ABSTRACT

Adding Machines

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We explore the endpoint structure of the inverse limit space of unimodal maps such that the restriction of the map to the ω-limit set of the critical point is topologically conjugate to an adding machine. These maps fall into the infinitely renormalizable unimodal family or the family of strange adding machines. A unimodal map, f, is renormalizable if there exists a restrictive interval J in its domain of period n such that J contains the critical point and $f^n : J \to J$ is a unimodal map. If the renomalization process can be repeated infinitely often, we have an infinitely renormalizable map. Strange adding machines, however, are not renormalizable, and understanding the dynamical differences in these two families of maps is one of our goals. We give a characterization of the kneading sequence structure for these strange adding machines, and use this characterization to provide an example of a strange adding machine for which the set of folding points and the set of endpoints are not equal in the inverse limit space. We show that in the case of infinitely renormalizable maps, these two sets will always coincide. We extend our endpoint result by considering maps on finite graphs.

Adding Machines

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CHAPTER ONE

Introduction

Adding machines are well-understood dynamical systems defined as follows; see [13], [28].

Definition 1.1. Let $\alpha = (p_1, p_2, \dots)$, where each p_i is an integer greater than or equal to 2. Let Δ_α denote the set of all sequences $(x_1, x_2, \dots)$ where $x_i \in \{0, 1, \dots, p_i - 1\}$ for each i, with the product topology. Define addition on Δ_α as follows:

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (z_1, z_2, \dots)$$

where $z_1 = x_1 + y_1 \mod p_1$, $z_2 = x_2 + y_2 + t_1 \mod p_2$, $\dots, z_i = x_i + y_i + t_{i-1} \mod p_i, \dots$. We set $t_1 = 0$ if $x_1 + y_1 < p_1$ and $t_1 = 1$ if $x_1 + y_1 \geq p_1$, and $t_i = 0$ if $x_i + y_i + t_{i-1} < p_i$ and $t_i = 1$ if $x_i + y_i + t_{i-1} \geq p_i$. This is addition with carry. We define $f_\alpha : \Delta_\alpha \to \Delta_\alpha$ by

$$f_\alpha(x_1, x_2, \dots) = (x_1, x_2, \dots) + (1, 0, 0, \dots)$$

with the addition operation above, and refer to the map f_α as an adding machine map.

We are interested in unimodal maps with the property that the function restricted to the ω-limit set of the critical point is topologically conjugate to an adding machine. For many years, adding machine dynamics were assumed to be a hallmark of infinite renormalization [23, p.236]. An example of an infinitely renormalizable map is the Feigenbaum map; see p.10, p.39. Its restriction to the ω-limit set of the critical point is topologically conjugate to the dyadic adding machine. In 2005, Block, Keesling and Misiurewicz discovered the existence of adding machine maps in non-renormalizable unimodal maps, which include the tent family [13]. They named these maps strange adding machines. Our quest began as a search for the answer to the following question posed by James Keesling:

> Is it the case that for a unimodal map, f, with critical point, c, such that $f|_{\omega(c)}$ is topologically conjugate to an adding machine, the endpoints in the inverse limit space of $\{I, f\}$ will be equal to $\varprojlim\{\omega(c), f|_{\omega(c)}\}$?

The set $\varprojlim\{\omega(c), f|_{\omega(c)}\}$ is often called the set of *folding points* because at each of these points, the inverse limit space is not homeomorphic to the product of a zero-dimensional set and an arc [25]. We show that the conjecture is true for all infinitely renormalizable maps (Theorem 39), and give a counterexample in the non-renormalizable case, p.51. Henk Bruin asked if this counterexample provided a map for which the set of endpoints is not closed in the inverse limit space. We answered his question through not only this specific example but with a necessary condition for the set of endpoints to be equal to the set of folding points in the inverse limit space of a tent map (Theorem 3.1). In our investigation, we developed a bigger result than expected. We completely characterize the kneading sequence structure of these newly discovered strange adding machines; (Theorems 2.3 and 2.4). With this characterization, we can construct kneading sequences of strange adding machines and investigate their behavior. Following ideas discussed in [19], and conditions set forth in [17], we use our techniques to construct the kneading sequence of a strange adding machine for which $\omega(c)$ is a wild attractor: a metric attractor which is not a topological attractor [30]. Wild attractors were shown to exist in unimodal maps on the interval by Bruin, Keller, Nowicki, and van Strien [20]; see also [14], [15].

We conjecture:

> Given any sequence, $\alpha = (p_1, p_2, \ldots)$, where each p_i is an integer greater than or equal to two, there exists a non-renormalizable unimodal map, f, such that $f|_{\omega(c)}$ is topologically conjugate to the adding machine f_α.

The familiarity of the mathematical community with the Feigenbaum map prompted curiosity as to whether or not a dyadic strange adding machine existed. The dyadic kneading sequence is potentially the most difficult to write because of the limited size of the building blocks and our need to have enough changes in the structure to prevent a renomalizable map. We were able to construct one; the existence of which is encouraging with respect to the validity of our conjecture. Our current work is on tiling spaces [1], [9], [32]. Substitution tilings of constant length are essentially r-adic adding machines. The maximal equicontinuous factor of the Prouhet-Thue-Morse substitution, for example, is the dyadic adding machine [24, Chp. 5]. Marcy Barge posed the following question:

> Does every pure, primitive, aperiodic substitution of constant length have an asymptotic cycle?

This question has implications for the Pisot conjecture which has been open for some time [2], [8], [10]. We prove a special case of this conjecture (Theorem 5.2). We found a counterexample to his conjecture in the general case. We are investigating proximal cycles and their usefulness in place of asymptotic cycles, [5], [6]. In Chapter 2 we give a characterization of the kneading sequence structure for strange adding machines, and provide an example. In Chapter 3 we show that Keesling's conjecture holds for infinitely renormalizable maps. We also give a necessary condition for the set of endpoints to be equal to the set of folding points in the inverse limit space of a tent map on the interval. We extend our results on Keesling's conjecture in the infinitely renormalizable case to maps on graphs (Theorem 3.1). This requires an extension of a Barge and Martin result, [11, Theorem 2.9], that recurrence of zero or one for a unimodal map implies the existence of an endpoint in the inverse limit space, from the interval to finite graphs (Proposition 3.6). In Chapter 4 we give the counterexample to Keesling's conjecture, and answer Bruin's question about the existence of a unimodal map for which the set of endpoints in the inverse limit space is not closed. In Chapter 5 we give our results which support Barge's Conjecture in the special case of a substitution for which the cardinality of the alphabet and the column height are the same.

CHAPTER TWO

Kneading Sequence Structure of Strange Adding Machines

In this chapter we give the complete characterization of the kneading sequence structure of strange adding machines. The starting point for our research on strange adding machines is a study of infinitely renormalizable maps. The kneading sequence of the Feigenbaum map, see p.10, is discussed in [16]. This sequence can be written as an infinite concatenation of a finite collection of words of length 2^j for all $j \in \mathbb{N}$. In fact, for each $j \in \mathbb{N}$, there are exactly two words of length 2^j which comprise the kneading sequence, W and $\hat{W}$, where $\hat{W}$ agrees with W in every position but the last. One of these words appears in every $2^j n^{th}$ position of the kneading sequence for $n \geq 0$. There is a specific pattern to the concatenation which is created by the cyclic nature of the adding machine.

For the family of strange adding machines, it is natural to expect a similar kneading sequence structure. However, because strange adding machines are not renormalizable, the kneading sequence cannot admit a breakdown as above. We still have sets consisting of a finite collection of words whose concatenation comprise the kneading sequence, but our collections are more interesting in that each one must contain two words that disagree in a position other than their last. The difference in infinitely renormalizable maps with embedded adding machines and strange adding machines can be seen not only in the kneading sequence structure, but in the dynamics of the map restricted to the ω-limit set of the critical point. We consider both in the exploration that follows. The main results of this chapter are Theorem 2.3 and Theorem 2.4, which specify the requirements for a kneading sequence to be that of a strange adding machine.

We use symbolic dynamics and begin with some basic definitions, see [4], [16], [21], [26] and [33]. We use I to represent the unit interval.

Definition 2.1. A continuous map $f : I \to I$ is *unimodal* if there exists $c \in (0,1)$ such that f is strictly increasing on $[0,c]$ and strictly decreasing on $[c,1]$.

Definition 2.2. Let $a \in [1,2]$. The map $T_a : I \to I$ such that $T_a(x) = ax$ for $x \in [0, \frac{1}{2}]$ and $T_a(x) = a(1-x)$ for $x \in [\frac{1}{2}, 1]$ is the *symmetric tent map* with slope a.

Definition 2.3. [16, Definition 3.2.1] Let (E, ρ) be a compact metric space and $f : E \to E$ be continuous. For $x \in E$, we define the *ω-limit set* of x under the map f as: $\omega(x,f) = \omega(x) = \{y \in E \mid \text{there exists } n_1 < n_2 < \dots \text{ with } f^{n_i}(x) \to y\}$. We call x *recurrent* provided $x \in \omega(x)$.

Definition 2.4. A *finite word*, W, is an element of $\{0, *, 1\}^n$ for some $n \in \mathbb{N}$. An *infinite word*, W, is an element of $\{0, *, 1\}^\infty$.

Definition 2.5. Let $f : I \to I$ be unimodal with critical point c. The *itinerary* of a point $x \in I$ under f, which we label $\tilde{x}$, is a sequence $b_0 b_1 b_2 \dots$ where $b_i = 0$ if $f^i(x) < c$, $b_i = 1$ if $f^i(x) > c$, and $b_i = *$ if $f^i(x) = c$.

Definition 2.6. Let $f : I \to I$ be a unimodal map with critical point c. The *kneading sequence* of f is the itinerary of $f(c)$.

Definition 2.7. A sequence S is *renormalizable* if $S = Wb_0Wb_1Wb_2 \dots$ or $Wb_0Wb_1 \dots W*$ for some word W and sequence $B = b_0b_1b_2 \dots$ or $B = b_0b_1 \dots b_{n-1}$, such that W and B are nonempty strings of 0's and 1's.

Definition 2.8. Let $\tilde{x} = s_0s_1s_2 \dots$. Define $\sigma(\tilde{x}) = s_1s_2 \dots$, and call σ the shift map.

Definition 2.9. Let $f : I \to I$ be a unimodal map with critical point c. Let v and w be two elements of I, with itineraries $\tilde{v}$ and $\tilde{w}$, such that $v \neq w$. Let p be the first position where $\tilde{v}$ and $\tilde{w}$ disagree, with symbols v_p and w_p in position p of $\tilde{v}$ and $\tilde{w}$

respectively. Define the *parity-lexicographical ordering* as $0 < * < 1$ if the number of 1's preceeding v_p is even and $1 < * < 0$ if the number of 1's preceeding v_p is odd. If under the parity-lexicographical ordering, $v_p < w_p$, then $\tilde{v}$ is *below* $\tilde{w}$, and we write $\tilde{v} \prec \tilde{w}$.

Definition 2.10. We say that a sequence K is *shift-maximal* if $\sigma^n(K) \preceq K$ for all $n \in \mathbb{N}$.

If K is an infinite sequence such that K is shift-maximal, $101^\infty \preceq K$, and K is not renormalizable, then K is the kneading sequence of a tent map [21, Lemma 3.1.6].

Definition 2.11. Given a word $W = w_1 w_2 \dots w_n$ such that $w_n \neq *$, let $W - 1 = w_1 w_2 \dots w_{n-1}$, let $\hat{W} = w_1 w_2 \dots (1 - w_n)$, and let $|W|$ be n.

Lemmas 2.1 and 2.2 follow from [25, Lemma 2.2].

Lemma 2.1. *Let $f : I \to I$ be a tent map. Let $x \in I$ such that $f^n(x) \neq c$ for all $n \geq 0$, and let $\epsilon > 0$. Then there exists an initial segment of $\tilde{x}$, W, such that if $y \in I$ and $\tilde{y}$ begins with W, then $y \in B_\epsilon(x)$.*

Lemma 2.2. *Let $f : I \to I$ be a tent map. Let $x \in I$ such that $f^n(x) = c$ for some $n \geq 0$, and let $\epsilon > 0$. Then there exists an initial segment of $\tilde{x}$, $W * V$, such that if $y \in I$ and $\tilde{y}$ begins with $W0V$ or $W1V$, then $y \in B_\epsilon(x)$.*

Definition 2.12. We say a collection of sets $C = \{C_i\}_{i \in \mathbb{N}}$ *refines* a collection $D = \{D_i\}_{i \in \mathbb{N}}$ provided for every $j \in \mathbb{N}$, with $C_j \in C$, there is a $k_j \in \mathbb{N}$ with $D_{k_j} \in D$ such that D_{k_j} contains the closure of C_j.

Definition 2.13. Let $f : X \to X$ and $g : Y \to Y$ be given. We say that f and g are *topologically conjugate* provided there is a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$.

Theorem 2.1. *[13, Theorem 1.1] Let* $\alpha = (p_1, p_2, \dots)$ *be a sequence of integers with* $p_i \geq 2$ *for each* i. *Let* $j_i = p_1 \cdot p_2 \cdot \dots \cdot p_i$ *for each* i. *Let* $f : X \to X$ *be a continuous map of a compact metric space* X. *Then* f *is topologically conjugate to the adding machine* f_α *if and only if the following conditions hold.*

(1) For each positive integer i, *there exists a cover* $\mathcal{P}_i$ *of* X *consisting of* j_i *pairwise disjoint, nonempty, clopen sets which are cyclically permuted by* f.

(2) For each positive i, $\mathcal{P}_{i+1}$ *refines* $\mathcal{P}_i$.

(3) If $\mathrm{mesh}(\mathcal{P}_i)$ *denotes the maximum diameter of an element of the cover* $\mathcal{P}_i$, *then* $\mathrm{mesh}(\mathcal{P}_i) \to 0$ *as* $i \to \infty$.

Nb: It is known that f, as described in Theorem 2.1, is a homeomorphism on X.

Definition 2.14. The point $x \in X$ is said to be *regularly recurrent* if for every neighborhood V of x, there is a positive integer n such that for every non-negative integer k, $f^{kn}(x) \in V$.

Definition 2.14 in conjunction with Lemmas 2.1 and 2.2 establish Lemmas 2.3 and 2.4.

Lemma 2.3. *Let* $f : I \to I$ *be a tent map, and let* $x \in I$ *such that* $f^n(x) \neq c$ *for all* $n \geq 0$. *If for every initial word of* $\tilde{x}$, *there exists a positive integer* n *such that for every non-negative integer* k, $\sigma^{kn}(\tilde{x})$ *begins with* W, *then* x *is regularly recurrent.*

Lemma 2.4. *Let* $f : I \to I$ *be a tent map, and let* $x \in I$ *such that* $f^n(x) = c$ *for some* $n \geq 0$. *Let* $W * V$ *be an initial segment of* $\tilde{x}$. *If there exists a positive integer* n *such that for every non-negative integer* k, $\sigma^{kn}(\tilde{x})$ *begins with* $W0V$ *or* $W1V$, *then* x *is regularly recurrent.*

Definition 2.15. [16, Definition 3.5.7] Let $f : E \to E$ be a continuous map of a compact metric space. We say $F \subset E$ is *minimal* provided $F \neq \emptyset$, F is closed, $f(F) \subseteq F$, and no proper subset of F has these three properties.

Theorem 2.2. *[12, Corollary 2.5] Let $f : X \to X$ be a continuous map of a compact Hausdorff space to itself. There is a sequence α of prime numbers such that f is topologically conjugate to the adding machine map f_α if and only if X is an infinite minimal set for f and each point of X is regularly recurrent.*

We developed the following definition to describe the general structure of the kneading sequence of a unimodal map for which the restriction of the map to the ω-limit set of the critical point is topologically conjugate to an adding machine.

Definition 2.16. Let K be an infinite sequence of 0's and 1's. Suppose K admits the following decomposition into finite words:

(1) There exists a set of words $\{V_1^1, \dots, V_{t_1}^1\}$ such that

$$K = W_{1,1}^1 W_{1,2}^1 W_{1,3}^1 \dots W_{1,t_1}^1 \; W_{2,1}^1 W_{2,2}^1 W_{2,3}^1 \dots W_{2,t_1}^1 \dots W_{i,1}^1 W_{i,2}^1 W_{i,3}^1 \dots W_{i,t_1}^1 \dots$$

where $W_{1,j}^1 = V_j^1$, for $j \in \{1, \dots, t_1\}$, and $W_{k,j}^1 \in \{V_j^1, \hat{V}_j^1\}$ for $j \in \{1, \dots, t_1\}$, and $k > 1$.

(2) Let $m \in \mathbb{N}$. Suppose we have a collection of words $\{V_1^{m-1}, \dots, V_{t_{m-1}}^{m-1}\}$ and $\{W_{i,1}^{m-1}, W_{i,2}^{m-1}, \dots, W_{i,t_{m-1}}^{m-1}\}_{i=1}^\infty$ such that $W_{1,j}^{m-1} = V_j^{m-1}$, for $j \in \{1, \dots, t_{m-1}\}$, and $W_{k,j}^{m-1} \in \{V_j^{m-1}, \hat{V}_j^{m-1}\}$ for $j \in \{1, \dots, t_{m-1}\}$ and $k > 1$. There exists a set of words $\{V_1^m, \dots, V_{t_m}^m\}$, such that

$$V_1^m = W_{1,1}^{m-1} \dots W_{1,t_{m-1}}^{m-1}$$

$$V_2^m = W_{2,1}^{m-1} \dots W_{2,t_{m-1}}^{m-1}$$

$$\vdots$$

$$V_{t_m}^m = W_{t_m,1}^{m-1} \dots W_{t_m,t_{m-1}}^{m-1}$$

$$K = W^m_{1,1}W^m_{1,2}W^m_{1,3}\dots W^m_{1,t_m}W^m_{2,1}W^m_{2,2}W^m_{2,3}\dots W^m_{2,t_m}\dots W^m_{i,1}W^m_{i,2}W^m_{i,3}\dots W^m_{i,t_m}\dots$$

where $W^m_{1,j} = V^m_j$, for $j \in \{1, \dots, t_m\}$, and $W^m_{k,j} \in \{V^m_j, \hat{V}^m_j\}$ for $j \in \{1, \dots, t_m\}$, and $k > 1$.

We call such a collection of words a *building block scheme* and refer to the finite collection of words, $\{V^m_1, \dots, V^m_{t_m}\}$ as the level m *building blocks*.

Notice that if K has a building block scheme as above then in position $1 + (n-1)|V^r_1V^r_2\dots V^r_{t_r}|$ we will have V^r_1 or $\hat{V}^r_1$, for all $n \in \mathbb{N}$, which we label $W^r_{n,1}$. In position $1 + |V^r_1| + (n-1)|V^r_1V^r_2\dots V^r_{t_r}|$ we will have V^r_2 or $\hat{V}^r_2$, for all $n \in \mathbb{N}$, which we label $W^r_{n,2}$. In position $1 + |V^r_1V^r_2\dots V^r_{i-1}| + (n-1)|V^r_1V^r_2\dots V^r_{t_r}|$ we will have V^r_i or $\hat{V}^r_i$, for $i \leq t_r$, and all $n \in \mathbb{N}$, which we label $W^r_{n,i}$.

