ABSTRACT

Spectral Functions for Generalized Piston Configurations

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In this work we explore various piston configurations with different types of potentials. We analyze Laplace-type operators $P = -g^{ij}\nabla_i^E\nabla_j^E + V$ where V is the potential. First we study delta potentials and rectangular potentials as examples of non-smooth potentials and find the spectral zeta functions for these piston configurations on manifolds $I \times \mathcal{N}$, where I is an interval and $\mathcal{N}$ is a smooth compact Riemannian $d-1$ dimensional manifold. Then we consider the case of any smooth potential with a compact support and develop a method to find spectral functions by finding the asymptotic behavior of the characteristic function of the eigenvalues for P. By means of the spectral zeta function on these various configurations, we obtain the Casimir force and the one-loop effective action for these systems as the values at $s = -1/2$ and the derivative at $s = 0$. Information about the heat kernel coefficients can also be found in the spectral zeta function in the form of residues, which provide an indirect way of finding this geometric information about the manifold and the operator.

Spectral Functions for Generalized Piston Configurations

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CHAPTER ONE

Spectral Functions

1.1 Finite Dimensional Setting

A central topic in the field of linear algebra is the study of invariant functions, for example, the trace and the determinant of a square matrix. Such functions go from the set of, say $n \times n$ matrices, into the field $\mathbb{K}$, usually taken to be a closed algebraic field. Keeping in mind the basic properties of these functions, we state that an invariant function is such that it is constant on the orbits of the action of $SU(n)$ on the set of matrices given by conjugation, i.e.

$$g.A = gAg^{-1} \tag{1.1}$$

for $A \in M_N(\mathbb{C})$ and $g \in SU(n)$. Another way of saying this is that the trace and the determinant are constant under a change of basis and their values are well defined in the sense that they do not depend on the particular representation of the matrix A.

Now, since the determinant is an invariant function, we have that the characteristic polynomial of a matrix defined by

$$p_A(t) = \det(I_n t - A) \tag{1.2}$$

where I_n is the $n \times n$ identity matrix, is also an invariant of A. Due to the ring homomorphism between $\mathbb{K}[t]$ and $\mathbb{K}[A]$ that formally exchanges the real variable t and the matrix variable A, we can make the characteristic polynomial to be real valued or matrix valued depending on the context.

If we take the field to be the complex numbers, we can factor the characteristic polynomial as a product of linear factors,

$$p_A(t) = (t - \lambda_1)(t - \lambda_2) \cdot \cdots \cdot (t - \lambda_n) \tag{1.3}$$

for some $\lambda_n \in \mathbb{C}$ not necessarily distinct. These values λ_i are called the *eigenvalues* of the matrix A. With the homomorphism between the polynomial rings $\mathbb{C}[t]$ and $\mathbb{C}[A]$ we have an important result due to Cayley and Hamilton, which states that the characteristic polynomial of a matrix is satisfied by the matrix itself [15].

Theorem 1.1 (Cayley-Hamilton). *If A is an $n \times n$ matrix with entries in a commutative ring R, then*

$$p_A(A) = 0\,. \tag{1.4}$$

Letting A be an $n \times n$ matrix over $\mathbb{C}$ and using the previous result gives us the existence of a vector $v_i \in \mathbb{C}^n$ such that

$$(A - \lambda_i I_n)\, v_i = 0\,, \tag{1.5}$$

which is the usual notion of an eigenvalue and an eigenvector. Looking at this, we find that the notion of a *characteristic equation* or *secular equation* plays a central role in spectral theory, for it provides a way of linking the study of an operator with the study of its eigenvalues. By the relation obtained in (1.3), we find a characterization of the eigenvalues of A as being the solutions of

$$p_A(\lambda) = 0\,. \tag{1.6}$$

Expanding the characteristic polynomial of a matrix A gives us that the coefficients of p_A are invariant, as p_A itself is invariant. Getting the expansion of the characteristic polynomial,

$$p_A(t) = t^n + \sum_{k=1}^{n} (-1)^k c_k(A) t^{n-k}\,, \tag{1.7}$$

gives these new invariants of A, namely $c_k(A)$. If we consider the case of $t = 0$, we find that

$$c_n(A) = \det(A)\,. \tag{1.8}$$

Via Jacobi's formula,

$$\frac{d}{dt}\det(tI_n - A) = \operatorname{tr}\left(\operatorname{adj}(tI_n - A)\right)\,, \tag{1.9}$$

and considering the polynomial $t^n P_A(1/t)$, we have that setting $t = 0$ gives

$$c_1(A) = \operatorname{tr}(A). \tag{1.10}$$

Exploiting the structure given by (1.3), we have that all the coefficients $c_k(A)$ can be expressed in terms of the eigenvalues λ_i by Vieta's formulas as the elementary symmetric polynomials in the eigenvalues,

$$c_k(A) = \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} \lambda_{i_1}\lambda_{i_2}\cdot\cdots\cdot\lambda_{i_k}\,. \tag{1.11}$$

Now, we can say that the functions $c_k : M_n \to \mathbb{C}$ are *spectral functions* in the sense that they are defined using the eigenvalues of the matrix. Thus, when we realize a matrix as an operator $A : \mathbb{C}^n \to \mathbb{C}^n$, the functions c_k are defined via the *spectrum* of the operator, i.e. by the eigenvalues of A.

1.2 Infinite Dimensional Setting

On the other hand, when in the infinite dimensional setting, the idea is to mimic what happens for the finite dimensional case. Yet, in order to study infinite dimensions, we will no longer work with vector spaces but with Hilbert spaces $\mathcal{H} = (H, \langle,\rangle)$ and with linear operators $L(\mathcal{H})$ from $\mathcal{H}$ into itself. Once we start working with an operator, we have the notion of an *eigenvalue* to be similar to the finite dimensional scenario.

Definition 1.1. Eigenvalue

Under the assumption that P is a linear operator defined on $\mathcal{H}$, a complex number λ is said to be an eigenvalue for P if $P-\lambda I$, where I is the identity operator, fails to be one-to-one. Since $P-\lambda I$ is not one-to-one, $n_\lambda = \dim\ker(P-\lambda I)$ is non-zero and is called the *geometric multiplicity* of λ.

Definition 1.2. Spectrum

Eigenvalues for a linear operator $P \in L(\mathcal{H})$ can be gathered in a set called the *point spectrum* of P, defined by

$$\sigma_p(P) = \{\lambda \in \mathbb{C} | \lambda \text{ is an eigenvalue of } P\}. \tag{1.12}$$

Each eigenvalue has a geometric multiplicity $n_\lambda = \dim\left(\ker(P-\lambda I)\right)$, with which we can define the *geometric point spectrum* of P to be the set

$$\tau(P) = \{(\lambda, n_\lambda) | \lambda \text{ is an eigenvalue of } P \text{ and } n_\lambda \text{ its geometric multiplicity}\}. \tag{1.13}$$

Knowing the geometric multiplicity of eigenvalues turns out to be of great importance in spectral theory, hence the definition of the geometric point spectrum of an operator is necessary in our discussion in order to include repeated eigenvalues.

Definition 1.3. Spectral Function

A function f taking linear operators in $B \subset L(\mathcal{H})$ into analytic functions defined in a region of the complex plane, is called a *spectral function* if there exists a function g that takes sequences of ordered pairs (λ, n_λ) into analytic functions defined in a region of the complex plane, such that

$$f(P) = g(\tau(P)) \tag{1.14}$$

for all $P \in B$.

Defining spectral functions in this fashion states that a spectral function f is determined only by the eigenvalues of P, including multiplicities.

Yet, this framework is to general for our study, so from here on we will focus on the case of self-adjoint operators, since most operators arising from physical applications tend to be self-adjoint.

Definition 1.4. Self-adjoint Operator

On $L(\mathcal{H})$, define the *adjoint* $P^* : \mathcal{H} \supset \text{Dom}\,(P^*) \to \mathcal{H}$ of the operator P to be such that

$$\langle Px, y\rangle = \langle x, P^* y\rangle \tag{1.15}$$

for all $x \in$ Dom (P) and $y \in$ Dom (P^*). Upon the case that $P = P^*$, we say that P is a *self-adjoint* or *hermitian* operator.

With this, we have one of the first important properties about the eigenvalues of self-adjoint operators [5].

Theorem 1.2 (Real spectrum). *In the class of self-adjoint operators, each operator has real spectrum.*

Among self adjoint operators, Laplace type operators are of special interest. A Laplace type operator is an operator P defined on a smooth compact Riemannian manifold $\mathcal{M}$ that has the form

$$P = -g^{ij}\nabla_i^E \nabla_j^E + V\,, \tag{1.16}$$

where we use Einstein summation for repeated indices, g^{ij} is the metric on $\mathcal{M}$, E is a smooth vector bundle over $\mathcal{M}$, and $V \in \text{End}(E)$.

For this type of operators, there are some nice properties about the general form of their spectrum.

Theorem 1.3. *If P is a Laplace type operator defined on a compact smooth Riemannian manifold, and λ_n is the nth eigenvalue of P counting multiplicities, we have that $\sigma_p(P)$ is bounded below and that the only limit point of $\sigma_p(P)$ is infinity.*

Now, for this type of operators, we have a nice growth behavior for the sequence of eigenvalues. Describing the behavior at infinity of the eigenvalues of a Laplace type operator in a smooth compact Riemannian manifold $\mathcal{M}$, we have Weyl's Asymptotic formula [2, 11].

Theorem 1.4 (Weyl's Law). *Keeping the assumptions as above, let $N(\lambda)$ be the number of eigenvalues that are smaller than a given real number λ. Near infinity, we have the asymptotic behavior*

$$N(\lambda) \sim \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)} \; Vol(\mathcal{M})\lambda^{d/2} \tag{1.17}$$

where $d = \dim(\mathcal{M})$.

One way of analyzing the eigenvalues of a differential operator indirectly is by means of a characteristic equation. Characteristic equations for a general operator can be thought of as a generalization of the equation (1.3). Knowing the previous properties of the spectrum of Laplace-type operators will ensure the existence of a characteristic function for their eigenvalues by using Weierstrass's product for a prescribed sequence of complex numbers [18], together with Theorem 1.3.

Theorem 1.5 (Weierstrass). *Any sequence $\{a_n\}$ of complex numbers with $|a_n| \to \infty$ as $n \to \infty$ defines an entire function f that vanishes at all $z = a_n$ and nowhere else. No other such function exists besides conformal products $f(z)e^{g(z)}$, where g is entire.*

Definitely, an important conclusion from this result is that, up to a conformal transformation, there is only one characteristic equation for the eigenvalues of a Laplace-type operator defined on a smooth compact Riemannian manifold.

1.3 Heat Kernel

The heat kernel of a differential operator P plays an important role for both analyzing theoretical properties about the underlying manifold $\mathcal{M}$ and the operator itself, and for finding solutions to the *heat equation* on $\mathcal{M}$ with a prescribed initial condition

$$\begin{aligned} &\partial_t f(t,x) = P f(t,x) \\ &f(0,x) = g(x) \end{aligned} \tag{1.18}$$

with $f \in L^2(\mathbb{R}^+ \times \mathcal{M})$ and $g \in L^2(\mathcal{M})$. Here L^2 is the space of square integrable functions with the underlying measure.

Establishing a formalism with distributions provides a way of solving this initial value problem in terms of the heat kernel.

Definition 1.5. Heat Kernel

Define the heat kernel $K_P(t,x,y) \in C^\infty(\mathbb{R}^+ \times \mathcal{M} \times \mathcal{M})$ for the operator P given as before, to be such that

$$\lim_{t\to 0} K_P(t,x,y) = \delta(x-y)\,, \tag{1.19}$$

in the sense of distributions, and if the integral operator $T_P : L^2(\mathcal{M}) \to L^2(\mathcal{M})$

$$g(x) \mapsto \int_{\mathcal{M}} dy\, K_P(t,x,y) g(y) \tag{1.20}$$

gives a solution for the initial value problem (1.18) for the initial condition $g(x)$.

One important result is that if the heat kernel exists, it can be described by the eigenvalues of the operator P, and hence this makes the heat kernel a spectral function [16].

Theorem 1.6 (Heat Kernel). *On the assumption that P is such that K exists, we have the pointwise convergence*

$$K_P(t,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \psi_n(x) \psi_n(y) \tag{1.21}$$

with $\{\psi_n\}$ an orthonormal basis for $L^2(\mathcal{M})$ and ψ_n an eigenfunction of P with eigenvalue λ_n.

Recall that the usual heat equation provides a way of studying geometric information about the underlying space by analyzing the heat distribution for short times. With the heat kernel defined in a more general setting, we also have that the small t behavior encodes important geometric invariants described by the coefficients of the asymptotic expansion [11, 16].

Theorem 1.7 (Heat Kernel Asymptotics). *If we consider the behavior of the heat kernel on the diagonal $x = y$, the heat trace for a Laplace-type operator over a smooth Riemannian manifold possibly with smooth boundary has an asymptotic expansion*

$$\int_{\mathcal{M}} dx\, K_P(t, x, x) \sim (4\pi t)^{d/2} \sum_{k=0,1/2,1,\ldots} a_k t^k \tag{1.22}$$

as $t \to 0$.

