ABSTRACT

Similarity of Blocks in Parabolic Category $\mathcal{O}$
and a Wonderful Correspondence for Modules of Covariants

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Let $(\mathfrak{g}, \mathfrak{k})$ be the pair of complexified Lie algebras corresponding to an irreducible Hermitian symmetric space of noncompact type. Then we have an associated triangular decomposition $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$, and $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$ is a maximal parabolic subalgebra of $\mathfrak{g}$ with abelian nilradical. Associated to the pair $(\mathfrak{g}, \mathfrak{q})$ we can define a highest weight category $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ which is a parabolic analogue of the BGG category $\mathcal{O}$. Each block in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ contains finitely many simple modules, whose highest weights form a nice partially ordered set. In this thesis a new notion, similarity of blocks, is introduced. Then, for the classical Hermitian symmetric pairs that are in the dual pair setting, a result is proved in each of the three classical cases that every block in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ is similar to some regular integral block.

As an application of similarity, a foray into classical invariant theory leads to a "wonderful correspondence" between certain modules of covariants. This correspondence, which was previously introduced for a few special cases, is extended to all modules of covariants (in the classical cases) that are Cohen-Macaulay.

Similarity of Blocks in Parabolic Category $\mathcal{O}$
and a Wonderful Correspondence for Modules of Covariants

by

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# CHAPTER ONE

# Introduction

## *1.1 Background*

This thesis is concerned with an application of representation theory of non-compact Lie groups, and related complex simple Lie algebras, to classical invariant theory.

The seeds of this work came from the 2004 paper [9] by Enright–Willenbring, where they introduced a correspondence between Hilbert series of highest weight representations in certain special cases. The work was extended by Enright–Hunziker in [3] to Hilbert series of rings of invariants in the context of Weyl's fundamental theorems of classical invariant theory.

Then in the 2014 paper [5] by Enright–Hunziker–Pruett, these results were explained using the structure of blocks in the BGG category $\mathcal{O}$, specifically a parabolic category $\mathcal{O}$ written $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$. Those authors also extended the correspondence to a "Wonderful Correspondence" that associated to every Wallach representation $L(\lambda)$ with Hilbert series $H_L(t)$ a finite-dimensional representation $E(\lambda')$ whose Hilbert series $H_E(t)$ is, up to a multiplicative constant, the numerator of the Hilbert series $H_L(t)$. In each of the three cases, the highest weight $\lambda'$ is obtained from $\lambda$ by Enright reduction.

## *1.2 Contributions*

In this thesis, we introduce a new notion, *similarity* of blocks in a parabolic category $\mathcal{O}$. We complete a proof that every block in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ is similar to a regular integral block, as Enright and Shelton had already shown part of this result without using the term "similar."

As an application of similarity, we are able to generalize the Wonderful Correspondence to all Cohen–Macaulay modules of covariants in the classical cases. The generalized correspondence still associates to each representation a finite-dimensional representation with a related Hilbert series, and the weight $\lambda'$ is still obtained from the weight $\lambda$ by Enright reduction. However, surprisingly $\lambda'$ can also be obtained using a partition that is the transpose of the one used to obtain $\lambda$.

# CHAPTER TWO

## Preliminaries

### *2.1 Lie Algebras of Hermitian type*

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank $n$. Let $\mathfrak{t}$ be a fixed Cartan subalgebra of $\mathfrak{g}$, and let $\Delta$ be the root system of $\mathfrak{g}$ relative to $\mathfrak{t}$. For each $\alpha \in \Delta$ there is a 1-dimensional root space

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{t}\}.$$

Fix a simple root system $\Pi \subseteq \Delta$, which determines a positive system $\Delta^+ \subseteq \Delta$. Then the standard Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is the direct sum of $\mathfrak{t}$ and the positive root spaces:

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

A *parabolic subalgebra* $\mathfrak{q}$ of $\mathfrak{g}$ is a subalgebra that contains some Borel subalgebra of $\mathfrak{g}$. If $\mathfrak{q}$ contains the standard Borel subalgebra above it is called a *standard parabolic subalgebra.*

Suppose $\mathfrak{g}_\mathbb{R}$ is a real simple Lie algebra and let $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}$ be its Cartan decomposition such that the center of $\mathfrak{k}_\mathbb{R}$ is nontrivial. Then $(\mathfrak{g}_\mathbb{R}, \mathfrak{k}_\mathbb{R})$ is a Hermitian symmetric pair of noncompact type. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. From the general theory, there is some $h_0 \in \mathfrak{z}(\mathfrak{k})$ such that $\mathfrak{z}(\mathfrak{k}_\mathbb{R}) = \mathbb{R}\sqrt{-1}h_0$, and the eigenvalues of $\mathrm{ad}(h_0)$ on $\mathfrak{g}$ are 0 and $\pm 1$. Now let

$$\mathfrak{p}^\pm = \{x \in \mathfrak{g} \mid [h_0, x] = \pm x\}.$$

Then the subalgebra $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$ is a maximal parabolic subalgebra of $\mathfrak{g}$, and its nilradical $\mathfrak{p}^+$ is abelian. Parabolic subalgebras $\mathfrak{q}$ that arise in this way are called parabolic subalgebras of *Hermitian type.*

**Example 2.1.1.** The emphasis here will be on the Lie algebras $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p,q)$, $\mathfrak{sp}(n,\mathbb{R})$, and $\mathfrak{so}^*(2n)$, which are the Lie algebras of the following Lie groups, respectively:

$$SU(p,q) = \left\{ g \in SL(p+q,\mathbb{C}) \middle| g \left( \begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array} \right) g^* = \left( \begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array} \right) \right\}$$

$$Sp(n,\mathbb{R}) = \left\{ g \in SU(n,n) \middle| g \left( \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right) g^t = \left( \begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array} \right) \right\}$$

$$SO^*(2n) = \left\{ g \in SU(n,n) \middle| g \left( \begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array} \right) g^t = \left( \begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array} \right) \right\}$$

The complexified Lie algebras are, respectively:

$$\mathfrak{su}(p,q)_{\mathbb{C}} = \mathfrak{sl}(p+q,\mathbb{C}) = \left\{ \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \middle| \operatorname{tr} A + \operatorname{tr} D = 0 \right\}$$

$$\mathfrak{sp}(n,\mathbb{R})_{\mathbb{C}} = \mathfrak{sp}(n,\mathbb{C}) = \left\{ \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \middle| A = -D^t, B = B^t, C = C^t \right\}$$

$$\mathfrak{so}^*(2n)_{\mathbb{C}} = \mathfrak{so}(2n,\mathbb{C}) = \left\{ \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \middle| A = -D^t, B = -B^t, C = -C^t \right\}$$

where $A, B, C, D$ are *complex* matrices of appropriate size. We will denote the complexified Lie algebra by $\mathfrak{g}$ in each case. In all three of these cases, the triangular decomposition $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ has the form

$$\mathfrak{p}^- = \left( \begin{array}{c|c} 0 & 0 \\ \hline C & 0 \end{array} \right), \quad \mathfrak{k} = \left( \begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right), \quad \mathfrak{p}^+ = \left( \begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array} \right),$$

where $A, B, C, D$ satisfy the conditions of the appropriate complexified Lie algebra. In the $\mathfrak{su}(p,q)$ case, $\mathfrak{k}$ is isomorphic to $(\mathfrak{gl}(p,\mathbb{C}) \oplus \mathfrak{gl}(q,\mathbb{C})) \cap \mathfrak{sl}(p+q,\mathbb{C})$, which we can write as $\mathfrak{s}(\mathfrak{gl}(p,\mathbb{C}) \oplus \mathfrak{gl}(q,\mathbb{C}))$. In the other two cases, since $D$ is determined by $A$, $\mathfrak{k}$ is isomorphic to $\mathfrak{gl}(n,\mathbb{C})$.

## 2.2 *Roots and Related Posets*

Fix $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t} \subseteq \mathfrak{k}$, and let $\Delta$ be the root system of $(\mathfrak{g}, \mathfrak{t})$. If $\mathfrak{a}$ is a $\mathfrak{t}$-invariant subspace of $\mathfrak{g}$, i.e. a subspace such that $[\mathfrak{t}, \mathfrak{a}] \subseteq \mathfrak{a}$, then let $\Delta(\mathfrak{a}) = \{\alpha \mid \mathfrak{g}_\alpha \subseteq \mathfrak{a}\}$. Choose $\Pi \subseteq \Delta$ a set of simple roots and $\Delta^+ \subseteq \Delta$ a set of positive roots that contains $\Delta(\mathfrak{p}^+)$, and let $\Delta^+(\mathfrak{k}) = \Delta^+ \cap \Delta(\mathfrak{k})$. Let $\mathcal{W}$ be the Weyl group of $(\mathfrak{g}, \mathfrak{t})$ and let $\mathcal{W}(\mathfrak{k})$ be the Weyl group of $(\mathfrak{k}, \mathfrak{t})$.

For $w \in \mathcal{W}$, define a length function $\ell(w)$ to be the smallest $r$ such that $w$ can be written as $w = s_1 s_2 \cdots s_r$ where each $s_i$ is equal to $s_\alpha$ for some $\alpha \in \Pi$.

Now for $w \in \mathcal{W}$, define $\Delta_w = \Delta^+ \cap w\Delta^-$. It is well known that $|\Delta_w| = \ell(w)$. Then, following Kostant ([18],(5.13.1)), define the set

$$^{\mathfrak{k}}\mathcal{W} = \{w \in \mathcal{W} \mid \Delta_w \subseteq \Delta(\mathfrak{p}^+)\}.$$

It is a result due to Kostant ([18], Prop. 5.13) that every $w \in \mathcal{W}$ can be written uniquely as $w = uv$, where $u \in \mathcal{W}(\mathfrak{k})$ and $v \in {}^{\mathfrak{k}}\mathcal{W}$, and thus ${}^{\mathfrak{k}}\mathcal{W}$ is the set of minimal length right coset representatives of $\mathcal{W}(\mathfrak{k})$ in $\mathcal{W}$.

We will view root systems as partially ordered sets using the standard ordering induced from the ordering on $\mathfrak{t}^*$: $\mu \leq \lambda$ iff $\lambda - \mu$ is a linear combination of positive roots with all coefficients being nonnegative integers. There is also a partial order on the Weyl group as follows.

For $w, w'$ in the Weyl group $\mathcal{W}$, write $w' \to w$ if $\ell(w) > \ell(w')$ and $w = w's_\alpha$ for some $\alpha \in \Delta^+$. Now define a partial ordering $\leq$ on $\mathcal{W}$ by $w' < w$ if there exists a sequence $w' = w_0 \to w_1 \to \cdots \to w_m = w$. This partial ordering is called the *Bruhat ordering* on $\mathcal{W}$. Similarly we can define the *weak Bruhat ordering* by requiring $\alpha \in \Pi$ rather than $\alpha \in \Delta^+$. It is a well-known result that if $\mathfrak{q}$ is of Hermitian type, the Bruhat ordering and the weak Bruhat ordering of ${}^{\mathfrak{k}}\mathcal{W}$ coincide. It is also well known that for $w, w' \in \mathcal{W}$, we have $w' \leq w$ in the weak Bruhat ordering (and thus, in our case, in the Bruhat ordering as well) if and only if $\Delta_v \subseteq \Delta_w$.

There is another interpretation of the sets of the form $\Delta_w$ due to Enright–Hunziker–Pruett.

**Definition 2.2.1.** For $\mathcal{P}$ a partially ordered set, a subset $\mathcal{I} \subseteq \mathcal{P}$ is a *lower-order ideal* of $\mathcal{P}$ if for all $x, y \in \mathcal{P}$, $x \in \mathcal{I}$ and $x \geq y$ implies $y \in \mathcal{I}$.

**Lemma 2.2.2** (Enright–Hunziker–Pruett, [5], Lemma 3.7)**.** *Suppose* $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$ *is a parabolic subalgebra of Hermitian type. Let* $w \in {}^{\mathfrak{k}}\mathcal{W}$. *Then* $\Delta_w$ *is a lower-order ideal of* $\Delta(\mathfrak{p}^+)$.

Viewing the sets $\Delta_w$ as lower-order ideals provides a convenient way to visualize them using Hasse diagrams. By a lemma of Jakobsen ([16], Lemma 4.1), if $\mathfrak{q}$ is of Hermitian type, the Hasse diagram of $\Delta(\mathfrak{p}^+)$ is 2-dimensional and it can be drawn on a square lattice. The diagrams are drawn on a square grid rotated 45°, with the simple root in $\Delta(\mathfrak{p}^+)$ at the bottom.

**Example 2.2.3.** Consider the Hermitian symmetric pair $(\mathsf{A}_6, \mathsf{A}_3 \times \mathsf{A}_2)$ which corresponds to $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(4,3)$. Let $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$ be the corresponding parabolic subalgebra of $\mathfrak{g} = \mathfrak{sl}(7, \mathbb{C})$. Then $\mathfrak{p}^+$ is isomorphic to the space of $4 \times 3$ complex matrices which, as in Example 2.1.1, is embedded in $\mathfrak{g}$ as a block in the top right.