An example of a kneading sequence with a building block scheme is the infinitely renormalizable Feigenbaum map. We describe this map here. An initial segment of this sequence is 1011 1010 1011 1011 1011 1010 1011 1010 1011 1010 1011 1011 1011 1010 1011 1011. There are an infinite number of ways to define a building block scheme for this sequence. One such scheme is to let the word 1011 1010 be the only building block on each level. Another scheme is as follows: For level one, we set

$$V^1_1 = W^1_{1,1} = 1$$

$$V^1_2 = W^1_{1,2} = 0$$

For level two, we set

$$V^2_1 = W^2_{1,1} = W^1_{1,1}W^1_{1,2} = 10$$

$$V^2_2 = W^2_{1,2} = W^1_{1,1}\hat{W}^1_{1,2} = 11$$

For level three, we set

$$V^3_1 = W^3_{1,1} = W^2_{1,1}W^2_{1,2} = 1011$$

$$V^3_2 = W^3_{1,2} = W^2_{1,1}\hat{W}^2_{1,2} = 1010$$

And for level i, we set

$$V_1^i = W_{1,1}^i = W_{1,1}^{i-1} W_{1,2}^{i-1}$$

$$V_2^i = W_{1,2}^i = W_{1,1}^{i-1} \hat{W}_{1,2}^{i-1}$$

We are interested in adding machines embedded in tent maps which cannot have such simple kneading sequence structures. The following proposition and remark allow us to consider only tent maps with slope greater than $\sqrt{2}$.

Proposition 2.1. *[16, Proposition 3.4.26]*

(1) Each symmetric tent map T_a with $a \in (1, \sqrt{2}]$ is renormalizable.

(2) If $\sqrt{2} < a^m \leq 2$ for some $m \in \{2, 2^2, 2^3, \dots\}$, then the symmetric tent map T_a is m times renormalizable, $m < \infty$.

(3) A symmetric tent map T_a with $a \in (\sqrt{2}, 2]$ is not renormalizable.

[16, Remark 3.3.7] For each $b \in (1, \sqrt{2}]$ there exists a $n_b \in \{2, 4, 8, 16, \dots\}$, an interval J_b containing the critical point, and a unique $a \in [\sqrt{2}, 2]$ such that $T_b^{n_b}|J_b$ is topologically conjugate to T_a. We therefore restrict our attention to tent maps which are not renormalizable.

2.1 Main Theorems

Definition 2.17. [16, Definition 3.4.12] We say a unimodal map f with critical point c is *locally eventually onto* provided that for every $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that, if U is an inverval with $|U| > \epsilon$ and if $n \geq M$, then $f^n(U) = [f^2(c), f(c)]$.

Tent maps are locally eventually onto, which in conjunction with Theorem 2.1, (1), made the discovery of strange adding machines unexpected. Lemma 2.5 tells us that more than one element of each cover, as described in Theorem 2.1, of $\omega(c)$ will contain c in its convex hull. This means the convex hull of our sets will fold more than once in a cycle, compensating for the stretching inherent to the map.

Definition 2.18. Let $A \subseteq [0,1]$. The *convex hull of* A, $\mathrm{conv}(A)$, is given by $\mathrm{conv}(A) = [\inf(A), \sup(A)]$.

Lemma 2.5. *Let $\alpha = (p_1, p_2, \dots)$ be a sequence of integers with $p_i \geq 2$ for each i. Let $j_i = p_1 \cdot p_2 \cdot \dots \cdot p_i$ for each i. Let $f : I \to I$, be a tent map with critical point c, such that $f|_{\omega(c)}$ is topologically conjugate to the adding machine f_α. Let $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ be the covers of $\omega(c)$ described below.*

(1) For each positive integer i, there exists a cover $\mathcal{P}_i$ of $\omega(c)$ consisting of j_i pairwise disjoint, nonempty, clopen sets which are cyclically permuted by f.

(2) For each positive i, $\mathcal{P}_{i+1}$ refines $\mathcal{P}_i$.

(3) $\mathrm{mesh}(\mathcal{P}_i) \to 0$ *as $i \to \infty$.*

Then for every $j \in \mathbb{N}$, $\mathcal{P}_j$ has more than one element that contains c in its convex hull.

Proof. Choose $j \in \mathbb{N}$. Suppose that $\mathcal{P}_j$ has only one element containing c in its convex hull, and call this element P_c^j. By [13, Proposition 1.3], $c \in \omega(c)$, and therefore $c \in P_c^j$. By (1), P_c^j has a finite orbit, say $\{P_c^j, f(P_c^j), \dots, f^t(P_c^j)\}$, where $t = |\mathcal{P}_j| - 1$. This means that $f^{t+1}(P_c^j) = P_c^j$, and none of the elements of the set $\{f(P_c^j), \dots, f^t(P_c^j)\}$ contains c in its convex hull. It is also true that $f(c) = 1 \in f(P_c^j)$. So the first word of K of length t, which we will call W, is determined by the position of the elements of $\{f(P_c^j), \dots, f^t(P_c^j)\}$ relative to c, as all of the points for a particular element of the set will lie on the same side of c. The symbol of K in position $t+1$ will be determined by $f^t(1)$. All of the elements of P_c^j are cyclically permuted and follow the same path as before. So K will have the structure, $Wb_0Wb_1Wb_2\dots$, where b_i is determined by the position of $f^{t+(t+1)i}(1)$, $i \geq 0$. This means that we have a kneading sequence which is renormalizable contradicting the definition of f. □

We now give the first of our two main theorems, providing one direction of our strange adding machine characterization.

Theorem 2.3. *If $f : I \to I$ is a tent map, with critical point c, and there exists a sequence α of prime numbers such that $f|_{\omega(c)}$ is topologically conjugate to the adding machine f_α, then the kneading sequence of f, K, will have the following properties:*

(1) K has a building block scheme.

(2) For each level, the set of building blocks contains at least two elements which differ in a position other than the last.

Proof. We begin with the proof of (1).

Every cover in $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ has a finite number of elements and therefore a finite number of elements that contain c in their convex hulls. We will label the unique element of $\mathcal{P}_j$ that contains c, $P^j_{c_1}$. We choose $P^j_{c_2}$ to be the first element in the orbit of $P^j_{c_1}$ not equal to $P^j_{c_1}$ that contains c in its convex hull. We choose $P^j_{c_3}$ to be the first element in the orbit of $P^j_{c_2}$ not equal to $P^j_{c_2}$ that contains c in its convex hull. We continue this process, obtaining the set of elements $\{P^j_{c_1}, \ldots, P^j_{c_{t_j}}\}$. Note that the first element in the orbit of $P^j_{c_{t_j}}$ not equal to $P^j_{c_{t_j}}$ that contains c in its convex hull is $P^j_{c_1}$. We also choose integers $m_{j,i}$ such that $f^{m_{j,i}}(P^j_{c_i}) = P^j_{c_{i+1}}$ for $i \in \{1, \ldots, t_j - 1\}$, and $f^{m_{j,t_j}}(P^j_{c_{t_j}}) = P^j_{c_1}$.

Choose $j_1 = j$, minimal, so that $m_{j,i} > 2$ for $i \in \{1, \ldots, t_j\}$. This is possible by (3) of Theorem 2.1. Consider the elements $\{P^j_{c_1}, f(P^j_{c_1}), \ldots, f^{m_{j,1}-1}(P^j_{c_1})\}$ of the orbit of $P^j_{c_1}$. Note that $f^{m_{j,1}}(P^j_{c_1}) = P^j_{c_2}$, none of the elements of $\{f(P^j_{c_1}), \ldots, f^{m_{j,1}-1}(P^j_{c_1})\}$ contains c in its convex hull, and $1 \in f(P^j_{c_1})$. So the first word of K of length $m_{j,1}-1$, is determined by the position of the elements of $\{f(P^j_{c_1}), \ldots, f^{m_{j,1}-1}(P^j_{c_1})\}$ relative to c, as all of the points for a particular element of the set will lie on the same side of c. The symbol of K in position $m_{j,1}$ will be determined by $f^{m_{j,1}-1}(1)$. We will call this initial word of K of length $m_{j,1}$, V^1_1.

We now follow the orbit of $P^j_{c_2}$, as it contains $f^{m_{j,1}-1}(1)$. Note that $f^{m_{j,2}}(P^j_{c_2}) = P^j_{c_3}$, and none of the elements of $\{f(P^j_{c_2}), \ldots, f^{m_{j,2}-1}(P^j_{c_2})\}$ contains c in its convex hull. We are therefore able to determine the $m_{j,2} - 1$ symbols of K that follow V^1_1, by the position of the elements of $\{f(P^j_{c_2}), \ldots, f^{m_{j,2}-1}(P^j_{c_2})\}$. The symbol of K in position $m_{j,1} + m_{j,2}$ will be determined by $f^{m_{j,2}}(f^{m_{j,1}-1}(1))$. We will call this word of K of length $m_{j,2}$, V^1_2. We have now established that the first segment of K is $V^1_1 V^1_2$.

We continue this process for $\{P^j_{c_3}, \ldots, P^j_{c_{t_j}}\}$, naming the words determined by the paths of these elements, $\{V^1_3, \ldots, V^1_{t_j}\}$ respectively. The initial segment of K is $V^1_1 V^1_2 \ldots V^1_{t_j}$. We will also refer to this initial segment of K using the notation $W^1_{1,1} W^1_{1,2} \ldots W^1_{1,t_j}$.

$f^{m_{j,1}+m_{j,2}+\cdots+m_{j,t_j}}(P^j_{c_1}) = P^j_{c_1}$. Thus the word following $V^1_1 V^1_2 \ldots V^1_{t_j}$ as it appears as the initial segment of K will be either V^1_1 or $\hat{V}^1_1$, as the first $m_{j,1} - 1$ symbols will be $V^1_1 - 1$ as when we followed $P^j_{c_1}$ before, and the last symbol is determined by the position of $f^{m_{j,1}}(f^{m_{j,1}+m_{j,2}+\cdots+m_{j,t_j}-1}(1))$. We will call this element $W^1_{2,1}$. Continuing this process, we see that

$$K = W^1_{1,1} W^1_{1,2} W^1_{1,3} \ldots W^1_{1,t_j}\; W^1_{2,1} W^1_{2,2} W^1_{2,3} \ldots W^1_{2,t_j} \ldots W^1_{i,1} W^1_{i,2} W^1_{i,3} \ldots W^1_{i,t_j} \ldots$$

where $W^1_{1,n} = V^1_n$, for $n \in \{1, \ldots, t_j\}$, and $W^1_{k,n} \in \{V^1_n, \hat{V}^1_n\}$ for $n \in \{1, \ldots, t_j\}$, and $k > 1$.

For the m^{th} step in our process we assume that the covers $\{\mathcal{P}_{j_i}\}$ for $i < m$ have been chosen. We choose j_m, minimal, so that each element of $\mathcal{P}_{j_m}$ containing c in its convex hull will be contained in $P^{j_{m-1}}_{c_1}$. We will label the elements of $\mathcal{P}_{j_m}$ that lie in $P^{j_{m-1}}_{c_1}$, $\{P^{j_m}_{d_1}, \ldots, P^{j_m}_{d_{s_{j_m}}}\}$, ordered as before. The elements of the set $\{P^{j_m}_{d_1}, \ldots, P^{j_m}_{d_{s_{j_m}}}\}$ establish the level m building blocks of K.

$K = W^m_{1,1}W^m_{1,2}W^m_{1,3}\dots W^m_{1,s_{j_m}}\ W^m_{2,1}W^m_{2,2}W^m_{2,3}\dots W^m_{2,s_{j_m}}\dots W^m_{i,1}W^m_{i,2}W^m_{i,3}\dots W^m_{i,s_{j_m}}\dots$ where $W^m_{1,n} = V^m_n$, for $n \in \{1,\dots,s_{j_m}\}$, and $W^m_{k,n} \in \{V^m_n, \hat{V}^m_n\}$ for $n \in \{1,\dots,s_{j_m}\}$, and $k > 1$.

We see that the pattern of K satisfies the definition of a building block scheme.

To prove item (2), we revisit the logic of Lemma 2.5. If for some level, K has only one building block, W, or only the building blocks, W, $\hat{W}$, then $K = (W-1)b_0(W-1)b_1(W-1)b_2\dots$ which contradicts the assumptions that f is not renormalizable and c is not periodic.

□

We now establish another description of K which will be useful in the proofs that follow in addition to the building block structure, where K is as described in Theorem 2.3. We consider the level m building blocks of K, where $m > 1$. All of the building blocks of level m are the same length, say α. Recall that the elements of $\mathcal{P}_{j_m}$ that lie in $P^{j_{m-1}}_{c_1}$ are $\{P^{j_m}_{d_1},\dots,P^{j_m}_{d_{s_{j_m}}}\}$. The subset of this set for which c is contained in the convex hull of each element we will label $\{P^{j_m}_{c_1},\dots,P^{j_m}_{c_{t_{j_m}}}\}$, where $c \in P^{j_m}_{c_1} = P^{j_m}_{d_1}$. If there exists $k \in \{2,\dots,s_{j_m}\}$ such that $P^{j_m}_{d_k} \neq P^{j_m}_{c_j}$ for any $j \in \{2,\dots,t_{j_m}\}$, then $P^{j_m}_{d_k}$ does not contain c in its convex hull. Therefore, all of its elements lie below or all lie above c. This means that $W^m_{i,d_k-1} = V^m_{d_k-1}$ for every $i \in \mathbb{N}$.

On the other hand, the word established by the orbit of $P^{j_m}_{c_1}$ as it travels to $P^{j_m}_{c_2}$ will be $V^m_1V^m_2\dots V^m_{n_1}$ for some $1 \le n_1 < s_{j_m}$, and W^m_{i,n_1} may be $V^m_{n_1}$ or $\hat{V}^m_{n_1}$. The strict inequality follows from Lemma 2.5. The length of $V^m_1V^m_2\dots V^m_{n_1}$ is αn_1. We call these blocks established by $\{P^{j_m}_{c_1},\dots,P^{j_m}_{c_{t_{j_m}}}\}$ *level m change blocks*, and we label them Q^m_i. All of the blocks on level one are change blocks. There are t_{j_m} change blocks for level m. If the length of Q^m_i is αn_i, then $\alpha s_{j_m} = \alpha(n_1 + \dots + n_{t_{j_m}})$.

$$Q_1^m Q_2^m \dots Q_{t_{j_m}}^m = V_1^m V_2^m \dots V_{s_{j_m}}^m,\ t_{j_m} \leq s_{j_m}.$$

We may alternately express K in terms of the level m change blocks.

$$K = Q_1^m Q_2^m \dots Q_{t_{j_m}}^m \ Q_{2,1}^m Q_{2,2}^m \dots Q_{2,t_{j_m}}^m \dots Q_{i,1}^m Q_{i,2}^m \dots Q_{i,t_{j_m}}^m \dots$$

where $Q_{k,n}^m \in \{Q_n^m, \hat{Q}_n^m\}$ for $n \in \{1, \dots, t_{j_m}\}$, and $k > 1$. We will see in Lemma 2.7 that for all $i \in \{1, \dots, t_{j_m}\}$, $\hat{Q}_i^m$ will appear in position $1 + \alpha(n_1 + \dots + n_{i-1}) + t\alpha s_{j_m}$ for some $t \in \mathbb{N}$, and therefore it will appear infinitely often.

A more detailed description of the level m building block structure of K is as follows:

$$K = W_{1,1}^m W_{1,2}^m W_{1,3}^m \dots W_{1,s_{j_m}}^m \ W_{2,1}^m W_{2,2}^m W_{2,3}^m \dots W_{2,s_{j_m}}^m \dots W_{i,1}^m W_{i,2}^m W_{i,3}^m \dots W_{i,s_{j_m}}^m \dots$$

where $W_{1,n}^m = V_n^m$, for $n \in \{1, \dots, s_{j_m}\}$. If $P_{d_n}^{j_m} = P_{c_q}^{j_m}$ for some $n \in \{2, \dots, s_{j_m}\}$, $q \in \{2, \dots, t_{j_m}\}$, then $W_{k,n-1}^m \in \{V_{n-1}^m, \hat{V}_{n-1}^m\}$; otherwise, $W_{k,n-1}^m = V_{n-1}^m$, $k \geq 1$. Recall that $P_{d_1}^{j_m} = P_{c_1}^{j_m}$. By Lemma 2.6, which follows, $W_{k,s_{j_m}}^m \in \{V_{s_{j_m}}^m, \hat{V}_{s_{j_m}}^m\}$.

Definition 2.19. If Q_i^m is a level m change block of K, K has t_{j_m} change blocks on level m, and the last symbol of Q_i^m sits in position β with $\alpha = |Q_1^m \dots Q_{t_{j_m}}^m|$, then we call $\beta + n\alpha$ a *change position* of K on level m for $n \geq 0$.

The structure of K described in Theorem 2.3 tells us that if we choose a building block level, say p, then every building block of level $p+2$ will begin with V_1^{p+1} or $\hat{V}_1^{p+1}$. Since there are at least two distinct building blocks on each level, both V_1^{p+1} and $\hat{V}_1^{p+1}$ will begin with V_1^p. This means that every building block of level $p+2$ will begin with V_1^p. The same is true for level $p+2$ change blocks and the word Q_1^p.

Lemma 2.6. *Let $f : I \to I$ be a tent map with critical point c such that there exists a sequence α of prime numbers with $f|_{\omega(c)}$ topologically conjugate to the adding* *machine f_α. Let $\{\mathcal{P}_{j_i}\}_{i\in\mathbb{N}}$ be the covers of $\omega(c)$ chosen from $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ that satisfy the following:*

(1) Choose $j_1 \in \mathbb{N}$.

(2) For $m > 1$, j_m is minimal such that each element of $\mathcal{P}_{j_m}$ containing c in its convex hull will be contained in $P_{c_1}^{j_{m-1}}$, the unique element of $\mathcal{P}_{j_{m-1}}$ which contains c.

Then for every $i \in \mathbb{N}$, $P_{c_1}^{j_i}$ will contain elements on both sides of c.

Proof. Suppose there exists an $m \in \mathbb{N}$ such that $P_{c_1}^{j_m}$, the unique element of $\mathcal{P}_{j_m}$ that contains c, does not contain elements on both sides of c. Then all elements of $P_{c_1}^{j_m}$ other than c are either below c or above c, and for all $k \in \mathbb{N}$, $W_{k,s_{j_m}}^m = V_{s_{j_m}}^m$, the last building block on level j_m. So for level $m+1$ all of the building blocks $\{V_1^{m+1}, \ldots, V_{s_{j_{m+1}}}^{m+1}\}$ will end in $V_{s_{j_m}}^m$, and therefore $W_{k,t}^{m+1} = V_t^{m+1}$, for $t \in \{1 \ldots s_{j_{m+1}}\}$ and all $k \in \mathbb{N}$. This forces level $m+2$ to have only one building block contradicting Theorem 2.3, (2).

