ABSTRACT

On the Goldbach Conjecture

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Goldbach's conjecture, proposed in 1742, is one of the oldest open questions in mathematics. Much work has been done on the problem, and despite significant progress, a solution remains elusive. The goal of this paper is to give an introduction to Goldbach's conjecture, discuss the history of the problem, summarize important papers on the subject, examine methodologies used to attack the problem, and explain related problems and consequences of the conjecture.

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# ON THE GOLDBACH CONJECTURE

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By

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For Nick Satullo. You’re the best.

# CHAPTER ONE

## Introduction

The Goldbach conjecture considers a relationship between all natural numbers greater than 2 and the prime numbers. A prime number $p$ is a natural number such that when $p = ab$, the only possible choices for $a$ and $b$ in the natural numbers are 1 and $p$. Goldbach originally considered 1 to be a prime, but today we consider 2 to be the smallest prime. The first few primes are $2, 3, 5, 7, 11, 13....$ In 1742, Christian Goldbach proposed Conjecture 1 in a letter to Leonhard Euler [39].

**Conjecture 1.** *Every integer greater than* 2 *can be written as the sum of at most three primes.*

**Example 1.** Some examples of Conjecture 1:

$3 = 1 + 1 + 1$

$5 = 2 + 2 + 1$

$13 = 5 + 5 + 3$

$27 = 13 + 7 + 7$

$100 = 91 + 7 + 2$

Euler replied that he believed the conjecture to be true, but he could not prove it. Later, Goldbach remarked that his original conjecture follows from what we now know as the strong Goldbach conjecture.

**Conjecture 2** (Strong Goldbach Conjecture)**.** *Every even integer greater than* 4 *can be written as the sum of two odd primes.*

**Example 2.** Some examples of Conjecture 2:

$6 = 3 + 3$

$34 = 31 + 3 = 29 + 5 = 23 + 11 = 17 + 17$

$66 = 61 + 3 = 59 + 7 = 53 + 13 = 47 + 19 = 43 + 23 = 37 + 29$

$100 = 97 + 3 = 89 + 11 = 83 + 17 = 71 + 29 = 59 + 41 = 53 + 47$

From the strong conjecture, an immediate corollary:

**Corollary 1.** *If Conjecture 2 is true, then every odd integer greater than* 7 *can be written as the sum of three odd primes.*

*Proof.* Suppose the strong Goldbach conjecture to be true. Then, we can write an even $n \geq 6$ as $n = p + q$, where $p$ and $q$ are odd primes. It follows then that every odd $k \geq 9$ can be written as $k = n + 3 = p + q + 3$. □

**Example 3.** Examples of Corollary 1:

$9 = 3 + 3 + 3$

$39 = 31 + 5 + 3$

$77 = 71 + 3 + 3$

The primes 3, 5, and 7 are omitted from Corollary 1 because they can not be represented as the sum of three odd primes. Since 3 is the smallest odd prime, $9 = 3 + 3 + 3$ is the smallest integer represented as the sum of 3 odd primes. The conclusion of Corollary 1 is known as the weak, or ternary, Goldbach conjecture.

By the same method as the proof of Corollary 1, if the weak Goldbach conjecture is proven independently, it will imply every even integer greater than 2 is the sum of at most four primes.

Currently, the weak Goldbach conjecture has been solved by Deshouillers et al. when assuming the generalized Riemann hypothesis and Terence Tao has demonstrated that all odd numbers are the sum of at most five odd primes without assuming the generalized Riemann hypothesis.

Olivier Ramaré has proven that all even numbers larger than 2 can be written as a sum of no more than six primes, independent of the generalized Riemann hypothesis.

*Biography of Christian Goldbach*

Christian Goldbach was born March 18, 1690 in Königsberg, part of modern-day Kaliningrad Oblast, a satellite of Russia located between Poland and Lithuania. He studied at Royal Albertus University and taught at the St. Petersburg Academy of Sciences. He also worked for the Russian Ministry of Foreign Affairs and died on November 20, 1764.

Goldbach is best known for the Goldbach conjecture, but he also proved several theorems and made significant progress on several more through correspondence with mathematicians such as Leonhard Euler, Gottfried Liebniz, and Nicholas Bernoulli [18].

*History*

The Goldbach conjecture dates back to a letter from Christian Goldbach to Leonhard Euler on June 7, 1742, although René Descartes may have been aware of this problem in the early 17th century [1].

A major breakthrough did not occur until Godfrey Hardy and John Littlewood [13] showed in 1923 that the weak Goldbach conjecture is true for all sufficiently large odd numbers when assuming the generalized Riemann hypothesis.

In 1930, Lev Schnirelmann [29] made progress on the strong Goldbach conjecture by showing that every even integer greater than 2 can be written as the sum of at most twenty primes, a result that has been improved several times.

Next, Ivan Vinogradov [36] improved Hardy and Littlewood's results in 1937 by proving that the weak conjecture holds for all sufficiently large odd numbers without assuming the Riemann hypothesis.

