ABSTRACT

A Finite Dimensional Approximation of a Density Dependent Mean Field Game

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The goal of this thesis is to establish the existence and uniqueness of a Nash equilibrium of a density dependent mean field game and approximate the solution with numerical methods. We first briefly introduce both mean field game theory and measure theory. Next, we define a game in which the final cost is the density of the equilibrium measure. Then we prove a unique solution exists by using the Browder-Minty Theorem. To conclude, we will show how Newton's method can be used to approximate a solution and look at some specific examples of this approximation in action.

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A FINITE DIMENSIONAL APPROXIMATION OF A DENSITY DEPENDENT MEAN FIELD GAME

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Contents

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To my family for supporting me in everything

1 Introduction of Relevant Concepts

Before we explore the specific problem presented in this paper, we will briefly introduce some relevant concepts necessary for understanding the context of this problem. We start with the topic of Mean Field Games Theory, specifically Optimal Control Theory, Game Theory, and Mean Field Games. Secondly, we will look at Measure Theory. Note that these topics will be covered only in brief: many relevant papers, such as Lasry and Lions, 2007, Gibbons, 1997, and others, are referenced at the end of the paper if the reader wishes to explore the topic in depth. The overall structure of this section was borrowed from Laurel, 2018.

Optimal Control Theory

In short, optimal control theory seeks to find the optimal strategy for players in some game such that each player incurs the least cost while achieving their goal. While this style of problem extends to many disciplines such as economics and physics, this paper will restrict its focus to the mathematics behind the theory of optimal control. We first define a control system as follows:

$$x'(t) = f(t, x(t), a(t)),\ \ x(t_0) = x_o,$$

where t is time, x is a function of t called the state function, a is the control input, and t_0 and x_0 corresponding to the initial time and state, respectively. This equation mathematically models the scenario of players playing the game from above, allowing us to study how to change and influence a system. Yet a control system does not give any information about optimization. For that, we need to define a cost function J:

$$J(x,t,a) = \int_{t_0}^{t_f} \mathcal{L}(t, x(t), a(t))\,\mathrm{d}t + T(t_f, x_f).$$

Here, t_f and x_f represent the final time and state, respectively, $\mathcal{L}$ is the Lagrangian of the state, and T is the final cost. Observe that $\mathcal{L}$ depends on all t's between t_0 and t_f, so that $\mathcal{L}$ will function as a sort of "running cost," whereas T only depends of the final conditions. The function J is that which we seek to optimize over all a in search of an optimal control. Though much more could be said on optimal control theory, this brief introduction will suffice for the sake of this paper.

Game Theory

The idea of a game substantially predates the mathematical theory behind it. Intellectual puzzles such as the famous Prisoner's Dilemma, the popular game Rock-Paper-Scissors, and even White Elephant Gift Exchanges, all are examples of so-called games. In general, a game involves multiple rational players under some constraints all trying to achieve goal, either together or individually. Note that the players must be rational, i.e. they must always behave in such a way that will maximize their chances of winning.

Mathematically, we can model a game with several control systems. To do so, suppose that there are N players in some game. Then

$$x_j'(t) = f_j(t, x_j(t), a_j(t)),\ \ x_j(t_0) = x_{j_0},\ \ j = 1, 2, \ldots, N$$

corresponds to a control system for the jth player. We can then express the cost function J for each player as:

$$J_j(a_1, \ldots, a_N) = \int_{t_0}^{t_f} \mathcal{L}_j(t, x_1(t), \ldots, x_N(t), a_1(t), \ldots, a_N(t)))\,\mathrm{d}t + T_j(t_f, x_{1f}, \ldots, x_{Nf}).$$

Clearly, this equation is much more complicated than the previous equation for J, as now each player must account for the decisions every other player will make in order to win themselves. If it is the case that every player, after taking into account the optimal actions for every other player, cannot improve their chances of winning, we arrive at a Nash equilibrium. In other words, in Nash equilibrium, every player doing the best that they individually can do (not necessarily "winning" the game). Not every game will have a Nash equilibrium, and often explicitly solving for the Nash equilibrium is impossible, but much of game theory deals with finding these Nash equilibria.

Mean Field Games

A mean field game is a specific type of game in which the number of players N is very large, so that the impact of a single player is negligible. Further, every player has the same goals and constraints and the actions of an individual player are unknown to all the other players. Thus, the need to specifically index each player's control system disappears, as each player will behave in the same way. To account for the now very large number of players, we introduce a probability density $m(x, t)$ which represents the density at x at time t. This turns our cost function J into:

$$J(x, t, a) = \int_{t_0}^{t_f} \mathcal{L}(t, x(t), a(t), m(t, x))\,\mathrm{d}t + T(t_f, x_f).$$

Though it remains very challenging to minimize this cost function J, much progress has been made in recent decades into mathematical theory on the equations (see Cardaliaguet, 2010). We will expand the topic of mean field games more in the next section. For now, we move onto the next pertinent topic.

Measure Theory

In short, measure theory looks at different ways to measure different quantities like length, mass, volume, etc. In this paper, we will deal with two of these ways: probability density and empirical measures. A probability density is a non-negative function whose integral is one and whose integral between two points is the probability of being between those two points. An empirical measure takes a finite number of points and assigns a particular proportion of the population to each point. To help understand the difference between these two measures, consider the following image:

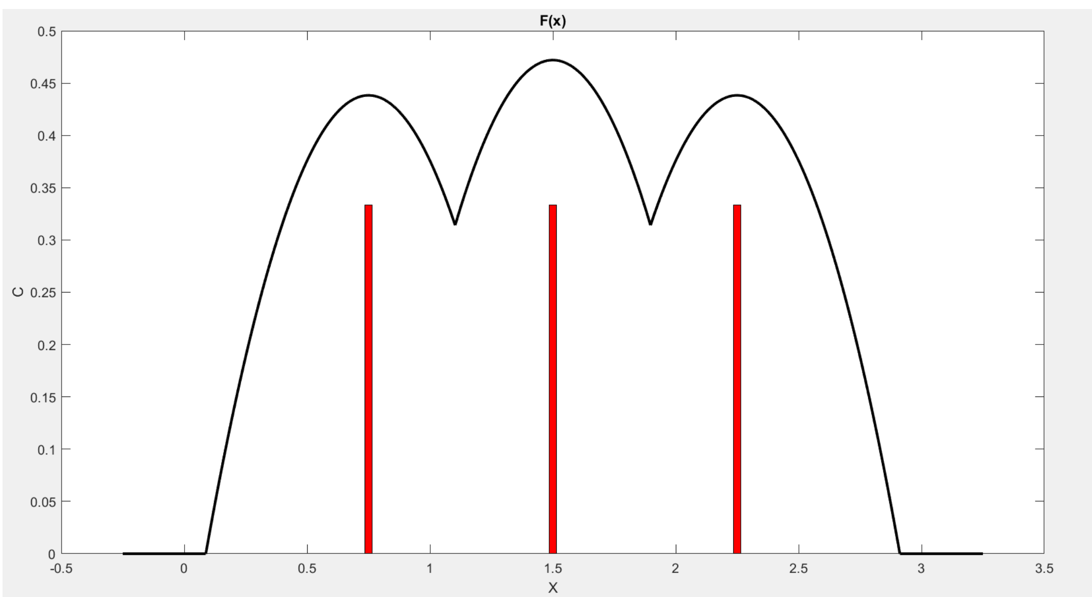

Figure 1: Simple Example

The three red bars represent an empirical measure in which each bar has a height of $\frac{1}{3}$, so exactly $\frac{1}{3}$ of the population is at 0.75, at 1.5, and at 2.25. On the other hand, the black curve corresponds to a probability density whose value at each point corresponds to the population density at that point.

In general, we can represent an empirical measure as $\sum_{j=1}^{n} a_j \delta_{x_j}$, where a_j tells the proportion of the population located at x_j, and δ_{x_j} is a Dirac Measure located at x_j. In the example above, for instance, x_1 is 0.75 and a_1 is $\frac{1}{3}$. Note that $\sum_{j=1}^{n} a_j$ will always be 1.

A probability density, on the other hand, can be represented generally by some non negative function f whose integral is 1 like the black curve in Figure 1. Note the fundamental difference between an empirical measure and a probability density: an empirical measure tells the exact position of the players and a probability density tells the probability of a player being in some interval.

2 Statement of Problem

Let $J(x, y, f) = (x - y)^2 + f(y)$. Then J is a cost function for players who start at x and move to y; f is a density function. Let $m = \sum_{j=1}^{N} a_j \delta_{x_j}$ with $a_1 + \cdots + a_N = 1, a_j \geq 0$ and $x_1, \ldots, x_N$ distinct points in $\mathbb{R}$. Then m is an empirical measure in which a_j is the proportion of players initially located at each x_j. In this case, a Nash equilibrium is a measure π on $\mathbb{R} \times \mathbb{R}$ such that:

1. $\pi(A \times \mathbb{R}) = m(A) = \sum_{j=1}^{N} a_j \delta_{x_j}(A)$, where $\delta_{x_j}(A) = \begin{cases} 1 \text{ if } x_j \notin A \\ 0 \text{ if } x_j \in A \end{cases}$, i.e. first

marginal of π is m,

2. $\pi(\mathbb{R} \times B) = \int_B f(x)\,\mathrm{d}x$ for a density f,

3. and for $\pi - a.e.(x, y), y \in \operatorname{argmin} J(x, \cdot, f)$, i.e. the second marginal of π is the density f.

In other words, given the cost function J, a density function f is chosen to possibly be an equilibrium measure. With the knowledge of this function f, the players choose the optimal strategy for themselves, resulting in some final distribution. If this final distribution matches f, then f is in fact an equilibrium. The idea of a "Self-Fulfilling Prophecy" can be helpful to understand this: f is "prophesied" to be an equilibrium measure, and, if it turns out to result in an equilibrium, then f in a sense "fulfilled its prophecy." The below diagram demonstrates this phenomenon.