□

We now use Lemma 2.6 and the adding machine dynamics outlined in Theorem 2.1 to provide additional information about the kneading sequence structure of a strange adding machine.

Lemma 2.7. *Let $f : I \to I$ be a tent map with critical point c such that there exists a sequence α of prime numbers with $f|_{\omega(c)}$ topologically conjugate to the adding machine f_α. Let K be the kneading sequence of f. If Q_k^m is a change block of K, then $\hat{Q}_k^m$ will appear in position $1 + |Q_1^m Q_2^m \ldots Q_{k-1}^m| + t|Q_1^m Q_2^m \ldots Q_{t_{j_m}}^m|$ for some $t \in \mathbb{N}$, where t_{j_m} is the total number of change blocks on level m.*

Proof. Let $j_m \in \mathbb{N}$ such that $\mathcal{P}_{j_m} \in \{\mathcal{P}_i\}_{i\in N}$, and j_m is chosen as in Lemma 2.6. Let the elements of $\mathcal{P}_{j_m}$ that contain points on both sides of c be $\{P_{c_1}^{j_m}, \ldots, P_{c_{t_{j_m}}}^{j_m}\}$, where

$c \in P_{c_1}^{j_m}$. The change words associated with these elements are $\{Q_1^m, \ldots, Q_{t_{j_m}}^m\}$. Let $n \in \{2, \ldots, t_{j_m}\}$. Consider $P_{c_n}^{j_m}$. Let e_l be the infimum of $P_{c_n}^{j_m}$, and e_r the supremum.

Let $\alpha = |Q_1^m \ldots Q_{t_{j_m}}^m|$ and $\beta = |Q_1^m \ldots Q_{n-1}^m|$.

Since the elements of the cover $\mathcal{P}_{j_m}$ are cyclically permuted, $f^{\beta+k\alpha}(P_{c_1}^{j_m}) = P_{c_n}^{j_m}$ for all $k \geq 0$.

There exists a refinement of $\mathcal{P}_{j_m}$, $\mathcal{P}_{j_s}$, where j_s is chosen as in Lemma 2.6 and e_l, e_r are in different elements of $\mathcal{P}_{j_s}$. Such a refinement exists by (3) of Theorem 2.1. Let $P_{e_l}^{j_s}$ and $P_{e_r}^{j_s}$ be those elements respectively, and let $c \in P_{c_1}^{j_s}$. Then $f^{v_l}(P_{c_1}^{j_s}) = P_{e_l}^{j_s}$ and $f^{v_r}(P_{c_1}^{j_s}) = P_{e_r}^{j_s}$ where $v_l = \beta + k_1\alpha$, $v_r = \beta + k_2\alpha$ for some $k_1, k_2 \geq 0$ such that $k_1 \neq k_2$. This means that Q_{n-1}^m and $\hat{Q}_{n-1}^m$ will appear in K.

For the case that $n = 1$, we let $\beta = 0$, and $n-1$ be t_{j_m}. That $Q_{c_1}^m$ has elements e_l and e_r such that $e_l \neq e_r \neq c$ is shown in Lemma 2.6.

$\square$

If W is an initial word of K, and the word $\hat{W}$ appears somewhere in K, then there exists an $n \in \mathbb{N}$ such that $\sigma^n(K)$ begins with $\hat{W}$. The requirement that a kneading sequence, K, be shift maximal tells us that $\sigma^n(K)$ must be below K with respect to the parity-lexicographical ordering. Consequently, by Lemma 2.7, the change blocks of K which are initial words of K, $\{Q_1^i\}_{i\in\mathbb{N}}$, have odd parity; see Lemma 2.15.

We now prove the converse of Theorem 2.3, and complete our characterization of the kneading sequence structure for strange adding machines. We begin with a fact about finite words.

Proposition 2.2. *If we have a word, W, such that $W = V^k$, the k-fold concatenation of V, $k > 1$, and V is the shortest such word, then V only begins in positions $1+t|V|$, $t \in \mathbb{N}$, with 1 being the first position of W.*

Proof. Suppose V begins in a position other than $1 + t|V|$ in V^k. We will call this position s, where s is minimal. There exists a word, V', that appears in positions 1 through $s-1$, $|V'| < |V|$. Since V begins in positions 1 and s, there is a copy of V' in positions s through $2(s-1)$. If $|V| > 2|V'|$, then V begins with $V'V'$. We may continue this process until we have written V in the form $V'^a B$, $|B| < |V'|$, $a \geq 1$. Since V begins in position s and $W = V^k$, where $k > 1$, the last $s-1$ symbols of this second occurrence of V in W will overlap the first $s-1$ symbols of the occurrence of V beginning in position $|V|+1$. This means that the last word of V is the first $s-1$ symbols of V, which is V'. In other words, the copy of V beginning in position s is shifted forward in the string V^k by $s-1$ symbols and V therefore begins and ends with the word V'. Using the technique above, we see that if $|V| > 2|V'|$, V will end in two copies of V'. Continuing this process we have $V = AV'^a$ where $|A| = |B|$. So $V = AV'^a = V'^a B$. Since $|A| < |V'|$, $V' = AD$ where D is not the empty word. Consider the first V in W. $V = V'^a B = V'V'^{(a-1)}B$. (If $a = 1$, then $V'^{(a-1)}$ is the empty word.) Beginning in position $s = |V'| + 1$ we have a copy of $V = V'^a B$ which has $V'^{(a-1)}B$ in positions s through $|V|$. So $V = V'^a B = V'^{(a-1)}BV'$ since a new copy of V begins in position $|V|+1$ with initial word V'. Since $|B| < |V'|$, $V' = BE$ for some word E. Further, $V' = AD = BE$ with $|A| = |B|$ giving us $A = B$. Now since $V = AV'^a$, the word in position $|A|+1$ through $|V|$ is V'^a. The word in position $|V| + 1$ through $|V| + |A|$ is A since it is the beginning word of V. So starting in position $|A| + 1$ we have $V'^a A = V'^a B = V$. Recall that $|A| < |V'| = s-1$, making $|A| + 1 \leq s - 1$ which contradicts our choice of s as the minimal position where V begins prior to $|V| + 1$ in W.

□

In the lemmas that follow we prove that for a tent map, f, with critical point, c, and kneading sequence, K, satisfying the conditions of Theorem 2.3, $\omega(c)$ will be infinite and minimal and all of its points will be regularly recurrent. The map, f, will therefore satisfy the conditions of Theorem 2.2, making it topologically conjugate to the adding machine, f_α, for some α.

Lemma 2.8. *Let $f : I \to I$ be a tent map with critical point c, and kneading sequence K. Suppose K has the following properties:*

(1) K has a building block scheme.

(2) For each level, the set of building blocks contains at least two elements that differ in a position other than the last.

Then $\omega(c)$ is infinite.

Proof. Since by construction, every initial segment of K repeats, K is not pre-periodic. We assume that K is periodic for the purpose of contradiction. Then there exists W, minimal, such that $K = W^\infty$. Let B be a building block of K on some level greater than 1, say z, such that B is an initial subword of K, and $|B| > |W|$. This implies that B and W will agree on $|W|$ symbols. For all $n \in \mathbb{N}$, $\sigma^{n|B||W|}(K)$ will begin with B and W. For a fixed number, say k, the initial string of K of length $k|B||W|$ will be $W^{k|B|}$. This same string may be written as $B_1 B_2 \dots B_{k|W|}$, where for each $i \in \{1, \dots, k|W|\}$, B_i is either a building block of level z or a building block with the last symbol changed. All level $z + 2$ building blocks begin with B, and have the same length. Suppose that in K, B only begins in positions $1 + n|W|$ for $n \geq 0$. Then each level $z + 2$ building block must begin in position $1 + n|W|$ for some $n \geq 0$. All level $z + 2$ building blocks are the same length. Therefore, every building block of level $z+2$ may be written as W^j for some $j \in \mathbb{N}$. This contradicts (2). So B begins in a position other than $1 + n|W|$, $n \in \mathbb{N}$. As W is a subword of B, it must also begin in a position of K other than $1 + n|W|$. This contradicts the minimality of W by Proposition 2.2.

□

Lemma 2.9. *Let $f : I \to I$ be a tent map with critical point c, and kneading sequence K. Suppose K has the following properties:*

(1) K has a building block scheme.

(2) For each level, the set of building blocks contains at least two elements that differ in a position other than the last.

Then $\omega(c)$ is minimal.

Proof. Let $x \in \omega(c)$. Then $\omega(x) \subseteq \omega(c)$. Now let $y \in \omega(c)$. Every initial segment of $\tilde{y}$ appears in K infinitely often. Let W be an initial segment of $\tilde{y}$. Let $p \in \mathbb{N}$ be minimal such that $\sigma^p(K)$ begins with W. Then there exists an N such that $\sigma^{kN+p}(K)$ begins with W for every $k \geq 0$. Such an N exists since this occurrence of W must occur as a subword of V_1^i for some i as V_1^i appears as an initial segment of K. V_1^i is the initial subword of all building blocks on level $i+2$. So $\sigma^{k|V_1^{i+2}|+p}(K)$ begins with W, although $|V_1^{i+2}|$ may not be minimal.

Since $x \in \omega(c)$, there exists $t_1 > p$ such that $\sigma^{t_1}(K)$ and $\tilde{x}$ agree on $3N$ initial symbols. This implies that W is a subword of $\tilde{x}$. To see that W appears infinitely often in $\tilde{x}$, we assume that there exists a position of $\tilde{x}$, say n, after which W no longer appears. But there exists a $t_2 > p$, such that $\sigma^{t_2}(K)$ and $\tilde{x}$ agree on $n + 3N$ initial symbols. This is a contradiction. As our choice of W was arbitrary, we have shown that every initial segment of $\tilde{y}$ appears in $\tilde{x}$ infinitely often. Hence $y \in \omega(x)$, and $\omega(c) \subseteq \omega(x)$. □

Definition 2.20. Let R be a word of a kneading sequence K. Let $p_1 \in \mathbb{N}$ be minimal such that $\sigma^{p_1}(K)$ has initial segment R. We define q to be a *regular return time* of R in K associated with p_1, provided $\sigma^{p_1+kq}(K)$ begins with R for all $k \in \mathbb{N}$. Let C_{R,p_1} be the least q for which this holds and call the ordered pair (p_1, C_{R,p_1}) a *cycle* of R. Let $p_i \in \mathbb{N}$ be minimal such that $\sigma^{p_i}(K)$ has initial segment R, $p_i \neq p_l + kC_{R,p_l}$ for any $k \in \mathbb{N}$, $l < i$. The *cycle* (p_i, C_{R,p_i}) of R is defined such that $\sigma^{p_i + kC_{R,p_i}}(K)$ begins with R for all $k \in \mathbb{N}$, and C_{R,p_i} is minimal.

Definition 2.21. Let $R = W * V$ be a word such that either $R_0 = W0V$ or $R_1 = W1V$ occur infinitely often in K. Let $p_1 \in \mathbb{N}$ be minimal such that $\sigma^{p_1}(K)$ has either R_0 or R_1 as its initial segment. We define q to be a *regular return time* of R in K associated with p_1, provided $\sigma^{p_1+kq}(K)$ begins with R_0 or R_1 for all $k \in \mathbb{N}$. Let C_{R,p_1} be the least q for which this holds, and call the ordered pair (p_1, C_{R,p_1}) a *cycle* of R. Let $p_i \in \mathbb{N}$ be minimal such that $\sigma^{p_i}(K)$ has either R_0 or R_1 as its initial segment, $p_i \neq p_l + kC_{R,p_l}$ for any $k \in \mathbb{N}$, $l < i$. The *cycle* (p_i, C_{R,p_i}) of R is defined such that $\sigma^{p_i+kC_{R,p_i}}(K)$ has initial segment R_0 or R_1 for all $k \in \mathbb{N}$, and C_{R,p_i} is minimal.

Lemma 2.10. *Let K be the kneading sequence of a tent map such that K has a building block scheme. If W is a subword of K, and m is a building block level of K for which there does not exist a change position, q, of level m, and $v \in \mathbb{N}$, such that $\sigma^v(K)$ begins with W and $v < q \leq v + |W|$, then W will have a finite number of cycles.*

If a word, W, satisfies the hypothesis, we say that W does not sit across a change position on level m.

Proof. We first write K in terms of its level m change blocks.

$$K = Q_1^m Q_2^m \dots Q_{t_{jm}}^m \; Q_{2,1}^m Q_{2,2}^m \dots Q_{2,t_{jm}}^m \dots Q_{i,1}^m Q_{i,2}^m \dots Q_{i,t_{jm}}^m \dots$$

where $Q_{k,n}^m \in \{Q_n^m, \hat{Q}_n^m\}$ for $n \in \{1, \dots, t_{jm}\}$, and $k > 1$.

Since W does not sit across a change position on level m, W is a subword of one of the elements of $\{Q_i^m - 1\}_{i \in \{1,\dots,t_{jm}\}}$. Each of these elements occurs with regularity $|Q_1^m Q_2^m \dots Q_{t_{jm}}^m|$ in K, which we will call α. This means that α is an upper bound on elements in the set $\{C_{W,p_i}\}$. Since there are a finite number of words in $\{Q_i^m - 1\}_{i\in\{1,\ldots,t_{jm}\}}$, W will have a finite number of cycles.

□

Proposition 2.3. *Let $f : I \to I$ be a tent map with critical point c and kneading sequence K. Let $x \in \omega(c)$, and let W be an initial word of $\tilde{x}$. If $W = W' * V$, then let $W_0 = W'0V$ and $W_1 = W'1V$. If W contains no $*$, then $W_0 = W = W_1$. If $\{C_{W,p_i}\}$, the set of regular return times of W in K, is finite, then there exists an M such that $\sigma^{kM}(\tilde{x})$ begins with W_0 or W_1 for all non-negative integers k.*

Proof. Let $x \in \omega(c)$ and let W be an initial word of $\tilde{x}$. There exists a sequence $\{y_k\}$ such that $y_k \to x$ and $y_k = f^{n_k}(c)$ for some $n_k \in \mathbb{N}$. Then there exists an N such that for $j > N$, W_0 or W_1 is an initial word of $\tilde{y}_j$. By the hypothesis, $\{C_{W,p_i}\}$ is finite and therefore has a least common multiple which we will call r. Then for $j > N$, $\sigma^{tr}(\tilde{y}_j)$ begins with W_0 or W_1 for all $t \in \mathbb{N}$. By the continuity of σ, $\sigma^{tr}(\tilde{y}_k) \to \sigma^{tr}(\tilde{x})$, implying that $\sigma^{tr}(\tilde{x})$ begins with W_0 or W_1 for all $k \in \mathbb{N}$. It follows that x is regularly recurrent.

□

Proposition 2.4. *Let $f : I \to I$ be a tent map with critical point c, and kneading sequence K. Suppose K has the following properties:*

(1) K has a building block scheme.

(2) For each level, the set of building blocks contains at least two elements that differ in a position other than the last.

Then for all $x \in \omega(c)$, x is regularly recurrent.

Proof. We will consider three cases:

(1) $x = f^n(c)$ for some $n \in \mathbb{N}$.

(2) $x \neq f^n(c)$, $f^k(x) \neq c$ for any $n, k \geq 0$.

(3) $f^k(x) = c$ for some $k \geq 0$.

Case 1: Suppose there exists an $n \in \mathbb{N}$ such that $x = f^n(c)$. Since every initial segment of K appears in K infinitely often, $f(c) \in \omega(c)$. It follows that $f^k(c) \in \omega(c)$ for all $k \in \mathbb{N}$. Let T be an initial word of $\tilde{x} = \sigma^{n-1}(K)$. As in the proof of Lemma 2.9, by the construction of K, there exists an M such that $\sigma^{kM+(n-1)}(K) = \sigma^{kM}(\tilde{x})$ begins with T for all $k \geq 0$ as desired.

Case 2: Suppose $x \in \omega(c)$ such that $x \neq f^n(c)$ and $f^k(x) \neq c$ for any $n, k \geq 0$. Let T be an initial word of $\tilde{x}$. As in the previous case, we wish to establish an M such that $\sigma^{kM}(\tilde{x})$ begins with T for every $k \in \mathbb{N}$.

Let $\epsilon_1 > 0$. Let W_1 be an initial word of $\tilde{x}$ such that if $\tilde{y} \in I$ and $\tilde{y}$ begins with W_1, then $y \in B_{\epsilon_1}(x)$. We continue this process and establish a set $\{W_i\}_{i \in \mathbb{N}}$ of initial words of $\tilde{x}$ such that for the j^{th} step we assume that $\epsilon_1, \ldots, \epsilon_{j-1}$ and $W_1, \ldots, W_{j-1}$ have been chosen. Choose $0 < \epsilon_j < \frac{\epsilon_{j-1}}{2}$ and let W_j be an initial word of $\tilde{x}$ such that if $\tilde{y} \in I$ and $\tilde{y}$ begins with W_j, then $y \in B_{\epsilon_j}(x)$.

If all of the initial segments of $\tilde{x}$ are initial segments of K, then we are in Case 1. So there exists an L such that for all $j > L$, W_j is not an initial segment of K. We consider $\{W_i\}_{i>L}$ in the following argument.

Let T be $a_1 \ldots a_n$. If for all i, W_i is equal to a_1 followed by an initial segment of K, then $\tilde{x} = *K$ which contradicts the hypothesis. So there exists an $N_1 > L$ such that $W_{N_1} = a_1 R$ where R is not an initial segment of K. Suppose that $a_1 R = a_1 a_2 R'$ and R' is an initial segment of K. If for all $i > N_1$, W_i is equal to $a_1 a_2$ followed by an initial segment of K, then $\tilde{x} = a_1 * K$ which is again a contradiction. We continue this process for $a_3, \ldots, a_n$, and obtain a word $W_{N_n} = a_1 a_2 \ldots a_n R_n$ where R_n is not an initial segment of K. It is also true that if we write W_{N_n} as $W_{N_n} = a_1 M_1 = a_1 a_2 M_2 = \cdots = a_1 \ldots a_n M_n$, the words $\{M_1, \ldots, M_n\}$ are not initial segments of K, where $M_n = R_n$.

Let z be a building block level of K, with V_1^z the first building block on level z, making it an initial segment of K, such that $|V_1^z| > |W_{N_n}|$. None of the words $\{W_{N_n}, M_1, \ldots, M_n\}$ are initial segments of V_1^z.

Suppose T sits across a change position on level $z+2$. Then T sits across infinitely many change positions of K on level $z+2$. Let $\{q_i\}_{i\in\mathbb{N}}$ be these change positions with q_1 the first, q_2 the next, and so forth. Let $\{S_i\}_{i\in\mathbb{N}}$ be the collection of words that begin with T such that T sits across position q_i as an initial segment of S_i, and $|S_i| = |W_{N_n}|$. In other words, for all i, there exists a $t_i \in \mathbb{N}$ such that $\sigma^{t_i}(K)$ begins with S_i and $t_i < q_i \leq t_i + |T|$.

By previous remarks, every change position on level $z+2$ will be followed by V_1^z, an initial segment of K which has length strictly greater than W_{N_n} and therefore greater than the words $M_1, \ldots, M_n$. It follows that $S_i \neq W_{N_n}$ for all $i \in \mathbb{N}$.

