ABSTRACT

Applied Left-Definite Theory: the Jacobi Polynomials, their Sobolev Orthogonality, and Self-Adjoint Operators

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It is well known that, for $-\alpha, -\beta, -\alpha - \beta - 1 \notin \mathbb{N}$, the Jacobi polynomials $\left\{ P_n^{(\alpha,\beta)}(x) \right\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to a bilinear form of the type

$$(f, g)_\mu = \int_{\mathbb{R}} f g d\mu,$$

for some measure μ. However, for negative integer parameters α and β, an application of Favard's theorem shows that the Jacobi polynomials cannot be orthogonal on the real line with respect to a bilinear form of this type for any positive or signed measure. But it is known that they are orthogonal with respect to a Sobolev inner product. In this work, we first consider the special case where $\alpha = \beta = -1$. We shall discuss the Sobolev orthogonality of the Jacobi polynomials and construct a self-adjoint operator in a certain Hilbert-Sobolev space having the entire sequence of Jacobi polynomials as eigenfunctions. The key to this construction is the left-definite theory associated with the Jacobi differential equation, and the left-definite spaces and operators will be constructed explicitly. The results will then be generalized to the case where $\alpha > -1, \beta = -1$.

Applied Left-Definite Theory: the Jacobi Polynomials, their Sobolev Orthogonality, and Self-Adjoint Operators

by

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DEDICATION

To my Dad

and

in loving memory of

my Mom

CHAPTER ONE

Summary

In 1929, S. Bochner classified all second order equations of hypergeometric type that have orthogonal polynomial eigenfunctions. Up to a complex linear change of variable, the only such equations are the Hermite, Laguerre, Jacobi, and the Bessel polynomial equations.

It has been well known that, for $-\alpha, -\beta, -\alpha-\beta-1 \notin \mathbb{N}$, the Jacobi polynomials $\left\{P_n^{(\alpha,\beta)}(x)\right\}_{n=0}^{\infty}$ are orthogonal on $\mathbb{R}$ with respect to a bilinear form of the type

$$(f,g)_\mu = \int_{\mathbb{R}} f\overline{g}d\mu, \tag{1.1}$$

for some measure μ [7]. However, for negative integer parameters α and β, an application of Favard's theorem shows that the Jacobi polynomials *cannot* be orthogonal on the real line with respect to a bilinear form of this type for *any* measure. But are they orthogonal with respect to some "natural" inner product? Indeed, they are orthogonal with respect to a Sobolev inner product [35]. We discuss this Sobolev orthogonality when $\alpha \geq -1$ and $\beta = -1$ and, by applying the left-definite spectral theory, we construct a self-adjoint operator that is generated from the Jacobi differential expression in a certain Hilbert space having the entire sequence of Jacobi polynomials as a complete set of eigenfunctions.

The left-definite theory can be traced back to Weyl [56] and the work of Schäfke and Schneider who coined the term *left-definite* in their 1965 paper, that is, the German *links-definit* [50]. In a recent paper, Littlejohn and Wellman develop a general left-definite theory. They show that any self-adjoint operator A in a Hilbert space H that is bounded below generates a continuum of Hilbert spaces and self-adjoint operators that are called the left-definite spaces and operators, respectively, associ-

ated with (H, A). Examples for which these left-definite spaces and operators have been specifically constructed include the Hermite [19], Legendre [21], and Laguerre [39] differential equations. The left-definite spectral analysis of the classical Jacobi differential expression, when $\alpha, \beta > -1$, has been discussed in [17]. The Laguerre differential equation for nonclassical parameters was studied in a left-definite setting in [20].

In this work, we study the Sobolev-orthogonality of the Jacobi polynomials for the non-classical parameters $\alpha = \beta = -1$ and $\beta = -1, \alpha > -1$. In the special case where $\alpha = \beta = -1$, the Jacobi differential expression reduces to

$$l_{-1,-1}[y](x) := (1 - x^2)\left(-(y'(x))' + k(1 - x^2)^{-1}y(x)\right)$$

for $x \in (-1, 1)$ and where $k \geq 0$ is a constant. The associated classical weight function is $w(x) = (1 - x^2)^{-1}$, and the right definite spectral setting is $L^2((-1, 1); w)$. The maximal domain associated with $l_{-1,-1}[\cdot]$ is

$$\Delta := \left\{f : (-1, 1) \longrightarrow \mathbb{C} \mid f, f' \in AC_{loc}(-1, 1); f, l[f] \in L^2\left((-1, 1); \left(1 - x^2\right)^{-1}\right)\right\}.$$

From the Glazman-Krein-Naimark theory, the operator

$$A : \mathcal{D}(A) \subset L^2\left((-1, 1); (1 - x^2)^{-1}\right) \longrightarrow L^2\left((-1, 1); (1 - x^2)^{-1}\right)$$

defined by

$$\begin{aligned} Af &= l_{-1,-1}[f] \\ f &\in \mathcal{D}(A) = \Delta \end{aligned}$$

is self-adjoint and bounded below by kI in $L^2((-1, 1); (1 - x^2)^{-1})$.

When considering the sequence of Jacobi polynomials $\left\{P_n^{(-1,-1)}\right\}_{n=0}^{\infty}$ in this setting, one quickly notices that the first Jacobi polynomial is degenerate, that is, $P_1^{(-1,-1)}(x) = 0$. However, any polynomial of degree 1 will be a solution of the equation $l_{-1,-1}[y](x) = 0$ and the degeneracy can be fixed by choosing a suitable first-degree polynomial. The next complication is that neither the Jacobi polynomial of degree 0 nor any non-trivial choice of $P_1^{(-1,-1)}(x)$ are in $L^2\left((-1,1);(1-x^2)^{-1}\right)$, due to the singularities in the weight function $w(x) = \left(1-x^2\right)^{-1}$. Although the Jacobi polynomials of degree ≥ 2 form a complete orthogonal set of eigenfunctions of A in $L^2\left((-1,1);(1-x^2)^{-1}\right)$, it is not possible for the entire sequence $\left\{P_n^{(-1,-1)}\right\}_{n=0}^{\infty}$ to be orthogonal on the real line with respect to any bilinear form of type (1.1) for any positive or signed measure μ. This is a simple application of Favard's theorem. However, upon choosing $P_1^{(-1,-1)}(x) = x/\sqrt{3}$, the entire sequence of polynomials $\left\{P_n^{(-1,-1)}\right\}_{n=0}^{\infty}$ can be normalized so that they form an orthonormal set with respect to the Sobolev inner product

$$\phi(f,g) := \frac{1}{2}f(-1)\overline{g}(-1) + \frac{1}{2}f(1)\overline{g}(1) + \int_{-1}^{1} f'(x)\overline{g}'(x)dx,$$

as shown in [35]. In fact, the set $\left\{P_n^{(-1,-1)}\right\}_{n=0}^{\infty}$ forms a complete orthonormal sequence in the Hilbert-Sobolev space

$$W_1 := \left\{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC\,[-1,1]\,; f' \in L^2(-1,1)\right\} \tag{1.2}$$

carrying the inner product $\phi(\cdot,\cdot)$. A central question in this dissertation is if there exists a self-adjoint operator in W_1 which is generated from the Jacobi differential expression $l_{-1,-1}[\cdot]$, that has the entire sequence of Jacobi polynomials $\left\{P_n^{(-1,-1)}\right\}_{n=0}^{\infty}$ as its eigenfunctions? We show that the answer is yes, and the left-definite spectral analysis associated with A will be the key in this construction.

The integral powers of $\ell_{-1,-1}[\cdot]$, the coefficients $c_j^{(-1,-1)}(n,k)$, the left-definite vector spaces $V_n^{(-1,-1)}$, and the left-definite inner products $(\cdot,\cdot)_n^{(-1,-1)}$ can be found in exactly the same fashion as in [17]; indeed, by letting $\alpha = \beta = -1$ in the formulae in [17], we obtain the necessary expressions, combinatorial numbers, spaces, and inner products. Indeed, for $n \in \mathbb{N}$, we shall see that the n^{th} left-definite Hilbert space associated with the pair $\left(L^2\left((-1,1);(1-x^2)^{-1}\right), A^{(-1,-1)}\right)$ is given by $W_n^{(-1,-1)} = (V_n^{(-1,-1)}, (\cdot,\cdot)_n^{(-1,-1)})$, where

$$V_n^{(-1,-1)} = \{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\mathrm{loc}}(-1,1); \qquad (1.3)$$
$$f^{(j)} \in L^2\left((-1,1);\left(1-x^2\right)^{j-1}\right), j = 0,1,..,n\}$$

and

$$(f,g)_n^{(-1,-1)} = \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k) \int_{-1}^{1} f^{(j)}(x)\overline{g}^{(j)}(x)(1-x^2)^{j-1}dx.$$

Moreover, the Jacobi polynomials $\{P_m^{(-1,-1)}\}_{m=2}^{\infty}$ form a complete orthogonal set in each $W_n^{(-1,-1)}$ and they satisfy the orthogonality relation

$$(P_m^{(-1,-1)}, P_r^{(-1,-1)})_n = (m(m-1)+k)^n \delta_{m,r.}$$

Furthermore, define $A_n^{(-1,-1)} : \mathcal{D}\left(A_n^{(-1,-1)}\right) \subset W_n^{(-1,-1)} \longrightarrow W_n^{(-1,-1)}$ by

$$A_n^{(-1,-1)} f := \ell_{-1,-1}[f] \qquad (f \in \mathcal{D}(A_n^{(-1,-1)}) := V_{n+2}^{(-1,-1)}).$$

Then the operator $A_n^{(-1,-1)}$ is the n^{th} left-definite operator associated with the pair $\left(L^2\left((-1,1);(1-x^2)^{-1}\right), A\right)$; this is a self-adjoint operator in $W_n^{(-1,-1)}$ with spectrum given by

$$\sigma(A_n^{(-1,-1)}) = \{m(m-1)+k \mid m \in \mathbb{N}_0\} = \sigma(A).$$

Moreover, the Jacobi polynomials $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ form a complete set of eigenfunctions of each $A_n^{(-1,-1)}$ in $W_n^{(-1,-1)}$.

To construct a self-adjoint operator T that is a realization of the Jacobi differential expression having the full sequence of Jacobi polynomials as a complete set of eigenfunctions in the space W_1, defined in (1.2), we consider the decomposition

$$W_1 = W_{1,1} \oplus W_{1,2},$$

where

$$W_{1,1} := \left\{f \in W^{(-1,-1)} \mid f(\pm 1) = 0\right\}$$
$$W_{1,2} := \left\{f \in W^{(-1,-1)} \mid f''(x) = 0\right\}.$$

It is the case that $\{P_m^{(-1,-1)}\}_{m=2}^{\infty}$ is a complete orthonormal set in $W_{1,1}$ and the set $\{P_m^{(-1,-1)}\}_{m=0}^{1}$ is complete and orthonormal in the two-dimensional space $W_{1,2}$. Furthermore, we show that

$$W_{1,1} = V_1^{(-1,-1)},$$

where $V_1^{(-1,-1)}$ denotes the first left-definite space defined in (1.3); moreover, the inner products $(\cdot,\cdot)_1^{(-1,-1)}$ and $\phi(\cdot,\cdot)$ are equivalent on $W_{1,1} = V_1^{(-1,-1)}$.

We will then show that the first left-definite operator

$$T_1 : \mathcal{D}(T_1) \subset W_{1,1} \longrightarrow W_{1,1}$$

given by

$$T_1 f = A_1^{(-1,-1)} f = \ell_{-1,-1}[f]$$
$$f \in \mathcal{D}(T_1) := V_3^{(-1,-1)}$$

is self-adjoint in $(W_{1,1}, \phi(\cdot,\cdot))$. It is easy to construct a self-adjoint operator T_2 in $W_{1,2}$ generated by $\ell_{-1,-1}[\cdot]$:

$$T_2 f = \ell_{-1,-1}[f],$$
$$\mathcal{D}(T_2) = \mathcal{P}_2.$$

For each $f \in W_1$, write $f = f_1 + f_2$ where $f_1 \in W_{1,1}$,and $f_2 \in W_{1,2}$. Define

$$T : \mathcal{D}(T) \subset W_1 \to W_1$$

by

$$Tf = T_1 f_1 + T_2 f_2 = \ell[f_1] + \ell[f_2] = \ell[f],$$

for

$$f \in \mathcal{D}(T) = \mathcal{D}(T_1) \oplus \mathcal{D}(T_2).$$

Then T is self-adjoint in $(W_1, \phi(\cdot,\cdot))$ and has the entire sequence of Jacobi polynomials $\{P_m^{(-1,-1)}\}_{m=0}^{\infty}$ as eigenfunctions. From the explicit determination of $\mathcal{D}(T_1)$ and $\mathcal{D}(T_2)$, it is not difficult to obtain the following characterization of $\mathcal{D}(T)$:

$$\begin{aligned}\mathcal{D}(T) &= \{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC[-1,1]; f', f'' \in AC_{\text{loc}}(-1,1); (1-x^2)f''', \\ &\qquad (1-x^2)^{1/2} f'', f' \in L^2(-1,1)\} \\ &= \{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC[-1,1]; f', f'' \in AC_{\text{loc}}(-1,1); \\ &\qquad (1-x^2)f''' \in L^2(-1,1)\big\}.\end{aligned}$$

Furthermore, the spectrum of T is given by $\sigma(T) = \{m(m-1) + k \mid m \in \mathbb{N}_0\}$ and T is bounded below by kI in $(W_1, \phi(\cdot,\cdot))$.

In chapter 6, these results will be extended to the general case where $\alpha > -1, \beta = -1$. This case is in many ways similar to the special case, but it is worth noting some fundamental differences: for fixed $\alpha = \beta = -1$, the set $\{P_m^{(-1,-1)}\}_{m=2}^{\infty}$ is complete in $L^2\left((-1,1);(1-x^2)^{-1}\right)$, i.e. the Jacobi polynomials starting with the polynomial of degree 2 form a maximal orthogonal set, whereas in the general case, $\{P_m^{(\alpha,-1)}\}_{m=1}^{\infty}$ forms a maximal orthogonal set in $L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$. Again, the left-definite theory will play a key role in constructing a self-adjoint operator in a certain Hilbert-Sobolev space having the entire sequence of Jacobi polynomials as eigenfunctions.

CHAPTER TWO

Right-Definite Spectral Theory

The purpose of this chapter is to summarize the theory of self-adjoint extensions of formally symmetric differential expressions. Our main source is [41] and references therein. Throughout this chapter, we shall assume that $I = (a, b) \subset \mathbb{R}$ is an open interval with $-\infty \leq a < b \leq \infty$, and that $a_j \in C^j(I, \mathbb{R})$, $j = 0, 1, ..., n$, with $a_n(x) \neq 0$ for all $x \in I$, and n is a positive integer. We consider the ordinary differential expression $l[\cdot]$ of order $2n$ defined by

$$l[y](x) := \sum_{j=0}^{n} (-1)^j \left(a_j(x) y^{(j)}(x)\right)^{(j)}, \qquad x \in I \tag{2.1}$$

and study certain linear operators in $L^2(I)$ generated from $l[\cdot]$. Two operators of interest are the maximal and the minimal operator associated with $l[\cdot]$. We will be concerned with constructing self-adjoint extensions (restrictions) of the minimal (maximal) operator, and we will study their spectra. In particular, we shall consider the eigenvalue problem

$$A[y] = \lambda y,$$

where A is one of these self-adjoint operators. Expression (2.1) is called a formally symmetric differential expression. We note that differential expressions with less smooth coefficients can be considered which leads to the concept of quasi-derivatives, as in [2],[43]. However, we will keep our smoothness assumptions and note that for any eigenvalue problem $l[y] = \lambda y$ having a sequence of orthogonal polynomial solutions, it is always the case that $a_j \in C^j(I, \mathbb{R})$.

Definition 2.1. The differential expression (2.1) is called regular if I is of finite length and the coefficients $\frac{1}{a_n}, a_{n-1}, ..., a_0 \in L(I)$. If $l[\cdot]$ is not regular, it is called singular. The endpoint a is called a regular point of $l[\cdot]$ if $a > -\infty$ and if there exists an $\varepsilon > 0$

such that $\frac{1}{a_n}, a_{n-1}, ..., a_0 \in L(a, a+\varepsilon)$. Otherwise, the point a is a singular point of $l[\cdot]$. There is a similar definition for the endpoint b.

The Jacobi differential expression is singular on $(-1, 1)$, and thus for the rest of this chapter, we will assume that $l[\cdot]$ is a singular differential expression unless otherwise stated.

Definition 2.2. Let $l[\cdot]$ be as in (2.1). The operator $\mathcal{L} : L^2(I) \longrightarrow L^2(I)$ defined by

$$\mathcal{L}[y] = l[y]$$

$$\mathcal{D}(\mathcal{L}) := \left\{y : I \longrightarrow \mathbb{C} | y^{(k)} \in AC_{loc}(I), k = 0, 1, ..., 2n-1; y, l[y] \in L^2(I)\right\}$$

is called the maximal operator generated by $l[\cdot]$ in $L^2(I)$.

The space $\mathcal{D}(\mathcal{L})$ is in fact the largest subspace in which $\mathcal{L}$ can be defined as an operator from $L^2(I)$ into $L^2(I)$.

For $f, g \in \mathcal{D}(\mathcal{L})$, and $[\alpha, \beta] \subset I$, it is easy to verify Green's formula by integration by parts:

$$\int_{\alpha}^{\beta} \{l[f]\overline{g} - l[\overline{g}]f\}\, dx = [f, g](x) \mid_{\alpha}^{\beta}$$

where the sesquilinear form $[f, g](\cdot)$ is defined by

$$[f, g](x) := \sum_{j=1}^{n} \sum_{m=1}^{j} \left\{ \left(a_j(x)\overline{g}^{(j)}(x)\right)^{(j-m)} f^{(m-1)}(x) \left(a_j(x) f^{(j)}(x)\right)^{(j-m)} \overline{g}^{(m-1)}(x) \right\}. \tag{2.2}$$

Observe that $[g, f](x) = -\overline{[f, g]}(x)$ for all $f, g \in \mathcal{D}(\mathcal{L})$ and $a < x < b$, and that the limits $[f, g](a) := \lim_{x \longrightarrow a^+} [f, g](x)$ and $[f, g](b) := \lim_{x \longrightarrow b^-} [f, g](x)$ both exist and are finite for all $f, g \in \mathcal{D}(\mathcal{L})$ by the definition of $\mathcal{D}(\mathcal{L})$ and Hölder's inequality.

Since $\mathcal{D}(\mathcal{L})$ is dense in $L^2(I)$, the adjoint operator $\mathcal{L}^*$ exists. If $T \subset \mathcal{L}$ is a densely defined linear operator in $L^2(I)$, then $\mathcal{L}^* \subset T^*$, so it is natural to call $\mathcal{L}_0 := \mathcal{L}^*$ the minimal operator generated by $l[\cdot]$.

Definition 2.3. The restriction of the maximal operator $\mathcal{L}$ to the (densely defined) subspace $\mathcal{D}_0'$ of all functions $f \in \mathcal{D}(\mathcal{L})$ with compact support in I will be denoted by $\mathcal{L}_0'$.

Definition 2.4. Let H be a Hilbert space with inner product $(\cdot, \cdot)$. A linear operator $S : H \longrightarrow H$ is symmetric in H if $\mathcal{D}(S)$ is dense in H and $(Sx, y) = (x, Sy)$ for all $x, y \in \mathcal{D}(S)$.

A densely defined operator S is symmetric in H if and only if $S \subset S^*$.

Definition 2.5. Let H be a Hilbert space. A linear operator $S : H \longrightarrow H$ is self-adjoint in H if $\mathcal{D}(S)$ is dense in H and $S = S^*$.

Theorem 2.1. *The operator $\mathcal{L}_0'$ is symmetric in $L^2(I)$.*

Definition 2.6. Let H be a Hilbert space and $T : H \longrightarrow H$ a linear operator with domain $\mathcal{D}(T)$. Then T is closed if whenever $\{x_n\} \subset \mathcal{D}(T)$ satisfies $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$, then $x \in \mathcal{D}(T)$ and $Tx = y$.

It is easy to see that the adjoint of a densely defined operator is closed. In particular, the minimal operator $\mathcal{L}_0$ is closed.

Definition 2.7. Let H be a Hilbert space and $T : H \longrightarrow H$ a linear operator. We say that T is closable if there exists a closed, linear extension S of T. If $T' : H \longrightarrow H$ is a closed linear extension of T and $T' \subset S$ for all closed linear extensions of T, then T' is called the closure of T and T is said to admit a closure. The closure of an operator T is denoted by $\overline{T}$.

Theorem 2.2. *Let H be a Hilbert space. A symmetric operator $S : H \longrightarrow H$ admits a closure. Moreover, this closure $\overline{S}$ is also symmetric in H.*

Proof. See [43], page 13. □

Consequently, $\mathcal{L}_0'$ has a symmetric closure $\overline{\mathcal{L}_0'}$.

Theorem 2.3. $\overline{(\mathcal{L}'_0)}^* = \mathcal{L}$.

Proof. See [43], page 68. □

It is well-known (e.g. [33]) that a closed, densely defined operator A in a Hilbert space H has the property that $A^{**} = A$. This fact, combined with the previous theorem yields:

Theorem 2.4. $\mathcal{L}_0 = \overline{\mathcal{L}'_0}$ *and* $\mathcal{L}_0^* = \mathcal{L}$. *In particular, the minimal operator* $\mathcal{L}_0$ *and the maximal operator* $\mathcal{L}$ *are closed operators, being adjoints of each other.*

The following theorem is a very useful criterion for determining whether or not an element $f \in \mathcal{D}(\mathcal{L})$ is in the minimal domain $\mathcal{D}(\mathcal{L}_0)$. It involves the sesquilinear form (2.2).

Theorem 2.5. *The domain* $\mathcal{D}(\mathcal{L}_0)$ *of the minimal operator* $\mathcal{L}_0$ *in* $L^2(I)$ *consists of all* $f \in \mathcal{D}(\mathcal{L})$ *satisfying* $[f,g](x) \mid_a^b = 0$, *for all* $g \in \mathcal{D}(\mathcal{L})$.

Proof. See [43], page 70. □

If one or both endpoints of I are regular, then the condition in the previous theorem simplifies further, see [43], page 71.

Remark 2.1. If A is a symmetric extension of the minimal operator $\mathcal{L}_0$ in $L^2(I)$, then $A \subset \mathcal{L}$, where $\mathcal{L}$ is the maximal operator. Indeed, this is an immediate consequence of Theorem 2.4:

$$\mathcal{L}_0 \subset A \subset A^* \subset \mathcal{L}_0^* = \mathcal{L}.$$

In particular, $A[y] = l[y]$ for all $y \in \mathcal{D}(A)$; i.e. A has the same form as the expression $l[\cdot]$ and A is the restriction of the maximal operator $\mathcal{L}$.

Remark 2.2. Note that the theory presented in this chapter can be applied *mutatis mutandis* to expressions of the form

$$m[y](x) = \frac{1}{f(x)} \sum_{j=0}^{n} (-1)^j \left(a_j(x) y^{(j)}(x)\right)^{(j)}, \qquad x \in I,$$

where $f(x) \in C^{2n}(I)$ and $f(x) > 0$ for all $x \in I$. Observe that $f(x)m[y]$ is then formally symmetric; in this case, we call the function $f(x)$ a symmetry factor for $m[\cdot]$, see [38]. The appropriate Hilbert space setting for the theory of self-adjoint extensions would be $L^2((a,b);f)$. We note that the maximal operator $\mathcal{L}$ in $L^2((a,b);f)$, generated by $m[\cdot]$, is defined to be

$$\begin{aligned}
\mathcal{L}[y] &= m[y] \\
\mathcal{D}(\mathcal{L}) &= \big\{ y : (a,b) \longrightarrow \mathbb{C} | y^{(k)} \in AC_{loc}(a,b), k = 0,1,...,2n-1; \\
&\quad y, m[y] \in L^2((a,b);f) \big\}.
\end{aligned}$$

Example 2.1. The classical Jacobi differential expression for $k \geq 0$ is defined by

$$\tau[y] := -(1-x^2)y'' + (\alpha - \beta + (\alpha+\beta+2)x)y' + ky, \qquad x \in (-1,1).$$

Although this expression cannot be directly put into the form (2.1), multiplication of $l[\cdot]$ by the symmetry factor $f(x) = (1-x)^{\alpha}(1+x)^{\beta}$ yields

$$\begin{aligned}
l[y](x) &:= (1-x)^{\alpha}(1+x)^{\beta}\tau[y] \\
&= -\left((1-x)^{\alpha+1}(1+x)^{\beta+1}y'\right)' + k(1-x)^{\alpha}(1+x)^{\beta}y(x).
\end{aligned}$$

For $\alpha = \beta = 0$, $\tau[\cdot]$ is called the Legendre expression. For the Jacobi expression, the proper right-definite setting is the weighted Lebesgue space

$$L^2((-1,1);(1-x)^{\alpha}(1+x)^{\beta}),$$

and the maximal and minimal operators in this space are generated from $\tau[\cdot] = (1-x)^{-\alpha}(1+x)^{-\beta}l[\cdot]$.

In 1929, von Neumann considered and solved the problem of when a symmetric operator in a Hilbert space H had self-adjoint extensions in H. The motivation for this study came from his interest in several unbounded operators that appear quite naturally in the theory of quantum mechanics. In 1939, Calkin presented his method for determining necessary and sufficient conditions when such self-adjoint extensions exist and proceeded to characterize the domains of each of these extensions in terms of general "boundary conditions". A well-written account of this elegant theory can be found in [12], pages 1222-1239 and 1268-1274. For our study, this theory has particularly important applications to the subject of symmetric differential operators. Indeed, the Russian mathematicians M. A. Naimark and I. M. Glazman are credited with applying and refining both van Neumann's theory and Calkin's method to the minimal operator $\mathcal{L}_0$ generated by $l[\cdot]$. We will now briefly describe von Neumann's results, followed by the Glazman-Naimark theory of self-adjoint extensions of $\mathcal{L}_0$.

Definition 2.8. Let A be a symmetric operator in a Hilbert space H. Let

$$\mathcal{D}_+ := \{f \in \mathcal{D}(A^*) \mid A^* f = if\}$$

$$\mathcal{D}_- := \{f \in \mathcal{D}(A^*) \mid A^* f = -if\},$$

where $i := \sqrt{-1}$. The space $\mathcal{D}_+$ is called the positive deficiency space of A, and $\mathcal{D}_-$ is called the negative deficiency space of A. The dimensions of these spaces are called the positive and negative deficiency indices of A, respectively, and we write $n_\pm := \dim(\mathcal{D}_\pm)$. The deficiency index of A in $L^2(I)$ is the ordered pair (n_+, n_-).

As shown in [12], page 1232, there is nothing special about using the complex number i in this definition: if $\lambda \in \mathbb{C}$ and $\operatorname{Im}(\lambda) > 0$, then it is the case that $\dim\{f \in \mathcal{D}(A^*) \mid A^* f = \lambda f\} = n_+$. A similar result holds for n_- and any $\lambda \in \mathbb{C}$ with $\operatorname{Im}(\lambda) < 0$. This is a result due to Weyl (1910, see [56] and [29], chapter 13) which he proved in the context of the classical second-order Sturm-Liouville differential expression.

If A is a symmetric operator in a Hilbert space H, we define a new inner product on $\mathcal{D}(A^*)$ by $(x, y)^* := (x, y) + (A^* x, A^* y)$. It can be shown (see [12], page 1225) that $\mathcal{D}(A^*)$ is a Hilbert space when equipped with this inner product. We are now in the position to state the following important theorem.

Theorem 2.6. *Let A be a symmetric operator in a Hilbert space H. Then $\mathcal{D}(\overline{A}), \mathcal{D}_+$, and $\mathcal{D}_-$ are closed orthogonal subspaces in $(\mathcal{D}(A^*), (\cdot,\cdot)^*)$ and*

$$\mathcal{D}(A^*) = \mathcal{D}(\overline{A}) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-.$$

This is known as von Neumann's formula.

Proof. See [12], page 1227. □

In the case of $A = \mathcal{L}_0$, the minimal operator in $L^2(I)$ generated by $l[\cdot]$, von Neumann's formula becomes

$$\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L}_0) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-. \tag{2.3}$$

Consequently, it is not surprising that the positive and negative deficiency spaces play a major role in determining the self-adjoint extensions of $\mathcal{L}_0$ in $L^2(I)$. In fact, we state the following theorem, [12] page 1228, to illustrate this influence.

Theorem 2.7. *Let A be a symmetric operator in a Hilbert space H. Let $\mathcal{D}'$ be a closed subspace of $\mathcal{D}_+ \oplus \mathcal{D}_-$ and set $\mathcal{D} = \mathcal{D}(\overline{A}) \oplus \mathcal{D}'$. Then the restriction of A^* to $\mathcal{D}$ is self-adjoint if and only if $\mathcal{D}'$ is the graph of an isometry mapping $\mathcal{D}_+$ onto $\mathcal{D}_-$.*

This implies the following key result:

Theorem 2.8. *Let A be a symmetric operator in a Hilbert space H. Then A has self-adjoint extensions in H if and only if its deficiency indices are equal. Furthermore, if $n_+ = n_- = 0$, then the only self-adjoint extension of A is its closure $\overline{A} = A^*$.*

Proof. See [12], page 1230. □

Although much more can be said about the characterizations of self-adjoint extensions of general symmetric operators in a Hilbert space, we return to our discussion of finding self-adjoint extensions of the minimal operator $\mathcal{L}_0$ in $L^2(I)$. Since for any complex number λ, the equation $l[y] = \lambda y$ has a basis of $2n$ solutions, the deficiency indices of $\mathcal{L}_0$ in $L^2(I)$ are both finite. In fact, these two indices are equal. Indeed, because the coefficients a_k of $l[\cdot]$ are real-valued, the function f is a solution of $l[y] = -iy$. This same argument shows that if $\{f_1, f_2, ..., f_m\}$ is a basis for the positive deficiency space $\mathcal{D}_+$, then $\left\{\overline{f_1}, \overline{f_2}, ..., \overline{f_m}\right\}$ is a basis for the negative deficiency space $\mathcal{D}_-$. However, we note that, in general, the deficiency indices $n_\pm$ need not be equal when the coefficients of $l[\cdot]$ are complex-valued.

Theorem 2.9. *Let $\mathcal{L}_0$ be the minimal operator in $L^2(I)$ generated by $l[\cdot]$, where $I = (a, b)$.*

(i) If both endpoints a and b are regular, then $n_\pm = 2n$.

(ii) If one of these endpoints is singular, then $0 \leq n_+ = n_- \leq 2n$. In fact, it is possible to construct $l[\cdot]$ so that $n_\pm = m$ for any integer m, $0 \leq m \leq 2n$. If exactly one of the endpoints is singular, then $n \leq n_+ = n_- \leq 2n$.

Proof. For the proof of (i), see [43], page 66. For the proof of (ii), see [43], pages 69 and 71. Furthermore, in [25], Glazman constructs examples to show that $m = n_\pm$ can actually take on all possible integer values between 0 and $2n$. □

Let $c \in I$; necessarily, c is a regular point of $l[\cdot]$. Let $\mathcal{L}_0^-$ denote the minimal operator generated by $l[\cdot]$ on (a, c) and let $\mathcal{L}_0^+$ denote the minimal operator generated by $l[\cdot]$ on (c, b). Let (m_-, m_-) and (m_+, m_+) denote the deficiency indices of $\mathcal{L}_0^-$ in $L^2(a, c)$ and $\mathcal{L}_0^+$ in $L^2(c, b)$, respectively.

Theorem 2.10. *The deficiency index of the minimal operator $\mathcal{L}_0$ in $L^2(I)$ is (m, m) where*

$$m = m_+ + m_- - 2n,$$

and $2n$ is the order of the expression $l[\cdot]$. Furthermore, m is independent of the choice of $c \in I$.

Proof. See [25], page 353. □

The importance of this theorem may need some explanation. Since the point c is a regular point, all solutions of $l[y] = \pm iy$ will belong to $L^2(c - \varepsilon, c]$ for all $0 < \varepsilon < c - a$. Consequently, the number m_- is precisely equal to the number of solutions of $l[y] = \pm iy$ that are in $L^2(a, a + \delta]$ for some sufficiently small $\delta > 0$. Similarly, the number m_+ is equal to the number of solutions of $l[y] = \pm iy$ that are in $L^2(b - \delta, b]$ for some small enough $\delta > 0$. This motivates the following.