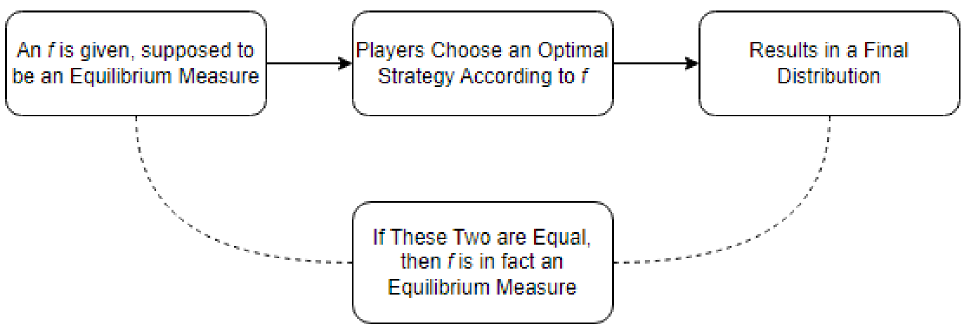

Figure 2: Mean Field Games Diagram

We now want to determine if there exists an equilibrium solution to J, and, if it does exist, to find the solution. To do so, we will assume that f is the equilibrium density, which allows us to define the following:

$$C_j = \min\Big\{y : (x_j - y)^2 + f(y)\Big\} \text{ and } E_j = \operatorname{argmin}\Big\{y : (x_j - y)^2 + f(y)\Big\}.$$

Notice that $\{f > 0\} \subset \bigcup_1^N E_j$, because every player must move into one of the sets E_j. On the other hand, for $y \in E_j$ we have $f(y) = C_j - (x_j - y)^2$. Thus, f is completely determined by C_j and E_j, and these in turn are coupled by the definition of E_j. So how do we determine C_j and E_j? A first clue is the following proposition, whose proof is elementary:

Proposition 2.1. $E_j = \Big\{y : C_j - (x_j - y)^2 \geq \big(C_k - (x_k - y)^2\big)_+ \ \forall k\Big\}$, *where* $x_+ := \max\{x, 0\}$.

Proof. If $y \in E_j$, then $(x_j - y)^2 + f(y) = C_j$, and at the same time $(x_k - y)^2 + f(y) \geq C_k$ for all k by definition of C_k. Recalling that $f \geq 0$, we see that $C_j - (x_j - y)^2 = f(y) \geq \big(C_k - (x_k - y)^2\big)_+$ for all k.

Conversely, suppose $y \notin E_j$. If $y \in E_k$ for some other k, then $C_j - (x_j - y)^2 < f(y) = C_k - (x_k - y)^2$. If $y \notin \bigcup_1^N E_k$, then $f(y) = 0$, hence $C_j - (x_j - y)^2 < 0$. It follows that $C_j - (x_j - y)^2 < \big(C_k - (x_k - y)^2\big)_+$ for at least one k. $\square$

Corollary 2.2. *Let* $f_j(y) = C_j - (x_j - y)^2$. *Then* $f(y) = \max\big\{0, f_1(y), \ldots, f_N(y)\big\}$ *and* $E_j = \big\{y : f(y) = f_j(y)\big\}$.

Intuitively, E_j should be the destination of all the players initially concentrated at x_j. We now make this intuition rigorous.

Definition 2.3. *The support of* π, *supp* π, *is the set of all* (x, y) *such that, for all* $\epsilon > 0$, $\pi(B_\epsilon(x, y)) > 0$, *where* $B_\epsilon(x, y) = \Big\{(z, w) | \sqrt{(x - z)^2 + (y - w)^2} < \epsilon\Big\}$.

Proposition 2.4. *Let* $\hat{E}_j = \{y : (x_j, y) \in \operatorname{supp}\pi\}$. *Then* $\hat{E}_j \subset E_j$, $\int_{E_j \setminus \hat{E}_j} f = 0$, *and* $\int_{E_j} f = a_j$.

Proof. If $(x_j, y) \in \operatorname{supp}\pi$, then by definition of equilibrium, it follows that $y \in E_j$. Thus $\hat{E}_j \subset E_j$.

To prove the remaining statements, start with the following inequality:

$$a_j = \pi\left(\{x_j\} \times \mathbb{R}\right) = \pi\left(\{x_j\} \times \hat{E}_j\right) \leq \pi\left(\mathbb{R} \times \hat{E}_j\right) = \int_{\hat{E}_j} f \leq \int_{E_j} f. \tag{2.1}$$

The second equality follows because $(\{x_j\} \times \mathbb{R}) \cap \operatorname{supp}\pi = \{x_j\} \times \hat{E}_j$, and the last inequality follows from $\hat{E}_j \subset E_j$. On the other hand, by Proposition 2.1, we see that ∂E_j has zero Lebesgue measure and the interiors of the sets E_j are disjoint. It follows that

$$1 = \int f = \sum_{j=1}^{N} \int_{E_j} f. \tag{2.2}$$

Thus all the inequalities above become equalities, and we get $\int_{E_j} f = a_j$ and $\int_{E_j \setminus \hat{E}_j} f = 0$, as desired. $\square$

With these results, we can now restate the problem using C_j and E_j. Given some m, we want to show that there exists a vector C such that

$$\int_{E_j} f_j(x) = \int_{E_j} C_j - (x - x_j)^2 = a_j.$$

To better explain this, consider again Figure 1. We want to find a function like the black one graphed such that the integral of the function over the interval on which the jth component of f is the max is equal to the height of the corresponding red bar. To do so, we will equivalently redefine some variables from the beginning as follows:

Definition 2.5. *Let* $f_j^{(C)}(x) = C_j - (x - x_j)^2$

Definition 2.6. *Let* $f^{(C)}(x) = \max\{f_j(x), 0\}$ *for* $j = 1, \dots, n$.

Definition 2.7. *Let* $E_j^{(C)} = \big\{x : f_j(x) = f(x)\big\}$.

Remark 2.8. *Whenever C is fixed, we will suppress the argument C.*

Definition 2.9. *Let* $F_j(C) = \int_{E_j} f(x)\,\mathrm{d}x$

Definition 2.10. *Let* $F(C) = \left(\int_{E_1} f(x)\,\mathrm{d}x, \ldots, \int_{E_N} f(x)\,\mathrm{d}x\right)$

To better understand these definitions, consider the following image:

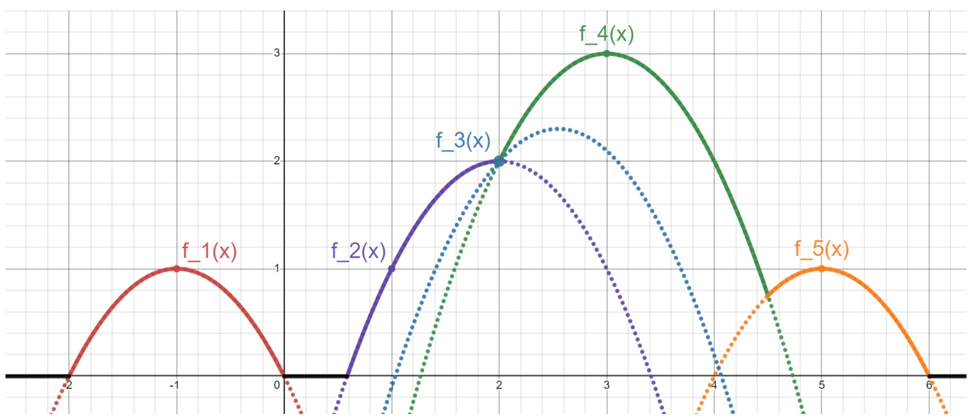

Figure 3: Helpful Visualization

In this example, we see that

1. $f_1(x) = C_1 - (x - x_1)^2 = 1 - (x+1)^2$ and $f_5(x) = C_5 - (x - x_5)^2 = 1 - (x-5)^2$,
2. $f(x) = \max\{f_1(x), f_2(x), f_3(x), f_4(x), f_5(x), 0\}$ is depicted by the graph of the solid line,
3. $E_2 = \big\{x : f_2(x) = f(x)\big\} = [2 - \sqrt{2}, 2]$ and $E_3 = \big\{x : f_3(x) = f(x)\big\} = \{2\}$, and
4. $F_4(x) = \int_{E_4} f(x)\,\mathrm{d}x = \int_2^{4.5} f(x)\,\mathrm{d}x \approx 6$.

Note that this graph is not allowed in this problem, as the total are under the curves is much greater than one, but it serves to aid in understanding the notation.

With these definitions in mind, we now state the problem we wish to solve as follows:

Problem Statement

Given a vector a, we want to find a vector C, whose components are all non-negative and sum to 1, such that $F(C) = a$. Then it will follow that $f(x)$ is the equilibrium density.

Theorem 2.11. *For each a such that $a_j > 0$ for $j = 1, \ldots, n$, there exists a unique C such that $F(C) = a$.*

We will prove Theorem 2.11 in two sections: first, in Chapter 4, we will show that a solution to the equation $F(C) = a$ exists, and then, in Chapter 5, we will show that there is at most one C such that $F(C) = a$. Before doing so, however, it is necessary to introduce some auxiliary lemmas.