Consider the sequence of shifts of K limiting to $\tilde{x}$, $\{\sigma^{k_i}(K)\}_{i\in\mathbb{N}}$, such that all terms in the sequence begin with T. Each of these shifts is regularly recurrent by Case 1. For a particular term of the sequence, say $\sigma^{k_j}(K)$, let $p_{k_j} = k_j + 1$, which is the position in K where the itinerary of $f^{k_j}(1)$ begins. Then T has a cycle which may be represented by $(p_{k_j}, C_{T,p_{k_j}})$ where $\sigma^{mC_{T,p_{k_j}}}(\sigma^{k_j}(K))$ begins with T for all $m \geq 0$.

There exists a D such that $i > D$ implies that W_{N_n} is an initial segment of $\sigma^{k_i}(K)$. Divide the elements of $\{\sigma^{k_i}(K)\}_{i>D}$ into equivalence classes where an element $\sigma^{k_a}(K)$ with starting position p_{k_a} in K, is related to an element $\sigma^{k_b}(K)$ with starting position p_{k_b}, if $C_{T,p_{k_a}} = C_{T,p_{k_b}}$. If we have a finite number of equivalence classes, then the collection $\{C_{T,p_{k_i}}\}_{i\in\mathbb{N}}$ has a least common multiple and we are done as in the proof of Proposition 2.3. So we assume that we have an infinite number of equivalence classes and therefore an infinite collection of regular return times to T for the set $\{\sigma^{k_i}(K)\}_{i>D}$. An infinite collection of regular return times implies that we have an infinite number of cycles for T as T appears as the initial word of each element in the set $\{\sigma^{k_i}(K)\}_{i>D}$. Hence, by Lemma 2.10, there exist $t, s \in \mathbb{N}$ such that T, as an initial word of $\sigma^{k_s}(K)$ sits across the change position q_t on level $z+2$. In other words, $k_s < q_t \leq k_s + |T|$. This means that S_t is an initial word of $\sigma^{k_s}(K)$ such that $|S_t| = |W_{N_n}|$, contradicting the fact that $S_t \neq W_{N_n}$.

Case 3: Let $x \in \omega(c)$, such that $f^k(x) = c$ for some $k \geq 0$. Then $\tilde{x} = Z * K$ for some word Z. Recall that $*$ is our symbol for c. Let T be an initial word of $\tilde{x}$. We may choose T such that $|T| > |Z*|$. Let $T_0 = Z0V$ or $T_1 = Z1V$.

Let $\{n_k\}_{k\in\mathbb{N}}$ be a sequence such that $\sigma^{n_k}(K) \to \tilde{x}$ and all elements in $\{\sigma^{n_k}(K)\}_{k\in\mathbb{N}}$ begin with T_0 or T_1.

By Proposition 2.3, if the set of all regular return times of T in K, $\{C_{T,p_i}\}_{i\in\mathbb{N}}$, is finite, then x is regularly recurrent. By Lemma 2.10, if there exists a building block level such that T_0 and T_1 do not sit across a change position of K on that level, then T has a finite number of cycles and we satisfy the hypothesis of Proposition 2.3. In fact, if $\sigma^{n_k}(K) \to \tilde{x}$, such that the set of regular return times of T for these shifts of K is finite, then x is regularly recurrent.

Let the regular return time to T associated with an element, $\sigma^{n_j}(K)$, of the set $\{\sigma^{n_k}(K)\}_{k\in\mathbb{N}}$ be $C_{T,p_{n_j}}$. Assume that the set $\{C_{T,p_{n_i}}\}_{i\in\mathbb{N}}$ is infinite.

Without loss of generality, assume $\{n_k\}_{k\in\mathbb{N}}$ are chosen such that no more than one element of $\{\sigma^{n_k}(K)\}_{k\in\mathbb{N}}$ agrees with $\tilde{x}$ for exactly n positions. Therefore, if $\sigma^{n_k}(K) \to \tilde{x}$, and there exists a subsequence $\sigma^{n_{k_i}}(K) \to \tilde{x}$, such that for some $M \in \mathbb{N}$, $\sigma^{Mt}(\sigma^{n_{k_i}}(K))$ begins with T_0 or T_1 for all $i \in \mathbb{N}, t \geq 0$, then $\sigma^{Mt}(\tilde{x})$ begins with T_0 or T_1 for all $t \geq 0$ by the continuity of σ.

We further assume that for all building block levels of K, all but finitely many elements in the set $\{\sigma^{n_k}(K)\}_{k\in\mathbb{N}}$ have their initial segment of length $|T|$, which will be T_0 or T_1, sitting across a change position. This implies that at least one of T_0 or T_1 will sit across infinitely many change positions on each level of K as an initial segment of a subsequence of $\{\sigma^{n_k}(K)\}_{k\in\mathbb{N}}$. Without loss of generality, we consider T_0. Let $T_0 = a_1a_2\dots a_n$. For some $j \in \{1,\dots,n\}$, a_j sits in an infinite number of change positions on every building block level as the j^{th} letter of T_0 and of each element in $\{\sigma^{n_{k_i}}(K)\}_{i\in\mathbb{N}}$, the subsequence of $\{\sigma^{n_k}(K)\}_{k\in\mathbb{N}}$ mentioned above. Since change positions are followed by increasingly longer initial segments of K as the building block levels increase, $\sigma^{n_{k_i}}(K) \to a_1a_2\dots a_{j-1} * K$. Recall that $\sigma^{n_{k_i}}(K) \to \tilde{x} = Z * K$, where Z is an initial word of T. This implies that $Z * K = a_1a_2\dots a_{j-1} * K$ and $Z = a_1a_2\dots a_{j-1}$, for otherwise $\sigma^l(K) = K$ for some $l \in \mathbb{N}$, contradicting Lemma 2.8.

Let q be minimal such that $|T| < |V_1^q|$ where V_1^q is the first building block on level q. Then V_1^q is the initial word of every building block on level $q+2$ and is preceeded by $V_{s_q}^q$ or $\hat{V}_{s_q}^q$, where $V_{s_q}^q$ is the last building block on level q.

Let $\alpha = |V_1^{q+2}|$ and $\beta = |Z *|$. Let

$$H_1 = \sigma^{\alpha-\beta}(K), H_2 = \sigma^{2\alpha-\beta}(K), \dots, H_i = \sigma^{i\alpha-\beta}(K), \dots.$$

Since a_j sits in an infinite number of change positions on every building block level as the j^{th} letter of T_0, and $|T_0| < |V_1^q|$, the first $j-1$ symbols of H_i for every $i \in \mathbb{N}$ will be $a_1 \dots a_{j-1}$ and the j^{th} symbol, which sits in the change positions on level $q+2$ may be a 0 or a 1. Thus the elements of $\{H_i\}_{i\in\mathbb{N}}$ begin with T_0 or T_1. Further, there exists a subsequence of $\{\sigma^{n_{k_i}}(K)\}_{i\in\mathbb{N}}$ which we will call $\{\sigma^{n_{k_{i_j}}}(K)\}_{j\in\mathbb{N}}$ that is a subset of $\{H_i\}_{i\in\mathbb{N}}$. So $\sigma^{s\alpha}(\sigma^{n_{k_{i_j}}}(K))$ begins with T_0 , $\sigma^{s\alpha}(\sigma^{n_{k_{i_j}}}(K)) \to \sigma^{s\alpha}(\tilde{x})$, and $\sigma^{s\alpha}(\tilde{x})$ begins with T_0 or T_1 for all $j \in \mathbb{N}, s \geq 0$.

$\square$

We now have the characterization theorem for the kneading sequence structure of strange adding machines.

Theorem 2.4. *Let $f : I \to I$ be a tent map with critical point c, and kneading sequence K. Suppose K has the following properties:*

(1) K has a building block scheme.

(2) For each level, the set of building blocks contains at least two elements that differ in a position other than the last.

Then there exists a sequence α of prime numbers such that $f|_{\omega(c)}$ is topologically conjugate to the adding machine f_α.

Proof. The proof follows from Theorem 2.2, Lemmas 2.8, 2.9, and Proposition 2.4. □

2.2 Example

The following is an example of a sequence, K, that satisfies the assumptions of Theorem 2.4. We show that this sequence is above 101^∞, non-renormalizable, and shift maximal, making K the kneading sequence of a tent map with slope greater than $\sqrt{2}$ on the interval $[0, 1]$, which we will call f. It then follows that $f|_{\omega(c)}$ is topologically conjugate to an adding machine map. We defined an adding machine map, f_α, for $\alpha = (p_1, p_2, \dots)$ where for each $i \in \mathbb{N}$, p_i is prime. However, if $\alpha' = (q_1, p_n, p_{n+1}, \dots)$ where $q_1 = p_1 \cdot p_2 \cdot \dots \cdot p_{n-1}$, then $f_{\alpha'}$ defined in the same manner as f_α is topologically conjugate to f_α. We will see that for our example, $f|_{\omega(c)}$ is topologically conjugate to the adding machine map f_α where $\alpha = (24, 3, 3, 3, \dots)$.

Let

$$V_1^1 = 10110110$$

$$V_2^1 = 11010111$$

$$V_3^1 = 10101111$$

and

$$V_1^i = V_1^{i-1}V_2^{i-1}V_3^{i-1}$$

$$V_2^i = \hat{V}_1^{i-1}\hat{V}_2^{i-1}\hat{V}_3^{i-1}$$

$$V_3^i = V_1^{i-1}\hat{V}_2^{i-1}V_3^{i-1}$$

where the initial segments of K are $V_1^iV_2^iV_3^i$, $i \in \mathbb{N}$. Consider the initial segment of K for $i = 3$, which we will call Z:

$$V_1^1V_2^1V_3^1\hat{V}_1^1\hat{V}_2^1\hat{V}_3^1V_1^1\hat{V}_2^1V_3^1V_1^1V_2^1\hat{V}_3^1\hat{V}_1^1\hat{V}_2^1V_3^1V_1^1\hat{V}_2^1\hat{V}_3^1V_1^1V_2^1V_3^1\hat{V}_1^1\hat{V}_2^1V_3^1V_1^1\hat{V}_2^1V_3^1$$

This initial segment may be written as a sequence of 0's and 1's as follows:

101101101101011110101111101101111101011010101110101101101101011010101111

101101101101011110101110101101111101011010101111101101101101011010101110

101101101101011110101111101101111101011010101111101101101101011010101111

Now we show this kneading sequence is shift maximal by proving the following lemmas regarding initial segments of K.

Lemma 2.11. *Every word of K of length 24 or less will be below or will agree with the initial segment of K of length 24, $V_1^1V_2^1V_3^1$, with respect to the parity lexicographical ordering.*

Proof. By the structure of K, every word of K of length 24 or less appears in Z. It can be easily verified that every word of length 24 or less in Z satisfies the Lemma. $\square$

Lemma 2.12. *Let $m \geq 0$. If an initial segment of K and a shift of K agree for $(3^m \cdot 24)$ positions, then they agree for $(2 \cdot 3^m \cdot 24 - 1)$ positions.*

Proof. Observe that $V_1^i V_2^i V_3^i$ will only begin in positions $(1 + 3k|V_1^i V_2^i V_3^i|)$ for $k \geq 0$, and will always be followed by $\hat{V}_1^i \hat{V}_2^i \hat{V}_3^i$ or $\hat{V}_1^i \hat{V}_2^i V_3^i$, which only differ in the last symbol. □

Lemma 2.13. *Let $m \geq 0$. If an initial segment of K and a shift of K agree for $(2 \cdot 3^m \cdot 24)$ positions, then they agree for $(3^{m+1} \cdot 24 - 1)$ positions.*

Proof. Observe that $V_1^i V_2^i V_3^i \hat{V}_1^i \hat{V}_2^i \hat{V}_3^i$ will only begin in positions $(1 + 3k|V_1^i V_2^i V_3^i|)$ for $k \geq 0$, and will always be followed by $V_1^i \hat{V}_2^i V_3^i$ or $V_1^i \hat{V}_2^i \hat{V}_3^i$, which only differ in the last symbol. □

Lemma 2.14. *Let $m \geq 0$. All initial segments of K of length $(3^m \cdot 24)$, $(2 \cdot 3^m \cdot 24)$, and $(3^{m+1} \cdot 24)$ have odd parity.*

Proof. We show that the parity of V_1^i, V_2^i, V_3^i is odd, even, even respectively for all $i \in \mathbb{N}$. Let $i = 1$. The parity of V_1^1, V_2^1, V_3^1 is odd, even, even respectively. Assume that for $i = n$, V_1^n, V_2^n, V_3^n have parity odd, even, even respectively. Then $V_1^{n+1} = V_1^n V_2^n V_3^n$ has odd parity. V_2^{n+1} has even parity since V_2^{n+1} and V_1^{n+1} disagree in exactly three positions. V_3^{n+1} has even parity since V_3^{n+1} and V_1^{n+1} disagree in exactly one position. It follows that V_1^i, $V_1^i V_2^i$, and $V_1^i V_2^i V_3^i$ have odd parity for all $i \in \mathbb{N}$. □

Lemma 2.15. *Let L be a sequence of 0's and 1's. Let W be an initial segment of L with odd parity. If there exists an $n \in \mathbb{N}$ such that $\sigma^n(L)$ begins with $\hat{W}$, then $\sigma^n(L) \prec L$.*

Proof. If W ends in a 0, then the parity of the word $W - 1$ is odd. By the definition of the parity-lexicographical ordering, an odd word followed by 0 is above the same odd word followed by a 1. If W ends in a 1, then the parity of the word $W - 1$

is even. An even word followed by 1 is above the same even word followed by a 0. Therefore, $\sigma^n(L) \prec L$. $\square$

Proposition 2.5. *K is shift maximal.*

Proof. The proof follows from Lemmas 2.11, 2.12, 2.13, 2.14, and 2.15. $\square$

Proposition 2.6. *K is the kneading sequence of a tent map with slope greater than $\sqrt{2}$.*

Proof. K is above 101^∞. K is not renormalizable because it will have two building blocks on each level that disagree in a position other than the last. K is shift maximal by Proposition 2.5. $\square$

Proposition 2.7. *$f|_{\omega(c)}$ is a homeomorphism that is topologically conjugate to an adding machine f_α.*

Proof. K satisfies the conditions of Theorem 2.4. $\square$

We wish to show that $\alpha = (24, 3, 3, \dots)$. We accomplish this by constructing a sequence of covers as described in Theorem 2.1.

$\mathcal{P}_1$ will have 24 elements defined as follows.

Let $P_1^1 = \{x \in \omega(c) |\ \tilde{x}$ has either 101101101101 or 101101111101 as its initial segment$\}$. Note that 101101101101 is an initial word of K. This set contains all points whose itineraries are $\sigma^{24n}(K)$ for $n \geq 0$ and all limit points of these elements.

Recursively define $P_i^1 = f(P_{i-1}^1)$, for $1 < i \leq 24$, which contains the points whose itineraries are $\sigma^{(24n+(i-1))}(K)$ for $n \geq 0$ and all limit points of these elements.

If we write K in terms of the words $V_1^1, V_2^1, V_3^1, \hat{V}_1^1, \hat{V}_2^1, \hat{V}_3^1$, the change positions in K are $8 + 24n, 16 + 24n$, and $24(n + 1)$ for $n \geq 0$ (see Definition 2.19). This is the level one description of K. By observation of Z above, we see that the initial segments of length twelve for $\sigma^n(K), 0 \leq n \leq 23$, are distinct. Further, no two initial segments of length twelve for $\sigma^n(K), 0 \leq n \leq 23$, agree in every position that is not a change position. For example: if $n_1, n_2 \in \{0, \ldots, 23\}, n_1 \neq n_2$, and $\sigma^{n_1}(K)$ begins with $A = a_1 a_2 \ldots a_{12}$, with a_2 and a_{10} sitting in change positions of K, and $\sigma^{n_2}(K)$ begins with $B = b_1 b_2 \ldots b_{12}$ with b_5 sitting in a change position of K, then the words A and B will disagree in a position other than $2, 5$, or 10.

Let $m \in \{0, \ldots, 23\}$. Then $\sigma^m(K)$ and $\sigma^{m+24k}(K), k \in \mathbb{N}$, are related as follows. Let the initial word of $\sigma^m(K)$ be $A = a_1 a_2 \ldots a_{12}$. Since the change positions on this level occur every eight symbols, A will contain either one or two symbols that sit in a change position. Without loss of generality, we may assume that a_2 and a_{10} sit in change positions of K on this level. Let $k \in \mathbb{N}$. Then the initial word of $\sigma^{m+24k}(K)$ of length twelve will be an element of $\{a_1 a_2 \ldots a_{10} a_{11} a_{12},\ a_1 \hat{a}_2 \ldots a_{10} a_{11} a_{12},$
$a_1 a_2 \ldots \hat{a}_{10} a_{11} a_{12},\ a_1 \hat{a}_2 \ldots \hat{a}_{10} a_{11} a_{12}\}$, where $\hat{a}_i = (1 - a_i)$.

By comments regarding the uniqueness of the initial words of $\sigma^n(K)$ for $n \in \{0, \ldots, 23\}$, we have that for $n_1 \neq n_2$, $P^1_{n_1} \neq P^1_{n_2}$. Let $\mathcal{P}_1 = \{P^1_i\}_{i=1}^{24}$.

By construction of our sets, for all $n \in \mathbb{N}$ there exists a $j \in \{1, \ldots, 24\}$ such that $f^n(c) \in P^1_j$, and for each $j \in \{1, \ldots, 24\}$, P^1_j is closed. Therefore, $\mathcal{P}_1$ is a cover of $\omega(c)$.

Assume for $0 < m \leq j$ we have constructed covers, $\mathcal{P}_m$, of $\omega(c)$ such that:

(1) Each $\mathcal{P}_m$ has $3^{m-1} \cdot 24$ many disjoint elements.

(2) $x \in P^m_1$ if and only if $\tilde{x}$ has $V_1^{m-1} V_2^{m-1} V_3^{m-1}$ or $V_1^{m-1} V_2^{m-1} \hat{V}_3^{m-1}$ as its initial segment.

(3) $f(P^m_{i-1}) = P^m_i$ for $2 \leq i \leq 3^{m-1} \cdot 24$.

(4) $\mathcal{P}_m$ refines $\mathcal{P}_{m-1}$.

Define P_1^{j+1} to be the elements whose itineraries are in the set $\{\sigma^{3^j \cdot 24k}(K)\}_{k=0}^{\infty}$, and the limit points of this set. These are the elements of $\omega(c)$ whose itineraries begin with $V_1^{j+1} = V_1^j V_2^j V_3^j$ or $\hat{V}_1^{j+1} = V_1^j V_2^j \hat{V}_3^j$. Recursively define $P_i^{j+1} = f(P_{i-1}^{j+1})$, $2 \leq i \leq 3^j \cdot 24$.

It is clear that for each $i \in \mathbb{N}$, the elements of $\mathcal{P}_i$ are disjoint and cyclically permuted, and mesh $(\mathcal{P}_i) \to 0$ as $i \to \infty$. It follows from Theorem 2.1 that $f|_{\omega(c)}$ is topologically conjugate to the adding machine map f_α, where $\alpha = (24, 3, 3, \dots)$.