Definition 2.9. The differential expression $l[\cdot]$ is said to be in the limit-p condition at $x = a$ in $L^2(I)$ if there exist exactly p solutions of $l[y] = \pm iy$ that belong to $L^2(a, a+\varepsilon)$ for some sufficiently small $\varepsilon > 0$. Similarly, $l[\cdot]$ is said to be in the limit-q condition at $x = b$ in $L^2(I)$ if there exist exactly q solutions of $l[y] = \pm iy$ that belong to $L^2(b - \varepsilon, b)$ for some sufficiently small $\varepsilon > 0$. Since $l[\cdot]$ is of order $2n$, it is clear that $0 \leq p, q \leq 2n$.

If the order of $l[\cdot]$ is two, the *limit-2 condition* is more commonly referred to as the *limit-circle condition*, while the *limit-1 condition* is known as the *limit-point condition*. This notion goes back to Weyl's seminal paper [56]. His analysis of the number of Lebesgue square integrable solutions of the second order Sturm-Liouville equation involved some key geometric arguments. The terms *limit-point* and *limit-circle* reflect the geometry used in his solution. In the second-order case, Weyl showed that if $l[y] = \lambda_0 y$ is limit-point (respectively, limit-circle) at a or b for a certain complex number λ_0, then $l[y] = \lambda y$ is limit-point (respectively, limit-circle) at a or b for a all complex numbers λ.

From the previous definition and theorem, it is clear that once we have determined the limit condition for each endpoint, then the deficiency index of the minimal operator $\mathcal{L}_0$ in $L^2(I)$ can be found. Fortunately, there is a method available for determining the limit condition of an endpoint when that endpoint is a regular singular point in the sense of Frobenius. Indeed, the so-called Method of Frobenius from ordinary differential equations (see [30], pages 396-404) can sometimes be used to determine the number of Lebesgue square integrable solutions near this singular endpoint.

Definition 2.10. Consider the differential equation

$$L[y](x) = \sum_{j=0}^{n} b_j(x)y^{(j)}(x) = 0, \qquad x \in J \tag{2.4}$$

where $J \subset \mathbb{R}$ is some open interval, $b_j : J \longrightarrow \mathbb{R}$, $j = 0, 1, ..., n$, $b_n(x) \neq 0$ for all $x \in J$. Suppose $a, b \in J$ with $a < b$. If $x = a > -\infty$, then $x = a$ is called a regular singular point of $L[\cdot]$ if

$$\frac{(x-a)^n L[y](x)}{b_n(x)} = \sum_{j=0}^{n}(x-a)^j c_j(x)y^{(j)}(x),$$

where $c_n(x) = 1$ and where each $c_j(x)$ is analytic in some neighborhood of $x = a$, $j = 0, 1, ..., n-1$. The definition of $x = b < \infty$ as a regular singular point is similar. If $a = -\infty$ or $(b = \infty)$ and $L[\cdot]$ can be put into the form

$$\sum_{j=0}^{n} t^j c_j(t)y^{(j)}(t),$$

under the transformation $x = 1/t$, where again $c_n(t) = 1$ and where each $c_j(t)$ is analytic in some neighborhood of $t = 0$, then we say that $x = \infty$ is a regular singular point of $L[\cdot]$. If an endpoint is not a regular singular point, it is called an irregular singular point.

Based on earlier work of Fuchs, Frobenius developed an ingenious tool for determining a basis of n solutions of the homogeneous equation (2.4), where each solution is expanded about a regular singular point. A key ingredient in this method is the indicial equation at $x = a$ associated with (2.4):

$$\sum_{j=0}^{n} P(r,j)c_j = 0, \tag{2.5}$$

where $c_j = c_j(a)$ and $P(r,j) = \frac{r!}{(r-j)!}$, $j = 0, 1, ..., n$. Evidently, this is a polynomial of degree exactly n. We will not describe this method here; it suffices to say that each of the n roots of the indicial equation (2.5) determines a solution of (2.4), even in the case of roots having multiplicity greater than one.

After the following definition, we will be in the position to state the important Glazman-Krein-Naimark theorem.

Definition 2.11. Let X be a vector space and $M_1 \subset M_2$ be subspaces of X. We say that the set $\{x_1, x_2, ..., x_n\} \subset M_2$ is linearly independent modulo M_1 if the condition

$$\sum_{j=0}^{n} \alpha_j x_j \in M_1$$

implies that $\alpha_j = 0$, $j = 1, 2, ..., n$. If $A \subset M_2$ is a maximal linearly independent set modulo M_1 and $\beta = card(A)$, we say that the dimension of M_2 is β modulo M_1.

It is not difficult to see that if $\{x_1, x_2, ..., x_n\} \subset M_2$ is a linearly independent set, then it is a maximal linearly independent modulo M_1 if and only if

$$M_2 = M_1 + sp\{x_1, x_2, ..., x_n\}. \tag{2.6}$$

Of course, any set of linearly independent vectors modulo M_1 is a linearly independent set in X; the converse of this is not necessarily true. This concept of linear independence modulo a subspace plays an important role in characterizing all self-adjoint extensions of $\mathcal{L}_0$ in $L^2(I)$. In view of (2.6) and the importance that von Neumann's formula (2.3) plays, this statement is not too surprising.

Theorem 2.11. *(Glazman-Krein-Naimark) Suppose the deficiency index of the minimal operator $\mathcal{L}_0$ in $L^2(a,b)$ generated by the expression $l[\cdot]$ is (m,m).*

(i) Let S be a self-adjoint extension of $\mathcal{L}_0$ in $L^2(a,b)$. Then there exists a set $\{w_1, w_2, ..., w_n\} \subset \mathcal{D}(S)$ that is linearly independent modulo $\mathcal{D}(\mathcal{L}_0)$ such that

$$S[y] = l[y]$$

$$\mathcal{D}(S) = \left\{ y \in \mathcal{D}(\mathcal{L}) \mid [w_j, y] \mid_a^b = 0, \ j = 1, 2, ..., m \right\}. \tag{2.7}$$

Here, $[\cdot,\cdot]$ *is the sesquilinear form defined in (2.2).*

(ii) Suppose $\{w_1, w_2, ..., w_n\} \subset \mathcal{D}(\mathcal{L})$ *is linearly independent modulo* $\mathcal{D}(\mathcal{L}_0)$ *with*

$$[w_j, w_k] \mid_a^b = 0, \ \ j,k = 1,2,...,m.$$

Define an operator S *in* $L^2(a,b)$ *by*

$$\begin{aligned} S[y] &= l[y] \\ \mathcal{D}(S) &= \left\{ y \in \mathcal{D}(\mathcal{L}) \mid [w_j, y] \mid_a^b = 0, \ \ j = 1,2,...,m \right\}. \end{aligned}$$

Then S *is a self-adjoint extension of* $\mathcal{L}_0$*.*

The conditions given in (2.7) are known as the Glazman boundary conditions and the functional $[w_j, \cdot] \mid_a^b : \mathcal{D}(\mathcal{L}) \longrightarrow \mathbb{C}$ is called a boundary value for $\mathcal{L}_0$. If for some j, $[w_j, y] \mid_a^b = 0$ is independent of a or b for all $y \in \mathcal{D}(S)$, then it is called a separated boundary condition; otherwise it is a mixed boundary condition. In [12], page 1234, a boundary value for a symmetric operator A is defined to be a continuous linear functional on $(\mathcal{D}(A^*), (\cdot,\cdot)^*)$ that vanishes on $\mathcal{D}(A)$. There is a generalization of the Glazman-Krein-Naimark theorem for arbitrary symmetric operators which can be found in [12], page 1239.

CHAPTER THREE

General Left-Definite Spectral Theory

In a recent paper [39], Littlejohn and Wellman developed a general abstract left-definite theory for a self-adjoint operator A that is bounded below in a Hilbert space $(H, (\cdot,\cdot))$. They show that there exists a continuum of unique Hilbert spaces $\{(W_r, (\cdot,\cdot)_r)\}_{r>0}$ and, for each $r > 0$, a unique self-adjoint restriction A_r of A in W_r. The Hilbert space W_r is called the r^{th} *left-definite Hilbert space* associated with the pair (H, A) and the operator A_r is called the r^{th} *left-definite operator* associated with (H, A). In this chapter, we discuss the main results in [39] and their relevance to the Jacobi equation that we study in this thesis.

The left-definite spectral theory has its roots in the work of Weyl [56] on formally symmetric second-order differential expressions. The terminology *left-definite* is due to Schäfke and Schneider who used the German *links-definit* [50] in 1965 to describe one of the Hilbert space settings in which certain formally symmetric differential expressions can be studied. As an example, let us consider the differential equation

$$L[y](t) = \lambda w(t)y(t) \qquad (t \in I; \lambda \in \mathbb{C}), \tag{3.1}$$

where $I = (a, b)$ is an open interval of the real line $\mathbb{R}$, w is Lebesgue-measurable, locally integrable and positive almost everywhere on I, and where $L[\cdot]$ is the formally symmetric differential expression

$$L[y](t) = \sum_{j=0}^{n} (-1)^j \left(b_j(t)y^{(j)}(t)\right)^{(j)} \qquad (t \in I),$$

with non-negative, infinitely differentiable coefficients $b_j(t)$ $(j = 0, 1, ..., n)$ on I. Then the classical Glazman-Krein-Naimark theory [43] applies to (3.1) and characterizes all self-adjoint extensions of the minimal operator $T_{\min}$ generated by $w^{-1}L[\cdot]$

in the weighted Hilbert space $L^2_w(I)$ of all Lebesgue-measurable functions $f : I \longrightarrow \mathbb{C}$ with inner product

$$(f, f) = \int_I |f(t)|^2 w(t)dt < \infty.$$

The space $L^2_w(I)$ is called the *right-definite Hilbert space* for $w^{-1}L[\cdot]$ because w appears on the right-hand side of (3.1). However, the differential expression $w^{-1}L[\cdot]$ can also be studied in a Hilbert space W generated by the Sobolev inner product

$$(f, g)_W = \sum_{j=0}^{n} b_j(t) f^{(j)}(t)\overline{g}^{(j)}(t) \qquad (f, g \in W).$$

Since this inner product is generated from the left-hand side of (3.1), we call W a *left-definite Hilbert space* and the spectral study of $w^{-1}L[\cdot]$ in W a *left-definite spectral setting*. It is worth noting that, although the motivation for the general left-definite theory developed in [39] arose from the study of certain self-adjoint differential operators, the left-definite theory can be applied to any self-adjoint operator that is bounded below. In what follows, we will give an overview of the general left-definite spectral theory as developed in [39].

Let V be a vector space over $\mathbb{C}$ with inner product $(\cdot, \cdot)$ such that $H := (V, (\cdot, \cdot))$ is a Hilbert space. Suppose that V_r is a vector subspace of V with inner product $(\cdot, \cdot)_r$ and let us denote this inner product space by $W_r := (V_r, (\cdot, \cdot)_r)$. Let $A : \mathcal{D}(A) \subset H \to H$ be a self-adjoint operator that is bounded below by rI for some $r > 0$, that is to say

$$(Ax, x) \geq r(x, x) \qquad (x \in \mathcal{D}(A)).$$

Then for any $s > 0$, the operator A^s is self-adjoint and bounded below in H by $r^s I$.

Definition 3.1. Let $s > 0$, let V_s be a vector subspace of the Hilbert space $H = (V, (\cdot, \cdot))$ with inner product $(\cdot, \cdot)_s$ and let $W_s := (V_s, (\cdot, \cdot)_s)$. We say that W_s is an s^{th} left-definite space associated with the pair (H, A) if

(i) W_s is a Hilbert space

(ii) $D(A^s)$ is a vector subspace of V_s

(iii) $D(A^s)$ is dense in W_s

(iv) $(x, x)_s \geq r^s(x, x)\ \forall x \in V_s$

(v) $(x, y)_s = (A^s x, y)\ \forall x \in \mathcal{D}(A^s), y \in V_s$.

Remark 3.1. In a sense, the most important property is (v), as it shows how the s^{th} left-definite space is generated by the s^{th} power of A.

Note that, at this point, the existence of such a left-definite space is certainly in question. However, Littlejohn and Wellman in [39] prove the following result.

Theorem 3.1. *Let $A : \mathcal{D}(A) \subset H \to H$ be a self-adjoint operator that is bounded below by rI for some $r > 0$. Let $s > 0$ and define $W_s := (V_s, (\cdot, \cdot)_s)$ by*

$$V_s = \mathcal{D}(A^{s/2})$$

and

$$(x, y)_s = (A^{s/2}x, A^{s/2}y) \qquad (x, y \in V_s).$$

Then W_s is the unique left-definite space associated with the pair (H, A).

Definition 3.2. For $s > 0$, let $W_s := (V_s, (\cdot, \cdot)_s)$ be the s^{th} left-definite space associated with (H, A). If there exists a self-adjoint operator $B_s : \mathcal{D}(B_s) \subset W_s \to W_s$ satisfying

$$B_s f = Af \qquad (f \in \mathcal{D}(B_s) \subset \mathcal{D}(A)),$$

we call such an operator an s^{th} left-definite operator associated with the pair (H, A).

Note that it is not immediately clear that such an operator exists. However, its existence and uniqueness is established as follows.

Theorem 3.2. *Let A be a self-adjoint operator in a Hilbert space H that is bounded below by rI for some $r > 0$. For any $s > 0$, let $W_s := (V_s, (\cdot,\cdot)_s)$ denote the s^{th} left-definite space associated with (H, A). Then there exists a unique left-definite operator B_s in W_s associated with (H, A). Furthermore,*

$$\mathcal{D}(B_s) = V_{s+2} \subset \mathcal{D}(A).$$

Theorem 3.3. *Suppose A is a self-adjoint operator in a Hilbert space H that is bounded below by rI for some $r > 0$. Let $\{H_s = (V_s, (\cdot,\cdot)_s)\}_{s>0}$ and $\{B_s\}_{s>0}$ be the left-definite spaces and operators associated with (H, A). Then the following hold:*

(1) Suppose A is bounded. Then, for each $s > 0$,

(i) $V = V_s$

(ii) the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_s$ are equivalent

(iii) $A = B_s$.

(2) Suppose A is unbounded. Then

(i) V_s is a proper subspace of V

(ii) V_s is a proper subspace of V_t whenever $0 < t < s$

(iii) the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_s$ are not equivalent for any $s > 0$

(iv) the inner products $(\cdot,\cdot)_t$ and $(\cdot,\cdot)_s$ are not equivalent for any $s, t > 0, s \neq t$

(v) $\mathcal{D}(B_s)$ is a proper subspace of $\mathcal{D}(A)$ for each $s > 0$

(vi) $\mathcal{D}(B_t)$ is a proper subspace of $\mathcal{D}(B_s)$ whenever $0 < s < t$.

Theorem 3.4. *For each $s > 0$, let B_s denote the s^{th} left-definite operator associated with the self-adjoint operator A that is bounded below by rI in H for some $s > 0$. Then* *(i) the point spectra of A and B_s are identical, i.e.* $\sigma_p(B_s) = \sigma_p(A)$

(ii) the continuous spectra of A and B_s coincide, i.e. $\sigma_c(B_s) = \sigma_c(A)$

(iii) the resolvent sets of A and B_s are equal, i.e. $\rho(B_s) = \rho(A)$.

CHAPTER FOUR

Spectral Analysis of the Jacobi Differential Equation ($\alpha, \beta > -1$)

4.1 The Classical Jacobi Differential Equation

The classical second-order Lagrange symmetrizable Jacobi differential expression is defined by

$$\begin{aligned} l_{\alpha,\beta}[y](x) &:= \frac{1}{\omega_{\alpha,\beta}(x)}\left[\left(\left(-(1-x)^{\alpha+1}(1+x)^{\beta+1}\right)y'(x)\right)' + k(1-x)^{\alpha}(1+x)^{\beta}y(x)\right] \\ &= -(1-x^2)y''(x) + (\alpha-\beta+(\alpha+\beta+2)x)y'(x) + ky(x) \end{aligned}$$

for $\alpha, \beta > -1$ and $x \in (-1,1)$, where

$$\omega_{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}$$

and $k \geq 0$ is a spectral parameter which is used to shift the spectrum of the self-adjoint operator $A_k^{\alpha,\beta}$ to a subset of the positive real line.

With

$$\lambda_{r,k}^{\alpha,\beta} := r(r+\alpha+\beta+1)+k, \qquad (r \in \mathbb{N}_0)$$

the Jacobi differential equation

$$l_{\alpha,\beta}[y](x) = \lambda_{r,k}^{\alpha,\beta} y(x)$$

has polynomial solutions $\left\{P_r^{(\alpha,\beta)}(x)\right\}_{r=0}^{\infty}$, where $P_r^{(\alpha,\beta)}(x)$ is the r^{th} Jacobi polynomial of exactly degree r, [46]:

$$P_r^{(\alpha,\beta)}(x) := k_r^{\alpha,\beta} \sum_{j=0}^{r} \frac{(1+\alpha)_r(1+\alpha+\beta)_{r+j}}{j!(r-j)!(1+\alpha)_j(1+\alpha+\beta)_r}\left(\frac{1-x}{2}\right)^j \tag{4.1}$$

and where

$$k_r^{\alpha,\beta} := \frac{(r!)^{1/2}(1+\alpha+\beta+2r)^{1/2}(\Gamma(\alpha+\beta+r+1))^{1/2}}{2^{(\alpha+\beta+1)/2}(\Gamma(\alpha+r+1))^{1/2}(\Gamma(\beta+r+1))^{1/2}}.$$

In fact, $\left\{P_r^{(\alpha,\beta)}(x)\right\}_{r=0}^{\infty}$ forms a complete orthonormal set in the weighted L^2-space $L^2_{\alpha,\beta}(-1,1) = L^2((-1,1);(1-x)^\alpha(1+x)^\beta)$, i.e.

$$(P_r^{(\alpha,\beta)}, P_n^{(\alpha,\beta)})_{\alpha,\beta} = \delta_{r,n} \quad (r, n \in \mathbb{N}_0).$$

The derivatives of the Jacobi polynomials satisfy the identity

$$\frac{d^j}{dx^j} P_r^{(\alpha,\beta)}(x) = a^{(\alpha,\beta)}(r,j) P_{r-j}^{(\alpha+j,\beta+j)}(x) \quad (r, j \in \mathbb{N}_0), \tag{4.2}$$

where

$$a^{(\alpha,\beta)}(r,j) = \frac{(r!)^{1/2}\left(\Gamma(\alpha+\beta+r+1+j)\right)^{1/2}}{((r-j)!)^{1/2}\left(\Gamma(\alpha+\beta+r+1)\right)^{1/2}} \quad (j = 0, 1, ..., r),$$

and $a^{(\alpha,\beta)}(r,j) = 0$ if $j > r$. Furthermore,

$$\begin{aligned} &\int_{-1}^{1} \frac{d^j\left(P_r^{(\alpha,\beta)}(x)\right)}{dx^j} \frac{d^j\left(P_n^{(\alpha,\beta)}(x)\right)}{dx^j} w_{\alpha+j,\beta+j}(x)dx \\ &= \frac{r!\Gamma(\alpha+\beta+r+1+j)}{(r-j)!\Gamma(\alpha+\beta+r+1)}\delta_{r,n} \quad (r, n, j \in \mathbb{N}_0). \end{aligned} \tag{4.3}$$

We remark that, equivalently, the Jacobi polynomials may be defined by

$$P_n^{(\alpha,\beta)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k} \quad (n \in \mathbb{N}_0) \tag{4.4}$$

as in [10]. Up to the normalization constant $k_r^{(\alpha,\beta)}$, these polynomials are identical to the ones in (4.1). The connection can be seen by using the ${}_2F_1-$representation for the Jacobi polynomials as in [46]. The set $\left\{P_n^{(\alpha,\beta)}(x)\right\}_{n=0}^{\infty}$ satisfies the orthogonality relation

$$\begin{aligned} &\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx \\ &= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{m,n} \end{aligned}$$

for $\alpha, \beta > -1$.

The Rodrigues' formula for the Jacobi polynomials (for $x \in (-1,1)$) is

$$P_n^{(\alpha,\beta)}(x) = (-2)^{-n}(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]$$

and many properties hold for arbitrary parameters α and β, but for integrability purposes one is restricted to $\alpha, \beta > -1$. The Jacobi polynomials satisfy the three-term recurrence relation

$$\begin{aligned} 2n(n+\alpha+\beta)(2n+\alpha+\beta-2)P_n^{(\alpha,\beta)}(x) = (2n+\alpha+\beta-1)\,&[(2n+\alpha+\beta)\times \\ &(2n+\alpha+\beta-2)x+\alpha^2-\beta^2]\,P_{n-1}^{(\alpha,\beta)}(x) \qquad (4.5) \\ &-2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)P_{n-2}^{(\alpha,\beta)}(x) \end{aligned}$$

for $n \geq 1$.

We note that for $\alpha = \beta = -1$, by Favard's theorem, the full sequence of Jacobi polynomials cannot be orthogonal on the real line with respect to an inner product of the form $\int_{\mathbb{R}} f(x)\overline{g}(x)d\mu$, where μ is a measure. This observation shall be the starting point for the study of the Sobolev orthogonality of the Jacobi polynomials for $\alpha = \beta = -1$ in chapter 5.

A generating function for the Jacobi polynomials is

$$2^{\alpha+\beta}R^{-1}(1-w+R)^{-\alpha}(1+w+R)^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)w^n$$

where

$$R := (1-2xw+w^2)^{1/2}.$$

A differentiation formula is

$$\begin{aligned} (2n+\alpha+\beta)(1-x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) &= n\left[\alpha-\beta-(2n+\alpha+\beta)x\right]P_n^{(\alpha,\beta)}(x) \\ &\quad + 2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(x) \end{aligned}$$

or

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

and the sequence of derivatives also forms an orthogonal polynomial system.

Special cases of the Jacobi polynomials are the Legendre polynomials ($\alpha = \beta = 0$), the Chebychev polynomials of the first kind ($\alpha = \beta = 1/2$) and of the second kind ($\alpha = \beta = -1/2$); in general, when $\alpha = \beta$, the Jacobi polynomials are also called Gegenbauer or ultraspherical polynomials.

Useful identities include

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$$
$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$$

and

$$\binom{n}{l} P_n^{(-l,\beta)}(x) = \binom{n+\beta}{l}\left(\frac{x-1}{2}\right)^l P_{n-l}^{(l,\beta)}(x) \qquad (l \in \mathbb{N}, 1 \leq l \leq n).$$

4.2 Combinatorics and Jacobi-Stirling Numbers

The key to constructing the left-definite spaces associated with the Jacobi differential expression is to determine the integral composite powers $l^n_{\alpha,\beta}[.]$ $(n \in \mathbb{N})$ of the Jacobi differential expression. In [17], the authors show that the Jacobi-Stirling numbers are closely connected to the explicit representation of the powers $l^n_{\alpha,\beta}[.]$. These results are purely algebraic and therefore hold for arbitrary parameters α and β. We shall apply these results in chapter 5 and 6 to construct the left-definite spaces associated with the Jacobi expression for $\alpha = \beta = -1$, so let us state the following definitions and theorems from [17].

Theorem 4.1. *Suppose $k \geq 0$ and $n \in \mathbb{N}$. For each $m \in \mathbb{N}_0$, the recurrence relations*

$$(m(m+\alpha+\beta+1)+k)^n = \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k)\frac{m!\Gamma(\alpha+\beta+m+1+j)}{(m-j)!\Gamma(\alpha+\beta+m+1)}$$

have a unique solution

$$\left(c_0^{(\alpha,\beta)}(n,k), c_1^{(\alpha,\beta)}(n,k), ..., c_n^{(\alpha,\beta)}(n,k)\right),$$

where each $c_j^{(\alpha,\beta)}(n,k)$ *is independent of* m*, given explicitly by*

$$c_0^{(\alpha,\beta)}(n,k) := \begin{cases} 0 & \textit{if } k = 0 \\ k^n & \textit{if } k > 0 \end{cases}, \tag{4.6}$$

and, for $j \in \{1, 2, ..., n\}$*,*

$$c_j^{(\alpha,\beta)}(n,k) := \begin{cases} P^{(\alpha,\beta)} & \textit{if } k = 0 \\ \sum_{s=0}^{n-j} \binom{n}{s} P^{(\alpha,\beta)} S_{n-s}^{(j)} k^s & \textit{if } k > 0 \end{cases}, \tag{4.7}$$

where each $P^{(\alpha,\beta)}S_n^{(j)}$ *is positive and given by*

$$P^{(\alpha,\beta)}S_n^{(j)} := \sum_{r=0}^{j} (-1)^{r+j} \frac{\Gamma(\alpha+\beta+r+1)\Gamma(\alpha+\beta+2r+2)[r(r+\alpha+\beta+1)]^n}{r!(j-r)!\Gamma(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+j+r+2)} \tag{4.8}$$

for each $n \in \mathbb{N}$ *and* $j \in \{1, 2, ..., n\}$*. The number* $P^{(\alpha,\beta)}S_n^{(j)}$ *is called the Jacobi-Stirling number of order* (n, j) *associated with* (α, β)*. This definition is extended by*

$$P^{(\alpha,\beta)}S_0^{(0)} := 1$$

$$P^{(\alpha,\beta)}S_n^{(j)} := 0 \quad \textit{if } j \in \mathbb{N} \textit{ and } 0 \le n \le j-1$$

$$P^{(\alpha,\beta)}S_n^{(0)} := 0 \textit{ for } n \in \mathbb{N}.$$

a) Let $k \ge 0$. For each $n \in \mathbb{N}$, the n^{th} composite power of the classical Jacobi differential expression $l_{\alpha,\beta}[.]$ is Lagrange symmetrizable, with symmetry factor $\omega_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$, and it is given explicitly by

$$\omega_{\alpha,\beta}(t) l_{\alpha,\beta}^n[y](t) = \sum_{j=0}^{n} (-1)^j \left(c_j^{(\alpha,\beta)}(n,k)(1-t)^{\alpha+j}(1+t)^{\beta+j} y^{(j)}(t) \right)^{(j)}$$

where $c_j^{(\alpha,\beta)}(n,k)$ is defined as in 4.6 and 4.7. Moreover, for $p, q \in P$, the following identity is valid:

$$\begin{aligned}(l^n_{\alpha,\beta}[p], q)_{\alpha,\beta} &= \int_{-1}^{1} l^n_{\alpha,\beta,k}[p](t)\overline{q}(t)\omega_{\alpha,\beta}(t)dt \\ &= \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k) \int_{-1}^{1} p^{(j)}(t)\overline{q}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}dt.\end{aligned}$$

b) For every $n \in \mathbb{N}$, the bilinear form $(.,.)_n^{(\alpha,\beta)}$ defined on $\mathcal{P} \times \mathcal{P}$ by

$$(p,q)_n^{(\alpha,\beta)} := \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k) \int_{-1}^{1} p^{(j)}(t)\overline{q}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}dt \qquad (p,q \in P)$$

is an inner product when $k > 0$, and, for each $k \geq 0$,

$$(l^n_{\alpha,\beta,k}[p], q)_{\alpha,\beta} = (p,q)_{n,k}^{(\alpha,\beta)} \qquad (p,q \in P).$$

c) For each $k \geq 0$, the Jacobi polynomials $\left\{P_m^{(\alpha,\beta)}\right\}_{m=0}^{\infty}$ are orthogonal with respect to $(.,.)_n^{(\alpha,\beta)}$:

$$(P_m^{(\alpha,\beta)}, P_r^{(\alpha,\beta)})_n^{(\alpha,\beta)} = (m(m+\alpha+\beta+1)+k)^n \delta_{m,r}.$$

4.3 Right-Definite Spectral Analysis

Here we shall briefly state the operator-theoretic properties of the classical Jacobi differential expression $l_{\alpha,\beta}[.]$ as found in [17] and references therein.

The maximal domain $\Delta^{(\alpha,\beta)}$ of $l_{\alpha,\beta}[.]$ in $L^2_{\alpha,\beta}(-1,1)$ is given by

$$\Delta^{(\alpha,\beta)} := \left\{ f \in L^2_{\alpha,\beta}(-1,1) \,\middle|\, f, f' \in AC_{loc}(-1,1); l_{\alpha,\beta}[f] \in L^2_{\alpha,\beta}(-1,1) \right\}.$$

Note that $\Delta^{(\alpha,\beta)}$ is a dense vector subspace of $L^2_{\alpha,\beta}(-1,1)$ since it contains the space of all polynomials $\mathcal{P}$. The maximal operator $T_{\max}^{(\alpha,\beta)}$ generated by $l_{\alpha,\beta}[.]$ in $L^2_{\alpha,\beta}(-1,1)$ is then defined by

$$T_{\max}^{(\alpha,\beta)}(f) := l_{\alpha,\beta}[f]$$

$$\mathcal{D}\left(T_{\max}^{(\alpha,\beta)}\right) := \Delta^{(\alpha,\beta)}.$$

The minimal operator $T_{\min}^{(\alpha,\beta)}$ is defined as the Hilbert space adjoint of $T_{\max}^{(\alpha,\beta)}$,

$$T_{\min}^{(\alpha,\beta)} := \left(T_{\max}^{(\alpha,\beta)}\right)^{*}.$$

The minimal operator is closed, symmetric, and satisfies

$$\left(T_{\min}^{(\alpha,\beta)}\right)^{*} = T_{\max}^{(\alpha,\beta)}.$$

The deficiency index $d\left(T_{\min}^{(\alpha,\beta)}\right)$ of $T_{\min}^{(\alpha,\beta)}$ depends on the values of α and β and is given by

$$d\left(T_{\min}^{(\alpha,\beta)}\right) = \begin{cases} (0,0) & \text{if } \alpha,\beta \geq 1 \\ (1,1) & \text{if } \alpha \geq 1 \text{ and } \beta \in (-1,1) \text{ or } \beta \geq 1 \text{ and } \alpha \in (-1,1) \\ (2,2) & \text{if } \alpha,\beta \in (-1,1) \end{cases}.$$

By the von-Neumann theory of self-adjoint extensions of symmetric operators [12], $T_{\min}^{(\alpha,\beta)}$ has self-adjoint extensions in $L^2_{\alpha,\beta}(-1,1)$ for all $\alpha,\beta > -1$. There is a unique self-adjoint extension when $\alpha,\beta \geq 1$ since the deficiency index is $(0,0)$.

The singular endpoints $x = \pm 1$ of the Lagrange symmetric differential expression $w_{\alpha,\beta} l_{\alpha,\beta}[.]$ satisfy the following limit-point/limit-circle criteria in $L^2_{\alpha,\beta}(-1,1)$:

(i) the endpoint $x = +1$ is limit-point if $\alpha \geq 1$; if $-1 < \alpha < 0$, $x = +1$ is regular, and if $0 \leq \alpha < 1$, $x = +1$ is limit-circle, non-oscillatory;

(ii) the endpoint $x = -1$ is limit-point if $\beta \geq 1$; if $-1 < \beta < 0$, $x = -1$ is regular, and if $0 \leq \beta < 1$, $x = -1$ is limit-circle, non-oscillatory.

From the Glazman-Krein-Naimark theory [2],[43], the operator

$$A^{(\alpha,\beta)} : \mathcal{D}\left(A^{(\alpha,\beta)}\right) \subset L^2_{\alpha,\beta}(-1,1) \longrightarrow L^2_{\alpha,\beta}(-1,1)$$
$$A^{(\alpha,\beta)}(f) := l_{\alpha,\beta}[f]$$

where $\mathcal{D}\left(A^{(\alpha,\beta)}\right) :=$

$$\begin{cases} \Delta^{(\alpha,\beta)} & \text{if } \alpha,\beta \geq 1 \\ \left\{ f \in \Delta^{(\alpha,\beta)} \left| \lim\limits_{x\to 1}(1-x)^{\alpha+1} f'(x) = 0 \right.\right\} & \text{if } |\alpha| < 1 \text{ and } \beta \geq 1 \\ \left\{ f \in \Delta^{(\alpha,\beta)} \left| \lim\limits_{x\to -1}(1+x)^{\beta+1} f'(x) = 0 \right.\right\} & \text{if } |\beta| < 1 \text{ and } \alpha \geq 1 \\ \left\{ f \in \Delta^{(\alpha,\beta)} \left| \lim\limits_{x\to \pm 1}(1-x)^{\alpha+1}(1+x)^{\beta+1} f'(x) = 0 \right.\right\} & \text{if } -1 < \alpha,\beta < 1. \end{cases}$$

is self-adjoint in $L^2_{\alpha,\beta}(-1,1)$. The Jacobi polynomials $\left\{P_n^{(\alpha,\beta)}\right\}_{n=0}^{\infty}$ form a complete set of eigenfunctions of $A^{(\alpha,\beta)}$ in $L^2_{\alpha,\beta}(-1,1)$, and the spectrum of $A^{(\alpha,\beta)}$ is given by

$$\sigma\left(A^{(\alpha,\beta)}\right) = \{n(n+\alpha+\beta+1)+k\,|n \in \mathbb{N}_0\}.$$

In particular,

$$\sigma\left(A^{(\alpha,\beta)}\right) \subset [k,\infty),$$

implying that $A^{(\alpha,\beta)}$ is bounded below by kI in $L^2_{\alpha,\beta}(-1,1)$, i.e.

$$\left(A^{(\alpha,\beta)}f,f\right)_{\alpha,\beta} \geq k\,(f,f)_{\alpha,\beta} \quad \left(f \in \mathcal{D}\left(A^{(\alpha,\beta)}\right)\right).$$

Consequently, the left-definite theory can be applied to this self-adjoint operator.