Looking at the *heat kernel coefficients* a_k we are able to find geometric invariants of the manifold $\mathcal{M}$, for instance, the first coefficient a_0 gives the volume of the manifold, $a_{1/2}$ gives the volume of the boundary and so on.

1.4 Zeta Function

Let us investigate another important spectral function, which is the spectral zeta function associated to a second order elliptic operator. Because we are mainly interested in self-adjoint operators and these have spectrum bounded from below, lets assume without loss of generality that the spectrum of the second order elliptic operator P defined on a d dimensional smooth compact Riemannian manifold $\mathcal{M}$, possibly with boundary, is strictly positive. Even in the case of having a negative part of the spectrum, second order elliptic operators will have spectrum bounded from below, which ensures having a positive spectrum just by considering a shift by a constant. Often these conditions are met by the operators arising from physical problems, and when it is not the case, a constant can be added in the form of a mass m, which ultimately is sent to zero to recover the initial problem.

Definition 1.6. Zeta Function

Putting the above restrictions, define the spectral zeta function associated with P to be

$$\zeta_P(s) = \sum_{n=1}^{\infty} \lambda_n^{-s} \,. \tag{1.23}$$

where s is a complex number.

Earlier we stated the asymptotic behavior near infinity of the eigenvalues for a Laplace-type operator. Now, by virtue of Theorem 1.4, we have that the above definition of the spectral zeta function gives a convergent sum for $\Re(s) > d/2$. Extending the convergence region to the whole complex plane is possible with the exception of simple poles [11].

Theorem 1.8. *Defining $\zeta_P(s)$ as above gives that it is holomorphic for $\Re(s) > d/2$. This can be analytically continued to the whole complex plane with at worst simple poles at $s = d/2, (d-1)/2, \ldots, 1/2, -(2l+1)/2,\ l \in \mathbb{N}$.*

On the other hand, using the Mellin transform, we can write the zeta function in the region $\Re(s) > d/2$ as

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{n=1}^{\infty} e^{-\lambda_n t} - \dim\ker P \right) dt \,, \tag{1.24}$$

which can be written using the heat kernel as

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\mathrm{tr}\ \ (T_P) - \dim\ker P\right) dt \,. \tag{1.25}$$

This expression can be split in order to obtain information about its meromorphic structure. One way of breaking the integral is to consider two pieces, one from 0 to 1, and another from 1 to ∞. Using this, the behavior for the integrand for large t behaves like $O(e^{-\lambda t})$, with λ being the smallest eigenvalue, we have that

$$\frac{1}{\Gamma(s)} \int_1^{\infty} t^{s-1} \left(\mathrm{tr}\ \ (T_P) - \dim\ker P\right) dt \tag{1.26}$$

is holomorphic for all s. Hence the meromorphic structure of ζ_P comes from the small t behavior, where we can use the heat kernel asymptotics

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\text{tr }\left(T_P\right)-\dim\ker P\right)dt \\ \sim \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left((4\pi t)^{-d/2}\sum_{k=0,1/2,1,\dots} a_k t^k-\dim\ker P\right)dt \\ =\frac{1}{(4\pi)^{d/2}\Gamma(s)}\sum_{k=0,1/2,1,\dots}\frac{a_k}{s+k-d/2}-\frac{\dim\ker P}{\Gamma(s)s}\end{aligned} \tag{1.27}$$

And therefore we have the following result stating the meromorphic structure of the zeta function [11, 16].

Theorem 1.9. *The residues of* $\zeta_P(s)$ *are located at* $s=d/2,(d-1)/2,\dots,1/2,-(2l+1)/2$, $l\in\mathbb{N}$ *with residues*

$$\mathit{Res}\ \zeta_P(s)=\frac{a_{d/2-s}}{(4\pi)^{d/2}\Gamma(s)}\,, \tag{1.28}$$

and we have the special values at $s=-n$ *for* $n\in\mathbb{N}$,

$$\zeta_P(0)=\frac{a_{d/2}}{(4\pi)^{d/2}}-\dim\ker P \tag{1.29}$$

and

$$\zeta_P(-n)=\frac{(-1)^n n!\,a_{d/2+n}}{(4\pi)^{d/2}}\,. \tag{1.30}$$

1.4.1 Integral Representation

One of the main problems of calculating spectral functions is that, in most cases, the explicit form of the spectrum of an operator is not known. One way of getting around this difficulty is by defining the spectral functions in an implicit way, by using a characteristic equation for the eigenvalues [11].

Usually, eigenvalues can be described as solutions of equations obtained by imposing boundary conditions in partial differential equations. Suppose that a boundary value problem of an elliptic operator P gives a characteristic equation

$$F(\lambda) = 0 \tag{1.31}$$

for the eigenvalues. It is then possible to define spectral functions implicitly using F and some complex analysis tools [1, 18].

Theorem 1.10 (Cauchy's Residue Theorem). *If f is a meromorphic function in a region U of the complex plane and $\gamma \subset U$ is a closed curve not passing through any pole of f, then*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{poles} I(\gamma, a) Res\ f(a)\,, \tag{1.32}$$

where $I(\gamma, a)$ is the winding number of γ around the pole a.

By analyzing the behavior at infinity of meromorphic functions, one can extend this result for closed contours passing through the projection of infinity in the Riemann sphere.

The function

$$\frac{d}{d\lambda}\log F(\lambda) = \frac{F'(\lambda)}{F(\lambda)} \tag{1.33}$$

has a simple pole at every zero of F with residue equal to its multiplicity. Hence the function

$$g(\lambda) = \lambda^{-s}\frac{d}{d\lambda}\log F(\lambda) \tag{1.34}$$

will have a simple pole at every zero λ_n of F with residue λ_n^{-s} times the multiplicity of λ_n, therefore by integrating along a contour that encloses all the eigenvalues λ_n, we have an integral representation of the spectral zeta function [11].

Theorem 1.11 (Integral Representation). *Let P be an elliptic operator and let $F(\lambda) = 0$ be a characteristic equation for its eigenvalues, then the spectral zeta function admits an integral representation for $\Re(s) > d/2$ given by*

$$\zeta_P(s) = \int_\gamma d\lambda\, \lambda^{-s}\frac{d}{d\lambda}\log F(\lambda) \tag{1.35}$$

where γ is a contour that encloses the eigenvalues of P.

This representation provides the advantage of writing the spectral zeta function without the explicit knowledge of the spectrum, but does not give any improvement on the region of convergence. As the behavior of the integrand near zero is finite, what prevents the integral to converge is the behavior at infinity. For this, subtracting asymptotic terms of the integrand will improve the convergence of the integral, and therefore will extend the convergence region to the left.

Therefore we can extend the convergence region of the spectral zeta function by writing it as

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{N-2} A_i(s)\,, \tag{1.36}$$

where $Z(s)$ is the integral with the first N asymptotic terms subtracted, and $A_i(s)$ is the contribution made by the λ^{-i} term in the asymptotic expansion when added back. The expression obtained in (1.36) will be now valid for $\Re(s) > d/2 - N/2 - 1$.

In most cases, we are going to deal with product manifolds of the form $\mathcal{M} = I \times \mathcal{N}$, where I is an interval. In this case, the Laplacian over $\mathcal{M}$ is separable, and hence the spectral zeta function associated with P can be expressed as a series over an angular part coming from the interval I, of integrals coming from the manifold $\mathcal{N}$,

$$\zeta_P(s) = \sum_{\nu} d_\nu \int_{\gamma_\nu} d\lambda\, \lambda^{-2s} \frac{d}{d\lambda} \log F_\nu(\lambda)\,, \tag{1.37}$$

where d_ν is the multiplicity of the angular modes and the γ_ν are contours that enclose the eigenvalues of the Laplacian on I. As the convergence in both, the integral and the series, is needed, we control the behavior of the angular part ν and the eigenvalues λ by means of a uniform asymptotic expansion. Then we have that for fixed ν, we write the angular part of the spectral zeta function as

$$Z^\nu(s) + \sum_{i=-1}^{N-2} A_i^\nu(s) \tag{1.38}$$

when N asymptotic terms are subtracted [11]. Thus we can perform the summation over ν by means of the uniform asymptotic expansion to calculate

$$\zeta_P(s) = \sum_\nu Z^\nu(s) + \sum_\nu \sum_{i=-1}^{N-2} A_i^\nu(s) \tag{1.39}$$

and obtain an analytic continuation for the spectral zeta function associated with P which is valid for $\Re(s) > d/2 - N/2$.

1.5 Casimir Effect and One-loop Effective Action

1.5.1 Van der Waals Forces

The first formulation of the Casimir effect was made by the Dutch physicist Hendrik Casimir and Dirk Polder in 1948, when studying Van der Waals forces in colloids [3, 6, 13].

Van der Waals forces are physical chemical forces due mainly to attraction and repulsion between dipoles in molecules, which were first introduced by another Dutch physicist, Johannes Diderik van der Waals, in his Doctoral thesis in 1873 [3, 6, 13]. Van der Waals was studying molecular forces that were present in the equation of state of a fluid,

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT\,, \tag{1.40}$$

where p is the pressure of the fluid, v is the volume of the container per particle, a is the measure of attraction between particles, b is the volume excluded from v by a particle, and R is the universal gas constant.

His work generalizes the study of ideal gases and gives an approximation of the behavior of real fluids [3, 6]. Van der Waals' major development was to include the parameters a and b into the equation of state in order to consider non-zero particle size, and with this, he considered the interaction between particles. Here the notion of scale plays a very important role, as in large scales an unporalized object will not present any of these forces, but when considering smaller scales, the molecules will have a polarization which will result in a non-zero manifestation of this effect. It was not until the 1930s when the German physicist Fritz London precisely described, with the aid of the recent developed quantum mechanics, the interaction between molecules due to the dipole interaction forces [6, 13].

This force appears as a *dispersion force* and hence it is non-linear. A connection with Heisenberg's uncertainty principle can therefore be realized, as it takes into account a non-zero dispersion value for the interaction Hamiltonian.

London found his expression for the dispersion of the Hamiltonian operator with a fourth-order perturbation theory, and the result he found was non relativistic, thus entirely quantum [3, 6]. In these formulations, Van der Waals and London assumed an instant reaction for the intermolecular forces, which Casimir and Polder were trying to generalize by including a retardation in their experimentation with colloids at Phillips Labs in the 1940s. This new perspective led them to find not only a quantum but a relativistic formulation of the molecule interaction phenomena.

The inclusion of the relativistic term into the interaction forces provided a way to explain that the interaction potential between two identical molecules with static polarizability is of the order of magnitude of r^{-7}, where r is the intermolecule separation, instead of the r^{-6} predicted by London's work [3, 13].

Casimir obtained a better insight on how to describe this phenomenon after Niels Bohr mentioned that it might had something to do with *zero-point energy*, and as Casimir himself puts it "...it put me on a new track..." [6].

1.5.2 Zero-point Energy

The fundamentals of the Casimir effect can be traced back to the early twentieth century with Planck's formulation for the energy of a single energy radiator. Planck introduced the notion of half-quanta when studying the black body radiation and later on Einstein and Stern interpreted this formulation in the case of the ground state of a physical system. The existence of these half-quantum states gives that the ground state of an oscillator will have a non zero energy, which Einstein and Stern called the *Nullpunktsenergie* or zero-point energy of the oscillator. This non zero energy comes from quantizing a field, and it is closely related with the Heisenberg uncertainty principle as well. This principle states that at any given time, one cannot simultaneously know a particle's position and momentum. Thus the particle can never be at the bottom of its potential well, as otherwise it would be possible to measure with high accuracy its position and momentum, hence its ground state energy must be non-zero [3, 6, 13].

In the case of the harmonic oscillator, the energy levels are given by

$$E = \hbar\omega \left(n + \frac{1}{2} \right) \tag{1.41}$$

where $\hbar$ is Planck's normalized constant, ω is the oscillator's frequency and n is the energy level. For the ground state, it is found that the zero-point energy is non-zero,

$$E_0 = \frac{\hbar\omega}{2}\,, \tag{1.42}$$

which then can be used to define what we understand by *quantum vacuum* [11, 13].

The first counterintuitive statement about vacuum viewed in a quantum setting is the fact it is not vacuous. When quantizing a field, one can view a field as a collection of harmonic oscillators placed in every position in space, and the field's strength can be represented by the displacement of the oscillator from its rest position. This is what is known as first quantization, which constitutes replacing classical quantities with operators. A second quantization comes from the fact that the frequencies of each harmonic oscillator are also quantized, i.e. discrete [3, 6].

Performing the field's quantization leads us to consider differential operators P, which will give us the Hamiltonian of the physical system being considered, together with its eigenvalues ω_n^2, which constitute the energy levels of the system. The value ω_n can be thought as the frequency of the nth mode of the system and for this reason, we have that the ground state of the vacuum will show contributions coming from all possible frequencies, in other words we will have that the energy of the quantum vacuum will be formally given by

$$E = \sum_{n=1}^{\infty} \frac{\hbar \omega_n}{2} \tag{1.43}$$

which clearly is divergent. This is one example of the many divergent quantities that appear throughout the study of quantum field theories. One way of dealing with this type of divergent expressions is by regularizing the expression, or in some sense, to take the finite part of a suitable regularized quantity [11–13]. A regularization is basically a way of defining a reference for the quantity that one wants to measure. Something similar happens when we look at electric potentials. It only makes sense to talk about a potential difference, not of an absolute potential. Usually in this case, one takes the difference with the potential at infinity and that becomes the reference point. In the case of quantum field theories, the problem is that usually these referencing quantities are to be infinite, due to the fact that we consider spaces with infinite volume. Thus taking the difference of the referencing point and the quantity studied will subtract infinities and a finite part can be recovered, although a more rigorous mathematical approach is needed in order to formally define this method of regularization.