Using methods developed by Vinogradov, over the next two years Nikolai Chudakov [4], J. G. van der Corput [34] and T. Estermann [9] proved that all but a finite number of even integers can be written as the sum of two primes.

In 1939, K. Borozdin [33] was the first to find an upper bound, $3^{14348907}$, for describing Vinogradov's "sufficiently large odd number."

The most recent improvement to Schnirelmann's results on the strong conjecture is by Olivier Ramaré [24]. In 1995, Ramaré demonstrated that even numbers larger than 2 can be written as the sum of at most six primes.

In the same year, Leszek Kaniecki [15] showed that every odd natural number is the sum of at most five primes when assuming the Riemann hypothesis, implying every even natural number is the sum of at most six primes.

In 1997, D. Zinoviev [41] proved that all odd integers greater than $10^{20}$ are the sum of at most three primes when assuming the generalized Riemann hypothesis. Immediately following this, J. M. Deshouillers, G. Effinger, H. te Riele, and D. Zinoviev [8] improved Kaniecki's result by demonstrating that the weak conjecture is true assuming the generalized Riemann hypothesis.

Y. Saouter [28] computationally proved in 1998 that the largest gap between two primes smaller than $10^{20}$ is less than $4 \times 10^{11}$, meaning all odd natural numbers less than $10^{20}$ are the sum of at most three primes. Combined with D. Zinoviev's paper the previous year, this provides a second argument for the truth of the weak Goldbach conjecture under the generalized Riemann hypothesis.

The upper bound found by Borozdin was lowered to $e^{3100}$ by Liu Ming-Chit and Wang Tian-Ze [17] in 2002.

In a recent 2012 preprint to appear in *Mathematics of Computation*, Terence Tao [32] improved Kaniecki's result by proving that all odd integers are the sum of at most five primes without the Riemann hypothesis.

Also in 2012, Tomás Olivera e Silva computationally verified that the strong conjecture is true up to $4 \times 10^{18}$.

As for related results, in 1953 Linnik [16] proved there is some constant $K$ such that every sufficiently large even number is the sum of two primes and at most $K$

powers of 2. Chen Jingrun [3] proved in 1973 that all sufficiently large even numbers can be written as either $p_1 + p_2$ or $p_1 + p_2p_3$ where $p_k$ is prime.

Also in 2002, Roger Heath-Brown and Jan-Christoph Schlage-Puchta [14] improved Linnik's result by showing $K = 13$ is sufficient and a year later Pintz and Ruzsa [22] showed only $K = 8$ is necessary.

## CHAPTER TWO

## Methods of Proof

An effective method for proving a conjecture about the nature of all the natural numbers is to partition the set of natural numbers into several finite subsets and one infinite subset and prove the conjecture for each of those subsets. Most recently using this method, Ramaré and Saouter (2003) [23] have demonstrated that integers less than $1.13 \times 10^{22}$ are the sum of at most three odd primes. Additionally, Liu and Wang (2002) [17] have shown that integers greater than $e^{3100} \approx 2.06 \times 10^{1346}$ are the sum of at most three odd primes. What remains is to determine how to handle the interval $[1.13 \times 10^{22}, e^{3100}]$.

### *Hardy-Littlewood Circle Method*

The Hardy-Littlewood method was originally developed to attack the Waring problem, which asks for each natural number $k$, what value must $s$ take such that every natural number is the sum of at most $s$ $k$th powers of natural numbers.

One starts with $\mathcal{A} = (a_m)$, a strictly increasing sequence of nonnegative integers. Let $F(z) = \sum_{m=1}^{\infty} z^{a_m}$. If $R_s(n)$ is the number of representations of natural number $n$ as the sum of $s$ members of $\mathcal{A}$, then

$$R_s(n) = \frac{1}{2\pi i} \int_{\mathcal{C}} F(z)^s z^{-(n+1)} dz,$$

where $\mathcal{C}$ is a circle centered at 0 with radius $0 < \rho < 1$ [35].

The Hardy-Littlewood method deals with how to evaluate the integral when the radius of the circle is precisely 1. In this form, one can not evaluate the contour integral over the circle of radius 1 because there exist singularities that lie on the circle when the radius is 1. For example, $\sum_{m=1}^{\infty} 1^{a_m}$ diverges. The solution is to break the circle into major arcs, which are intervals of (0,1] containing the singularities of the function $F$ on the unit circle, and minor arcs, which contain whatever is left over. The integral when $r = 1$ is then estimated by separately estimating the integrals over the major and minor arcs and summing the two.