Referring to Figure 2.1, the bottom node in the Hasse diagram is $\alpha_4 = \varepsilon_4 - \varepsilon_5$, the unique simple root in $\Delta(\mathfrak{p}^+)$, and the top node is $\sum_{i=1}^{6} \alpha_i = \varepsilon_1 - \varepsilon_7$. For adjacent nodes corresponding to roots $\alpha$ and $\beta$, with the node of $\alpha$ lower in the diagram than that of $\beta$, the edge between the nodes is labeled with the subscript of the simple root $\beta - \alpha$. For example, the leftmost of the nodes adjacent to the bottom node, that of $\alpha_4$, is reached by moving along an edge labelled "3" from that bottom node; thus this node corresponds to $\alpha_4 + \alpha_3$. The missing labels can be quickly filled in by the rule that opposite sides of any square in the diagram have the same label. For example, the other (rightmost) node adjacent to the bottom one corresponds to $\alpha_4 + \alpha_5$.

$$\mathfrak{p}^+ = \left\{ \left( \begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\} \qquad \Delta(\mathfrak{p}^+) =$$

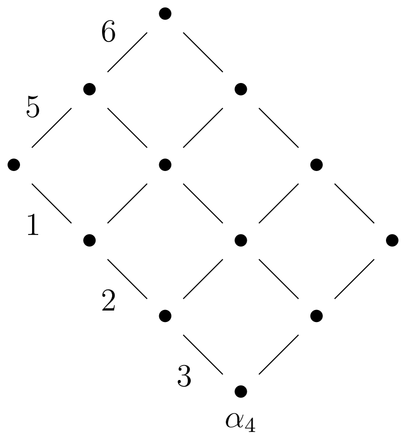



Figure 2.1: The Hasse diagram of $\Delta(\mathfrak{p}^+)$ for $\mathfrak{su}(p, q)$, shown here in the case $p = 4$ and $q = 3$

Now let $w \in {}^{\mathfrak{k}}\mathcal{W}$ and consider the lower-order ideal $\Delta_w \subseteq \Delta(\mathfrak{p}^+)$, which corresponds to some abelian ideal $\mathfrak{a}_w$ of $\mathfrak{b}$ that is contained in $\mathfrak{p}^+$. (Specifically, $\Delta_w = \Delta(\mathfrak{p}^+) - \Delta(\mathfrak{a}_w)$.) The Hasse diagram of $\Delta_w$ can be viewed as a subdiagram of the Hasse diagram of $\Delta(\mathfrak{p}^+)$ such that if a node of the diagram of $\Delta(\mathfrak{p}^+)$ belongs to the diagram of $\Delta_w$, then all nodes adjacent to it below in the diagram of $\Delta(\mathfrak{p}^+)$ also belong to the diagram of $\Delta_w$. Figure 2.2 shows an example abelian ideal $\mathfrak{a}_w$ and the Hasse diagram of its corresponding lower-order ideal $\Delta_w$ as a subdiagram of the one in Figure 2.1. Open circles represent nodes that are absent from the diagram of $\Delta_w$.

Furthermore, we can associate a generalized Young diagram to every lower-order ideal $\Delta_w$. Take the Hasse diagram of $\Delta_w$ and replace each node by a square, such that two squares share a side if and only if the corresponding nodes were connected. Then rotate the resulting diagram 135° clockwise, so that the box corresponding to the simple root, which was at the bottom, is now the top left box. Figure 2.3 shows the resulting Young diagram from the same element $w$ as in Figure 2.2.

The remaining figures in this chapter depict an example of the Hasse diagram and the generalized Young diagram of the longest element in ${}^{\mathfrak{k}}\mathcal{W}$ for each of the

$$\mathfrak{a}_w = \left\{ \left( \begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\} \qquad \Delta_w =$$

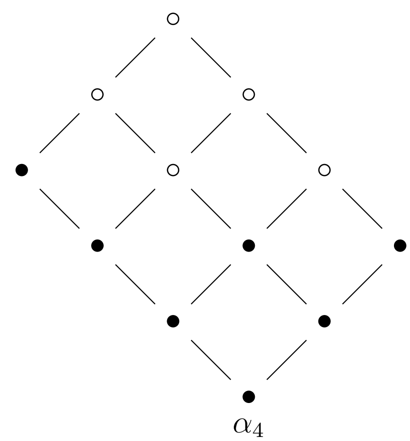



Figure 2.2: The Hasse diagram of a lower-order ideal $\Delta_w \subseteq \Delta(\mathfrak{p}^+)$ for $\mathfrak{su}(p,q)$, shown here in the case $p = 4$ and $q = 3$

$\Delta_w =$

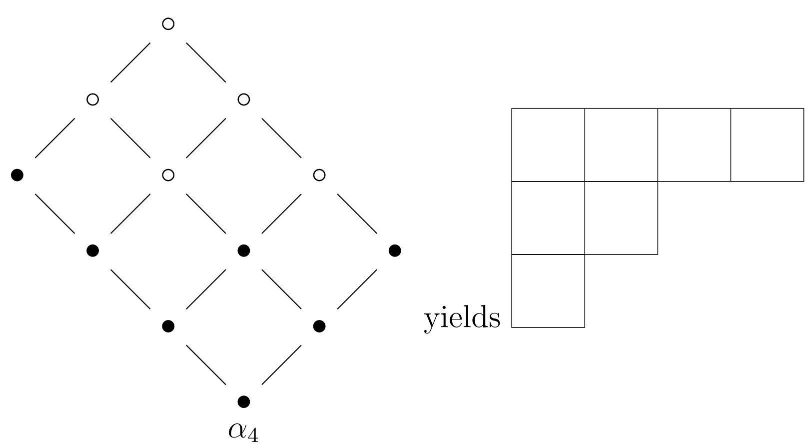



Figure 2.3: The Young diagram associated to the lower-order ideal $\Delta_w$ of Figure 2.2

other two classical cases being considered here. Figure 2.4 gives the diagrams for type $(\mathsf{D}_n, \mathsf{A}_{n-1})$ (corresponding to $\mathfrak{sp}(n, \mathbb{R})$) in the case $n = 5$, and Figure 2.5 gives the diagrams for type $(\mathsf{D}_n, \mathsf{A}_{n-1})$ (corresponding to $\mathfrak{so}^*(2n)$) in the case $n = 6$. The diagrams work by the same rules as above, i.e. opposite sides of a square have the same label, etc.

### *2.3 Blocks in Parabolic Category $\mathcal{O}$*

Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$ be a standard parabolic subalgebra. We will denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. We can define a *parabolic category* $\mathcal{O}$, written $\mathcal{O}^{\mathfrak{q}}$ or $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$, by analogous axioms to those used to define category $\mathcal{O}$.

**Definition 2.3.1.** The category $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ is defined as the full subcategory of the category of $U(\mathfrak{g})$-modules whose objects $M$ satisfy the following conditions:

(1) $M$ is a finitely generated $U(\mathfrak{g})$-module.

(2) $M$ is a semisimple $U(\mathfrak{k})$-module, i.e. $M$ is a direct sum of finite-dimensional simple modules.

(3) $M$ is locally $\mathfrak{p}^+$-finite, i.e. for all $v \in V$, $U(\mathfrak{p}^+)v$ is finite-dimensional.

The key objects in this category are the parabolic Verma modules. The simple modules in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ are parametrized by the set of $\mathfrak{k}$-*dominant integral weights*:

$$\Lambda^+(\mathfrak{k}) = \{\lambda \in \mathfrak{t}^* \mid (\lambda + \rho, \alpha^\vee) \in \mathbb{Z}_{>0}\ \forall \alpha \in \Delta^+(\mathfrak{k})\}.$$

For $\lambda \in \Lambda^+(\mathfrak{k})$, let $F(\mathfrak{k}, \lambda)$ be the simple finite-dimensional $\mathfrak{k}$-module with highest weight $\lambda$, which can also be viewed as a $\mathfrak{q}$-module by letting $\mathfrak{p}^+$ act by 0. Then the *parabolic Verma module* with highest weight $\lambda$ is the induced module $N(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F(\mathfrak{k}, \lambda)$.

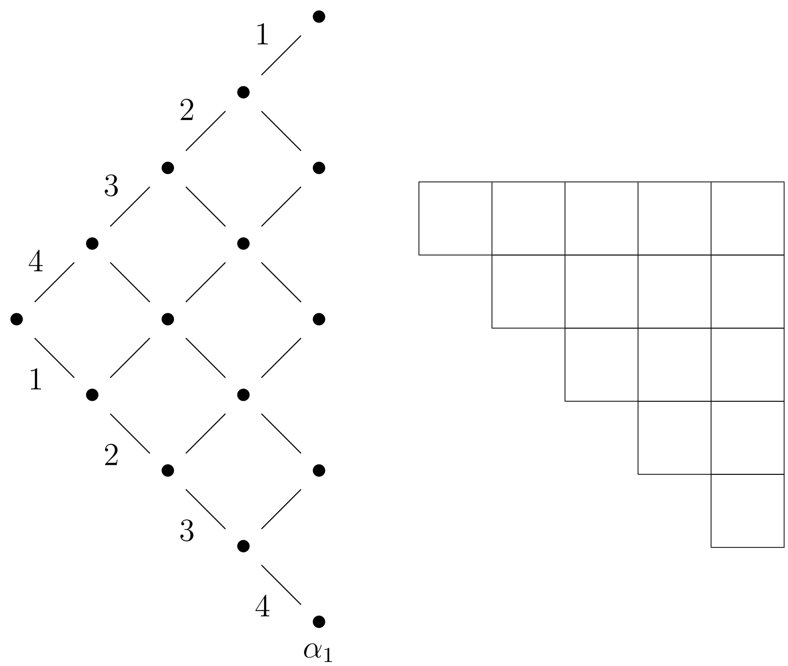



Figure 2.4: Diagrams for $\mathfrak{sp}(n, \mathbb{R})$, shown here in the case $n = 5$

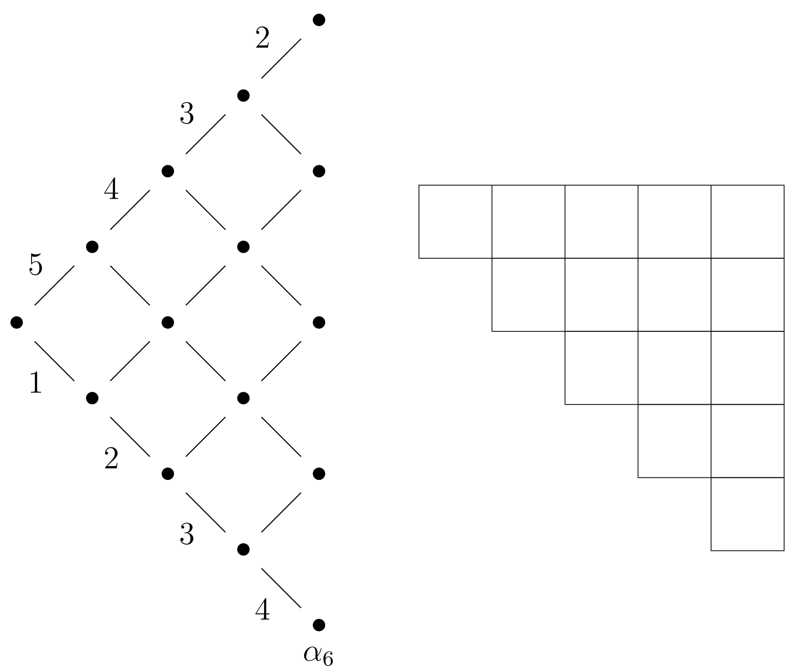



Figure 2.5: Diagrams for $\mathfrak{so}^*(2n)$, shown here in the case $n = 6$

If $N(\lambda)$ is a parabolic Verma module, let $L(\lambda)$ be its unique simple quotient. Note that both $L(\lambda)$ and $N(\lambda)$ are quotients of an ordinary Verma module $M(\lambda)$ with highest weight $\lambda$.

**Definition 2.3.2.** An *infinitesimal character* $\chi_\lambda$ of a module $M(\lambda)$ is a character of $Z(U(\mathfrak{g}))$, the center of $U(\mathfrak{g})$, such that $zv = \chi_\lambda(z)v$ for all $z \in Z(U(\mathfrak{g}))$ and all $v \in M(\lambda)$.

Since the Verma module $M(\lambda)$ admits an infinitesimal character, so do its quotients including $L(\lambda)$. For $\lambda \in \Lambda^+(\mathfrak{k})$, let $\chi_\lambda$ be the infinitesimal character of $L(\lambda)$. Define $\mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda)$ to be the full subcategory of $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ that contains exactly the modules whose composition factors have infinitesimal character $\chi_\lambda$.

For each $\lambda$, the simple modules in the category $\mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda)$ are exactly the modules $L(\mu)$ with $\mu \in \Lambda^+(\mathfrak{k}) \cap \mathcal{W} \cdot \lambda$, where $\mathcal{W} \cdot \lambda$ is the orbit of $\lambda$ under the "dot action," defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$. Thus the number of simple modules in $\mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda)$ is finite.