There are different ways of regularizing expressions, the most common ones being using spectral functions, finite parts or principal values, dimensional regularizations, normal orderings, Green's functions methods, and cut-off functions among others [11–13].

1.5.3 Casimir Effect

The idea of looking at the pressure present in the vacuum led Casimir to his formulation of the Casimir effect [6]. In order to calculate a pressure, one needs to take into consideration the geometry present in the vacuum, and this was Casimir's first formulation [3, 6, 13].

Consider two uncharged perfectly conducting parallel plates in vacuum. By the previous formulation, all possible wave lengths are present in the quantum vacuum, but due to the fact of the geometric configuration of the parallel plates and due to the boundary conditions, only those wavelengths that divide evenly the separation between the plates are allowed to be between them. Roughly speaking, this provides a higher density of wave lengths on the region outside the plates thus creating a pressure over the plates resulting in an attractive force. This is what is known as the Casimir force for this configuration [3, 6].

Casimir proposed this experiment in 1948, but it was not until 1997 when Lamoreaux was able to experimentally detect this force unambiguously. The first attempt of experimentally observing this force was due to Sparnaay in 1958, where he tried to measure the force by implementing Casimir's parallel plate configuration. Sparnaay could measure an attractive force although the uncertainty of the experiment was of 100%, which means that the experimental error was of the order of magnitude of the force itself, thus the experiment was inconclusive to experimentally prove the existence of the force. The subsequential experiments made use of an indirect calculation called the *proximity force approximation*, where the parallel plate configuration is replaced by a plate-sphere configuration with the separation between the two being much smaller than the radius of the sphere. With this approximation, it is possible to suppress experimental errors due to the lack of parallelism between the two plates, and exploid the plane-like geometry of the sphere for small separations. In his experiment, Lamoreaux was able to confirm the prescence of the Casimir force with a 15% error, which predicted a force density of

$$F = -\frac{\hbar c \pi^2}{240 a^4}\,, \tag{1.44}$$

where a is the plate separation [3, 6, 11, 13].

This gives a hint about the geometric nature of this effect. As a finite result can be obtained by perturbing the position of the plates, this suggest a high dependence on the geometry and the boundary conditions.

1.5.4 Piston Configuration

The problem with regularizing the energy arises from the fact that we have a divergent quantity. This appears due to the infinite eigenfrequencies present in the quantum vacuum and the unbounded nature of the ground state energies associated with them. By considering a change in the geometry of the configuration it is possible to arrive at a finite expression which can give some insight into the nature of this phenomenon. As mentioned before, in order to make mathematical and practical sense of these divergent quantities, one can use one of the many regularization methods in order to achieve a finite answer [4, 11, 12, 19]. Basically, these regularization methods take care of the infinities by taking the difference of two points in configuration space. The same idea is customary when dealing with potentials, where only taking a potential difference makes sense, as the potential itself is ill defined. Usually this reference for the potential is taken to be the potential at infinity, and hence all the quantities are referenced with the difference of the potential at a given point and the potential at infinity.

The same idea can be used to consider the difference between the energy of a given configuration and the energy of empty space. Here, the energy of empty space becomes the reference point and usually is infinite, as the volume of the entire space that is being considered is usually infinite, hence the necessity of taking this difference in order to obtain a finite result.

Another way of dealing with this phenomenon is by considering the force or pressure that an object experiences. By considering a piston configuration, it is possible to unambiguously find the force exerted over the piston, as the difference of the forces made by the two chambers over the piston is a well defined quantity [11].

This approach was first suggested by Boyer in the late 1960s when he considered the Casimir force of spherical shells with the ultimate goal of letting the outside radius tend to infinity [4]. With this, he took into account the contributions made by the inside of the sphere and the outside space. The same idea was then used by N. F. Svaiter and B. F. Svaiter in 1992 where they considered rectangular geometries and used the notion of a middle plate as a piston [19]. Basically they introduced a third plate in the parallel plate configuration and their final objective was to send that third plate to infinity in order to include the contributions coming from the outside of the two plates.

Their motivation for this kind of configuration was also due to mathematical arguments. They considered two rectangular configurations, one of them taking into account both the inside and the outside regions and the other one due to a confined field just in the inside of the region. They questioned which configuration was actually taken into account by performing the zeta function regularization, or in other words, if the zeta function knows anything about the outside region. The initial zeta function approach only took into account the inside region, which led to divergences and ambiguity in the definition of the energy. When using the piston formalism one arrives at a well defined expression of the energy between the plates, which comes from the regularization due to considering both regions.

Therefore, the zeta function regularization provides an excellent framework to work with piston configurations, as all the geometric structure is captured within the analytical nature of it [11, 12, 19]. More recent developments of the use of zeta function regularizations aim to exploit this built in geometric information by means of considering geometric information, such as boundaries, in an analytical way, by means of a potential [11,12]. In this way, boundaries that typify ideal materials can be replaced by potentials that can model real materials. These potentials appear when considering Laplace-type differential operators in the form of endomorphisms of vector bundles over the underlying manifold,

$$P = -g^{jk}\nabla_j^E \nabla_k^E - V\,. \tag{1.45}$$

In many practical applications, the vector bundle is taken to be a line bundle, in which case the endomorphism becomes a distribution over the underlying manifold. This consideration provides a way of analyzing real material and obtain results that can be numerically computed and compared with experiments.

1.5.5 Zeta Function Regularization

Theoretically, one encounters the difficulty of ill-defined quantities when dealing with quantum field theories in general. As mentioned before, in the context of the Casimir effect and the zero-point energy, one finds the necessity of evaluating divergent expressions. The different regularization methods attempt to solve the problem of having divergent expressions. The main goal of these regularizations is to obtain a finite answer that still matches the theory [11–13].

The zeta function regularization aims to make sense of divergent sums and products by means of exploiting the analytical structure of the expressions studied. This type of regularization method first appeared in a number theoretical context, when various people studied divergent series coming from problems in number theory [20].

Probably the most celebrated example of this type of regularizations is the Riemann zeta function. This function was first introduced by Euler who considered only a real parameter, as complex analysis was not yet developed at the time [8].

Although the idea of a complex number had been floating around since Cardano studied how to solve a general cubic, it was until the 19th century when Cauchy and Gauss among others developed the theory of complex analysis.

Euler was the first to systematically study the behavior of infinite series [20] and he defined this function in order to study the old problem of analyzing the *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots . \tag{1.46}$$

After establishing the divergence of the series, Euler started studying the *prime harmonic series*

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots \tag{1.47}$$

to see if this was divergent too. He introduced the zeta function

$$1^{+}\frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \dots \tag{1.48}$$

for values of positive integers k in order to solve this problem by establishing his famous relation between prime numbers and the zeta function [9]

$$\zeta(k) = \prod_p \frac{p^k}{p^k - 1} \tag{1.49}$$

for k a positive integer [8]. Then, by writing the harmonic series in terms of the prime harmonic series and using other results he had about the infinite product of prime numbers, he established the divergency of the prime harmonic series.

Dirichlet generalized the idea of this type of series also when studying number theoretical problems [20]. He was studying arithmetic sequences and posed the question if any arithmetic progression $\{a + dn\}_{n=0}^{\infty}$ with $\gcd(a, d) = 1$ will contain infinitely many primes. Here the $\gcd(p, q)$ gives the greatest common divisor of the numbers p and q.

For treating this, Dirichlet introduced the notion of a *Dirichlet series*

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{1.50}$$

where χ is a Dirichlet character, that is, a function χ from $\mathbb{Z}_d$, the set of integers modulo d, to the complex numbers, that has support on the multiplicative group of $\mathbb{Z}_b$ and is a group homomorphism between this group and the multiplicative group of the nonzero complex numbers. Most of Euler's results for infinite series can be expressed as relations between these Dirichlet series for different kinds of characters.

It was almost twenty years after Euler introduced the zeta function that he started talking about the concept of regularization. He somehow started formalizing the treatment of divergent series by assigning them values such that it would be possible to treat them in a more formal way. In his paper *De Seribvs Divergentibus* (On divergent series), Euler addresses the problem of dealing with divergent expressions and the controversy about the statement first made by of Leibniz when he stated that the alternating sum $(-1)^n$ could be interpreted as to have a finite value [9],

$$1-1+1-1+1-1+1-1+\cdots=\frac{1}{2}\,. \tag{1.51}$$

Euler defended Leibniz's position by making a distinction between the idea of the value of a sum when the series was convergent and when it was divergent. He envisioned the value of a divergent sum as the *value that gave rise to the sum* instead of the value that the partial sums approached. This could be the birth of analytic continuations, where one finds the value of divergent expressions by finding a function that generates the desired expression and then evaluating that at a special point.

This is the main idea involved in the zeta function regularization, where in order to make sense of a divergent sum

$$\sum_{n=1}^{\infty}\lambda_n \tag{1.52}$$

one defines an auxiliary function

$$\zeta_P(s)=\sum_{n=1}^{\infty}\lambda_n^{-s} \tag{1.53}$$

which would typically be defined for all complex numbers lying in a half plane $\Re(s) > \sigma$, where σ depends on the growth conditions of the sequence $\{\lambda_n\}_{n=1}^{\infty}$. Then, the function $\zeta_P(s)$ admits a meromorphic continuation on the entire complex plane, making it available to calculate values of the series outside the convergence region by means of the analytic continuation

$$\sum_{n=1}^{\infty} \lambda_n := \zeta_P(-1)\,. \tag{1.54}$$

The same idea can be used then for regularizing infinite products,

$$\prod_{n=1}^{\infty} \lambda_n\,, \tag{1.55}$$

which can be found by considering instead the series of logarithms

$$\sum_{n=1}^{\infty} \log \lambda_n \tag{1.56}$$

and using the derivative of the zeta function

$$\zeta_P'(s) = -\sum_{n=1}^{\infty} \lambda^{-s} \log \lambda_n\,. \tag{1.57}$$

It is possible to find the regularization of the logarithmic series as

$$\sum_{n=1}^{\infty} \log \lambda_n := -\zeta_P'(0)\,. \tag{1.58}$$

Hence the infinite product can be regularized to be

$$\prod_{n=1}^{\infty} \lambda_n := \exp(-\zeta_P(0))\,. \tag{1.59}$$

In many applications it is useful to think of λ_n as the eigenvalues of an elliptic (pseudo) differential operator, therefore the zeta function regularization provides, as mentioned before, a way to extend the notions of trace and determinant from the finite dimensional setting [11, 12].

If P is an elliptic (pseudo) differential operator with eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, it is possible to define P^{-s}, a complex power of P, by means of the spectral theorem, and define its zeta regularized trace to be

$$\text{tr}\ (P^{-s}) := \zeta_P(s)\,. \tag{1.60}$$

Also, it is possible to define a generalization of the usual determinant as the regularized product of the eigenvalues of P by

$$\det(P) := \exp(-\zeta_P'(0)) \quad . \tag{1.61}$$

On the other hand, it is also possible to define a functional determinant for an operator P arising from a physical system as a path integral

$$\left(\int_{\mathcal{M}} \mathcal{D}\phi \, e^{-\langle \phi, P\phi \rangle}\right)^{-2} \tag{1.62}$$

where $\mathcal{D}\phi$ is a measure in the configuration space [10]. This definition turns out to be divergent as it is the definition using the eigenvalues of the operator, which reinforces the necessity of using a regularization method. Usually instead of looking at determinants it is common to look at ratios of determinants. In this way the divergences cancel out leaving just a finite part. This can be thought as the equivalent of setting a reference as done when dealing with potential functions.

It was proved by Osgood, Phillips, and Sarnak that these two definitions of a functional determinant agree, in other words, they proved that both methods of regularizing the functional determinant lead to the same result [14].

1.5.6 One-loop Effective Action

The concept of a functional determinant becomes important for the dynamical description of a physical system. It appears as part of the effective action of a system in the first order quantum correction provided by the one-loop effective action. The action of a physical system is a functional that can be used to describe the behavior of the theory, as for instance, the equations of motion can be obtained from the principle of least action or stationary action, which basically states that the motion can be thought of as geodesics, where the system moves in such a way that the action is minimized [10].

The action of the system is usually defined by summing over all the states in between two points in the configuration space. Classically, this will just depend on time and the action can be realized as a time integral between the two states

$$S = \int \mathcal{L} dt \,, \tag{1.63}$$

where $\mathcal{L}$ is the Lagrangian of the system. When considering a quantum theory, the action is then defined instead by a path integral, which involves all the possible states between two configurations. Here the possible paths between the two points are usually not unique as in the classical case, leading to divergent expressions.