This example allows us to better understand the dynamics of a strange adding machine by demonstrating the properties of an adding machine described in Theorem 2.1 in the setting a a non-renormalizable map. Further, the rich body of work using kneading sequence theory, much of which is due to Henk Bruin ([18], [17]), is now at our disposal. We were able, for example, to find a wild attractor in a non-renormalizable unimodal map, and we show in Chapter 4, the existence of a map for which the set of endpoints in the inverse limit space of a unimodal map is not closed.

CHAPTER THREE

Endpoints

Endpoints of an inverse limit space are a topological invariant and help us better understand the structure of the space. Endpoints and their relationship to the ω-limit set of the critical point have been extensively studied, [11], [18], [31]. In this chapter, we address Keesling's conjecture by considering the question: For a map with critical point c, when is it the case that all points in $\varprojlim \{\omega(c), f|_{\omega(c)}\}$ are endpoints? We give an answer for a family of maps that are a subset of the continuous maps, f, with $f|_{\omega(c)}$ topologically conjugate to an adding machine. We also provide a general result, Proposition 3.1, on a necessary condition for equality of the set of endpoints and the set of folding points. In the last section we generalize the results to adding machines embedded in maps on graphs.

3.1 Preliminary Theorems and Definitions

We give preliminary definitions for graphs and local endpoints.

Definition 3.1. [33, Definition 28.1] A *continuum* is a compact, connected metric space.

Definition 3.2. A *graph*, G, is a connected union of finitely many arcs, $\{A_1, A_2, \ldots, A_k\}$, that intersect only at their endpoints. These endpoints, V, are called the *vertices* of G. If $v \in V$, the *degree* of v is the number of arcs, A_j, that contain v as an endpoint. If the degree of v is greater than or equal to three, we call v a *branch point*.

Definition 3.3. A *chain* is a finite collection $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ of open sets called links such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The *mesh* of a chain $\mathcal{C}$ is $\max\{diameter(C) : C \text{ is a link of } \mathcal{C}\}$. An ϵ-*chain* is a chain whose mesh is not greater than ϵ. A continuum is *chainable* if and only if it can be covered by an ϵ-chain for each positive ϵ.

Definition 3.4. Let X be a chainable continuum, and $x \in X$. We say that x is an *endpoint* of X provided that for every A, B which are subcontinua of X with $x \in A \cap B$, either $A \subseteq B$ or $B \subseteq A$.

Definition 3.5. Sets H and K in X are *mutually separated* in X if and only if $H \cap \overline{K} = \overline{H} \cap K = \emptyset$

Definition 3.6. A continuum X is called a *triod* provided there exists a subcontinuum Z of X such that $X \setminus Z$ is the union of three non-empty sets, each two of which are mutually separated in X.

Inverse limit spaces of unimodal maps are known to be chainable. We are, however, also interested in inverse limit spaces of graphs which may contain a triod. We therefore provide the following definition of a local endpoint.

Definition 3.7. Let X be a compact metric space, and $x \in X$. We say that x is a *local endpoint* provided there exists $\epsilon > 0$ such that if $A, B \subseteq B_\epsilon(x)$ are subcontinua and $x \in A \cap B$ then either $A \subseteq B$ or $B \subseteq A$.

Definition 3.8. We define the *inverse limit space*, X, of a map $f : I \to I$ as $X = \{(x_0, x_1, x_2, \dots) |\ x_i \in I \text{ and } x_i = f(x_{i+1})\}$.

Let $A \subseteq \varprojlim\{I, f\}$ be a subset. We adopt the notation $\pi_n(A) = A_n$ for the n^{th} projection of A. We use the notation A° for the interior of A.

Definition 3.9. Let $f : E \to E$ be a continuous map of a compact metric space. Let $x \in E$. We say x is *uniformly recurrent* provided that, for every open set U containing x, there exists an $M \in \mathbb{N}$ such that $f^j(x) \in U$, $j \geq 0$, implies $f^{j+k}(x) \in U$ for some $0 < k \leq M$.

3.2 *Unimodal Maps*

It is known that for a unimodal map $f : I \to I$, the set of endpoints in $\varprojlim\{I, f\}$ is a subset of the set of folding points. This result follows from [11, Theorem 1.4]; see also [18, Corollary 2], and [31, Lemma 6.6]. Proposition 3.1 tells us that if the set of endpoints is equal to the set of folding points, then the critical point must be uniformly recurrent under f.

Definition 3.10. [16, Definition 3.4.7] Let $f : E \to E$ be a continuous map of a compact metric space. We say a nondegenerate open set $U \subset E$ is *wandering* provided:

(1) $f^n(U) \cap U = \emptyset$ for all $n > 0$, and

(2) U does not tend towards a periodic orbit, that is , $\bigcup_{x \in U} \omega(x)$ is not a single periodic orbit.

Proposition 3.1. *Let $f : I \to I$ be a unimodal map, with recurrent critical point c, such that c is not periodic, f has no wandering intervals or periodic attractors, and $\varprojlim\{\omega(c), f|_{\omega(c)}\}$ is equal to the set of endpoints of $\varprojlim\{I, f\}$. Then c is uniformly recurrent.*

Proof. Assume that the hypothesis holds and that c is not uniformly recurrent. Then there exists an open set U containing c, and sets, $\{f^{t_j}(c)\}_{j\in\mathbb{N}}$ and $\{f^{t_j+n_j}(c)\}_{j\in\mathbb{N}}$, such that for each j, $f^{t_j}(c) \in U$ and $f^{t_j+n_j}(c) \in U$, but $f^{t_j+s}(c) \notin U$ for $1 \le s < n_j$, and $n_j \to \infty$ as $j \to \infty$.

Then the sequence $\{f^{t_j+n_j-1}(c)\}_{j\in\mathbb{N}}$ has a convergent subsequence, $f^{t_{j,x_0}+n_{j,x_0}-1}(c) \to x_0 \in \omega(c)$. Further, the sequence $\{f^{t_{j,x_0}+n_{j,x_0}-2}(c)\}$ has a subsequence $\{f^{t_{j,x_1}+n_{j,x_1}-2}(c)\}$ which will converge to some point $x_1 \in \omega(c)$ such that $f(x_1) = x_0$. Continuing this process, we define a point in $\varprojlim\{\omega(c), f|_{\omega(c)}\}$, $\hat{x} = (x_0, x_1, x_2, \dots)$.

By assumption, $\hat{x}$ is an endpoint. Proposition 2 of [18] states that if $\hat{x}$ is an endpoint, then there exists a sequence of coordinates of $\hat{x}$, $\{x_{k_j}\}_{j\in\mathbb{N}}$ such that $x_{k_j} \to 1$. Under the conditions of the hypothesis, we are guaranteed that $x_{k_j+1} \to c$. In particular, there exists an integer, N, such that the coordinate of $\hat{x}$, x_N, is contained in U. Then all but finitely many elements of the sequence $\{f^{t_{j,x_N}+n_{j,x_N}+(-N-1)}(c)\}$ must also lie in U. However, for $n_{j,x_N} > N+1$, this is a contradiction.

□

The following result is similar to that of Bruin [18, Proposition 2], and follows from Barge and Martin [11, Theorem 2.9].

Proposition 3.2. *Let $f : I \to I$ be a unimodal map with critical point c. Let $X = \varprojlim\{\omega(c), f|_{\omega(c)}\}$. If $\hat{x} = (x_0, x_1, \dots) \in X$ such that for infinitely many $j \in \mathbb{N}$ we have that for all $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $f^k([x_{j+k}, 1]) \subset B_\epsilon(x_j)$, then $\hat{x}$ is an endpoint of $\varprojlim\{I, f\}$.*

Proof. Let $\hat{x}$ be a point in X satisfying the hypotheses above and assume that $\hat{x}$ is not an endpoint of $\varprojlim\{I, f\}$. Then there exist A and B, subcontinua of $\varprojlim\{I, f\}$ such that $\hat{x} \in A\cap B$ and $A-B \neq \emptyset, B-A \neq \emptyset$. Further, there exists an M such that for all $j \geq M$, $A_j - B_j \neq \emptyset$ and $B_j - A_j \neq \emptyset$. Let $j > M$, $\epsilon > 0$ such that $B_\epsilon(x_j) \subset (A_j \cup B_j)^\circ$, and $a_j \in A_j - B_j$ and $b_j \in B_j - A_j$. Then there exists a $k \in \mathbb{N}$ such that $f^k(x_{j+k}) = x_j$ and $f^k([x_{j+k}, 1]) \subset B_\epsilon(x_j)$. Since $f^k(A_{j+k}) = A_j$, $f^k(B_{j+k}) = B_j$, a_j has a preimage in $A_{j+k} - B_{j+k}$ and b_j has a preimage in $B_{j+k} - A_{j+k}$, which we will call a_{j+k}, b_{j+k} respectively. Also, $x_{j+k} \in A_{j+k}\cap B_{j+k}$, and each of A_{j+k}, B_{j+k}, and $A_{j+k}\cup B_{j+k}$ is an arc. Since a_{j+k} and b_{j+k} are not in $[x_{j+k}, 1]$, it must be that $b_{j+k} < a_{j+k} < x_{j+k} < 1$ or $a_{j+k} < b_{j+k} < x_{j+k} < 1$. In the first case, $a_{j+k} \in [b_{j+k}, x_{j+k}] \subseteq B_{j+k}$, which is a contradiction. In the second case, $b_{j+k} \in [a_{j+k}, x_{j+k}] \subseteq A_{j+k}$, which is also a contradiction. □

Proposition 3.3. *Let $\alpha = (p_1, p_2, \dots)$ be a sequence of integers with $p_i \geq 2$ for each i. Let $j_i = p_1 \cdot p_2 \cdot \dots \cdot p_i$ for each i. Let $f : I \to I$, be a unimodal map with critical point c, such that $f|_{\omega(c)}$ is topologically conjugate to the adding machine f_α. If there exist covers, $\{\mathcal{P}_i\}_{i \in \mathbb{N}}$, of $\omega(c)$ such that for all $i \in \mathbb{N}$, we have:*

(1) $\mathcal{P}_i$ is a cover of $\omega(c)$ consisting of j_i pairwise disjoint, nonempty, clopen sets which are cyclically permuted by f.

(2) $\mathcal{P}_{i+1}$ refines $\mathcal{P}_i$.

(3) $\mathrm{mesh}(\mathcal{P}_i) \to 0$ as $i \to \infty$.

(4) there exists an N such that for all $m \geq N$, $\mathcal{P}_m$ has only one element which contains c in its convex hull

then $\varprojlim\{\omega(c), f|_{\omega(c)}\}$ is equal to the set of endpoints of $\varprojlim\{I, f\}$.

Proof. Let $\hat{x} = (x_0, x_1, \dots)$ be a point in $\varprojlim\{\omega(c), f|_{\omega(c)}\}$, and let $\epsilon > 0$. Consider x_j, a coordinate of $\hat{x}$, such that $x_j \neq 1$, $x_j \neq c$. We wish to find k such that $f^k([x_{j+k}, 1]) \subset B_\epsilon(x_j)$. Without loss of generality, we may assume $N = 1$ in the hypothesis, and thus for each cover in $\{\mathcal{P}_i\}_{i \in \mathbb{N}}$, only one element contains c in its convex hull. Then by (3), there exists an integer m, such that the element of the cover $\mathcal{P}_m$ which contains x_j, P_j^m, is contained in $B_\epsilon(x_j)$. Let P_1^m and P_c^m be the elements of $\mathcal{P}_m$ which contain 1 and c respectively. We may assume that m is large enough so that P_1^m, P_j^m, and P_c^m are distinct. Note that $f(P_c^m) = P_1^m$. There exists a minimal $k \in \mathbb{N}$ such that $f^k(P_1^m) = P_j^m$, by (1). Then P_1^m contains a preimage of x_j under f^k, and since $f|_{\omega(c)}$ is a homeomorphism, this element is x_{j+k}, a coordinate of $\hat{x}$. Let s be minimal such that $f^s(P_c^m) = P_c^m$. Such an s exists by (1), and $1 \leq k < (s-1)$. By (4) since only the element P_c^m contains c in its convex hull, $f^t(\mathrm{conv}(P_c^m)) = \mathrm{conv}(f^t(P_c^m))$ for $1 \leq t < s$. Further, for each $t \in \{1, \dots, s-1\}$ all of the points of $\mathrm{conv}(f^t(P_c^m))$, lie on the same side of c. This means that f^k is monotone on $[x_{j+k}, 1]$, giving us $f^k([x_{j+k}, 1]) \subset B_\epsilon(x_j)$, which satisfies the conditions of Proposition 3.2.

$\square$

We now describe a family of maps for which the conditions of the hypothesis of Proposition 3.3 are met.

Definition 3.11. [16, p.39] Let $f : I \to I$ be continuous and onto. An interval $J \subset I$ is called *restrictive* if $J \neq I$ and there exists an $n > 1$, minimal, such that $f^n(J) \subset J$.

Definition 3.12. [23, p.139] The unimodal map $f : I \to I$ is *renormalizable* if there exists a restrictive interval $J \subset I$ of period n such that J contains the critical point and $f^n : J \to J$ is a unimodal map. f folds J onto $f(J)$ and maps $f^i(J)$ homeomorphically onto $f^{i+1}(J)$ for $i = 1, \dots, n-1$.

If the renormalization process can be repeated infinitely often, we have an *infinitely renormalizable* map. This map possesses a nested sequence of intervals $c \in \cdots \subset J_2 \subset J_1 \subset I$ such that for every $n \in \mathbb{N}$, J_n is a restrictive interval as described in Definition 3.12, which has period q_n. Let $K_n = \bigcup\limits_{k=0}^{q_n-1} f^k(J_n)$. Under sufficient smoothness conditions $C = \bigcap\limits_{n=1}^{\infty} K_n$ is a minimal Cantor set on which f is topologically conjugate to a (p_i)-adic adding machine, with the sequence p_i defined by $q_k = \prod\limits_{i=1}^{k}(p_i)$, [19] and [23, p.236].

Given an infinitely renormalizable unimodal map, $f : I \to I$, we can construct covers $C_n = \omega(c) \cap K_n$ that satisfy the conditions of Proposition 3.3. We therefore may conclude that the set of endpoints in $\varprojlim\{I, f\}$ is equal to the set $\varprojlim\{\omega(c), f|_{\omega(c)}\}$.

For strange adding machines, each cover of $\omega(c)$ will contain more than one element with c in its convex hull; see Lemma 2.5. The conditions of Proposition 3.3 are therefore not met by these maps.

3.3 Local Endpoints

We turn our attention to finite graphs. We consider adding machine dynamics in this more general setting and address Keesling's conjecture. The conditions of Proposition 3.2 are undefined on finite graphs. We therefore introduce the convex hull property (Definition 3.18), as our tool for establishing that a folding point is an endpoint. We give conditions in Theorem 3.1, which include the convex hull property, under which the set of folding points are equal to the set of endpoints in the inverse limit space of a map on a graph.

Definition 3.13. Let G be a graph, and $f : G \to G$ a continuous function. We say that f is *monotone* if $f^{-1}(x)$ is connected for all $x \in G$.

Definition 3.14. Let G be a graph, and $f : G \to G$ a continuous function. We say that f is *piecewise monotone* if there exist finitely many compact connected subsets of G, $\{A_1, A_2, \ldots, A_k\}$, such that $G = A_1 \cup \cdots \cup A_k$ and $f|_{A_i}$ is monotone for $i \in \{1, \ldots, k\}$.

Definition 3.15. Let G be a graph, and $f : G \to G$ a continuous function. A *critical point*, c, is a point such that for every $\epsilon > 0$, $f|_{B_\epsilon(c)}$ is not monotone, or $f(c)$ is an endpoint.

Definition 3.16. A *tree* is a graph with no simple closed curves.

Definition 3.17. Let G be a graph and $A \subseteq G$. Let $\epsilon > 0$ such that $\operatorname{diam}(A) < \epsilon$. Then the *$\epsilon$-convex hull of A* is $\operatorname{conv}_\epsilon(A) = \bigcap\{K \supseteq A : K \subseteq G$ is compact and connected, $\operatorname{diam}(K) < \epsilon\}$.

Lemma 3.1 is necessary to ensure that we can avoid simple closed curves in our use of the convex hull of a set.

Lemma 3.1. *Let G be a graph. There exists an $\epsilon > 0$, such that for $A \subseteq G$ with* $\operatorname{diam}(A) < \epsilon$, $\operatorname{conv}_\epsilon(A)$ *is a tree.*

Proof. Let the simple closed curves of G be $\{l_1, \ldots, l_t\}$. Let the arc-length of l_i be m_i, for $i \in \{1, \ldots, t\}$. Let $m = \min\{m_1, \ldots, m_t\}$. Since G has a finite number of branch points, there exists a minimum distance between any two branch points, which we will call d. Let $\epsilon = \min\{d, \frac{1}{2}m\}$, and let $A \subseteq G$ be such that $\mathrm{diam}(A) < \epsilon$. We have the following cases:

(1) There exists an arc, J, which contains A such that the length of J is less than ϵ. Then $\mathrm{conv}_\epsilon(A) = \bigcap\{K \supseteq A : K \subseteq J,\ K \text{ is compact and connected }\}$, which is a tree.

(2) A is contained in a finite number of arcs, $\{J_1, \ldots, J_k\}$, each of length less than ϵ, which are connected at a single branch point. Let $J = \bigcup J_i$ for $i \in \{1, \ldots, k\}$. Then $\mathrm{conv}_\epsilon(A) = \bigcap\{K \supseteq A : K \subseteq J,\ K \text{ is compact and connected }\}$, which is a tree.

□

In what follows we write $\mathrm{conv}(A)$ for $\mathrm{conv}_\epsilon(A)$ where ϵ is small enough to guarantee that $\mathrm{conv}_\epsilon(A)$ is a tree.

Let G be a graph with at least one endpoint, e. Let $f : G \to G$ be a continuous function with finitely many critical points,$\{c_1, c_2, \ldots, c_m\}$. Suppose that for one of the critical points, $c_k = c$, $f(c) = e$. Also assume that c is recurrent and that there exists a sequence α of prime numbers, such that $f|_{\omega(c)}$ is topologically conjugate to the adding machine f_α. Let $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ be the covers of $\omega(c)$ described in Theorem 2.1. There exists an M such that for $i > M$, the convex hull of each element of $\mathcal{P}_i$, $P^i_t \subseteq G$, is connected and contains no simple closed curves. We assume without loss of generality that $M = 1$. Let $P^i_{m_{i,c}}$ be the element of $\mathcal{P}_i$ which contains c.

Note that for i chosen as above, $\mathrm{conv}(f^t(P^i_{m_{i,c}}))$ is contained in $f^t(\mathrm{conv}(P^i_{m_{i,c}}))$ for $t \in \{1, \ldots, |\mathcal{P}_i|\}$, although they will not necessarily be equal.