3 Structure of $F(C)$

Before proving existence or uniqueness, we first need to explore and compute a variety of properties of $f(x)$ and $F(C)$. We will start the following:

Definition 3.1. *Let* $\gamma_{ij} = \dfrac{C_i - C_j}{2(x_j - x_i)} + \dfrac{x_j + x_i}{2}, i \neq j.$

Lemma 3.2. *γ_{ij} is the unique solution to $f_i(\gamma_{ij}) = f_j(\gamma_{ij})$ for any $i \neq j$.*

Proof. We simply observe that:

$$C_i - (\gamma_{ij} - x_i)^2 = C_j - (\gamma_{ij} - x_j)^2 \Leftrightarrow \gamma_{ij} = \frac{C_i - C_j}{2(x_j - x_i)} + \frac{x_j + x_i}{2} \tag{3.1}$$

Since the implication goes both directions, the solution exists and is unique. Further, we find a formula for γ_{ij}, which is the x value at which $f_i(x)$ intersects $f_j(x)$. □

Remark 3.3. *As a result of the previous Lemma, and since $f(x)$ is continuous, we observe that if $f_i(x) > f_j(x)$ for all for $x < \gamma_{ij}$, then $f_i(x) < f_j(x)$ for all $x > \gamma_{ij}$, and likewise if $f_i(x) < f_j(x)$ for all for $x < \gamma_{ij}$, then $f_i(x) > f_j(x)$ for all $x > \gamma_{ij}$.*

With this in mind, we proceed to the following section:

3.1 Properties of E_j

Lemma 3.4. *E_j is either empty, a singleton, or a closed, bounded interval.*

Proof. We first note that, since E_j is by definition a set, E_j is empty, a singleton, or contains at least two points. Hence, all we need to prove is that if E_j contains more than one point, then E_j is a closed, bounded interval. So, suppose that E_j contains at least two points. Call two of these points $\alpha, \beta \in E_j$. Let $x \in (\alpha, \beta)$. To show that E_j is an interval, we need to show that $x \in E_j$, i.e. $f_j(x) \geq f_k(x)$ and 0 for all $k \neq j$. Since f is a concave function and both $f_j(\alpha)$ and $f_j(\beta)$ are greater than or equal to 0, $f_j(x) \geq 0$. To prove $f_j(x) \geq f_k(x)$ for all $k \neq j$, it suffices to show that there is no $\gamma_{jk} \in (\alpha, \beta)$. This can be shown simply using proof by contradiction: Suppose that there exists a $\gamma_{jk} \in (\alpha, \beta)$. Then, because of Remark 2.4, either $f_j(\alpha) > f_k(\alpha)$ and $f_j(\beta) < f_k(\beta)$, or $f_j(\alpha) < f_k(\alpha)$ and $f_j(\beta) > f_k(\beta)$. However, since both $\alpha_j, \beta_j \in E_j$, neither of these possibilities can hold. Thus, E_j must contain all the points between

α and β. Next, to show that E_j is bounded, we simply note that $E_j \subseteq [\gamma_{ij}, \gamma_{jk}]$ for any $i < j < k$, and therefore is bounded. Lastly, to show E_j is closed, we define $\{x_k\}$ to be a sequence in E_j such that $x_k \to x$. Thus, by definition of E_j, $f_j(x_k) = f(x_k)$. Since both $f_j(x)$ and $f(x)$ are continuous, letting $k \to \infty, f_j(x) = f(x)$. Therefore, if E_j contains at least two points, then E_j is a closed, bounded interval. □

In light of this, we need the following definition:

Definition 3.5. *In the case that E_j is an interval, we define α_j and β_j as the left and right end points of E_j, i.e. $[\alpha_j, \beta_j] = E_j$.*

Definition 3.6. *Let $\|E_j\|$ = the length of E_j, or $\beta_j - \alpha_j$.*

Before proceeding, we observe that we cannot obtain a unique solution for arbitrary vectors a, but only vectors a with only positive components. So, to study uniqueness, we need to restrict the domain of F to a set $\mathbb{G}$ defined as follows:

Definition 3.7. $\mathbb{G} = \{C_j : F(C_j) > 0 \text{ for } j = 1, \ldots, n\}$.

Corollary 3.8. *If $C \in \mathbb{G}$, then E_j is an interval for all $j = 1, \ldots, n$.*

Proof. First, by definition of $\mathbb{G}$, if $C \in \mathbb{G}$, then $F_j(C) > 0$ for all $j = 1, \ldots, n$. Further, by 2.9, it is clear that, in order for $F_j(C)$ to be greater than 0, then E_j cannot be empty or a singleton. Thus, by 3.4, if $F(C) \in \mathbb{G}$, then E_j must be an interval for all $j = 1, \ldots, n$. □

Lemma 3.9. *If $C \in \mathbb{G}$, α_j and β_j are a continuous function of C.*

Proof. As $f = \max(f_1, f_2, \ldots, f_n, 0)$, α_j is either $\gamma_{(j-1)j}$ if $f_j(\alpha_j) > 0$ or the leftmost root of $f_j(x)$. Likewise, β_j is either $\gamma_{j(j+1)}$ or the rightmost root of $f_j(x)$. We already have an equation for γ_{ij} in 3.2, so all that remains is to find when $f_j(x) = 0$, which we do as follows:

$$f_j(x) = 0 \Rightarrow C_j - (x - x_j)^2 = 0 \Rightarrow x = x_j \pm \sqrt{C_j}.$$

Thus, by definition of α_j and β_j and Definition 3.2, we have that, for $C \in \mathbb{G}$,

$$\alpha_j = \begin{cases} \dfrac{C_{j-1} - C_j}{2(x_j - x_{j-1})} + \dfrac{x_j + x_{j-1}}{2}, \text{if } f_j(\alpha_j) > 0 \\ x_j - \sqrt{C_j}, \text{ if } f_j(\alpha_j) = 0 \end{cases} \qquad \beta_j = \begin{cases} \dfrac{C_j - C_{j+1}}{2(x_{j+1} - x_j)} + \dfrac{x_{j+1} + x_j}{2}, \text{ if } f_j(\beta_j) > 0 \\ x_j + \sqrt{C_j}, \text{ if } f_j(\beta_j) = 0 \end{cases}$$

In either case, it is clear that α_j and β_j are continuous with respect to C. □

We notice something special if E_j is defined only at a point:

Lemma 3.10. *If $E_j = \{p\}$ with $f_j(p) > 0$, then $\exists\ i < j < k$ such that $f_i(p) = f_j(p) = f_k(p)$ and $\alpha_i \neq p, \beta_k \neq p$.*

Remark 3.11. *To motivate this Lemma, observe that, in Figure 3, E_3 is just a single point. We want to show that the only way this can happen, aside from the trivial case of $C_j = 0$, is in the exact manner demonstrated in Figure 3.*

Proof. Suppose, like has been said, that $E_j = \{p\}$. Further, suppose, without loss of generality, that $x_j < p$ (the case for $x_j > p$ is similar). We wish to show that $\exists\ i < j < k$ such that $f_i(p) = f_j(p) = f_k(p)$, or possibly that $f_k(p) = 0$, with the property that both E_i and E_k are intervals. To do so, pick a sequence $\xi_N \downarrow p$. We know that $\exists\ k_N \in \{1, \ldots, n\} - \{j\}$ such that $f_{k_N}(\xi_N) = f(\xi_N), k_N \neq j$ We can now extract the specific sub-sequence N_i (which we can do since we have an infinite sequence in a finite space) such that $k_{N_i} = k\ \forall i$. So, $\forall i$, we have that $f_k(\xi_{Ni}) = f(\xi_{N_i})$, and when

$i \to \infty$, we get that $f_k(p) = f(p) = f_j(p)$, which is the very property we seek. Fix a particular $\xi = \xi_{N1} > \gamma$. We wish to show that $f_k(x) = f(x)\ \forall x \in (p, \xi)$ To do so, we simply take the difference of F_k and F_j:

$$(C_k - (x - x_k)^2) - (C_j - (x - x_j)^2) = C_k - C_j + x_j^2 - x_k^2 + (x_k - x_j)x$$

Since this is a linear function with respect to x, and since we know this difference is 0 at p, it must hold that $f_k(x)$ is positive on (p, ξ_{N1}).

By the same reasoning, we could establish that there exists an $f_i(x) = f(x)$ to the left of p as well. Thus, every time that E_j is defined only at a single point p, there must exist two functions f_i and $f_k, i < j < k$, such that E_i and E_k are intervals and $f_i(p) = f_j(p) = f_k(p)$. □

3.2 Properties of $\mathbb{G}$

In this section, we want to show that $\mathbb{G}$ is non-empty and convex.

Lemma 3.12. $\mathbb{G}$ *is non-empty.*

Proof. To show $\mathbb{G}$ is non-empty, by 3.7, we need a C such that $F_j(C) > 0$ for all $j = 1, \ldots, n$. So, let $\epsilon > 0$. Choose ϵ small enough so that, if $C_j = \epsilon$ for all $j = 1, \ldots, n$, then $f_j(\alpha_j) = 0 = f_j(\beta_j)$ for all $j = 1, \ldots, n$, i.e. no f_j intersects any $f_i, i \neq j$ above the x-axis. Then $E_j = [x_j - \sqrt{\epsilon}, x_j + \sqrt{\epsilon}]$ for all $j = 1, \ldots, n$. Thus, we have (as will be calculated explicitly in 3.15)

$$F_j(C) = \frac{4}{3}\epsilon^{3/2}, \tag{3.2}$$

which is greater than 0. Thus, $\mathbb{G}$ is non-empty. □

Before showing convexity, we need to carefully consider what the relations between the intersection points of f_j's with their neighbors, i.e. the γ_{ij}'s, must be in order for C to be in $\mathbb{G}$. Well, if $C \in \mathbb{G}$, then E_j is an interval for all $j = 1, \ldots, n$. So, if $C \notin \mathbb{G}$, then one of the following inequalities must be false:

1. $\gamma_{j(j+1)} < \gamma_{(j-1)j}$
2. $\gamma_{(j-1)j} > x_j + \sqrt{C_j}$
3. $\gamma_{j(j+1)} < x_j - \sqrt{C_j}$

Based on these inequalities, we define a function

$$k_j(C) = \max\{\gamma_{j-1,j}(C), x_j - \sqrt{C_j}\} - \min\{\gamma_{j,j+1}(C), x_j + \sqrt{C_j}\}, \qquad (3.3)$$

so that, if $C \in \mathbb{G}$, then $k_j(C) < 0$. Using this, we will now prove convexity:

Lemma 3.13. *The set* $\mathbb{G} = \{C | k_j(C) < 0 \ \forall j\}$ *is convex.*

Proof. Let $C, \tilde{C} \in \mathbb{G}, t \in [0, 1]$. Then $k_j(C) < 0$ and $k_j(\tilde{C}) < 0$, so
$k_j(tC + (1-t)\tilde{C}) \leq k_j(C) + (1-t)k_j(\tilde{C}) < 0$. Thus, $tC + (1-t)\tilde{C} \in \mathbb{G} \ \forall j$. □

Thus, the set $\mathbb{G}$ is both non-empty and convex.