For $f,g \in \mathcal{D}\left(A^{(\alpha,\beta)}\right)$, we have the well-known Dirichlet-identity for $A^{(\alpha,\beta)}$

$$\begin{aligned} \left(A^{(\alpha,\beta)}f,g\right)_{\alpha,\beta} &= \int_{-1}^{1} l_{\alpha,\beta}[f](x)\overline{g(x)}(1-x)^{\alpha}(1+x)^{\beta}dx \\ &= \int_{-1}^{1} \left\{(1-x)^{\alpha+1}(1+x)^{\beta+1}f'(x)\overline{g}'(x)\right. \qquad (4.9) \\ &\qquad \left. +k(1-x)^{\alpha}(1+x)^{\beta}f(x)\overline{g}(x)\right\}dx \end{aligned}$$

as a consequence of the strong limit-point condition on the domain $\mathcal{D}\left(A^{(\alpha,\beta)}\right)$:

$$\lim_{x\to\pm 1}(1-x)^{\alpha+1}(1+x)^{\beta+1}f(x)\overline{g}'(x) = 0 \quad \left(f,g \in \mathcal{D}\left(A^{(\alpha,\beta)}\right)\right).$$

Note that Dirichlet's identity holds on $\mathcal{D}\left(A^{(\alpha,\beta)}\right)$, and not in general on the maximal domain $\Delta^{(\alpha,\beta)}$. Furthermore, when $k > 0$, the right-hand side of 4.9 satisfies the conditions of an inner product. We define the inner product $(\cdot,\cdot)_1^{(\alpha,\beta)}$ on $\mathcal{D}\left(A^{(\alpha,\beta)}\right)\times\mathcal{D}\left(A^{(\alpha,\beta)}\right)$ by

$$(f,g)_1^{(\alpha,\beta)} := \int_{-1}^{1} \left\{(1-x)^{\alpha+1}(1+x)^{\beta+1}f'(x)\overline{g}'(x) + k(1-x)^{\alpha}(1+x)^{\beta}f(x)\overline{g}(x)\right\} dx$$

for all $f,g\in\mathcal{D}\left(A^{(\alpha,\beta)}\right)$. The authors extend this inner product to the set $V_1^{(\alpha,\beta)}\times V_1^{(\alpha,\beta)}$ where $V_1^{(\alpha,\beta)}$ is the first left-definite space associated with $\left(L^2_{\alpha,\beta}(-1,1),A^{(\alpha,\beta)}\right)$. The inner product $(\cdot,\cdot)_1^{(\alpha,\beta)}$ is called the first left-definite inner product in the literature.

4.4 Left-Definite Spectral Analysis

This section will give a summary of the left-definite results by Everitt, Kwon, Littlejohn, Wellman and Yoon [17]. In the following, we shall write

$$L^2_{\alpha,\beta}(-1,1) := L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{\beta}\right).$$

Definition 4.1. Let $k>0$. For each $n\in\mathbb{N}$, define

$$V_n^{(\alpha,\beta)} := \left\{f:(-1,1)\to\mathbb{C}\mid f\in AC_{loc}^{(n-1)}(-1,1);\, f^{(j)}\in L^2_{\alpha+j,\beta+j}(-1,1),\, j=0,1,...,n\right\}$$

and let $(.,.)_n^{(\alpha,\beta)}$ and $\|.\|_n^{(\alpha,\beta)}$ denote the inner product

$$(f,g)_n^{(\alpha,\beta)} := \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k)\int_{-1}^{1} f^{(j)}(t)\overline{g}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j}dt \qquad (f,g\in V_n^{(\alpha,\beta)}),$$

and the norm $\|f\|_n^{(\alpha,\beta)} := ((f,f)_n^{(\alpha,\beta)})^{1/2}$, where the numbers $c_j^{(\alpha,\beta)}(n,k)$ are defined as in 4.6 and 4.7. Let

$$W_{n,k}^{(\alpha,\beta)}(-1,1) := (V_n^{(\alpha,\beta)},(\cdot,\cdot)_n^{(\alpha,\beta)}).$$

Note that, from the non-negativity of each of the numbers $c_j^{(\alpha,\beta)}(n,k)$, $j=0,1,...,n$, we have

$$\begin{aligned}\left(\|f\|_n^{(\alpha,\beta)}\right)^2 &= \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n,k)\left\|f^{(j)}\right\|^2_{\alpha+j,\beta+j}\\ &\geq c_j^{(\alpha,\beta)}(n,k)\left\|f^{(j)}\right\|^2_{\alpha+j,\beta+j} \qquad (j=0,1,...,n;\, f\in V_n^{(\alpha,\beta)}).\end{aligned}$$

In particular, if $j = 0$, we see that

$$(f, f)_n^{(\alpha,\beta)} \geq k^n (f, f)_{\alpha,\beta} \qquad\qquad (f \in W_n^{(\alpha,\beta)}(-1, 1)),$$

so the left-definite theory can be applied.

Theorem 4.2. *Let $k > 0$. For each $n \in \mathbb{N}$, $W_{n,k}^{(\alpha,\beta)}(-1, 1)$ is a Hilbert space.*

Theorem 4.3. *Let $k > 0$. The Jacobi polynomials $\left\{P_n^{(\alpha,\beta)}(x)\right\}_{n=0}^{\infty}$ form a complete orthogonal set in the space $W_{n,k}^{(\alpha,\beta)}(-1, 1)$. Equivalently, the space $\mathcal{P}$ of polynomials is dense in $W_{n,k}^{(\alpha,\beta)}(-1, 1)$.*

We are now ready to state the main result in [17].

Theorem 4.4. *For $k > 0$, let*

$$A_k^{(\alpha,\beta)} : \mathcal{D}\left(A_k^{(\alpha,\beta)}\right) \subset L^2_{\alpha,\beta}(-1, 1) \longrightarrow L^2_{\alpha,\beta}(-1, 1)$$

be the Jacobi self-adjoint operator having the Jacobi polynomials $\left\{P_m^{(\alpha,\beta)}\right\}_{m=0}^{\infty}$ as eigenfunctions. For each $n \in \mathbb{N}$, let

$$V_n^{(\alpha,\beta)} := \left\{ f : (-1, 1) \longrightarrow \mathbb{C} \,\middle|\, f \in AC_{loc}^{(n-1)};\, f^{(j)} \in L^2_{(\alpha+j,\beta+j)}(-1, 1), j = 0, ..., n \right\}$$

and

$$(f, g)_{n,k}^{(\alpha,\beta)} := \sum_{j=0}^{n} c_j^{(\alpha,\beta)}(n, k) \int_{-1}^{1} f^{(j)}(t)\overline{g}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{\beta+j} dt \qquad (f, g \in V_n^{(\alpha,\beta)}).$$

Then $W_{n,k}^{(\alpha,\beta)}(-1, 1) := \left(V_n^{(\alpha,\beta)}, (\cdot, \cdot)_{n,k}^{(\alpha,\beta)}\right)$ is the n^{th} left-definite space associated with $\left(L^2_{\alpha,\beta}(-1, 1), A_k^{(\alpha,\beta)}\right)$. Moreover, the Jacobi polynomials $\left\{P_m^{(\alpha,\beta)}\right\}_{m=0}^{\infty}$ form a complete orthogonal set in each $W_{n,k}^{(\alpha,\beta)}(-1, 1)$, and they satisfy the orthogonality relation

$$\left(P_m^{(\alpha,\beta)}, P_l^{(\alpha,\beta)}\right)_{n,k} = (m(m-1) + k)^n \delta_{m,l}.$$

Furthermore, define

$$B_{n,k}^{(\alpha,\beta)} := \mathcal{D}\left(B_{n,k}^{(\alpha,\beta)}\right) \subset W_{n,k}^{(\alpha,\beta)}(-1, 1) \longrightarrow W_{n,k}^{(\alpha,\beta)}(-1, 1)$$

by

$$B_{n,k}^{(\alpha,\beta)} f := l\,[f] \qquad \left(f \in \mathcal{D}\left(B_{n,k}^{(\alpha,\beta)}\right) := V_{n+2}^{(\alpha,\beta)}\right).$$

Then $B_{n,k}^{(\alpha,\beta)}$ *is the* n^{th} *left-definite operator associated with* $\left(L^2_{\alpha,\beta}(-1,1), A_k^{(\alpha,\beta)}\right)$. *Lastly, the spectrum of* $B_{n,k}^{(\alpha,\beta)}$ *is given by*

$$\sigma\left(B_{n,k}^{(\alpha,\beta)}\right) = \{m(m-1) + k \mid m \in \mathbb{N}_0\} = \sigma\{A_k^{(\alpha,\beta)}\},$$

and the Jacobi polynomials $\left\{P_m^{(\alpha,\beta)}\right\}_{m=0}^{\infty}$ *form a complete set of eigenfunctions of each* $B_{n,k}^{(\alpha,\beta)}$.

CHAPTER FIVE

Spectral Analysis of the Jacobi Differential Equation ($\alpha, \beta = -1$)

The art of doing mathematics consists in finding that special case which contains all the seeds of generality.

- David Hilbert

In this chapter, the Jacobi differential equation will be considered for the non-classical parameters $\alpha = \beta = -1$. The authors in [17] study the Jacobi differential equation for classical parameters ($\alpha, \beta > -1$) and develop the left-definite spectral analysis associated with the self-adjoint Jacobi operator which has the full sequence of Jacobi polynomials as a complete set of eigenfunctions. Note that, for non-classical parameters, the full sequence of Jacobi polynomials cannot be orthogonal on $\mathbb{R}$ with respect to any bilinear form of type $(f, g)_\mu = \int_{\mathbb{R}} f\overline{g}d\mu$, for some positive or signed measure μ; this is an application of Favard's theorem (see [10]). However, it is known that the Jacobi polynomials for parameters $\alpha = \beta = -1$ are orthogonal with respect to a Sobolev inner product [35],

$$\phi\left(f, g\right) := \frac{1}{2}f(-1)\overline{g}(-1) + \frac{1}{2}f(1)\overline{g}(1) + \int_{-1}^{1} f'(x)\overline{g}'(x)dx,$$

that is to say,

$$\phi\left(P_n^{(-1,-1)}, P_m^{(-1,-1)}\right) = \delta_{nm} \qquad (n, m \in \mathbb{N}_0).$$

This observation is the starting point for this work, and a proof is included below (see theorem 5.3). It is a natural question to ask if there exists a self-adjoint operator in a certain Hilbert space which is equipped with this Sobolev inner product that has the full sequence of Jacobi polynomials as a complete set of eigenfunctions. Here, this question will be answered in the affirmative; the self-adjoint operator and its domain will be constructed at the end of this chapter. The left-definite spectral analysis will play a key role in this construction. These results will be extended to the more general case of parameters $\alpha > -1, \beta = -1$ in the next chapter. Let us begin by establishing some (right-definite) spectral results for the Jacobi differential equation for non-classical parameters.

5.1 Right-Definite Spectral Analysis

For $\alpha = \beta = -1$, the Jacobi differential expression reduces to

$$l_{-1,-1}[y](x) := (1-x^2)\left(-(y'(x))' + k(1-x^2)^{-1}y(x)\right) \tag{5.1}$$

for $x \in (-1,1)$ and where $k \geq 0$ is a constant. For brevity, let us define

$$L^2_{-1,-1}(-1,1) := L^2\left((-1,1);(1-x^2)^{-1}\right), \tag{5.2}$$

and, more generally,

$$L^2_{\alpha,\beta}(-1,1) := L^2\left((-1,1);(1-x)^\alpha(1+x)^\beta\right).$$

The maximal domain associated with $l_{-1,-1}[.]$ in $L^2_{-1,-1}(-1,1)$ is

$$\Delta := \left\{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); f, l[f] \in L^2_{-1,-1}(-1,1)\right\}.$$

For $f, g \in \Delta$ and $[a,b] \subset (-1,1)$, we have Dirichlet's formula:

$$\int_a^b l_{-1,-1}[f](x)\overline{g}(x)(1-x^2)^{-1}dx = -f'(x)\overline{g}(x)\mid_a^b$$

$$+\int_a^b \left[f'(x)\overline{g}'(x) + k(1-x^2)^{-1}f(x)\overline{g}(x)\right]dx$$

and Green's formula:

$$\int_a^b l_{-1,-1}[f](x)\overline{g}(x)(1-x^2)^{-1}dx = [f(x)\overline{g}'(x) - f'(x)\overline{g}(x)]\mid_a^b$$

$$+\int_a^b f(x)\overline{l_{-1,-1}[g]}(x)(1-x^2)^{-1}dx.$$

Dirichlet's formula for $a \longrightarrow -1, b \longrightarrow 1$ is a key result in constructing the left-definite inner products and in establishing that the (right-definite) self-adjoint operator from the GKN theory is bounded below for the left-definite theory to apply. Green's formula in turn shows that this operator is indeed Hermitian if the integrated out terms vanish as $a \longrightarrow -1, b \longrightarrow 1$. However, initially there is no reason to expect $\int_a^b f'(x)\overline{g}'(x)dx$ to be finite or the integrated out terms to vanish in the limit as $a \longrightarrow -1, b \longrightarrow 1$. Thus we shall prove the following result.

Theorem 5.1. *The Jacobi differential expression (5.1) is strong limit-point (SLP) and Dirichlet at $x = \pm 1$, i.e.*

(i) (Dirichlet) $\int_0^1 |f'(t)|^2 \, dt < \infty$ and $\int_{-1}^0 |f'(t)|^2 \, dt < \infty$ for all $f \in \Delta$ and

(ii) (SLP) $\lim_{x \to \pm 1} f'(x)\overline{g}(x) = 0$ for all $f, g \in \Delta$.

Note that strong limit-point implies that the Jacobi expression is in the limit-point condition in the sense of Weyl's second theorem.

The proof is via the following three lemmas. We begin by rewriting the maximal domain as

$$\Delta := \left\{ f : (-1,1) \longrightarrow \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); \frac{f}{\sqrt{1-x^2}}, \sqrt{1-x^2} f'' \in L^2(-1,1) \right\}$$

and by recalling a result by Chisholm and Everitt, [8], [9].

Theorem 5.2. *(Chisholm-Everitt) Let $(a,b) \subseteq \mathbb{R}$, $c \in (a,b)$, and assume that*

$$\varphi \in L^2(a,c)$$

$$\psi \in L^2(c,b).$$

Define the two linear operators

$$S, T : L^2(a,b) \longrightarrow L^2_{loc}(a,b)$$

by

$$\left(Sf\right)(x) := \varphi(x) \int_x^b \psi(x) f(x) dx$$

$$\left(Tf\right)(x) := \psi(x) \int_a^x \varphi(x) f(x) dx.$$

Then S and T are bounded operators into $L^2(a,b)$ if and only if there exists $K > 0$ such that

$$\int_a^x |\varphi(x)|^2 \, dx \cdot \int_x^b |\psi(x)|^2 \, dx \leq K \qquad \forall x \in (a,b).$$

Lemma 5.1 (Dirichlet). *$f' \in L^2(-1,1)$ $\forall f \in \Delta$. In particular, $f \in AC[-1,1]$.*

Proof. Write

$$f'(x) = f(0) + \int_0^x \frac{f''(t)\sqrt{1-t^2}}{\sqrt{1-t^2}} dt \qquad (x \in [0,1)) \tag{5.3}$$

and apply Chisholm-Everitt with $\varphi(x) = 1$ and $\psi(x) = \frac{1}{\sqrt{1-x^2}}$. Since

$$\int_0^x \psi^2(t) dt \int_0^x \varphi^2(t) dt = \frac{1}{2}(1-x)\ln(\frac{1+x}{1-x})$$

is bounded on $[0,1)$, we see that $\int_0^x \frac{f''(t)\sqrt{1-t^2}}{\sqrt{1-t^2}} dt \in L^2[0,1)$. Hence, $f' \in L^2[0,1)$. Similarly, $f' \in L^2(-1,0)$. □

Lemma 5.2. *$f(\pm 1) = 0$ for all $f \in \Delta$.*

Proof. Note that from the previous lemma, $f \in AC[-1,1]$ and thus we may define

$$f(\pm 1) := \lim_{x \to \pm 1} f(x),$$

and the limits exist and are finite. First let us consider $f(1)$ and suppose that $f(1) \neq 0$. Without loss of generality, we may assume that $f(1) > 0$. By continuity, there exists $x^* \in (0,1)$ such that

$$f(x) > \frac{f(1)}{2} \qquad \text{for } x \in [x^*, 1).$$

Then

$$\infty > \int_0^1 \frac{|f(t)|^2}{1-t^2}dt \geq \int_{x^*}^1 \frac{|f(t)|^2}{1-t^2}dt \geq \frac{(f(1))^2}{4}\int_{x^*}^1 \frac{dt}{1-t^2} = \infty,$$

a contradiction. A similar argument shows that $f(-1) = 0$. $\square$

Lemma 5.3 (Strong limit-point). $\lim\limits_{x\to 1^-} f(x)g'(x) = 0$ *for all* $f, g \in \Delta$.

Proof. Let $f, g \in \Delta$, and assume that f, g are both real-valued. Note that $\frac{f}{\sqrt{1-x^2}}$ and $\sqrt{1-x^2}g'' \in L^2(-1,1)$, which implies $fg'' \in L^1(-1,1)$. Now

$$\int_0^x f(t)g''(t)dt = f(t)g'(t)\mid_0^x - \int_0^x f'(t)g'(t)dt.$$

By our first lemma, $\lim\limits_{x\to 1^-}\int_0^x f'(t)g'(t)dt$ exists and is finite. Since $\lim\limits_{x\to 1^-}\int_0^x f(t)g''(t)dt$ exists and is finite, we see that $\lim\limits_{x\to 1^-} f(x)g'(x)$ exists and is finite. We will now show that necessarily $\lim\limits_{x\to 1^-} f(x)g'(x) = 0$ for all $f, g \in \Delta$. Suppose that $\lim\limits_{x\to 1^-} f(x)g'(x) = c > 0$; we may assume that, for x close to 1,

$$f(x) > 0 \text{ and } g'(x) > 0.$$

Hence, there exists $x^* \in [0,1)$ such that $g'(x) \geq \frac{\widetilde{c}}{f(x)}$ for $x \in [x^*, 1)$, where $\widetilde{c} = \frac{c}{2} > 0$. Therefore,

$$|f'(x)g'(x)| \geq \widetilde{c}\frac{|f'(x)|}{f(x)} \qquad (x \in [x^*, 1)).$$

Integrate to obtain

$$\int_{x^*}^x |f'(t)g'(t)|\,dt \geq \widetilde{c}\int_{x^*}^x \frac{|f'(t)|}{f(t)}dt \geq \widetilde{c}\left|\int_{x^*}^x \frac{f'(t)}{f(t)}dt\right| = \widetilde{c}\,|\ln f(x)| + k.$$

Now let $x \to 1^-$; from lemma 5.1,

$$\infty > \int_{x^*}^1 |f'(t)g'(t)|\,dt \geq k + \widetilde{c}\lim_{x\to 1^-}|\ln f(x)| = \infty,$$

a contradiction by the second lemma. It can be shown in a similar fashion that the Jacobi differential expression (for $\alpha, \beta = -1$) is strong limit-point at $x = -1$. $\square$

This completes the proof of the theorem, i.e. the Jacobi differential expression (for $\alpha, \beta = -1$) is both strong limit-point and Dirichlet at $x = \pm 1$ on Δ.
We now define the operator

$$A : D(A) \subset L^2_{-1,-1}(-1,1) \longrightarrow L^2_{-1,-1}(-1,1)$$

by

$$Af = l_{-1,-1}[f]$$
$$f \in D(A) := \Delta.$$

By the Glazman-Krein-Naimark theory, since $x = \pm 1$ are SLP (in fact, all that is necessary is that $x = \pm 1$ are LP), A is self-adjoint (and, in fact, is the same as the minimal or the maximal operator generated by $l_{-1,-1}[.]$ in $L^2_{-1,-1}(-1,1)$). By our theorem, we have Green's formula,

$$(Af, g)_{L^2_{-1,-1}(-1,1)} = \int_{-1}^{1} l_{-1,-1}[f](x)\overline{g}(x)(1-x^2)^{-1}dx = (f, Ag)_{L^2_{-1,-1}(-1,1)}$$

(i.e. A is Hermitian), and Dirichlet's formula,

$$(Af, g)_{L^2_{-1,-1}(-1,1)} = \int_{-1}^{1} \left[f'(x)\overline{g}'(x) + k(1-x^2)^{-1}f(x)\overline{g}(x)\right] dx =: (f,g)_1$$

and we will see that this is the first left-definite inner product generated from $l_{-1,-1}[\cdot]$. Note, in particular, that

$$(Af, f)_{L^2_{-1,-1}(-1,1)} = \int_{-1}^{1} \left[|f'(x)|^2 + k(1-x^2)^{-1}|f(x)|^2\right] dx$$
$$\geq k\,(f,f)_{L^2_{-1,-1}(-1,1)}$$

i.e. A is bounded below in $L^2_{-1,-1}(-1,1)$ by kI, so that the left-definite theory can be applied.

For $\alpha = \beta = -1$, we note again that the Jacobi differential equation is given by

$$-(1-x^2)y'' + ky = \lambda y \tag{5.4}$$

where we now assume that $k > 0$ and $x \in (-1, 1)$. We shall study the second-order differential equation $l_{-1,-1}[y] = \lambda y$, where $l_{-1,-1}[\cdot]$ is as in (5.1), and the initial Hilbert space setting is $L^2_{-1,-1}(-1,1)$ as defined in (5.2).

Based on the (equivalent) definitions of the Jacobi polynomials in (4.1) and (4.4), we define the Jacobi polynomials for $\alpha = \beta = -1$ as

$$P_n^{(-1,-1)}(x) := \sum_{j=0}^{n} \binom{n-1}{j} \binom{n-1}{n-j} \left(\frac{x-1}{2}\right)^j \left(\frac{x+1}{2}\right)^{n-j}.$$

We see immediately that there is degeneracy for the polynomial of degree 1 : $P_0^{(-1,-1)}(x) = 1$ and $P_1^{(-1,-1)}(x) = 0$. However, it is important to note that any first degree polynomial will be a solution of equation (5.4). Therefore, we redefine $P_1^{(-1,-1)}(x)$ and normalize the sequence of Jacobi polynomials as follows:

Definition 5.1. Define the Jacobi polynomials for $\alpha = \beta = -1$ as

$$P_0^{(-1,-1)}(x) := 1$$
$$P_1^{(-1,-1)}(x) := \frac{x}{\sqrt{3}},$$

and, for $n \geq 2$,

$$P_n^{(-1,-1)}(x) := \sqrt{\frac{2n(2n-1)}{n-1}} \sum_{j=0}^{n} \binom{n-1}{n-j} \binom{n-1}{j} \left(\frac{x-1}{2}\right)^j \left(\frac{x+1}{2}\right)^{n-j}.$$

With this definition of the Jacobi polynomials, it is the case that $\left\{P_n^{(-1,-1)}\right\}_{n=2}^{\infty}$ forms a complete *orthonormal* set in $L^2\left((-1,1);(1-x^2)^{-1}\right)$, that is to say,

$$\left(P_m^{(-1,-1)}, P_n^{(-1,-1)}\right)_{-1,-1} = \delta_{mn} \quad (m, n \geq 2),$$

see lemma 5.6.

For the remainder of this chapter, we shall write $P_n^{(-1,-1)}(x)$ to mean the n^{th} Jacobi polynomial normalized as in the definition above.

To show that the Jacobi polynomials are *orthonormal* with respect to a Sobolev inner product, we renormalize the Jacobi polynomials as follows for our next two results:

Definition 5.2. Define the Jacobi polynomials for $\alpha = \beta = -1$ as

$$\widetilde{P}_0^{(-1,-1)}(x) := 1 \tag{5.5}$$

$$\widetilde{P}_1^{(-1,-1)}(x) := \frac{x}{\sqrt{3}} \tag{5.6}$$

and, for $n \geq 2$,

$$\widetilde{P}_n^{(-1,-1)}(x) := \frac{(2n-1)^{\frac{1}{2}}}{2^{-1/2}\,(n-1)} \sum_{j=0}^{n} \binom{n-1}{n-j}\binom{n-1}{j}\left(\frac{x-1}{2}\right)^{j}\left(\frac{x+1}{2}\right)^{n-j}.$$

Lemma 5.4. *For $n \geq 2$,*

$$\widetilde{P}_n^{(-1,-1)}(x) = \kappa_n (x^2-1)\widetilde{P}_{n-2}^{(1,1)}(x) \tag{5.7}$$

where

$$\kappa_n = \frac{2^{1/2}(2n-1)^{1/2}}{4(n-1)}.$$

Proof. Note that

$$\widetilde{P}_{n-2}^{(1,1)}(x) = \left(\frac{1}{2}\right)^{n-2} \sum_{j=0}^{n-2} \binom{n-1}{j}\binom{n-1}{n-2-j}(x-1)^{n-2-j}(x+1)^{j}.$$

Now,

$$\begin{aligned}\widetilde{P}_n^{(-1,-1)}(x) &= \frac{(2n-1)^{\frac{1}{2}}}{2^{n-1/2}\,(n-1)} \sum_{j=0}^{n} \binom{n-1}{j}\binom{n-1}{n-j}(x-1)^{n-j}(x+1)^{j} \\ &= \frac{(2n-1)^{\frac{1}{2}}}{2^{n-1/2}\,(n-1)} \sum_{j=1}^{n-1} \binom{n-1}{j}\binom{n-1}{n-j}(x-1)^{n-j}(x+1)^{j},\end{aligned}$$

since $\binom{n-1}{n} = 0$. Shifting the index from j to $j+1$ yields

$$\widetilde{P}_n^{(-1,-1)}(x) = \frac{(2n-1)^{\frac{1}{2}}}{2^{n-1/2}\,(n-1)} \sum_{j=0}^{n-2} \binom{n-1}{j+1}\binom{n-1}{n-j-1}(x-1)^{n-j-1}(x+1)^{j+1}.$$

Note that $\binom{n-1}{n-j-1} = \binom{n-1}{j}$, and $\binom{n-1}{j+1} = \binom{n-1}{n-2-j}$ to obtain

$$\begin{aligned}\widetilde{P}_n^{(-1,-1)}(x) &= \frac{(2n-1)^{\frac{1}{2}}}{2^{n-1/2}\,(n-1)} \sum_{j=0}^{n-2} \binom{n-1}{j} \binom{n-1}{n-2-j} (x-1)^{n-j-1}(x+1)^{j+1} \\ &= \frac{2^{1/2}(2n-1)^{1/2}}{n-1} (x^2-1)2^{-2}\widetilde{P}_{n-2}^{(1,1)}(x),\end{aligned}$$

which agrees with (5.7) if we choose

$$\kappa_n = \frac{2^{1/2}(2n-1)^{1/2}}{4(n-1)}.$$

□

We shall use this lemma to prove that the Jacobi polynomials for $\alpha = \beta = -1$ are orthonormal with respect to a Sobolev inner product.

Theorem 5.3. *The Jacobi polynomials* $\left\{ \widetilde{P}_n^{(-1,-1)}(x) \right\}_{n=0}^{\infty}$ *as given in (5.6), are orthonormal with respect to the Sobolev inner product*

$$\phi\,(f,g) := \frac{1}{2} f(-1)\overline{g}(-1) + \frac{1}{2} f(1)\overline{g}(1) + \int_{-1}^{1} f'(x)\overline{g}'(x)dx,$$

i.e.

$$\phi\left(\widetilde{P}_n^{(-1,-1)}, \widetilde{P}_m^{(-1,-1)}\right) = \delta_{nm} \qquad (n,m \in \mathbb{N}_0).$$

Proof. A calculation shows that

$$\phi\left(\widetilde{P}_0^{(-1,-1)}, \widetilde{P}_0^{(-1,-1)}\right) = \phi\left(\widetilde{P}_1^{(-1,-1)}, \widetilde{P}_1^{(-1,-1)}\right) = 1.$$

For $n = 0, m = 1$,

$$\phi\left(\widetilde{P}_0^{(-1,-1)}, \widetilde{P}_1^{(-1,-1)}\right) = 0.$$

Let $n = 0, m \geq 2$, and use lemma 5.4 to see that

$$\phi\left(\widetilde{P}_0^{(-1,-1)}, \widetilde{P}_m^{(-1,-1)}\right) = 0$$

and

$$\phi\left(\widetilde{P}_1^{(-1,-1)}, \widetilde{P}_m^{(-1,-1)}\right) = 0.$$

For $n, m \geq 2$,

$$\phi\left(\widetilde{P}_n^{(-1,-1)}, \widetilde{P}_m^{(-1,-1)}\right) = \frac{1}{2}\widetilde{P}_n^{(-1,-1)}(-1)\overline{\widetilde{P}_m^{(-1,-1)}}(-1) + \frac{1}{2}\widetilde{P}_n^{(-1,-1)}(1)\overline{\widetilde{P}_m^{(-1,-1)}}(1)$$
$$+ \int_{-1}^{1} \left(\widetilde{P}_n^{(-1,-1)}(x)\right)' \left(\overline{\widetilde{P}_m^{(-1,-1)}}(x)\right)' dx.$$

The first two summands vanish by the previous lemma. Note that $\left(\widetilde{P}_n^{(-1,-1)}\right)'$ reduces to a Legendre polynomial (that is, $\alpha = \beta = 0$, and the n^{th} Legendre polynomial is denoted by $\widetilde{P}_n(x)$) by the following well known identity (see [10] page 149)

$$\frac{d}{dx}\widetilde{P}_n^{(-1,-1)}(x) = \frac{1}{2}(n-1)\widetilde{P}_{n-1}^{(0,0)}(x) = \frac{1}{2}(n-1)\widetilde{P}_n(x)$$

so that

$$\begin{aligned}
\phi\left(\widetilde{P}_n^{(-1,-1)}, \widetilde{P}_m^{(-1,-1)}\right) &= \frac{2^{1/2}(2n-1)^{1/2}}{n-1}\frac{2^{1/2}(2m-1)^{1/2}}{m-1} \\
&\quad \times \frac{1}{4}(n-1)(m-1)\int_{-1}^{1} \widetilde{P}_n(x)\widetilde{P}_m(x)dx \\
&= (2n-1)^{1/2}(2m-1)^{1/2}\frac{1}{2n-1}\delta_{nm} \\
&= \delta_{nm}.
\end{aligned}$$

□

From the theory of classical orthogonal polynomials, it is well known that the Jacobi polynomials for $\alpha = \beta = 1$ are dense in a weighted L^2-space:

Lemma 5.5. *The sequence* $\left\{P_n^{(1,1)}(x)\right\}_{n=0}^{\infty}$ *forms a complete orthogonal set in the Hilbert space* $L^2\left((-1,1);(1-x^2)\right)$.

We shall use this lemma to establish the following result.

Lemma 5.6. *The sequence* $\left\{P_n^{(-1,-1)}(x)\right\}_{n=2}^{\infty}$ *forms a complete orthogonal set in the Hilbert space* $L^2\left((-1,1);(1-x^2)^{-1}\right)$.

Equivalently, the set of all polynomials $\mathcal{P}_{-1}[-1,1]$ of degree ≥ 2 satisfying $p(\pm 1) = 0$ is dense in $L^2\left((-1,1);(1-x^2)^{-1}\right)$. These statements are equivalent because a complete orthogonal set is dense, and by lemma 5.4, they satisfy $p(\pm 1) = 0$. Moreover, the Jacobi polynomials $\left\{P_n^{(-1,-1)}(x)\right\}_{n=2}^{\infty}$ form a complete orthonormal set in $L^2\left((-1,1);(1-x^2)^{-1}\right)$. In fact, for each $j \in \mathbb{N}_0$, the Jacobi polynomials $\left\{P_n^{(j-1,j-1)}(x)\right\}_{n=2}^{\infty}$ form a complete orthonormal set in the Hilbert space $L^2_{j-1,j-1}(-1,1)$.