Definition 3.18. We say that f has the *convex hull property* if

$$\mathrm{mesh}\{f(\mathrm{conv}(P^i_{m_{i,c}})), \ldots, f^{|\mathcal{P}_i|}(\mathrm{conv}(P^i_{m_{i,c}}))\} \to 0 \textit{ as } i \to \infty.$$

Proposition 3.4. *Let G be a graph and $f : G \to G$ a continuous function with critical point c, such that c is uniformly recurrent under f. Let $\hat{x} \in \underleftarrow{lim}\{\omega(c), f|_{\omega(c)}\}$. Then for $z \in \omega(c)$, there exists a subsequence $\{x_{n_{j,z}}\}_{j\in\mathbb{N}}$ of coordinates of $\hat{x}$ such that $x_{n_{j,z}} \to z$.*

Proof. Let $z \in \omega(c)$ such that there does not exist a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of coordinates of $\hat{x}$ with $x_{n_j} \to z$. Then there exists an open set U that contains z such that $x_j \notin U$ for all $j \in \mathbb{N}$. Since $z \in \omega(c)$, there exists a subsequence of the orbit of c, $\{f^{m_j}(c)\}_{j\in\mathbb{N}}$ which converges to z. Further, there exists an M such that for all $k \geq M$, $f^{m_k}(c) \in U$. Since c is uniformly recurrent, $\omega(c)$ is minimal. Thus for every i, $\omega(f^i(c)) = \omega(c)$, and $\omega(f^i(c))$ is minimal. It follows that since $f^i(c) \in \omega(f^i(c))$, $f^i(c)$ is uniformly recurrent [16, p.43]. The uniform recurrence of the elements of the set $\{f^{m_j}(c)\}_{j\in\mathbb{N}}$ tells us that for $k \geq M$ there exists a J_k such that $f^{m_k+t_{i,k}}(c) \in U$ for $t_{1,k} \leq J_k$, $t_{l+1,k} - t_{l,k} \leq J_k$.

Choose $a \in \mathbb{N}$ such that $f^a(c) \in \{f^{m_j}(c)\}_{j\geq M}$, and for simplicity, let $J_a = J$. Let $s \in \mathbb{N}$, where x_s is a coordinate of $\hat{x}$. Since $x_{s+J+1} \in \omega(c) = \omega(f^a(c))$, there exists a sequence $\{b_i\}$ such that $f^{a+b_i}(c) \to x_{s+J+1}$ as $i \to \infty$. Since $f^t(x_{s+J+1})$ is not in U for $t \in \{0, \ldots, J+1\}$, there is a finite collection of open sets $\{U_0, U_1, \ldots, U_{J+1}\}$, with $f^t(x_{s+J+1}) \in U_t$, such that $\bigcup\limits_t U_t$ will not meet U. Note that $x_{s+J+1} \in U_0$. By the continuity of f, there exists W_1, open, such that $x_{s+J+1} \in W_1 \subseteq U_0$ and $f(W_1) \subseteq U_1$. Likewise, there exists W_r, open, such that $x_{s+J+1} \in W_r \subseteq U_0$ and $f^r(W_r) \subseteq U_r$ for $r \in \{2, \ldots, J+1\}$. Then $x_{s+J+1} \in \bigcap\limits_r W_r$, $r \in \{1, \ldots, J+1\}$, and this intersection is an open set which we call W. There exists an N such that for $i \geq N$, $f^{a+b_i}(c) \in W$.

In particular, there exists $v \geq N$ such that $f^{a+b_v}(c) \in W \subseteq U_0$, $f^{a+b_v+t}(c) \in U_t$ for $t \in \{1, \ldots, J+1\}$. This implies $f^{a+b_v+t}(c) \notin U$ for $t \in \{0, \ldots, J+1\}$, which is a contradiction.

$\square$

Proposition 3.5. *Let G be a graph and $f : G \to G$ a continuous function. Let $X = \varprojlim\{G, f\}$. Let $V = \{v_1, v_2, \ldots, v_t\}$ be the vertices of G of degree greater than two. Let e be an endpoint of G such that $e \notin \overline{\mathrm{orb}(v_i)}$ for all $i \in \{1, \ldots, t\}$. Then there exists a $\delta > 0$ such that if $\pi_j^{-1}(B_\delta(e)) = U \subset X$ for some $j \in \mathbb{N}$, then for any subcontinuum $A \subset U$, $\pi_n(A)$ is a subcontinuum of an arc for all $n \geq j$.*

Proof. Since $e \notin \overline{\mathrm{orb}(v_i)}$, there exists a δ_i for each $i \in \{1, \ldots, t\}$ such that $B_{\delta_i}(e) \cap \overline{\mathrm{orb}(v_i)}$ is empty. Let $\delta = \min\{\delta_i\}_{i=1}^{t}$. Choose $j \in \mathbb{N}$. Let A be a subcontinuum of $\pi_j^{-1}(B_\delta(e))$. We wish to show that $\pi_n(A)$ contains no vertices with degree greater than two for all $n \geq j$. Suppose there exists $v_m \in \{v_1, v_2, \ldots, v_t\}$, and $r > j$, such that $v_m \in \pi_r(A)$. Then $f^{r-j}(v_m) \in B_\delta(e)$ which contradicts our choice of δ.

$\square$

It is a result of Barge and Martin [11, Theorem 2.9] that for $h : I \to I$, a continuous function, if either 0 or 1 is recurrent under h, then the inverse limit space of h has at least one endpoint. We generalize this result in Proposition 3.6 for a continuous map on a graph satisfying the hypothesis of Proposition 3.5.

Proposition 3.6. *Let G be a graph and $f : G \to G$ a continuous function. Let $\{v_1, v_2, \ldots, v_t\}$ be the vertices of G of degree greater than two. If G has an endpoint, e, such that e is recurrent and $e \notin \overline{\mathrm{orb}(v_i)}$ for all $i \in \{1, \ldots, t\}$, then $\varprojlim\{G, f\}$ has at least one local endpoint.*

Proof. Let e be an endpoint of G such that e is recurrent under f. There exists a δ_i for each $i \in \{1, \ldots, t\}$ such that $B_{\delta_i}(e) \cap \overline{\mathrm{orb}(v_i)}$ is empty by assumption. Let $\delta = \min\{\delta_i\}_{i=1}^{t}$.

Let $n_1 = 1$ and $\epsilon_1 = \delta$. For all $i > 1$ we assume that $[e, e + \epsilon_i] \subseteq [e, e + \delta)$ where the ϵ_i are chosen below. Choose n_2 so that $f^{n_2}(e) \in B_{\epsilon_1}(e)$ and $f^{n_1+n_2}(e) \in B_{\frac{\delta}{2}}(f^{n_1}(e))$. The first choice is possible by the recurrence of e and the second by the continuity of f^{n_1}. Let $\epsilon_2 > 0$ be chosen so that $f^{n_2}([e, e + \epsilon_2]) \subset [e, e + \epsilon_1)$ and $f^{n_1+n_2}([e, e + \epsilon_2]) \subset B_{\frac{\delta}{2}}(f^{n_1}(e))$. By $[e, e + \epsilon_i]$, $i \in \mathbb{N}$, we mean the arc in G with endpoints e and some point z such that the arclength distance from e to z is ϵ_i. We are guaranteed that this interval does not contain a branch point of G by our choice of δ.

We now assume that positive integers $n_1, n_2, \ldots, n_k$ and positive numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ have been defined. Let n_{k+1} be such that:

$$f^{n_{k+1}}(e) \in [e, e + \epsilon_k)$$

$$f^{n_k+n_{k+1}}(e) \in [e, e + \epsilon_{k-1}) \cap B_{\frac{\delta}{2^{k+1}}}(f^{n_k}(e))$$

$$f^{n_{k-1}+n_k+n_{k+1}}(e) \in [e, e + \epsilon_{k-2}) \cap B_{\frac{\delta}{2^{k+1}}}(f^{n_{k-1}+n_k}(e))$$

$$\vdots$$

$$f^{n_2+\cdots+n_k+n_{k+1}}(e) \in [e, e + \epsilon_1) \cap B_{\frac{\delta}{2^{k+1}}}(f^{n_2+\cdots+n_k}(e))$$

$$f^{n_1+\cdots+n_{k+1}}(e) \in B_{\frac{\delta}{2^{k+1}}}(f^{n_1+\cdots+n_k}(e)).$$

Let $\epsilon_{k+1} > 0$ be chosen so that

$$f^{n_{k+1}}([e, e + \epsilon_{k+1}]) \subset [e, e + \epsilon_k)$$

$$f^{n_k+n_{k+1}}([e, e + \epsilon_{k+1}]) \subset [e, e + \epsilon_{k-1}) \cap B_{\frac{\delta}{2^{k+1}}}(f^{n_k}(e))$$

$$f^{n_{k-1}+n_k+n_{k+1}}([e, e + \epsilon_{k+1}]) \subset [e, e + \epsilon_{k-2}) \cap B_{\frac{\delta}{2^{k+1}}}(f^{n_{k-1}+n_k}(e))$$

$$\vdots$$

$$f^{n_2+\cdots+n_k+n_{k+1}}([e, e + \epsilon_{k+1}]) \subset [e, e + \epsilon_1) \cap B_{\frac{\delta}{2^{k+1}}}(f^{n_2+\cdots+n_k}(e))$$

$$f^{n_1+\cdots+n_{k+1}}([e, e + \epsilon_{k+1}]) \subset B_{\frac{\delta}{2^{k+1}}}(f^{n_1+\cdots+n_k}(e)).$$

The sequence $\{f^{n_{k+1}}(e), f^{n_{k+1}+n_{k+2}}(e), \ldots, f^{n_{k+1}+n_{k+2}+\cdots+n_{k+i}}(e), \ldots\}$, where

$$f^{n_{k+1}+n_{k+2}}(e) \in [e, e+\epsilon_k) \cap B_{\frac{\delta}{2^{k+2}}}(f^{n_{k+1}}(e))$$

$$\vdots$$

$$f^{n_{k+1}+n_{k+2}+\cdots+n_{k+i}}(e) \in [e, e+\epsilon_k) \cap B_{\frac{\delta}{2^{k+i}}}(f^{n_{k+1}+\cdots+n_{k+i-1}}(e))$$

$$\vdots$$

is a Cauchy sequence and we designate x_{N_k} to be its limit, where $N_k = n_1+n_2+\cdots+n_k$. Note that as each element is in $[e, e+\epsilon_k)$, $x_{N_k} \in [e, e+\epsilon_k]$. Further, the sequence $\{f^{n_{k+j+1}}(e), f^{n_{k+j+1}+n_{k+j+2}}(e), \ldots, f^{n_{k+j+1}+n_{k+j+2}+\cdots+n_{k+j+i}}(e), \ldots\}$ converges to some point which we will call $x_{N_{k+j}}$. Continuity gives us $f^{n_{k+j}+\cdots+n_{k+1}}(x_{N_{k+j}}) = x_{N_k}$. By construction, $f^{n_{k+1}+\cdots+n_{k+j}}([e, e+\epsilon_{k+j}]) \subset [e, e+\epsilon_k) \cap B_{\frac{\delta}{2^{k+j}}}(f^{n_{k+1}+\cdots+n_{k+j-1}}(e))$ and the distance from x_{N_k} to $f^{n_{k+1}+\cdots+n_{k+j-1}}(e)$ is less than $\frac{\delta}{2^{k+j}}$. Therefore, $f^{n_{k+1}+\cdots+n_{k+j}}([e, e+\epsilon_{k+j}]) \subset B_{\frac{\delta}{2^{k+j-1}}}(x_{N_k})$ for $k, j \in \mathbb{N}$.

Let $\hat{x} \in \varprojlim\{G, f\}$ be the point defined by $\pi_{N_k}(\hat{x}) = x_{N_k}$. We claim that $\hat{x}$ is a local endpoint. For the purpose of contradiction, assume otherwise. Let $\epsilon > 0$ such that $B_\epsilon(\hat{x}) \subseteq \pi_{N_1}^{-1}(B_\delta(e))$. Let $\hat{x} \in A \cap B$ where A and B are subcontinua of $\varprojlim\{G, f\}$ and $A \cup B \subseteq B_\epsilon(\hat{x})$, $A - B \neq \emptyset$, $B - A \neq \emptyset$, . Then there exists an M such that for all $j \geq M$, $A_j - B_j \neq \emptyset$ and $B_j - A_j \neq \emptyset$. Choose $N_l > M$. Then $\pi_{N_l}(A \cup B)$ is a subcontinuum of an arc by Proposition 3.5, and $\pi_{N_l}(\hat{x}) = x_{N_l} \in A_{N_l} \cap B_{N_L}$. Let $a_{N_l} \in A_{N_l} - B_{N_l}$ and $b_{N_l} \in B_{N_l} - A_{N_l}$. Let $\gamma > 0$ be such that $B_\gamma(x_{N_l}) \subseteq \pi_{N_l}(A \cup B)$ and $a_{N_l} \notin B_\gamma(x_{N_l})$, $b_{N_l} \notin B_\gamma(x_{N_l})$. Then there exists $m > l$ such that the following hold:

(1) $\frac{\delta}{2^{m-1}} < \gamma$

(2) $f^{n_m+n_{m-1}+\cdots+n_{l+1}}(x_{N_m}) = x_{N_l}$,

(3) $x_{N_m} \in [e, e+\epsilon_m]$ and

(4) $f^{n_m+n_{m-1}+\cdots+n_{l+1}}([e, e+\epsilon_m]) \subset B_{\frac{\delta}{2^{m-1}}}(x_{N_l})$.

$x_{N_m} \in A_{N_m} \cap B_{N_m}$ and $\pi_{N_m}(A \cup B)$ is a subcontinuum of an arc. The point a_{N_l} has a preimage in $A_{N_l} - B_{N_l}$, while b_{N_l} has a preimage in $B_{N_l} - A_{N_l}$, which we will call a_{N_m}, b_{N_m} respectively. By (4) above, $a_{N_m} \notin [e, e+\epsilon_m]$, $b_{N_m} \notin [e, e+\epsilon_m]$, while by (3), $x_{N_m} \in [e, e+\epsilon_m]$. Then either $e < x_{N_m} < a_{N_m} < b_{N_m}$ or $e < x_{N_m} < b_{N_m} < a_{N_m}$, where by $y < z$ on $[e, e+\epsilon]$ we mean $d(e, y) < d(e, z)$ with d representing arclength distance, which is a contradiction.

□

Theorem 3.1. *Let G be a graph with at least one endpoint, e. Let $f : G \to G$ be a continuous function such that f has finitely many critical points, $\{c_1, c_2, \ldots, c_m\}$. Suppose that for one of the critical points, $c_k = c$, $f(c) = e$. Let the vertices of G of degree greater than two be $\{v_1, v_2, \ldots, v_t\}$. Further assume that the following hold:*

(1) $e \notin \overline{\mathrm{orb}(v_i)}$ for all $i \in \{1, \ldots, t\}$,

(2) c is recurrent,

(3) there exists a sequence α of prime numbers, such that $f|_{\omega(c)}$ is topologically conjugate to the adding machine f_α, and

(4) the convex hull property holds for the covers $\{\mathcal{P}_i\}_{i \in \mathbb{N}}$ of $\omega(c)$.

Let $\hat{x} \in \varprojlim\{\omega(c), f|_{\omega(c)}\}$. Then $\hat{x}$ is a local endpoint.

Proof. Let δ be chosen as in Proposition 3.5. Then by Theorem 2.2 and Proposition 4, there exists a subsequence $\{x_{n_j}\}_{j \in \mathbb{N}}$ of the coordinates of $\hat{x}$ that limit to e. So there exists an M such that for $k \geq M$, $x_{n_k} \in B_\delta(e)$. By Proposition 3.5, if T is a subcontinuum of $\pi_{n_M}^{-1}(B_\delta(e))$, then $\pi_m(T)$ is a subcontinuum of an arc for all $m \geq n_M$. Notice that $\hat{x} \in \pi_{n_M}^{-1}(B_\delta(e))$. Let $\epsilon > 0$ such that $B_\epsilon(\hat{x}) \subseteq \pi_{n_M}^{-1}(B_\delta(e))$.

Since $\omega(c)$ is an infinite minimal set containing e, we know that e is not periodic. Therefore, there exists an L such that $x_i \neq f^k(e)$ for any k with $i \geq L$. Let $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ be the covers of $\omega(c)$ described in Theorem 2.1, and let the individual elements of each cover be denoted P^i_t.

Suppose $\hat{x}$ is not a local endpoint. Let A and B be two subcontinua in the inverse limit space which contain $\hat{x}$, such that $A \cup B \subseteq B_\epsilon(\hat{x})$, and $A \nsubseteq B$, $B \nsubseteq A$. Then there exists an N such that for all $n \geq N$, $A_n \nsubseteq B_n$ and $B_n \nsubseteq A_n$. Let $j \geq \max\{L, N, n_M\}$, with x_j a coordinate of $\hat{x}$. For every i, let $m_{i,j} \in \mathbb{N}$ such that $x_j \in P^i_{m_{i,j}}$ and let $m_{i,e} \in \mathbb{N}$ such that $e \in P^i_{m_{i,e}}$. For every i, there exists a k_i such that $f^{k_i}(P^i_{m_{i,e}}) = P^i_{m_{i,j}}$ since the elements of $\mathcal{P}_i$ are cyclically permuted. $x_j \in \bigcap\limits_{i\in\mathbb{N}} f^{k_i}(\mathrm{conv}(P^i_{m_{i,e}}))$ with the diameter of the elements in $\{f^{k_i}(\mathrm{conv}(P^i_{m_{i,e}}))\}_{i\in\mathbb{N}}$ going to zero. So there exists a u such that $f^{k_u}(\mathrm{conv}(P^u_{m_{u,e}}))$ is contained in the interior of $A_j \cup B_j$ and $f^{k_u}(P^u_{m_{u,e}}) = P^u_{m_{u,j}}$. We may choose u large enough so that e is not an element of $P^u_{m_{u,j}}$. Since $f^{k_u}(P^u_{m_{u,e}}) = P^u_{m_{u,j}}$, $P^u_{m_{u,e}}$ will contain e and x_{j+k_u}, the preimage of x_j under $f^{-k_u}|_{\omega(c)}$. Recall that since $f|_{\omega(c)}$ is topologically conjugate to an adding machine, f is a homeomorphism when acting on $\omega(c)$.