3.3 Computing $F_j(C)$ and its Derivatives

In this section, we want to calculate $F_j(C)$ and all of its derivatives for $C \in \mathbb{G}$. We have the following:

Theorem 3.14. $F_j(C) = C_j(\beta_j - \alpha_j) - \frac{1}{3}(\beta_j - x_j)^3 + \frac{1}{3}(\alpha_j - x_j)^3$.

Proof. We can compute this explicitly:

$$F_j(C) = \int_{\alpha_j}^{\beta_j} C_j - (x - x_j)^2 \,\mathrm{d}x = C_j(\beta_j - \alpha_j) - \frac{1}{3}(\beta_j - x_j)^3 + \frac{1}{3}(\alpha_j - x_j)^3 \tag{3.4}$$

□

Theorem 3.15. $\dfrac{\partial F_j(C)}{\partial C_j} = \|E_j\| + \dfrac{f_j(\beta_j)}{2(x_{j+1} - x_j)} + \dfrac{f_j(\alpha_j)}{2(x_j - x_{j-1})}$

Proof. With $F(C)$ now calculated, we can easily calculate $\dfrac{\partial F_j(C)}{\partial C_j}$, taking into account the different possibilities for α_j and β_j (see 3.2):

$$F_j(C) = \int_{\alpha_j}^{\beta_j} C_j - (x - x_j)^2 \,\mathrm{d}x = C_j(\beta_j - \alpha_j) - \frac{1}{3}(\beta_j - x_j)^3 + \frac{1}{3}(\alpha_j - x_j)^3$$

If $f_j(\alpha_j), f_j(\beta_j) > 0$,

$$= C_j \left(\frac{C_j - C_{j+1}}{2(x_{j+1} - x_j)} + \frac{x_{j+1} + x_j}{2} \right) - \frac{1}{3} \left(\frac{C_j - C_{j+1}}{2(x_{j+1} - x_j)} + \frac{x_{j+1} + x_j}{2} - x_j \right)^3$$

$$- C_j \left(\frac{C_{j-1} - C_j}{2(x_j - x_{j-1})} + \frac{x_j + x_{j-1}}{2} \right) + \frac{1}{3} \left(\frac{C_{j-1} - C_j}{2(x_j - x_{j-1})} + \frac{x_j + x_{j-1}}{2} - x_j \right)^3,$$

or if $f_j(\alpha_j), f_j(\beta_j) = 0$,

$$= \left(C_j(x_j + \sqrt{C_j}) - \frac{1}{3}(x_j + \sqrt{C_j} - x_j)^3 \right) - \left(C_j(x_j - \sqrt{C_j}) - \frac{1}{3}(x_j - \sqrt{C_j} - x_j)^3 \right) = \frac{4}{3} C_j^{3/2}.$$

The other two cases, if $f_j(\alpha_j) > 0, f_j(\beta_j) = 0$ and $f_j(\alpha_j) = 0, f_j(\beta_j) > 0$ are straightforward variations of two cases. They are explicitly calculated in the appendix 8.2.

We now differentiate these formulas:

$$
\begin{aligned}
&\text{If } f_j(\alpha_j), f_j(\beta_j) > 0,\\
\frac{\partial F_j(C)}{\partial C_j} =& \left(\frac{C_j - C_{j+1}}{2(x_{j+1}-x_j)} + \frac{x_{j+1}+x_j}{2}\right) + \left(\frac{C_j}{2(x_{j+1}-x_j)}\right) - \frac{1}{2(x_{j+1}-x_j)}\left(\frac{C_j - C_{j+1}}{2(x_{j+1}-x_j)} + \frac{x_{j+1}-x_j}{2}\right)^2\\
&- \left(\frac{C_{j-1} - C_j}{2(x_j-x_{j-1})} + \frac{x_j+x_{j-1}}{2}\right) + \left(\frac{C_i}{2(x_j-x_{j-1})}\right) - \frac{1}{2(x_j-x_{j-1})}\left(\frac{C_{j-1} - C_j}{2(x_j-x_{j-1})} - \frac{x_j-x_{j-1}}{2}\right)^2\\
=&\left\|E_j\right\| + \frac{f_j(\beta_j)}{2(x_{j+1}-x_j)} + \frac{f_j(\alpha_j)}{2(x_j-x_{j-1})},\\
&\text{or if } f_j(\alpha_j), f_j(\beta_j) = 0,\\
\frac{\partial F_j(C)}{\partial C_j} =& \ 2C_j.
\end{aligned}
\tag{3.5}
$$

Again, the other two cases are included in the appendix. In any case, we have:

$$\frac{\partial F_j(C)}{\partial C_j} = \left\|E_j\right\| + \frac{f_j(\beta_j)}{2(x_{j+1}-x_j)} + \frac{f_j(\alpha_j)}{2(x_j-x_{j-1})}. \tag{3.6}$$

Notice that this holds for every case, as if $f_j(\alpha_j), f_j(\beta_j) = 0$, then those components of the partial derivative would be 0, leaving $\left\|E_j\right\|$, which is in fact $2C_j$ in that case. □

Theorem 3.16. $\dfrac{\partial F_i(C)}{\partial C_j} = \dfrac{-f_i(\gamma_{ij})}{2(x_j - x_i)}$ *and* $\dfrac{\partial F_j(C)}{\partial C_i} = \dfrac{-f_j(\gamma_{ij})}{2(x_j - x_i)}$

Proof. We now seek to calculate all $\dfrac{\partial F_j(C)}{\partial C_i}$ for $i \neq j$. Without loss of generality, assume $i < j$. So, we only need to calculate $\dfrac{\partial F_j(C)}{\partial C_i}$ and $\dfrac{\partial F_i(C)}{\partial C_j}$. Further, if $f_j(x)$ does not touch $f_i(x)$ on or above the x-axis, then $\dfrac{\partial F_j(C)}{\partial C_i} = 0$, as slight changes in C_i won't change C_j. So we only consider the case if $f_j(\alpha_j), f_j(\beta_j) > 0$. We simply differentiate:

$$\frac{\partial F_i(C)}{\partial C_j} = \frac{1}{2(x_j - x_i)}\left(\left(\frac{C_i - C_j}{2(x_j - x_i)} + \frac{x_j - x_i}{2}\right)^2 - C_i\right) = \frac{-f_i(\gamma_{ij})}{2(x_j - x_i)} \tag{3.7}$$

$$\frac{\partial F_j(C)}{\partial C_i} = \frac{1}{2(x_j - x_i)}\left(\left(\frac{C_i - C_j}{2(x_j - x_i)} - \frac{x_j - x_i}{2}\right)^2 - C_j\right) = \frac{-f_j(\gamma_{ij})}{2(x_j - x_i)} = \frac{-f_i(\gamma_{ij})}{2(x_j - x_i)} \tag{3.8}$$

□

We notice the following:

Remark 3.17.

$$\frac{\partial F_i(C)}{\partial C_j} = \frac{\partial F_j(C)}{\partial C_i}$$

Additionally, we observe that $\dfrac{\partial F_i(C)}{\partial C_j}$ for any i and j depends on at most three components of C, as α_j and β_j each only depend on one C_i other than C_j itself. With this in mind, we can say:

Lemma 3.18. $\dfrac{\partial F_i(C)}{\partial C_j} \neq 0$ *for at most three components of* C*:* C_{j-1}, C_j*, and* C_{j+1}

3.4 Coercivity

Lemma 3.19. $F(C)$ *is coercive: i.e.* $\displaystyle\lim_{\|C\|\to\infty} \frac{F(C)\cdot C}{\|C\|} = \infty$

Definition 3.20. *Let* $\mathbb{R}^n_{\geq 0}$ *denote the set of all vectors* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *such that* $x_j \geq 0$ *for all* $j = 1 \ldots n$.

Remark 3.21. *Since* $F(C) : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$*, all norms are equivalent.*

Proof. To show that $F(C)$ is coercive, we need to show that $\lim_{\|C\| \to \infty} \frac{F(C) \cdot C}{\|C\|} = \infty$. First, choose a C_j such that $C_j = \max\{C_1, ..., C_n\} = \|C\|_\infty$. Then, we know that $F(C) \cdot C \geq F_j(C) \cdot C_j$. We now seek to explicitly compute $F_j(C)$. Note that E_j must be an interval. The smallest E_j could be is if $\beta_{j-1} = \alpha_j$ and $\beta_j = \alpha_{j+1}$. In this case, we can compute α_j, β_j, and $\beta_j - \alpha_j$:

$$\alpha_j = \frac{C_{j-1} - C_j}{2(x_j - x_{j-1})} + \frac{x_{j-1} + x_j}{2} \ , \ \beta_j = \frac{C_j - C_{j+1}}{2(x_{j+1} - x_j)} + \frac{x_j + x_{j+1}}{2}$$

$$\begin{aligned} \beta_j - \alpha_j &= \frac{C_j - C_{j+1}}{2(x_{j+1} - x_j)} + \frac{x_j + x_{j+1}}{2} - \frac{C_{j-1} - C_j}{2(x_j - x_{j-1})} - \frac{x_{j-1} + x_j}{2} \\ &= \frac{C_j - C_{j+1}}{2(x_{j+1} - x_j)} + \frac{C_j - C_{j-1}}{2(x_j - x_{j-1})} + \frac{x_{j+1} - x_{j-1}}{2} > \frac{x_{j+1} - x_{j-1}}{2} \end{aligned}$$