The categories $\mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda)$ are further broken down into *blocks*. We define blocks in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ as usual, by separating indecomposable modules that are homologically unrelated using the Ext functor. Every block in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ is a subcategory of some category $\mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda)$, and for every $\lambda \in \mathfrak{t}^*$, the category $\mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda)$ is a block in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$, although it may be a zero block containing only the zero module.

# CHAPTER THREE

## Similarity of Blocks

### *3.1 Posets of Regular Blocks*

**Definition 3.1.1.** A weight $\lambda$ is called *regular* if $(\lambda+\rho,\alpha^\vee) \neq 0$ for all $\alpha \in \Delta^+$.

**Definition 3.1.2.** A highest weight $\lambda$ is called *integral* if $(\lambda+\rho,\alpha^\vee)$ is an integer for all $\alpha \in \Delta^+$.

If $\lambda$ is regular and integral, then $\mathcal{O}(\mathfrak{g},\mathfrak{q},\lambda)$ is a block, called a *regular integral block*. By the Jantzen-Zuckerman translation principle, the categories $\mathcal{O}(\mathfrak{g},\mathfrak{q},\lambda)$ are equivalent for all regular integral $\lambda$. We will denote such a category by $\mathcal{O}_{\mathrm{reg}}(\mathfrak{g},\mathfrak{q})$.

Using the standard partial ordering induced from the ordering on $\mathfrak{t}^*$, we can view the set of highest weights of simple modules in a block $\mathcal{B}$ as a poset, which we will call *the poset of* $\mathcal{B}$. If $\mathcal{B}$ is a regular integral block, then the poset of $\mathcal{B}$ is isomorphic to the set ${}^{\mathfrak{k}}\mathcal{W}$ in the Bruhat ordering (cf. [7]). In particular, the poset of $\mathcal{B}$ has a unique maximal element and a unique minimal element. Furthermore, in the case of Hermitian symmetric pairs, the posets are distributive lattices [19].

### *3.2 Posets of Singular Blocks and Enright Reduction*

If $(\lambda+\rho,\alpha^\vee) = 0$ for some $\alpha \in \Delta$, then we will say that $\lambda+\rho$ contains a *singularity*, and we call $\alpha$ a *singular root* with respect to $\lambda$. In the $\mathfrak{su}(p,q)$ case, this occurs when two coordinates $a_i$ and $a_j$ of $\lambda+\rho$ are equal (with $i \le p$ and $j > p$), for then the root $\varepsilon_i - \varepsilon_j$ will make the above dot product equal to 0. In the other two cases, a singularity occurs when two coordinates $a_i$ and $a_j$ of $\lambda+\rho$ are negatives of each other, and the root $\varepsilon_i+\varepsilon_j$ is the culprit.

**Definition 3.2.1.** The process of *Enright reduction* obtains a regular weight (plus

$\rho$) from one that contains singularities by removing all coordinates that are involved in singularities, leaving a lower-dimensional weight plus a lower-dimensional $\rho$.

Thus, if a coordinate is repeated in an $\mathfrak{su}(p,q)$ case, Enright reduction will remove both instances of that coordinate; in the other two cases if two coordinates are negatives of each other, both are removed.

**Proposition 3.2.2** (cf. Enright–Shelton [7]). *The poset of a block obtained from another by applying Enright reduction to each element is isomorphic to the poset of the original block.*

**Example 3.2.3.** Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(4,3)$ and let $\lambda + \rho = (5,4,3,2;2,1,0)$. Enright reduction of $\lambda+\rho$ will result in $\lambda' + \rho' = (5,4,3;1,0)$, corresponding to $\mathfrak{g}'_{\mathbb{R}} = \mathfrak{su}(3,2)$. The isomorphic posets of the two blocks are illustrated in Figure 3.1.

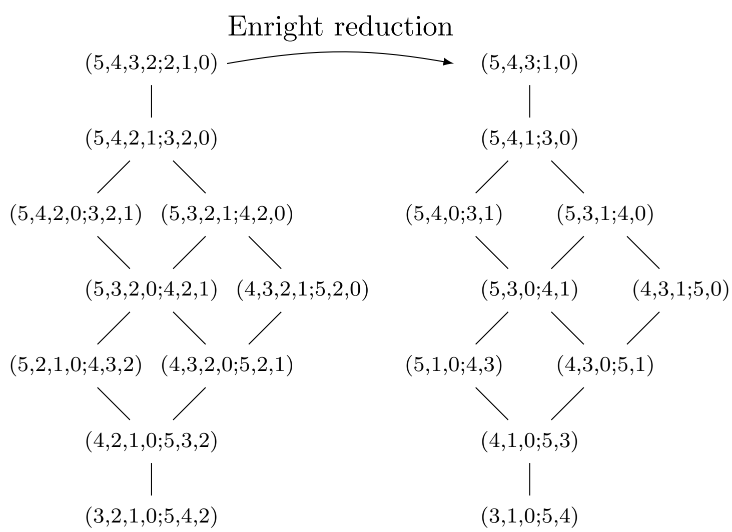



Figure 3.1: Isomorphic Posets

## 3.3 Enright–Shelton Equivalences

Enright and Shelton have shown that each category $\mathcal{O}(\mathfrak{g},\mathfrak{q},\lambda)$ decomposes into at most two blocks, and they also provide simple conditions for when there are exactly two blocks in the decomposition.

**Theorem 3.3.1** (cf. Enright–Shelton [7, 8]). *Suppose* $\mathfrak{q}$ *is of Hermitian type, and let* $\mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda)$ *be the corresponding block. Then:*

*(1) If* $\Delta$ *has one root length, or* $\Delta$ *has two root lengths and all the singular simple roots are short, then there exists an equivalence of categories*

$$\mathcal{E} : \mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda) \to \mathcal{O}_{\mathrm{reg}}(\mathfrak{g}', \mathfrak{q}')$$

*where* $\mathfrak{q}'$ *is a parabolic subalgebra of Hermitian type of a complex simple Lie algebra* $\mathfrak{g}'$ *of rank less than or equal to the rank of* $\mathfrak{g}$.

*(2) If* $\Delta$ *has two root lengths and there is a long singular simple root, then there exists an equivalence of categories*

$$\mathcal{E} : \mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda) \to \mathcal{O}_{\mathrm{reg}}(\mathfrak{g}', \mathfrak{q}') \oplus \mathcal{O}_{\mathrm{reg}}(\mathfrak{g}', \mathfrak{q}')$$

*where* $\mathfrak{q}'$ *is a parabolic subalgebra of Hermitian type of a complex simple Lie algebra* $\mathfrak{g}'$ *of rank less than or equal to the rank of* $\mathfrak{g}$.

### *3.4 Similarity of Blocks*

**Definition 3.4.1.** If $\mathcal{B}$ is a block in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ and $\mathcal{B}'$ is a block in $\mathcal{O}'(\mathfrak{g}', \mathfrak{q}')$, we say that $\mathcal{B}$ is *similar* to $\mathcal{B}'$ if

(1) The posets of $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic via a map that sends each $\lambda$ in the poset of $\mathcal{B}$ to some $\lambda'$ in the poset of $\mathcal{B}'$.

(2) For all $\mu, \nu$ in the poset of $\mathcal{B}$, $\mathrm{Ext}^j_{\mathcal{O}}(N(\nu), L(\mu)) \cong \mathrm{Ext}^j_{\mathcal{O}'}(N(\nu'), L(\mu'))$.

(3) The ratio $\dim F(\mathfrak{k}, \mu)/\dim F(\mathfrak{k}', \mu')$ is constant for all $\mu$ in the poset of $\mathcal{B}$.

We say that $\mathcal{B}$ is *congruent* to $\mathcal{B}'$ if the constant in (3) is equal to 1.

This brings us to the main result.

**Theorem 3.4.2.** *Let $\mathcal{B}$ be any block in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$, where $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ of Hermitian type as above. Then $\mathcal{B}$ is similar to some regular integral block $\mathcal{B}' = \mathcal{O}(\mathfrak{g}', \mathfrak{q}', \lambda')$, where $\mathfrak{q}'$ is a parabolic subalgebra (of Hermitian type) of some complex simple Lie algebra $\mathfrak{g}'$ of generally smaller rank than $\mathfrak{g}$.*

For the classical cases as in example 2.1.1, this theorem has been proven in part by Enright–Shelton [7]; they showed that for any block $\mathcal{B}$ in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$, there is some regular integral block $\mathcal{B}'$ such that (1) and (2) of the definition of similarity are satisfied. In the following sections it will be shown that (3) is satisfied as well, completing the proof (for those classical cases). The regular integral block $\mathcal{B}'$ will be obtained in each case by applying Enright reduction to the highest weight (plus $\rho$) of each simple module in $\mathcal{B}$.

A couple of examples will give some idea of what similar blocks look like.

**Example 3.4.3.** Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(4,3)$ and $\mathfrak{g}'_{\mathbb{R}} = \mathfrak{su}(3,2)$. Using $\lambda + \rho$ as in Example 3.2.3:

$$\lambda + \rho = (5,4,3,2;2,1,0) \text{ and } \lambda' + \rho' = (5,4,3;1,0).$$

In the $\mathfrak{su}(p,q)$ case, for $\mu \in \Lambda^+(\mathfrak{k})$, $L(\mu)$ is in the same block as $L(\lambda)$ if and only if $\mu + \rho$ can be obtained from $\lambda + \rho$ by exchanging some number of coordinates from before the semicolon with an equal number after the semicolon. For this $\lambda$, one such $\mu$ gives

$$\mu + \rho = (5,2,1,0;4,3,2).$$

Applying Enright reduction to the latter yields

$$\mu' + \rho' = (5,1,0;4,3).$$

Using the Weyl dimension formula to compute both $\dim F(\mathfrak{k}, \lambda)/\dim F(\mathfrak{k}', \lambda')$ and $\dim F(\mathfrak{k}, \mu)/\dim F(\mathfrak{k}', \mu')$ yields the value 1 for both ratios; these blocks are in fact congruent.

**Example 3.4.4.** Suppose $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(n, \mathbb{R})$ and let $\lambda \in \Lambda^+(\mathfrak{k})$ be a half-integral weight. So $\lambda + \rho = (a_1, \ldots, a_n)$, where each $a_i$ is a half-integer. Let $(b_1, \ldots, b_m)$ be the sequence obtained from $(a_1, \ldots, a_n)$ by Enright reduction. Then as long as $m \geq 3$, if $\mathfrak{g}'_{\mathbb{R}} = \mathfrak{so}^*(2m)$ and $\lambda' \in \Lambda^+(\mathfrak{k})$ such that $\lambda' + \rho' = (b_1, \ldots, b_m)$, then the block $\mathcal{B}$ in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$ that contains $L(\lambda)$ is similar to the regular integral block $\mathcal{B}' = \mathcal{O}(\mathfrak{g}', \mathfrak{q}', \lambda')$. For a specific example, let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(10, \mathbb{R})$ and let

$$\lambda + \rho = \left( \frac{15}{2}, \frac{13}{2}, \frac{11}{2}, \frac{9}{2}, \frac{5}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{5}{2}, -\frac{9}{2}, -\frac{11}{2} \right).$$

Under Enright reduction, this becomes

$$\lambda' + \rho' = \left( \frac{15}{2}, \frac{13}{2}, \frac{3}{2}, -\frac{1}{2} \right)$$

which corresponds to $\mathfrak{g}'_{\mathbb{R}} = \mathfrak{so}^*(8)$.

In the $\mathfrak{sp}(n, \mathbb{R})$ case, for $\mu \in \Lambda^+(\mathfrak{k})$, $L(\mu)$ is in the same block as $L(\lambda)$ if and only if $\mu + \rho$ is a signed permutation of $\lambda + \rho$ with an even number of sign changes. For this $\lambda$, one such $\mu$ gives

$$\mu + \rho = \left( \frac{13}{2}, \frac{11}{2}, \frac{9}{2}, \frac{5}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{9}{2}, -\frac{11}{2}, -\frac{15}{2} \right).$$

Applying Enright reduction to $\mu + \rho$ yields

$$\mu' + \rho' = \left( \frac{13}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{15}{2} \right).$$

Using the Weyl dimension formula to compute both $\dim F(\mathfrak{k}, \lambda)/ \dim F(\mathfrak{k}', \lambda')$ and $\dim F(\mathfrak{k}, \mu)/ \dim F(\mathfrak{k}', \mu')$ yields the value 117,975 for each ratio.

**Remark 3.4.5.** The notion of similarity of blocks makes sense for arbitrary parabolic subalgebras, not only those of Hermitian type. The general question of which blocks in maximal parabolic subalgebras have a poset isomorphic to the poset of some regular integral block is still open.

## 3.5 *Singular Blocks for* $\mathfrak{su}(p, q)$

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(p, q)$, so $\mathfrak{k} \cong \mathfrak{s}(\mathfrak{gl}(p, \mathbb{C}) \oplus \mathfrak{gl}(q, \mathbb{C}))$ as in Example 2.1.1. Let $\lambda \in \Lambda^+(\mathfrak{k})$, so $\lambda + \rho = (a_1, \ldots, a_p; b_1, \ldots, b_q)$; following Enright–Willenbring [9], define:

$\Theta^s$ the set of all coordinates $c$ for which $c = a_i = b_j$ for some $i, j$

$\Theta_+$ the set of all coordinates $a_i \notin \Theta^s$ for which $a_i > b_j$ for some $b_j \notin \Theta^s$

$\Theta_-$ the set of all coordinates $b_j \notin \Theta^s$ for which $a_i > b_j$ for some $a_i \notin \Theta^s$

$\Psi_+$ the set of all coordinates $a_i$ that are not in $\Theta_+$ or $\Theta^s$

$\Psi_-$ the set of all coordinates $b_j$ that are not in $\Theta_-$ or $\Theta^s$.