This method must be adapted slightly in order to attack the Goldbach conjecture. When working on the weak conjecture, Vinogradov [36] found that if he replaced $F(z)$ with $f(\alpha) = \sum_{p \leq n} \log(p) e^{2\pi i \alpha p}$ and let $\mathcal{A}$ be the set of all primes less than or equal to a natural number $n$, which changes the infinite summation to a finite one, then one can derive

$$(1) \qquad \int_{\mathcal{M}} f(\alpha)^3 e^{-2\pi i n \alpha} d\alpha + \int_{m} f(\alpha)^3 e^{-2\pi i n \alpha} d\alpha = \sum_{\substack{p_1, p_2, p_3 \\ p_1 + p_2 + p_3 = n}} \log(p_1) \log(p_2) \log(p_3).$$

Here, $\mathcal{M}$ and $m$ represent the major and minor arcs respectively. Clearly, if (1) is greater than 0, then $n$ can be represented as the sum of three primes. In evaluating the integrals of the major and minor arcs, Vinogradov found that there exists some natural number $n$ such that (1) is greater than 0 for all natural numbers bigger than $n$ [35].

*Selberg Sieve*

Another method employed in theorems related to the Goldbach conjecture is the Selberg sieve. Atle Selberg developed his sieve in the 1940s to estimate the size of a sifted set, which is a set of positive integers whose members meet some list of conditions.

Suppose $\mathcal{A}$ is a finite set of natural numbers and $\mathcal{P}$ is a set of primes. Let $\mathcal{A}_p$ be a subset of $\mathcal{A}$ for every $p \in \mathcal{P}$. Suppose $z$ is a positive real number and set $P(z) := \prod_{\substack{p \in \mathcal{P} \\ p < z}} p$.

The purpose of the sieve is to give an upper bound to the size of

$$S(\mathcal{A}, \mathcal{P}, z) = \mathcal{A} \setminus \bigcup_{p \mid P(z)} \mathcal{A}_p.$$

Next, let $d$ represent a squarefree number that is the product of members of $\mathcal{P}$, and define $\mathcal{A}_d := \cup_{p|d}\mathcal{A}_p$. Assume there exist $X > 0$ and a multiplicative function $f$ that satisfies $f(p) > 0 \ \forall p \in \mathcal{P}$, and for every squarefree $d$ there is a real number $R_d$ with

$$|\mathcal{A}_d| = \frac{X}{f(d)} + R_d.$$

A multiplicative function $f(n), n \in \mathbb{N}$ is one such that $f(1) = 1$ and whenever $gcd(a, b) = 1$, then $f(ab) = f(a)f(b)$. Then define

$$f(n) = \sum_{d|n} g(d),$$

where $g$ is some multiplicative function uniquely determined by $f$. Since both $f$ and $g$ are multiplicative, we can use Möbius inversion [40] to get

$$g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d}),$$

where $\mu(d)$ is the Möbius function:

$$\mu(d) = \begin{cases} -1 & \text{if d square free with an odd number of prime factors} \\ 1 & \text{if d square free with an even number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

Then, if we define

$$V(z) = \sum_{\substack{d \leq z \\ d|P(z)}} \frac{\mu^2(d)}{g(d)},$$

by Selberg (1947), we can conclude

**Theorem 1** (Selberg Sieve)**.**

$$S(\mathcal{A}, \mathcal{P}, z) \leq \frac{X}{V(z)} + \mathcal{O}\left( \sum_{\substack{d_1, d_2 \leq z \\ d_1, d_2 | P(z)}} |R_{\gcd(d_1,d_2)}| \right),$$

where $\mathcal{O}(f(x))$ represents a function $g(x)$ such that $g(x) \leq cf(x)$ for some constant $c$, and $X > 0$. This is known as big O notation and describes how a function $f(x)$ behaves asymptotically as $x$ tends towards infinity.

**Example 4.** Use the Selberg sieve to give an upper bound for the prime counting function, $\pi(x; k, a) = \#\{p \leq x \,|\, p \equiv a (\mathrm{mod}\, k)\}$.

Let

$$\mathcal{A} = \{n \leq x \,|\, n \equiv a (\mathrm{mod}\, k)\},$$

and

$$\mathcal{P} = \{p \,|\, \gcd(p, k) = 1\}.$$

Then we see that

$$S(\mathcal{A}, \mathcal{P}, z) = \#\{n \leq x \,|\, n \equiv a (\mathrm{mod}\, k), p \nmid n \; \forall p \in \mathcal{P}\}.$$

That is, $S(\mathcal{A}, \mathcal{P}, z)$ contains the integers less than $x$ and congruent to $a$ modulo $k$ that are also not multiples of primes in $\mathcal{P}$. Thus, an upper bound of $S(\mathcal{A}, \mathcal{P}, z)$ is also an upper bound of $\pi(x; k, a)$. Because a $p \in \mathcal{P}$ is coprime to $k$,

$$|\mathcal{A}_d| = \frac{x}{kd} + \mathcal{O}(1).$$

Using the notation described above, let $X = \frac{x}{k}$, $f(d) = d$, $g(d) = \phi(d)$, and $R_d = \mathcal{O}(1)$. Then

$$V(z) = \sum_{\substack{d \le z \\ (d,k)=1}} \frac{\mu^2(d)}{\phi(d)},$$

and so

$$\pi(x;k,a) \le S(\mathcal{A},\mathcal{P},z) \le \frac{x}{kV(z)} + \mathcal{O}(z^2).$$

This example is part of a proof of the Brun-Titchmarsh theorem [7]. A little more manipulation will give an explicit formula for an upper bound to $\pi(x;k,a)$.

