kneading map $Q(k)$ and $S_{Q(k)} \leq M$ for all k. Then (I, f) has no strictly increasing folding patterns.*

Proof. Suppose e is a backwards itinerary with strictly increasing folding pattern $\{\alpha_i\}$. Pick k so that $\alpha_i > M$ for all $i > k$. Fix $i - 2 > k$. Then

$$R^{i-1} e = \dots e_{-\alpha_i} V_{(\alpha_i - \alpha_{i-1} - 1)} e_{-\alpha_{i-1}} V_{(\alpha_{i-1} - 1)}$$
$$= \dots e_{-\alpha_i} V_{(\alpha_i - 1)}$$

Since $\alpha_{i-1} - 1 > M$, Proposition 7.5 implies $\alpha_i - \alpha_{i-1}$ is not a cutting time. This contradicts Proposition 7.8. Hence, the hypothesized backwards itinerary e cannot exist. □

It is worth pointing out that there do exists strictly increasing folding patterns in the non-longbranched case. We constructed such an example for the Fibonacci combinatorics in the previous section.

The final proposition in this section gives a characterization, based on the folding pattern, for when a single arc-component can contain points whose respective backwards itineraries are discrepant infinitely often.

Proposition 7.10. *An arc-component C in (I, f) has backwards itineraries e and $\widetilde{e}$ such that $e_* \neq \widetilde{e}_*$ if and only if there exists a folding pattern $\{\alpha_i\}$ on C with* $\liminf \alpha_i = \infty$.

Proof. This follows from Proposition 4.4, having one of the parameterizations map onto a single point. □

Corollary 7.2. *Suppose (I, f) contains a backwards itinerary with folding pattern $\{\alpha_i\}$ so that* $\liminf \alpha_i = \infty$. *Then $Q(k)$ is unbounded.*

Proof. This follows from the previous proposition and Theorem 2.1. □

CHAPTER EIGHT

Conclusion

This dissertation began by giving some basic results concerning inverse limits of continua. Our journey then took us through the work Bruin did for asymptotic arc-components arising in inverse limits of the interval, and then through Baldwin's work with dendrites and continuous itinerary functions. We then successfully fused the two theories, culminating in the presentation of sufficient conditions for arc-components, arising in the inverse limit spaces of a class of dendrites, to be asymptotic. However, an abundance of questions remain. For example:

- Are these the only asymptotic arc-components for the spaces considered?
- Can we extend our results to the non-periodic case?
- Is there an upper bound for the number of asymptotic arc-components in $\hat{D}_\tau$?
- What more can we say of the structure of asymptotic arc-components in $\hat{D}_\tau$?
- What other results from (I, f) can be generalized to $\hat{D}_\tau$?
- If $\tau \neq \nu$, can $\hat{D}_\tau$ and $\hat{D}_\nu$ be homeomorphic? In particular, must there be some topological difference in the structure of their respective asymptotic arc-components?

Determining whether a backwards itinerary in (I, f) is admissible is quite laborious, and provides a stumbling block for many qustions for this class of spaces. As this dissertation shows, reconsidering such problems in the context of $\hat{D}_\tau$ may alleviate such problems. We intend to keep exploring this line of inquiry, hoping to answer some, if not all, of the above questions. Indeed, it is foreseen that these questions should keep us occupied for some time to come. We look forward to the adventure!

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