Thus,

$$F_j(C) \cdot C_j \geq C_j \int_{\alpha_j}^{\beta_j} C_j - (x - x_j)^2 \, \mathrm{d}x = {C_j}^2(\beta_j - \alpha_j) - C_j \left(\frac{\beta_j - x_j}{3} - \frac{\alpha_j - x_j}{3} \right) = (\beta_j - \alpha_j)({C_j}^2 - \frac{C_j}{3})$$

Finally, we have

$$\begin{aligned} &\lim_{\|C\| \to \infty} \frac{F(C) \cdot C}{\|C\|} \geq \lim_{\|C\|_\infty \to \infty} \frac{F_j(C) \cdot C_j}{\|C\|} \geq \lim_{\|C\|_\infty \to \infty} \frac{(\beta_j - \alpha_j)({C_j}^2 - \frac{C_j}{3})}{\|C\|_\infty} \\ &\geq \lim_{\|C\|_\infty \to \infty} \left(\frac{\min\left\{x_{j+1} - x_{j-1}\right\}}{2} \right) \left(\frac{({C_j}^2 - \frac{C_j}{3})}{\|C\|_\infty} \right) \\ &= \lim_{\|C\|_\infty \to \infty} \left(\frac{\min\left\{x_{j+1} - x_{j-1}\right\}}{2} \right) \left(\frac{(\|C\|_\infty{}^2 - \frac{\|C\|_\infty}{3})}{\|C\|_\infty} \right) \\ &= \left(\frac{\min\left\{x_{j+1} - x_{j-1}\right\}}{2} \right) - \frac{1}{3} + \infty = \infty. \end{aligned}$$

Thus, $F(C)$ is coercive. ∎

3.5 Continuity

With coercivity proved, we move to prove that $F(C)$ is continuous. Before doing so, we need to explore what happens in $E_j = \varnothing$.

Lemma 3.22. *Let $I \subseteq \{1, \ldots, n\}$ and $\mathcal{C}_I$ be the set of all $C \in \mathbb{R}^n_{\geq 0}$ such that $E_j(C) = \varnothing$ for all $j \in I$. Then $\mathcal{C}_I$ is open, i.e. for every $C \in \mathcal{C}_I$, there exists $\epsilon > 0$ such that, if $|\tilde{C} - C| < \epsilon$, then $\tilde{C} \in \mathcal{C}_I$.*

Proof. Let $C \in \mathcal{C}_I$. Define $\tilde{f}^{(C)}(x) = \max\{f_j^{(C)}(x) : j \in I\}$. Then $f^{(C)}(x) > \tilde{f}^{(C)}(x)$ for all x, because all of the sets $E_j(C)$, $j \in I$, are empty. Since all of the $f_j^{(C)}(x)$ are parabolas that go to $-\infty$ as $|x| \to \infty$, there exists $M > 0$ such that if $|x| > M$, then $f^{(C)}(x) - \tilde{f}^{(C)}(x) \geq 1$. Now $f^{(C)}(x) - \tilde{f}^{(C)}(x)$ is a continuous and positive function, hence it has a minimum $\delta > 0$ on the compact set $[-M, M]$. Without loss of generality, we can assume $\delta \leq 1$; in particular, we have $f^{(C)}(x) - \tilde{f}^{(C)}(x) \geq \delta$ for every x. Now assume $\left|\tilde{C}_j - C_j\right| < \delta/2$ for $j = 1, 2, \ldots, n$. Then $\left|f_j^{(\tilde{C})}(x) - f_j^{(C)}(x)\right| < \delta/2$ for every j and every x. It follows that $\left|f^{(\tilde{C})}(x) - f^{(C)}(x)\right| < \delta/2$ and $\left|\tilde{f}^{(\tilde{C})}(x) - \tilde{f}^{(C)}(x)\right| < \delta/2$, so

$$\begin{aligned} f^{(\tilde{C})}(x) - \tilde{f}^{(\tilde{C})}(x) &= f^{(\tilde{C})}(x) - f^{(C)}(x) + f^{(C)}(x) - \tilde{f}^{(C)}(x) + \tilde{f}^{(C)}(x) - \tilde{f}^{(\tilde{C})}(x) \\ &> -\delta/2 + \delta - \delta/2 = 0. \end{aligned} \tag{3.9}$$

It follows that all the sets $E_j(\tilde{C})$, $j \in I$, are empty, and so $\tilde{C}$ must also be in $\mathcal{C}_I$. Thus, $\mathcal{C}_I$ is open. □

With this in mind, we can now prove continuity:

Theorem 3.23. *$F(C)$ is continuous.*

Proof. By Lemma 3.5, we know that E_j is empty, a singleton, or is a closed, bounded interval. First, consider the case if $E_j = \varnothing$. Then $C \in \mathcal{C}_I$, as defined in Lemma 3.21. Take a sequence of vectors $C^{(k)} \to C$. Then, for k sufficiently large, $C^{(k)} \in \mathcal{C}_I$ also. So $F_j(C^{(k)}) = F_j(C) = 0$ for every $j \in I$. Therefore, all the $j \in I$ will not effect the continuity of $F(C)$. In light of this, we can relabel the sequence $\{1, \ldots, m\} = \{1, \ldots, n\} \setminus I$. Then $f(x)$ will still be the $\max\{f_1(x), f_2(x), \ldots, f_m(x), 0\}$, as those indices which may have been removed will not affect the max. Thus, we can assume for the remainder of this proof that $E_j(C) \neq \varnothing$, i.e. $j = 1, \ldots, m$.

Next, suppose that $E_j(C)$ is an interval. Choose a sequence of vectors $C^{(k)} \to C$. By equation (3.3), it is clear that, as $F_j(C)$ is simply a polynomial in terms of C, that $F_j(C^{(k)}) \to F_j(C)$. Thus, if $E_j(C)$ is an interval, $F_j(C)$ is continuous.

Lastly, suppose E_j is a single point. Then, by the contra-positive of Corollary 3.8, $F_j(C) = 0$. Again, we choose a sequence of vectors $C^{(k)} \to C$. So, we need to show $F_j(C^{(k)}) \to 0$. Define

$$k_j(C) = \max\{\gamma_{j-1,j}(C), x_j - \sqrt{C_j}\} - \min\{\gamma_{j,j+1}(C), x_j + \sqrt{C_j}\}. \tag{3.10}$$

Then $k_j(C)$ is a continuous function of C. Then $\|E_j(C)\|$ is at most $k_j(C)$. Further, since $k_j(C)$ is continuous, if $k_j(C) = 0$, then $E_j(C)$ must only be a single point. So, since $C^{(k)} \to C$, then $k_j(C^{(k)}) \to 0$ as $k \to \infty$. Therefore, whenever $E_j(C^{(k)})$ is an interval, $\|E_j(C)\| \to 0$ as $k \to \infty$, so $F_j(C^{(k)}) \to 0$.

Thus, since $F(C)$ is continuous in each of these 3 cases, then $F(C)$ is continuous. $\square$

We have now sufficiently explored various properties of $f(x)$ and $F(C)$. We proceed to our first goal:

4 Proving Existence of a Solution

In this chapter, we seek to show that a solution C to the equation $F(C) = a$ exists for all a. Before we begin the proof, we state Brouwer Fixed Point Theorem:

Lemma 4.1. *Every continuous function from a nonempty convex compact subset K of a Euclidean space to K itself has a fixed point.*

For the proof of this lemma, see Florenzano, 2003. We now begin the main result:

Theorem 4.2. *A solution C to the equation $F(C) = a$ exists for all a.*

Proof. Let $M_r = \overline{B(0,r)} \cap \mathbb{R}^n_{\geq 0}$, where $\overline{B(0,r)}$ is the closure of a ball of radius r centered at the origin. Then M_r is closed and bounded, so M_r is compact. Also, M_r is non-empty, as an n-dimensional vector whose components are all $\frac{r}{2}$ will be in M_r for any $r > 0$. Lastly, M_r is convex, as it is the intersection of two convex sets. Then, by 4.1, any function $g : M_r \to M_r$ has a fixed point.

In order to show that a solution to $F(C) = a$ exists, is suffices to show that $F(C) = 0$ has a solution. To see this, let $H(C) = F(C) - a$. Then, since $F(C)$ is coercive, $H(C)$ is coercive, as

$$\frac{H(C) \cdot C}{\|C\|} = \frac{F(C) \cdot C}{\|C\|} - \frac{C \cdot a}{\|C\|} \geq \frac{F(C) \cdot C}{\|C\|} - \frac{\|C\|\|a\|}{\|C\|} = \frac{F(C) \cdot C}{\|C\|} - \|a\| \tag{4.1}$$

So, if a solution to $F(C) = 0$ exists, then it follows that a solution to $F(C) = a$ will also exists for all $a \in \mathbb{R}^n_{\geq 0}$.