These difficulties for calculating the action of a system motivate seeking for alternative ways of approaching the description of the action. It is customary to approximate the behavior of the system by means of the classical action and then do perturbative analysis to find quantum corrections to the theory. This can be thought of as a semiclassical approximation to the action, where one describes the action as an asymptotic series in terms of a quantum parameter, which is taken to be small.

The effective action then can be thought of as the action arising from the classical action and the quantum corrections

$$\Gamma = S + \Sigma \,, \tag{1.64}$$

where Σ can be described as an asymptotic series in the quantum parameter $\hbar$,

$$\Sigma \sim \sum_{n=1}^{\infty} \Gamma_{(n)} \hbar^n \,, \tag{1.65}$$

for small values of $\hbar$. Here $\Gamma_{(n)}$ are the quantum corrections of order n [10]. In practice, it is customary to consider only the smallest corrections, the first order correction being the most important one. This is called the one-loop effective action and it is given by

$$\Gamma_{(1)} = \int_{\mathcal{M}} \mathcal{D}\phi \, e^{-\frac{i}{2}\phi^i \mathcal{L}_{ij} \phi^j} \,, \tag{1.66}$$

where $G = \mathcal{L}_{ij}^{-1}$ is the propagator, also called the Green's function of the Hamiltonian, between the states ϕ^i and ϕ^j [10]. From (1.62) one can see that the one-loop effective action is then given by the functional determinant of the inverse Green's function, which formally is given by the Hamiltonian.

Similarly, the next order quantum corrections can be found by considering higher correlation functions, which can be found using the partition function

$$Z[J] = e^{-iE[J]} \tag{1.67}$$

where $E[J]$ is the energy functional and J is the source field, and then taking the Taylor expansion in the source field.

CHAPTER TWO

Delta Potentials

Different potentials can describe piston configurations where the boundary conditions model the boundaries of the piston and the middle plate is modeled by a potential. One of such potentials is a delta potential, which is associated with materials sensitive just to some frequencies, also called semitransparent materials. For this discussion we consider the localized potential

$$V(x) = \sigma\delta(x - a) \tag{2.1}$$

with σ a positive constant. One important fact is that when the coupling constant σ is zero, the parallel plate configuration is recovered, and when $\sigma \to \infty$ the case with 3 perfectly conducting parallel plates is recovered, so this discussion generalizes these two configurations.

2.1 Differential Equation

The piston configuration can be modeled by a boundary value problem. For a massless scalar field in the manifold $\mathcal{M} = [0, L] \times \mathcal{N}$, where $0 < a < L$ and $\mathcal{N}$ is a smooth compact $d - 1$ dimensional Riemannian manifold , the second order differential operator

$$P = -\frac{\partial^2}{\partial x^2} - \Delta_{\mathcal{N}} + V(x) \tag{2.2}$$

gives the Hamiltonian of the piston configuration and provides the energy eigenvalues by considering the eigenvalue problem

$$P\mu_\lambda = \lambda^2\mu_\lambda \tag{2.3}$$

with Dirichlet boundary conditions at $x = 0$ and $x = L$, and requiring μ_λ to be continuous.

One way to explicitly solve the eigenvalue problem is to use separation of variables and write

$$\mu(x,n) = X(x)N(n) \tag{2.4}$$

with $x \in [0,L]$ and $n \in \mathcal{N}$. Recall that the Laplacian is separable in $\mathcal{M}$ and thus we have that if ν_k^2 are the eigenvalues for the Laplacian on the interval $[0,L]$ and η_l^2 are the eigenvalues for the Laplacian on $\mathcal{N}$, we can rewrite the eigenvalue problem on $\mathcal{M}$ as

$$P\mu_{k,l} = (\nu_k^2 + \eta_l^2)\mu_{k,l}\,. \tag{2.5}$$

Solving the eigenvalue problem on $[0,L]$,

$$-X''(x) + \sigma\delta(x-a)X(x) = \nu^2 X(x)\,, \tag{2.6}$$

gives the solutions for the eigenfunctions on the interval

$$X_k(x) = \begin{cases} X_{1,k}(x) = A\sin(\nu_k x) & 0 < x < a \\ X_{2,k}(x) = B\sin(\nu_k(L-x)) & a < x < L \end{cases} \tag{2.7}$$

where A, B are constants.

A natural condition is to require continuity of μ, which makes X to be continuous at $x = a$ and hence

$$A\sin(\nu_k a) = B\sin(\nu_k(L-a))\,. \tag{2.8}$$

The presence of the term $\sigma\delta(x-a)$ in (2.2) makes the derivative of X_k to have a jump discontinuity at $x = a$,

$$X'_{2,k}(a) - X'_{1,k}(a) = \sigma X_k(a)\,. \tag{2.9}$$

Excluding the case where the wave vanishes at the piston, for technical reasons we are going to assume incommensurable lengths, i.e.

$$\frac{a}{L} \notin \mathbb{Q}\,, \tag{2.10}$$

where the case of commensurable lengths can be obtained back by continuity. This gives $X_k(a) \neq 0$ and thus we have

$$\frac{X'_{2,k}(a)}{X_{2,k}(a)} - \frac{X'_{1,k}(a)}{X_{1,k}(a)} = \sigma \, . \tag{2.11}$$

With this we have the characteristic equation

$$-\nu \cot(\nu a) - \nu \cot(\nu(L-a)) = \sigma \tag{2.12}$$

that will give the eigenvalues for the Laplacian on $[0, L]$. In order to do the spectral zeta function formalism for this problem, it is possible to modify the behavior of the characteristic equation, as the left hand side of this equation has singularities at

$$x = \frac{\pi n}{a} \quad \text{and} \quad x = \frac{\pi n}{L-a} \, , \tag{2.13}$$

for $n \in \mathbb{Z}^+$, hence we can improve this by multiplying the characteristic equation by $\sin(a\nu) \sin\left((L-a)\nu\right)$ which makes the characteristic equation to be

$$\frac{1}{\nu^2}(\sigma \sin(\nu a) \sin(\nu(L-a)) + \nu \sin(\nu L)) = 0 \, . \tag{2.14}$$

In this equation, no singularities in the complex plane occur and it is non zero at $\nu = 0$.

2.2 *Associated Zeta Function*

Since P is a second order differential operator, we can define the spectral zeta functions as

$$\zeta_P(s) = \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \lambda_{k,l}^{-2s} = \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} (\nu_k^2 + \eta_l^2)^{-s} \tag{2.15}$$

for $\Re(s) > d/2$.

Defining the spectral function in this fashion and using (2.14) together with the Cauchy Residue Theorem enables us to write an integral representation for (2.15) as [11]

$$\zeta_P(s) = \frac{1}{2\pi i} \sum_{l=1}^{\infty} \int_{\gamma_l} d\nu (\nu^2 + \eta_l^2)^{-s} \frac{d}{d\nu} \log(F(\nu)) \tag{2.16}$$

where the function $F(\nu)$ is given by

$$F(\nu) = \sigma \frac{\sin(\nu a)\sin(\nu(L-a))}{\nu^2} + \frac{\sin(\nu L)}{\nu} \tag{2.17}$$

and γ_l is a path enclosing the eigenvalues ν_k.

2.3 Contour Deformation

On analyzing the l behavior of (2.16), we are going to deform each contour γ_l to be the imaginary axis. Mainly, the idea is to fix l and analyze each summand separately. A contour deformation in (2.16) is possible since F does not have any singularities in the complex plane besides the eigenvalues, has a non zero behavior near zero, and the zeta function does not present any residue at infinity. Now, for this purpose, we have to determine the convergence region where the integral representation is valid by considering the behavior of the integrand near zero and infinity. Dealing with the asymptotic behavior near infinity and near zero will enable us to freely deform γ_l onto the imaginary axis. Since we have a power of a complex number, we have to analyze the phase that the term $(\nu^2 + \eta_l^2)^{-s}$ will have when $\Im(\nu) > \eta_l$ and $\Im(\nu) < \eta_l$.

Hence we begin by analyzing the asymptotic behavior of $F(\imath x)$ for large x. Examining the case when $x \to \infty$, we have that the asymptotic behavior of the characteristic function

$$F(\imath x) \sim \left(\frac{1}{2x} + \frac{\sigma}{4x^2}\right) e^{Lx} \tag{2.18}$$

which makes the integral defined in (2.16) to converge for $\Re(s) > 1/2$.

With this we control the convergence at infinity of the integral. In order to ensure full convergence, the next restriction is given by the behavior near zero. Looking at the behavior of F at zero, we find that the only restriction for convergence comes from the term $\nu^2 + \eta_l^2$, as the rest of the integrand in non-zero. Let $\nu = \imath\eta_l$, and since F is an even function, it is enough to analyze this case. Putting the contribution of the integral near this point to be zero provides an upper limit for the real part of s in the convergence region of this integral representation.

Regarding this, consider a small semi circle C_ϵ of radius $\epsilon > 0$ around the point $\nu = \imath\eta_l$. On this semicircle we have that the contribution of the integral is given by

$$\int_{C_\epsilon} d\nu (\nu^2 + \eta_l^2)^{-s} \frac{d}{d\nu} \log F(\nu)\,, \tag{2.19}$$

where we parametrize C_ϵ by $\nu = \imath\eta_l + \epsilon e^{i\theta}$, where $-\pi/2 \leq \theta \leq \pi/2$.

Therefore, we have that

$$\begin{aligned} &\left| \int_{C_\epsilon} d\nu (\nu^2 + \eta_l^2)^{-s} \frac{d}{d\nu} \log F(\nu) \right| \\ &\leq \int_{C_\epsilon} \left| (\nu^2 + \eta_l^2)^{-s} \frac{d}{d\nu} \log F(\nu) \right| d\nu \\ &\leq \int_{-\pi/2}^{\pi/2} \left| i\epsilon e^{i\theta} \left(2i\eta_l \epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}\right)^{-s} C_{\eta_l} \right| d\theta \\ &= C_{\eta_l} \epsilon^{1-s} \int_{-\pi/2}^{\pi/2} \left| \left(2i\eta_l e^{i\theta} + \epsilon e^{2i\theta}\right)^{-s} \right| d\theta\,, \end{aligned} \tag{2.20}$$

where C_{η_l} is the supremum of $\left| \frac{d}{d\nu} \log F(\nu) \right|$ on C_ϵ. Earlier we stated that F is meromorphic and does not vanish on the imaginary axis. Combining this with the fact that C_{η_l} is bounded as $\epsilon \to 0$ will make the integral tend to zero as ϵ goes to zero whenever we have that $1 - \Re(s) > 0$.

Thus we can deform the path γ_l to pass through the points $\nu = \pm i\eta_l$ without affecting the value of the integral, giving the integral in (2.16) to be convergent in the region $1/2 < \Re(s) < 1$.

Yet we have to analyze the argument of $(\nu^2 + \eta_l^2)^{-s}$ as ν approaches the imaginary axis. On analyzing this we will consider the case when ν has a positive and a negative argument.

Under the assumption that ν approaches the positive imaginary axis, ν will have a positive argument tending to $\pi/2$ and we can write $\nu = xe^{\imath\pi/2}$. Looking at this we have that for $0 < x < \eta_l$,

$$(\nu^2 + \eta_l^2)^{-s} = (\eta_l^2 - x^2)^{-s} \tag{2.21}$$

and for $x > \eta_l$,

$$(\nu^2 + \eta_l^2)^{-s} = (x^2 - \eta_l^2)^{-s} e^{-\imath\pi s} . \tag{2.22}$$

On the other hand, for ν approaching the negative imaginary axis, we can write $\nu = x e^{-\imath\pi/2}$ and then for $0 < x < \eta_l$,

$$(\nu^2 + \eta_l^2)^{-s} = (\eta_l^2 - x^2)^{-s} \tag{2.23}$$

and for $x > \eta_l$,

$$(\nu^2 + \eta_l^2)^{-s} = (x^2 - \eta_l^2)^{-s} e^{\imath\pi s} . \tag{2.24}$$

Via this formalism we find that the integral in (2.16) takes the form

$$\begin{aligned}
&\frac{e^{-\imath\pi s}}{2\pi\imath} \int_{\infty}^{\eta_l} dx (x^2 - \eta_l^2)^{-s} \frac{d}{dx} \log(F(\imath x)) + \frac{1}{2\pi\imath} \int_{\eta_l}^{0} dx (\eta_l^2 - x^2)^{-s} \frac{d}{dx} \log(F(\imath x)) \\
&+ \frac{1}{2\pi\imath} \int_{0}^{\eta_l} dx (\eta_l^2 - x^2)^{-s} \frac{d}{dx} \log(F(\imath x)) + \frac{e^{\imath\pi s}}{2\pi\imath} \int_{\eta_l}^{\infty} dx (x^2 - \eta_l{}^2)^{-s} \frac{d}{dx} \log(F(\imath x)) \\
=&\frac{\sin(\pi s)}{\pi} \int_{\eta_l}^{\infty} dx (x^2 - \eta_l^2)^{-s} \frac{d}{dx} \log(F(\imath x)) .
\end{aligned} \tag{2.25}$$

2.4 Analytic Continuation

Extending the convergence region of the above representation from $1/2 < \Re(s) < 1$ to the left is important in order to obtain a suitable expression for calculating the residues and special values of the spectral zeta function. Here we perform the analytic continuation by subtracting the behavior at infinity of the characteristic function from the integral representation of the zeta function.