Similar methods are used in proofs concerning the Goldbach conjecture, some of which use the Selberg sieve to approximate primes in arithmetic progression, like in the example, while others use it to approximate the prime counting function. Generally, more precise values for $f(x)$ and $X$ are chosen in order to attain a more accurate upper bound for $S(\mathcal{A},\mathcal{P},z)$.

*The Riemann Hypothesis*

The Riemann hypothesis is one of the millennium problems determined by the Clay Institute and has seen significant progress, but no proof. The nontrivial zeroes of the Riemann zeta function are intimately connected with the prime numbers, and thus the Riemann hypothesis and Goldbach conjecture are related.

In 1859, Bernhard Riemann [26] conjectured:

**Conjecture 3** (Riemann hypothesis)**.** *All nontrivial zeros of the Riemann zeta function lie on the critical line, $\Re(s) = \frac{1}{2} + it$, where $\frac{1}{2}$ is the real component and $it$ is the imaginary component.*

The Riemann zeta function is defined for all complex numbers $s \neq 1$ as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Euler demonstrated an equivalence from the Riemann zeta function to

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}},$$

where the product is taken over the primes, $p$. Because of this equivalence, the Riemann hypothesis is connected with the distribution of primes and the number of zeros of the Riemann zeta function less than some natural number $x$ is an approximation of the number of primes less than $x$, known as $\pi(x)$.

Riemann also suggested that the prime counting function, $\pi(x)$, is approximated by $Li(x) = \int_0^x \frac{dt}{\log(t)}$, but this was not proven until later.

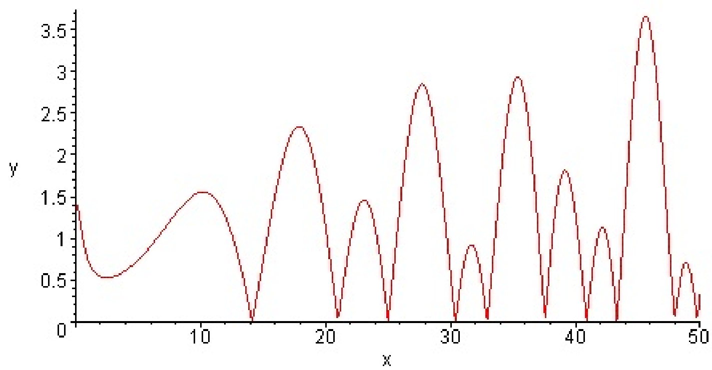


FIGURE 1. $\zeta(s)$ for $0 \le s \le 50$

Seen in Figure 1, the first few zeros of the Riemann zeta function occur at imaginary components 14.135, 21.022, 25.011, 30.425, 32.935, 37.586, 40.919, 43.327, 48.005, and 49.774 [19].

The generalized Riemann hypothesis concerns Dirichlet L-functions, of which the Riemann Zeta function is a special case. Suppose $\chi_k(n)$ is a multiplicative, arithmetic function that has period $k$. An arithmetic function for natural numbers $n$ and $m$, $\psi(n)$ is one such that $\psi(n+m) = \psi(\psi(n) + \psi(m))$ and $\psi(nm) = \psi(\psi(n)\psi(m))$. A period of $k$ means $\psi(n) = \psi(n + km)$, where $n$, $k$, and $m$ are natural numbers. The Dirichlet L-function of $\chi_k$ is defined as

$$L_k(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s},$$

when $s$ is a complex number [38].

**Conjecture 4** (Generalized Riemann Hypothesis)**.** *If $L_k(s, \chi) = 0$, then $s$ does not have a real part larger than $\frac{1}{2}$.*

Kaniecki [15] notes that it is not known how closely the Riemann hypothesis and Goldbach conjecture are related, since the distribution of zeroes of the Riemann zeta function gained by assuming the Riemann hypothesis is not immediately useful in proving theorems relating to Goldbach's problem. Deshouillers et al. [8] show that the generalized Riemann hypothesis is much more useful, since one is not constrained to the case when $\chi_k(n) = 1$, like in the case of the standard Riemann hypothesis.

*Computing*

With the rise in computing power, computers have played an increasingly important role in the verification of theorems. Perhaps the most famous theorem to be proven with the assistance of a computer is the four color theorem, which states that at most four colors are needed to color any map of contiguous regions such that no two regions of the same color are adjacent. Concerning the Goldbach conjecture, Olivera e Silva (2012) [20] has determined that even natural numbers up to $4 \times 10^{18}$ are the sums of two odd primes, Saouter [28] has shown that all odd numbers up to $10^{20}$ are the sum of three primes, and Wedeniwski [37] has verified that all the nontrivial zeroes of the Riemann zeta function in the interval $[0, 3.29 \times 10^9]$ lie on the critical line, $\Re(s) = \frac{1}{2}$.