**Definition 3.5.1.** We call $\lambda$ *quasi-dominant* if it is the unique maximal element in the poset of the block that $L(\lambda)$ belongs to; i.e. for $\mu \neq \lambda$ in $\mathcal{W} \cdot \lambda \cap \Lambda^+(\mathfrak{k})$, the difference $\lambda - \mu$ is a nonnegative integer linear combination of simple roots.

**Lemma 3.5.2.** *If $\lambda$ is quasi-dominant, then for $\lambda + \rho$ written out as above, each $a_i$ not in $\Theta^s$ is greater than all $b_j$ not in $\Theta^s$.*

*Proof.* If $\lambda$ is higher than $\mu$ in a $\mathfrak{u}(p, q)$ case, then the first nonzero coordinate of $\lambda - \mu$ (or $(\lambda + \rho) - (\mu + \rho)$) is positive and the last nonzero coordinate is negative. Thus, for $\lambda + \rho$ to be higher than everything else in its Weyl group orbit, its largest coordinate must be its first coordinate $a_1$, and its smallest coordinate must be its last coordinate $b_q$. But if $\lambda + \rho$ is being compared to something that has the same first coordinate, then for $\lambda + \rho$ to be higher the second coordinate of the difference must be nonnegative, and similarly the second-to-last coordinate of the difference must be nonpositive. Continuing in this manner, we get that except for the coordinates in $\Theta^s$ which are not permuted, the coordinates of $\lambda + \rho$ must be in descending order. □

Now suppose $\lambda + \rho$ is quasi-dominant. (Thus $\Psi_+$ and $\Psi_-$ are empty.) For each pair of subsets $\Phi = (\Phi_+, \Phi_-)$ with $\Phi_+ \subseteq \Theta_+$ and $\Phi_- \subseteq \Theta_-$ and $|\Phi_+| = |\Phi_-|$, let

$\lambda_\Phi + \rho$ be obtained from $\lambda + \rho$ by switching the elements of $\Phi_+$ with the elements of $\Phi_-$. Let $F(\mathfrak{k}, \lambda_\Phi)$ denote the finite dimensional $\mathfrak{k}$-module with highest weight $\lambda_\Phi$.

**Theorem 3.5.3.** *The ratio*

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}', \lambda'_\Phi)}$$

*is constant for all* $\Phi$.

*Proof.* Let $\lambda_\Phi + \rho = (c_1, \dots, c_p; d_1, \dots, d_q)$. Let $D_m = \prod_{1 \le j \le m-1} j!$. Then the Weyl dimension formula gives $\dim F(\mathfrak{k}, \lambda_\Phi) = \frac{1}{D_p D_q} \prod_{1 \le i < j \le p}(c_i - c_j) \prod_{1 \le i < j \le q}(d_i - d_j)$.

For any pair of subsets $\Phi = (\Phi_+, \Phi_-)$ as above, let $(\Phi_{+,1}, \dots, \Phi_{+,r})$ denote the coordinates of $\Theta_+$ listed in descending order after the elements of $\Phi_+$ have been replaced by the elements of $\Phi_-$, and let $(\Phi_{-,1}, \dots, \Phi_{-,s})$ denote the coordinates of $\Theta_-$ listed in descending order after the elements of $\Phi_-$ have been replaced by the elements of $\Phi_+$.

Now by the Weyl dimension formula, as above, we have

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}, \lambda_\varnothing)} = \frac{\prod_{1 \le i < j \le p}(c_i - c_j) \prod_{1 \le i < j \le q}(d_i - d_j)}{\prod_{1 \le i < j \le p}(a_i - a_j) \prod_{1 \le i < j \le q}(b_i - b_j)}.$$

But since the elements of $\Theta^s$ appear in all four of these products, and each element of $\Theta_+$ and $\Theta_-$ appears in exactly one of the products in the numerator and exactly one of the products in the denominator, every term in the fraction containing some $a \in \Theta^s$ cancels, and we have

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}, \lambda_\varnothing)} = \frac{\prod_{1 \le i < j \le r}(\Phi_{+,i} - \Phi_{+,j}) \prod_{1 \le i < j \le s}(\Phi_{-,i} - \Phi_{-,j})}{\prod_{1 \le i < j \le r}(\varnothing_{+,i} - \varnothing_{+,j}) \prod_{1 \le i < j \le s}(\varnothing_{-,i} - \varnothing_{-,j})}.$$

Let $\lambda' + \rho'$ be the Euclidean coordinate expression of the representation obtained from $\lambda + \rho$ by Enright reduction. So $\lambda' + \rho'$ contains only the coordinates in $\Theta_+$ and $\Theta_-$. As above, let $\lambda'_\Phi + \rho'$ be obtained from $\lambda' + \rho'$ by switching the elements of $\Phi_+$ with the elements of $\Phi_-$. Let $F(\mathfrak{k}', \lambda'_\Phi)$ denote the finite-dimensional $\mathfrak{k}'$-module with highest weight $\lambda'_\Phi$. Then, since in the above ratio of dimensions all terms that include some $a \in \Theta^s$ cancel, and $\lambda' + \rho'$ is just $\lambda + \rho$ with the elements of $\Theta^s$ removed, we immediately have

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}, \lambda_\varnothing)} = \frac{\dim F(\mathfrak{k}', \lambda'_\Phi)}{\dim F(\mathfrak{k}', \lambda'_\varnothing)}.$$

Thus, the ratio

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}', \lambda'_\Phi)}$$

is constant for all $\Phi$. □

### *3.6 Singular Blocks for $\mathfrak{sp}(n, \mathbb{R})$*

Let $\mathfrak{g}_\mathbb{R} = \mathfrak{sp}(n, \mathbb{R})$, so $\mathfrak{k} \cong \mathfrak{gl}(n, \mathbb{C})$ as in example 2.1.1. Let $\lambda \in \Lambda^+(\mathfrak{k})$ where $\lambda$ is either integral or half-integral, so $\lambda + \rho = (a_1, \ldots, a_n)$; following Enright–Willenbring [9], define:

$\Theta^s$ the set of all positive coordinates $a$ for which $-a$ is also a coordinate

$\Theta$ the set of all positive coordinates $c$ not in $\Theta^s$, along with all negative coordinates $d$ not in $\Theta^s$ such that $|d| < c$ for some positive $c$ not in $\Theta^s$

$\Psi$ the set of all nonzero coordinates of $\lambda + \rho$ that are not in $\Theta$, $\Theta^s$, or $-\Theta^s$.

**Definition 3.6.1.** We call $\lambda$ *quasi-dominant* if it the unique maximal element in the poset of the block that $L(\lambda)$ belongs to; i.e. for $\mu \neq \lambda$ in $\mathcal{W} \cdot \lambda \cap \Lambda^+(\mathfrak{k})$ such that the number of positive coordinates in $\mu + \rho$ has the same parity as the number of positive coordinates in $\lambda + \rho$, the difference $\lambda - \mu$ is a nonnegative integer linear combination of simple roots.

**Lemma 3.6.2.** *If $\lambda$ is quasi-dominant, then $\lambda + \rho$ contains at most one negative coordinate, and if there is a negative coordinate, that coordinate has the smallest nonzero absolute value of all the coordinates.*

*Proof.* Suppose we have $(a_1, \ldots, a_n)$ with $a_{n-1}$ and $a_n$ negative. Let $(b_1, \ldots, b_n)$ be the result of negating $a_{n-1}$ and $a_n$ and writing the resulting coordinates in descending order. Let $c_k = b_k - a_k$ for $1 \leq k \leq n$.

For $k < n-1$, we have three possible cases:

(1) if $-a_n \leq a_k$, then $c_k = 0$;

(2) if $-a_{n-1} < a_k < -a_n$, then $a_k = b_{k+1}$ so $c_k > 0$;

(3) if $a_k \leq -a_{n-1}$, then $a_k = b_{k+2}$ so $c_k > 0$.

If $k = n-1$ or $k = n$, then either:

(4) $b_k = a_j$ for some $j < n-1$ (so $c_k > 0$) or

(5) $b_k$ is $-a_{n-1}$ or $-a_n$ so $c_k$ is a positive number minus a negative number and thus is positive.

In all cases, $c_k \geq 0$. Also note that $c_k$ is an integer for every $k$ since the $a$ and $b$ coordinates are either all integers or all odd half-integers.

Now if $\{\alpha_1, \ldots, \alpha_n\}$ is the standard set of simple roots, then $(c_1, \ldots, c_n)$ can be written as

$$c_1\alpha_1 + (c_1 + c_2)\alpha_2 + (c_1 + c_2 + c_3)\alpha_3 + \cdots + (c_1 + \cdots + c_{n-1})\alpha_{n-1} + \frac{c_1 + \cdots + c_n}{2}\alpha_n.$$

Note that all these coefficients are integers: $c_1 + \cdots + c_n$ is equal to $-2a_{n-1} - 2a_n$ so it is even. Also all the coefficients are nonnegative since $c_i \geq 0$ for all $i$. Therefore $(a_1, \ldots, a_n)$ is lower than $(b_1, \ldots, b_n)$ and thus not quasi-dominant.

It remains to show that if $\lambda + \rho$ is quasi-dominant and has a negative coordinate, that coordinate has the smallest nonzero absolute value of all the coordinates. Suppose we have $(a_1, \ldots, a_n)$ with $a_n$ the only negative coordinate, and $0 < |a_j| < |a_n|$ for some $j$. Let $a_m$ be the coordinate of smallest nonzero absolute value. Let $(b_1, \ldots, b_n)$ be the result of negating $a_m$ and $a_n$ and writing the resulting coordinates in descending order. There is exactly one negative coordinate, namely $b_n = -a_m$, which has the smallest nonzero absolute value of all the coordinates.

Note that $a_m = a_{n-2}$ if there is a zero coordinate present, and otherwise $a_m = a_{n-1}$.

Let $c_k = b_k - a_k$ for $1 \le k \le n$.

*If there is no zero coordinate present:*

For $k < n - 1$, there are two possible cases:

(1) if $-a_n \le a_k$, then $c_k = 0$;

(2) if $a_k < -a_n$, then $a_k = b_{k+1}$ so $c_k > 0$.

For $k = n - 1$, we have

(3) $c_k \ge 0$ since $b_k$ and $a_k$ are nonnegative and $a_{n-1}$ had the smallest absolute value of all the coordinates.

For $k = n$,

(4) $c_k$ is a negative number with the smallest absolute value minus a negative number with a larger absolute value, so it is positive.

*If there is a zero coordinate present:*

For $k < n - 2$, there are two possible cases:

(5) if $-a_n \le a_k$, then $c_k = 0$;

(6) if $a_k < -a_n$, then $a_k = b_{k+1}$ so $c_k > 0$.

For $k = n - 2$, we have

(7) $c_k \ge 0$ since $b_k$ and $a_k$ are positive and $a_{n-2}$ had the smallest nonzero absolute value of all the coordinates.

For $k = n - 1$, we have

(8) $b_k = a_k = 0$ so $c_k = 0$.

For $k = n$,

(9) $c_k$ is a negative number with the smallest absolute value minus a negative number with a larger absolute value, so it is positive.

In all cases, $c_k \geq 0$.

Since the $c_k$ are all nonnegative, the same argument as above works to show that $(a_1, \dots, a_n)$ is lower than $(b_1, \dots, b_n)$ and thus not quasi-dominant. So if $\lambda + \rho$ is quasi-dominant and contains a negative coordinate, it is the coordinate with the smallest absolute value. □

Now suppose $\lambda + \rho = (a_1, \dots, a_n)$ is quasi-dominant. (Thus $\Psi$ is empty.) For each subset $\Phi \subseteq \Theta$ of even cardinality, let $\lambda_\Phi + \rho$ be obtained from $\lambda + \rho$ by negating each element of $\Phi$. Let $F(\mathfrak{k}, \lambda_\Phi)$ denote the finite dimensional $\mathfrak{k}$-module with highest weight $\lambda_\Phi$.

**Theorem 3.6.3.** *The ratio*

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}', \lambda'_\Phi)}$$

*is constant for all* $\Phi$.

*Proof.* Let $\lambda_\Phi + \rho = (c_1, \dots, c_n)$. Let $D_m = \prod_{1 \leq j \leq m-1} j!$. Then the Weyl dimension formula gives

$$\dim F(\mathfrak{k}, \lambda_\Phi) = \frac{1}{D_n} \prod_{1 \leq i < j \leq n} (c_i - c_j).$$

For any subset $\Phi$ of $\Theta$, let $(\Phi_1, \dots, \Phi_r)$ denote the coordinates of $\Theta$ listed in descending order after the elements of $\Phi$ have been negated.