By our choice of j, $A_j - B_j \neq \emptyset, B_j - A_j \neq \emptyset$. As $\hat{x} \in A \cap B$ and A, B are continua, $\pi_j(A \cup B)$ is compact and $A_j \cup B_j$ is an arc. Let $a_j \in (A_j - B_j) - f^{k_u}(\mathrm{conv}(P^u_{m_{u,e}}))$, $b_j \in (B_j - A_j) - f^{k_u}(\mathrm{conv}(P^u_{m_{u,e}}))$. Such an a_j and b_j exist by our choice of j and the fact that $f^{k_u}(\mathrm{conv}(P^u_{m_{u,e}})) \subseteq (A_j \cup B_j)^\circ$. Let $a_{j+k_u} \in A_{j+k_u}$, such that $f^{k_u}(a_{j+k_u}) = a_j$, and let $b_{j+k_u} \in B_{j+k_u}$, such that $f^{k_u}(b_{j+k_u}) = b_j$. Then $a_{j+k_u} \in A_{j+k_u} - B_{j+k_u}$, $b_{j+k_u} \in B_{j+k_u} - A_{j+k_u}$, and a_{j+k_u}, $b_{j+k_u} \notin \mathrm{conv}(P^u_{m_{u,e}})$. Also, $x_{j+k_u} \in A_{j+k_u} \cap B_{j+k_u}$, and each of A_{j+k_u}, B_{j+k_u}, and $A_{j+k_u} \cup B_{j+k_u}$ is an arc. So either $a_{j+k_u} < b_{j+k_u} < x_{j+k_u} < e$ or $b_{j+k_u} < a_{j+k_u} < x_{j+k_u} < e$. In the first case, $b_{j+k_u} \in [a_{j+k_u}, x_{j+k_u}] \subseteq A_{j+k_u}$, which is a contradiction. In the second case, $a_{j+k_u} \in [b_{j+k_u}, x_{j+k_u}] \subseteq B_{j+k_u}$, which is also a contradiction. $\square$

An example of a function, f, and graph, G, which satisfy the hypotheses of Theorem 3.1 is the following:

(1) G contains an edge, A, with endpoint e,

(2) $f|_A$ is an infinitely renormalizable logistic map with recurrent critical point c,

(3) $f(c) = e$, and

(4) the vertices of G of degree greater than two are fixed under f.

CHAPTER FOUR

Endpoints of a Strange Adding Machine

In this chapter we show that the strange adding machine in the example section of Chapter 2 is a map for which the set of endpoints in the inverse limit space is strictly contained in the set of folding points, thus providing a counterexample to Keesling's conjecture. We also prove that the set of endpoints is dense in the set of folding points, which in conjunction with the counterexample, answers Bruin's question about the existence of a map for which the set of endpoints is not closed.

In [18], Bruin gives necessary and sufficient conditions for a point $\hat{x} \in \varprojlim\{I, f\}$ to be an endpoint. The following definitions are taken from this reference; see also [21]. Given a point $\hat{x} = (\ldots, x_{-2}, x_{-1}, x_0)$ in $\varprojlim\{I, f\}$, we define its symbolic representation by $\tilde{x} = (\ldots, \tilde{x}_{-2}, \tilde{x}_{-1}, \tilde{x}_0)$, where

$$\tilde{x}_i = \begin{cases} 0 & x_i < c \\ * & x_i = c \\ 1 & x_i > c \end{cases}$$

We similarly define the *itinerary* of x_0, $x_i = f^i(x_0)$, $i \geq 0$ as follows:

$$\tilde{x}_i = \begin{cases} 0 & f^i(x_0) < c \\ * & f^i(x_0) = c \\ 1 & f^i(x_0) > c \end{cases}$$

If $\hat{x}$ is an endpoint, there must exist an infinite sequence $\{\tilde{x}_{-k_i}\}_{i\in\mathbb{N}}$ such that the word $\tilde{x}_{-k_j} \ldots \tilde{x}_0$ is an initial word of the kneading sequence of f, [18, Proposition 2]. This means that x_0 has an infinite sequence of preimages $\{f^{-k_i}|_{\omega(c)}(x_0)\}_{i\in\mathbb{N}}$ such that for $j \in \mathbb{N}$, $f^{-k_j+m}|_{\omega(c)}(x_0)$ and $f^{m+1}(c)$ lie on the same side of c for $0 \leq m \leq k_j$. Additionally, the images of c, $\{f^{k_j+1}(c)\}_{j \in \mathbb{N}}$ must be limiting to x_0.

We first attempt to develop some intuition as to how a non-endpoint might exist in the set of folding points for a strange adding machine, f.

It is known that $c \in \omega(c)$ for all strange adding machines, [13, Proposition 3]. Let $j \in \mathbb{N}$ such that $\mathcal{P}_j$ is a cover as described in Theorem 2.1, and let $P^j_{c_1}$ be the unique element in the set $\mathcal{P}_j$ that contains c. From Lemma 2.5, we know that there exists at least one other element in $\mathcal{P}_j$, which we will call $P^j_{c_2}$, such that $c \in \text{conv}(P^j_{c_2})$, the convex hull of $P^j_{c_2}$. By Theorem 2.1, (1), there exists k_0 such that $f^{k_0}(P^j_{c_1}) = P^j_{c_2}$. We assume that if $P^j_{c_3}$ is distinct from $P^j_{c_1}$ and $P^j_{c_2}$, with $c \in \text{conv}(P^j_{c_3})$, and $f^k(P^j_{c_1}) = P^j_{c_3}$ for some $k \in \mathbb{N}$, then $k > k_0$.

Note that none of the elements in $f^n(P^j_{c_1})$, $0 < n < k_0$, contains elements on both sides of c. $f^{k_0}(c) \in P^j_{c_2}$ and we may assume without loss of generality that $f^{k_0}(c)$ is to the right of c. Let $I^L_{j,2} = P^j_{c_2} \cap [0, c)$ and $I^R_{j,2} = P^j_{c_2} \cap (c, 1]$, both of which are non-empty.

Let $z_0 \in f^{k_0+1}(P^j_{c_1})$ such that $f^{-1}|_{\omega(c)}(z_0) \in I^L_{j,2}$. Then $f^{-(k_0+1)}|_{\omega(c)}(z_0) \in P^j_{c_1}$. The word created by $f^{-(k_0)}(z_0) \ldots z_0$, which we represent as $\tilde{z}_{-k_0} \ldots \tilde{z}_0$, disagrees with the first word of K of length $k_0 + 1$ in position k_0, where $f^{k_0}(c)$ is to the right of c and $f^{k_0}(z_0)$ is to the left of c. We say that the $(k_0+1)^{st}$ pre-image of z and c are separated by c in their ${k_0}^{th}$ image.

Consider the refinements of $\mathcal{P}_j$, $\{\mathcal{P}_{j+m}\}_{m \in \mathbb{N}}$ given by Theorem 2.1, and the elements $\{P^{j+m}_{c_1}\}_{m \in \mathbb{N}}$, where for a particular r, $P^{j+r}_{c_1}$ is the unique element of $\mathcal{P}_{j+r}$ containing c. Let $\{z_{-(k_m+1)}\}_{m \in \mathbb{N}}$ be the sequence of preimages of z_0 in $\omega(c)$ such that $z_{-(k_m+1)} \in P^{j+m}_{c_1}$, and $f^{k_m+1}(P^{j+m}_{c_1}) = P^{j+m}_{z_0}$, the unique element of $\mathcal{P}_{j+m}$ containing z_0, where $k_m + 1$ is minimal. By (3), we know that $z_{-(k_m+1)} \to c$ and $f^{k_m+1}(c) \to z_0$ as $m \to \infty$, making the elements in the sequence $\{z_{-(k_m+1)}\}_{m \in \mathbb{N}}$ good candidates to establish $(\ldots, z_{-2}, z_{-1}, z_0)$ as an endpoint in the inverse limit space. But suppose that for all $m \in \mathbb{N}$, $f(c)$ and $z_{-(k_m)}$ are separated on their initial journey to $P_{z_0}^{j+m}$ by passage through an element of $\mathcal{P}_{j+m}$ that contains points on both sides of c, a split element, with the forward image of $f(c)$ on one side and the corresponding forward image of $z_{-(k_m)}$ on the other. Then the word created by $z_{-k_m} \dots z_0$ is not an initial segment of K, and we have a chance that $(\dots, z_{-2}, z_{-1}, z_0)$ is not an endpoint. (We should still be concerned that another sequence of preimages of z_0 might exist which satisfies the conditions for $(\dots, z_{-2}, z_{-1}, z_0)$ to be an endpoint.)

We now present the kneading sequence, K, of the strange adding machine for which the set of endpoints is not equal to $\varprojlim\{\omega(c), f|_{\omega(c)}\}$. It is shown in Chapter 2 that K is the kneading sequence of a tent map, f, such that $f|_{\omega(c)}$ is topologically conjugate to the adding machine f_α where $\alpha = (24, 3, 3, \dots)$.

$$W_1^1 = 10110110$$

$$W_2^1 = 11010111$$

$$W_3^1 = 10101111$$

$$W_1^i = W_1^{i-1} W_2^{i-1} W_3^{i-1}$$

$$W_2^i = \hat{W}_1^{i-1} \hat{W}_2^{i-1} \hat{W}_3^{i-1}$$

$$W_3^i = W_1^{i-1} \hat{W}_2^{i-1} W_3^{i-1}$$

The initial segments of K are $W_1^i W_2^i W_3^i$, $i \in \mathbb{N}$.

We construct a bi-infinite sequence with the "." in the initial segments shown below coming prior to the starting position of the itinerary of a point, $z_0 \in \omega(c)$. Recall that $f|_{\omega(c)}$ is a homeomorphism.

Let $\hat{z} = (\dots, f^{-3}|_{\omega(c)}(z_0), f^{-2}|_{\omega(c)}(z_0), f^{-1}|_{\omega(c)}(z_0), z_0) = (\dots, z_{-3}, z_{-2}, z_{-1}, z_0)$. Note that $\hat{z} \in \varprojlim\{\omega(c), f|_{\omega(c)}\}$.

We begin with $\hat{W}_1^1.\hat{W}_2^1 W_3^1$ which is equal to $\hat{W}_2^2$. We now extend this word in both directions to

$$\hat{W}_1^2\hat{W}_2^2 W_3^2 = W_1^1 W_2^1 \hat{W}_3^1 \hat{W}_1^1.\hat{W}_2^1 W_3^1 W_1^1 \hat{W}_2^1 W_3^1$$

which is the word $\hat{W}_2^3$.

For the i^{th} step we have the word $\hat{W}_2^i$, and we prefix this word with $\hat{W}_1^i$ and follow it with W_3^i. Note that for $j \in \mathbb{N}$, $|W_k^j| = 8 \cdot 3^{j-1}$ for $k = \{1, 2, 3\}$.

Proposition 4.1. *Let V_k^i be the word formed by placing* 10 *at the end of W_k^i, for $i > 1$, $k = \{1, 2, 3\}$. Define $\hat{V}_k^i$ similarly. Then no subword of the words $\{\hat{V}_1^i, V_2^i, \hat{V}_2^i, V_3^i, \hat{V}_3^i\}$ beginning in position $1, \ldots, |W_1^i|$ and ending in position $|V_1^i|$ is an initial subword of K. Further, V_1^i has no subword beginning in position $2, \ldots, |W_1^i|$ and ending in position $|V_1^i|$ which is an initial subword of K.*

Proof. We begin with the case $i = 2$. $W_1^2 =$ 10110110 11010111 10101111 and is always followed by 10. We may easily check that no subwords of
10110110 11010111 10101111 10 which start in positions $2, \ldots, |W_1^2|$ and end in position $|V_1^2|$ are initial subwords of K. We list the remaining words to consider for this case.

$\hat{V}_1^2 = \hat{W}_1^2 10 =$ 10110110 11010111 10101110 10

$V_2^2 = W_2^2 10 =$ 10110111 11010110 10101110 10

$\hat{V}_2^2 = \hat{W}_2^2 10 =$ 10110111 11010110 10101111 10

$V_3^2 = W_3^2 10 =$ 10110110 11010110 10101111 10

$\hat{V}_3^2 = \hat{W}_3^2 10 =$ 10110110 11010110 10101110 10

We see that for no subword beginning in position $1, \ldots, |W_1^2|$ of a word in the set $\{\hat{V}_1^2, V_2^2, \hat{V}_2^2, V_3^3, \hat{V}_3^3\}$ and ending in position $|V_1^i|$ of the respective word will be an initial subword of K.

Let $n \in \mathbb{N}$, and assume that V_1^n has no subwords that start in positions $2, \ldots, |W_1^n|$ and end in position $|V_1^n|$ that are initial subwords of K. Further assume that the words $\{\hat{V}_1^n, V_2^n, \hat{V}_2^n, V_3^n, \hat{V}_3^n\}$ have no subwords starting in positions $1 \ldots |W_1^n|$ and ending in position $|V_1^n|$ that are initial subwords of K.

We now wish to establish the same properties for $n+1$:

(1) No subword starting in position $2 \ldots |W_1^{n+1}|$ of V_1^{n+1} and ending in position $|V_1^{n+1}|$ is an initial subword of K.

(2) No subword starting in position $1 \ldots |W_1^{n+1}|$ of any of $\{\hat{V}_1^{n+1}, V_2^{n+1}, \hat{V}_2^{n+1}, V_3^{n+1}, \hat{V}_3^{n+1}\}$ and ending in position $|V_1^{n+1}|$ respectively, is an initial subword of K.

Recall that:

$$W_1^{n+1} = W_1^n W_2^n W_3^n$$

$$\hat{W}_1^{n+1} = W_1^n W_2^n \hat{W}_3^n$$

$$W_2^{n+1} = \hat{W}_1^n \hat{W}_2^n \hat{W}_3^n$$

$$\hat{W}_2^{n+1} = \hat{W}_1^n \hat{W}_2^n W_3^n$$

$$W_3^{n+1} = W_1^n \hat{W}_2^n W_3^n$$

$$\hat{W}_3^{n+1} = W_1^n \hat{W}_2^n \hat{W}_3^n$$

Notice that $V_1^{n+1}, \hat{V}_1^{n+1}, V_3^{n+1}$, and $\hat{V}_3^{n+1}$ all begin with V_1^n. We begin our analysis with V_1^{n+1}. By the induction hypothesis, subwords of V_1^{n+1} beginning in positions $2, \ldots, |W_1^n|$ and ending in position $|V_1^n|$, are not initial subwords of K. Therefore, subwords of V_1^{n+1} beginning in positions $2, \ldots, |W_1^n|$ and ending in position $|V_1^{n+1}|$ are not initial subwords of K. Similarly, subwords of V_1^{n+1} beginning in positions $|W_1^n|+1, \ldots, |W_1^{n+1}|$ and ending in position $|V_1^{n+1}|$ are handled by the induction hypothesis.

It is immediate that $\hat{V}_1^{n+1} = \hat{W}_1^{n+1}10$ is not an initial segment of K. For V_3^{n+1} and $\hat{V}_3^{n+1}$, although W_1^n is an initial subword of K, $W_1^n \hat{W}_2^n$ is not. Thus the subwords of V_3^{n+1} and $\hat{V}_3^{n+1}$ beginning in position 1 and ending in position $|V_3^{n+1}|$ fail to be initial subwords of K. The other subwords of $\hat{V}_1^{n+1}$, V_3^{n+1} and $\hat{V}_3^{n+1}$ to be considered, as well as all subwords of V_2^{n+1} and $\hat{V}_2^{n+1}$ of concern, are handled by the induction hypothesis.

□

We next prove Keesling's conjecture is false.

Theorem 4.1 (p.51). *$\hat{z} = (\ldots, z_{-3}, z_{-2}, z_{-1}, z_0)$ is not an endpoint.*

Proof. We wish to establish that there does not exist a sequence $\{z_{-k_i}\}_{i\in\mathbb{N}}$, such that $f^{-k_i}|_{\omega(c)}(z_0) = z_{-k_i}$ and the words $\{\tilde{z}_{-k_i} \ldots \tilde{z}_0\}_{i\in\mathbb{N}}$ are initial segments of K. Recall that an initial subword of the itinerary of z_0 is $\hat{W}_2^1 W_3^1$ and an initial subword of the itinerary of z_{-8} is $\hat{W}_1^1.\hat{W}_2^1 W_3^1$, with the "." indicating the beginning of the itinerary of z_0. We continue this pattern of prefixing $\hat{W}_1^i$ with $\hat{W}_1^{i+1}$, $i \to \infty$, and establish the initial subwords of the itineraries of $\{z_{-n}\}_{n\in\mathbb{N}}$. For $i > 1$, $\hat{W}_1^i$ will always be followed by 10 in this construction, forming the word $\hat{V}_1^i$. By Proposition 4.1, no subword beginning in position $1, \ldots, |W_1^i|$ of $\hat{V}_1^i$ and ending in position $|\hat{V}_1^i|$ is an initial subword of K. Therefore, none of the words $\{\tilde{z}_{-n} \ldots \tilde{z}_0\}$ for $n > 8$ are initial subwords of K. By [18, Proposition 2], $\hat{z}$ is not an endpoint.

□

Lemma 4.1. *Let $f : I \to I$ be a tent map with critical point c such that c is recurrent. Let $F = \varprojlim\{\omega(c), f|_{\omega(c)}\}$ be the set of folding points in $X = \varprojlim\{I, f\}$. Let E be the set of endpoints in X. Then E is a dense subset of F.*

Proof. Let $\hat{x} = (\ldots, x_{-2}, x_{-1}, x_0)$ be a point in F. Let U' be an open set in X and let $U = F \cap U'$, such that $\hat{x} \in U$. As previously noted, E is non-empty and $E \subseteq F$. Let $\hat{z} = (\ldots, z_{-2}, z_{-1}, z_0)$ be an element of E. Note that for all $k \in \mathbb{N}$, $\hat{f}^k(z) \in E$, where $\hat{f}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, z_{-2}, z_{-1})$. We are using the product topology: $U = (I^\infty \times U_{-n} \times \cdots \times U_0) \cap F$. Let $V \subset U_{-n}$ be open such that $x_{-n} \in V$ and $f^k(V) \subset U_j$, for $k = 1, \ldots, n$ and corresponding j values, $j = -n+1, \ldots 0$. Then for any $y_{-n} \in V \cap \omega(c)$, $\hat{y} = (\ldots, y_{-n}, \ldots, y_{-1}, y_0)$ will be in U by the definition of V and the fact that $\omega(c)$ is an invariant set. Since $x_{-n} \in \omega(c)$, there exists an $m \in \mathbb{N}$ such that $f^m(c) = c_m \in V$. Since z is an endpoint, by [18, Proposition 2], there exists a sequence $\{k_i\}_{i\in\mathbb{N}}$ such that $z_{-k_i} \to c$, and $f^m(z_{-k_i}) \to c_m$ by the continuity of f. Thus there exists a $t > n$ such that $z_{-t} \in V \cap \omega(c)$. This implies that $z' = (\ldots, z_{-(t-n)-1}, z_{-(t-n)}) \in U$ as desired.

$\square$

It follows from Lemma 4.1 that if $f : I \to I$ is a tent map with critical point c such that c is recurrent and $F - E \neq \emptyset$, the set of endpoints is not closed.

From 3.1, we know that if $f : I \to I$ is a tent map with critical point c such that c is recurrent and $F = E$, then c is uniformly recurrent. This implies that for a tent map with a critical point that is recurrent but not uniformly recurrent, the set of endpoints is not closed.