Proof. Note that

$$\int_{-1}^{1} |f(x)|^2 (1-x^2)^{-1} dx = \int_{-1}^{1} \left|(1-x^2)^{-1} f(x)\right|^2 (1-x^2) dx,$$

i.e. $f \in L^2\left((-1,1);(1-x^2)^{-1}\right) \iff (1-x^2)^{-1} f \in L^2\left((-1,1);(1-x^2)\right)$, and in this case,

$$\|f\|_{L^2((-1,1);(1-x^2)^{-1})} = \left\|(1-x^2)^{-1} f\right\|_{L^2((-1,1);(1-x^2))}.$$

Let $f \in L^2\left((-1,1);(1-x^2)^{-1}\right)$, and let $\epsilon > 0$. Hence

$$(1-x^2)^{-1} f \in L^2\left((-1,1);(1-x^2)\right),$$

so by lemma 5.5, there exists $q \in \mathcal{P}[-1,1]$ such that

$$\left\|(1-x^2)^{-1} f - q\right\|_{L^2((-1,1);(1-x^2))} < \epsilon.$$

Let $p(x) := (1-x^2)q(x)$. Then p is a polynomial of degree ≥ 2, and we may write $q(x) = (1-x^2)^{-1}p(x)$. Hence

$$\begin{aligned}
\epsilon &> \left\|(1-x^2)^{-1} f - (1-x^2)^{-1} p\right\|_{L^2((-1,1);(1-x^2))} \\
&= \left\|(1-x^2)^{-1}(f-p)\right\|_{L^2((-1,1);(1-x^2))} \\
&= \|f-p\|_{L^2((-1,1);(1-x^2)^{-1})}
\end{aligned}$$

which completes the proof of the lemma. $\square$

Definition 5.3. For each $n \in \mathbb{N}$, define

$$W_{n,k}^{(-1,-1)}(-1,1) := \left(V_n^{(-1,-1)}; (.,.)_{n,k}^{(-1,-1)}\right),$$

where

$$V_n^{(-1,-1)} := \Big\{f : (-1,1) \longrightarrow \mathbb{C} \mid f \in AC_{loc}^{(n-1)}(-1,1); \tag{5.8}$$
$$f^{(j)} \in L^2\left((-1,1); \left(1-x^2\right)^{j-1}\right), j = 0, 1, .., n\Big\}$$

and

$$(f,g)_{n,k}^{(-1,-1)} := \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k) \int_{-1}^{1} f^{(j)}(x)\overline{g}^{(j)}(x)(1-x^2)^{j-1}dx.$$

We shall show that $W_{n,k}^{(-1,-1)}(-1,1)$ is the nth left-definite space associated with the pair $\left(L^2\left((-1,1);(1-x^2)^{-1}\right); A_k^{(-1,-1)}\right)$.

Theorem 5.4. *Let $k > 0$. For each $n \in \mathbb{N}$, $W_{n,k}^{(-1,-1)}(-1,1)$ is a Hilbert space.*

Proof. Let $n \in \mathbb{N}$, and let $\{f_m\}_{m=1}^{\infty}$ be a Cauchy sequence in $W_{n,k}^{(-1,-1)}(-1,1)$. Then, since the numbers $c_j^{(-1,-1)}(n,k) \geq 0$,

$$\begin{aligned}\left(\|f_m - f_r\|_{n,k}^{(-1,-1)}\right)^2 &= \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k) \left\|f_m^{(j)} - f_r^{(j)}\right\|_{(1-x^2)^{j-1}}^2 \\ &\geq c_n^{(-1,-1)}(n,k) \left\|f_m^{(n)} - f_r^{(n)}\right\|_{(1-x^2)^{n-1}}^2 \end{aligned} \tag{5.9}$$

so $\left\{f_m^{(n)}\right\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^2\left((-1,1);(1-x^2)^{n-1}\right)$, and hence there exists a $g_{n+1} \in L^2\left((-1,1);(1-x^2)^{n-1}\right)$ such that

$$f_m^{(n)} \longrightarrow g_{n+1} \tag{5.10}$$

in $L^2\left((-1,1);(1-x^2)^{n-1}\right)$ as $m \longrightarrow \infty$. In particular, $g_{n+1} \in L_{loc}^1(-1,1)$: fix $t, t_0 \in (-1,1)$ such that $t_0 \leq t$. Then, by Hölder's inequality,

$$\int_{t_0}^{t} \left|f_m^{(n)}(u) - g_{n+1}(u)\right| du = \int_{t_0}^{t} \left|f_m^{(n)}(u) - g_{n+1}(u)\right| (1-u^2)^{(n-1)/2}(1-u^2)^{-(n-1)/2}du$$

$$\leq \left(\int_{t_0}^{t} \left| f_m^{(n)}(u) - g_{n+1}(u) \right| (1-u^2)^{n-1} du \right)^{1/2} \left(\int_{t_0}^{t} (1-u^2)^{1-n} du \right)^{1/2}$$

$$= M(t,t_0) \left(\int_{t_0}^{t} \left| f_m^{(n)}(u) - g_{n+1}(u) \right| (1-u^2)^{n-1} du \right)^{1/2} \longrightarrow 0$$

by (5.10), i.e.

$$\int_{t_0}^{t} f_m^{(n)}(u)du \longrightarrow \int_{t_0}^{t} g_{n+1}(u)du \tag{5.11}$$

as $m \longrightarrow \infty$. Now, since $f_m^{(n-1)} \in AC_{loc}(-1,1)$, we can integrate in (5.11):

$$f_m^{(n-1)}(t) - f_m^{(n-1)}(t_0) = \int_{t_0}^{t} f_m^{(n)}(u)du \longrightarrow \int_{t_0}^{t} g_{n+1}(u)du. \tag{5.12}$$

Also, from (5.9), it follows that $\left\{ f_m^{(n-1)} \right\}_{m=1}^{\infty}$ is Cauchy in $L^2\left((-1,1);(1-x^2)^{n-2}\right)$. Hence, there exists a $g_n \in L^2\left((-1,1);(1-x^2)^{n-2}\right)$ such that

$$f_m^{(n-1)} \longrightarrow g_n$$

in $L^2\left((-1,1);(1-x^2)^{n-2}\right)$.

Repeating the above argument, we see that $g_n \in L^1_{loc}(-1,1)$, and, for $t, t_1 \in (-1,1)$,

$$f_m^{(n-2)}(t) - f_m^{(n-2)}(t_1) = \int_{t_1}^{t} f_m^{(n-1)}(u)du \longrightarrow \int_{t_1}^{t} g_n(u)du. \tag{5.13}$$

By the Riesz-Fischer theorem, there exists a subsequence $\left\{ f_{m_k}^{(n-1)} \right\}_{m=1}^{\infty}$ of $\left\{ f_m^{(n-1)} \right\}_{m=1}^{\infty}$ such that

$$f_{m_k}^{(n-1)}(t) \longrightarrow g_n(t)$$

for a.e. $t \in (-1,1)$. Choose $t_0 \in (-1,1)$ in (5.12) such that $f_{m_k}^{(n-1)}(t_0) \longrightarrow g_n(t_0)$ and then pass through the subsequence in (5.12) to obtain

$$g_n(t) - g_n(t_0) = \int_{t_0}^{t} g_{n+1}(u)du$$

for a.e. $t \in (-1,1)$. This is to say that $g_n \in AC_{loc}(-1,1)$, and

$$g_n'(t) = g_{n+1}(t)$$

for a.e. $t \in (-1,1)$. Again, from (5.9), we see that $\left\{f_m^{(n-2)}\right\}_{m=1}^{\infty}$ is Cauchy in $L^2\left((-1,1);(1-x^2)^{n-3}\right)$, implying that there exists a $g_{n-1} \in L^2\left((-1,1);(1-x^2)^{n-3}\right)$ such that

$$f_m^{(n-2)} \longrightarrow g_{n-1}$$

in $L^2\left((-1,1);(1-x^2)^{n-3}\right)$. Moreover, for any $t, t_2 \in (-1,1)$,

$$f_m^{(n-3)}(t) - f_m^{(n-3)}(t_2) = \int_{t_2}^{t} f_m^{(n-2)}(u)du \longrightarrow \int_{t_2}^{t} g_{n-1}(u)du$$

and there exists a subsequence $\left\{f_{m_k}^{(n-2)}\right\}_{m=1}^{\infty}$ of $\left\{f_m^{(n-2)}\right\}_{m=1}^{\infty}$ such that

$$f_{m_k}^{(n-2)}(t) \longrightarrow g_{n-1}(t)$$

for a.e. $t \in (-1,1)$. In (5.13), choose t_1 such that $f_{m_k}^{(n-2)}(t_1) \longrightarrow g_{n-1}(t_1)$ and then pass through the subsequence in (5.13) to get

$$g_{n-1}(t) - g_{n-1}(t_1) = \int_{t_1}^{t} g_n(u)du$$

for a.e. $t \in (-1,1)$, i.e. $g_{n-1} \in AC_{loc}^{(1)}(-1,1)$, and

$$g_{n-1}''(t) = g_n'(t) = g_{n+1}(t)$$

for a.e. $t \in (-1,1)$. Continuing in this manner, we obtain $n+1$ functions $g_{n-j+1} \in L^2\left((-1,1);(1-x^2)^{n-j-1}\right)$ for $j = 0,1,...,n$ such that

(1) $f_m^{(n-j)} \longrightarrow g_{n-j+1}$ in $L^2\left((-1,1);(1-x^2)^{n-j-1}\right)$, for $j = 0,1,...,n$

(2) $g_1 \in AC_{loc}^{(n-1)}(-1,1)$, $g_2 \in AC_{loc}^{(n-2)}(-1,1)$, ..., $g_n \in AC_{loc}(-1,1)$

(3) $g_{n-j}'(t) = g_{n-j+1}'(t)$ for a.e. $t \in (-1,1)$, $j = 0,1,...,n-1$

(4) $g_1^{(j)} = g_{j+1}$, $j = 0, 1, ..., n$.

In particular,

$$f_m^{(j)} \longrightarrow g_1^{(j)}$$

in $L^2\left((-1,1);(1-x^2)^{j-1}\right)$ for $j = 0, 1, ..., n$ and $g_1 \in V_n^{(-1,-1)}$. Hence,

$$\begin{aligned}\left(\|f_m - g_1\|_{n,k}^{(-1,-1)}\right)^2 &= \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k) \int_{-1}^{1} \left|f_m^{(j)}(u) - g_1^{(j)}(u)\right|^2 (1-u^2)^{j-1} du \\ &= \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k) \left\|f_m^{(j)} - g_1^{(j)}\right\|^2_{(1-x^2)^{j-1}} \longrightarrow 0\end{aligned}$$

as $m \longrightarrow \infty$, i.e. $W_{n,k}^{(-1,-1)}(-1,1)$ is complete. □

Definition 5.4. $W_1 := \{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC\,[-1,1]\,; f' \in L^2(-1,1)\}$

Lemma 5.7. $V_1^{(-1,-1)} \subseteq W_{1,1} := \{f \in W_1 \mid f(\pm 1) = 0\}$.

Proof. Let $f \in V_1^{(-1,-1)}$. In particular, $f \in AC_{loc}(-1,1)$ and $f' \in L^2(-1,1)$, so $f' \in L^1(-1,1)$. For $0 \leq x < 1$,

$$\int_0^x f'(t)dt = f(x) - f(0)$$

and

$$\int_0^x f'(t)dt \longrightarrow \int_0^1 f'(t)dt$$

which implies that $\lim\limits_{x \to 1^-} f(x)$ exists and is finite. Similarly, $\lim\limits_{x \to -1^+} f(x)$ exists and is finite. Define

$$f(\pm 1) := \lim_{x \to \pm 1^{\mp}} f(x),$$

so $f \in AC\,[-1,1]$. It suffices to show that $f(\pm 1) = 0$. Suppose that $f(1) \neq 0$. Hence, for some $c > 0$, there exists $0 < \delta < 1$ such that

$$|f(x)| \geq c > 0$$

for all $x \in [\delta, 1]$. Since $f \in L^2\left((-1,1);(1-x^2)^{-1}\right)$, we see that

$$\begin{aligned}\infty &> \int_0^1 |f(x)|^2 (1-x^2)^{-1}dx \\ &\geq \int_0^\delta |f(x)|^2 (1-x^2)^{-1}dx \geq c^2 \int_0^\delta (1-x^2)^{-1}dx = \infty,\end{aligned}$$

a contradiction. Hence, $f(1) = 0$, and, similarly, $f(-1) = 0$, so $f \in W_{1,1}$. $\square$

Theorem 5.5. *The Jacobi polynomials* $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ *form a complete orthogonal set in the first left-definite space* $W_{1,k}^{(-1,-1)}(-1,1)$.

Proof. Let $f \in W_{1,k}^{(-1,-1)}(-1,1)$, so $f' \in L^2(-1,1)$. Since the Legendre polynomials $\{P_m\}_{m=0}^{\infty}$ are complete and orthonormal in $L^2(-1,1)$, we know

$$\sum_{m=0}^{r} c_{m,1}^{(-1,-1)} P_m \to f' \quad \text{as } r \to \infty \text{ in } L^2(-1,1)$$

where $c_{m,1}^{(-1,-1)}$ are the Fourier coefficients given by

$$c_{m,1}^{(-1,-1)} := \int_{-1}^{1} f'(t) P_m(t) dt.$$

Note that $c_{0,1}^{(-1,-1)} = 0$ by lemma 5.7. For $r \geq 1$ define

$$p_r(t) := \sum_{m=2}^{r} \frac{c_{m-1,1}^{(-1,-1)}}{(m(m-1))^{1/2}} P_m^{(-1,-1)}(t).$$

Then

$$p_r'(t) = \sum_{m=2}^{r} \frac{c_{m-1,1}^{(-1,-1)}}{(m(m-1))^{1/2}} (m(m-1))^{1/2} P_{m-1}(t) = \sum_{m=2}^{r} c_{m-1,1}^{(-1,-1)} P_{m-1}(t),$$

since

$$\frac{d}{dt} P_m^{(-1,-1)}(t) = (m(m-1))^{1/2} P_{m-1}(t).$$

Shifting the index of summation from m to $m-1$ yields

$$p_r'(t) = \sum_{m=1}^{r-1} c_{m,1}^{(-1,-1)} P_m \to f' \quad \text{as } r \to \infty \text{ in } L^2(-1,1).$$

Furthermore, by Riesz-Fischer, there exists a subsequence $\left\{p'_{r_j}\right\}$ of $\{p'_r\}$ such that

$$p'_{r_j} \to f' \qquad \text{for a.e. } t \in (-1,1). \tag{5.14}$$

Since, by Dirichlet's test (see [4]), the sequence

$$\left\{ \frac{c_{m-1,1}^{(-1,-1)}}{(m(m-1))^{1/2}} \right\}_m \in \ell^2$$

and $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ is complete in $L^2\left((-1,1);(1-x^2)^{-1}\right)$, we see that there exists a $g \in L^2\left((-1,1);(1-x^2)^{-1}\right)$ such that

$$p_r \to g \quad \text{ in } L^2\left((-1,1);(1-x^2)^{-1}\right) \text{ as } r \longrightarrow \infty. \tag{5.15}$$

From (5.14), we see that, for $a, t \in (-1,1)$,

$$\int_a^t p'_{r_j}(u)du \longrightarrow \int_a^t f'(u)du.$$

Now integrate both sides to obtain

$$p_{r_j}(t) = f(t) + c \qquad \text{for a.e. } t \in (-1,1),$$

implying that

$$g(t) = f(t) + c \qquad \text{for a.e. } t \in (-1,1)$$

by (5.15). Define $\pi_r(t) := p_r(t) - c$. Then

$$\begin{aligned} \|f - \pi_r\|^2_{W_{1,k}^{(-1,-1)}} &= \int_{-1}^{1} \left\{ |f'(t) - \pi'_r(t)|^2 + k(1-t^2)^{-1} |f(t) - \pi_r(t)|^2 \right\} dt \\ &= \int_{-1}^{1} \left\{ |f'(t) - p'_r(t)|^2 + k(1-t^2)^{-1} |f(t) - p_r(t) + c|^2 \right\} dt \\ &\longrightarrow 0 \end{aligned}$$

as $r \to \infty$, i.e. the Jacobi polynomials $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ are complete in the first left-definite space $W_{1,k}^{(-1,-1)}(-1,1)$. $\square$

In the next theorem, we generalize this result and prove that in fact the Jacobi polynomials $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ form a complete orthogonal set in *each* left-definite space $W_{n,k}^{(-1,-1)}(-1,1)$, $n \in \mathbb{N}$.

Theorem 5.6. *The Jacobi polynomials $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ form a complete orthogonal set in each left-definite space $W_{n,k}^{(-1,-1)}(-1,1)$, $n \in \mathbb{N}$.*

Proof. Fix $n \in \mathbb{N}$, and let $f \in W_{n,k}^{(-1,-1)}(-1,1)$, so $f^{(n)} \in L^2\left((-1,1);(1-x^2)^{n-1}\right)$. Since $\left\{P_m^{(n-1,n-1)}\right\}_{m=0}^{\infty}$ is complete and orthonormal in $L^2\left((-1,1);(1-x^2)^{n-1}\right)$, we know

$$\sum_{m=0}^{r} c_{m,n}^{(-1,-1)} P_m^{(n-1,n-1)} \to f^{(n)} \quad \text{as } r \to \infty \text{ in } L^2\left((-1,1);(1-x^2)^{n-1}\right) \tag{5.16}$$

where $c_{m,n}^{(-1,-1)}$ are the Fourier coefficients given by

$$c_{m,n}^{(-1,-1)} = \int_{-1}^{1} f^{(n)}(t) P_m^{(n-1,n-1)}(t)(1-t^2)^{n-1} dt.$$

For $r \geq n$ define

$$p_r(t) := \sum_{m=\max\{2,n\}}^{r} \frac{c_{m-n,n}^{(-1,-1)} \left((m-n)!\right)^{1/2} \left((m-2)!\right)^{1/2}}{(m!)^{1/2} \left((m+n-2)!\right)^{1/2}} P_m^{(-1,-1)}(t).$$

From the differentiation formula (4.2),

$$\frac{d^j}{dt^j} P_m^{(-1,-1)}(t) = \frac{(m!)^{1/2} \left((m+j-2)!\right)^{1/2}}{\left((m-j)!\right)^{1/2} \left((m-2)!\right)^{1/2}} P_{m-j}^{(j-1,j-1)}(t),$$

we see that, for $j = 0, 1, ..., n$,

$$p_r^{(j)}(t) = \sum_{m=\max\{2,n\}}^{r} \frac{c_{m-n,n}^{(-1,-1)} \left((m-n)!\right)^{1/2} \left((m+j-2)!\right)^{1/2}}{\left((m+n-2)!\right)^{1/2} \left((m-j)!\right)^{1/2}} P_{m-j}^{(j-1,j-1)}(t).$$

In particular, by (5.16),

$$\begin{aligned} p_r^{(n)}(t) &= \sum_{m=\max\{2,n\}}^{r} c_{m-n,n}^{(-1,-1)} P_{m-n}^{(n-1,n-1)} \\ &= \sum_{l=0}^{r-\max\{2,n\}} c_{l,n}^{(-1,-1)} P_l^{(n-1,n-1)} \\ &= \sum_{m=0}^{s} c_{m,n}^{(-1,-1)} P_m^{(n-1,n-1)} \to f^{(n)} \quad \text{as } s \to \infty \text{ in } L^2\left((-1,1);(1-x^2)^{n-1}\right). \end{aligned}$$

Furthermore, by Riesz-Fischer, there exists a subsequence $\left\{p_{r_j}^{(n)}\right\}$ of $\left\{p_r^{(n)}\right\}$ such that

$$p_{r_j}^{(n)} \to f^{(n)} \qquad \text{for a.e. } t \in (-1,1).$$

From Dirichlet's test, the sequence

$$\left\{ \frac{c_{m-n,n}^{(-1,-1)} \left((m-n)!\right)^{1/2} \left((m+j-2)!\right)^{1/2}}{\left((m+n-2)!\right)^{1/2} \left((m-j)!\right)^{1/2}} \right\} \in \ell^2,$$

so there exists a $g_j \in L^2\left((-1,1);(1-x^2)^{j-1}\right)$ such that

$$p_r^{(j)} \longrightarrow g_j \qquad \text{in } L^2\left((-1,1);(1-x^2)^{j-1}\right). \tag{5.17}$$

For a.e. $a, t \in (-1,1)$,

$$\int_a^t p_{r_j}^{(n)}(u)du \longrightarrow \int_a^t f^{(n)}(u)du.$$

Integrate both sides and obtain

$$p_{r_j}^{(n-1)}(t) \longrightarrow f^{(n-1)}(t) + c_1 \qquad \text{for a.e. } t \in (-1,1) \tag{5.18}$$

for some constant c_1. Passing through the subsequence implies

$$g_{n-1}(t) = f^{(n-1)}(t) + c_1 \qquad \text{for a.e. } t \in (-1,1).$$

From (5.18), we see that

$$\int_a^t p_{r_j}^{(n-1)}(u)du \longrightarrow \int_a^t f^{(n-1)}(u)du + c_1 \int_a^t du,$$

i.e.

$$p_{r_j}^{(n-2)}(t) \longrightarrow f^{(n-2)}(t) + c_1 t + c_2 \qquad \text{for a.e. } t \in (-1,1)$$

or

$$g_{n-2}(t) = f^{(n-2)}(t) + c_1 t + c_2 \qquad \text{for a.e. } t \in (-1,1).$$

Continue this process to see that for $j \in \{0, 1, ..., n-1\}$,

$$g_j(t) = f^{(j)}(t) + q_{n-j+1} \qquad \text{for a.e. } t \in (-1,1),$$

where q_{n-j-1} is a polynomial of degree $\leq n-j-1$ and

$$q'_{n-j-1} = q_{n-j-2}.$$

Hence, from (5.17),

$$p_r^{(j)} \longrightarrow f^{(j)} + q_{n-j-1} \qquad \text{in } L^2\left((-1,1);(1-x^2)^{j-1}\right). \tag{5.19}$$

For $r \geq n$, define

$$\pi_r(t) := p_r(t) - q_{n-1}(t).$$

Note that, from (5.19),

$$\pi_r^{(j)}(t) = p_r^{(j)}(t) - q_{n-1}^{(j)}(t) = p_r^{(j)}(t) - q_{n-j-1}(t) \longrightarrow f^{(j)}(t)$$

in $L^2\left((-1,1);(1-x^2)^{j-1}\right)$. Now,

$$\left(\|f-\pi_r\|_{n,k}^{(-1,-1)}\right)^2 = \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k) \int_{-1}^{-1} \left|f^{(j)}(t) - \pi_r^{(j)}(t)\right|^2 (1-t^2)^{j-1} dt \longrightarrow 0$$

as $r \longrightarrow \infty$. $\square$

The following lemma should be for $n \geq 2$!!

Lemma 5.8. *For $p, q \in \mathcal{P}$,*

$$(p,q)_{n,k}^{(-1,-1)} = \left(\left(A_k^{(-1,-1)}\right)^n p, q\right)_{-1,-1}.$$

Proof. First we note that this may be restated as

$$\begin{aligned}
\left(l_{-1,-1}^n[p], q\right)_{-1,-1} &= \int_{-1}^{1} l_{-1,-1}^n[p](x)\overline{q}(x) w_{-1,-1}(x) dx \\
&= \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k) p^{(j)}(x) \overline{q}^{(j)}(x) (1-x)^{j-1}(1+x)^{j-1} dx.
\end{aligned} \tag{5.20}$$

Since the Jacobi polynomials form a basis for $\mathcal{P}$, it suffices to prove (5.20) for $p = P_m^{(-1,-1)}$ and $q = P_r^{(-1,-1)}$ for arbitrary $m, r \in \mathbb{N}_0$. From

$$l_{-1,-1}^n[P_m^{(-1,-1)}](x) = (m(m-1)+k)^n P_m^{(-1,-1)}(x) \quad (m \in \mathbb{N}_0)$$

and

$$\left(P_r^{(-1,-1)}, P_m^{(-1,-1)}\right)_{-1,-1} = \delta_{r,m} \quad (r, m \in \mathbb{N}_0),$$

the left-hand side of (5.20) becomes

$$\begin{aligned}\left(l_{-1,-1}^n[P_m^{(-1,-1)}], P_r^{(-1,-1)}\right)_{-1,-1} &= \int_{-1}^{1} l_{-1,-1}^n[P_m^{(-1,-1)}](x)\overline{P_r^{(-1,-1)}}(x)w_{-1,-1}(x)dx \\ &= (m(m-1)+k)^n \delta_{r,m}. \qquad (5.21)\end{aligned}$$

Upon using (4.3) for $\alpha = \beta = -1$ and the recurrence relation for the $c_j^{(-1,-1)}(n,k)$, that is,

$$(m(m-1)+k)^n = \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k)\frac{m!(m+j-2)!}{(m-j)!(m-2)!}$$

the right-hand side of (5.20) becomes

$$\begin{aligned}&\sum_{j=0}^{n} c_j^{(-1,-1)}(n,k)\left(P_m^{(-1,-1)}(x)\right)^{(j)}(x)\left(\overline{P_r^{(-1,-1)}}(x)\right)^{(j)}(x)(1-x)^{j-1}(1+x)^{j-1}dx \\ &= \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k)\frac{m!(m+j-2)!}{(m-j)!(m-2)!}\delta_{r,m} \qquad (5.22) \\ &= (m(m-1)+k)^n\delta_{r,m}.\end{aligned}$$

Comparing (5.21) and (5.22) completes the proof of the lemma. □

Theorem 5.7. *For $k > 0$, let*

$$A_k^{(-1,-1)} : \mathcal{D}\left(A_k^{(-1,-1)}\right) \subset L^2\left((-1,1);(1-x^2)^{-1}\right) \longrightarrow L^2\left((-1,1);(1-x^2)^{-1}\right)$$

be the Jacobi self-adjoint operator having the Jacobi polynomials $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ as eigenfunctions as discussed in section 5.1. For each $n \in \mathbb{N}$, let

$$\begin{aligned}V_n^{(-1,-1)} := \Big\{f : (-1,1) \longrightarrow \mathbb{C} \mid f \in AC_{loc}^{(n-1)}(-1,1); \\ f^{(j)} \in L^2\left((-1,1);(1-x^2)^{j-1}\right), j = 0,1,..,n\Big\}\end{aligned}$$

and

$$(f,g)_{n,k}^{(-1,-1)} := \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k)\int_{-1}^{1} f^{(j)}(x)\overline{g}^{(j)}(x)(1-x^2)^{j-1}dx.$$

Then $W_{n,k}^{(-1,-1)}(-1,1) := \left(V_n^{(-1,-1)}, (\cdot,\cdot)_{n,k}^{(-1,-1)}\right)$ *is the* n^{th} *left-definite space associated with* $\left(L^2\left((-1,1);(1-x^2)^{-1}\right), A_k^{(-1,-1)}\right)$*. Moreover, the Jacobi polynomials* $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ *form a complete orthogonal set in each* $W_{n,k}^{(-1,-1)}(-1,1)$*, and they satisfy the orthogonality relation*

$$\left(P_m^{(-1,-1)}, P_l^{(-1,-1)}\right)_{n,k} = (m(m-1)+k)^n \delta_{m,l}.$$

Furthermore, define

$$B_{n,k}^{(-1,-1)} := \mathcal{D}\left(B_n^{(-1,-1)}\right) \subset W_{n,k}^{(-1,-1)}(-1,1) \longrightarrow W_{n,k}^{(-1,-1)}(-1,1)$$

by

$$B_{n,k}^{(-1,-1)} f := l_{-1,-1}[f] \qquad \left(f \in \mathcal{D}\left(B_{n,k}^{(-1,-1)}\right) := V_{n+2}^{(-1,-1)}\right).$$

Then $B_{n,k}^{(-1,-1)}$ *is the* n^{th} *left-definite operator associated with the pair* $\left(L^2_{-1,-1}(-1,1), A_k^{(-1,-1)}\right)$*. Lastly, the spectrum of* $B_{n,k}^{(-1,-1)}$ *is given by*

$$\sigma\left(B_{n,k}^{(-1,-1)}\right) = \{m(m-1)+k \mid m \in \mathbb{N}_0\} = \sigma\{A_k^{(-1,-1)}\},$$

and the Jacobi polynomials $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ *form a complete set of eigenfunctions of each* $B_{n,k}^{(-1,-1)}$*.*

Proof. Let $n \in \mathbb{N}$. We need to show that $W_{n,k}^{(-1,-1)}(-1,1)$ satisfies the five properties given in definition 3.1.

(i) $W_{n,k}^{(-1,-1)}(-1,1)$ is a Hilbert space (see theorem 5.4).

(ii) We need to show: $\mathcal{D}\left(\left(A_k^{(-1,-1)}\right)^n\right) \subset W_{n,k}^{(-1,-1)}(-1,1)$.

Let $f \in \mathcal{D}\left(\left(A_k^{(-1,-1)}\right)^n\right)$. Since the Jacobi polynomials $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ form a complete orthonormal set in $L^2\left((-1,1);(1-x^2)^{-1}\right)$, we see that

$$p_j \longrightarrow f \qquad \text{in } L^2\left((-1,1);\left(1-x^2\right)^{-1}\right) \text{ as } j \longrightarrow \infty \tag{5.23}$$

where

$$p_j(t) := \sum_{m=0}^{j} c_m^{(-1,-1)} P_m^{(-1,-1)}(t) \qquad (t \in (-1,1)),$$

and

$$c_m^{(-1,-1)} := \left(f, P_m^{(-1,-1)}\right)_{-1,-1} = \int_{-1}^{1} f(t) P_m^{(-1,-1)}(t) \left(1-t^2\right)^{-1} dt \qquad (m \in \mathbb{N}_0).$$

Since $\left(A_k^{(-1,-1)}\right)^n f \in L^2\left((-1,1);(1-x^2)^{-1}\right)$, we see that

$$\sum_{m=0}^{j} \widetilde{c}_m^{(-1,-1)} P_m^{(-1,-1)} \longrightarrow \left(A_k^{(-1,-1)}\right)^n f \qquad \text{in } L^2\left((-1,1);(1-x^2)^{-1}\right)$$

as $j \longrightarrow \infty$, where

$$\begin{aligned}
\widetilde{c}_m^{(-1,-1)} &:= \left(\left(A_k^{(-1,-1)}\right)^n f, P_m^{(-1,-1)}\right)_{-1,-1} = \left(f, \left(A_k^{(-1,-1)}\right)^n P_m^{(-1,-1)}\right)_{-1,-1} \\
&= (m(m-1)+k)^n \left(f, P_m^{(-1,-1)}\right)_{-1,-1} \\
&= (m(m-1)+k)^n c_m^{(-1,-1)},
\end{aligned}$$

i.e.

$$\left(A_k^{(-1,-1)}\right)^n p_j \longrightarrow \left(A_k^{(-1,-1)}\right)^n f \qquad \text{in } L^2\left((-1,1);\left(1-x^2\right)^{-1}\right) \text{ as } j \longrightarrow \infty.$$

Moreover, by lemma 5.8,

$$\begin{aligned}
\left(\|p_j - p_r\|_{n,k}^{(-1,-1)}\right)^2 &= \left(\left(A_k^{(-1,-1)}\right)^n [p_j - p_r], p_j - p_r\right)_{-1,-1} \\
&\longrightarrow 0 \qquad \text{as } j, r \longrightarrow \infty
\end{aligned}$$

i.e. $\{p_j\}_{j=0}^{\infty}$ is Cauchy in $W_{n,k}^{(-1,-1)}(-1,1)$. Since $W_{n,k}^{(-1,-1)}(-1,1)$ is a Hilbert space (by theorem 5.4), there exists a

$$g \in W_{n,k}^{(-1,-1)}(-1,1) \subset L^2\left((-1,1);\left(1-x^2\right)^{-1}\right)$$

such that

$$p_j \longrightarrow g \qquad \text{in } W_{n,k}^{(-1,-1)}(-1,1) \text{ as } j \longrightarrow \infty.$$

Furthermore, since

$$(f,f)_{n,k}^{(-1,-1)} \geq k^n (f,f)_{-1,-1} \quad \left(f \in W_{n,k}^{(-1,-1)}(-1,1)\right),$$

[this is due to

$$\begin{aligned}(f,f)_{n,k}^{(-1,-1)} &= \sum_{j=0}^{n} c_j^{(-1,-1)}(n,k) \left\| f^{(j)} \right\|_{j-1,j-1}^2 \\ &\geq c_0^{(-1,-1)}(n,k) \left\| f^{(j)} \right\|_{-1,-1}^2 \\ &= k^n (f,f)_{-1,-1} \quad \left(f \in W_{n,k}^{(-1,-1)}(-1,1)\right)\end{aligned}$$

from the positivity of the coefficients $c_j^{(-1,-1)}(n,k)$], we see that

$$\|p_j - g\|_{-1,-1} \leq k^{-n/2} \|p_j - g\|_{n,k}^{(-1,-1)},$$

and hence,

$$p_j \longrightarrow g \qquad \text{in } L^2\left((-1,1);(1-x^2)^{-1}\right). \tag{5.24}$$

Comparing (5.23) and (5.24),

$$f = g \in W_{n,k}^{(-1,-1)}(-1,1).$$

(iii) We need to show: $\mathcal{D}\left(\left(A_k^{(-1,-1)}\right)^n\right)$ is dense in $W_{n,k}^{(-1,-1)}(-1,1)$. Since the set of polynomials is contained in $\mathcal{D}\left(\left(A_k^{(-1,-1)}\right)^n\right)$ and is dense in the n^{th} left-definite space $W_{n,k}^{(-1,-1)}(-1,1)$ (by theorem 5.6), $\mathcal{D}\left(\left(A_k^{(-1,-1)}\right)^n\right)$ is dense in $W_{n,k}^{(-1,-1)}(-1,1)$. Furthermore, from theorem 5.6, the Jacobi polynomials $\left\{P_m^{(-1,-1)}\right\}_{m=2}^{\infty}$ form a complete orthonormal set in $W_{n,k}^{(-1,-1)}(-1,1)$.