Expanding the logarithm of the characteristic function at infinity, we find the asymptotic expansion of the characteristic function $F(\imath x)$ for large x

$$\log\left(\left(\frac{1}{2x} + \frac{\sigma}{4x^2}\right) e^{Lx}\right) \sim Lx - \log(2x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{\sigma}{2x}\right)^n . \tag{2.26}$$

Regarding the behavior of a fixed mode l, we have that the integral can be written as

$$Z^l(s) + \sum_{i=-1}^{N-2} A_i^l(s) \tag{2.27}$$

where $Z^l(s)$ is the finite part of the integral with N asymptotic terms subtracted, and A_i^l is the asymptotic term where the x^{-i} behavior is subtracted, that is

$$\begin{aligned} Z^l(s) = \frac{\sin(\pi s)}{\pi} \int_{\eta_l}^{\infty} & dx (x^2 - \eta_l^2)^{-s} \\ & \times \frac{d}{dx} \left(\log F(\imath x) - Lx + \log(2x) - \sum_{n=1}^{N-2} \frac{(-1)^{n+1}}{n} \left(\frac{\sigma}{2x} \right)^n \right) \end{aligned} \tag{2.28}$$

and the i-th contributions are given by

$$A_{-1}^l(s) = \frac{\sin(\pi s)}{\pi} \int_{\eta_l}^{\infty} dx (x^2 - \eta_l^2)^{-s} \frac{d}{dx} (Lx) \tag{2.29}$$

$$A_0^l(s) = \frac{\sin(\pi s)}{\pi} \int_{\eta_l}^{\infty} dx (x^2 - \eta_l^2)^{-s} \frac{d}{dx} (-\log(2x)) \tag{2.30}$$

$$A_i^l(s) = \frac{\sin(\pi s)}{\pi} \int_{\eta_l}^{\infty} dx (x^2 - \eta_l^2)^{-s} \frac{d}{dx} \left(\frac{(-1)^{i+1}}{i} \left(\frac{\sigma}{2x} \right)^i \right) . \tag{2.31}$$

After performing the summation over l, we find the analytic continuation of the integral representation for the zeta function obtained in (2.16) to be

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{N} A_i(s) , \tag{2.32}$$

with $Z(s)$ given by

$$\begin{aligned} Z(s) = \sum_{l=1}^{\infty} Z^l(s) = \sum_{l=1}^{\infty} \frac{\sin(\pi s)}{\pi} \int_{\eta_l}^{\infty} & dx (x^2 - \eta_l^2)^{-s} \\ & \times \frac{d}{dx} \left(\log F(\imath x) - Lx + \log(2x) - \sum_{n=1}^{N-2} \frac{(-1)^{n+1}}{n} \left(\frac{\sigma}{2x} \right)^n \right) \end{aligned} \tag{2.33}$$

and with the asymptotic contributions

$$A_{-1}(s) = \sum_{l=1}^{\infty} A_{-1}^l(s) = \sum_{l=1}^{\infty} \eta_l^{1-2s} \frac{L\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)} \tag{2.34}$$

$$A_0(s) = \sum_{l=1}^{\infty} A_0^l(s) = -\frac{1}{2} \sum_{l=1}^{\infty} \eta_l^{-2s} \tag{2.35}$$

$$A_i(s) = \sum_{l=1}^{\infty} A_i^l(s) = \sum_{l=1}^{\infty} (-1)^i \eta_l^{-2(s+i/2)} \left(\frac{\sigma}{2}\right)^i \frac{\Gamma(s+i/2)}{2\Gamma(1+i/2)\Gamma(s)} \,. \tag{2.36}$$

Now, writing $\zeta_{\mathcal{N}}(s)$ for the zeta function arising from the Laplacian on $\mathcal{N}$, we have that the asymptotic terms can be written as

$$A_{-1}(s) = \zeta_{\mathcal{N}}(s-1/2) \frac{L\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)} \,, \tag{2.37}$$

$$A_0(s) = -\frac{1}{2}\zeta_{\mathcal{N}}(s) \,, \tag{2.38}$$

$$A_i(s) = (-1)^i \zeta_{\mathcal{N}}(s+i/2) \left(\frac{\sigma}{2}\right)^i \frac{\Gamma(s+i/2)}{2\Gamma(1+i/2)\Gamma(s)} \,. \tag{2.39}$$

2.5 Functional Determinant and Casimir Force

Due to the fact that the integral representation (2.16) for the zeta function is convergent for $\Re(s) > d/2$, in order to include the points $s = 0$ and $s = -1/2$ to calculate the functional determinant for the one-loop effective action and the Casimir force, we need to subtract at least $d+1$ and $d+2$ asymptotic terms respectively to have a convergent expression.

2.5.1 Functional Determinant

Subtracting $d+1$ asymptotic terms makes the representation for the spectral zeta function to have a convergence region including the point $s = 0$, which enables us to calculate the derivative at this point. Hence we use

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{d-1} A_i(s) \tag{2.40}$$

which is valid in the half plane $\Re(s) > -1/2$, where we are only including the first $d+1$ asymptotic terms in the definition of $Z(s)$,

$$Z(s) = \sum_{l=1}^{\infty} \frac{\sin(\pi s)}{\pi} \int_{\eta_l}^{\infty} dx (x^2 - \eta_l^2)^{-s}$$

$$\times \frac{d}{dx}\left(\log F(\imath x) - Lx + \log(2x) - \sum_{i=1}^{d-1} \frac{(-1)^{i+1}}{i}\left(\frac{\sigma}{2x}\right)^i\right) . \quad (2.41)$$

Exploiting this representation, we can find $\zeta_P'(0)$ by finding the derivatives of the finite part and the asymptotic terms. With the finite part, we find the derivative to be

$$Z'(0) = -\sum_{l=1}^{\infty}\left(\log F(\imath\eta_l) - L\eta_l + \log(2\eta_l) - \sum_{i=1}^{d-1} \frac{(-1)^{i+1}}{i}\left(\frac{\sigma}{2\eta_l}\right)^n\right) . \quad (2.42)$$

In order to find the contributions coming from the asymptotic terms we use the meromorphic structure of the A_i given in (2.37).

Looking at the structure of A_i makes us require a detailed analysis of its meromorphic structure together with the properties of the zeta function on $\mathcal{N}$ in order to be able to calculate the derivative at zero. Let k be a complex number. We have that around $s = 0$

$$\zeta_{\mathcal{N}}(s+k) = \frac{\text{Res } \zeta_{\mathcal{N}}(k)}{s} + \text{FP } \zeta_{\mathcal{N}}(k) + O(s) \quad (2.43)$$

where Res $\zeta_{\mathcal{N}}(k)$ would be zero if k is not a pole of $\zeta_{\mathcal{N}}$. Also, we have that

$$\frac{1}{\Gamma(s)} = s + \gamma s^2 + O(s^3) \quad (2.44)$$

where γ is Euler-Mascheroni's constant. Thus we have that

$$\frac{\zeta_{\mathcal{N}}(s+k)}{\Gamma(s)} = \text{Res } \zeta_{\mathcal{N}}(k) + (\text{FP } \zeta_{\mathcal{N}}(k) + \gamma\text{Res } \zeta_{\mathcal{N}})\, s + O(s^2) \quad (2.45)$$

which gives

$$\frac{d}{ds}\left.\frac{\zeta_{\mathcal{N}}(s+k)}{\Gamma(s)}\right|_{s=0} = \text{FP } \zeta_{\mathcal{N}}(k) + \gamma\text{Res } \zeta_{\mathcal{M}} \quad (2.46)$$

and therefore

$$\frac{d}{ds}\left.\frac{\zeta_{\mathcal{N}}(s+k)\Gamma(s+k)}{\Gamma(s)}\right|_{s=0} = (\text{FP } \zeta_{\mathcal{N}}(k) + \gamma\text{Res } \zeta_{\mathcal{N}}(k))\, \Gamma(k) + \text{Res } \zeta_{\mathcal{N}}(k)\Gamma'(k) . \quad (2.47)$$

With this expression we are able to find the derivatives of the asymptotic terms A_i. The meromorphic structure of $\zeta_{\mathcal{N}}$ gives that it has simple poles at $s = \frac{d-1}{2}, \frac{d-2}{2}, \ldots, \frac{1}{2}, -\frac{2j+1}{2}$ for $j \in \mathbb{N}$, with residues

$$\text{Res } \zeta_{\mathcal{N}}(k) = \frac{a_{(d-1)/2-k}}{\Gamma(k)} \tag{2.48}$$

where the a_n are the heat kernel coefficients of the Laplacian for the manifold $\mathcal{N}$. Therefore we can find the derivatives of the asymptotic terms to be

$$\begin{aligned} A'_{-1}(0) = \frac{L}{2\sqrt{\pi}} & \left[\left(\text{FP } \zeta_{\mathcal{N}}(-1/2) + \gamma \text{Res } \zeta_{\mathcal{N}}(-1/2) \right) \Gamma(-1/2) \right. \\ & \left. + \text{Res } \zeta_{\mathcal{N}}(-1/2) \Gamma'(-1/2) \right] , \end{aligned} \tag{2.49}$$

$$A'_0(0) = -\frac{1}{2} \zeta'_{\mathcal{N}}(0) , \tag{2.50}$$

$$A'_i(0) = \frac{(-1)^i \sigma^i}{2^{i+1}} \left(\text{FP } \zeta_{\mathcal{N}}(i/2) + \gamma \text{Res } \zeta_{\mathcal{N}}(i/2) \right) \Gamma(i/2) + \text{Res } \zeta_{\mathcal{N}}(i/2) \Gamma'(i/2) , \tag{2.51}$$

and hence the functional determinant for P is found to be

$$\exp(-\zeta'_{\mathcal{M}}(0)) = \exp\left(-Z'(0) - \sum_{i=-1}^{d-1} A'_i(0) \right) . \tag{2.52}$$

This is the most general result that can be obtained without explicitly specifying the manifold $\mathcal{N}$. Once $\mathcal{N}$ is given, a more detailed result can be computed, either analytically, if the zeta function for $\mathcal{N}$ is given, or numerically.

2.5.2 Casimir Force

To calculate the Casimir force we need to subtract $d+2$ asymptotic terms in order to include the point $s = -1/2$ in the convergence region. We use the representation

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{d} A_i(s) \tag{2.53}$$

which is valid for $\Re(s) > -1$, where $Z(s)$ only includes the first $d+2$ asymptotic terms

$$Z(s) = \sum_{l=1}^{\infty} \frac{\sin(\pi s)}{\pi} \int_{\eta_l}^{\infty} dx (x^2 - \eta_l^2)^{-s}$$
$$\times \frac{d}{dx} \left(\log F(\imath x) - Lx + \log(2x) - \sum_{i=1}^{d} \frac{(-1)^{i+1}}{i} \left(\frac{\sigma}{2x} \right)^i \right) . \quad (2.54)$$

The associated force is then given by

$$F_{Cas} = -\frac{1}{2} \frac{\partial}{\partial a} \zeta_P \left(-\frac{1}{2} \right) \quad (2.55)$$

which will only include te contribution from $Z(s)$ as the asymptotic terms do not depend on the position of the piston. Thus

$$F_{Cas} = \frac{1}{2\pi} \frac{\partial}{\partial a} \sum_{l=1}^{\infty} \int_{\eta_l}^{\infty} dx (x^2 - \eta_l^2)^{1/2}$$
$$\times \frac{d}{dx} \left(\log F(\imath x) - Lx + \log(2x) - \sum_{n=1}^{d} \frac{(-1)^{n+1}}{n} \left(\frac{\sigma}{2x} \right)^n \right) \quad (2.56)$$

and using the Leibniz Integral Rule we can bring the partial derivative inside the integral which gives

$$F_{Cas} = \frac{1}{2\pi} \sum_{l=1}^{\infty} \int_{\eta_l}^{\infty} dx (x^2 - \eta_l^2)^{1/2} \frac{\partial}{\partial a} \frac{d}{dx} \log(F(\imath x))$$
$$= \frac{1}{2\pi} \sum_{l=1}^{\infty} \int_{\eta_l}^{\infty} dx (x^2 - \eta_l^2)^{1/2} \left(\frac{F_a'(\imath x) F(\imath x) - F_a(\imath x) F'(\imath x)}{F^2(\imath x)} \right) . \quad (2.57)$$

This expression gives the Casimir force in terms of the characteristic equation (2.14). It is possible to find a definite sign of the force by analyzing the behavior of the characteristic function F. Consider

$$\frac{F_a(\imath x)}{F(\imath x)} = \frac{\sigma x \sinh((L-2a)x)}{x \sinh(Lx) + \sigma \sinh(ax) \sinh((L-a)x)}$$
$$= \left(\frac{\sinh(Lx)}{\sigma \sinh((L-2a)x)} + \frac{\sinh(ax) \sinh((L-a)x)}{x \sinh((L-2a)x)} \right)^{-1} \quad (2.58)$$

as a function of a for fixed x. Our goal is to find the behavior of the previous expression for $0 < a < L/2$ and $L/2 < a < L$.