Let a be given. Define $L : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ as $L(C) := C - F(C)$. Then $L(C) \cdot C =$

$\|C\|^2 - F(C) \cdot C$. Since $F(C)$ is coercive, we can choose r large enough so that

$$L(C) \cdot C < \|C\|^2 \text{ whenever } \|C\| \geq r. \tag{4.2}$$

Now define a function

$$P(D) := \begin{cases} D, \|D\| \leq r \\ \frac{rD}{\|D\|}, \|D\| > r. \end{cases} \tag{4.3}$$

Finally, let $Q(C) := P(L(C))$. Then Q is continuous, as it is the composition of continuous functions, and it maps M_r to itself. Thus, by 4.1, Q has a fixed point $C \in M_r$. If $\|C\| = r$, then

$$r = \|C\| = \left\|Q(C)\right\| = \left\|P(L(C))\right\|. \tag{4.4}$$

By construction of P,

$$\frac{r}{\left\|L(C)\right\|} \leq 1 \tag{4.5}$$

But by 4.2,

$$\|C\|^2 = C \cdot C = Q(C) \cdot C = \frac{r}{\left\|L(C)\right\|} L(C) \cdot C < \frac{r}{\left\|L(C)\right\|} \|C\|^2 < \|C\|^2, \tag{4.6}$$

which is a contradiction, so $\|C\| < r$. Thus,

$$r > \|C\| = \left\|Q(C)\right\| = \left\|P(L(C))\right\|. \tag{4.7}$$

Hence, $P(L(C)) = L(C)$, so finally we have

$$C = Q(C) = P(L(C)) = L(C) = C - F(C). \tag{4.8}$$

Therefore,

$$F(C) = 0. \tag{4.9}$$

$\square$

5 Proving Uniqueness of a Solution

In this section, we seek to show that the C satisfying $F(C) = a$ is unique. Notice that, if one of the $a_j = 0$, then there will not be a unique solution. Thus, throughout this section, we assume $a_j > 0$, and so we will look for a solution $C \in \mathbb{G}$.

By 4.2, we know that a solution to $F(C) = a$ exists. All that remains is to show that the solution is unique. To do so, we need to show that $F(C)$ is strictly monotone.

Lemma 5.1. *On $\mathbb{G}$, $F(C)$ is strictly monotone, i.e. for any $C, \tilde{C} \in \mathbb{G}, (F(C) - F(\tilde{C}))(C - \tilde{C}) > 0$.*

Proof. To show F is strictly monotone, since $\mathbb{G}$ is convex, it suffices to show that $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial F_i(C)}{\partial C_j} \cdot (v_i v_j) > 0 \ \forall\ C, v \in \mathbb{R}^n$ with $v \neq 0$.

Assume for the sake of notation that $v_0 = 0 = v_{n+1}$. By Lemma 2.11

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial F_i(C)}{\partial C_j}v_iv_j = \sum_{j=1}^{n}\left(\frac{\partial F_j(C)}{\partial C_j}(v_j)^2 + \frac{\partial F_j(C)}{\partial C_{j-1}}(v_{j-1}v_j) + \frac{\partial F_j(C)}{\partial C_{j+1}}(v_jv_{j+1})\right)$$

Using the derivative formulas from 3.7,

$$= \sum_{j=1}^{n}\left(\|E_j\| + \frac{f_j(\alpha_j)}{2(x_j - x_{j-1})} + \frac{f_j(\beta_j)}{2(x_{j+1}-x_j)}\right)(v_j)^2$$
$$-\left(\frac{f_j(\alpha_j)}{2(x_j - x_{j-1})}\right)(v_{j-1}v_j) - \left(\frac{f_j(\beta_j)}{2(x_{j+1}-x_j)}\right)(v_jv_{j+1}).$$

Since $f_1(\alpha_1) = 0 = f_n(\beta_n)$,

$$= \sum_{j=1}^{n}\|E_j\|{v_j}^2 + \sum_{j=2}^{n}\left(\frac{f_j(\alpha_j)}{2(x_j - x_{j-1})}(v_j)^2 - \left(\frac{f_j(\alpha_j)}{2(x_j - x_{j-1})}\right)(v_{j-1}v_j)\right)$$
$$+\sum_{j=1}^{n-1}\left(\frac{f_j(\beta_j)}{2(x_{j+1}-x_j)}(v_j)^2 - \left(\frac{f_j(\beta_j)}{2(x_{(j+1)}-x_j)}\right)(v_jv_{j+1})\right).$$

Since $f_{j+1}(\alpha_{j+1}) = f_j(\beta_j)$ or 0,

$$= \sum_{j=1}^{n}\|E_j\|{v_j}^2 + \sum_{j=1}^{n-1}\frac{f_j(\beta_j)}{2(x_{j+1}-x_j)}{v_{j+1}}^2 - \frac{f_j(\beta_j)}{2(x_{j+1}-x_j)}(v_jv_{j+1}) + \frac{f_j(\beta_j)}{2(x_{j+1}-x_j)}{v_j}^2 - \frac{f_j(\beta_j)}{2(x_{j+1}-x_j)}(v_jv_{j+1})$$
$$= \sum_{j=1}^{n}\|E_j\|{v_j}^2 + \sum_{j=1}^{n-1}\frac{f_j(\beta_j)}{2(x_{j+1}-x_j)}(v_{j+1})^2 - \frac{2f_j(\beta_j)}{2(x_{j+1}-x_j)}(v_jv_{j+1}) + \frac{f_j(\beta_j)}{2(x_{j+1}-x_j)}(v_j)^2$$
$$= \sum_{j=1}^{n}\|E_j\|{v_j}^2 + \sum_{j=1}^{n-1}\left(\frac{f_j(\beta_j)}{2(x_{j+1}-x_j)}\right)(v_{j+1}-v_j)^2 > 0.$$

Thus, $F(C)$ is strictly monotone on $\mathbb{G}$. □

We now have everything we need to prove uniqueness.

Theorem 5.2. *The solution to the equation $F(C) = a$ has a unique solution.*

Proof. Suppose, for the sake of contradiction, that $C_1, C_2 \in \mathbb{G}$ with $C_1 \neq C_2$ are both solutions to $F(C) = a$. Then $F(C_1) = F(C_2)$. So $(F(C_1) - F(C_2)(C_1 - C_2) = 0$. But this contradicts 5.1. Thus, the solution to $F(C)$ is unique. $\square$

We have now shown that a unique solution to $F(C) = a$ exists and is unique. We now move onto the next section in which we seek to numerically approximate solutions to $F(C) = a$.

6 Approximating a Solution

Now that we have shown a unique solution exists to $F(C) = a$, we want to explore what the solution will look like. Though we cannot explicitly solve for an equilibrium solution, we can use a multidimensional variant of Newton's Method to approximate it. Newton's Method is an iterative algorithm which finds a root of a function $f(x)$ using an initial guess x_0 as to what the root might be. With these in hand, one then takes the initial guess x_0 and calculates where the tangent line to that point crosses the x axis, producing a new point x_1. This can be done with the simple following formula:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \tag{6.1}$$

This new point x_1 becomes the next guess and the process is then repeated indefinitely, using the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{6.2}$$

If all goes well, the sequence of points x_n will converge to the root of the function nearest the initial guess. However, in order to use Newton's Method to find the roots of $F(C) - a$, we need to modify the above formula to be n-dimensional. We can do

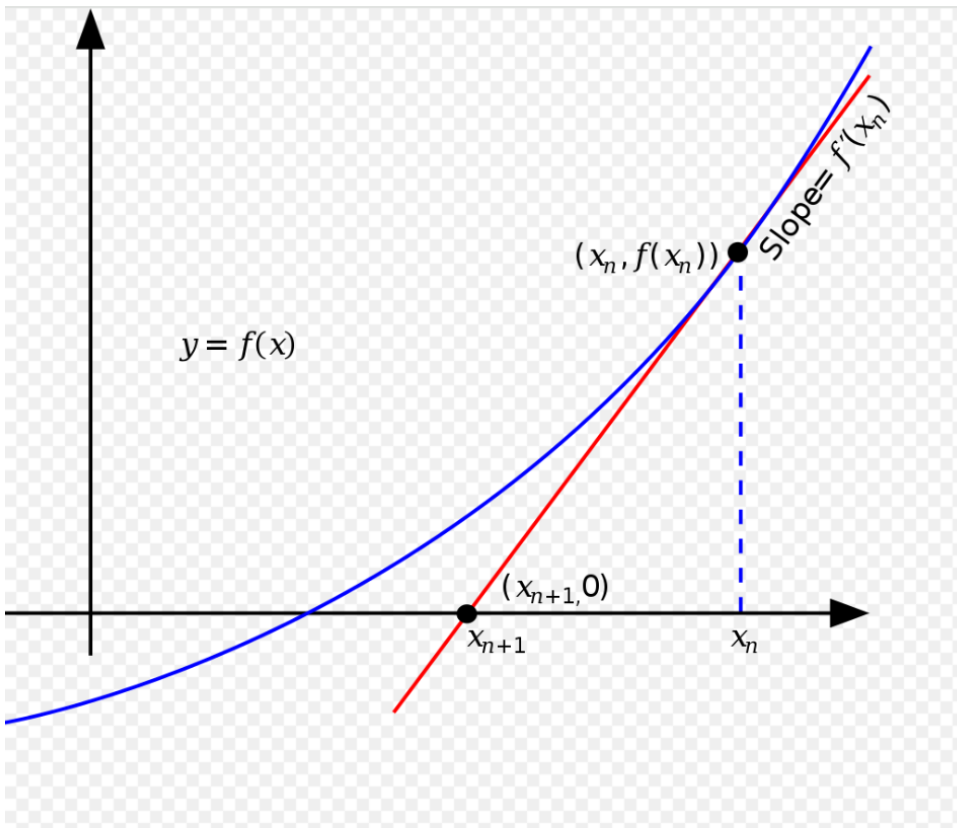

Figure 4: Newton's Method in Action

so quite simply by replacing the single dimensional values with n dimensional vectors and using the Jacobian matrix as the derivative. This gives us the following iterative formula:

$$\vec{x_{n+1}} = \vec{x_n} - J^{-1}(\vec{x_n})f(\vec{x_n}) \tag{6.3}$$

Using this iterative formula whose components can be expressly calculated from formulas aforementioned, I have written a code in Matlab which takes an initial guess and finds the roots of $F(C) - a$, which in turn finds the solution to $F(C) = a$. Below is first a visualization of Newton's Method, then several graphs produced by this program.