Some concern has been raised over the validity of computer-assisted proofs. Much of what a computer does can not reasonably be verified by a human being, so there is uncertainty about whether the program and its output are truly error free. This can be overcome by defining what constitutes a computer-based proof and by verifying results in multiple computer languages, compilers, and with different hardware. An alternative resolution is to use computers to verify human work, instead of using them to do work that can not reasonably be done otherwise.

Another objection to using computers for brute-force proof of theorems is that they provide no insight into new or useful concepts, since they simply exhaust possibilities. For many purposes, the Goldbach conjecture and Riemann hypothesis included, modern computers are not powerful enough to come close to providing a complete proof. However, they are capable of providing useful lower bounds with which to build theorems [5].

On the other hand, the likelihood that a top-tier theorem proving program will return a false positive that is not caught by human inspection as part of the verification process is much lower than the error rates in prestigious math journals [12]. A computer-assisted proof can always be re-verified once a new method is developed and retracted if it is discovered to be a false positive, just like purely human-constructed proofs.

# CHAPTER THREE

## Summaries of Major Papers

### *On the Vinogradov bound in the three primes Goldbach conjecture*

M. Liu and T. Wang [17] use the Hardy Littlewood circle method to prove:

**Theorem 2.** *Every odd integer* $\geq e^{3100}$ *is a sum of three odd primes.*

The authors employ several new methods to improve previous bounds, including a refinement of certain numerical estimates and a slight change to the general Hardy-Littlewood circle method. Usually with the circle method, the unit interval is broken into several disjoint subsets, for which Liu and Wang use $M_j, 1 \leq j \leq 4$, where $M_1 \cup M_2$ contain the major arcs, and $M_3 \cup M_4$ are the minor arcs. They split these unions and treat $M_2$ as a minor arc in order to obtain a better lower bound for the integral over the major arc, $M_1$, better upper bounds for the integral over the minor arcs $M_2$ and $M_3$, and a better Vinogradov estimate, which approximates sums of the form $\sum\limits_{p \leq N} e^{2i\alpha p}$ where $\alpha$ is real and $p$ a prime, for $M_4$.

Combining these modified methods and computational work done on Mathematica, Liu and Wang find that the value of the integral found using the circle method has a sufficiently large lower bound, which then allows them to conclude Theorem 2, which is independent of the truth of the generalized Riemann hypothesis.

### *On Šnirel'man's Constant*

Olivier Ramaré [24] focuses most of his paper on proving

**Theorem 3.** *For* $x \geq e^{67}$*,*

$$|\{n \in [x, 2x] : \exists p_1, p_2 \ \textit{primes, with } n = p_1 + p_2\}| \geq \frac{x}{5}$$

which he then uses to prove

**Theorem 4.** *Every even integer is the sum of at most 6 odd primes.*

To prove Theorem 3, Ramaré uses the Selberg sieve to determine an upper bound for the number of representations of an even integer as the sum of two primes. He shows that this upper bound is asymptotic to a function, which is then used to construct an estimate for the size of $|\{n \in [x, 2x] : \exists p_1, p_2 \text{ primes, with } n = p_1 + p_2\}|$. This estimate is split into three disjoint pieces and Ramaré proves bounds for each piece, giving a bound for the whole estimate.

Ramaré then proves a theorem concerning sums of certain sequences of integers and combines it with Theorem 3, allowing him to conclude that every integer larger than $3.006 \times 10^{30}$ is the sum of at most 6 odd primes. Ramaré completes his proof by giving an algorithm that shows any even integer $n \leq 3.006 \times 10^{30}$ is the sum of at most 6 odd primes, proving Theorem 4.

*All odd integers greater than one are the sum of at most five primes*

The primary method Terence Tao [32] uses to prove that all odd numbers greater than 1 is the sum of at most five primes is to represent a number $n$ as the sum of three primes and an integer between 2 and $N_0$ where $N_0 := 4 \times 10^{14}$. By Richstein [25], this integer between 2 and $N_0$ can be represented as the sum of at most two primes, therefore every odd integer greater than 1 can be written as the sum of at most five primes. There are several major theorems proven by other mathematicians that Tao utilizes in his paper. Richstein (2000) [25] has verified

**Theorem 5.** *All even numbers between 4 and $N_0$ are the sum of two primes.*

Ramaré and Saouter (2003) [23] demonstrated

**Theorem 6.** *Even natural numbers less than $1.13 \times 10^{22}$ are the sum of two primes.*

While Liu and Wang (2002) [17] have shown

**Theorem 7.** *Natural numbers greater than $e^{3100}$ are the sum of at most three primes.*

Tao uses the method developed by Ramaré and Saouter to determine that every odd integer between 3 and $8.7 \times 10^{36}$ is the sum of at most five primes, then uses the Hardy-Littlewood circle method to show an odd $n$ is the sum of at most five primes when $8.7 \times 10^{36} \leq n \leq e^{3100}$.