(iv) We need to show: $(f,f)_{n,k}^{(-1,-1)} \geq k^n (f,f)_{-1,-1}$ for all $f \in V_n^{(-1,-1)}$. This follows immediately by the definition of $(\cdot,\cdot)_{n,k}^{(-1,-1)}$

(v) We show: $(f,g)_{n,k}^{(-1,-1)} = \left(\left(A_k^{(-1,-1)}\right)^n f, g\right)_{-1,-1}$ for $f \in \mathcal{D}\left(\left(A_k^{(-1,-1)}\right)^n\right)$ and $g \in V_n^{(-1,-1)}$. This is true for any $f,g \in \mathcal{P}$ by lemma 5.8.

Let $f \in \mathcal{D}\left(\left(A_k^{(-1,-1)}\right)^n\right) \subset W_{n,k}^{(-1,-1)}(-1,1)$, $g \in W_{n,k}^{(-1,-1)}(-1,1)$.

Since the set of polynomials is dense in both $W_{n,k}^{(-1,-1)}(-1,1)$ and in the space $L^2\left((-1,1);(1-x^2)^{-1}\right)$, and since (by (iv)), convergence in $W_{n,k}^{(-1,-1)}(-1,1)$ implies convergence in $L^2\left((-1,1);(1-x^2)^{-1}\right)$, there exist sequences $\{p_j\}_{j=0}^{\infty}$ and $\{q_j\}_{j=0}^{\infty}$ such that

$$p_j \longrightarrow f \qquad \text{in } W_{n,k}^{(-1,-1)}(-1,1) \text{ as } j \longrightarrow \infty$$

$$\left(A_k^{(-1,-1)}\right)^n p_j \longrightarrow \left(A_k^{(-1,-1)}\right)^n f \quad \text{in } L^2\left((-1,1);\left(1-x^2\right)^{-1}\right) \text{ as } j \longrightarrow \infty$$

and

$$q_j \longrightarrow g \qquad \text{in } W_{n,k}^{(-1,-1)}(-1,1) \text{ and } L^2\left((-1,1);\left(1-x^2\right)^{-1}\right) \text{ as } j \longrightarrow \infty.$$

Hence, from lemma 5.8,

$$\begin{aligned}
\left(\left(A_k^{(-1,-1)}\right)^n f, g\right)_{-1,-1} &= \lim_{j\longrightarrow\infty} \left(\left(A_k^{(-1,-1)}\right)^n p_j, q_j\right)_{-1,-1} \\
&= \lim_{j\longrightarrow\infty} \left(p_j, q_j\right)_{n,k} \\
&= (f,f)_{n,k}^{(-1,-1)}.
\end{aligned}$$

The results listed in the theorem on $B_{n,k}^{(-1,-1)}$ and the spectrum of $B_{n,k}^{(-1,-1)}$ follow immediately from the general left-definite theory.

□

5.4 Self-Adjoint Operators

Definition 5.5. Define

$$W_1 := \left\{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC\,[-1,1]\,;\, f' \in L^2(-1,1)\right\}$$

$$\phi\,(f,g) := \frac{1}{2}f(-1)\overline{g}(-1) + \frac{1}{2}f(1)\overline{g}(1) + \int_{-1}^{1} f'(x)\overline{g}'(x)dx \quad (f,g \in W_1)$$

and

$$\|f\|_{\phi} := \phi(f,f)^{1/2} \quad (f \in W_1).$$

Theorem 5.8. $(W_1, \phi(\cdot,\cdot))$ *is a Hilbert space.*

Proof. Let $\{f_n\} \subset W_1$ be a Cauchy sequence. Hence

$$\|f_n - f_m\|_\phi^2 = \frac{1}{2}\left|f_n(-1) - f_m(-1)\right|^2 + \frac{1}{2}\left|f_n(1) - f_m(1)\right|^2 + \int_{-1}^{1} \left|f_n'(x) - f_m'(x)\right|^2 dx$$

$$\longrightarrow 0 \qquad \text{as } n, m \longrightarrow \infty.$$

In particular, since

$$\int_{-1}^{1} \left|f_n'(x) - f_m'(x)\right|^2 dx \leq \|f_n - f_m\|_\phi^2 ,$$

we see that $\{f_n'\}$ is Cauchy in $L^2(-1,1)$. Since $L^2(-1,1)$ is complete, there exists $g \in L^2(-1,1)$ such that

$$f_n' \longrightarrow g \qquad \text{as } n \longrightarrow \infty \ \text{ in } L^2(-1,1). \tag{5.25}$$

Also, since

$$\frac{1}{2}\left|f_n(-1) - f_m(-1)\right|^2 \leq \|f_n - f_m\|_\phi^2 \quad \text{and}$$
$$\frac{1}{2}\left|f_n(1) - f_m(1)\right|^2 \leq \|f_n - f_m\|_\phi^2 ,$$

we see that the sequences $\{f_n(\pm 1)\}$ are both Cauchy in $\mathbb{C}$ and, hence, there exists $A_{\pm 1} \in \mathbb{C}$ such that

$$f_n(1) \longrightarrow A_1 \tag{5.26}$$

$$f_n(-1) \longrightarrow A_{-1} \tag{5.27}$$

Furthermore, since $f_n \in AC\,[-1,1]$ $(n \in \mathbb{N})$, we see that

$$\int_{-1}^{1} g(t)dt \longleftarrow \int_{-1}^{1} f_n'(t)dt = f_n(1) - f_n(-1) \longrightarrow A_1 - A_{-1},$$

i.e.

$$A_1 = A_{-1} + \int_{-1}^{1} g(t)dt. \tag{5.28}$$

Define $f : [-1, 1] \longrightarrow \mathbb{C}$ by

$$f(x) = A_{-1} + \int_{-1}^{x} g(t)dt.$$

It is clear that $f \in AC[-1, 1]$ and $f'(x) = g(x) \in L^2(-1, 1)$ for a.e. $x \in [-1, 1]$, so $f \in W_1$. Furthermore, $f(-1) = A_{-1}$ and $f(1) = A_{-1} + \int_{-1}^{1} g(t)dt = A_1$ by (5.28). Now

$$\begin{aligned}
\|f_n - f\|_\phi^2 &= \frac{1}{2}|f_n(-1) - f(-1)|^2 + \frac{1}{2}|f_n(1) - f(1)|^2 + \int_{-1}^{1} |f_n'(t) - f'(t)|^2 dt \\
&= \frac{1}{2}|f_n(-1) - A_{-1}|^2 + \frac{1}{2}|f_n(1) - A_1|^2 + \int_{-1}^{1} |f_n'(t) - g(t)|^2 dt \\
&\longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$ by (5.25), (5.26) and (5.27). Thus, $(W_1, \phi(\cdot, \cdot))$ is complete. $\square$

Theorem 5.9. *Let W_1 and $\phi(\cdot, \cdot)$ be as before, and*

$$W_{1,1} := \{f \in W_1 \mid f(\pm 1) = 0\}$$
$$W_{1,2} := \{f \in W_1 \mid f''(x) = 0\}.$$

Then $W_{1,1}$ and $W_{1,2}$ are closed, orthogonal subspaces of $(W_1, \phi(\cdot, \cdot))$ and

$$W_1 = W_{1,1} \oplus W_{1,2}.$$

Proof. Since $W_{1,2}$ is 2-dimensional, it is a closed subspace of W_1. The orthogonal complement of $W_{1,2}$ is given by

$$W_{1,2}^{\perp} := \{f \in W_1 \mid (f, g)_1 = 0 \quad (g \in W_{1,2})\}.$$

To see that $W_{1,1} \subset W_{1,2}^{\perp}$, let $f \in W_{1,1}$, $g \in W_{1,2}$ and consider

$$\phi(f, g) = \frac{1}{2}f(-1)\overline{g}(-1) + \frac{1}{2}f(1)\overline{g}(1) + \int_{-1}^{1} f'(x)\overline{g}'(x)dx.$$

The first two summands vanish because $f \in W_{1,1}$, and $\overline{g}'(x) = c$ for some constant c since $g \in W_{1,2}$, and we see that

$$\begin{aligned}
\phi(f,g) &= \int\limits_{-1}^{1} f'(x)\overline{g}'(x)dx = c\int\limits_{-1}^{1} f'(x)dx \\
&= c\left(f(1) - f(-1)\right) \\
&= 0,
\end{aligned}$$

so $f \in W_{1,2}^{\perp}$.

Now let $f \in W_1$. We need to find $f_1 \in W_{1,1}$ and $f_2 \in W_{1,2}$ such that $f = f_1 + f_2$. To this end, let

$$f_2(x) := Ax + B,$$

A, B to be determined. Clearly, $f_2 \in W_{1,2}$. Let

$$f_1(x) := f(x) - f_2(x).$$

We show that $f_1 \in W_{1,1}$ by appropriate choice of A, B. For any choice of A, B, $f_1 \in W_1$. Now set

$$\begin{aligned}
f_1(1) &= f(1) - A - B \stackrel{!}{=} 0 \\
f_1(-1) &= f(-1) + A - B \stackrel{!}{=} 0
\end{aligned}$$

and add the two equations to find

$$\begin{aligned}
A &= \frac{f(1) - f(-1)}{2} \\
B &= \frac{f(1) + f(-1)}{2},
\end{aligned}$$

i.e. with the choice of

$$\begin{aligned}
f_1(x) &:= f(x) - f_2(x) \\
f_2(x) &:= \frac{f(1) - f(-1)}{2}x + \frac{f(1) + f(-1)}{2}
\end{aligned}$$

every $f \in W_1$ can indeed be written as $f = f_1 + f_2$, where $f_1 \in W_{1,1}$ and $f_2 \in W_{1,2}$. $\square$

The next result shows that the space $W_{1,1}$ is precisely the first left-definite space.

Theorem 5.10. *$W_{1,1} = V_1^{(-1,-1)}$, where $V_1^{(-1,-1)}$ is defined as in (5.8).*

Proof. (1) $V_1^{(-1,-1)} \subseteq W_{1,1}$ was proved in lemma 5.7.

(2) $W_{1,1} \subseteq V_1^{(-1,-1)}$:

Let $f \in W_{1,1}$. It suffices to show that $f \in L^2\left((-1,1);(1-x^2)^{-1}\right)$. For $-1 < x < 0$,

$$(1-x^2)^{-1/2} \int_{-1}^{x} f'(t)dt = (1-x^2)^{-1/2} f(x)$$

since $f(-1) = 0$. We use Chisholm-Everitt on $(-1,0)$ with

$$\psi(x) = (1-x^2)^{-1/2}$$

$$\varphi(x) = 1.$$

Clearly, ψ is L^2 near 0, and φ is L^2 near -1. In this case,

$$\int_{-1}^{x} dt \int_{x}^{0} \frac{dt}{1-t^2} \leq \int_{-1}^{x} dt \int_{x}^{0} \frac{dt}{1+t}$$

$$= -(x+1)\ln(1+x),$$

and this is a bounded function on $(-1,0)$. By Chisholm-Everitt, we have $f \in L^2\left((-1,0);(1-x^2)^{-1}\right)$. For $0 \leq x < 1$,

$$(1-x^2)^{-1/2} \int_{x}^{1} f'(t)dt = -(1-x^2)^{-1/2} f(x).$$

We again apply Chisholm-Everitt on $[0,1)$ with

$$\varphi(x) = (1-x^2)^{-1/2}$$

$$\psi(x) = 1.$$

In this case,

$$\int_{0}^{x} (1-t^2)^{-1} dt \int_{x}^{1} dt \leq \int_{0}^{x} \frac{dt}{1-t} \int_{x}^{1} dt$$

$$= -(1-x)\ln(1-x),$$

which is also bounded on $(0,1)$. By Chisholm-Everitt, $-(1-x^2)^{-1/2}f \in L^2(0,1)$, or, equivalently, $f \in L^2\left((0,1);(1-x^2)^{-1}\right)$. $\square$

Theorem 5.11. *The inner products $\phi(\cdot,\cdot)$ and $(\cdot,\cdot)_1$ are equivalent on $W_{1,1} = V_1^{(-1,-1)}$.*

Proof. First of all, $(W_{1,1}, \phi(\cdot,\cdot))$ is a Hilbert space, and, by definition, $\left(V_1^{(-1,-1)}, (\cdot,\cdot)_1\right)$ is a Hilbert space. Let $f \in W_{1,1} = V_1^{(-1,-1)}$. Then

$$\begin{aligned}\|f\|_\phi^2 &= \int_{-1}^{1} |f'(x)|^2\,dx \leq \int_{-1}^{1} \left[|f'(x)|^2 + |f(x)|^2\left(1-x^2\right)^{-1}\right]dx \\ &= \|f\|_1^2\,.\end{aligned}$$

By the open mapping theorem, these inner products must be equivalent. $\square$

Note that T_2 is self-adjoint in $W_{1,2}$ since it is defined on the whole two-dimensional space.

We now need to consider T_1 in the space $W_{1,1}$. Recall that by theorem 5.10, $V_1^{(-1,-1)} = W_{1,1}$. We also know that the operator

$$B_{1,k}^{(-1,-1)} : \mathcal{D}\left(B_{1,k}^{(-1,-1)}\right) := V_3^{(-1,-1)} \subset V_1^{(-1,-1)} \longrightarrow V_1^{(-1,-1)}$$

namely, the first left-definite operator associated with $(A_k, L^2\left((-1,1);(1-x^2)^{-1}\right))$, is self-adjoint and given by

$$B_{1,k}^{(-1,-1)}[f](x) = l_{-1,-1}[f](x) = -(1-x^2)f''(x) + kf(x)$$

$$\begin{aligned}f \in \mathcal{D}\left(B_{1,k}^{(-1,-1)}\right) = V_3^{(-1,-1)} = \{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f', f'' \in AC_{loc}(-1,1);\\ (1-x^2)f''', (1-x^2)^{1/2}f'', f', (1-x^2)^{-1/2}f \in L^2(-1,1)\big\}\,.\end{aligned}$$

More specifically, $B_{1,k}^{(-1,-1)}$ is self-adjoint with respect to the first left-definite inner product $(\cdot,\cdot)_1$ which we know is equivalent to the inner product $\phi(\cdot,\cdot)$. We shall prove that the operator

$$T_1 : \mathcal{D}(T_1) \subset W_{1,1} \longrightarrow W_{1,1}$$

given by

$$T_1 f = B_{1,k}^{(-1,-1)} f = l_{-1,-1}[f]$$

$$f \in \mathcal{D}(T_1) := V_3^{(-1,-1)}$$

is self-adjoint in $(W_{1,1}, \phi(\cdot,\cdot))$.

Theorem 5.12. *Let $f, g \in V_3^{(-1,-1)}$. Then*

$$\lim_{x \longrightarrow \pm 1} (1-x^2) f''(x) \overline{g}'(x) = 0.$$

We shall prove this result for $x \longrightarrow +1^-$.

Proof. Let $f, g \in V_3^{(-1,-1)}$. Without loss of generality, assume that f, g are both real-valued. Since $V_3^{(-1,-1)} \subset V_1^{(-1,-1)}$ and $T_1 f \in V_1^{(-1,-1)}$, we see that

$$f', (T_1 f)', g' \in L^2(-1,1).$$

Hence $(T_1 f)' g', f' g' \in L^1(-1,1)$. For $0 \leq x < 1$,

$$\int_0^x (T_1 f)'(t) g'(t) dt = -\int_0^x \left((1-t^2) f''(t)\right)' g'(t) dt + k \int_0^x f'(t) g'(t) dt.$$

It follows that

$$\lim_{x \longrightarrow 1} \int_0^x \left((1-t^2) f''(t)\right)' g'(t) dt \tag{5.29}$$

exists and is finite. An integration by parts step shows that

$$\int_0^x \left((1-t^2) f''(t)\right)' g'(t) dt = (1-t^2) f''(t) g'(t) \mid_0^x - \int_0^x (1-t^2) f''(t) g''(t) dt.$$

Since $(1-x^2)^{1/2} f''(x), (1-x^2)^{1/2} g''(x) \in L^2(-1,1)$, this implies that

$$\lim_{x \longrightarrow 1} \int_0^x (1-t^2) f''(t) g''(t) dt$$

exists and is finite. It follows that

$$\lim_{x \longrightarrow 1} (1-x^2) f''(x) g'(x)$$

exists and is finite. Suppose

$$\lim_{x \longrightarrow 1} (1 - x^2) f''(x) g'(x) =: 2c$$

where we assume that $c \neq 0$. Without loss of generality, assume $c > 0$. Then there exists $x_0 \in [0, 1)$ such that

$$(1 - x^2) f''(x) g'(x) \geq c \tag{5.30}$$

$$f''(x) > 0, g'(x) > 0 \qquad \forall x \in [x_0, 1),$$

implying that

$$(1 - x^2) f''(x) \left| g''(x) \right| \geq c \frac{\left| g''(x) \right|}{g'(x)} \qquad \forall x \in [x_0, 1).$$

Hence,

$$\begin{aligned} \int_{x_0}^{x} (1 - t^2) f''(t) \left| g''(t) \right| dt &\geq c \int_{x_0}^{x} \frac{\left| g''(x) \right|}{g'(x)} dt \\ &= c \left| \ln\left(g'(t)\right) \right|_{x_0}^{x} \qquad (5.31) \\ &= c \left| \ln\left(g'(x)\right) \right| - c_1 \qquad \forall x \in [x_0, 1). \end{aligned}$$

Therefore,

$$\limsup_{x \longrightarrow 1} \left| \ln\left(g'(x)\right) \right| < \infty.$$

Claim: There exist constants M_1, M_2 such that

$$M_1 < g'(x) < M_2 \qquad \forall x \in [x_0, 1).$$

Otherwise, if $g'(x)$ is unbounded above, there exists a sequence $\{x_n\}_{n \geq 1} \subset [x_0, 1)$ such that

$$g'(x_n) \longrightarrow \infty.$$

Then it follows from (5.31) that

$$(1 - x^2) f''(x) g''(x) \notin L^1(-1, 1),$$

so $M_2 > 0$ exists as claimed. If M_1 doesn't exist, then there exists a sequence $\{y_n\}_{n\geq 1} \subset [x_0, 1)$ such that

$$g'(y_n) \longrightarrow 0.$$

Again, it follows from (5.31) that

$$(1-x^2)f''(x)g''(x) \notin L^1(-1,1),$$

a contradiction. From the claim, it now follows from (5.30) that

$$(1-x^2)f''(x) \geq \frac{c}{g'(x)} > \frac{c}{M_2} =: \widetilde{c} \qquad \forall x \in [x_0, 1).$$

This implies

$$(1-x^2)^2 \left(f''(x)\right)^2 > \widetilde{c}^{\,2}$$

so

$$(1-x^2)\left(f''(x)\right)^2 > \frac{\widetilde{c}^{\,2}}{1-x^2} \qquad \forall x \in [x_0, 1).$$

Integrating over $[x_0, 1)$ and using the fact that

$$(1-x^2)^{1/2} f''(x) \in L^2(-1,1),$$

we see that

$$\infty > \int\limits_{x_0}^{1} (1-t^2) f''(t)dt > \widetilde{c}^{\,2} \int\limits_{x_0}^{1} \frac{dt}{1-t^2} = \infty.$$

It follows that $c = 0$. $\square$

Lemma 5.9. T_1 *is densely defined in* $(W_{1,1}, \phi(\cdot,\cdot))$.

Proof. T_1 has the Jacobi polynomials $\left\{P_n^{(-1,-1)}\right\}_{n=2}^{\infty}$ as its eigenfunctions, and they are dense in $\mathcal{D}(T_1)$. $\square$

Theorem 5.13. T_1 *is symmetric in* $(W_{1,1}, \phi(\cdot,\cdot))$.

Proof. From the previous lemma, it suffices to show that T_1 is Hermitian. Let $f, g \in \mathcal{D}(T_1) = V_3^{(-1,-1)}$. Since $V_3^{(-1,-1)} \subset V_1^{(-1,-1)}$ and $T_1 f, T_1 g \in V_1^{(-1,-1)}$, we see that

$$f(\pm 1) = g(\pm 1) = 0 = T_1 f(\pm 1) = T_1 g(\pm 1).$$

Hence,

$$\begin{aligned}
(T_1 f, g)_\phi &= \int_{-1}^{1} (T_1 f)'(x) \overline{g}'(x) dx \\
&= \int_{-1}^{1} \left[-\left((1-x^2) f''(x) \right)' + k f'(x) \right] \overline{g}'(x) dx \\
&= -(1-x^2) f''(x) \overline{g}'(x) \mid_{-1}^{1} + \int_{-1}^{1} \left[(1-x^2) f''(x) \overline{g}''(x) + k f'(x) \overline{g}'(x) \right] dx \\
&= (f, T_1 g)_\phi
\end{aligned}$$

since $-(1-x^2) f''(x) \overline{g}'(x) \mid_{-1}^{1} = 0$ by theorem 5.12. A similar calculation shows that

$$\begin{aligned}
(f, T_1 g)_\phi &= \int_{-1}^{1} \left[-\left((1-x^2) \overline{g}''(x) \right)' + k \overline{g}'(x) \right] f'(x) dx \\
&= -(1-x^2) \overline{g}''(x) f'(x) \mid_{-1}^{1} + \int_{-1}^{1} \left[(1-x^2) f''(x) \overline{g}''(x) + k f'(x) \overline{g}'(x) \right] dx \\
&= (T_1 f, g)_\phi
\end{aligned}$$

since $f, g \in V_3 \implies (1-x^2) \overline{g}''(x) f'(x) \longrightarrow 0$ as $x \longrightarrow \pm 1$. $\square$

Theorem 5.14. *The operator T_1 has the following properties:*

(i) T_1 is self-adjoint in $(W_1, \phi(\cdot,\cdot))$.

(ii) $\sigma(T_1) = \{ n(n-1) + k \mid n \geq 2 \}$.

(iii) $\left\{ P_n^{(-1,-1)} \right\}_{n \geq 2}$ is a complete orthonormal set of eigenfunctions of T_1 in $(W_1, \phi(\cdot,\cdot))$.

(iv) T_1 is bounded below by kI in $(W_1, \phi(\cdot,\cdot))$.

Proof. For (iii): We know that $\left\{P_n^{(-1,-1)}\right\}_{n\geq 0}$ is a complete orthonormal set in $(W_1, \phi(\cdot,\cdot))$ and we know that $W_1 = W_{1,1} \oplus W_{1,2}$.
Also, $W_{1,2} = span\left\{P_0^{(-1,-1)}, P_1^{(-1,-1)}\right\}$ and so $W_{1,1} = W_{1,2}^{\perp} = span\left\{P_n^{(-1,-1)}\right\}_{n\geq 2}$.
We shall now prove that T_1 is closed in $(W_1, \phi(\cdot,\cdot))$. Take a sequence $\{f_n\} \subseteq \mathcal{D}(T_1) = V_3^{(-1,-1)}$ such that

$$f_n \longrightarrow f \qquad \text{in } (W_1, \phi(\cdot,\cdot))$$
$$T_1 f_n \longrightarrow g \qquad \text{in } (W_1, \phi(\cdot,\cdot)).$$

We show that $f \in \mathcal{D}(T_1)$ and $T_1 f = g$. We know that A_1 is self-adjoint and hence closed in $(W_1, (\cdot,\cdot)_1)$, and we know, since $\phi(\cdot,\cdot)$ and $(\cdot,\cdot)_1$ are equivalent, there exist constants c_1 and c_2 such that

$$c_1 \|f\|_\phi \leq \|f\|_1 \leq c_2 \|f\|_\phi \qquad \forall f \in W_{1,1} = V_1^{(-1,-1)}.$$

Hence,

$$\|f_n - f\|_1 \leq c_2 \|f_n - f\|_\phi \longrightarrow 0$$

i.e.

$$f_n \longrightarrow f \qquad \text{in } (W_1, (\cdot,\cdot)_1)$$

and

$$\|T_1 f_n - g\|_1 \leq c_2 \|T_1 f_n - g\|_\phi \longrightarrow 0$$

i.e.

$$T_1 f_n \longrightarrow g \qquad \text{in } (W_1, (\cdot,\cdot)_1)$$

and since T_1 is closed in $(W_1, (\cdot,\cdot)_1)$, we see that $f \in \mathcal{D}(T_1)$ and $T_1 f = g$. Also, we know that, for $n \geq 2$,

$$\begin{aligned}\left(T_1 P_n^{(-1,-1)}\right)(x) &= l_{-1,-1}[P_n^{(-1,-1)}](x)\\ &= (n(n-1)+k)P_n^{(-1,-1)}(x).\end{aligned}$$

This implies

$$\{n(n-1)+k \mid n \geq 2\} \subseteq \sigma(T_1).$$

Since $\left\{P_n^{(-1,-1)}\right\}_{n\geq 2}$ is complete and $\lambda_n := n(n-1)+k \longrightarrow \infty$, we know that

$$\sigma(T_1) = \{n(n-1)+k \mid n \geq 2\}$$

by a result due to Riesz-Nagy, which proves (ii) and (iii). To summarize: T_1 is a closed, symmetric operator with a complete set of eigenfunctions. From Naimark's book, T_1 is self-adjoint. This proves (i). To prove (iv), let $f \in \mathcal{D}(T_1)$. Then, since

$$T_1 : V_3^{(-1,-1)} \subset V_1^{(-1,-1)} \longrightarrow V_1^{(-1,-1)},$$

and by (5.29),

$$\begin{aligned}
(T_1 f, f)_\phi &= \frac{1}{2}\left(T_1 f\right)(-1)\overline{f}(-1) + \frac{1}{2}\left(T_1 f\right)(1)\overline{f}(1) \\
&\quad + \int_{-1}^{1} \left(T_1 f\right)'(x)\overline{f}'(x)dx \\
&= \int_{-1}^{1} \left(T_1 f\right)'(x)\overline{f}'(x)dx \\
&\underset{5.29}{=} \int_{-1}^{1} \left[(1-x^2)\left|f''(x)\right|^2 + k\left|f'(x)\right|^2\right] dx \\
&\geq k\int_{-1}^{1} \left|f'(x)\right|^2 dx \\
&= \frac{k}{2}\left|f(-1)\right|^2 + \frac{k}{2}\left|f(1)\right|^2 + k\int_{-1}^{1} \left|f'(x)\right|^2 dx \\
&= k\,(f,f)_\phi\,.
\end{aligned}$$

□

We now construct the self-adjoint operator T in $(W_1, \phi(\cdot,\cdot))$ that is generated by the Jacobi differential expression $l_{-1,-1}[.]$, having the *entire* set of Jacobi polynomials $\left\{P_n^{(-1,-1)}\right\}_{n\geq 0}$ as eigenfunctions and having spectrum $\sigma(T) = \{n(n-1)+k \mid n \in \mathbb{N}_0\}$. For $f \in W_1$, write

$$f = f_1 + f_2$$

where $f_1 \in W_{1,1}$, and $f_2 \in W_{1,2}$. Define

$$T : \mathcal{D}(T) \subset W_1 \longrightarrow W_1$$

by

$$Tf = T_1 f_1 + T_2 f_2 = l_{-1,-1}[f_1] + l_{-1,-1}[f_2] = l_{-1,-1}[f],$$

$$\mathcal{D}(T) = \mathcal{D}(T_1) \oplus \mathcal{D}(T_2).$$

Theorem 5.15. *T is self-adjoint in $(W_1, \phi(\cdot,\cdot))$ and*

$$\begin{aligned}\mathcal{D}(T) &= \{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC[-1,1]; f', f'' \in AC_{loc}(-1,1);\\ &\quad (1-x^2)f''', (1-x^2)^{1/2} f'', f' \in L^2(-1,1)\}\\ &= \{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC[-1,1]; f', f'' \in AC_{loc}(-1,1;)\\ &\quad (1-x^2)f''' \in L^2(-1,1)\}.\end{aligned}$$

Furthermore, $\sigma(T) = \{n(n-1)+k \mid n \in \mathbb{N}_0\}$ and T is bounded below by kI in $(W_1, \phi(\cdot,\cdot))$.

For the following theorem let us recall the spaces

$$\begin{aligned}V_1^{(-1,-1)} &= \left\{f : (-1,1) \longrightarrow \mathbb{C} \mid f \in AC_{loc}(-1,1); (1-x^2)^{1/2} f, f' \in L^2(-1,1)\right\}\\ &= \left\{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC[-1,1]; f' \in L^2(-1,1); f(\pm 1) = 0\right\}\\ &= W_{1,1}\end{aligned}$$

$$
\begin{aligned}
V_3^{(-1,-1)} &= \mathcal{D}(T_1) = \{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f', f'' \in AC_{loc}(-1,1); \\
&\quad (1-x^2)f''', (1-x^2)^{1/2}f'', f', (1-x^2)^{-1/2}f \in L^2(-1,1)\big\} \\
&= \big\{f \in V_1 \mid f', f'' \in AC_{loc}(-1,1); (1-x^2)f''', (1-x^2)^{1/2}f'' \in L^2(-1,1)\big\} \\
&= \{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC[-1,1]; f', f'' \in AC_{loc}(-1,1); \\
&\quad f(\pm 1) = 0; (1-x^2)f''', (1-x^2)^{1/2}f'', f' \in L^2(-1,1)\big\}.
\end{aligned}
$$

Note that the space $\mathcal{D}$ below is $V_3^{(-1,-1)}$ minus the condition $f(\pm 1) = 0$, so $V_3^{(-1,-1)} \subseteq \mathcal{D}$.

Theorem 5.16. *Let*

$$
\begin{aligned}
\mathcal{D} := \{f : [-1,1] \longrightarrow \mathbb{C} \mid f \in AC[-1,1]; f', f'' \in AC_{loc}(-1,1); \\
(1-x^2)f''', (1-x^2)^{1/2}f'', f' \in L^2(-1,1)\big\}.
\end{aligned}
$$

Then $\mathcal{D}(T) = \mathcal{D}$.

Proof. First show $\mathcal{D}(T) \subseteq \mathcal{D}$: Let $f \in \mathcal{D}(T) = \mathcal{D}(T_1) \oplus \mathcal{D}(T_2)$. Write

$$
f = f_1 + f_2
$$

where $f_1 \in \mathcal{D}(T_1) = V_3^{(-1,-1)} \subseteq \mathcal{D}$, $f_2 \in \mathcal{D}(T_2) \subseteq \mathcal{D}$. Then $f \in \mathcal{D}$. To show that $\mathcal{D} \subseteq \mathcal{D}(T)$, let $f \in \mathcal{D}$. Write

$$
\begin{aligned}
f(x) = &\left[f(x) - \left(\frac{f(1) - f(-1)}{2}\right)x - \left(\frac{f(1) + f(-1)}{2}\right)\right] \\
&+ \left[\left(\frac{f(1) - f(-1)}{2}\right)x + \left(\frac{f(1) + f(-1)}{2}\right)\right]
\end{aligned}
$$

with

$$
\begin{aligned}
f_1(x) &:= f(x) - \left(\frac{f(1) - f(-1)}{2}\right)x - \left(\frac{f(1) + f(-1)}{2}\right) \\
f_2(x) &:= \left(\frac{f(1) - f(-1)}{2}\right)x + \left(\frac{f(1) + f(-1)}{2}\right).
\end{aligned}
$$

Then $f_1 \in \mathcal{D}$, and $f_1(\pm 1) = 0$, i.e. $f_1 \in V_3 = \mathcal{D}(T_1)$. Also, $f_2''(x) = 0$, i.e. $f_2 \in \mathcal{D}(T_2)$. Together, $f \in \mathcal{D}(T)$. □

To summarize, we have studied the Sobolev orthogonality of the Jacobi polynomials for $\alpha = \beta = -1$ in depth, and, through the left-definite theory, we have constructed a self-adjoint operator T in a suitable Hilbert space having the full sequence of Jacobi polynomials $\left\{P_n^{(-1,-1)}\right\}_{n=0}^{\infty}$ as eigenfunctions. This completes the discussion of the special case where $\alpha = \beta = -1$.

CHAPTER SIX

Spectral Analysis of the Jacobi Differential Equation ($\alpha > -1, \beta = -1$)

One should always generalize.