Now by the Weyl dimension formula, as above, we have

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}, \lambda_\varnothing)} = \frac{\prod_{1 \leq i < j \leq n} (c_i - c_j)}{\prod_{1 \leq i < j \leq n} (a_i - a_j)}.$$

But for all $a, b \in \Theta^s \cup -\Theta^s$ with $a \geq b$, the term $(a - b)$ will immediately cancel from this fraction, and if $c \in \Theta - \Phi$ and $a \in \Theta^s$, the term $(c - a)$ will also immediately cancel. Furthermore, if $c \in \Phi$ and $a \in \Theta^s$, the terms $(c - a)$ and $(c + a)$ in the denominator will cancel with the terms $(a-c)$ and $(-a-c)$ in the numerator, leaving two negative signs which cancel each other out. Thus every term in the fraction containing some $a \in \Theta^s \cup -\Theta^s$ cancels, and we have

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}, \lambda_\varnothing)} = \frac{\prod_{1 \le i < j \le r}(\Phi_i - \Phi_j)}{\prod_{1 \le i < j \le r}(\varnothing_i - \varnothing_j)}.$$

Let $\lambda' + \rho'$ be the Euclidean coordinate expression of the representation obtained from $\lambda + \rho$ by Enright reduction. So $\lambda' + \rho'$ contains only the coordinates in $\Theta$. As above, let $\lambda'_\Phi + \rho'$ be obtained from $\lambda' + \rho'$ by negating each element of $\Phi$. Let $F(\mathfrak{k}', \lambda'_\Phi)$ denote the finite-dimensional $\mathfrak{k}'$-module with highest weight $\lambda'_\Phi$. Then, since in the above ratio of dimensions all terms including some $a \in \Theta^s \cup -\Theta^s$ cancel, and $\lambda' + \rho'$ is just $\lambda + \rho$ with the elements of $\Theta^s \cup -\Theta^s$ removed, we immediately have

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}, \lambda_\varnothing)} = \frac{\dim F(\mathfrak{k}', \lambda'_\Phi)}{\dim F(\mathfrak{k}', \lambda'_\varnothing)}.$$

Thus, the ratio

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}', \lambda'_\Phi)}$$

is constant for all $\Phi$. □

### *3.7 Singular Blocks for* $\mathfrak{so}^*(2n)$

Let $\mathfrak{g}_\mathbb{R} = \mathfrak{so}^*(2n)$, so $\mathfrak{k} \cong \mathfrak{gl}(n, \mathbb{C})$ as in example 2.1.1. Let $\lambda \in \Lambda^+(\mathfrak{k})$, so $\lambda + \rho = (a_1, \ldots, a_n)$; following Enright–Willenbring [9] with a slight modification, define:

$\Theta^s$ the set of all positive coordinates $a$ for which $-a$ is also a coordinate

$\Theta$ the set of all nonnegative coordinates $c$ not in $\Theta^s$ along with one negative coordinate of smallest absolute value if the number of negative coordinates is odd

$\Psi$ the set of all coordinates of $\lambda + \rho$ that are not in $\Theta$, $\Theta^s$, or $-\Theta^s$.

**Definition 3.7.1.** We call $\lambda$ *quasi-dominant* if it is the unique maximal element in the poset of the block that $L(\lambda)$ belongs to; i.e. for $\mu \neq \lambda$ in $\mathcal{W} \cdot \lambda \cap \Lambda^+(\mathfrak{k})$, the difference $\lambda - \mu$ is a nonnegative integer linear combination of simple roots.

**Lemma 3.7.2.** *If $\lambda$ is quasi-dominant, then $\lambda + \rho$ contains at most one negative coordinate, and if there is a negative coordinate, that coordinate has the smallest absolute value of all the coordinates. (In particular, there is no zero coordinate.)*

*Proof.* Suppose we have $(a_1, \ldots, a_n)$ with $a_{n-1}$ and $a_n$ negative. Let $(b_1, \ldots, b_n)$ be the result of negating $a_{n-1}$ and $a_n$ and writing the resulting coordinates in descending order. Let $c_k = b_k - a_k$ for all $1 \leq k \leq n$.

For $k < n - 1$, we have three possible cases:

(1) if $-a_n \leq a_k$, then $c_k = 0$;

(2) if $-a_{n-1} < a_k < -a_n$, then $a_k = b_{k+1}$ so $c_k > 0$;

(3) if $a_k \leq -a_{n-1}$, then $a_k = b_{k+2}$ so $c_k > 0$.

If $k = n - 1$ or $k = n$, then either:

(4) $b_k = a_j$ for some $j < n - 1$ (so $c_k > 0$) or

(5) $b_k$ is $-a_{n-1}$ or $-a_n$ so $c_k$ is a positive number minus a negative number and thus is positive.

In all cases, $c_k \geq 0$. Also note that $c_k$ is an integer for every $k$ since the $a$ and $b$ coordinates are all integers.

Now if $\{\alpha_1, \ldots, \alpha_n\}$ is the standard set of simple roots for $\mathfrak{so}^*(2n)$, then $(c_1, \ldots, c_n)$ can be written as

$$c_1\alpha_1 + (c_1 + c_2)\alpha_2 + (c_1 + c_2 + c_3)\alpha_3 + \cdots + (c_1 + \cdots + c_{n-2})\alpha_{n-2} + \frac{c_1 + \cdots + c_{n-1} - c_n}{2}\alpha_{n-1} + \frac{c_1 + \cdots + c_n}{2}\alpha_n.$$

Note that all these coefficients are integers: $c_1 + \cdots + c_n$ is equal to $-2a_{n-1} - 2a_n$ so it is even, and $c_1 + \cdots + c_{n-1} - c_n$ is also even since it would equal $-2a_{n-1} - 2a_n - 2c_n$. Since $c_i$ is nonnegative for all $i$, if we show that the coefficient on $\alpha_{n-1}$ is nonnegative (i.e., $c_1 + \cdots + c_{n-1} \geq c_n$), then $(a_1, \ldots, a_n)$ will be lower than $(b_1, \ldots, b_n)$ and thus not quasi-dominant.

Suppose we have

$$(b_1 - a_1) + \cdots + (b_{n-1} - a_{n-1}) - (b_n - a_n) < 0.$$

Since $(b_1 - a_1) + \cdots + (b_n - a_n) = -2a_{n-1} - 2a_n$, subtracting yields

$$(b_n - a_n) + (b_n - a_n) > -2a_{n-1} - 2a_n,$$

or $b_n > -a_{n-1}$. This is a contradiction since $-a_{n-1} = b_j$ for some $j$ and $b_n$ is the smallest of the $b$ coordinates. Thus, if $\lambda + \rho$ is quasi-dominant it can have no more than one negative coordinate.

It remains to show that if $\lambda+\rho$ is quasi-dominant and has a negative coordinate, that coordinate has the smallest absolute value of all the coordinates. Suppose we have $(a_1, \ldots, a_n)$ with $a_n$ the only negative coordinate, and $|a_j| < |a_n|$ for some $j$. Since the coordinates are in descending order, $|a_{n-1}| < |a_n|$. Let $(b_1, \ldots, b_n)$ be the result of negating $a_{n-1}$ and $a_n$ and writing the resulting coordinates in descending order. There is exactly one nonpositive coordinate, namely $b_n = -a_{n-1}$, which has the smallest absolute value of all the coordinates. Let $c_k = b_k - a_k$ for all $1 \leq k \leq n$. For $k < n - 1$, there are two possible cases:

(1) if $-a_n \leq a_k$, then $c_k = 0$;

(2) if $a_k < -a_n$, then $a_k = b_{k+1}$ so $c_k > 0$.

For $k = n - 1$, we have

(3) $c_k \geq 0$ since $b_k$ and $a_k$ are nonnegative and $a_{n-1}$ had the smallest absolute value of all the coordinates.

For $k = n$,

(4) $c_k$ is a nonpositive number with the smallest absolute value minus a negative number with a larger absolute value, so it is positive.

In all cases, $c_k \geq 0$.

Since the $c_k$ are all nonnegative, the same argument as above works to show that $c_1 + \cdots + c_{n-1} \geq c_n$. Thus $(a_1, \ldots, a_n)$ is lower than $(b_1, \ldots, b_n)$ and thus not quasi-dominant. So if $\lambda + \rho$ is quasi-dominant and contains a negative coordinate, it is the coordinate with the smallest absolute value. □

Now suppose $\lambda + \rho = (a_1, \ldots, a_n)$ is quasi-dominant. (Thus $\Psi$ is empty.) For each subset $\Phi \subseteq \Theta$ of even cardinality, let $\lambda_\Phi + \rho$ be obtained from $\lambda + \rho$ by negating each element of $\Phi$. Let $F(\mathfrak{k}, \lambda_\Phi)$ denote the finite dimensional $\mathfrak{k}$-module with highest weight $\lambda_\Phi$.

**Theorem 3.7.3.** *The ratio*

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}', \lambda'_\Phi)}$$

*is constant for all* $\Phi$.

*Proof.* Let $\lambda_\Phi + \rho = (c_1, \ldots, c_n)$. Let $D_m = \prod_{1 \leq j \leq m-1} j!$. Then the Weyl dimension formula gives

$$\dim F(\mathfrak{k}, \lambda_\Phi) = \frac{1}{D_n} \prod_{1 \leq i < j \leq n} (c_i - c_j).$$

For any subset $\Phi$ of $\Theta$, let $(\Phi_1, \ldots, \Phi_r)$ denote the coordinates of $\Theta$ listed in descending order after the elements of $\Phi$ have been negated.

Now by the Weyl dimension formula, as above, we have

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}, \lambda_\varnothing)} = \frac{\prod_{1 \leq i < j \leq n} (c_i - c_j)}{\prod_{1 \leq i < j \leq n} (a_i - a_j)}.$$

But for all $a, b \in \Theta^s \cup -\Theta^s$ with $a \geq b$, the term $(a - b)$ will immediately cancel from this fraction, and if $c \in \Theta - \Phi$ and $a \in \Theta^s$, the term $(c - a)$ will also immediately cancel. Furthermore, if $c \in \Phi$ and $a \in \Theta^s$, the terms $(c - a)$ and $(c + a)$ in the denominator will cancel with the terms $(a - c)$ and $(-a - c)$ in the numerator, leaving two negative signs which cancel each other out. Thus every term in the fraction containing some $a \in \Theta^s \cup -\Theta^s$ cancels, and we have

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}, \lambda_\varnothing)} = \frac{\prod_{1 \leq i < j \leq r} (\Phi_i - \Phi_j)}{\prod_{1 \leq i < j \leq r} (\varnothing_i - \varnothing_j)}.$$

Let $\lambda' + \rho'$ be the Euclidean coordinate expression of the representation obtained from $\lambda + \rho$ by Enright reduction. So $\lambda' + \rho'$ contains only the coordinates in $\Theta$. As above, let $\lambda'_\Phi + \rho'$ be obtained from $\lambda' + \rho'$ by negating each element of $\Phi$. Let $F(\mathfrak{k}', \lambda'_\Phi)$ denote the finite-dimensional $\mathfrak{k}'$-module with highest weight $\lambda'_\Phi$. Then, since in the above ratio of dimensions all terms including some $a \in \Theta^s \cup -\Theta^s$ cancel, and $\lambda' + \rho'$ is just $\lambda + \rho$ with the elements of $\Theta^s \cup -\Theta^s$ removed, we immediately have

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}, \lambda_\varnothing)} = \frac{\dim F(\mathfrak{k}', \lambda'_\Phi)}{\dim F(\mathfrak{k}', \lambda'_\varnothing)}.$$

Thus, the ratio

$$\frac{\dim F(\mathfrak{k}, \lambda_\Phi)}{\dim F(\mathfrak{k}', \lambda'_\Phi)}$$

is constant for all $\Phi$. $\square$

## CHAPTER FOUR

## A Wonderful Correspondence for Modules of Covariants

### *4.1 Weyl's Fundamental Theorems of Classical Invariant Theory*

There is a natural way to identify invariant rings and modules of covariants with unitarizable highest weight modules using Howe duality.

**Definition 4.1.1.** Let $H$ be a complex reductive group and $W$ a finite-dimensional representation of $H$. The *coordinate ring* $\mathbb{C}[W]$ is the ring of polynomial functions from $W$ to $\mathbb{C}$.

**Definition 4.1.2.** Let $H$ and $W$ be as above. The *ring of invariants* $\mathbb{C}[W]^H$ is defined as

$$\mathbb{C}[W]^H = \{f \in \mathbb{C}[W] \mid f(hw) = f(w) \quad \forall h \in H, w \in W\}$$

**Definition 4.1.3.** Let $W$ be a representation of $H$ as above and let $U$ be another representation of $H$. The *module of covariants of* $W$ *of type* $U$ is defined as

$$\mathrm{Cov}_H(W, U) = \{\phi : W \to U \mid \phi(hw) = h\phi(w) \quad \forall h \in H, w \in W\}$$

where the $\phi$ are polynomials.