The author examines $n$ in the interval $[8.7 \times 10^{36}, e^{3100}]$ and breaks this interval up into various arcs that encompass the entire interval for use with the Hardy-Littlewood circle method. He then proves that his theorem holds true in each of the arcs he constructs. The majority of the paper is spent proving theorems used for determining the various arcs.

Tao's result is independent of the truth of the Riemann hypothesis since he uses a bound proven true by Sebastian Wedeniwski (2003) [37]:

**Theorem 8.** *All nontrivial zeros of the Riemann zeta function in the interval* $[0, 3.29 \times 10^9]$ *lie on the line* $\Re(s) = \frac{1}{2}$.

Tao also mentions in several places that bounds or constants used could be improved, but that they were sufficient for his theorem.

From this result, an immediate corollary:

**Corollary 2.** *Every even integer* $n \geq 5$ *is the sum of at most 6 odd primes.*

*Proof.* Suppose every odd integer $n \geq 3$ is the sum of at most 5 odd primes. Then $n + 3$ is even and the sum of at most 6 odd primes. $\square$

This is an alternate proof of Theorem 4 [24].

*On Vinogradov's constant in Goldbach's ternary problem*

In this paper, Dmitrii Zinoviev [41] proves

**Theorem 9.** *Assuming the generalized Riemann hypothesis, every odd number greater than* $10^{20}$ *is a sum of three prime numbers.*

The primary goal of the paper is to demonstrate that a function,

$$J(N) = \sum_{p_1+p_2+p_3=N} \ln(p_1)\ln(p_2)\ln(p_3),$$

where $p_i$ are prime and $N \geq 10^{20}$, has a lower bound greater than zero. Clearly, if $J(N) > 0$ for some N, then there must exist $p_1$, $p_2$, and $p_3$ such that $N = p_1 + p_2 + p_3$. An equivalent formulation of $J(N)$ is shown and allows the author to work more easily with the bounds on $J(N)$.

Assuming the generalized Riemann hypothesis allows Zinoviev to estimate values for the gamma function and for $\pi(x; k, l)$, which is the prime counting function for the number of primes less than $x$ and congruent to $l$ modulo $k$. The generalized Riemann hypothesis is necessary for these estimations because they deal with sums over zeroes of Dirichlet L-functions, of which the Riemann Zeta function is a special case.

These estimations, as well as several others, allow the author to determine that $J(N)$ is greater than 0 when $N \geq 10^{20}$, proving the theorem.

This paper brought Goldbach's weak conjecture, when assuming the generalized Riemann hypothesis, within a computationally realistic distance. Soon after Zinoviev published this paper, Deshouillers, Effinger, te Riele, and Zinoviev [8] published a joint paper that proved Goldbach's weak conjecture when assuming the generalized Riemann hypothesis. Deshoulliers and te Riele proved that every even number $4 \leq m \leq 10^{13}$ is the sum of two prime numbers. Finally, it was demonstrated that there exists a prime $p$ such that for every natural number $6 \leq n \leq 10^{20}$, $n - p \leq 1.615 \times 10^{12}$. The conclusion is that $n$ is the sum of at most 3 primes.

Y. Saouter [28] computationally proved that all odd natural numbers less than $10^{20}$ can be represented as the sum of three primes, which gives a second proof of the weak Goldbach conjecture assuming the generalized Riemann hypothesis when combined with this paper.

*On Šnirelman's constant under the Riemann hypothesis*

Leszek Kaniecki [15] assumes the Riemann hypothesis in order to show

**Theorem 10.** *Every odd natural number can be written as the sum of at most five odd primes.*

The Riemann hypothesis allows Kaniecki to place an upper bound on

$$\lambda(x) = \begin{cases} 2 & \text{if } 0 < x \leq 7 \\ \max_{p \leq x}(p' - p) & \text{if } x > 7 \end{cases}$$

where $p$ and $p'$ are consecutive prime numbers. The function $\lambda(x)$ measures the largest gap between successive prime numbers in the interval $[1, x]$.

Next, Kaniecki uses $\lambda(x)$ to show that there exists some constant $h$ for which every interval of the form $[x, x+h]$ when $0 \leq x \leq e^{e^{11503}}$ contains some number that is the sum of two primes, $p_1$ and $p_2$. He then splits the interval $[0, e^{e^{11503}}]$ into several disjoint subintervals and proves that $h < 1.405 \times 10^{12}$ in all cases.

This implies that $3 < m = n - (p_1 + p_2) < 1.405 \times 10^{12}$ for all $9 < n \leq e^{e^{11503}}$. Therefore, because of work done by Sinalso, Young, and Potler [31], [21], $m$ is the sum of 2 or 3 prime numbers, implying $n$ is the sum of 4 or 5 primes, completing Theorem 10.

Kaniecki notes that his results could be improved by the same argument to show that every even number can be represented as the sum of at most four odd primes, but it would require a lot of time by a computer.

## CHAPTER FOUR

## Related Conjectures and Consequences

### *Twin Primes Conjecture*

The twin prime conjecture states that there are infinitely many primes $p$ and $q$ such that $q = p + 2$.