For the first case, let $0 < a < L/2$. By the Appendix A we have that the functions

$$\frac{\sinh(Lx)}{\sinh((L-2a)x)} \qquad \text{and} \qquad \frac{\sinh((L-a)x)}{\sinh((L-2a)x)} \tag{2.59}$$

are increasing. Also, we have that the function

$$\frac{\sinh(ax)}{x} \tag{2.60}$$

is increasing, so we have that

$$\frac{\sinh(Lx)}{\sigma \sinh((L-2a)x)} + \frac{\sinh(ax)\sinh((L-a)x)}{x \sinh((L-2a)x)} \tag{2.61}$$

is increasing and hence (2.58) is decreasing for $0 < a < L/2$.

Since the function defined in (2.58) is odd with respect to $a = L/2$, we have that for the case when $L/2 < a < L$ (2.58) has the same magnitude for a as for $L - a$ but with a negative sign making it increasing. Therefore

$$\frac{d}{dx}\frac{F_a(\imath x)}{F(\imath x)} = \frac{\partial}{\partial a}\frac{d}{dx}\log(F(\imath x)) \tag{2.62}$$

is negative for $0 < a < L/2$ and it is positive for $L/2 < a < L$. Thus, the Casimir force is negative for $0 < a < L/2$ and positive for $L/2 < a < L$ making the piston to move toward the closest wall.

Notice that no explicit knowledge of the manifold $\mathcal{N}$ is needed to achieve this result, hence making the behavior of the piston being independent of the extra dimensions contained in the manifold $\mathcal{N}$.

CHAPTER THREE

Rectangular Potentials

Here we are going to consider a piston configuration where the potential is given by a rectangular function

$$V(x) = \begin{cases} \sigma, & a-w<x<a+w \\ 0, & \text{elsewhere} \end{cases} \tag{3.1}$$

with $\sigma > 0$, $0 < a < L$ and $0 < w < \min(a, L-a)$. This provides in some sense a generalization of the discussion made in the previous chapter, as the result obtained for the delta potentials can be seen as having the restriction that σw is a constant and then making $w \to 0$. Rectangular potentials arise from analyzing potential wells that are common, for instance, when studying semiconductors.

3.1 One Dimension

First we consider the one dimensional case where the manifold the piston lives in the interval $[0, L]$. By studying this case we can afterwards generalize the results obtained in this section to higher dimensions by exploiting the separability of the Laplacian on products of compact manifolds and by calculating more asymptotic terms of the characteristic equation for large eigenvalues.

3.1.1 Differential Equation

Consider a piston on the interval $[0, L]$ with a rectangular potential given by (3.1), and analyze the eigenvalue problem

$$P\mu_\lambda = \lambda^2 \mu_\lambda\,, \tag{3.2}$$

where P is the second order differential operator

$$P = -\frac{\partial^2}{\partial x^2} + V(x)\,, \tag{3.3}$$

and where we consider Dirichlet boundary conditions at $x = 0$ and $x = L$.

In order to do the zeta function formalism, consider a mass $m > 0$, so the eigenvalue problem (3.2) is equivalent to

$$P_m \mu_\nu = \nu^2 \mu_\nu\,, \tag{3.4}$$

with $m^2 > \sigma$, $\nu^2 = \lambda^2 + m^2$, and the second order differential operator P_m is given by

$$P_m = -\frac{\partial^2}{\partial x^2} + V(x) + m^2\,. \tag{3.5}$$

This consideration is made in order to avoid the case of negative eigenvalues. With this consideration, we can study the behavior of P_m and then take $m \to 0$ to recover the original set up with P.

To solve the eigenvalue problem (3.2), label the regions $0 < x < a - w$, $a - w < x < a + w$ and $a - w < x < L$ by I, II and III respectively. Solving the differential equation in I and III gives

$$\mu_I(x) = A_I \mu_{I,1}(x) \quad \text{and} \quad \mu_{III}(x) = A_{III} \mu_{III,1}(x)\,, \tag{3.6}$$

with A_I, A_{III} constants, $\mu_{I,1} = \sin(\lambda x)$, and $\mu_{III,1}(x) = \sin(\lambda(L - x))$. On the other hand, solving the equation in the middle region gives

$$\mu_{II}(x) = A_{II} \mu_{II,1}(x) + B_{II} \mu_{II,2}(x) \tag{3.7}$$

with A_{II}, B_{II} constants, $\mu_{II,1}(x) = \sin\left(\sqrt{\lambda^2 - \sigma} x\right)$, and $\mu_{II,2}(x) = \cos\left(\sqrt{\lambda^2 - \sigma} x\right)$.

Now, we require the solutions to be continuous and with continuous first derivatives, i.e. we need to have

$$F(\lambda)\begin{pmatrix} A_I \\ A_{II} \\ B_{II} \\ A_{III} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.8}$$

with

$$F(\nu) = \begin{pmatrix} -\mu_{I,1}(a-w) & \mu_{II,1}(a-w) & \mu_{II,2}(a-w) & 0 \\ 0 & \mu_{II,1}(a+w) & \mu_{II,2}(a+w) & -\mu_{III,1}(a+w) \\ -\mu'_{I,1}(a-w) & \mu'_{II,1}(a-w) & \mu'_{II,2}(a-w) & 0 \\ 0 & \mu'_{II,1}(a+w) & \mu'_{II,2}(a+w) & -\mu'_{III,1}(a+w) \end{pmatrix}. \tag{3.9}$$

In order to have non-trivial solutions for μ, we require (3.8) to have non-trivial solutions for $A_I, A_{II}, B_{II}, A_{III}$. To achieve this, we must require the determinant of $F(\nu)$ to be zero, which will give a characteristic equation for the eigenvalues

$$f(\lambda) = \det\left(F(\lambda)\right) = 0\,. \tag{3.10}$$

3.1.2 Associated Zeta Function

With this characteristic equation it is possible to write the integral representation for the zeta function associated with the massive case P_m as

$$\zeta_{P_m}(s) = \frac{1}{2\pi\imath}\int_\gamma d\lambda\,(\lambda^2+m^2)^{-2s}\frac{d}{d\lambda}\log(f(\lambda)) \tag{3.11}$$

for γ a contour enclosing the eigenvalues ν.

Using the relation between ν and λ, we find the zeta function in terms of λ after performing the contour deformation to be

$$\zeta_{P_m}(s) = \frac{\sin(\pi s)}{\pi}\int_m^\infty d\lambda\,(\lambda^2-m^2)^{-s}\frac{d}{d\lambda}\log(f(\imath\lambda))\,, \tag{3.12}$$

where we used the fact that $\nu^2 = \lambda^2 + m^2$.

3.1.3 Analytic Continuation

The expression obtained in (3.12) is only valid for $\Re(s) > 1/2$. Then, to find the analytic continuation for the zeta function, we find the asymptotic behavior of $f(\imath\lambda)$ for $\lambda \to \infty$ and then we can perform the analytic continuation by subtracting asymptotic terms, which will extend the convergence region to the left.

We find the asymptotic expansion of the logarithm of the characteristic function to be

$$\log(f(\imath\lambda)) \sim L\lambda - \log(2) + 2\log(\lambda) - \frac{w\sigma}{2}\lambda^{-1} + O\left(\lambda^{-2}\right) \tag{3.13}$$

and therefore we can write the zeta function as

$$\zeta_{P_m}(s) = Z(s) + \sum_{i=-1}^{1} A_i(s)\,, \tag{3.14}$$

where we have the finite part given by

$$Z(s) = \frac{\sin(\pi s)}{\pi}\int_m^\infty d\lambda\,(\lambda^2 - m^2)^{-s} \times \frac{d}{d\lambda}\left(\log(f(\imath\lambda)) - \left(L\lambda + \log(\lambda^2/2) - \frac{w\sigma}{2}\lambda^{-1}\right)\right)\,, \tag{3.15}$$

and the first asymptotic terms are given by

$$A_{-1}(s) = \frac{\sin(\pi s)}{\pi}\int_m^\infty d\lambda\,(\lambda^2 - m^2)^{-s}\frac{d}{d\lambda}\left(L\lambda\right)\,, \tag{3.16}$$

$$A_0(s) = \frac{\sin(\pi s)}{\pi}\int_m^\infty d\lambda\,(\lambda^2 - m^2)^{-s}\frac{d}{d\lambda}\left(\log(\lambda^2/2)\right)\,, \tag{3.17}$$

$$A_1(s) = \frac{\sin(\pi s)}{\pi}\int_m^\infty d\lambda\,(\lambda^2 - m^2)^{-s}\frac{d}{d\lambda}\left(-\frac{w\sigma}{2}\lambda^{-1}\right)\,. \tag{3.18}$$

After performing the integration, we have that the asymptotic terms have the form

$$A_{-1}(s) = \frac{Lm^{1-2s}\Gamma\left(s - \frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(s)}\,, \tag{3.19}$$

$$A_0(s) = m^{-2s}\,, \tag{3.20}$$

$$A_1(s) = \frac{w\sigma m^{-1-2s}\Gamma\left(s+\frac{1}{2}\right)}{2\sqrt{\pi}\Gamma(s)}\,, \tag{3.21}$$

which make the expression for the zeta function obtained in (3.14) to be valid for $\Re(s) > -1$.

3.1.4 Functional Determinant and Casimir Force

As before, in order to calculate the functional determinant and the Casimir force, we need to subtract 2 and 3 asymptotic terms respectively.

3.1.5 Functional Determinant

With these expressions we can calculate the derivative of the zeta function at $s = 0$ by evaluating the finite part and the asymptotic parts,

$$Z'(0) = -\log(f(m)) + \left(Lm + \log(m^2/2)\right)\,, \tag{3.22}$$

and

$$A'_{-1}(0) = -Lm\,, \tag{3.23}$$

$$A'_0(0) = -2\log m\,. \tag{3.24}$$

Hence the functional determinant is given by

$$\exp\left(-\zeta'_{P_m}(0)\right) = \exp\left(\log(f(m)) + \log(2)\right) = 2f(m) \tag{3.25}$$

for the massive case. Therefore the massless case can be obtained by taking $m \to 0$, which gives the functional determinant for the operator P to be

$$\exp\left(-\zeta'_P(0)\right) = \lim_{m\to 0} 2f(m)\,. \tag{3.26}$$

3.1.6 Casimir Force

The Casimir force is defined as usual by

$$F_{Cas} = -\frac{1}{2}\frac{\partial}{\partial a}\zeta_P\left(-\frac{1}{2}\right) , \tag{3.27}$$

thus by exchanging the integral and differential sign, we find that the force is given by

$$F_{Cas} = \frac{1}{2\pi}\left(\int_m^\infty d\lambda(\lambda^2 - m^2)^{1/2} \right. \\ \left. \times\frac{\partial}{\partial a}\frac{d}{d\lambda}\left(\log(f(\imath\lambda)) - L\lambda + \log(2) - 2\log(\lambda) + \frac{w\sigma}{2}\lambda^{-1}\right)\right) . \tag{3.28}$$

Analyzing the behavior of $\log(f(\imath\lambda))$ as a function of λ for fixed a gives the sign of the integrand in (3.28) and hence it determines the sign of the Casimir force.

In order to determine whether $\frac{\partial}{\partial a}\log(f(\imath\lambda)) = \frac{f_a(\imath\lambda)}{f(\imath\lambda)}$ is increasing or decreasing in λ, we can work instead for simplicity with

$$\frac{f(\imath\lambda)}{f_a(\imath\lambda)} = \frac{\sinh[(L-w)\lambda]}{2\lambda\sigma\sinh[(2a-L)\lambda]}\left((2\lambda^2-\sigma)\coth[(L-w)\lambda] \right. \\ \left. +2\lambda\sqrt{\lambda^2-\sigma}\coth[w\sqrt{\lambda^2-\sigma}] + \sigma\frac{\cosh[(2a-L)\lambda]}{\sinh[(L-w)\lambda]}\right) . \tag{3.29}$$

We are going to analyze this function in parts and determine whether each of the parts is increasing or decreasing.

First, we note that the previous expression is odd with respect to $L/2$ as a function of a, hence it suffices to analyze the case where $2a - L > 0$.

Consider then that $2a - L > 0$. Since $L - w > 2c - L$, we have that

$$\frac{\sinh[(L-w)\lambda]}{\sinh[(2c-L)\lambda]} \tag{3.30}$$

is positive and increasing in λ.

With the remaining factor, we are going to replace it by something that decays faster and prove that it still is increasing. Replace the remaining part

$$a(\lambda) = (2\lambda^2-\sigma)\coth[(L-w)\lambda] + \sigma\frac{\cosh[(2c-L)\lambda]}{\sinh[(L-w)\lambda]} \tag{3.31}$$

by $kb(\lambda)$, where

$$b(\lambda) = 2\lambda^2 \coth[(L-w)\lambda] \tag{3.32}$$

and k is a constant that we will determine later.