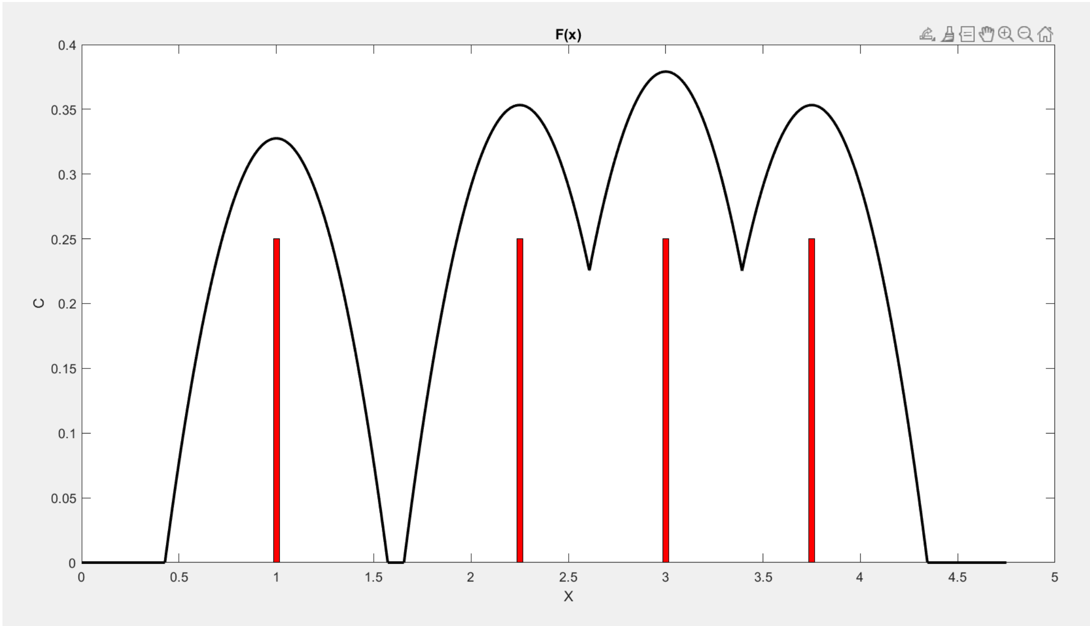

Figure 5: Example 1: $x = (1, 2.25, 3, 3.75), a = (0.25, 0.25, 0.25, 0.25)$

The graph above displays two components of the problem, the initial empirical measure, shown by the red bars, and the final probability density, represented by the black graph. In this graph, we can see a visual representation of the problem thoroughly explored in the preceding sections: the population which initially resides in varying proportions at specific locations after playing the game spreads out with a parabolic probability density. Notice how even though all the heights of the red bars, or the weights of the empirical measure, are equal, the heights of $F(x)$ differ. This difference results from the relative positions of the populations to each other: the leftmost portion is far enough away from rest that that quarter of the population can spread out without colliding with the others, whereas the other three are close enough to each other that more players will not move, resulting in a narrower probability density.

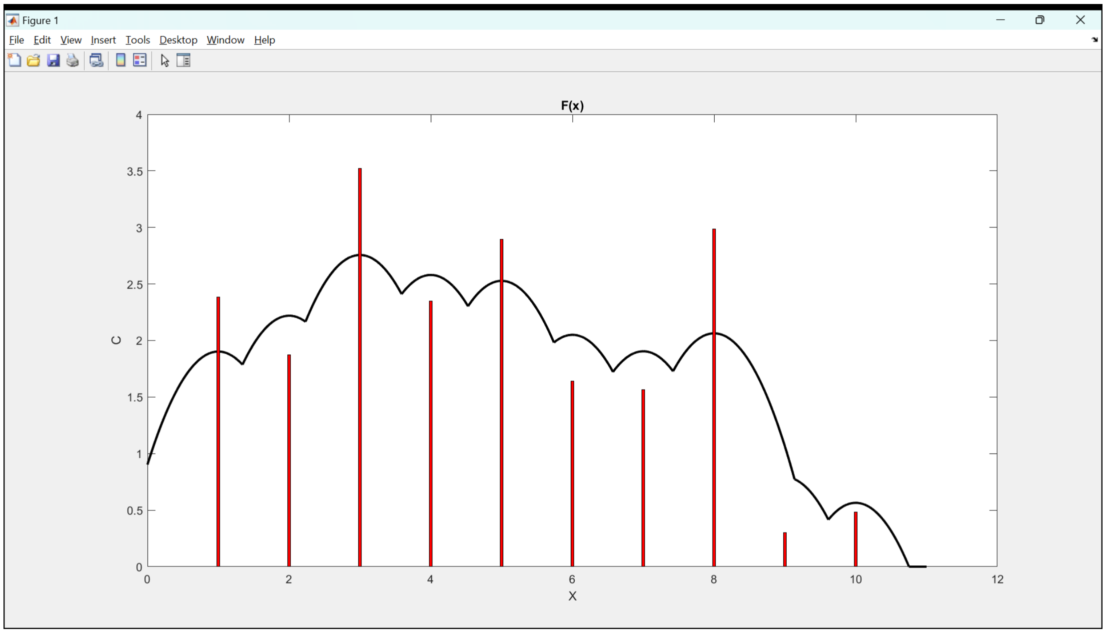

Figure 6: Example 2: $x = (1, 2, \ldots, 10), a = random$

In this sample, as opposed to the previous example, I let Δx be constant and let a be random. Note that the choice for a is not allowed in this problem as the sum of all the components of a do not add to one, but we still consider it for the sake of example. Notice how in this sample the probability density has a very interesting contour. In particular, the "bubble" corresponding to $x = 9$ at which point the function barely appears because it is crowded by neighboring bubbles. This demonstrates how the different weights change the shapes of their neighbors.

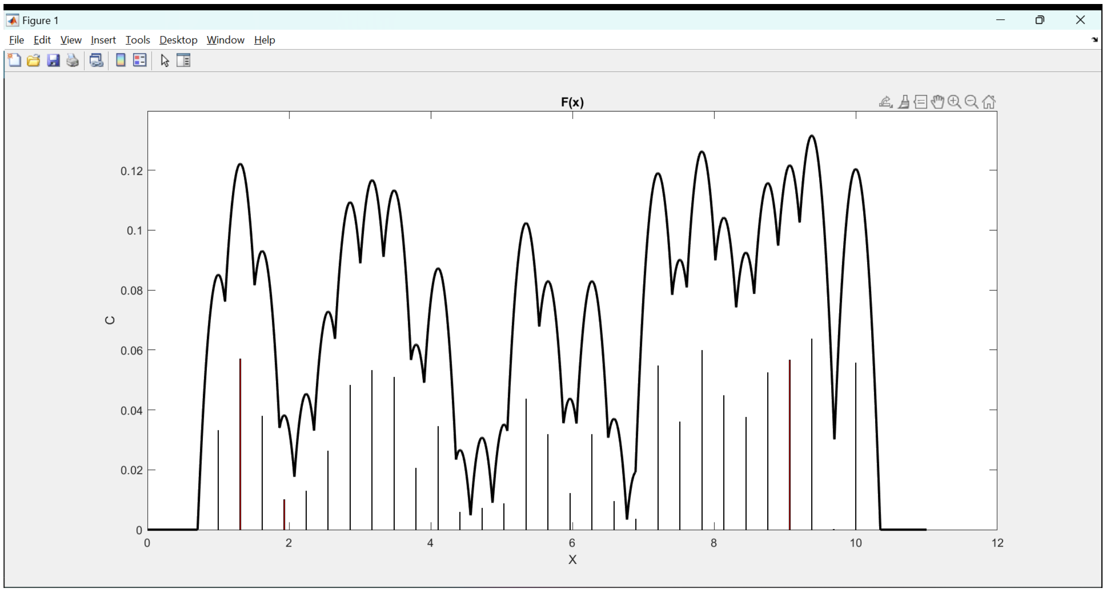

Figure 7: Example 3: $x = random$, $a = random$

This example is designed to mimic a realistic playing out of this game. The x and a are random, mimicking a data sample of some sort. The total area under $f(x)$ and the sum of all the components of a is one, as they are densities, which accurately models a real life situation such as traffic flow out of a building.

7 Conclusion

In conclusion, this thesis has addressed the fundamental question of establishing the existence and uniqueness of a Nash equilibrium within the context of density-dependent mean field games. Through a systematic exploration, we have merged concepts from mean field game theory and measure theory to construct a framework that encapsulates the dynamics of strategic interactions among a large number of agents whose behaviors are influenced by the density of the population.

Our approach involved defining a game structure where the ultimate cost function is intricately linked to the equilibrium density measure. Leveraging the Browder-Minty Theorem, we have rigorously demonstrated the existence of a unique solution to this game, underlining the stability and reliability of our theoretical model.

Furthermore, recognizing the practical limitations of analytical solutions, we have delved into the realm of numerical approximation. By employing Newton's method, we have showcased an effective strategy to approximate the Nash equilibrium, bridging the gap between theoretical insights and computational implementation.

As evidenced by the examples presented, our approach not only offers theoretical clarity but also practical utility in understanding and analyzing complex social and economic systems governed by density-dependent interactions. Moving forward, the insights gleaned from this thesis provide a solid foundation for further exploration and application of mean field game theory in diverse fields ranging from economics to biology and beyond. Through continued refinement and innovation in both theoretical frameworks and computational techniques, we stand poised to unlock new frontiers in understanding and navigating the dynamics of large-scale interactive systems.

8 Appendix

In this section, we list all the explicit formulas for E_j, $F(C)$, and $\dfrac{\partial F_j(C)}{\partial C_i}$ on the boundary of $\mathbb{G}$.

8.1 Formulas for E_j

We begin with the following definition:

Definition 8.1. *Let i^* be the index such that $\gamma_{i^*j} = \max\left\{\gamma_{ij} : i < j\right\}$ and an index k^* such that $\gamma_{jk^*} = \min\left\{\gamma_{jk} : k > j\right\}$. (If there are two or more choices for i^* or k^*, choose the one furthest from j).*

We will divide into different cases based on the sign of $f_j(\gamma_{i^*j})$ and $f_j(\gamma_{jk^*})$, so that we can explicitly compute formulas for E_j by reasoning where $f_j(x)$ will be a maximum in each case:

a.) $f_j(\gamma_{i^*j}) \geq 0$ and $f_j(\gamma_{jk^*}) \geq 0$.