- Carl Gustav Jacobi

Following Jacobi's advice, we shall now extend the results from the previous chapter to the more general case where $\alpha > -1, \beta = -1$. There are many similarities between the two cases, and the main techniques can be modified to apply to the general case. However, it is worth noting that there is a fundamental difference: unlike in the special case where the set $\left\{P_n^{(-1,-1)}(x)\right\}_{n=2}^{\infty}$ is complete in $L^2\left((-1,1);(1-x^2)^{-1}\right)$, see lemma 5.6, in the general case, the set $\left\{P_n^{(\alpha,-1)}(x)\right\}_{n=1}^{\infty}$ is complete in $L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$, that is, the maximal orthogonal set contains the first Jacobi polynomial!

6.1 Right-Definite Spectral Analysis

In this section, we show that for $\alpha \geq 1, \beta = -1$, both endpoints $x = \pm 1$ are in the limit-point condition, and thus the right-definite GKN self-adjoint operator is unique. No boundary conditions are necessary. For $-1 < \alpha < 1$, the endpoint $x = -1$ is in the limit-point condition, whereas $x = +1$ is in the limit-circle condition. Therefore, one boundary condition is needed to define the right-definite GKN self-adjoint operator.

For $\alpha > -1, \beta = -1$, the Jacobi differential expression becomes

$$\begin{aligned} l_{\alpha,-1}[y](x) &= \frac{1}{w_{\alpha,-1}(x)}\left[-\left((1-x)^{\alpha+1}y'(x)\right)' + k(1-x)^{\alpha}(1+x)^{-1}y(x)\right] \qquad (6.1) \\ &= -(1-x^2)y'' + (\alpha+1)(x+1)y' + ky. \end{aligned}$$

Let $k = 0$, so

$$l_{\alpha,-1}[y](x) = -(1-x^2)y'' + (\alpha+1)(x+1)y'.$$

Consider the endpoint $x = 1$.

Multiply $l_{\alpha,-1}[y](x)$ by $\frac{(x-1)^2}{1-x^2}$ to see that $x = 1$ is a regular singular point in the sense of Frobenius:

$$\frac{(x-1)^2}{1-x^2} l_{\alpha,-1}[y](x) = -(x-1)^2 y'' - (\alpha+1)(x-1)y' = 0$$

or

$$(x-1)^2 y'' + (\alpha+1)(x-1)y' = 0.$$

Then the indicial equation for $x = 1$ is

$$r(r-1) + r(\alpha+1) = 0.$$

Thus,

$$r_1 = 0 \sim 1 =: y_1(x)$$

$$r_2 = -\alpha \sim (1-x)^{-\alpha} =: y_2(x).$$

Determine whether y_1 and y_2 are in $L^2((-1,1), \frac{(1-x)^\alpha}{1+x})$ near $x = 1$:

$$\int_0^1 y_1^2(x) \frac{(1-x)^\alpha}{1+x} dx = \int_0^1 \frac{(1-x)^\alpha}{1+x} dx < \infty$$

Since $\frac{1}{1+x} < 1$ for $x \in (0,1)$, we have

$$\int_0^1 y_2^2(x) \frac{(1-x)^\alpha}{1+x} dx = \int_0^1 \frac{(1-x)^{-\alpha}}{1+x} dx < \int_0^1 (1-x)^{-\alpha} dx$$
$$\begin{cases} < \infty & \text{if } -1 < \alpha < 1 \\ = \infty & \text{if } \alpha \geq 1 \end{cases},$$

i.e. the endpoint $x = 1$ is limit-point if $-1 < \alpha < 1$ and limit-circle if $\alpha \geq 1$.

Now consider the endpoint $x = -1$.

Multiply $l_{\alpha,-1}[y](x)$ by $\frac{(x+1)^2}{1-x^2}$ to see that $x = -1$ is a regular singular point in the sense of Frobenius:

$$\frac{(x+1)^2}{1-x^2} l_{\alpha,-1}[y](x) = -(x+1)^2 y'' - (\alpha+1)\frac{(x+1)^2}{x-1} y' = 0$$

or

$$(x+1)^2y'' + (\alpha+1)\frac{(x+1)^2}{x-1}y' = 0.$$

Then the indicial equation for $x = -1$ is

$$r(r-1) = 0.$$

Thus,

$$r_1 = 0 \sim 1 =: y_1(x)$$

$$r_2 = 1 \sim (1+x) =: y_2(x).$$

Determine whether y_1 and y_2 are in $L^2((-1,1), \frac{(1-x)^\alpha}{1+x})$ near $x = -1$:

$$\int_{-1}^{0} y_1^2(x)\frac{(1-x)^\alpha}{1+x}dx > \int_{-1}^{0} \frac{1}{1+x}dx = \infty$$

since $(1-x)^\alpha > 0$ on $(-1, 0)$. For the same reason,

$$\int_{-1}^{0} y_2^2(x)\frac{(1-x)^\alpha}{1+x}dx = \int_{-1}^{0} (1+x)(1-x)^\alpha dx < \infty,$$

which makes $x = -1$ limit-point for any $\alpha > -1$. No boundary conditions are necessary at $x = -1$.

Lemma 6.1. *For all* $n \in \mathbb{N}$,

$$P_n^{(\alpha,-1)}(x) = \frac{n+\alpha}{2n}(x+1)P_{n-1}^{(\alpha,1)}(x).$$

Proof.

$$\begin{aligned}
P_n^{(\alpha,-1)}(x) &= \sum_{j=0}^{n} \binom{n+\alpha}{j}\binom{n-1}{n-j}\left(\frac{x-1}{2}\right)^{n-j}\left(\frac{x+1}{2}\right)^{j} \\
&= \frac{1}{2}(x+1)\sum_{j=0}^{n} \binom{n+\alpha}{j}\binom{n-1}{n-j}\left(\frac{x-1}{2}\right)^{n-j}\left(\frac{x+1}{2}\right)^{j-1} \\
&\underset{\binom{n-1}{n}=0}{=} \frac{1}{2}(x+1)\sum_{j=1}^{n} \binom{n+\alpha}{j}\binom{n-1}{n-j}\left(\frac{x-1}{2}\right)^{n-j}\left(\frac{x+1}{2}\right)^{j-1} \\
&= \frac{1}{2}(x+1)\sum_{j=0}^{n-1} \binom{n+\alpha}{j+1}\binom{n-1}{n-j-1}\left(\frac{x-1}{2}\right)^{n-j-1}\left(\frac{x+1}{2}\right)^{j} \\
&= \frac{1}{2}\frac{n+\alpha}{n}(x+1)P_{n-1}^{(\alpha,1)}(x).
\end{aligned}$$

□

We now turn to the discussion of the operator theoretic properties of the Jacobi differential expression (6.1). The appropriate right-definite setting is given by $L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$, and the maximal domain of $l_{\alpha,-1}[\cdot]$ in this space is

$$\begin{aligned}
\Delta := \{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); \\
f, l_{\alpha,-1}[f] \in L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)\}.
\end{aligned}$$

For $f, g \in \Delta$ and $[a,b] \subset (-1,1)$, we have Dirichlet's formula:

$$\begin{aligned}
\int_a^b l_{\alpha,-1}[f](x)\overline{g}(x)(1-x)^{\alpha}(1+x)^{-1}dx &= -(1-x)^{\alpha+1}f'(x)\overline{g}(x)\mid_a^b \\
&+ \int_a^b \left[(1-x)^{\alpha+1}f'(x)\overline{g}'(x) + k(1-x)^{\alpha}(1+x)^{-1}f(x)\overline{g}(x)\right]dx
\end{aligned}$$

and Green's formula:

$$\begin{aligned}
\int_a^b l_{\alpha,-1}[f](x)\overline{g}(x)(1-x)^{\alpha}(1+x)^{-1}dx &= \left[(1-x)^{\alpha+1}\left(f(x)\overline{g}'(x) - f'(x)\overline{g}(x)\right)\right]\mid_a^b \\
&+ \int_a^b f(x)\overline{l_{\alpha,-1}[g]}(x)(1-x)^{\alpha}(1+x)^{-1}dx.
\end{aligned}$$

Let us first consider the endpoint $x = -1, \alpha > -1, \beta = -1$. We begin by showing that the Jacobi differential expression ($k = 0$)

$$l_{\alpha,-1}[y](x) = \frac{1}{(1-x)^{\alpha}(1+x)^{-1}}\left[-\left((1-x)^{\alpha+1}y'(x)\right)'\right]$$

is Dirichlet at $x = -1$. Note that the maximal domain can be written as

$$\Delta = \begin{cases} \Big\{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); (1-x)^{\frac{\alpha}{2}}(1+x)^{-\frac{1}{2}}f \in L^2(-1,1), \\ (1-x)^{-\frac{\alpha}{2}}(1+x)^{\frac{1}{2}}\left[(1-x)^{\alpha+1}f'(x)\right]' \in L^2(-1,1)\Big\}, \quad \text{if } \alpha \geq 1 \\ \Big\{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); (1-x)^{\frac{\alpha}{2}}(1+x)^{-\frac{1}{2}}f \in L^2(-1,1), \\ (1-x)^{-\frac{\alpha}{2}}(1+x)^{\frac{1}{2}}\left[(1-x)^{\alpha+1}f'(x)\right]' \in L^2(-1,1), \\ \lim\limits_{x\to 1^-}(1-x)^{\alpha+1}f'(x) = 0\Big\}, \quad \text{if } -1 < \alpha < 1. \end{cases}$$

For $f, g \in \Delta$ and $-1 < x \leq 0$, we have Green's formula

$$\int_x^0 l_{\alpha,-1}[f](t)\overline{g}(t)(1-t)^{\alpha}(1+t)^{-1}dt = \int_x^0 \left[(1-t)^{\alpha+1}f'(t)\right]'\overline{g}(t)dt$$

$$= f'(0)\overline{g}(0) - (1-x)^{\alpha+1}f'(x)\overline{g}(x) - \int_x^0 (1-t)^{\alpha+1}f'(t)\overline{g}'(t)dt.$$

Let $f = g$ be real-valued, then Green's formula becomes

$$\int_x^0 (1-t)^{\alpha+1}\left(f'(t)\right)^2 dt = f'(0)f(0) - (1-x)^{\alpha+1}f'(x)f(x) \tag{6.2}$$

$$- \int_x^0 l_{\alpha,-1}[f](t)f(t)(1-t)^{\alpha}(1+t)^{-1}dt. \tag{6.3}$$

Lemma 6.2. $(1-x)^{\frac{\alpha+1}{2}}f' \in L^2(-1,0)$ *for all* $f \in \Delta$.

Proof. By contradiction. We assume that f is real-valued on $(-1,1)$. Now suppose that

$$\lim_{x\to -1}\int_x^0 (1-t)^{\alpha+1}\left(f'(t)\right)^2 dt = \infty.$$

Then, from (6.2), $\lim\limits_{x\to -1}(1-x)^{\alpha+1}f'(x)f(x) = -\infty$, and consequently, $\lim\limits_{x\to -1} f'(x)f(x) = -\infty$. Then for any $N \in \mathbb{N}$, there exists $x_N \in (-1,0)$ such that for $-1 < x < x_N$, we have $f'(x)f(x) \leq -N$. Choose any such N, e.g. $N = 1$. Integrate from x to x_N (where $-1 < x < x_N$) to obtain

$$\frac{f^2(x_N)}{2} - \frac{f^2(x)}{2} = \int_x^{x_N} f'(t)f(t)dt \leq -N(x_N - x).$$

It follows that

$$-\frac{f^2(x)}{2} \leq -N(x_N - x) - \frac{f^2(x_N)}{2} \leq -N(x_N - x)$$

i.e.

$$-\frac{f^2(x)}{2} \leq -N(x_N - x), \qquad (-1 < x < x_N)$$

or

$$f^2(x) \geq 2N(x_N - x), \qquad (-1 < x < x_N).$$

Hence,

$$\int_x^{x_N} \left(f'(t)\right)^2 (1-t)^\alpha (1+t)^{-1} dt \geq 2N \int_x^{x_N} (x_N - t)(1-t)^\alpha (1+t)^{-1} dt.$$

Here, we distinguish between two cases, $\alpha \geq 0$ and $-1 < \alpha < 0$. First, for $\alpha \geq 0$,

$$2N \int_x^{x_N} (x_N - t)(1-t)^\alpha (1+t)^{-1} dt \geq 2N \int_x^{x_N} \frac{(x_N - t)}{(1+t)} dt$$

$$\begin{aligned}
&= 2N \int_x^{x_N} \frac{x_N - 1 + 1 - t}{(1+t)} dt \\
&= 2N \int_x^{x_N} \left[-1 + \frac{1 + x_N}{1+t}\right] dt \\
&= -2N(x_N - x) + 2N(1 + x_N)\left[\ln(1 + x_N) - \ln(1+x)\right] \\
&= -2N(x_N - x) - 2N(1 + x_N)\ln(1+x) + c_N \longrightarrow \infty
\end{aligned}$$

with $x \longrightarrow -1^+$, contradicting that $(1-x)^{\frac{\alpha}{2}}(1+x)^{-\frac{1}{2}} f \in L^2(-1,1)$. For $-1 < \alpha < 0$, we have

$$2N \int_x^{x_N} (x_N - t)(1-t)^\alpha (1+t)^{-1} dt \geq 2^{\alpha+1} N \int_x^{x_N} \frac{(x_N - t)}{(1+t)} dt$$

$$= 2^{\alpha+1}N\int\limits_{x}^{x_N}\frac{x_N-1+1-t}{(1+t)}dt$$

$$= 2^{\alpha+1}N\int\limits_{x}^{x_N}\frac{x_N-1+1-t}{(1+t)}dt$$

$$= 2^{\alpha+1}N\int\limits_{x}^{x_N}\left[-1+\frac{1+x_N}{1+t}\right]dt$$

$$= -2^{\alpha+1}N(x_N-x)+2^{\alpha+1}N(1+x_N)\left[\ln(1+x_N)-\ln(1+x)\right]$$

$$= -2^{\alpha+1}N(x_N-x)-2^{\alpha+1}N(1+x_N)\ln(1+x)+c_N\longrightarrow\infty$$

with $x\longrightarrow -1^{+}$, contradicting that $(1-x)^{\frac{\alpha}{2}}(1+x)^{-\frac{1}{2}}f\in L^2(-1,1)$. This completes the proof of the lemma. □

Lemma 6.3. *$f(-1)=0$ for all $f\in\Delta$.*

Proof. Let $f\in\Delta$, and suppose that $f(-1)\neq 0$. We may assume that $f(-1)>0$. By continuity, there exists $x^*\in(-1,0)$ such that $f(x)>\frac{f(-1)}{2}$ for $x\in(-1,x^*]$. Then, since$(1-t)^{\alpha}$ is bounded below by some $K>0$,

$$\infty>\int\limits_{-1}^{0}|f(t)|^2(1-t)^{\alpha}(1+t)^{-1}dt>K\frac{f^2(-1)}{4}\int\limits_{-1}^{0}\frac{dt}{1+t}=\infty,$$

a contradiction, and hence, $f(-1)=0$ for all $f\in\Delta$. □

Together, the previous two lemmas imply that the Jacobi expression is Dirichlet at the endpoint $x=-1$ for $\alpha>-1,\beta=-1$. We now proceed to show that the Jacobi expression is strong limit-point at $x=-1$ for $\alpha>-1,\beta=-1$.

Lemma 6.4. $\lim\limits_{x\to-1^{+}}(1-x)^{\alpha+1}f(x)g'(x)=0$ *for all $f,g\in\Delta$.*

Proof. From Green's formula,

$$\int\limits_{x}^{0}l_{\alpha,-1}[f](t)\overline{g}(t)(1-t)^{\alpha}(1+t)^{-1}dt= \tag{6.4}$$

$$f'(0)\overline{g}(0) - (1-x)^{\alpha+1}f'(x)\overline{g}(x) - \int_x^0 (1-t)^{\alpha+1}f'(t)\overline{g}'(t)dt,$$

we see that $\lim_{x\to -1^+}(1-x)^{\alpha+1}f'(x)\overline{g}(x)$ exists and is finite (since the left-hand side of (6.4) is finite, $f'(0)\overline{g}(0)$ is a constant and the integral on the right-hand side of (6.4) is finite by the previous lemma). Now suppose that $\lim_{x\to -1^+}(1-x)^{\alpha+1}f(x)g'(x) = c > 0$. We may assume that, for x close to -1,

$$f(x) > 0 \text{ and } g'(x) > 0.$$

Hence, there exists $x^* \in (-1, 0]$ such that $g'(x) \geq \frac{\widetilde{c}}{f(x)}$ for $x \in (-1, x^*]$, where $\widetilde{c} = \frac{c}{2} > 0$. Therefore,

$$|f'(x)g'(x)| \geq \widetilde{c}\frac{f'(x)}{f(x)} \qquad (x \in (-1, x^*]).$$

Integrate to obtain

$$\begin{aligned}\int_x^{x^*} |f'(t)g'(t)|\, dt &\geq \widetilde{c}\int_x^{x^*} \frac{|f'(t)|}{f(t)}dt \\ &\geq \widetilde{c}\left|\int_x^{x^*} \frac{f'(t)}{f(t)}dt\right| \\ &= \widetilde{c}\left|K - \ln|f(x)|\right|.\end{aligned}$$

Knowing that $\int_{-1}^{x^*} |f'(t)g'(t)|\, dt < \infty$ (since $f' \in L^2(-1,0)$), we let $x \to -1^+$ to see that

$$\infty > \int_{-1}^{x^*} |f'(t)g'(t)|\, dt \geq \widetilde{c}\left|K - \lim_{x\to -1^+} \ln|f(x)|\right| = \infty,$$

a contradiction. $\square$

We now turn our attention to the endpoint $x = 1$.

Lemma 6.5. $\lim_{x\to 1^-}(1-x)^{\alpha+1}f'(x) = 0$ *for all* $f \in \Delta$ *and for all* $\alpha > -1$.

Proof. For $-1 < \alpha < 1$, this holds true due to the boundary condition in Δ.

For $\alpha \geq 1$, the Jacobi expression is limit-point at $x = 1$, and, from Green's formula, the Wronskian must vanish:

$$W[f, g](x) = (1-x)^{\alpha+1}[f'(x)\overline{g}(x) - f(x)\overline{g}'(x)] = 0.$$

By Naimark's patching lemma, we can find a function $g \in \Delta$ which is 1 near 1 and 0 near -1 :

$$g(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ -16x^3 + 12x^2, & 0 < x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

It is easy to see that $g \in C^2 \cap \Delta$. □

Lemma 6.6. *Let* $\alpha > -1, f \in \Delta$. *Then* $(1-x)^{\frac{\alpha+1}{2}} f' \in L^2(0, 1)$.

Proof. From the previous lemma, we know that

$$\lim_{x \to 1^-} (1-x)^{\alpha+1} f'(x) = 0 \tag{6.5}$$

for $f \in \Delta$ and for all $\alpha > -1$. For $f = g$, Green's formula is

$$\int_0^x (1-t)^{\alpha+1} \left(f'(t)\right)^2 dt = (1-x)^{\alpha+1} f'(x) f(x) - f'(0) f(0)$$
$$- \int_0^x l[f](t) f(t) (1-t)^{\alpha} (1+t)^{-1} dt.$$

Assume that $\int_0^x (1-t)^{\alpha+1} \left(f'(t)\right)^2 dt = \infty$. Then $\lim_{x \to 1^-} (1-x)^{\alpha+1} f'(x) f(x) = \infty$, since the remaining terms on the right-hand side are known to be finite. Hence, there exists $x^* \in [0, 1)$ such that $(1-x)^{\alpha+1} f'(x) f(x) \geq 1$ for all $x \in [x^*, 1)$. Assume, without loss of generality, that

$$f(x) > 0 \text{ and } (1-x)^{\alpha+1} f'(x) > 0$$

on $[x^*, 1)$. It follows that

$$\left|\left((1-t)^{\alpha+1}f'(t)\right)'\right| f(t) \geq \frac{\left|\left((1-t)^{\alpha+1}f'(t)\right)'\right|}{(1-t)^{\alpha+1}f'(t)} \qquad \text{on } [x^*, 1).$$

Integrate to obtain

$$\begin{aligned}
\infty > \int_{-1}^{1} |l[f](t)|\, f(t)(1-t)^{\alpha}(1+t)^{-1}dt &= \int_{-1}^{1} \left|\left((1-t)^{\alpha+1}f'(t)\right)'\right| f(t)dt \\
\geq \int_{x^*}^{x} \left|\left((1-t)^{\alpha+1}f'(t)\right)'\right| f(t)dt &\geq \int_{x^*}^{x} \frac{\left|\left((1-t)^{\alpha+1}f'(t)\right)'\right|}{(1-t)^{\alpha+1}f'(t)}dt \\
\geq \left|\int_{x^*}^{x} \frac{\left((1-t)^{\alpha+1}f'(t)\right)'}{(1-t)^{\alpha+1}f'(t)}dt\right| &= \left|\ln\left((1-t)^{\alpha+1}f'(t)\right)\right| \Big|_{x^*}^{x} \longrightarrow \infty
\end{aligned}$$

as $x \to 1^-$ by (6.5), a contradiction. □

Lemma 6.7. *Let $\alpha > -1$, $f, g \in \Delta$. Then*

$$\lim_{x \to 1^-} (1-x)^{\alpha+1} f'(x)\overline{g}(x) = 0.$$

Proof. From Dirichlet's formula,

$$\begin{aligned}
\int_0^x l_{\alpha,-1}[f](t)\overline{g}(t)(1-t)^{\alpha}(1+t)^{-1}dt &= \int_0^x \left((1-t)^{\alpha+1}f'(t)\right)'\overline{g}(t)dt \\
&= (1-x)^{\alpha+1}f'(x)\overline{g}(x) - f'(0)\overline{g}(0) \\
&\quad - \int_0^x (1-t)^{\alpha+1}f'(t)\overline{g}'(t)dt,
\end{aligned}$$

we see that $\lim_{x\to 1^-}(1-x)^{\alpha+1}f'(x)\overline{g}(x)$ exists and is finite, since all the other terms are finite (the last integral is finite for $x \longrightarrow 1^-$ from the previous lemma). Assume, without loss of generality, that $f, g \in \Delta$ are both real-valued, and suppose

$$\lim_{x \to 1^-} (1-x)^{\alpha+1} f'(x)g(x) = c > 0.$$

Then we may assume that, for x close to 1,

$$(1-x)^{\alpha+1}f'(x) > 0 \text{ and } g(x) > 0.$$

Hence, there exists $x^* \in [0,1)$ such that

$$(1-x)^{\alpha+1} f'(x) \geq \frac{\widetilde{c}}{g(x)} \tag{6.6}$$

where $\widetilde{c} := \frac{c}{2} > 0$, and consequently,

$$\left| \left((1-x)^{\alpha+1} f'(x) \right)' \right| g(x) \geq \widetilde{c} \frac{\left| \left((1-x)^{\alpha+1} f'(x) \right)' \right|}{(1-x)^{\alpha+1} f'(x)} \tag{6.7}$$

on $[x^*, 1)$. Integrate:

$$\begin{aligned}
\infty > \int_{-1}^{1} l_{\alpha,-1}[f](t) g(t) (1-t)^{\alpha} (1+t)^{-1} dt &= \int_{-1}^{1} \left| \left((1-t)^{\alpha+1} f'(t) \right)' \right| g(t) dt \\
\geq \int_{x^*}^{x} \left| \left((1-t)^{\alpha+1} f'(t) \right)' \right| g(t) dt \underset{(6.7)}{\geq} &\int_{x^*}^{x} \widetilde{c} \frac{\left| \left((1-t)^{\alpha+1} f'(t) \right)' \right|}{(1-t)^{\alpha+1} f'(t)} dt \\
\geq \left| \int_{x^*}^{x} \widetilde{c} \frac{\left((1-t)^{\alpha+1} f'(t) \right)'}{(1-t)^{\alpha+1} f'(t)} dt \right| = &\left| \ln((1-t)^{\alpha+1} f'(t)) \right| \Big|_{x^*}^{x} \longrightarrow \infty
\end{aligned}$$

by the previous lemma, a contradiction. $\square$

This completes the proof of the following theorem.

Theorem 6.1. *The Jacobi expression (6.1) is strong limit-point and Dirichlet at $x = \pm 1$, i.e.*

(i) $\int_{0}^{1} |f'(t)|^2 (1-t)^{\alpha+1} dt < \infty$ and $\int_{-1}^{0} |f'(t)|^2 (1-t)^{\alpha+1} dt < \infty$ for all $f \in \Delta$ and

(ii) $\lim_{x \to \pm 1} (1-t)^{\alpha+1} f'(x) \overline{g}(x) = 0$ for all $f, g \in \Delta$.

The remainder of this section will be devoted to the (right-definite) self-adjoint operator that is generated by the Jacobi differential expression $l_{\alpha,-1,k}[\cdot]$. To this end, recall that $L^2_{\alpha,-1}(-1,1)$ denotes the space $L^2\left((-1,1); (1-x)^{\alpha}(1+x)^{-1}\right)$, and the maximal domain $\Delta_k^{(\alpha,-1)}$ of $l_{\alpha,-1,k}[\cdot]$ in $L^2_{\alpha,-1}$ is defined to be

$$\Delta_k^{(\alpha,-1)} := \left\{ f \in L^2_{\alpha,-1}(-1,1) \,\middle|\, f, f' \in AC_{loc}(-1,1); l_{\alpha,-1,k}[f] \in L^2_{\alpha,-1}(-1,1) \right\}$$

The maximal operator $T^{(\alpha,-1)}_{\max,k}$ associated with $l_{\alpha,-1}[\cdot]$ is given by

$$T^{(\alpha,-1)}_{\max,k}(f) := l_{\alpha,-1}[f]$$
$$\mathcal{D}(T^{(\alpha,-1)}_{\max}) := \Delta^{(\alpha,-1)}_{k}.$$

The minimal operator is defined as $T^{(\alpha,-1)}_{\min,k} := (T^{(\alpha,-1)}_{\max,k})^{*}$, the Hilbert space adjoint of $T^{(\alpha,-1)}_{\max,k}$. The operator $T^{(\alpha,-1)}_{\min,k}$ is closed, symmetric and satisfies

$$(T^{(\alpha,-1)}_{\min,k})^{*} = T^{(\alpha,-1)}_{\max,k}.$$

The deficiency index $d(T^{(\alpha,-1)}_{\min,k})$ of $T^{(\alpha,-1)}_{\min,k}$ is

$$d(T^{(\alpha,-1)}_{\min,k}) = \begin{cases} (0,0) & \text{if } \beta = -1,\ \alpha \geq 1 \\ (1,1) & \text{if } \beta = -1,\ \ -1 < \alpha < 1. \end{cases}$$

This can be seen from the limit-point/limit-circle classification of the singular endpoints $x = \pm 1$:

(i) $x = \pm 1$ are limit-point if $\beta = -1$, $\alpha \geq 1$ and

(ii) $x = -1$ is limit-point, $x = 1$ is limit-circle if $\beta = -1$, $-1 < \alpha < 1$.

Consequently, by von Neumann's theory of self-adjoint extensions of symmetric operators ([12], chapter XII), $T^{(\alpha,-1)}_{\min,k}$ has self-adjoint extensions in $L^2_{\alpha,-1}(-1,1)$ for $\beta = -1$, $\alpha > -1$. If $\alpha \geq 1$, there is a unique self-adjoint extension in $L^2_{\alpha,-1}(-1,1)$. From the Glazman-Krein-Naimark theory [2], [43], the self-adjoint operator $A^{(\alpha,-1)}_{k}$: $\mathcal{D}(A^{(\alpha,-1)}_{k}) \subset L^2_{\alpha,-1}(-1,1) \longrightarrow L^2_{\alpha,-1}(-1,1)$ defined by

$$A^{(\alpha,-1)}_{k} f := l_{\alpha,-1}[f] \tag{6.8}$$

for $f \in \mathcal{D}(A_k^{(\alpha,-1)}) =$

$$\begin{cases} \Big\{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); (1-x)^{\frac{\alpha}{2}}(1+x)^{-\frac{1}{2}} f \in L^2(-1,1), \\ \qquad (1-x)^{-\frac{\alpha}{2}}(1+x)^{\frac{1}{2}} \left[(1-x)^{\alpha+1} f'(x)\right]' \in L^2(-1,1)\Big\}, \qquad \text{if } \alpha \geq 1 \\ \Big\{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f' \in AC_{loc}(-1,1); (1-x)^{\frac{\alpha}{2}}(1+x)^{-\frac{1}{2}} f \in L^2(-1,1), \\ \qquad (1-x)^{-\frac{\alpha}{2}}(1+x)^{\frac{1}{2}} \left[(1-x)^{\alpha+1} f'(x)\right]' \in L^2(-1,1), \\ \qquad \lim_{x \to 1^-} (1-x)^{\alpha+1} f'(x) = 0\Big\}, \qquad \text{if } -1 < \alpha < 1 \end{cases}$$

is self-adjoint in $L^2_{\alpha,-1}(-1,1)$. By theorem 6.1, we have Green's formula

$$\begin{aligned} (A_k^{(\alpha,-1)} f, g)_{\alpha,-1} &= \int_{-1}^{1} l_{\alpha,-1}[f](x)\overline{g}(x)(1-x)^{\alpha}(1+x)^{-1} dx \\ &= \int_{-1}^{1} f(x)\overline{l_{\alpha,-1}[g]}(x)(1-x)^{\alpha}(1+x)^{-1} dx \\ &= (f, A_k^{(\alpha,-1)} g)_{\alpha,-1} \end{aligned}$$

and Dirichlet's formula

$$\begin{aligned} (A_k^{(\alpha,-1)} f, g)_{\alpha,-1} &= \int_{-1}^{1} l_{\alpha,-1}[f](x)\overline{g}(x)(1-x)^{\alpha}(1+x)^{-1} dx \\ &= \int_{-1}^{1} \left[(1-x)^{\alpha+1} f'(x)\overline{g}'(x) + k(1-x)^{\alpha}(1+x)^{-1} f(x)\overline{g}(x)\right] dx. \end{aligned}$$

In particular,

$$\begin{aligned} (A_k^{(\alpha,-1)} f, f)_{\alpha,-1} &= \int_{-1}^{1} \left[(1-x)^{\alpha+1} \left|f'(x)\right|^2 + k(1-x)^{\alpha}(1+x)^{-1} \left|f(x)\right|^2\right] dx \\ &\geq k(f, f)_{\alpha,-1}, \end{aligned}$$

for all $f, g \in \mathcal{D}(A_k^{(\alpha,-1)})$, i.e. $A_k^{(\alpha,-1)}$ is bounded below in $L^2_{\alpha,-1}(-1,1)$ by kI. (Another way to see this is to observe that $\sigma(A_k^{(\alpha,-1)}) \subset [k, \infty)$.) Thus, the left-definite theory can be applied.

6.2 Completeness Results

The initial setting is the weighted Hilbert space

$$L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right) =: L^2_{\alpha,-1}(-1,1).$$

Define

$$l[y] := \frac{1}{(1-x)^{\alpha}(1+x)^{-1}}\left[-\left((1-x)^{\alpha+1}y'(x)\right)' + k(1-x)^{\alpha}(1+x)^{-1}y(x)\right].$$

We study the second-order differential equation $l[y] = \lambda y$ in $L^2_{\alpha,-1}(-1,1)$. With $\alpha > -1, \beta = -1$, the definition of the Jacobi polynomials in (4.4) becomes

$$P_n^{(\alpha,-1)}(x) := \sum_{j=0}^{n}\binom{n+\alpha}{j}\binom{n-1}{n-j}\left(\frac{x-1}{2}\right)^{j}\left(\frac{x+1}{2}\right)^{n-j},$$

and we note that the first Jacobi polynomial $P_1^{(\alpha,-1)}(x)$ is degenerate. However, any multiple of the first degree polynomial $y = x+1$ will solve the Jacobi differential equation. Therefore, we redefine $P_1^{(\alpha,-1)}(x)$ and normalize the sequence of Jacobi polynomials as follows:

Definition 6.1.

$$P_0^{(\alpha,-1)}(x) := 1$$

$$P_1^{(\alpha,-1)}(x) := \sqrt{\frac{(\alpha+1)(\alpha+2)}{2^{\alpha+2}}}\,(x+1)$$

and, for $n \geq 2$,

$$P_n^{(\alpha,-1)}(x) := \sqrt{\frac{n\,(2n+\alpha)}{2^{\alpha}\,(n+\alpha)}}\sum_{j=0}^{n}\binom{n+\alpha}{n-j}\binom{n-1}{j}\left(\frac{x-1}{2}\right)^{j}\left(\frac{x+1}{2}\right)^{n-j}.$$

With this definition of the Jacobi polynomials for $\alpha > -1, \beta = -1$, the sequence $\left\{P_n^{(\alpha,-1)}\right\}_{n=1}^{\infty}$ forms a complete *orthonormal* set in $L^2_{\alpha,-1}(-1,1)$, see lemma 6.10. Note that $P_0^{(\alpha,-1)}(x) \notin L^2_{\alpha,-1}(-1,1)$ due to the singularity in the weight function, but unlike in the special case, $P_1^{(\alpha,-1)}(x) \in L^2_{\alpha,-1}(-1,1)$.