**Example 5.** A few twin primes:

$5 = 3 + 2$

$7 = 5 + 2$

$73 = 71 + 2$

The largest twin primes known were discovered in December of 2011: $3756801695685 \times 2^{666669} - 1$ and $3756801695685 \times 2^{666669} - 3$. Each of these has over $200,000$ digits [2].

Larry Gerstein (1993) [10] has shown that the Goldbach conjecture is related to the twin prime conjecture.

**Proposition 1.** *The strong Goldbach conjecture is true if and only if, for each integer $n \geq 2$, there exist integers $k, p$, and $q$, where $0 \leq k \leq n - 2$ and $p$ and $q$ are prime, such that $n^2 - k^2 = pq$.*

*Proof.* By assumption, for all integers $n \geq 2$ there exist at least two primes $p \leq q$ such that $2n = p + q$. If $k \geq n - 2$ is a natural number, then $p = n - k$ and $q = n + k$, and therefore $pq = (n - k)(n + k) = n^2 - k^2$. Conversely, suppose for every integer $n \geq 2$ there exists some natural number $k \leq n - 2$ and primes $p$ and $q$ with $p \leq q$ such that $n^2 - k^2 = pq$. Then $(n + k)(n - k) = pq$, so $p + q = n + k + n - k = 2n$. □

Gerstein then proposes setting $k = 1$ and makes the following conjecture:

**Conjecture 5.** *There are infinitely many integers $n \geq 2$ for which there exist primes $p$ and $q$ such that $n^2 - 1 = pq$.*

If $n^2 - 1 = (n+1)(n-1) = pq$, then for $q > p$, $q = n+1$ and $p = n-1$. Therefore, $q = p + 2$, meaning Conjecture 5 is equivalent to the twin primes conjecture.

*Goldbach's Number*

Goldbach's number, $G(n)$, is the number of unique combinations of primes $p + q$ that satisfy the Goldbach conjecture for some $n$. Fliegel and Robertson [6] note, when considering a graph of $G(n)$, that $G(n)$ tends to form bands, prompting the authors to entitle it Goldbach's comet. They find that these bands arise from the number of different combinations of prime numbers that satisfy the Goldbach conjecture for $n$.

For example, if $n \equiv 0 \pmod 6$, then $n \pmod 6 \equiv 5 + 1$ or $1 + 5$. Alternatively, when $n \equiv 2 \pmod 6$, then $n \pmod 6 \equiv 1 + 1$, and when $n \equiv 4 \pmod 6$, then $n \pmod 6 \equiv 5 + 5$. Since there are twice as many combinations for $n \equiv 0 \pmod 6$ than in either of the other two cases above, $G(n)$ is approximately twice as large for numbers divisible by 6 than the base tail (those numbers with only powers of 2 as factors) on the comet, as seen in Figure 2.

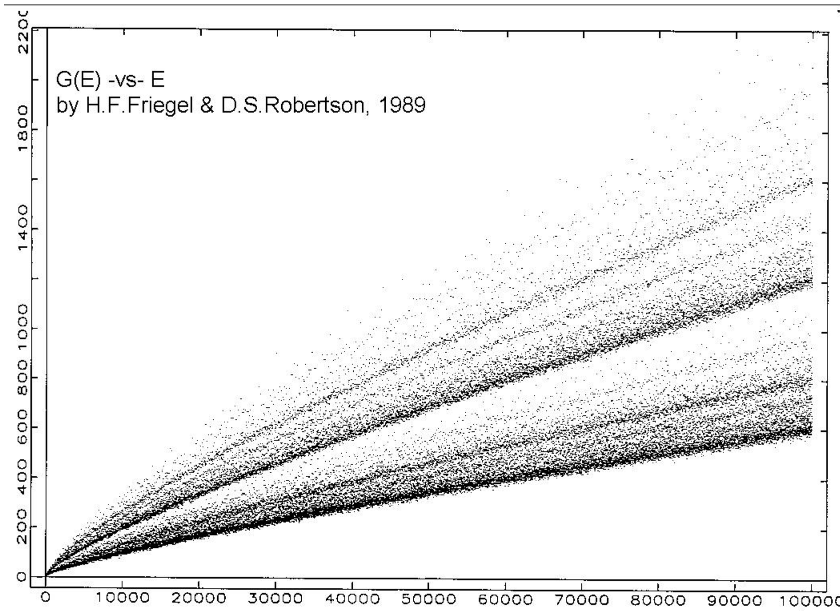


FIGURE 2. G(E) for $1 \leq E \leq 100,000$. [6]

Here are some Goldbach combinations and their corresponding Goldbach number:

**Example 6.**

$$4 = 2 + 2 \qquad G(4) = 1$$

$$10 = 5 + 5 = 7 + 3 \qquad G(10) = 2$$

$$34 = 31 + 3 = 29 + 5 = 23 + 11 = 17 + 17 \qquad G(34) = 4$$

$$66 = 61 + 3 = 59 + 7 = 53 + 13 = 47 + 19 = 43 + 23 = 37 + 29 \qquad G(66) = 6$$

$$100 = 97 + 3 = 89 + 11 = 83 + 17 = 71 + 29 = 59 + 41 = 53 + 47 \qquad G(100) = 6$$