To show this, consider

$$\begin{aligned}\frac{a'(\lambda)}{b'(\lambda)} &= \frac{2(L-w)\,(2\lambda^2-\sigma)}{4\lambda((L-w)\lambda - \sinh[2(L-w)\lambda]} \\ &+ \frac{2(L-(c+w/2))\sigma\cosh[(2c-w)\lambda] + 2(c-w/2)\sigma\cosh[(2c-2L+w)\lambda]}{4\lambda((L-w)\lambda - \sinh[2(L-w)\lambda])} \\ &- \frac{4\lambda\sinh[2(L-w)\lambda]}{4\lambda((L-w)\lambda - \sinh[2(L-w)\lambda]}\,. \end{aligned} \tag{3.33}$$

Replacing $\cosh[(2c-w)\lambda]$ and $\cosh[(2c-2L+w)\lambda]$ by 1 in the previous expression, we obtain that $\dfrac{a'(\lambda)}{b'(\lambda)}$ grows faster than 1, which means that it is non-decreasing, i.e.

$$\frac{a'(\lambda)}{b'(\lambda)} \geq k := \frac{a'(m)}{b'(m)} > 0, \tag{3.34}$$

hence replacing $a(\lambda)$ by $kb(\lambda)$ makes the original expression to grow slower.

Therefore replacing by $a(\lambda)$ by $kb(\lambda)$ on the characteristic equation, we find that

$$\tilde{f}(\lambda) = \frac{\sinh[(L-w)\lambda]}{\sigma\sinh[(2a-L)\lambda]}\left(k\lambda\coth[(L-w)\lambda] + \sqrt{\lambda^2-\sigma}\coth[w\sqrt{\lambda^2-\sigma}]\right) \tag{3.35}$$

decays faster than $\dfrac{f}{f_a}$.

Moreover, by the Appendix A we have that the functions

$$\lambda\coth[(L-w)\lambda] \quad \text{and} \quad \sqrt{\lambda^2-\sigma}\coth[w\sqrt{\lambda^2-\sigma}] \tag{3.36}$$

are increasing, and therefore

$$\lambda\coth[(L-w)\lambda] + \sqrt{\lambda^2-\sigma}\coth[w\sqrt{\lambda^2-\sigma}] \tag{3.37}$$

is an increasing positive function. Hence for $\sigma > 0$, $\tilde{f}$ is increasing and positive for $2a-L > 0$, and it is decreasing and negative for $2a-L < 0$. Thus $\dfrac{f}{f_a}$ has that same behavior as $\tilde{f}(\lambda)$ in each of these regions and hence $\frac{f_a}{f}$ is decreasing for $a > L/2$ and increasing for $a < L/2$. This will make the force to be positive for $a > L/2$ and negative for $a < L/2$, making the piston to be attracted to the closest wall.

As this result does not depend on the mass m, we have that the force will have the same behavior for the massless case.

3.2 Higher Dimensions

With the result from the one dimensional case it is possible to analyze the case when we consider the piston to be defined in the manifold $\mathcal{M} = [0, L] \times \mathcal{N}$, where $\mathcal{N}$ is a $d-1$ dimensional smooth compact Riemannian manifold and the piston moves along the interval $[0, L]$.

3.2.1 Differential Equation

Just as in the one dimensional case, the eigenvalue problem is given by

$$P\mu_\lambda = \lambda^2 \mu_\lambda \tag{3.38}$$

where P is the second order differential operator

$$P = -\frac{\partial^2}{\partial x^2} - \Delta_{\mathcal{N}} + V(x) \tag{3.39}$$

with $V(x)$ the potential defined in (3.1) and where we consider Dirichlet boundary conditions at $x = 0$ and $x = L$.

3.2.2 Associated Zeta Function

In order to obtain the zeta function for the higher dimensional case, we follow the same argument used in the previous chapter. If we call $\{\eta_l\}_{l=1}^{\infty}$ the spectrum of the Laplacian on $\mathcal{N}$, we analyze the behavior for fixed l obtaining that the integral representation of the zeta function in this case is given by

$$\zeta_P(s) = \frac{1}{2\pi\imath}\sum_{l=1}^{\infty}\int_{\gamma_l}^{\infty} d\lambda\,(\lambda^2-\eta_l^2)^{-s}\frac{d}{d\lambda}\log(f(\imath\lambda))\,, \tag{3.40}$$

for γ_l a contour enclosing the eigenvalues λ, and where the characteristic function $f(\lambda)$ the same as in the previous case.

3.2.3 Analytic Continuation

In order to obtain an analytic continuation for the integral representation of the zeta function, we subtract asymptotic terms of the characteristic function to extend the convergence region to the left. Thus for fixed l we obtain an analytic continuation for the integral in (3.40) by

$$Z^l(s) + \sum_{i=-1}^{N-2} A_i^l(s)\,, \tag{3.41}$$

where $Z^l(s)$ is the finite part obtained by subtracting N asymptotic terms of the characteristic function

$$Z^l(s) = \frac{\sin(\pi s)}{\pi}\int_{\eta_l}^{\infty} d\lambda\,(\lambda^2-\eta_l^2)^{-s} \times \frac{d}{d\lambda}\left(\log(f(\imath\lambda)) - L\lambda + \log(2) - 2\log(\lambda) - \sum_{i=1}^{N-2} d_i\lambda^{-i}\right) \tag{3.42}$$

with the coefficients d_i coming from the asymptotic behavior of $g(\imath\lambda)$ as $|\lambda|\to\infty$,

$$f(\imath\lambda) = L\lambda - \log(2) + 2\log(\lambda) + \sum_{i=1}^{N-2} d_i\lambda^{-i} + O(\lambda^{-N+1})\,, \tag{3.43}$$

which can be explicitly calculated by analyzing $f(\imath\lambda)$. The asymptotic contributions to the integral are given by

$$A_{-1}^l(s) = \frac{\sin(\pi s)}{\pi}\int_{\eta_l}^{\infty} d\lambda\,(\lambda^2-\eta_l^2)^{-s}\frac{d}{d\lambda}\left(L\lambda\right)\,, \tag{3.44}$$

$$A_0^l(s) = \frac{\sin(\pi s)}{\pi}\int_{\eta_l}^{\infty} d\lambda\,(\lambda^2-\eta_l^2)^{-s}\frac{d}{d\lambda}\left(\log(\lambda^2/2)\right)\,, \tag{3.45}$$

$$A_i^l(s) = \frac{\sin(\pi s)}{\pi} \int_{\eta_l}^{\infty} d\lambda \, (\lambda^2 - \eta_l^2)^{-s} \frac{d}{d\lambda} \left(d_i \lambda^{-i} \right) \,, \tag{3.46}$$

which after performing the integration read

$$A_{-1}^l(s) = \frac{L\eta_l^{1-2s}\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)} \,, \tag{3.47}$$

$$A_0^l(s) = \eta^{-2s} \,, \tag{3.48}$$

$$A_i^l(s) = -\frac{d_i \eta^{-i-2s}\Gamma(s+i/2)}{\Gamma(i/2)\Gamma(s)} \,. \tag{3.49}$$

Therefore after summing over l, we get the analytic continuation for the zeta function to be

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{N-2} A_i(s) \tag{3.50}$$

with $Z(s)$ given by

$$Z(s) = \sum_{l=1}^{\infty} Z^l(s)$$
$$= \frac{\sin(\pi s)}{\pi} \sum_{l=1}^{\infty} \int_{\eta_l}^{\infty} d\lambda \, (\lambda^2 - \eta_l^2)^{-s} \frac{d}{d\lambda} \left(\log(f(\imath\lambda)) - L\lambda + \log(2) - 2\log(\lambda) - \sum_{i=1}^{N-2} d_i \lambda^{-i} \right) \tag{3.51}$$

and the asymptotic terms are

$$A_{-1}(s) = \frac{L\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)} \zeta_{\mathcal{N}}(s-1/2) \,, \tag{3.52}$$

$$A_0(s) = \zeta_{\mathcal{N}}(s) \,, \tag{3.53}$$

$$A_i(s) = -\frac{d_i \Gamma(s+i/2)}{\Gamma(i/2)\Gamma(s)} \zeta_{\mathcal{N}}(s+i/2) \,, \tag{3.54}$$

with $\zeta_{\mathcal{N}}(s)$ the zeta function associated with the Laplacian on $\mathcal{N}$. This representation is then valid for $\Re(s) > d/2 - N/2$.

3.2.4 Functional Determinant and Casimir Force

To find the functional determinant and the Casimir force associated with this piston configuration we need to include the points $s = 0$ and $s = -1/2$ into the convergence region for the representation of the zeta function that we derived. In order to do this we take $N = d + 1$ and $N = d + 2$ respectively.

3.2.5 Functional Determinant

We have that the functional determinant is given by

$$\exp(-\zeta_P'(0)) = \exp\left(-Z'(0) - \sum_{i=-1}^{d-1} A_i'(0)\right) \tag{3.55}$$

where we have that

$$Z'(0) = -\sum_{l=1}^{\infty}\left(\log(f(\imath\eta_l)) - L\eta_l + \log(2) - 2\log(\eta_l) - \sum_{i=1}^{d-1} d_i \eta_l^{-i}\right), \tag{3.56}$$

and

$$A_{-1}'(0) = L\zeta_{\mathcal{N}}(-1/2), \tag{3.57}$$

$$A_0'(0) = \zeta_{\mathcal{N}}'(0), \tag{3.58}$$

and using the formalism developed in the previous chapter, we find the derivative for the other asymptotic terms to be

$$A_i'(0) = -d_i \text{FP } \zeta_{\mathcal{N}}(i/2) + \gamma \text{Res } \zeta_{\mathcal{N}}(i/2) - d_i \frac{\Gamma'(i/2)}{\Gamma(i/2)} \text{Res } \zeta_{\mathcal{N}}(i/2). \tag{3.59}$$

3.2.6 Casimir Force

The evaluation for the Casimir force in the higher dimensional case follows from the one dimensional setting. The force can be found by calculating

$$F_{Cas} = \lim_{\epsilon\to 0} -\frac{\partial}{\partial a}\zeta_P(\epsilon - 1/2) \tag{3.60}$$

which can be found by just evaluating $\frac{\partial}{\partial a} Z(\epsilon - 1/2)$ as the asymptotic terms do not depend on the position of the piston. As the resulting force has the same behavior for every fixed l, we have that after performing the l summation we have the definite behavior of the force being negative for $a < L/2$ and positive for $a > L/2$, that is, the piston is attracted to the closest wall.

CHAPTER FOUR

Smooth Potentials

In this chapter we continue considering the use of potentials to model the behavior of real materials on piston configurations. Here we consider a general smooth manifold and provide a method to do the spectral analysis within this setting. First, we study the case of a one dimensional piston and introduce the method for finding the spectral zeta function in this problem, which is based on the use of the WKB method to find the asymptotic behavior to solutions of differential equations. Secondly we analyze the two dimensional case where an extra condition on the asymptotic expansion is needed in order to control the behavior of the large eigenvalue expansion and the large angular momentum behavior.

4.1 One Dimension

Consider a one dimensional piston configuration where the piston is modeled by a smooth potential $V(r)$ which is compactly supported on $(a-w, a+w) \subset [b, c]$, with $w > 0$.

4.1.1 Differential Equation

Then the piston configuration can be defined by the eigenvalue problem

$$P\mu_\lambda = \lambda^2 \mu_\lambda \tag{4.1}$$

with the second order differential operator P defined by

$$P = -\frac{\partial^2}{\partial r^2} + V(r) \tag{4.2}$$

on the interval $[b, c]$ with Dirichlet boundary conditions at $r = b$ and $r = c$.

The eigenvalue problem given in (4.1) can be written as an initial value problem so we can obtain a characteristic equation for the eigenvalues of P.

Consider the second order ordinary differential equation

$$R''(r;\lambda) - \left(V(r) - \lambda^2\right) R(r;\lambda) = 0 \tag{4.3}$$

subject to the initial conditions $R(b;\lambda) = 0$ and $R'(b;\lambda) = 1$. Thus the eigenvalues for (4.1) are determined by the characteristic equation

$$R(c;\lambda) = 0\,. \tag{4.4}$$

4.1.2 Zeta Function

Thus with the previous characteristic equation we can find the integral representation for the spectral function associated to P as

$$\zeta_P(s) = \frac{1}{2\pi\imath}\int_\gamma d\lambda\,\lambda^{-2s}\frac{d}{d\lambda}\ln R(c;\lambda) \tag{4.5}$$

where γ is a contour enclosing the eigenvalues λ. This representation is valid for $\Re(s) > 1/2$.

4.1.3 Analytic Continuation

In order to extend the region of convergence of the integral representation obtained in (4.5) we are going to find the asymptotic behavior of the characteristic function $R(c;\lambda)$ by means of the WKB approximation.

Consider the function

$$S(r;\lambda) = \frac{\partial}{\partial r}\ln R(r;\imath\lambda) \tag{4.6}$$

which will satisfy the first order differential equation

$$S'(r;\lambda) = \lambda^2 + V(r) - S^2(r;\lambda)\,. \tag{4.7}$$

Let S have the asymptotic expansion as $\lambda \to \infty$

$$S(r;\lambda) \sim \sum_{i=-1}^{\infty} S_i(r)\lambda^{-i} \tag{4.8}$$

where the coefficients $S_i(r)$ can be found recursively by substituting the asymptotic expansion into the differential equation (4.7) and equating orders in λ. Performing this we find the recurrence relation

$$S_{i+1}(r) = -\frac{1}{2S_{-1}(r)}\left(S_i'(r) + \sum_{j=1}^{i} S_k(r)S_{i-j}(r) + f_{i+1}(r)\right) \tag{4.9}$$

with the initial condition

$$S_{-1}(r) = \pm 1 \tag{4.10}$$

and the functions f_i given by

$$f_{-1}(r) = 1, \quad f_0(r) = 0, \quad f_1(r) = -V(r) \tag{4.11}$$

and $f_i(r) = 0$ for $i > 1$. With this recursion we can calculate as many asymptotic terms of (4.7) as needed and thus it provides an asymptotic expansion for the logarithmic derivative of the characteristic function appearing in the integral representation of the zeta function associated with P.