To see that the Jacobi polynomials are orthonormal with respect to a Sobolev inner product, we renormalize them for the next two results:

Definition 6.2.

$$\widetilde{P}_0^{(\alpha,-1)}(x) := 1$$

$$\widetilde{P}_1^{(\alpha,-1)}(x) := \left(\frac{\alpha+2}{2^{\alpha+2}}\right)^{1/2}(x+1)$$

and, for $n \geq 2$,

$$\widetilde{P}_n^{(\alpha,-1)}(x) := \frac{(2n+\alpha)^{\frac{1}{2}}}{2^{n+\alpha/2}\,(n+\alpha)}\sum_{j=0}^{n}\binom{n+\alpha}{n-j}\binom{n-1}{j}\left(\frac{x-1}{2}\right)^{j}\left(\frac{x+1}{2}\right)^{n-j}.$$

Lemma 6.8. *For $n \geq 2$,*

$$\widetilde{P}_n^{(\alpha,-1)}(x) = \frac{(n+\alpha)!(n-1)!}{2n!(n+\alpha-1)!}(x+1)\widetilde{P}_{n-1}^{(\alpha,1)}(x).$$

In particular, $\widetilde{P}_n^{(\alpha,-1)}(-1) = 0$.

We shall use this lemma to prove that the Jacobi polynomials for $\beta = -1, \alpha > -1$ are orthonormal with respect to a Sobolev inner product.

Theorem 6.2. *The Jacobi polynomials $\left\{P_n^{(\alpha,-1)}(x)\right\}_{n=0}^{\infty}$ are orthonormal with respect to the Sobolev inner product*

$$\phi\,(f,g) := f(-1)\overline{g}(-1) + \int_{-1}^{1}(1-x)^{\alpha+1}f'(x)\overline{g}'(x)dx,$$

i.e.

$$\phi\left(\widetilde{P}_n^{(\alpha,-1)}, \widetilde{P}_m^{(\alpha,-1)}\right) = \delta_{nm} \qquad (n,m \in \mathbb{N}_0).$$

Proof. A calculation shows that

$$\phi\left(\widetilde{P}_0^{(\alpha,-1)}, \widetilde{P}_0^{(\alpha,-1)}\right) = \phi\left(\widetilde{P}_1^{(\alpha,-1)}, \widetilde{P}_1^{(\alpha,-1)}\right) = 1.$$

For $n = 0, m = 1$,

$$\phi\left(\widetilde{P}_0^{(\alpha,-1)}, \widetilde{P}_1^{(\alpha,-1)}\right) = 0.$$

Let $n = 0, m \geq 2$, and use lemma 6.1 to see that

$$\phi\left(\widetilde{P}_0^{(\alpha,-1)}, \widetilde{P}_m^{(\alpha,-1)}\right) = 0.$$

For $n = 1, m \geq 2$: we recall from [46] that

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} x^j P_m^{(\alpha,\beta)}(x)dx = 0$$

for $j = 0, 1, ..., m-1$ and $\alpha, \beta > -1$. Applying this result and lemma 6.1 shows that

$$\phi\left(\widetilde{P}_1^{(\alpha,-1)}, \widetilde{P}_m^{(\alpha,-1)}\right) = 0.$$

For $n, m \geq 2$,

$$\begin{aligned}\phi\left(\widetilde{P}_n^{(\alpha,-1)}, \widetilde{P}_m^{(\alpha,-1)}\right) &= \widetilde{P}_n^{(\alpha,-1)}(-1)\overline{\widetilde{P}_m^{(\alpha,-1)}}(-1) \\ &\quad + \int_{-1}^{1} (1-x)^{\alpha+1} \left(\widetilde{P}_n^{(\alpha,-1)}(x)\right)' \left(\overline{\widetilde{P}_m^{(\alpha,-1)}}(x)\right)' dx\end{aligned}$$

The first summand vanishes by the previous lemma. Note that $\left(\widetilde{P}_n^{(\alpha,-1)}\right)'$ reduces to a Jacobi polynomial with classical parameters [10], p. 149,

$$\frac{d}{dx}\widetilde{P}_n^{(\alpha,-1)}(x) = \frac{1}{2}(n+\alpha)\widetilde{P}_{n-1}^{(\alpha+1,0)}(x)$$

so that

$$\begin{aligned}\phi\left(\widetilde{P}_n^{(\alpha,-1)}, \widetilde{P}_m^{(\alpha,-1)}\right) &= \frac{(2n+\alpha)^{1/2}}{2^{\alpha/2}\,(n+\alpha)} \frac{(2m+\alpha)^{1/2}}{2^{\alpha/2}\,(m+\alpha)} \\ &\quad \times \int_{-1}^{1} (1-x)^{\alpha+1} \left(\widetilde{P}_n^{(\alpha,-1)}(x)\right)' \left(\overline{\widetilde{P}_m^{(\alpha,-1)}}(x)\right)' dx \\ &= \left(\frac{n+\alpha}{2}\right)^2 \frac{(2n+\alpha)^{1/2}}{2^{\alpha/2}\,(n+\alpha)} \frac{(2m+\alpha)^{1/2}}{2^{\alpha/2}\,(m+\alpha)} \\ &\quad \times \int_{-1}^{1} (1-x)^{\alpha+1} \left(\widetilde{P}_{n-1}^{(\alpha+1,0)}(x)\right)' \left(\overline{\widetilde{P}_{m-1}^{(\alpha+1,0)}}(x)\right)' dx \\ &= \begin{cases} 0 & \text{if } n \neq m \\ \left(\frac{n+\alpha}{2}\right)^2 \frac{2n+\alpha}{2^{\alpha}(n+\alpha)^2} \frac{2^{\alpha+2}}{2n+\alpha} = 1 & \text{if } n = m \end{cases} \\ &= \delta_{nm}.\end{aligned}$$

□

From the theory of orthogonal polynomials, it is well known that the classical Jacobi polynomials are dense in a corresponding Hilbert space:

Lemma 6.9. *The sequence* $\left\{P_n^{(\alpha,1)}(x)\right\}_{n=0}^{\infty}$ *forms a complete orthogonal set in the Hilbert space* $L^2\left((-1,1);(1-x)^\alpha(1+x)\right)$.

We shall use this result to prove that the truncated sequence of non-classical Jacobi polynomials are dense in $L^2\left((-1,1);(1-x)^\alpha(1+x)^{-1}\right)$.

Lemma 6.10. *The sequence* $\left\{P_n^{(\alpha,-1)}(x)\right\}_{n=1}^{\infty}$ *forms a complete orthogonal set in the Hilbert space* $L^2\left((-1,1);(1-x)^\alpha(1+x)^{-1}\right)$. *Equivalently, the set of all polynomials* $P\in\mathcal{P}[-1,1]$ *of degree* ≥ 1 *satisfying* $p(-1)=0$ *is dense in the space* $L^2\left((-1,1);(1-x)^\alpha(1+x)^{-1}\right)$.

Proof. We have

$$\int_{-1}^{1}|f(x)|^2(1-x)^\alpha(1+x)^{-1}dx=\int_{-1}^{1}\left|(1+x)^{-1}f(x)\right|^2(1-x)^\alpha(1+x)dx,$$

i.e.

$$f\in L^2\left((-1,1);(1-x)^\alpha(1+x)^{-1}\right)\iff(1+x)^{-1}f\in L^2\left((-1,1);(1-x)^\alpha(1+x)\right),$$

and in this case,

$$\|f\|_{L^2((-1,1);(1-x)^\alpha(1+x)^{-1})}=\left\|(1+x)^{-1}f\right\|_{L^2((-1,1);(1-x)^\alpha(1+x))}.$$

Let $f\in L^2\left((-1,1);(1-x)^\alpha(1+x)^{-1}\right)$, and let $\epsilon>0$. Hence

$$(1+x)^{-1}f\in L^2\left((-1,1);(1-x)^\alpha(1+x)\right),$$

so by lemma 6.9, there exists $q\in\mathcal{P}[-1,1]$ such that

$$\left\|(1+x)^{-1}f-q\right\|_{L^2((-1,1);(1-x)^\alpha(1+x))}<\epsilon.$$

Let $p(x) := (1+x)q(x)$, so $\deg(p) \geq 1$ and $p(-1) = 0$. Then $q(x) = (1+x)^{-1}p(x)$. Hence

$$\begin{aligned}\epsilon &> \left\|(1+x)^{-1}f - (1+x)^{-1}p\right\|_{L^2((-1,1);(1-x)^{\alpha}(1+x))} \\ &= \left\|(1+x)^{-1}(f-p)\right\|_{L^2((-1,1);(1-x)^{\alpha}(1+x))} \\ &= \|f-p\|_{L^2((-1,1);(1-x)^{\alpha}(1+x)^{-1})}\,.\end{aligned}$$

□

Remark 6.1. We note that this property, that is the completeness of $\left\{P_n^{(\alpha,-1)}(x)\right\}_{n=1}^{\infty}$ in $L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$, distinguishes the special case where $\alpha = \beta = -1$ from the general case $\alpha > -1, \beta = -1$ which is considered here. In the special case, the set $\left\{P_n^{(-1,-1)}(x)\right\}_{n=2}^{\infty}$ is complete in $L^2\left((-1,1);(1-x^2)^{-1}\right)$, see lemma 5.6.

6.3 Left-Definite Spectral Analysis

Definition 6.3. Let $k > 0$. For each $n \in \mathbb{N}$, define

$$V_n^{(\alpha,-1)} := \left\{f : (-1,1) \longrightarrow \mathbb{C} \,\middle|\, f \in AC_{loc}^{(n-1)};\, f^{(j)} \in L^2_{(\alpha+j,j-1)}(-1,1),\, j = 0,...,n\right\}$$

and let $(\cdot,\cdot)_{n,k}^{(\alpha,-1)}$ and $\|\cdot\|_{n,k}^{(\alpha,-1)}$ denote the Sobolev inner product

$$(f,g)_{n,k}^{(\alpha,-1)} := \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k)\int_{-1}^{1} f^{(j)}(t)\overline{g}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{j-1}dt \qquad (f,g \in V_n^{(\alpha,-1)})$$

and the norm $\|f\|_{n,k}^{(\alpha,-1)} := \left((f,f)_{n,k}^{(\alpha,-1)}\right)^{1/2}$, where the numbers $c_j^{(\alpha,-1)}(n,k)$ are defined in (4.6) and (4.7) in section 4.2. Let

$$W_{n,k}^{(\alpha,-1)}(-1,1) := \left(V_n^{(\alpha,-1)}, (\cdot,\cdot)_{n,k}^{(\alpha,-1)}\right).$$

In this section, it is our goal to show that $W_{n,k}^{(\alpha,-1)}(-1,1)$ is the n^{th} left-definite space associated with the pair $\left(L^2_{\alpha,-1}(-1,1), A_k^{(\alpha,-1)}\right)$, where $A_k^{(\alpha,-1)}$ is the self-adjoint Jacobi operator defined in (6.8).

Theorem 6.3. *Let $k > 0$. For each $n \in \mathbb{N}$, $W_{n,k}^{(\alpha,-1)}(-1,1)$ is a Hilbert space.*

Proof. Let $n \in \mathbb{N}$, and let $\{f_m\}_{m=1}^{\infty}$ be a Cauchy sequence in $W_{n,k}^{(\alpha,-1)}(-1,1)$. Then, since the numbers $c_j^{(\alpha,-1)}(n,k) \geq 0$,

$$\begin{aligned} \left(\|f_m - f_r\|_{n,k}^{(\alpha,-1)}\right)^2 &= \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k) \left\|f_m^{(j)} - f_r^{(j)}\right\|_{\alpha+j,j-1}^2 \\ &\geq c_n^{(\alpha,-1)}(n,k) \left\|f_m^{(n)} - f_r^{(n)}\right\|_{\alpha+j,j-1}^2 \end{aligned} \tag{6.9}$$

for any $j = 0, 1, ..., n$ and $f \in V_n^{(\alpha,-1)}$, so $\left\{f_m^{(n)}\right\}_{m=1}^{\infty}$ is a Cauchy sequence in $L^2_{\alpha+n,n-1}(-1,1)$, and hence there exists a $g_{n+1} \in L^2_{\alpha+n,n-1}(-1,1)$ such that

$$f_m^{(n)} \longrightarrow g_{n+1} \tag{6.10}$$

in $L^2_{\alpha+n,n-1}(-1,1)$ as $m \longrightarrow \infty$. In particular, $g_{n+1} \in L^1_{loc}(-1,1)$. Fix $t, t_0 \in (-1,1)$ such that $t_0 \leq t$. Then, by Hölder's inequality,

$$\begin{aligned} &\int_{t_0}^{t} \left|f_m^{(n)}(u) - g_{n+1}(u)\right| du \\ &= \int_{t_0}^{t} \left|f_m^{(n)}(u) - g_{n+1}(u)\right| (1-u)^{(\alpha+n)/2}(1+u)^{(n-1)/2}(1-u)^{-(\alpha+n)/2}(1+u)^{-(n-1)/2} du \\ &\leq \left(\int_{t_0}^{t} \left|f_m^{(n)}(u) - g_{n+1}(u)\right| (1-u)^{\alpha+n}(1+u)^{n-1} du\right)^{1/2} \\ &\qquad \times \left(\int_{t_0}^{t} (1-u)^{-(\alpha+n)}(1+u)^{-(n-1)} du\right)^{1/2} \\ &= M(t-t_0) \left(\int_{t_0}^{t} \left|f_m^{(n)}(u) - g_{n+1}(u)\right| (1-u)^{\alpha+n}(1+u)^{n-1} du\right)^{1/2} \longrightarrow 0 \end{aligned}$$

as $m \longrightarrow \infty$ by (6.10), i.e.

$$\int_{t_0}^{t} f_m^{(n)}(u) du \longrightarrow \int_{t_0}^{t} g_{n+1}(u) du \tag{6.11}$$

as $m \longrightarrow \infty$. Now, since $f_m^{(n-1)} \in AC_{loc}(-1,1)$, we can integrate in (6.11):

$$f_m^{(n-1)}(t) - f_m^{(n-1)}(t_0) = \int_{t_0}^{t} f_m^{(n)}(u)du \longrightarrow \int_{t_0}^{t} g_{n+1}(u)du. \tag{6.12}$$

Also, from (6.9), it follows that $\left\{ f_m^{(n-1)} \right\}_{m=1}^{\infty}$ is Cauchy in $L^2_{\alpha+n-1,n-2}(-1,1)$. Hence, there exists a $g_n \in L^2_{\alpha+n-1,n-2}(-1,1)$ such that

$$f_m^{(n-1)} \longrightarrow g_n$$

in $L^2_{\alpha+n-1,n-2}(-1,1)$. Repeating the above argument we see that $g_n \in L^1_{loc}(-1,1)$, and, for $t, t_1 \in (-1,1)$,

$$f_m^{(n-2)}(t) - f_m^{(n-2)}(t_1) = \int_{t_1}^{t} f_m^{(n-1)}(u)du \longrightarrow \int_{t_1}^{t} g_n(u)du. \tag{6.13}$$

By Riesz-Fischer, there exists a subsequence $\left\{ f_{m_{k,n-1}}^{(n-1)} \right\}_{m=1}^{\infty}$ of $\left\{ f_m^{(n-1)} \right\}_{m=1}^{\infty}$ such that

$$f_{m_{k,n-1}}^{(n-1)}(t) \longrightarrow g_n(t)$$

for a.e. $t \in (-1,1)$. Choose $t_0 \in (-1,1)$ in (6.12) such that $f_{m_{k,n-1}}^{(n-1)}(t_0) \longrightarrow g_n(t_0)$ and then pass through the subsequence in (6.12) to obtain

$$g_n(t) - g_n(t_0) = \int_{t_0}^{t} g_{n+1}(u)du$$

for a.e. $t \in (-1,1)$. This is to say that $g_n \in AC_{loc}(-1,1)$, and

$$g_n'(t) = g_{n+1}(t)$$

for a.e. $t \in (-1,1)$. Again, from (6.9), we see that $\left\{ f_m^{(n-2)} \right\}_{m=1}^{\infty}$ is Cauchy in $L^2_{\alpha+n-2,n-3}(-1,1)$, implying that there exists a $g_{n-1} \in L^2_{\alpha+n-2,n-3}(-1,1)$ such that

$$f_m^{(n-2)} \longrightarrow g_{n-1}$$

in $L^2_{\alpha+n-2,n-3}(-1,1)$. Again, $g_{n-1} \in L^1_{loc}(-1,1)$, and, for any $t, t_2 \in (-1,1)$,

$$f_m^{(n-3)}(t) - f_m^{(n-3)}(t_2) = \int_{t_2}^{t} f_m^{(n-2)}(u)du \longrightarrow \int_{t_2}^{t} g_{n-1}(u)du$$

and there exists a subsequence $\left\{f_{m_{k,n-2}}^{(n-2)}\right\}_{m=1}^{\infty}$ of $\left\{f_m^{(n-2)}\right\}_{m=1}^{\infty}$ such that

$$f_{m_k,n-2}^{(n-2)}(t) \longrightarrow g_{n-1}(t)$$

for a.e. $t \in (-1,1)$. In (6.13), choose t_1 such that $f_{m_k,n-2}^{(n-2)}(t_1) \longrightarrow g_{n-1}(t_1)$ and then pass through the subsequence in (6.13) to get

$$g_{n-1}(t) - g_{n-1}(t_1) = \int_{t_1}^{t} g_n(u)du$$

for a.e. $t \in (-1,1)$, i.e. $g_{n-1} \in AC_{loc}^{(1)}(-1,1)$, and $g_{n-1}''(t) = g_n'(t) = g_{n+1}(t)$ for a.e. $t \in (-1,1)$. Continuing in this manner, we obtain $n+1$ functions $g_{n-j+1} \in L^2_{\alpha+n-j,n-j-1}(-1,1)$ for $j = 0,1,...,n$ such that

(1) $f_m^{(n-j)} \longrightarrow g_{n-j+1}$ in $L^2_{\alpha+n-j,n-j-1}(-1,1)$, for $j = 0,1,...,n$

(2) $g_1 \in AC_{loc}^{(n-1)}(-1,1)$, $g_2 \in AC_{loc}^{(n-2)}(-1,1), ..., g_n \in AC_{loc}(-1,1)$

(3) $g_{n-j}'(t) = g_{n-j+1}'(t)$ for a.e. $t \in (-1,1)$, $j = 0,1,...,n-1$

(4) $g_1^{(j)} = g_{j+1}, j = 0,1,...,n.$

In particular, $f_m^{(j)} \longrightarrow g_1^{(j)}$ in $L^2_{\alpha+j,j-1}(-1,1)$ for $j = 0,1,...,n$ and $g_1 \in V_n^{(\alpha,-1)}$. Hence,

$$\begin{aligned}\left(\|f_m - g_1\|_{n,k}^{(\alpha,-1)}\right)^2 &= \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k) \\ &\quad \times \int_{-1}^{1} \left|f_m^{(j)}(u) - g_1^{(j)}(u)\right|^2 (1-u)^{\alpha+j}(1+u)^{j-1}du \\ &= \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k) \left\|f_m^{(j)} - g_1^{(j)}\right\|^2_{\alpha+j,j-1} \longrightarrow 0\end{aligned}$$

as $m \longrightarrow \infty$, i.e. $W_{n,k}^{(\alpha,-1)}(-1,1)$ is complete.

□

Definition 6.4.

$$W_\alpha := \left\{ f : [-1,1) \longrightarrow \mathbb{C} \mid f \in AC\,[-1,1)\,; f' \in L^2\left((-1,1);(1-x)^{\alpha+1}\right)\right\}$$

Lemma 6.11. $V_1 \subseteq W_{\alpha,1} := \{f \in W_\alpha \mid f(-1) = 0\}$

Proof. $V_1 \subseteq W_{\alpha,1}$: Let $f \in V_1$. We know that $V_1 \subset \Delta$, and by lemma 6.3 $f(-1) = 0$ for all $f \in \Delta$, so $f \in W_{\alpha,1}$. □

Theorem 6.4. *The Jacobi polynomials* $\left\{P_m^{(\alpha,-1)}\right\}_{m=1}^{\infty}$ *form a complete orthogonal set in each* $W_{n,k}^{(\alpha,-1)}(-1,1)$. *Equivalently, the set of polynomials,* $\mathcal{P}$, *is dense in* $W_{n,k}^{(\alpha,-1)}(-1,1)$.

Proof. Fix $n \in \mathbb{N}$, and let $f \in W_{n,k}^{(\alpha,-1)}(-1,1)$, so

$$f^{(n)} \in L^2\left((-1,1);(1-x)^{\alpha+n}(1+x)^{n-1}\right).$$

Since $\left\{P_m^{(\alpha+n,n-1)}\right\}_{m=0}^{\infty}$ is a complete orthonormal set in $L^2_{\alpha+n,n-1}(-1,1)$, we know

$$\sum_{m=0}^{r} c_{m,n}^{(\alpha,-1)} P_m^{(\alpha+n,n-1)} \to f^{(n)} \quad \text{as } r \to \infty \text{ in } L^2\left((-1,1);(1-x)^{\alpha+n}(1+x)^{n-1}\right) \tag{6.14}$$

where $c_{m,n}^{(\alpha,-1)}$ are the Fourier coefficients given by

$$c_{m,n}^{(\alpha,-1)} = \int_{-1}^{1} f^{(n)}(t) P_m^{(\alpha+n,n-1)}(t)(1-t)^{\alpha+n}(1+t)^{n-1}dt$$

for $m \in \mathbb{N}_0$. For $r \geq n$ define the polynomials

$$p_r(t) := \sum_{m=\max\{2,n\}}^{r} \frac{c_{m-n,n}^{(\alpha,-1)}\left((m-n)!\right)^{1/2}\left(\Gamma(\alpha+m)\right)^{1/2}}{(m!)^{1/2}\left(\Gamma(\alpha+m+n)!\right)^{1/2}} P_m^{(\alpha,-1)}(t).$$

From

$$\frac{d^j}{dt^j} P_m^{(\alpha,-1)}(t) = \frac{(m!)^{1/2}\left(\Gamma(\alpha+m+j)\right)^{1/2}}{\left((m-j)!\right)^{1/2}\left(\Gamma(\alpha+m)\right)^{1/2}} P_{m-j}^{(\alpha+j,j-1)}(t),$$

we see that, for $j = 0, 1, ..., n$,

$$p_r^{(j)}(t) = \sum_{m=\max\{2,n\}}^{r} \frac{c_{m-n,n}^{(\alpha,-1)}\left((m-n)!\right)^{1/2}\left(\Gamma(\alpha+m+j)!\right)^{1/2}}{\left(\Gamma(\alpha+m+n)!\right)^{1/2}\left((m-j)!\right)^{1/2}} P_{m-j}^{(\alpha+j,j-1)}(t).$$

In particular, by (6.14),

$$\begin{aligned} p_r^{(n)}(t) &= \sum_{m=\max\{2,n\}}^{r} c_{m-n,n}^{(\alpha,-1)} P_{m-n}^{(\alpha+n,n-1)} \\ &= \sum_{l=0}^{r-\max\{2,n\}} c_{l,n}^{(\alpha,-1)} P_l^{(\alpha+n,n-1)} \\ &= \sum_{m=0}^{s} c_{m,n}^{(\alpha,-1)} P_m^{(\alpha+n,n-1)} \to f^{(n)} \end{aligned}$$

as $r \to \infty$ in $L^2\left((-1,1);(1-x)^{\alpha+n}(1+x)^{n-1}\right)$. Furthermore, by Riesz-Fischer, there exists a subsequence $\left\{p_{r_j}^{(n)}\right\}$ of $\left\{p_r^{(n)}\right\}$ such that

$$p_{r_j}^{(n)} \to f^{(n)} \qquad \text{for a.e. } t \in (-1,1).$$

By Dirichlet's test, the sequence

$$\left\{ \frac{c_{m-n,n}^{(\alpha,-1)} \left((m-n)!\right)^{1/2} \left(\Gamma(\alpha+m+j)!\right)^{1/2}}{\left(\Gamma(\alpha+m+n)!\right)^{1/2} \left((m-j)!\right)^{1/2}} \right\} \in \ell^2,$$

so there exists a $g_j \in L^2\left((-1,1);(1-x)^{\alpha+j}(1+x)^{j-1}\right)$ such that

$$p_{r_j} \longrightarrow g_j \qquad \text{in } L^2\left((-1,1);(1-x)^{\alpha+j}(1+x)^{j-1}\right). \tag{6.15}$$

For a.e. $a, t \in (-1,1)$,

$$\int_a^t p_{r_j}^{(n)}(u)du \longrightarrow \int_a^t f^{(n)}(u)du.$$

Integrate both sides and obtain

$$p_{r_j}^{(n-1)}(t) \longrightarrow f^{(n-1)}(t) + c_1 \qquad \text{for a.e. } t \in (-1,1) \tag{6.16}$$

for some constant c_1. Passing through the subsequence implies

$$g_{n-1}(t) = f^{(n-1)}(t) + c_1 \qquad \text{for a.e. } t \in (-1,1).$$

From (6.16), we see that

$$\int_a^t p_{r_j}^{(n-1)}(u)du \longrightarrow \int_a^t f^{(n-1)}(u)du + c_1 \int_a^t du,$$

i.e.

$$p_{r_j}^{(n-2)}(t) \longrightarrow f^{(n-2)}(t) + c_1 t + c_2 \qquad \text{for a.e. } t \in (-1,1)$$

or

$$g_{n-2}(t) = f^{(n-2)}(t) + c_1 t + c_2 \qquad \text{for a.e. } t \in (-1,1).$$

Continue this process to see that for $j \in \{0, 1, ..., n-1\}$,

$$g_j(t) = f^{(j)}(t) + q_{n-j+1} \qquad \text{for a.e. } t \in (-1,1),$$

where q_{n-j-1} is a polynomial of degree $\leq n-j-1$ and where

$$q'_{n-j-1} = q_{n-j-2}.$$

Hence, with (6.15),

$$p_r^{(j)} \longrightarrow f^{(j)} + q_{n-j-1} \qquad \text{in } L^2\left((-1,1); (1-x)^{\alpha+j}(1+x)^{j-1}\right). \tag{6.17}$$

For $r \geq n$, define

$$\pi_r(t) := p_r(t) - q_{n-1}(t).$$

Note that, with (6.17),

$$\pi_r^{(j)}(t) = p_r^{(j)}(t) - q_{n-1}^{(j)}(t) = p_r^{(j)}(t) - q_{n-j-1}(t) \longrightarrow f^{(j)}(t).$$

Now,

$$\left(\|f - \pi_r\|_{n,k}^{(\alpha,-1)}\right)^2 = \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k) \times \int_{-1}^{-1} \left|f^{(j)}(t) - \pi_r^{(j)}(t)\right|^2 (1-t)^{\alpha+j}(1+t)^{j-1} dt \longrightarrow 0$$

as $r \longrightarrow \infty$. □

The following lemma holds for $n \geq 1$!!

Lemma 6.12. *For $p, q \in \mathcal{P}$,*

$$(p,q)_{n,k}^{(\alpha,-1)} = \left(\left(A_k^{(\alpha,-1)}\right)^n p, q\right)_{\alpha,-1}.$$

Proof. First we note that this may be restated as

$$\left(l^n_{\alpha,-1}[p],q\right)_{\alpha,-1} = \int\limits_{-1}^{1} l^n_{\alpha,-1}[p](x)\overline{q}(x)w_{\alpha,-1}(x)dx$$
$$= \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k)p^{(j)}(x)\overline{q}^{(j)}(x)(1-x)^{j+\alpha}(1+x)^{j-1}dx. \qquad (6.18)$$

Since the Jacobi polynomials form a basis for $\mathcal{P}$, it suffices to prove (6.18) for $p = P_m^{(\alpha,-1)}$ and $q = P_r^{(\alpha,-1)}$ for arbitrary $m, r \in \mathbb{N}_0$. From

$$l^n_{\alpha,-1}[P_m^{(\alpha,-1)}](x) = (m(m-1)+k)^n P_m^{(\alpha,-1)}(x) \quad (m \in \mathbb{N}_0)$$

and

$$\left(P_r^{(\alpha,-1)}, P_m^{(\alpha,-1)}\right)_{\alpha,-1} = \delta_{r,m} \quad (r, m \in \mathbb{N}_0)\,,$$

the left-hand side of (6.18) becomes

$$\left(l^n_{\alpha,-1}[P_m^{(\alpha,-1)}], P_r^{(\alpha,-1)}\right)_{\alpha,-1} = \int\limits_{-1}^{1} l^n_{\alpha,-1}[P_m^{(\alpha,-1)}](x)\overline{P_r^{(\alpha,-1)}}(x)w_{\alpha,-1}(x)dx$$
$$= (m(m-1)+k)^n\delta_{r,m}. \qquad (6.19)$$

Upon using (4.3) for $\alpha > -1, \beta = -1$ and the recurrence relation for the $c_j^{(\alpha,-1)}(n,k)$, that is,

$$(m(m+\alpha)+k)^n = \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k)\frac{m!(m+\alpha+j-1)!}{(m-j)!(m+\alpha-1)!}$$

the right-hand side of (6.18) becomes

$$\sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k)\left(P_m^{(\alpha,-1)}(x)\right)^{(j)}(x)\left(\overline{P_r^{(\alpha,-1)}}(x)\right)^{(j)}(x)(1-x)^{j-1}(1+x)^{j-1}dx$$
$$= \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k)\frac{m!(m+\alpha+j-1)!}{(m-j)!(m+\alpha-1)!}\delta_{r,m} \qquad (6.20)$$
$$= (m(m+\alpha)+k)^n\delta_{r,m}.$$

Comparing (6.19) and (6.20) completes the proof of the lemma. $\square$

Theorem 6.5. *For $k>0$, let*

$$A_k^{(\alpha,-1)}:\mathcal{D}\left(A_k^{(\alpha,-1)}\right)\subset L^2_{\alpha,-1}(-1,1)\longrightarrow L^2_{\alpha,-1}(-1,1)$$

be the Jacobi self-adjoint operator having the Jacobi polynomials $\left\{P_m^{(\alpha,-1)}\right\}_{m=1}^{\infty}$ as eigenfunctions. For each $n\in\mathbb{N}$, let

$$V_n^{(\alpha,-1)}:=\left\{f:(-1,1)\longrightarrow\mathbb{C}\,\middle|\,f\in AC_{loc}^{(n-1)};f^{(j)}\in L^2_{(\alpha+j,j-1)}(-1,1),j=0,...,n\right\}$$

and

$$(f,g)_{n,k}^{(\alpha,-1)}:=\sum_{j=0}^{n}c_j^{(\alpha,-1)}(n,k)\int_{-1}^{1}f^{(j)}(t)\overline{g}^{(j)}(t)(1-t)^{\alpha+j}(1+t)^{j-1}dt\qquad(f,g\in V_n^{(\alpha,-1)}).$$

Then $W_{n,k}^{(\alpha,-1)}(-1,1):=\left(V_n^{(\alpha,-1)},(\cdot,\cdot)_{n,k}^{(\alpha,-1)}\right)$ is the n^{th} left-definite space associated with $\left(L^2_{\alpha,-1}(-1,1),A_k^{(\alpha,-1)}\right)$. Moreover, the Jacobi polynomials $\left\{P_m^{(\alpha,-1)}\right\}_{m=1}^{\infty}$ form a complete orthogonal set in each $W_{n,k}^{(\alpha,-1)}(-1,1)$, and they satisfy the orthogonality relation

$$\left(P_m^{(\alpha,-1)},P_l^{(\alpha,-1)}\right)_{n,k}=(m(m-1)+k)^n\delta_{m,l}.$$

Furthermore, define

$$B_{n,k}^{(\alpha,-1)}:=\mathcal{D}\left(B_{n,k}^{(\alpha,-1)}\right)\subset W_{n,k}^{(\alpha,-1)}(-1,1)\longrightarrow W_{n,k}^{(\alpha,-1)}(-1,1)$$

by

$$B_{n,k}^{(\alpha,-1)}f:=l\,[f]\qquad\left(f\in\mathcal{D}\left(B_{n,k}^{(\alpha,-1)}\right):=V_{n+2}^{(\alpha,-1)}\right).$$

Then $B_{n,k}^{(\alpha,-1)}$ is the n^{th} left-definite operator associated with $\left(L^2_{\alpha,-1}(-1,1),A_k^{(\alpha,-1)}\right)$. Lastly, the spectrum of $B_{n,k}^{(\alpha,-1)}$ is given by

$$\sigma\left(B_{n,k}^{(\alpha,-1)}\right)=\{m(m-1)+k\mid m\in\mathbb{N}_0\}=\sigma\{A_k^{(\alpha,-1)}\},$$

and the Jacobi polynomials $\left\{P_m^{(\alpha,-1)}\right\}_{m=1}^{\infty}$ form a complete set of eigenfunctions of each $B_{n,k}^{(\alpha,-1)}$.