Note that if $U = \mathbb{C}$ (the trivial representation) then $\mathrm{Cov}_H(W, U) = \mathbb{C}[W]^H$. In general, $\mathrm{Cov}_H(W, U)$ is a module over $\mathbb{C}[W]^H$ via multiplication of functions (i.e., $(f\phi)(w) = f(w)\phi(w)$). In the case where $H \subseteq GL(V)$ is one of the classical groups, this leads us to Hermann Weyl's Fundamental Theorems of classical invariant theory [21]. Consider the following table:

Table 4.1: Setup for Weyl's Fundamental Theorems

| $H$ | $W$ | $\mathfrak{g}_{\mathbb{R}}$ | $\mathfrak{p}^+$ |
|---|---|---|---|
| $GL(V)$ | $(V^*)^p \oplus V^q$ | $\mathfrak{su}(p,q)$ | $M(p \times q, \mathbb{C})$ |
| $O(V)$ | $V^n$ | $\mathfrak{sp}(n,\mathbb{R})$ | $\mathrm{Sym}(n,\mathbb{C})$ |
| $Sp(V)$ | $V^n$ | $\mathfrak{so}^*(2n)$ | $\mathrm{Alt}(n,\mathbb{C})$ |

In each case of the table, define an $H$-invariant polynomial map $\pi : W \to \mathfrak{p}^+$ of degree two such that the comorphism $\pi^* : \mathbb{C}[\mathfrak{p}^+] \to \mathbb{C}[W]^H$ sends the matrix coefficient $z_{ij}$ to the invariant $f_{ij}$, where $f_{ij}$ is given by:

$$f_{ij}(\lambda_1, \ldots, \lambda_p, v_1, \ldots, v_q) = \lambda_i(v_j) \text{ if } H = GL(V);$$

$$f_{ij}(v_1, \ldots, v_n) = b(v_i, v_j) \text{ if } H = O(V, b);$$

$$f_{ij}(v_1, \ldots, v_n) = \omega(v_i, v_j) \text{ if } H = Sp(V, \omega).$$

Then Weyl's First Fundamental Theorem for $H$ says that $\mathbb{C}[W]^H = \mathbb{C}[f_{ij}]$, or equivalently that the map $\pi^*$ above is surjective. The image of the map $\pi$ is either all of $\mathfrak{p}^+$ or the determinantal variety that consists of all matrices in $\mathfrak{p}^+$ of rank less than or equal to $\dim V$. Weyl's Second Fundamental Theorem says that $\ker \pi^*$ is generated by certain determinants (minors), but it will not play a role here.

## *4.2 Howe Duality and Modules of Covariants*

**Definition 4.2.1.** Let $\mathcal{D}(W)$ be the Weyl algebra of polynomial differential operators on $W$, and let $\mathcal{D}(W)^H$ be the subalgebra consisting of all operators in $\mathcal{D}(W)$ that are $H$-invariant.

Following Howe [12], there exists an injective Lie algebra homomorphism

$$\phi : \mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+ \to \mathcal{D}(W)^H$$

such that $\phi(\mathfrak{g})$ generates $\mathcal{D}(W)^H$. Furthermore, for every choice of a nondegenerate invariant bilinear form $\kappa$ on $\mathfrak{g}$, we have an isomorphism $\mathfrak{p}^- \cong (\mathfrak{p}^+)^*$ given by sending $x \in \mathfrak{p}^-$ to the function $\kappa(x, \_)$, and $\phi$ can be chosen such that for $x \in \mathfrak{p}^-$, $\phi(x) = \pi^*(\kappa(x, \_))$.

**Example 4.2.2.** Let $H = O(V)$ and identify $V$ with $\mathbb{C}^k$. Let $b$ be the standard dot product. Then the image of $\mathfrak{g}$ in $\mathcal{D}(W)^H$ is spanned by the $H$-invariant differential operators

$$f_{ij} = \sum_{\nu=1}^{k} x_{i\nu} x_{j\nu}, \quad E_{ij} = -\sum_{\nu=1}^{k} x_{i\nu} \frac{\partial}{\partial x_{j\nu}} - \frac{k}{2}\delta_{ij}, \quad \Delta_{ij} = -\frac{1}{4}\sum_{\nu=1}^{n} \frac{\partial^2}{\partial x_{i\nu} \partial x_{j\nu}}.$$

(Note that $f_{ij}$ simply gives the result of the dot product, $v_i \cdot v_j$.) The operators for the other groups are similar.

Now let $\mathfrak{g}$ act on the coordinate ring $\mathbb{C}[W]$ via the injective map $\phi$ above. For every irreducible $H$-representation $U$, we also have $\mathfrak{g}$ acting on $\mathrm{Cov}_H(W, U) = \mathrm{Hom}_H(U^*, \mathbb{C}[W])$. Explicitly, for $\phi \in \mathrm{Hom}_H(U^*, \mathbb{C}[W])$ and $x \in \mathfrak{g}$, the action is given by $(x\phi)(u^*) = x\phi(u^*)$. Thus, $\mathrm{Cov}_H(W, U)$ is a $\mathfrak{g}$-module and is in Category $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$, where $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$ as before.

Let $\Sigma = \{\sigma \in \widehat{H} \mid \mathrm{Cov}_H(W, U_\sigma) \neq 0\}$, where $\widehat{H}$ is the set of isomorphism classes of simple $H$-modules (here $H$ is viewed as an algebraic group), and $U_\sigma$ is a simple $H$-module in the class $\sigma$. Then by the work of Howe [12] and Kashiwara–Vergne [17], we have the following theorem:

**Theorem 4.2.3** (Howe duality, cf. [12, 17])**.** *As an $H \times \mathfrak{g}$-module,*

$$\mathbb{C}[W] = \bigoplus_{\sigma \in \Sigma} U_\sigma \otimes \mathrm{Cov}_H(W, U_\sigma).$$

*Moreover, for each $\sigma \in \Sigma$ we have that*

$$\mathrm{Cov}_H(W, U_\sigma) \cong L(\lambda)$$

*for some $\lambda \in \Lambda^+(\mathfrak{k})$, and furthermore the map that sends each $\sigma$ to its corresponding $\lambda$ is injective. The modules $L(\lambda)$ that arise in this way are unitarizable (i.e., there exists a positive definite $\mathfrak{g}_{\mathbb{R}}$-invariant Hermitian inner product on $L(\lambda)$).*

**Example 4.2.4.** Let $H = GL(V)$, $k = \dim V$, $W = \overbrace{V^* \oplus \cdots \oplus V^*}^{p \text{ copies}} \oplus \overbrace{V \oplus \cdots \oplus V}^{q \text{ copies}}$, where $p, q \geq 1$. Identify $\widehat{H}$ with the set of weakly decreasing sequences of integers of length $k$. Then an element $\sigma \in \widehat{H}$ can be written as

$$\sigma = (n_1, n_2, \ldots, 0, 0, \ldots, -m_2, -m_1)$$

(with $k$ entries) where $n_1 \geq n_2 \geq \cdots \geq 0$ and $m_1 \geq m_2 \geq \cdots \geq 0$. Define a function $l$ on infinite sequences that are eventually zero by letting $l(x_1, x_2, \ldots)$ equal the largest $i$ such that $x_i \neq 0$. Now define two partitions $\sigma_+ = (n_1, n_2, \ldots)$ and $\sigma_- = (m_1, m_2, \ldots)$. Then $\sigma \in \Sigma$ if and only if $l(\sigma_+) \leq q$ and $l(\sigma_-) \leq p$ (see Kashiwara–Vergne [17]). The injective map above that sends each $\sigma \in \Sigma$ to some $\lambda \in \Lambda^+(\mathfrak{k})$ is given by

$$\lambda = (-k, -k, \ldots, -k - m_2, -k - m_1; n_1, n_2, \ldots, 0, 0)$$

It is a result due to Enright and Parthasarathy [6] that every simple unitary highest weight Harish-Chandra module of $G_{\mathbb{R}} = SU(p, q)$ is isomorphic to $L(\lambda)$ for some $\lambda$ of this form.

Any module of covariants $\mathrm{Cov}_H(W, U_\sigma)$ can be viewed as a graded $\mathbb{C}[\mathfrak{p}^+]$-module via the surjective homomorphism $\pi^*$ from the previous section. Note that $\mathbb{C}[\mathfrak{p}^+]$ can be identified with the symmetric algebra $S(\mathfrak{p}^-)$ via a nondegenerate invariant bilinear form as above. But since $\mathfrak{p}^-$ is abelian, $S(\mathfrak{p}^-)$ can be identified with the universal enveloping algebra $U(\mathfrak{p}^-)$.

Furthermore, by the PBW theorem, for each $\lambda \in \Lambda^+(\mathfrak{k})$ we have

$$N(\lambda) = S(\mathfrak{p}^-) \otimes F(\mathfrak{k}, \lambda)$$

as a left $S(\mathfrak{p}^-)$-module. This means that $N(\lambda)$ is a free $S(\mathfrak{p}^-)$-module of rank $\dim F(\mathfrak{k}, \lambda)$. Also note that $\mathfrak{g}$ is graded via the Cartan decomposition as in section 2.1. This allows us to compute minimal graded free resolutions (or syzygies) of these modules of covariants via BGG resolutions in terms of parabolic Verma modules:

**Theorem 4.2.5** (Enright–Hunziker, [4])**.** *Let $\lambda \in \Lambda^+(\mathfrak{k})$ such that $L(\lambda)$ is a unitarizable highest weight module in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$. Then $L(\lambda)$ has a BGG resolution in terms of parabolic Verma modules in $\mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda)$. This resolution is a minimal graded free resolution of $L(\lambda)$ as an $S(\mathfrak{p}^-)$-module.*

## *4.3 Cohen–Macaulay Modules of Covariants*

**Definition 4.3.1.** Let $R$ be a finitely generated graded $\mathbb{C}$-algebra, and let $M$ be a finitely generated graded $R$-module. The module $M$ is called *Cohen–Macaulay* if the depth of $M$ is equal to the dimension of $M$.

(Here, and throughout this chapter, the unqualified term *dimension* and the operator "dim" refer to Krull dimension.)

Moreover, if $R$ is a quotient of a graded polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$, then $M$ can be viewed as a graded $S$-module. In this case, $M$ is Cohen–Macaulay as an $R$-module if and only if it is Cohen–Macaulay as an $S$-module.

One may compute the depth of $M$ using the graded version of the Auslander–Buchsbaum Theorem, which says that the depth of $M$ is equal to $\dim S - \mathrm{hd}_S(M)$, where $\mathrm{hd}_S(M)$ is the homological dimension of $M$ as an $S$-module, which is equal to the length of a minimal graded free resolution of $M$ as an $S$-module.

Now suppose $L(\lambda)$ is a unitary highest weight module in $\mathcal{O}(\mathfrak{g}, \mathfrak{q})$. Then by Theorem 4.2.5, $L(\lambda)$ has a BGG resolution in $\mathcal{O}(\mathfrak{g}, \mathfrak{q}, \lambda)$, and it was shown in [3] that this resolution is a minimal free resolution of $L(\lambda)$ as a graded $S(\mathfrak{p}^-)$-module.

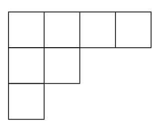
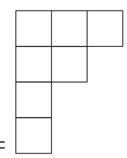
Thus we have

$$\mathrm{hd}_{S(\mathfrak{p}^-)}\, L(\lambda) = \max\{i \mid H^i(\mathfrak{p}^+, L(\lambda)) \neq 0\}.$$

Therefore, Enright's formula [2] for the cohomology modules $H^i(\mathfrak{p}^+, L(\lambda))$ allows computation of the homological dimension, and hence the depth, of $L(\lambda)$ as an $S(\mathfrak{p}^-)$-module. Since the dimension (Krull dimension) of $L(\lambda)$ as an $S(\mathfrak{p}^-)$-module is also known, it is possible to determine which unitary highest weight modules are Cohen–Macaulay $S(\mathfrak{p}^-)$-modules. In particular, in the context of this chapter, one can determine for which $\sigma \in \Sigma$ the module of covariants $\mathrm{Cov}_H(W, U_\sigma)$ is Cohen–Macaulay.

The following defintion will be needed in the remainder of the chapter.

**Definition 4.3.2.** If $\sigma$ is an integer partition with corresponding Young diagram, we define $\sigma'$ to be the *transpose partition*, i.e. the one whose Young diagram is the transpose of the Young diagram of $\sigma$. For example:

$\sigma = (4, 2, 1) =$  $\sigma' = (3, 2, 1, 1) =$ 

**Example 4.3.3.** Let $H = O(V)$, $k = \dim V$, and $W = V^n$, with $2k < n$. Then $\Sigma = \widehat{H}$, and we can identify $\Sigma$ with the set of integer partitions $\sigma = (\sigma_1, \sigma_2, \dots)$ such that $\sigma'_1 + \sigma'_2 \leq k$. With the above method of computing depth using cohomology modules, one can prove that $\mathrm{Cov}_H(W, U_\sigma)$ is Cohen–Macaulay if and only if $\sigma_1 \leq n - k + 1$.

Table 4.2 summarizes the conditions under which modules of covariants are Cohen–Macaulay in each of the three classical cases. Note that with the conditions on $k$ listed in the $W$ column, we have $\Sigma = \widehat{H}$ in all cases.

**Remark 4.3.4.** If the given conditions on $k$ fail, then Alexander–Hunziker [1] have shown that the module is Cohen–Macaulay if and only if it is free. This case is not being considered here.