- i.) If $\gamma_{i^*j} < \gamma_{jk^*}$, then $E_j = [\gamma_{i^*j}, \gamma_{jk^*}]$.
- ii.) If $\gamma_{i^*j} = \gamma_{jk^*}$, then $E_j = \left\{\gamma_{i^*j}\right\}$.
- iii.) If $\gamma_{i^*j} > \gamma_{jk^*}$, then $E_j = \varnothing$.

b.) $f_j(\gamma_{i^*j}) < 0$ and $f_j(\gamma_{jk^*}) \geq 0$

- i.) If $\gamma_{i^*j} < \gamma_{jk^*}$, then $E_j = [x_j - \sqrt{C_j}, \gamma_{jk^*}]$.
- ii.) If $\gamma_{i^*j} > \gamma_{jk^*}$, $E_j = \varnothing$

c.) $f_j(\gamma_{i^*j}) \geq 0$ and $f_j(\gamma_{jk^*}) < 0$.

i.) If $\gamma_{i^*j} > \gamma_{jk^*}$, then $E_j = \varnothing$

ii.) If $\gamma_{i^*j} < \gamma_{jk^*}$, then $E_j = [\gamma_{i^*j}, x_j + \sqrt{C_j}]$.

d.) $f_j(\gamma_{i^*j}) < 0$ and $f_j(\gamma_{jk^*}) < 0$

i.) If $\gamma_{i^*j} < x_j$ and $\gamma_{jk^*} > x_j$, then $E_j = [x_j - \sqrt{C_j}, x_j + \sqrt{C_j}]$.

ii.) If either $\gamma_{i^*j} > x_j$ or $\gamma_{jk^*} < x_j$, then $E_j = \varnothing$.

8.2 Formulas for $F(C)$ in the other two cases of E_j

$$F_j(C) = \int_{\alpha_j}^{\beta_j} C_j - (x - x_j)^2 \,\mathrm{d}x = C_j(\beta_j - \alpha_j) - \frac{1}{3}(\beta_j - x_j)^3 + \frac{1}{3}(\alpha_j - x_j)^3$$

If $f_j(\alpha_j) = 0, f_j(\beta_j) > 0$,

$$= C_j \left(\frac{C_j - C_{j+1}}{2(x_{j+1} - x_j)} + \frac{x_{j+1} + x_j}{2} \right) - \frac{1}{3} \left(\frac{C_j - C_{j+1}}{2(x_{j+1} - x_j)} + \frac{x_{j+1} + x_j}{2} - x_j \right)^3 - \frac{2}{3} C_j^{3/2}.$$

If $f_j(\alpha_j) > 0, f_j(\beta_j) = 0$,

$$= \frac{2}{3} C_j^{3/2} - C_j \left(\frac{C_{j-1} - C_j}{2(x_j - x_{j-1})} + \frac{x_j + x_{j-1}}{2} \right) + \frac{1}{3} \left(\frac{C_{j-1} - C_j}{2(x_j - x_{j-1})} + \frac{x_j + x_{j-1}}{2} - x_j \right)^3.$$

8.3 Formulas for $\dfrac{\partial F_j(C)}{\partial C_i}$ on the Boundary of $\mathbb{G}$

In this section, we will compute the partial derivatives of $F_j(C)$ the boundary of $\mathbb{G}$, i.e. when E_j. We can only compute only compute one-sided derivatives in this case. In order to do so, we first need to define the following:

Definition 8.2. *Let* $\dfrac{\partial F_j(C)}{\partial C_i^+}$ *be the right handed partial derivative and* $\dfrac{\partial F_j(C)}{\partial C_i^-}$ *be the left handed.*

Now, by subdividing all the different ways that some E_j can be a singleton (as explored in 8.1), we can begin computing the one-sided derivatives. We will assume throughout this section that there is a point γ such that $E_k = \{\gamma\}$ for one or more $k \in \{1, ..., n\}$.

CASE 1:

$\alpha_i < \gamma < \beta_j$

For this case to occur, it must be the case that $k \neq i, j$. Otherwise, E_k would be greater than just one point. We note the following: Increasing C_i or C_j by a sufficiently small amount will not result in f_k being the max anywhere, since, by Remark 2.4, if $x_k < \gamma$, then $f_i(x) > f_k(x)\ \forall x < \gamma$, and similarly if $x_k > \gamma$, then $f_j(x) > f_k(x)\ \forall x > \gamma$. Thus, we can use the same math as prior to assert:

$$\frac{\partial F_i(C)}{\partial C_j^+} = \frac{-f_i(\beta_i)}{2(x_j - x_i)} = \frac{\partial F_j}{\partial C_i^+}(C)$$

As for the left hand derivatives, we use a similar line of reasoning to conclude that a decrease in C_i will not change the interval on which $f_j(x)$ is a maximum, and vice versa. Thus,

$$\frac{\partial F_j(C)}{\partial C_i^-} = 0 = \frac{\partial F_i(C)}{\partial C_j^-}$$

We now proceed to the next possibility:

CASE 2:

$\alpha_i = \gamma < \beta_j$ Here we have that $k = i$. We first note the obvious:

$$\frac{\partial F_j(C)}{\partial C_i^-} = 0 = \frac{\partial F_i(C)}{\partial C_j^+}$$

We will need to use the definition of a derivative to evaluate the other two partials. For $\delta > 0$,

$$\frac{\partial F_j(C)}{\partial C_i^+} = \lim_{\delta \to 0} \frac{F_j(C_i + \delta) - F_j(C_i)}{\delta} = \lim_{\delta \to 0} -\frac{1}{\delta} \int_\gamma^{\gamma + \frac{\delta}{2(x_k - x_j)}} f_j(x)\,\mathrm{d}x$$

$$= \frac{-f_j(\gamma)}{2(x_k - x_j)}$$

and similarly,

$$\frac{\partial F_i(C)}{\partial C_j^-} = \lim_{\delta \to 0} \frac{F_i(C_j - \delta) - F_i(C_j)}{\delta} = \lim_{\delta \to 0} \frac{1}{\delta} \int_\gamma^{\gamma + \frac{\delta}{2(x_k - x_j)}} f_i(x)\,\mathrm{d}x$$

$$= \frac{f_i(\gamma)}{2(x_k - x_j)} = \frac{f_j(\gamma)}{2(x_k - x_j)}$$

CASE 3:

$\alpha_i < \gamma = \beta_j$ This will be analogous to case 2, but with $k = j$:

$$\frac{\partial F_i(C)}{\partial C_j^-} = 0 = \frac{\partial F_j(C)}{\partial C_i^-}$$

$$\frac{\partial F_i(C)}{\partial C_j^+} = \frac{-f_i(\gamma)}{2(x_k - x_j)}$$

$$\frac{\partial F_j(C)}{\partial C_i^-} = \frac{f_j(\gamma)}{2(x_k - x_j)} = \frac{f_i(\gamma)}{2(x_k - x_j)}$$

CASE 4:

$\alpha_i = \gamma = \beta_j$ In this case, $k = i, j$. Notice that for this case to occur, $f_m(x), m < i$, such that $f_m(x) > f_i(x) > f_j(x)$ for all $x < \gamma$, and likewise a $f_n(x), n > j$, such that

$f_n(x) > f_j(x) > f_i(x)$ for all $x > \gamma$. Thus, any change in C_i or C_j will not change $f(x)$. Therefore,

$$\frac{\partial F_i}{\partial C_j} = 0 = \frac{\partial F_j}{\partial C_i}.$$

Observe that, rather trivially, $F(C)$ is in fact differentiable in this case. All that remains is to compute $\dfrac{\partial F_j(C)}{\partial C_j^+}$ and $\dfrac{\partial F_j(C)}{\partial C_j^-}$. Obviously, if E_j is only a single point, then decreasing C_j will result in $E_j = \varnothing$. Thus,

$$\frac{\partial F_j(C)}{\partial C_j^-} = 0$$

Lastly, to compute $\dfrac{\partial F_j(C)}{\partial C_j^+}$, we need to again use the definition of a derivative. For $\delta > 0$,

$$\frac{\partial F_j}{\partial C_j^+} = \lim_{\delta \to 0} \frac{F_j(C_j + \delta) - F_j(C_j)}{\delta}$$

We know that $F_j(C_j) = 0$. This leaves us with the following:

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_a^b f_j(x)\, \mathrm{d}x,$$

where α and β are simply γ shifted:

$$a = \gamma - \frac{\delta}{2(x_j - x_i)} \text{ and} b = \gamma + \frac{\delta}{2(x_k - x_j)}$$

This yields the following:

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{\gamma - \frac{\delta}{2(x_j - x_i)}}^{\gamma + \frac{\delta}{2(x_k - x_j)}} f_j(x)\, \mathrm{d}x$$

We can apply the F.T.C. to this expression (twice), giving:

$$\frac{\partial F_j}{\partial C_j^+} = f_j(\gamma)\left(\frac{1}{2(x_k - x_j)} + \frac{1}{2(x_j - x_i)}\right).$$

We have now computed all the possible partial derivatives of $F(\vec{C})$. In summary, we have the following:

Lemma 8.3. *If $f(x)$ is not differentiable, then $\frac{\partial F_j}{\partial C_j^+} > 0$ and $\frac{\partial F_j}{\partial C_j^-} = 0$, and if $\alpha_i < \gamma < \beta_j$, $\frac{\partial F_i}{\partial C_j^\pm} + \frac{\partial F_j}{\partial C_i^\pm} = 0$, or otherwise, $\frac{\partial F_i}{\partial C_j^\pm} + \frac{\partial F_j}{\partial C_i^\mp} = 0$*

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