Lemma 4.2. *Let $f : I \to I$ be a tent map with critical point c such that c is uniformly recurrent. Let $F = \varprojlim\{\omega(c), f|_{\omega(c)}\}$ be the set of folding points in $X = \varprojlim\{I, f\}$. Let E be the set of endpoints in X. If $\hat{z} = (\ldots, z_{-2}, z_{-1}, z_0) \in F$, and $\hat{f}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, z_{-2}, z_{-1})$, then the orbit of $\hat{z}$ under $\hat{f}$ will be dense in F. In particular, if $F - E \neq \emptyset$, $F \backslash E$ is dense in F.*

Proof. It is known from 4 that if c is uniformly recurrent under f and $z \in F$, there exists a subsequence $\{z_{-n_i}\}_{i\in\mathbb{N}}$ of coordinates of z such that $z_{-n_i} \to c$. The proof therefore mirrors the proof of Lemma 4.1. $\square$

It follows from Lemma 4.2 that if $f : I \to I$ is a tent map with critical point c such that c is uniformly recurrent and $F - E \neq \emptyset$, the set $F - E$ is not closed. The critical point of a strange adding machine is regularly recurrent [12, Corollary

2.5]. Therefore, the strange adding machine presented in this paper provides an example of a map, f, such that the sets E and $F\backslash E$ are not closed with respect to $F = \varprojlim\{\omega(c), f|_{\omega(c)}\}$.

CHAPTER FIVE

Current Research

In this chapter, we present our current research in tiling spaces. We are interested in tiling spaces that are associated with substitution systems. We introduce a one-dimensional tiling space for its simplicity, but note that our work on substitution systems has implications for tiling spaces in higher dimensions. The following definitions may be found in [5], [6].

We begin with an alphabet $\mathcal{A} = \{1, \ldots, n\}$ and a substitution ϕ that sends each letter of $\mathcal{A}$ to a finite word which is a collection of letters from $\mathcal{A}$. For example, the Fibonacci substitution has the alphabet $\{1, 2\}$ and the substitution $\phi(1) = 12$, $\phi(2) = 1$. The substitution rule naturally extends to a mapping of words to words.

Definition 5.1. Let ϕ be a substitution on the alphabet $\mathcal{A} = \{1, \ldots, n\}$. We say that a bi-infinite word, w, is *allowed* if each finite subword of w is a subword of $\phi^n(i)$ for some $n \in \mathbb{N}$ and $i \in \mathcal{A}$.

Definition 5.2. Let $w = \ldots w_{-1}.w_0w_1\ldots$ be an allowed bi-infinite word for ϕ, and let W_ϕ be the collection of all such bi-infinite words. The *shift map*, $\sigma : W_\phi \to W_\phi$, is defined by the action $\sigma(\ldots w_{-1}.w_0w_1\ldots) = (\ldots w_{-1}w_0.w_1\ldots)$.

Associated with a substitution ϕ is an incidence matrix A_ϕ, where the entry in position a_{ij} is the number of times the letter i occurs in $\phi(j)$. For the Fibonacci substitution, the incidence matrix is $A_\phi = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition 5.3. A substitution, ϕ, is said to be *primitive* provided there exists an $n \in \mathbb{N}$ such that for all $i \in \mathcal{A}$, $\phi^n(i)$ contains each letter of $\mathcal{A}$. Similarly, [29, p. 678], a matrix A with non-negative entries is said to be *primitive* if there exists an $n \in \mathbb{N}$ such that A^n contains strictly positive entries.

Nb: A primitive substitution will have a primitive incidence matrix.

Definition 5.4. A bi-infinite word, w, for a substitution, ϕ, is said to be *ϕ-periodic* if there exists an m such that $\phi^m(w) = w$.

Definition 5.5. A substitution is *aperiodic* if it does not contain a ϕ-periodic word, w, which is periodic under the shift map. In other words, for every ϕ-periodic word, w, there does not exist a k such that $\sigma^k(w) = w$.

Definition 5.6. A substitution, ϕ, is a *constant length substitution* of constant length l, if for all $i \in \mathcal{A}$, $\phi(i)$ is a word of length l.

Definition 5.7. [22, Definition 2.8] Let ϕ be a primitive substitution of constant length l with ϕ-periodic word $w = \ldots w_{-1}.w_0w_1\ldots$. The *height* of the substitution, $h(\phi)$, is $\max\{n \geq l : n$ and l are relatively prime and n divides $\gcd\{a : w_a = w_0, a \geq 0\}\}$.

Nb: Under the assumptions of the definition, the height of ϕ is independent of the ϕ-periodic word chosen.

Every primitive substitution has at least one allowed ϕ-periodic word [6].

Definition 5.8. [22, Definition 2.18] A primitive substitution ϕ is *pure* if $h(\phi) = 1$.

Definition 5.9. [22] Let ϕ be a substitution of constant length l on the alphabet $\mathcal{A} = \{1, \ldots, n\}$. Let $i \in \mathcal{A}$. Note that the length of $\phi^k(i)$ will be l^k, and so we write $\phi^k(i) = \phi^k(i)_1\phi^k(i)_2\ldots\phi^k(i)_{l^k}$. For a fixed $r \leq l^k$, the set $\{\phi^k(1)_r, \phi^k(2)_r, \ldots, \phi^k(n)_r\}$, $k \geq 1$, is called a *column* of ϕ.

Definition 5.10. [22, Definition 3.1] Let $c_{k,r}$ represent the cardinality of the column $\{\phi^k(1)_r, \phi^k(2)_r, \ldots, \phi^k(n)_r\}$. The *column number*, $c(\phi)$, is $\min\{c_{k,r}\}$, $1 \leq r \leq l^k$, $k \geq 1$.

We are interested in asymptotic cycles and proximal cycles in tiling spaces that arise from substitution systems. The well-known Perron-Frobenius Theorem is important to our transition from a substitution system to a tiling space [24, p.10], [27, p.109], [29, p.667].

Theorem 5.1 (Perron-Frobenius). *Let A be a primitive matrix. Then A has a strictly positive eigenvalue λ with associated positive left and right eigenvectors, such that for any other eigenvalue, μ, $|\mu| < \lambda$.*

We call this leading eigenvalue the Perron-Frobenius eigenvalue.

The primary source for our tiling space description is [6]. Let ϕ be a primitive substitution on n letters with Perron-Frobenius eigenvalue λ of its associated transition matrix. Let ω be the left eigenvector associated with λ, and denote the entries of ω by $(\omega_1, \ldots, \omega_n)$. The *prototiles* for ϕ are the intervals $P_i = [0, \omega_i]$, $i \in \{1, \ldots, n\}$. A tiling of the real line, $T = \{T_i\}_{-\infty}^{\infty}$, is a covering of the line with tiles such that each T_i is a translate of one of $\{P_i\}_{i=1}^{n}$, and two tiles, T_i, T_j, $i \neq j$, intersect in at most one point. We say that a tile T_i from a tiling T is of type j if it is a translate of the prototile P_j. We may therefore associate with $T = \{T_i\}_{-\infty}^{\infty}$ a bi-infinite word, $\ldots w_{-1}w_0w_1\ldots$, where $w_i = j$ if T_i is of type j. We define our tiling space $\mathcal{T}_\phi$ to be the collection of tilings such that for any $T \in \mathcal{T}_\phi$, the bi-infinite word associated with T is an allowed bi-infinite word for ϕ. The flow on $\mathcal{T}_\phi$ is translation by $t \in \mathbb{R}$, where $T - t = \{T_i - t\}_{-\infty}^{\infty}$. The flow acting on $\mathcal{T}_\phi$ induces an equivalence relation where two tilings, T and S, are equivalent if there exists $t \in \mathbb{R}$ such that $T = S - t$.

Let Σ_ϕ represent the collection of all tilings of $\mathbb{R}$. There is a natural topology on Σ_ϕ, where two tilings, $T = \{T_i\}_{-\infty}^{\infty}$ and $T' = \{T'_i\}_{-\infty}^{\infty}$, are "close" if there exists an ϵ near 0 such that $\{T_i\}_{-\infty}^{\infty}$ and $\{T'_i + \epsilon\}_{-\infty}^{\infty}$ are identical in a large neighborhood of the origin. More formally:

Definition 5.11. [1] Let $T, T' \in \Sigma_\phi$. The *distance*, $d(T, T')$, is $\inf\{\epsilon \mid T + u$ and $T' + v$ agree on the neighborhood $[-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$ about the origin for some $|u|, |v| < \epsilon\}$.

The space Σ_ϕ is compact and metrizable with this topology [7].

Definition 5.12. [33, p.208] Let K be a continuum. For $p \in K$, consider the set C_p of all points x of K such that a proper subcontinuum of K contains both p and x. We call C_p the *composant* of p.

For a primitive substitution, the composants and arc components of $\mathcal{T}_\phi$ are the same. If $\mathcal{C}$ is a composant of $\mathcal{T}_\phi$, then $\mathcal{C}$ contains the points of $\mathcal{T}_\phi$ that are in an orbit equivalence class induced by the flow.

Definition 5.13. Two tilings $T, T' \in \mathcal{T}_\phi$ are *forward asymptotic* if $\lim_{t\to\infty} dist(T - t, T' - t) = 0$. If composants contain forward asymptotic tilings, the composants are said to be *forward asymptotic*. *Backward asymptotic* tilings and composants are similarly defined.

Definition 5.14. Two tilings $T, T' \in \mathcal{T}_\phi$ are *proximal* if the infimum over $t \in \mathbb{R}$ of $dist(T - t, T' - t)$ is zero. If composants contain proximal tilings, the composants are said to be *proximal*.

The distinction between asymptotic composants and proximal composants is that proximal composants are allowed to have gaps of separation of arbitrary distance.

Definition 5.15. An *asymptotic cycle* is a collection of composants, $\{C_1, C_2, \ldots, C_{2n}\}$ such that C_1 is forward asymptotic to C_2, C_2 is backward asymptotic to C_3, C_3 is forward asymptotic to C_4, $\ldots$, C_{2n} is backward asymptotic to C_1.

Our initial goal was to answer the following question posed by Marcy Barge: If ϕ is a pure, primitive, aperiodic substitution of constant length with column height greater than one, does $\mathcal{T}_\phi$ contain an asymptotic cycle?

Let ϕ be a substitution and $\mathcal{A} = \{1, \ldots, n\}$ the alphabet on which ϕ is defined. If $\phi(a) = a\ldots$ for some $a \in \mathcal{A}$, then we say that a is a *starting rule* of the substitution. If $\phi(b) = \ldots b$ for some $b \in \mathcal{A}$, then we say that b is a *stopping rule* of the substitution. Suppose that b is a stopping rule, a is a starting rule, and the word ba appears as a subword of $\phi^k(i)$ for some $i \in \mathcal{A}$, $k \geq 1$, i.e. ba is an allowed word of ϕ, and $\phi(b) = ub$, $\phi(a) = av$ for words u, v. The word $w = \ldots \phi^2(u)\phi(u)ub.av\phi(v)\phi^2(v)\ldots$ is both allowed for the substitution and fixed by ϕ [5]. We say that the fixed word w is generated by $b.a$.

On a basic level, if we can identify $2n$ generators for fixed words of a substitution, ϕ, of the form $\{b_1.a_1, b_2.a_1, b_2.a_2, \ldots b_1.a_n\}$, then $\mathcal{T}_\phi$ has an asymptotic cycle; see [6]. This is enough to prove the existence of an asymptotic cycle in the case that ϕ is a pure, primitive, aperiodic substitution of constant length acting on an alphabet of n letters with $c(\phi) = n$.

Lemma 5.1. *Let $\mathcal{A} = \{1, \ldots, n\}$. If θ is a constant length substitution on $\mathcal{A}$ such that $c(\theta) = n$, then for all $a \in \mathcal{A}$, there exists m such that $\theta^m(a)$ is a word that begins and ends with a.*

Proof. Suppose that there exists $a_1 \in \mathcal{A}$ such that $\theta(a_1) = a_2w$, $a_1 \neq a_2$ and w is a word. Then since for all $k \geq 1$, the columns of $\theta^k(1), \ldots, \theta^k(n)$ have cardinality n, $\theta(a_2)$ does not begin with a_2. Let the first letter of $\theta(a_2)$ be a_3. This process will continue until we reach a_t such that $\theta(a_t)$ begins with a_1. Let $\{a_1, \ldots, a_t\}$ be a set of length t, where $\theta(a_i)$ begins with a_{i+1}, for $i \in \{1, \ldots, (t-1)\}$, and $\theta(a_t)$ begins with a_1. We call such a set a *cycle*. If we consider θ^2, then $\theta^2(a_i)$ begins with a_{i+2} for $i \in \{1, \ldots, (t-2)\}$, and $\theta^2(a_{t-1})$ begins with a_1, $\theta^2(a_t)$ begins with a_2. If we consider θ^j, then $\theta^j(a_i)$ begins with a_{i+j} for $i \in \{1, \ldots, (t-j)\}$, and $\theta^j(a_{(t-j)+1)})$ begins with a_1, $\ldots$, $\theta^j(a_t)$ begins with a_j.

Setting $j = t$, we see that $\theta^t(a_i)$ begins with a_i for $a_i \in \{a_1, \ldots, a_t\}$. Since $\mathcal{A}$ is a finite alphabet, there exist a finite number of cycles, each associated with some t; the least common multiple of which we will call m_1. Then for all $a \in \mathcal{A}$, $\theta^{m_1}(a)$ begins with a.

By a similar argument, there exists an m_2 such that for all $a \in \mathcal{A}$, $\theta^{m_2}(a)$ ends with a. If we let $m = m_1 \cdot m_2$, then $\theta^m(a) = aw_a a$ for all $a \in \mathcal{A}$. $\square$

Theorem 5.2. *Let $\mathcal{A} = \{1, \ldots, n\}$. If θ is a pure, primitive, aperiodic, constant length substitution on $\mathcal{A}$ such that $c(\theta) = n$, then θ has an asymptotic cycle.*

Proof. Since if θ^m has an asymptotic cycle, θ has an asymptotic cycle, we may assume that $m = 1$ in Lemma 5.1. It follows that the lists of starting and stopping rules for θ will each be $\{1, \ldots, n\}$.

For the purpose of establishing our asymptotic cycle, we look at pairs of letters that are adjacent under $\theta^k(a)$ for all $k \geq 1$, and all $a \in \mathcal{A}$.

Suppose that for $a \in \{1, \ldots, n\}$, a is followed by only one letter b_a. Then we have the adjacent pairs: $\{(1\ b_1)\ , (2\ b_2)\ , \ldots (n\ b_n)\}$. By assumption, the first column of our substitution is $1, 2, \ldots, n$. We must eventually have a column that is different than column one, so we assume without loss of generality that column 2 is different than column 1. This establishes the first two letters of $\theta(1)$ as $1\ b_1$. Since we have column height n, if $b_1 \neq 1$, then we may assume that $b_1 = 2$. If $b_2 = 1$, then we will say that the elements of $\{1, 2\}$ form a cluster. If $b_2 \neq 1$, then we may assume that $b_2 = 3$. If $b_3 = 1$, then the elements of $\{1, 2, 3\}$ form a cluster; otherwise we continue until we reach a t such that $b_t = 1$. If $t = n$, then θ is periodic, since as $k \to \infty$, $\theta^k(1) = 1\ 2\ \ldots n\ 1\ 2 \ldots n \ldots$. If $t < n$, then θ is not primitive since for all k, $\theta^k(1)$ begins with 1 and will only witness elements from its cluster.

This means that for some letter $a \in \mathcal{A}$, a is followed by something other than b_a. Without loss of generality we may assume that 1 is followed by b_1 and c_1 where $b_1 \neq c_1$. There must exist two columns under θ^k for some k such that the adjacent pair $(1\ c_1)$ is witnessed. Let r be the column, which by assumption contains each of $\{1, \ldots, n\}$, so that 1 is in position r of $\theta^k(p_1)$ for some k and p_1, and c_1 is in position $r + 1$ of $\theta^k(p_1)$. Since $c_1 \neq b_1$, then $c_1 = b_{q_1}$, $q_1 \neq 1$. Without loss of generality, we may assume that $q_1 = 2$. This means that for 2 in position r of $\theta^k(p_2)$, b_2 is not in position $r + 1$ of $\theta^k(p_2)$. Let c_2 be in position $r + 1$ of $\theta^k(p_2)$. Therefore, 2 appears in the horizontal pair $(2\ b_2)$ for $\theta(2)$ by a previous argument, and 2 appears in the horizontal pair $(2\ c_2)$ under $\theta^k(p_2)$, $b_2 \neq c_2$.

Since $c_1 = b_2$, and the pairs $(1\ \ b_1), (1\ \ c_1)$ have been witnessed, if $c_2 = b_1$, then we have an asymptotic cycle consisting of the fixed words generated by $1.b_1, 1.c_1, 2.b_1, 2.c_1$.

If $c_2 \neq b_1$, it must be the case that $c_2 = b_{q_2}$ for some $q_2 \neq 1, q_2 \neq 2$. We continue this process until we reach t such that $c_t = b_1$, and the pairs $\{(1\ c_1), (2\ c_2), \ldots, (t\ c_t)\}$ are witnessed under θ^k. Further, $c_i = b_{i+1}$ for $i \in \{1, \ldots, (t-1)\}$, and $c_t = b_1$. Recall that we have already witnessed the pairs $\{(1\ \ b_1), (2\ \ b_2), \ldots, (t\ \ b_t)\}$. We therefore have the asymptotic cycle consisting of the fixed words generated by $\{1.b_1, 1.c_1, 2.c_1, 2.c_2, 3.c_2, 3.c_3, \ldots, t.c_{t-1}, t.b_1\}$.

$\square$

The existence of an asymptotic cycle in the tiling spaces of all pure, primitive, aperiodic substitutions of constant length with column height greater than one did not turn out to be true. We found a counterexample to Barge's conjecture when we investigated the case of substitutions on four-letter alphabets for which $c(\phi) = 2$.

We are currently exploring the existence and usefulness of proximal cycles in substitutions where no asymptotic cycle exists.

CHAPTER SIX

Conclusion

In this thesis we considered adding machine dynamics as they appeared in unimodal maps, finite graphs and tiling spaces. We gave a complete characterization of the kneading sequence structure for the recently discovered strange adding machines. We addressed the question of James Keesling, showing that for infinitely renormalizable unimodal maps the set of folding points of the inverse limit space will be equal to the set of endpoints of the inverse limit space, and for non-renormalizable maps, this equality does not always hold. We showed that uniform recurrence of the critical point is a necessary condition for equality of the set of endpoints and the set of folding points in the inverse limit space of a tent map with recurrent critical point. Further, since the set of endpoints is a dense subset of the set of folding points, we have identified a class of tent maps for which the set of endpoints is not closed, answering a question posed by Henk Bruin.

In our current research on tiling spaces, we proved Barge's conjecture on the existence of an asymptotic cycle in constant length substitutions in the special case where the column height of the substitution and the cardinality of the alphabet on which the substitution is acting are the same, and found a counterexample in the general case.

We remain interested in inverse limit spaces of unimodal maps. We would like to find sufficient conditions for a unimodal map, f, such that the set of endpoints of the inverse limit space of f and the set of folding points of the inverse limit space of f are equal. Another goal is to better understand the structure of the inverse limit space of a strange adding machine and the implications of our results on this structure.

We also have a continued interest in tiling spaces. We are working to determine the relationship between the contributions of an asymptotic cycle and a proximal cycle for a pure, primitive, aperiodic, constant length substitution to the degree of the Čech cohomology of the associated tiling space.

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