Table 4.2: Cohen–Macaulay modules

| $H$ | $\sigma \in \widehat{H}$ | $W$ | Cohen–Macaulay iff |
|---|---|---|---|
| $GL(V)$, $\dim V = k$ | $\sigma = (n_1, n_2, \ldots, 0, \ldots, -m_2, -m_1)$ | $(V^*)^p \oplus V^q$, $k < \min\{p, q\}$ | $n_1 \leq p-k$, $m_1 \leq q-k$ |
| $O(V)$, $\dim V = k$ | $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$; $\sigma_1' + \sigma_2' \leq k$ | $V^n$, $k < n$ | $n_1 \leq n - k + 1$ |
| $Sp(V)$, $\dim V = 2k$ | $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ | $V^n$, $2k < n$ | $n_1 \leq n - 2k - 1$ |

### *4.4 Hilbert Series and the Wonderful Correspondence*

In [5], Enright–Hunziker–Pruett, extending results by Enright–Willenbring [9], introduced a correspondence that associates to every Wallach representation a finite-dimensional representation of a complex simple Lie algebra.

**Definition 4.4.1.** Let $S$ be a polynomial ring over $\mathbb{C}$ in finitely many variables. Let $M = \bigoplus_{i \geq 0} M_i$ be a finitely generated graded $S$-module. Then the *Hilbert series* of $M$ is the formal power series $H_M(t) = \sum_{i \geq 0} (\dim_{\mathbb{C}} M_i) t^i$.

If $M$ is generated by the first-degree elements, then Hilbert's syzygy theorem implies that $H_M(t)$ is of the form

$$H_M(t) = \frac{g(t)}{(1-t)^{\dim M}}$$

where $g(t) \in \mathbb{Z}[t]$ and $g(1) \neq 0$. Also, if $M$ is Cohen-Macaulay, then the coefficients of the numerator $f(t)$ are nonnegative integers.

Now let $M = \mathrm{Cov}_H(W, U_\sigma)$ be a module of covariants as in the previous section. Our goal is to give an interpretation of the coefficients of the numerator polynomials $g(t)$. In [5], Enright–Hunziker–Pruett handled the case of Wallach representations, i.e. when $U_\sigma$ is the trivial representation. Returning to the map $\pi$ from section 4.1, note again that the image $\pi(W)$ is either all of $\mathfrak{p}^+$ or a determinantal variety, and that $\mathbb{C}[\pi(W)] \cong \mathbb{C}[W]^H$ by the map $\pi^*$.

By a result of Hochster and Eagon [11], the ring $\mathbb{C}[\pi(W)]$ is Cohen-Macaulay, so it has a Hilbert series of the form $g(t)/(1-t)^{\dim \mathbb{C}[W]^H}$ where $f(t)$ has nonnegative integer coefficients. The "wonderful" result is that those coefficients can be interpreted as the Hilbert series of a certain finite-dimensional representation that corresponds to the Wallach representation on $\mathbb{C}[\pi(W)]$. The results are detailed in Table 4.3, assuming the same restrictions on $k$ as are given in the $W$-column of Table 4.2. In each case of the table, $\mathbb{C}[W]^H \cong L(\lambda)$ as a $\mathfrak{g}$-module.

Table 4.3: The Wonderful Correspondence for Wallach representations

| $H$ | $\mathfrak{g}_{\mathbb{R}}$ | $\lambda = -kc\zeta$ | $\mathfrak{g}'_{\mathbb{R}}$ | $\lambda' = k\zeta'$ |
|---|---|---|---|---|
| $GL(k,\mathbb{C})$ | $\mathfrak{su}(p,q)$ | $-k\omega_p$ | $\mathfrak{su}(p-k,q-k)$ | $k\omega'_{p-k}$ |
| $O(k,\mathbb{C})$ | $\mathfrak{sp}(n,\mathbb{R})$ | $-\frac{k}{2}\omega_n$ | $\mathfrak{so}^*(2n-2k+2)$ | $k\omega'_{n-k+1}$ |
| $Sp(k,\mathbb{C})$ | $\mathfrak{so}^*(2n)$ | $-2k\omega_n$ | $\mathfrak{sp}(n-2k-1,\mathbb{R})$ | $k\omega'_{n-2k-1}$ |

To each Wallach representation $L(\lambda)$ in the table, we associate an irreducible finite-dimensional representation $E(\lambda')$ of a complex simple Lie algebra $\mathfrak{g}'$ which is the complexification of the given $\mathfrak{g}'_{\mathbb{R}}$. Then the block $\mathcal{B}$ in $\mathcal{O}(\mathfrak{g},\mathfrak{q})$ that contains $L(\lambda)$ is congruent, as in Definition 3.4.1, to the regular integral block $\mathcal{B}' = \mathcal{O}(\mathfrak{g}',\mathfrak{q}',\lambda')$, and the poset isomorphism of Definition 3.4.1 sends the weight $\lambda$ to $\lambda'$. We also have the following theorem:

**Theorem 4.4.2** ([5], Theorem 8.1)**.** *The numerator polynomial $h(t)$ of the Hilbert series of $E(\lambda')$ is equal to the numerator polynomial $g(t)$ of the Hilbert series of $L(\lambda)$.*

In the remainder of this chapter, the Wonderful Correspondence will be generalized to all Cohen-Macaulay modules of covariants under the given conditions on $k$. Tables 4.4 and 4.5 outline the general correspondence. Let $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}'_{\mathbb{R}}$ be as in Table 4.3; the elements $\sigma \in H$ that correspond to Cohen-Macaulay modules of covariants, and the restrictions on $k$, are as in Table 4.2.

Table 4.4: The Wonderful Correspondence for modules of covariants

| $H$ | $\lambda$ |
|---|---|
| $GL(V)$ | $(-k-m_p, -k-m_{p-1}, \ldots, -k-m_2, -k-m_1; n_1, n_2, \ldots, n_{q-1}, n_q)$ |
| $O(V)$ | $\left(-\frac{k}{2}-\sigma_n, -\frac{k}{2}-\sigma_{n-1}, \ldots, -\frac{k}{2}-\sigma_2, -\frac{k}{2}-\sigma_1\right)$ |
| $Sp(V)$ | $(-k-\sigma_n, -k-\sigma_{n-1}, \ldots, -k-\sigma_2, -k-\sigma_1)$ |

Table 4.5: The Wonderful Correspondence for modules of covariants

| $H$ | $\lambda'$ |
|---|---|
| $GL(V)$ | $(k-n'_{p-k}, k-n'_{p-k-1}, \ldots, k-n'_2, k-n'_1; m'_1, m'_2, \ldots, m'_{q-k-1}, m'_{q-k})$ |
| $O(V)$ | $\left(\frac{k}{2}-\sigma'_{n-k+1}, \frac{k}{2}-\sigma'_{n-k}, \ldots, \frac{k}{2}-\sigma'_2, \frac{k}{2}-\sigma'_1\right)$ |
| $Sp(V)$ | $(k-\sigma'_{n-2k-1}, k-\sigma'_{n-2k-2}, \ldots, k-\sigma'_2, k-\sigma'_1)$ |

In each case of the tables, we associate to the representation $L(\lambda)$ a finite-dimensional representation $E(\lambda')$ as with the Wallach representations. In the following sections we will prove in each case that $\lambda' + \rho'$ is obtained from $\lambda + \rho$ by Enright reduction, thus showing that the block $\mathcal{B}$ containing $L(\lambda)$ is similar (as per Definition 3.4.1) to the regular integral block $\mathcal{B}' = \mathcal{O}(\mathfrak{g}', \mathfrak{q}', \lambda')$. We also have:

**Theorem 4.4.3.** *If $H_L(t) = \frac{g(t)}{(1-t)^{\dim L}}$ and $H_E(t) = \frac{h(t)}{(1-t)^{\dim E}}$ are the respective Hilbert series of $L(\lambda)$ and $E(\lambda')$, then*

$$g(t) = \frac{\dim F(\mathfrak{k}, \lambda)}{\dim F(\mathfrak{k}', \lambda')} h(t)$$

*where the fraction is the constant of similarity from Definition 3.4.1.*

This theorem follows easily from the proof given in [5] for Theorem 4.4.2.

## *4.5 Proof of the Wonderful Correspondence for $GL(V)$*

Let $H = GL(V)$ and set up the elements of $\widehat{H}$ that correspond to Cohen-Macaulay modules of covariants as sequences of integers:

Let $p$ and $q$ be positive integers, and $k$ a positive integer with $k \le p$ and $k \le q$. Let $\sigma = (n_1, n_2, \dots, 0, 0, \dots, -m_2, -m_1)$ of length $k$, $\sigma_+ = (n_1, n_2, \dots)$, and $\sigma_- = (m_1, m_2, \dots)$, with $l(\sigma_+) + l(\sigma_-) \le k$, $n_1 \le p - k$, and $m_1 \le q - k$. Let $\sigma'_+ = (n'_1, n'_2, \dots)$ and $\sigma'_- = (m'_1, m'_2, \dots)$ be the conjugate partitions to $\sigma_+$ and $\sigma_-$, respectively, each corresponding to the Young diagram that is the transpose of the one associated with the original. Now let:

$$\lambda = (-k - m_p, -k - m_{p-1}, \dots, -k - m_2, -k - m_1; n_1, n_2, \dots, n_{q-1}, n_q)$$

$$\rho = (p + q - 1, p + q - 2, \dots, 1, 0)$$

$$\lambda' = (k - n'_{p-k}, k - n'_{p-k-1}, \dots, k - n'_2, k - n'_1; m'_1, m'_2, \dots, m'_{q-k-1}, m'_{q-k})$$

$$\rho' = (p + q - 2k - 1, p + q - 2k - 2, \dots, 1, 0).$$

Thus:

$$\lambda + \rho = (p + q - 1 - k - m_p, p + q - 2 - k - m_{p-1}, \dots, q + 1 - k - m_2, q - k - m_1; \\ q - 1 + n_1, q - 2 + n_2, \dots, 1 + n_{q-1}, n_q)$$

$$\lambda' + \rho' = (p + q - k - 1 - n'_{p-k}, p + q - k - 2 - n'_{p-k-1}, \dots, q + 1 - n'_2, q - n'_1; \\ q - k - 1 + m'_1, q - k - 2 + m'_2, \dots, 1 + m'_{q-k-1}, m'_{q-k})$$

**Theorem 4.5.1.** *$\lambda' + \rho'$ is obtained from $\lambda + \rho$ by Enright reduction.*

*Proof.* Suppose $\sigma = (0, 0, \dots)$. So $\sigma_+ = \sigma_- = (0, 0, \dots) = \sigma'_+ = \sigma'_-$. Then

$$\lambda = (-k, \dots, -k; 0, \dots, 0) \text{ and } \lambda' = (k, \dots, k; 0, \dots, 0).$$

So we have

$$\lambda + \rho = (p + q - 1 - k, p + q - 2 - k, \dots, q + 1 - k, q - k; q - 1, q - 2, \dots, 1, 0);$$

$$\lambda' + \rho' = (p+q-k-1, p+q-k-2, \ldots, q+1, q; q-k-1, q-k-2, \ldots, 1, 0).$$

Note that all (and only) the integers from $q-k$ to $q-1$ are repeated in $\lambda+\rho$ and are missing in $\lambda'+\rho'$. Thus $\lambda'+\rho'$ is obtained from $\lambda+\rho$ by Enright reduction.

Now assume we have a $\sigma$ with $i-1$ entries in $\sigma_+$ that reduces as desired, and add an $n_i$, assuming this is possible under the given restrictions.

The previous $i-1$ entries in $\sigma_+$ have the effect of adding to the entries of $\lambda+\rho$ that in the empty case read $q-1$ through $q-i+1$, so they do not affect the entry that reads $q-i$. Adding the $n_i$ entry changes the $q-i$ to a $q-i+n_i$. Thus $q-i$ is no longer repeated in $\lambda+\rho$, however $q-i+n_i$ is now repeated (it was already present in $\lambda+\rho$ since $n_1$, and thus all the entries of $\sigma_+$, are less than or equal to $p-k$).

Meanwhile, in $\lambda'+\rho'$, the introduction of $n_i$ will subtract 1 from the last $n_i$ entries before the semicolon, which creates a new entry of $q-i$ and removes an entry of $q-i+n_i$. Thus $\lambda+\rho$ still reduces to $\lambda'+\rho'$.

Now assume we have a $\sigma$ with $j-1$ entries in $\sigma_-$ that reduces as desired, and add an $m_j$, assuming this is possible under the given restrictions.

The previous $j-1$ entries in $\sigma_-$ have the effect of subtracting from the entries of $\lambda+\rho$ that in the empty case read $q+j-k-2$ through $q-k$, so they do not affect the entry that reads $q+j-k-1$. Adding the $m_j$ entry changes the $q+j-k-1$ to a $q+j-k-1-m_j$. Thus $q+j-k-1$ is no longer repeated in $\lambda+\rho$, however $q+j-k-1-m_j$ is now repeated (it was already present in $\lambda+\rho$ since $m_1$, and thus all the entries of $\sigma_-$, are less than or equal to $q-k$).