Proof. Let $n \in \mathbb{N}$. We need to show that $W_{n,k}^{(\alpha,-1)}(-1,1)$ satisfies the five properties in definition 3.1.

(i) $W_{n,k}^{(\alpha,-1)}(-1,1)$ is a Hilbert space (see theorem 6.3).

(ii) We need to show:

$$\mathcal{D}\left(\left(A_k^{(\alpha,-1)}\right)^n\right) \subset W_{n,k}^{(\alpha,-1)}(-1,1) \subset L^2\left((-1,1);(1-x)^\alpha(1+x)^{-1}\right).$$

Let $f \in \mathcal{D}\left(\left(A_k^{(\alpha,-1)}\right)^n\right)$. Since the Jacobi polynomials $\left\{P_m^{(\alpha,-1)}\right\}_{m=1}^{\infty}$ form a complete orthonormal set in $L^2\left((-1,1);(1-x)^\alpha(1+x)^{-1}\right)$, we see that

$$p_j \longrightarrow f \qquad \text{in } L^2\left((-1,1);(1-x)^\alpha(1+x)^{-1}\right) \text{ as } j \longrightarrow \infty \tag{6.21}$$

where

$$p_j(t) := \sum_{m=0}^{j} c_m^{(\alpha,-1)} P_m^{(\alpha,-1)}(t) \qquad (t \in (-1,1)),$$

$$c_m^{(\alpha,-1)} := \left(f, P_m^{(\alpha,-1)}\right)_{\alpha,-1} = \int_{-1}^{1} f(t)P_m^{(\alpha,-1)}(t)(1-t)^\alpha(1+t)^{-1}dt \qquad (m \in \mathbb{N}_0).$$

Since $\left(A_k^{(\alpha,-1)}\right)^n f \in L^2\left((-1,1);(1-x)^\alpha(1+x)^{-1}\right)$, we see that

$$\sum_{m=0}^{j} \widetilde{c}_m^{(\alpha,-1)} P_m^{(\alpha,-1)} \longrightarrow \left(A_k^{(\alpha,-1)}\right)^n f \qquad \text{in } L^2\left((-1,1);(1-x)^\alpha(1+x)^{-1}\right)$$

as $j \longrightarrow \infty$, where

$$\begin{aligned}
\widetilde{c}_m^{(\alpha,-1)} &:= \left(\left(A_k^{(\alpha,-1)}\right)^n f, P_m^{(\alpha,-1)}\right)_{-1,-1} = \left(f, \left(A_k^{(\alpha,-1)}\right)^n P_m^{(\alpha,-1)}\right)_{\alpha,-1} \\
&= (m(m+\alpha)+k)^n \left(f, P_m^{(\alpha,-1)}\right)_{\alpha,-1} \\
&= (m(m+\alpha)+k)^n c_m^{(\alpha,-1)},
\end{aligned}$$

i.e.

$$\left(A_k^{(\alpha,-1)}\right)^n p_j \longrightarrow \left(A_k^{(\alpha,-1)}\right)^n f$$

in $L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$ as $j \longrightarrow \infty$. Moreover, by lemma 6.12,

$$\left(\|p_j - p_r\|_{n,k}^{(\alpha,-1)}\right)^2 = \left(\left(A_k^{(\alpha,-1)}\right)^n [p_j - p_r], p_j - p_r\right)_{\alpha,-1}$$

$$\longrightarrow 0 \qquad \text{as } j, r \longrightarrow \infty$$

i.e. $\{p_j\}_{j=0}^{\infty}$ is Cauchy in $W_{n,k}^{(\alpha,-1)}(-1,1)$. Since $W_{n,k}^{(\alpha,-1)}(-1,1)$ is a Hilbert space (theorem 6.3), there exists

$$g \in W_{n,k}^{(\alpha,-1)}(-1,1) \subset L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$$

such that

$$p_j \longrightarrow g \qquad \text{in } W_{n,k}^{(\alpha,-1)}(-1,1) \text{ as } j \longrightarrow \infty.$$

Furthermore, since

$$(f,f)_{n,k}^{(\alpha,-1)} \geq k^n (f,f)_{\alpha,-1} \qquad \left(f \in W_{n,k}^{(\alpha,-1)}(-1,1)\right),$$

[this is due to

$$\begin{aligned}(f,f)_{n,k}^{(\alpha,-1)} &= \sum_{j=0}^{n} c_j^{(\alpha,-1)}(n,k) \left\|f^{(j)}\right\|_{j+\alpha,j-1}^2 \\ &\geq c_0^{(\alpha,-1)}(n,k) \left\|f^{(j)}\right\|_{\alpha,-1}^2 \\ &= k^n (f,f)_{\alpha,-1} \quad \left(f \in W_{n,k}^{(\alpha,-1)}(-1,1)\right)\end{aligned}$$

from the positivity of the coefficients $c_j^{(\alpha,-1)}(n,k)$], we see that

$$\|p_j - g\|_{\alpha,-1} \leq k^{-n/2} \|p_j - g\|_{n,k}^{(\alpha,-1)},$$

and hence,

$$p_j \longrightarrow g \qquad \text{in } L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right). \tag{6.22}$$

Comparing (6.21) and (6.22),

$$f = g \in W_{n,k}^{(\alpha,-1)}(-1,1).$$

(iii) We need to show: $\mathcal{D}\left(\left(A_k^{(\alpha,-1)}\right)^n\right)$ is dense in $W_{n,k}^{(\alpha,-1)}(-1,1)$. Since the set of polynomials is contained in $\mathcal{D}\left(\left(A_k^{(\alpha,-1)}\right)^n\right)$ and is dense in $W_{n,k}^{(\alpha,-1)}(-1,1)$ (by theorem 6.4), $\mathcal{D}\left(\left(A_k^{(\alpha,-1)}\right)^n\right)$ is dense in $W_{n,k}^{(\alpha,-1)}(-1,1)$. Furthermore, from theorem 6.4, the Jacobi polynomials $\left\{P_m^{(\alpha,-1)}\right\}_{m=1}^{\infty}$ form a complete orthonormal set in $W_{n,k}^{(\alpha,-1)}(-1,1)$.

(iv) We need to show that $(f,f)_{n,k}^{(\alpha,-1)} \geq k^n (f,f)_{\alpha,-1} \quad \forall f \in V_{n,k}^{(\alpha,-1)}$. This follows immediately from the definition of $(\cdot,\cdot)_{n,k}^{(\alpha,-1)}$.

(v) We need to show: $(f,g)_{n,k}^{(\alpha,-1)} = \left(\left(A_k^{(\alpha,-1)}\right)^n f, g\right)_{\alpha,-1}$ for $f \in \mathcal{D}\left(\left(A_k^{(\alpha,-1)}\right)^n\right)$ and $g \in V_{n,k}^{(\alpha,-1)}$. This is true for any $f,g \in P$ by lemma 6.12. Let $f \in D\left(\left(A_k^{(\alpha,-1)}\right)^n\right) \subset W_{n,k}^{(\alpha,-1)}(-1,1)$, $g \in W_{n,k}^{(\alpha,-1)}(-1,1)$. Since the set of polynomials is dense in both $W_{n,k}^{(\alpha,-1)}(-1,1)$ and $L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$, and since convergence in $W_{n,k}^{(\alpha,-1)}(-1,1)$ implies convergence in the space $L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$ (by (iv)), there exist sequences $\{p_j\}_{j=0}^{\infty}$ and $\{q_j\}_{j=0}^{\infty}$ such that

$$p_j \longrightarrow f \qquad \text{in } W_{n,k}^{(\alpha,-1)}(-1,1) \text{ as } j \longrightarrow \infty$$
$$\left(A_k^{(\alpha,-1)}\right)^n p_j \longrightarrow \left(A_k^{(\alpha,-1)}\right)^n f$$

in $L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$ as $j \longrightarrow \infty$ and

$$q_j \longrightarrow g$$

in $W_{n,k}^{(\alpha,-1)}(-1,1)$ and $L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$ as $j \longrightarrow \infty$. Hence, by lemma 6.12,

$$\begin{aligned}\left(\left(A_k^{(\alpha,-1)}\right)^n f, g\right)_{-1,-1} &= \lim_{j\longrightarrow\infty}\left(\left(A_k^{(\alpha,-1)}\right)^n p_j, q_j\right)_{\alpha,-1} \\ &= \lim_{j\longrightarrow\infty}(p_j,q_j)_{n,k} \\ &= (f,f)_{n,k}^{(\alpha,-1)}.\end{aligned}$$

The results listed in the theorem on $B_{n,k}^{(\alpha,-1)}$ and the spectrum of $B_{n,k}^{(\alpha,-1)}$ follow immediately from the general left-definite theory.

$\square$

6.4 Self-Adjoint Operators

Definition 6.5. Define

$$W_\alpha := \left\{ f : [-1,1) \longrightarrow \mathbb{C} \mid f \in AC\,[-1,1)\,;\, f' \in L^2\left((-1,1);(1-x)^{\alpha+1}\right)\right\}$$

$$\phi\,(f,g) := f(-1)\overline{g}(-1) + \int_{-1}^{1} f'(x)\overline{g}'(x)(1-x)^{\alpha+1}dx.$$

Theorem 6.6. $(W_\alpha, \phi(\cdot,\cdot))$ *is a Hilbert space.*

Proof. Let $\{f_n\} \subset W_1$ be a Cauchy sequence. Hence

$$\|f_n - f_m\|_\phi^2 = |f_n(-1) - f_m(-1)|^2 + \int_{-1}^{1} |f_n'(x) - f_m'(x)|^2 (1-x)^{\alpha+1}dx$$

$$\longrightarrow 0 \qquad \text{as } n,m \longrightarrow \infty.$$

In particular, since

$$\int_{-1}^{1} |f_n'(x) - f_m'(x)|^2 (1-x)^{\alpha+1}dx \le \|f_n - f_m\|_\phi^2\,,$$

we see that $\{f_n'\}$ is Cauchy in $L^2\left((-1,1);(1-x)^{\alpha+1}\right)$. Since $L^2\left((-1,1);(1-x)^{\alpha+1}\right)$ is complete, there exists $g \in L^2\left((-1,1);(1-x)^{\alpha+1}\right)$ such that

$$f_n' \longrightarrow g \qquad \text{as } n \longrightarrow \infty \;\text{ in } L^2\left((-1,1);(1-x)^{\alpha+1}\right). \tag{6.23}$$

Also, since

$$|f_n(-1) - f_m(-1)|^2 \le \|f_n - f_m\|_\phi^2$$

we see that the sequence $\{f_n(-1)\}$ is Cauchy in $\mathbb{C}$ and, hence, there exists $A \in \mathbb{C}$ such that

$$f_n(-1) \longrightarrow A. \tag{6.24}$$

Furthermore, since $f_n \in AC[-1,1)$ $(n \in \mathbb{N})$, we see that

$$\int_{-1}^{1} f_n'(t)(1-t)^{\alpha+1}dt \longrightarrow \int_{-1}^{1} g(t)(1-t)^{\alpha+1}dt,$$

Since $g \in AC[-1,1)$, we may define $f : [-1,1) \longrightarrow \mathbb{C}$ by

$$f(x) = A + \int_{-1}^{x} g(t)dt.$$

It is clear that $f \in AC[-1,1)$ and $f'(x) = g(x) \in L^2((-1,1);(1-x)^{\alpha+1})$ for a.e. $x \in [-1,1)$, so $f \in W_\alpha$. Furthermore, $f(-1) = A$. Now

$$\begin{aligned}
\|f_n - f\|_\phi^2 &= |f_n(-1) - f(-1)|^2 + \int_{-1}^{1} |f_n'(t) - f'(t)|^2 (1-t)^{\alpha+1}dt \\
&= |f_n(-1) - A|^2 + \int_{-1}^{1} |f_n'(t) - g(t)|^2 (1-t)^{\alpha+1}dt \\
&\longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$ by (6.23) and (6.24). Thus, $(W_\alpha, \phi(\cdot,\cdot))$ is complete. □

Theorem 6.7. *Let W_α and $\phi(\cdot,\cdot)$ be as before, and*

$$\begin{aligned}
W_{\alpha,1} &:= \{f \in W_\alpha \mid f(-1) = 0\} \\
W_{\alpha,2} &:= \{f \in W_\alpha \mid f'(x) = 0\}.
\end{aligned}$$

Then $W_{\alpha,1}$ and $W_{\alpha,2}$ are closed, orthogonal subspaces of W_α and

$$W_\alpha = W_{\alpha,1} \oplus W_{\alpha,2}.$$

Proof. Since $W_{\alpha,2}$ is one-dimensional, it is a closed subspace of W_α. The orthogonal complement of $W_{\alpha,2}$ is given by

$$W_{\alpha,2}^{\perp} := \{f \in W_\alpha \mid \phi(f,g) = 0 \ \ \forall g \in W_{\alpha,2}\}.$$

To see that $W_{\alpha,1} \subset W_{\alpha,2}^{\perp}$, let $f \in W_{\alpha,1}$, $g \in W_{\alpha,2}$ and consider

$$\phi(f,g) = f(-1)\overline{g}(-1) + \int_{-1}^{1} f'(x)\overline{g}'(x)(1-x)^{\alpha+1}dx = 0.$$

The first summand vanishes because $f \in W_{\alpha,1}$, and the integral is 0 because $g \in W_{\alpha,2}$.

Now Let $f \in W_{\alpha}$. We need to find $f_1 \in W_{\alpha,1}$ and $f_2 \in W_{\alpha,2}$ such that $f = f_1 + f_2$. To this end, let

$$f_2(x) := C,$$

C to be determined. Clearly, $f_2 \in W_{\alpha,2}$. Let

$$f_1(x) := f(x) - f_2(x).$$

We show that $f_1 \in W_{\alpha,1}$ by appropriate choice of C. For any choice of C, $f_1 \in W_{\alpha}$. Now set

$$f_1(-1) = f(-1) + C \stackrel{!}{=} 0$$

to find

$$C = -f(-1)$$

i.e. with the choice of

$$f_1(x) := f(x) + f(-1)$$

$$f_2(x) := -f(-1)$$

every $f \in W_{\alpha}$ can indeed be written as $f = f_1 + f_2$, where $f_1 \in W_{\alpha,1}$ and $f_2 \in W_{\alpha,2}$. □

The next result shows that the space $W_{\alpha,1}$ is precisely the first left-definite space.

Theorem 6.8. $W_{\alpha,1} = V_1$.

Proof. (1) $V_1 \subseteq W_{\alpha,1}$: This was shown in lemma 6.11.

(2) $W_{\alpha,1} \subseteq V_1$: Let $f \in W_{\alpha,1}$.

It suffices to show that $f \in L^2\left((-1,1);(1-x)^{\alpha}(1+x)^{-1}\right)$. For $-1 < x < 0$,

$$\begin{aligned}(1-x)^{\alpha/2}(1+x)^{-1/2}\int_{-1}^{x} f'(t)dt &= (1-x)^{\alpha/2}(1+x)^{-1/2}\left[f(x)-f(-1)\right]\\ &= (1-x)^{\alpha/2}(1+x)^{-1/2}f(x)\end{aligned}$$

since $f(-1)=0$. We use Chisholm-Everitt on $(-1,0)$ with

$$\psi(x) = (1-x)^{\alpha/2}(1+x)^{-1/2}$$

$$\varphi(x) = 1.$$

Clearly, ψ is L^2 near 0, and φ is L^2 near -1. In this case,

$$\begin{aligned}\int_{-1}^{x} dt \int_{x}^{0} (1-t)^{\alpha}(1+t)^{-1}dt &\le c\int_{-1}^{x} dt \int_{x}^{0} \frac{dt}{1+t}\\ &= -c(x+1)\ln(1+x),\end{aligned}$$

and this is a bounded function on $(-1,0)$. By Chisholm-Everitt, we have $f \in L^2\left((-1,0);(1-x)^{\alpha}(1+x)^{-1}\right)$.

For $0 \le x < 1$,

$$(1-x)^{\alpha/2}(1+x)^{-1/2}\int_{x}^{1} f'(t)dt = -(1-x)^{\alpha/2}(1+x)^{-1/2}f(x).$$

We again apply Chisholm-Everitt on $[0,1)$ with

$$\varphi(x) = (1-x)^{\alpha/2}(1+x)^{-1/2}$$

$$\psi(x) = 1.$$

In this case,

$$\int_0^x (1-t)^\alpha (1+t)^{-1} dt \int_x^1 dt \leq c \int_0^x (1-t)^\alpha dt \int_x^1 dt$$
$$= -\frac{c}{\alpha+1}(1-t)^{\alpha+1} \Big|_0^x (1-x)$$
$$= -\frac{c}{\alpha+1}(1-x)^{\alpha+2}$$

which is bounded on $(0,1)$. By Chisholm-Everitt, $f \in L^2\left((0,1);(1-x)^\alpha(1+x)^{-1}\right)$.

□

Theorem 6.9. *The inner products $\phi(\cdot,\cdot)$ and $(\cdot,\cdot)_1$ are equivalent on $W_{\alpha,1} = V_1$.*

Proof. First of all, $(W_{\alpha,1}, \phi(\cdot,\cdot))$ is a Hilbert space, and, by definition, $(V_1,(\cdot,\cdot)_1)$ is a Hilbert space. Let $f \in W_{\alpha,1} = V_1$. Then

$$\begin{aligned} \|f\|_\phi^2 &= \int_{-1}^1 |f'|^2 (1-x)^{\alpha+1} dx \\ &\leq \int_{-1}^1 \left[|f'|^2 (1-x)^{\alpha+1} + k(1-x)^\alpha (1+x)^{-1} |f|^2\right] dx \\ &= \left(\|f\|_1\right)^2. \end{aligned}$$

By the open mapping theorem, these inner products must be equivalent. □

Note that T_2 is self-adjoint in $W_{\alpha,2}$ since it is defined on the whole one-dimensional space.

We now need to consider T_1 in the space $W_{\alpha,1}$. Recall that by theorem 6.8, $V_1 = W_{\alpha,1}$. We also know that the operator

$$B_{1,k}^{(\alpha,-1)} : \mathcal{D}\left(B_{1,k}^{(\alpha,-1)}\right) := V_3 \subset V_1 \longrightarrow V_1$$

namely, the first left-definite operator associated with $\left(A, L^2_{\alpha,-1}(-1,1)\right)$, is self-adjoint and given by

$$B_{1,k}^{(\alpha,-1)}[f](x) = l_{\alpha,-1}[f](x)$$
$$f \in \mathcal{D}\left(B_{1,k}^{(\alpha,-1)}\right) = V_3 = \{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f', f'' \in AC_{loc}(-1,1);$$
$$(1-x)^{(\alpha+3)/2}(1+x)f''', (1-x)^{(\alpha+2)/2}(1+x)^{1/2}f'', (1-x)^{(\alpha+1)/2}f',$$
$$(1-x)^{\alpha/2}(1+x)^{-1/2}f \in L^2(-1,1)\big\}$$

More specifically, $B_{1,k}^{(\alpha,-1)}$ is self-adjoint with respect to the first left-definite inner product $(\cdot,\cdot)_1$ which we know is equivalent to the inner product $\phi(\cdot,\cdot)$. We shall prove that the operator

$$T_1 : \mathcal{D}(T_1) \subset W_{\alpha,1} \longrightarrow W_{\alpha,1}$$

given by

$$T_1 f = B_{1,k}^{(\alpha,-1)} f = l_{\alpha,-1}[f]$$
$$f \in \mathcal{D}(T_1) := V_3$$

is self-adjoint in $(W_{\alpha,1}, \phi(\cdot,\cdot))$.

Proof. Let $f, g \in V_3$. □

Lemma 6.13. *T_1 in $(W_{\alpha,1}, \phi(\cdot,\cdot))$ is densely defined.*

Proof. T_1 is defined through the first left-definite operator,

$$T_1 f = B_{1,k}^{(\alpha,-1)} f$$

with domain V_3. The Jacobi polynomials $\left\{P_n^{(\alpha,-1)}\right\}_{n=1}^{\infty}$ are its eigenfunctions and they are dense in V_3. □

Theorem 6.10. *T_1 is symmetric in $(W_{\alpha,1}, \phi(\cdot,\cdot))$.*

Proof. From the previous lemma, it suffices to show that T_1 is Hermitian. Let $f, g \in \mathcal{D}(T_1) = V_3$. Since $V_3 \subset V_1$ and $T_1 f, T_1 g \in V_1$, we know that

$$f(-1) = g(-1) = 0 = T_1 f(-1) = T_1 g(-1).$$

Integration by parts shows that

$$\begin{aligned}(T_1 f, g)_\phi &= \int_{-1}^{1} (T_1 f)'(x) \overline{g}'(x) dx \\ &= (f, T_1 g)_\phi\end{aligned}$$

□

Theorem 6.11. *The operator T_1 has the following properties:*

(i) T_1 is self-adjoint in $(W_\alpha, \phi(\cdot,\cdot))$.

(ii) $\sigma(T_1) = \{n(n+\alpha) + k \mid n \geq 2\}$.

(iii) $\left\{P_n^{(\alpha,-1)}\right\}_{n\geq 1}$ is a complete orthonormal set of eigenfunctions of T_1 in the space $(W_\alpha, \phi(\cdot,\cdot))$.

(iv) T_1 is bounded below by kI in $(W_\alpha, \phi(\cdot,\cdot))$.

Proof. For (iii): We know that $\left\{P_n^{(\alpha,-1)}\right\}_{n\geq 0}$ is a complete orthonormal set in $(W_\alpha, \phi(.,.))$ and we know that $W_\alpha = W_{\alpha,1} \oplus W_{\alpha,2}$. Also, $W_{\alpha,2} = span\left\{P_0^{(\alpha,-1)}\right\}$ and so $W_{\alpha,1} = W_{\alpha,2}^\perp = span\left\{P_n^{(\alpha,-1)}\right\}_{n\geq 1}$. We next prove that T_1 is closed in $(W_\alpha, \phi(\cdot,\cdot))$. Take a sequence $\{f_n\} \subseteq \mathcal{D}(T_1) = V_3$ such that

$$\begin{aligned} f_n &\longrightarrow f \quad \text{ in } (W_\alpha, \phi(\cdot,\cdot)) \\ T_1 f_n &\longrightarrow g \quad \text{ in } (W_\alpha, \phi(\cdot,\cdot)). \end{aligned}$$

We show that $f \in \mathcal{D}(T_1)$ and $T_1 f = g$. We know that $B_{1,k}^{(\alpha,-1)}$ is self-adjoint and hence closed in $(W_{\alpha,1}, (\cdot,\cdot)_1)$, and we know, since $\phi(\cdot,\cdot)$ and $(\cdot,\cdot)_1$ are equivalent, there exist constants c_1 and c_2 such that

$$c_1 \|f\|_\phi \leq \|f\|_1 \leq c_2 \|f\|_\phi \qquad \forall f \in W_{\alpha,1} = V_1.$$

Hence,

$$\|f_n - f\|_1 \leq c_2 \|f_n - f\|_\phi \longrightarrow 0$$

i.e.

$$f_n \longrightarrow f \qquad \text{in } \left(W_{\alpha,1}, (\cdot,\cdot)_1\right)$$

and

$$\|T_1 f_n - g\|_1 \leq c_2 \|T_1 f_n - g\|_\phi \longrightarrow 0$$

i.e.

$$T_1 f_n \longrightarrow g \qquad \text{in } \left(W_{\alpha,1}, (\cdot,\cdot)_1\right)$$

and since T_1 is closed in $(W_{\alpha,1}, (\cdot,\cdot)_1)$, we see that $f \in \mathcal{D}(T_1)$ and $T_1 f = g$. Also, we know that, for $n \geq 2$,

$$\begin{aligned}\left(T_1 P_n^{(\alpha,-1)}\right)(x) &= l_{\alpha,-1}[P_n^{(\alpha,-1)}](x) \\ &= (n(n+\alpha)+k)P_n^{(\alpha,-1)}(x).\end{aligned}$$

This implies

$$\{n(n+\alpha)+k \mid n \geq 2\} \subseteq \sigma(T_1).$$

Since $\left\{P_n^{(\alpha,-1)}\right\}_{n\geq 1}$ is complete and $\lambda_n := n(n+\alpha)+k \longrightarrow \infty$, we know that

$$\sigma(T_1) = \{n(n+\alpha)+k \mid n \geq 2\}$$

by a result due to Riesz-Nagy, which proves (ii) and (iii). To summarize: T_1 is a closed, symmetric operator with a complete set of eigenfunctions. From Naimark's book, T_1 is self-adjoint. This proves (i). To prove (iv), let $f \in \mathcal{D}(T_1)$. Then, since

$T_1 : V_3 \subset V_1 \longrightarrow V_1,$

$$\begin{aligned}
(T_1 f, f)_\phi &= (T_1 f)(-1)\overline{f}(-1) + \int_{-1}^{1} (T_1 f)'(x)\overline{f}'(x)(1-x)^{\alpha+1}dx \\
&= \int_{-1}^{1} (T_1 f)'(x)\overline{f}'(x)(1-x)^{\alpha+1}dx \\
&= \int_{-1}^{1} \left[\frac{1}{(1-x)^{\alpha}(1+x)^{-1}} \left|\left((1-x)^{\alpha+1}f'(x)\right)'\right|^2 + k\left|f'(x)\right|^2(1-x)^{\alpha+1}\right]dx \\
&\geq k\int_{-1}^{1} \left|f'(x)\right|^2(1-x)^{\alpha+1}dx \\
&= k\left|f(-1)\right|^2 + k\int_{-1}^{1} \left|f'(x)\right|^2(1-x)^{\alpha+1}dx \\
&= k\,(f,f)_\phi\,.
\end{aligned}$$

□

We now construct the self-adjoint operator T in $(W_\alpha, \phi(\cdot,\cdot))$ that is generated by the Jacobi differential expression $l_{\alpha,-1}[.]$, having the *entire* set of Jacobi polynomials $\left\{P_n^{(\alpha,-1)}\right\}_{n\geq 0}$ as eigenfunctions and having spectrum

$$\sigma(T) = \{n(n+\alpha)+k \mid n \in \mathbb{N}_0\}\,.$$

For $f \in W_\alpha$, write

$$f = f_1 + f_2$$

where $f_i \in W_{1,i}, (i = 1, 2)$. Define

$$T : \mathcal{D}(T) \subset W_\alpha \longrightarrow W_\alpha$$

by

$$Tf = T_1 f_1 + T_2 f_2 = l_{\alpha,-1}[f_1] + l_{\alpha,-1}[f_2] = l_{\alpha,-1}[f],$$

$$\mathcal{D}(T) = \mathcal{D}(T_1) \oplus \mathcal{D}(T_2).$$

Theorem 6.12. *T is self-adjoint in $(W_\alpha, \phi(\cdot,\cdot))$ and*

$$\begin{aligned}\mathcal{D}(T) = \{f : [-1,1) \longrightarrow \mathbb{C} \mid f \in AC[-1,1); f', f'' \in AC_{loc}(-1,1); \\ (1-x)^{(\alpha+3)/2}(1+x)f''', (1-x)^{(\alpha+2)/2}(1+x)^{1/2}f'', \\ (1-x)^{(\alpha+1)/2}f' \in L^2(-1,1)\big\} \\ = \{f : [-1,1) \longrightarrow \mathbb{C} \mid f \in AC[-1,1); f', f'' \in AC_{loc}(-1,1); \\ (1-x)^{(\alpha+3)/2}(1+x)f''' \in L^2(-1,1)\big\}.\end{aligned}$$

Furthermore, $\sigma(T) = \{n(n+\alpha)+k \mid n \in \mathbb{N}_0\}$ and T is bounded below by kI in $(W_\alpha, \phi(\cdot,\cdot))$.

For the following theorem let us recall the definitions of the first and third left-definite spaces:

$$\begin{aligned}V_1 &= \{f : (-1,1) \longrightarrow \mathbb{C} \mid f \in AC_{loc}(-1,1); \\ &\quad (1-x)^{\alpha/2}(1+x)^{-1/2}f, (1-x)^{(\alpha+1)/2}f' \in L^2(-1,1)\big\} \\ &= \big\{f : [-1,1) \longrightarrow \mathbb{C} \mid f \in AC\,[-1,1)\,; (1-x)^{(\alpha+1)/2}f' \in L^2(-1,1); f(-1)=0\big\} \\ &= W_{\alpha,1}\end{aligned}$$

$$\begin{aligned}V_3 = \mathcal{D}(T_1) &= \{f : (-1,1) \longrightarrow \mathbb{C} \mid f, f', f'' \in AC_{loc}(-1,1); \\ &\quad (1-x)^{(\alpha+3)/2}(1+x)f''', (1-x)^{(\alpha+2)/2}(1+x)^{1/2}f'', (1-x)^{(\alpha+1)/2}f', \\ &\quad (1-x)^{\alpha/2}(1+x)^{-1/2}f \in L^2(-1,1)\big\} \\ &= \big\{f \in V_1 \mid f', f'' \in AC_{loc}(-1,1); (1-x)^{(\alpha+3)/2}(1+x)f''' \\ &\quad (1-x)^{(\alpha+2)/2}(1+x)^{1/2}f'' \in L^2(-1,1)\big\} \\ &= \{f : [-1,1) \longrightarrow \mathbb{C} \mid f \in AC[-1,1); f', f'' \in AC_{loc}(-1,1); \\ &\quad f(-1)=0; (1-x)^{(\alpha+3)/2}(1+x)f''', (1-x)^{(\alpha+2)/2}(1+x)^{1/2}f'', \\ &\quad (1-x)^{(\alpha+1)/2}f' \in L^2(-1,1)\big\}.\end{aligned}$$

Note that the space $\mathcal{D}$ below is V_3 minus the condition $f(-1)=0$, so $V_3 \subseteq \mathcal{D}$.

Theorem 6.13. *Let*

$$\mathcal{D} := \{f : [-1,1) \longrightarrow \mathbb{C} \mid f \in AC[-1,1); f', f'' \in AC_{loc}(-1,1);$$
$$(1-x)^{(\alpha+3)/2}(1+x)f''', (1-x)^{(\alpha+2)/2}(1+x)^{1/2}f'', (1-x)^{(\alpha+1)/2}f' \in L^2(-1,1)\big\}.$$

Then $\mathcal{D}(T) = \mathcal{D}$.

Proof. First show $\mathcal{D}(T) \subseteq \mathcal{D}$: Let $f \in \mathcal{D}(T) = \mathcal{D}(T_1) \oplus \mathcal{D}(T_2)$. Write

$$f = f_1 + f_2$$

where $f_1 \in \mathcal{D}(T_1) = V_3 \subseteq \mathcal{D}$, $f_2 \in \mathcal{D}(T_2) \subseteq \mathcal{D}$. Then $f \in \mathcal{D}$. To show that $\mathcal{D} \subseteq \mathcal{D}(T)$, let $f \in \mathcal{D}$. Write

$$f(x) = [f(x) + f(-1)] - f(-1)$$

with

$$f_1(x) := f(x) + f(-1)$$
$$f_2(x) := -f(-1).$$

Then $f_1 \in \mathcal{D}$, and $f_1(-1) = 0$, i.e. $f_1 \in V_3 = \mathcal{D}(T_1)$. Also, $f_2''(x) = 0$, i.e. $f_2 \in \mathcal{D}(T_2)$. Together, $f \in \mathcal{D}(T)$. $\square$

CHAPTER SEVEN

Further Work

The self-adjoint operator T which was constructed in sections 5.4 and 6.4, respectively, is bounded below so that the left-definite theory can be applied again to this operator.

In the study of angular momentum in quantum mechanics, the Jacobi polynomials occur quite naturally for negative integer parameters [6]. Usually, this is treated by using identities relating the Jacobi polynomials for negative integer parameters to those for positive integer parameters. This application is of particular interest, now that orthogonality and spectral results are available for the Jacobi polynomials for non-classical parameters.

The left-definite theory has never been applied to difference equations or partial differential equations. A natural place to start with difference equations would be the Charlier difference equation, as the integral powers are known for the corresponding difference expression.

In the orthogonal polynomial examples, the set of polynomials is dense in every left-definite space, while these left-definite spaces are proper subsets of one another. It seems natural to ask what functions are contained in the intersection of all left-definite spaces. In the case of the orthogonal polynomial examples, our conjecture is that the intersection consists of the set of all infinitely differentiable functions.

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