The authors note that for $n$ divisible by prime $p$, $E$ is approximately $\frac{p-1}{p-2}$ times larger than numbers not divisible by $p$. This does not apply when $p = 2$ since the base tail is made of those numbers only divisible by 2. When $n$ has multiple prime factors, $E$ is approximately $\frac{(p_1-1)(p_2-1)(p_3-1)\ldots}{(p_1-2)(p_2-2)(p_3-2)\ldots}$ times larger than the base tail when $p_i \geq 3$. The scaling in Figure 2 is not sufficient to distinguish between the base tail and bands that are close to it, such as $n = 2^{e_2} \cdot 101^{e_{101}}$, which would be $\frac{101-1}{101-2} = \frac{100}{99} \approx 1.01$ times as high as the base tail. However, one can see the band created by those numbers $n = 2^{f_2} \cdot 3^{f_3} \cdot 5^{f_5}$, which is about $\frac{(3-1)(5-1)}{(3-2)(5-2)} = \frac{2 \cdot 4}{1 \cdot 3} = \frac{8}{3} \approx 2.67$ times as high as the base tail.

These results are an attempt to explain the tails on the graph on $G(n)$ and are not rigorous as they are only approximations based on the number of $p$ and $q$ that could satisfy $n = p + q$. The authors note that if one were to find a rigorous lower bound of the base tail, then the Goldbach conjecture would surely be quickly proven.

*Untouchable Numbers*

An untouchable number $k$ is a number that cannot be represented as the sum of the proper divisors of any integer. That is, $k$ can not be written as $\sigma(n) - n$, where $\sigma(n)$ is the sum of the divisors of $n$. The natural number $n$ is subtracted from $\sigma(n)$ because $n$ is not considered a proper divisor of itself.

**Example 7.** Some touchable and untouchable numbers:

$\sigma(8) - 8 = \frac{2^4-1}{2-1} - 8 = 26 - 8 = 7$, therefore 7 is a touchable number.

$\sigma(6) - 6 = (\frac{2^2-1}{2-1})(\frac{3^2-1}{3-1}) - 6 = 3 * 4 - 6 = 6$, so 6 is also a touchable number.

However, 5 is an untouchable number. $5 = 1 + 4 = 2 + 3$ are the only ways to write 5 as the sum of unique natural numbers and neither of the two ways can be the sum of natural divisors of any number.

A proof of the Goldbach conjecture that demonstrates every even number greater than 4 is the sum of two unique primes would prove that 5 is the only odd untouchable number, as Richard Guy [11] notes:

**Proposition 2.** *Suppose $2n = p + q$, $p$ and $q$ distinct primes, for $6 \leq n$. Then 5 is the only odd untouchable number.*

*Proof.* Assume $2n = p + q$, $p$ and $q$ distinct primes, for $6 \leq n$.

Consider $2n + 1 = p + q + 1$ and the composite $pq$:

$$\begin{aligned}
\sigma(pq) - pq &= \left(\frac{p^2-1}{p-1}\right)\left(\frac{q^2-1}{q-1}\right) - pq \\
&= (p+1)(q+1) - pq \\
&= pq + p + q + 1 - pq \\
&= p + q + 1 \\
&= 2n + 1
\end{aligned}$$

Therefore, 5 is the only odd untouchable number. $\square$

# CHAPTER FIVE

## Outlook

While a complete proof for Goldbach's conjecture still may be far off, significant progress has been made towards a solution, especially on the weak conjecture. Unfortunately, there have also been many false or questionable proofs. One notable case is that of Henry Pogorzelski, who claimed to have proven the strong conjecture in 1977 [39]. His proof assumes three open questions that have not been acknowledged to be resolved. Perhaps Pogorzelski has proven Goldbach's conjecture, but the mathematical community is not yet ready to accept his proof as valid.

Moving forward, there are several discoveries that would certainly help the effort to prove Goldbach's conjecture. Firstly, the factoring power of quantum computers would help immensely in discovering primes and raising the upper computed bound for integers that are the sum of two or three primes. Secondly, a better understanding of the distribution of primes, particularly in the interval below Chen's upper bound of $e^{3100}$ and above what has been calculated by computer. This would especially help in a proof of the weak conjecture where it is sufficient to find a prime $p_3$ such that $n - p_3$ lies within some interval for which it is known that $n - p_3 = p_1 + p_2$. A similar technique was used by O. Ramaré in his proof that every even number is the sum of at most 6 primes. Unfortunately, the usefulness of this technique declines when considering the strong conjecture and there is no longer the luxury of simply placing $n - p_1$ within some interval that has already been calculated. However, a better understanding about the location of a prime number is certainly is not detrimental to proving Goldbach's conjecture.

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