Meanwhile, in $\lambda'+\rho'$, the introduction of $m_j$ will add 1 to the first $m_j$ entries after the semicolon, which creates a new entry of $q-k-1+j$ and removes an entry of $q-k-1+j-m_j$. Thus $\lambda+\rho$ still reduces to $\lambda'+\rho'$. □

### 4.6 Proof of the Wonderful Correspondence for $O(V)$

Let $H = O(V)$ and set up the elements of $\widehat{H}$ that correspond to Cohen-Macaulay modules of covariants as sequences of integers:

Let $\sigma = (\sigma_1, \sigma_2, \dots)$ be a weakly decreasing sequence of nonnegative integers, eventually zero. Let $\sigma' = (\sigma'_1, \sigma'_2 \dots)$ be the sequence corresponding to the transpose of the Young diagram that $\sigma$ corresponds to. Let $n$ be a positive integer and $k$ a non-negative integer less than $n$, satisfying $\sigma_1 \leq n-k+1$, $\sigma'_1+\sigma'_2 \leq k$, and $l(\sigma) \leq n$. (Note $l(\sigma) = \sigma'_1$.) Let

$$\lambda = \left(-\frac{k}{2} - \sigma_n, -\frac{k}{2} - \sigma_{n-1}, \dots, -\frac{k}{2} - \sigma_2, -\frac{k}{2} - \sigma_1\right)$$

$$\rho = (n, n-1, \dots, 2, 1)$$

$$\lambda' = \left(\frac{k}{2} - \sigma'_{n-k+1}, \frac{k}{2} - \sigma'_{n-k}, \dots, \frac{k}{2} - \sigma'_2, \frac{k}{2} - \sigma'_1\right)$$

$$\rho' = (n-k, n-k-1, \dots, 1, 0)$$

So we have:

$$\lambda + \rho = \left(n - \frac{k}{2} - \sigma_n, n-1-\frac{k}{2} - \sigma_{n-1}, \dots, 2 - \frac{k}{2} - \sigma_2, 1 - \frac{k}{2} - \sigma_1\right)$$

$$\lambda' + \rho' = \left(n-k+\frac{k}{2} - \sigma'_{n-k+1}, n-k-1+\frac{k}{2} - \sigma'_{n-k}, \dots, 1 + \frac{k}{2} - \sigma'_2, \frac{k}{2} - \sigma'_1\right)$$

**Theorem 4.6.1.** *Enright reduction reduces $\lambda+\rho$ to $\lambda'+\rho'$, except if $\lambda+\rho$ does not contain a 0, then $\lambda'+\rho'$ is the Enright-reduced form of $\lambda+\rho$ with a 0 inserted.*

*Proof.* Suppose $\sigma = (0, 0, \dots)$. Then

$$\lambda + \rho = \left(n - \frac{k}{2}, n-1-\frac{k}{2}, \dots, 2 - \frac{k}{2}, 1 - \frac{k}{2}\right);$$

$$\lambda' + \rho' = \left(n-k+\frac{k}{2}, n-k-1+\frac{k}{2}, \dots, 1 + \frac{k}{2}, \frac{k}{2}\right)$$

where the difference between consecutive entries is always 1 in both.

If $k = 0$, then $\lambda+\rho = (n, n-1, \dots, 2, 1)$ and $\lambda'+\rho' = (n, n-1, \dots, 2, 1, 0)$ as desired.

If $k > 0$ and $k$ is even: $\lambda + \rho$ has a 0 in the $\frac{k}{2}$-to-last slot, and $\frac{k}{2} - 1$ negative numbers to its right that create singularities with $\frac{k}{2} - 1$ positive numbers to its left. So Enright reduction will remove the last $2\left(\frac{k}{2} - 1\right) + 1$ entries in $\lambda + \rho$, leaving a string $n - k + 1$ long which is $\lambda' + \rho'$.

If $k$ is odd: $\lambda + \rho$ has $\frac{k-1}{2}$ negative numbers at the end, creating singularities with $\frac{k-1}{2}$ positive numbers. So Enright reduction will remove the $k - 1$ right-most entries in $\lambda + \rho$, leaving a string $n - k + 1$ long which is $\lambda' + \rho'$.

Now take a $\sigma$ with $m - 1$ entries that reduces as desired, and add a positive $m$th entry, assuming that this is possible under the given restrictions on $\sigma_1$, $\sigma_1'$, and $\sigma_2'$.

**Case** $k > 2m$:

Introducing a positive $\sigma_m$ as the $m$th entry of $\sigma$ will have the effect of subtracting $\sigma_m$ from the $n - m + 1$ ($m$-to-last) entry of $\lambda + \rho$, which is unchanged from the empty case above and is negative since it equals $m - \frac{k}{2}$. (Thus in this case positive entries of $\lambda + \rho$ are not affected by introducing the new entry or by any previous entry, so they are identical to the empty case.) This leaves the positive $n-k+m+1$ entry nonsingular and creates a singularity with the positive $n - k + m + 1 - \sigma_m$ entry.

Meanwhile in $\lambda' + \rho'$, the introduction of $\sigma_m$ has the effect of subtracting 1 from the last $\sigma_m$ entries (which are all 1 apart), leaving the previous $n - k + 2 - \sigma_m$ ($\sigma_m$-to-last) entry missing and creating a new entry 1 less than the previous $n-k+1$ (last) entry. It remains to show, then, that

(1) the $n - k + m + 1$ entry of $\lambda + \rho$ is 1 less than the last entry of the previous $\lambda' + \rho'$, and

(2) the $n - k + m + 1 - \sigma_m$ entry of $\lambda + \rho$ is equal to the previous $n - k + 2 - \sigma_m$ entry of $\lambda' + \rho'$.

To see (1), note that the last entry of the previous $\lambda' + \rho'$ is simply $\frac{k}{2} - m + 1$, (1 has been subtracted from the empty case's $\frac{k}{2}$, $m - 1$ times) while the $n - k + m + 1$ entry of $\lambda + \rho$ is positive and thus unchanged from the empty case, so it equals $k - m - \frac{k}{2}$, 1 less than the $\frac{k}{2} - m + 1$ above.

Now (2) can be seen by noting that the difference between the two positions is $m - 1$, and introducing an $m$th row does not change the positive $n - k + m + 1 - \sigma_m$ entry of $\lambda + \rho$. In the previous version of $\sigma$, where everything reduces as desired by assumption, the $m - 1$ negative numbers to the right of our current $m$-to-last position in $\lambda + \rho$ cause $m - 1$ positive numbers to the left of the $n - k + m + 1 - \sigma_m$ entry to be missing from $\lambda' + \rho'$, resulting in (2).

**Case** $k = 2m$:

In this case, the $m$-to-last entry of $\lambda + \rho$ is a 0, so subtracting $\sigma_m$ from it will remove a 0 and introduce a negative number, which creates a singularity, as in the above case, with the positive $n - k + m + 1 - \sigma_m$ entry. As above, the $n - k + 1 - \sigma_m$ entry of the previous $\lambda' + \rho'$ was equal to that $n - k + m + 1 - \sigma_m$ and ends up missing.

We also need to show that a 0 has been introduced to $\lambda' + \rho'$. From the formula above for $\lambda' + \rho'$, its last entry after adding an $m$th row will be $\frac{k}{2} - m$ which is 0 since $k = 2m$.

**Case** $k < 2m$:

Here, instead of adding one entry at a time, we will start with a $\sigma$ with $\frac{k}{2}$ (for $k$ even) or $\frac{k-1}{2}$ (for $k$ odd) entries that reduces as desired, and introduce $m - \frac{k}{2}$ or $m - \frac{k-1}{2}$ new positive entries at once. Due to the restriction on $\sigma'_1 + \sigma'_2$, each of these new entries must equal 1.

*Subcase* $k$ even:

In the previous $\frac{k}{2}$-entry version, $\lambda + \rho$ contains no 0 but contains all positive integers up to $n - \frac{k}{2}$, while $\lambda' + \rho'$ has a 0 as its last entry. Introducing the $m - \frac{k}{2}$

entries each equal to 1 will remove the $n-m+1$ entry of $\lambda+\rho$ which is the positive integer $m-\frac{k}{2}$, and replace it with a 0; meanwhile it will subtract $m-\frac{k}{2}$ from the last entry of $\lambda'+\rho'$ which was a 0, yielding the negative integer $\frac{k}{2}-m$. Thus, we need to show that $\frac{k}{2}-m$ appears in the previous $\lambda+\rho$. But assuming that the increase to $m$ entries is legal under our restrictions, $\sigma_2'$ cannot be greater than $k-m$; thus, $\sigma_{k-m+1}=1$, so the $n-k+m$ entry of $\lambda+\rho$ is 1 less than in the empty case; namely, it is $(k-m+1)-\frac{k}{2}-1=\frac{k}{2}-m$.

*Subcase* $k$ odd:

In the previous $\frac{k-1}{2}$-entry version, $\lambda+\rho$ contains no $-\frac{1}{2}$ but contains all positive odd half-integers up to $n-\frac{k}{2}$, while $\lambda'+\rho'$ has a $\frac{1}{2}$ as its last entry. Introducing the $m-\frac{k-1}{2}$ entries each equal to 1 will remove the $n-m+1$ entry of $\lambda+\rho$ which is the positive number $m-\frac{k}{2}$, and replace it with a $-\frac{1}{2}$; meanwhile it will subtract an $m-\frac{k-1}{2}$ from the last entry of $\lambda'+\rho'$ which was a $\frac{1}{2}$, yielding the negative number $\frac{k}{2}-m$. Now if $\frac{k}{2}-m<-\frac{1}{2}$, then it appears in the previous $\lambda+\rho$ using the same proof as above, and we are done. However, if $\frac{k}{2}-m=-\frac{1}{2}$, then the $\sigma_{k-m+1}$ entry is the one we are adding, so the above proof does not work. In this case, the reduction still goes through as desired since we have lost a $\frac{1}{2}$ and gained a $-\frac{1}{2}$ with no other changes. □

### *4.7 Proof of the Wonderful Correspondence for* $Sp(V)$

Let $H=Sp(V)$ and set up the elements of $\widehat{H}$ that correspond to Cohen-Macaulay modules of covariants as sequences of integers:

Let $n$ be a positive integer and $k$ a non-negative integer with $2k<n$. Let $\sigma=(\sigma_1,\sigma_2,\dots)$ be a weakly decreasing sequence of non-negative integers, eventually zero, with $l(\sigma)\leq n$, and let $\sigma'=(\sigma_1',\sigma_2'\dots)$ be the sequence corresponding to the transpose of the Young diagram that $\sigma$ corresponds to. Suppose $\sigma_1\leq n-2k-1$.

Let:

$$\lambda = (-k - \sigma_n, -k - \sigma_{n-1}, \dots, -k - \sigma_2, -k - \sigma_1)$$

$$\rho = (n-1, n-2, \dots, 1, 0)$$

$$\lambda' = (k - \sigma'_{n-2k-1}, k - \sigma'_{n-2k-2}, \dots, k - \sigma'_2, k - \sigma'_1)$$

$$\rho' = (n-2k-1, n-2k-2, \dots, 2, 1).$$

So we have:

$$\lambda + \rho = (n - 1 - k - \sigma_n, n - 2 - k - \sigma_{n-1}, \dots, 1 - k - \sigma_2, -k - \sigma_1);$$

$$\lambda' + \rho' = (n - k - 1 - \sigma'_{n-2k-1}, n - k - 2 - \sigma'_{n-2k-2}, \dots, k + 2 - \sigma'_2, k + 1 - \sigma'_1).$$

**Theorem 4.7.1.** *Enright reduction reduces $\lambda + \rho$ to $\lambda' + \rho'$.*

*Proof.* Empty case: Suppose $\sigma = (0, 0, \dots) = \sigma'$. Then

$$\lambda + \rho = (n - 1 - k, n - 2 - k, \dots, 1 - k, -k);$$

$$\lambda' + \rho' = (n - k - 1, n - k - 2, \dots, k + 2, k + 1).$$

Since no integers are skipped in $\lambda + \rho$, Enright reduction will remove everything from $-k$ to $k$, leaving $\lambda' + \rho'$.

Now suppose we have a $\sigma$ with $m-1$ entries that reduces as desired, and insert a $\sigma_m$, assuming the given restrictions allow this. (Note that the restrictions force $m \leq k$). This insertion subtracts $\sigma_m$ from the $m$-to-last entry in $\lambda + \rho$, which is negative since $m \leq k$. So positive entries of $\lambda + \rho$ are never affected. This removes a negative entry which equaled $m - 1 - k$, removing the singularity with the positive entry $k + 1 - m$, and creates a new negative entry $m - 1 - k - \sigma_m$ which causes a singularity (our restrictions guarantee the positive entry $k + 1 - m + \sigma_m$ is present).

Meanwhile, in $\lambda' + \rho'$, the new $\sigma_m$ causes 1 to be added to the first $\sigma_m$ entries of $\sigma'$, which has the effect of subtracting 1 from each of the last $\sigma_m$ entries of $\lambda' + \rho'$. This removes an entry that read $k - (m - 1) + \sigma_m$, and introduces an entry that reads $k + 1 - m$. Thus Enright reduction still reduces $\lambda + \rho$ to $\lambda' + \rho'$. □

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