Therefore we find $R(r;\lambda)$ to be a linear combination of two exponential terms

$$R(r;\imath\lambda) = A^+ \exp\left(\int_b^r dt\, S^+(t;\lambda)\right) + A^- \exp\left(\int_b^r dt\, S^-(t;\lambda)\right) \tag{4.12}$$

where $S^+(t;\lambda)$ and $S^-(t;\lambda)$ are the exponentially growing and decaying solutions of (4.7) respectively. After imposing the initial conditions given in (4.3) we find that

$$A^+ = -A^-, \text{ and } A^+ = \frac{1}{S^+(b;\lambda) - S^-(b;\lambda)}\,. \tag{4.13}$$

With this we find that $R(r;\imath\lambda)$ reads

$$R(r;\imath\lambda) = A^+ \int_b^r dt\, S^+(t;\lambda) + E(\lambda) \tag{4.14}$$

where $E(\lambda)$ are exponentially damped terms in λ. Therefore , for large λ, we have the asymptotic behavior

$$\log R(r;\lambda) \sim \log\left(A^{+}\right) + \int_{b}^{r} dt\, S^{+}(r;\lambda)\,. \tag{4.15}$$

We find A^{+} by noticing that $S_i^{-}(r) = (-1)^i S_i^{+}(r)$, so then

$$A^{+} = 2\sum_{j=0}^{\infty} S_{2j-1}(b)\lambda^{-2j+1} \tag{4.16}$$

and hence

$$\log A^{+} = -\log\left(2\lambda S_{-1}^{+}(b)\right) - \log\left(1 + \sum_{j=1}^{\infty} \frac{S_{2j-1}^{+}(b)}{S_{-1}^{+}(b)}\lambda^{-2j}\right) \tag{4.17}$$

from which we obtain the asymptotic expansion for the logarithm of the characteristic function

$$\log R(c;\lambda) \sim \log A^{+} + \sum_{i=-1}^{\infty} \lambda^{-i} \int_{b}^{c} dt\, S_i^{+}(t) \tag{4.18}$$

and hence we find the analytic continuation for the integral representation of the zeta function associated with P by subtracting asymptotic terms

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{N-2} A_i(s) \tag{4.19}$$

with the finite part given as usual by

$$\begin{aligned} Z(s) = {} & \frac{\sin(\pi s)}{\pi} \int_{0}^{1} d\lambda\, \lambda^{-2s} \frac{d}{d\lambda} \log R(c;\imath\lambda) \\ & + \frac{\sin(\pi s}{\pi} \int_{1}^{\infty} d\lambda\, \lambda^{-2s} \frac{d}{d\lambda} \left(\log R(c;\imath\lambda) - \log A^{+} - \sum_{i=-1}^{N-2} \lambda^{-i} \int_{b}^{c} dt\, S_i^{+}(t)\right) \end{aligned} \tag{4.20}$$

and the asymptotic terms $A_i(s)$ come from considering the contributions of λ^{-i} arising from both $\log A^{+}$ and S_i^{+}.

4.2 Two Dimensions

The two dimensional scenario is similar to the one dimensional case, with the additional requirement of handling the behavior of the large angular momentum together with the large eigenvalue asymptotics. This can be done by performing a uniform asymptotic expansion in both parameters. For technical reasons, we need to separate the cases where we have a zero angular momentum from the case of a nonzero one.

4.2.1 Differential Equation

Consider the second order differential operator P acting on the plane given by

$$P = -\Delta + V(r) \tag{4.21}$$

where $V(r)$ is a radial smooth compactly supported potential on $(a-w, a+w) \subset [b, c]$. The associated eigenvalue problem is then

$$P\mu_\lambda = \lambda^2 \mu_\lambda \tag{4.22}$$

subject to Dirichlet boundary conditions at $r = b$ and $r = c$.

Writing the Laplacian in polar coordinates we have that (4.22) can be written as

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mu}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 \mu}{\partial \theta^2} + V(r)\mu = \lambda^2 \mu \tag{4.23}$$

and by using separation of variables $\mu(r, \theta) = R(r)\Theta(\theta)$ this equation can be reduced to

$$-\frac{r^2}{R(r)}R''(r) - \frac{r}{R(r)}R'(r) + r^2(V(r) - \lambda^2) - \frac{1}{\Theta(\theta)}\Theta''(\theta) = 0 \tag{4.24}$$

which gives the system of ordinary differential equations

$$-\frac{r^2}{R(r)}R''(r) - \frac{r}{R(r)}R'(r) + r^2(V(r) - \lambda^2) = -k^2 \tag{4.25}$$

$$\frac{1}{\Theta(\theta)}\Theta''(\theta) = -k^2\,. \tag{4.26}$$

In order to have periodicity in the angular part Θ, from (4.26) we obtain that $k \in \mathbb{Z}$, and thus we have that the eigenvalues λ can be found by looking at the solutions of (4.25). For this, just as for the one dimensional case, consider the initial value problem

$$R''(r) + \frac{1}{r}R'(r) - R(r)\left(V(r) - \lambda^2 + \frac{k^2}{r^2}\right) = 0 \tag{4.27}$$

for fixed $k \in \mathbb{Z}$ and with initial conditions $R_k(b;\lambda) = 0$ and $R'_k(b;\lambda) = 1$. Hence the values of λ satisfy the characteristic equation

$$R_k(c;\lambda) = 0\,. \tag{4.28}$$

4.2.2 Zeta Function

The zeta function integral representation can be found as usual to be

$$\zeta(s) = \frac{1}{2\pi\imath}\sum_{k=-\infty}^{\infty}\int_{\gamma_\kappa} d\lambda\,\lambda^{-2s}\frac{\partial}{\partial\lambda}\log R_k(c;\lambda)\,. \tag{4.29}$$

In order to find an analytic continuation, we can subtract the asymptotic behavior of the $R_k(c;\imath\lambda)$ for $|\lambda| \to \infty$ and $|k| \to \infty$ by considering WKB asymptotics.

4.2.3 WKB Asymptotics

In order to achieve convergence in the series in k, a treatment of the convergence for both the integral in λ and the series in k is needed. For this matter, we find the asymptotic expansion of the zero case $k = 0$ as $|\lambda| \to \infty$, and in the case $k \neq 0$ we use the uniform asymptotic expansion for big $|\lambda|$ and k setting

$$\lambda = ku\,. \tag{4.30}$$

4.2.4 Zero Case

For the case $k = 0$, let

$$\tilde{S}(r,\lambda) = \frac{\partial}{\partial r}\log R_0(r;\imath\lambda) \tag{4.31}$$

which satisfies the first order differential equation

$$\tilde{S}'(r,\lambda) = \lambda^2 + V(r) - \tilde{S}^2(r,\lambda) - \frac{1}{r}\tilde{S}(r,\lambda) = 0\,. \tag{4.32}$$

As $|\lambda| \to \infty$, $\tilde{S}$ has an asymptotic expansion

$$\tilde{S}(r,\lambda) \sim \sum_{i=-1}^{\infty} \tilde{S}_i(r)\lambda^{-i} \tag{4.33}$$

where $\tilde{S}_{i+1}$ can be found by (4.32) to satisfy

$$\tilde{S}_{i+1}(r) = -\frac{1}{2\tilde{S}_{-1}(r)}\left(\tilde{S}_i'(r) + \frac{1}{r}\tilde{S}_i(r) + \sum_{j=0}^{i}\tilde{S}_j(r)\tilde{S}_{i-j}(r) + \tilde{f}_{i+1}(r)\right) \tag{4.34}$$

with

$$\tilde{S}_{-1}(r) = \pm 1 \tag{4.35}$$

$$\tilde{f}_{-1}(r) = 1, \qquad \tilde{f}_0(r) = 0, \qquad \tilde{f}_1(r) = -V(r) \tag{4.36}$$

and $\tilde{f}_j(r) = 0$ for $j > 1$.

4.2.5 Non-zero Case

Similarly, for $k \neq 0$ using (4.30) and (4.27) we have

$$R''(r) + \frac{1}{r}R'(r) - R(r)\left(V(r) + k^2\left(u^2 + \frac{1}{r^2}\right)\right) = 0\,. \tag{4.37}$$

As before, let

$$\hat{S}(r,k) = \frac{\partial}{\partial r}\ln R_k(r;\imath k u) \tag{4.38}$$

which satisfies the first order differential equation

$$\hat{S}'(r,k) = k^2\left(u^2 + \frac{1}{r^2}\right) + V(r) - \hat{S}^2(r,k) - \frac{1}{r}\hat{S}(r,k)\,. \tag{4.39}$$

As $k \to \infty$, $\hat{S}$ has the asymptotic form

$$\hat{S}(r,k) = \sum_{i=-1}^{\infty} \hat{S}_i(r)k^{-i} \tag{4.40}$$

where $\hat{S}_i$ can be found recursively by

$$\hat{S}_{i+1}(r) = -\frac{1}{2\hat{S}_{-1}(r)}\left(\hat{S}_i'(r) + \frac{1}{r}\hat{S}_i(r) + \sum_{j=0}^{i}\hat{S}_j(r)\hat{S}_{i-j}(r) + \hat{f}_{i+1}(r)\right) \tag{4.41}$$

with

$$\hat{S}_{-1}(r) = \pm\sqrt{u^2 + \frac{1}{r^2}}\,, \tag{4.42}$$

and the functions $\hat{f}_j$ given by

$$\hat{f}_{-1}(r) = u^2 + \frac{1}{r^2}, \quad \hat{f}_0(r) = 0, \quad \hat{f}_1(r) = -V(r) \tag{4.43}$$

and $\hat{f}_j(r) = 0$ for $j > 1$.

4.2.6 Characteristic Function Asymptotic Expansion

By the previous discussion we find $R_k(r;\imath\lambda)$ to be a combination of exponentially growing and decaying terms

$$R_k(r,\imath\lambda) = A^+ \exp\left(\int_b^r dt\, S^+(t,k)\right) + A^- \exp\left(\int_b^r dt\, S^-(t,k)\right) \tag{4.44}$$

where we have

$$S(t) = \begin{cases} \tilde{S}(t,\lambda), & \text{for } k = 0 \\ \hat{S}(t) = S(t,\lambda), & \text{for } k \neq 0 \end{cases}. \tag{4.45}$$

This, together with the initial conditions given in (4.27), shows that

$$A^+ = -A^-, \qquad A^+ = \frac{1}{S^+(b) - S^-(b)} \tag{4.46}$$

and therefore we have the asymptotic behavior of $R_k(r;\imath\lambda)$ given by

$$R_k(c,\imath\lambda) = A^+ \exp\left(\int_b^c dt\, S^+(t)\right) + E(k) \tag{4.47}$$

where $E(k)$ represents exponentially damped terms in k. Thus the characteristic function has the form

$$\log R_k(c;\imath\lambda) = \log\left(A^+\right) + \int_b^c dt\, S^+(t) + E(k)\,, \tag{4.48}$$

from where we obtain the asymptotic expansion for the logarithm of the characteristic function to be

$$\log R_k(c;\imath\lambda) \sim \begin{cases} \log A^+ + \sum_{i=-1}^{\infty} \lambda^{-i} \int_b^c dt\, \tilde{S}_i^+(t) & \text{for } k=0 \\ \log A^+ + \sum_{i=-1}^{\infty} k^{-i} \int_b^c dt\, \hat{S}_i^+(t) & \text{for } k\neq 0 \end{cases}, \tag{4.49}$$

where we have that the asymptotic terms satisfy

$$S_i^-(r) = (-1)^i S_i^+(r) \tag{4.50}$$

and hence

$$\log A^+ = \begin{cases} \log\left(2\lambda \tilde{S}_{-1}^+(b)\right) - \log\left(1 + \sum_{j=1}^{\infty} \frac{\tilde{S}_{2j-1}^+(b)}{\tilde{S}_{-1}^+(b)} \lambda^{-2j}\right) & \text{for } k=0 \\ \log\left(2k \hat{S}_{-1}^+(b)\right) - \log\left(1 + \sum_{j=1}^{\infty} \frac{\hat{S}_{2j-1}^+(b)}{\hat{S}_{-1}^+(b)} k^{-2j}\right) & \text{for } k\neq 0 \end{cases}. \tag{4.51}$$

4.2.7 Analytic Continuation

With the asymptotic expansion for the characteristic function we can obtain an analytic continuation for the zeta function associated with P as usual by

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{N-2} A_i(s) \tag{4.52}$$

where each term can be separated into a zero and non-zero case pieces as

$$Z(s) = \tilde{Z}(s) + \hat{Z}(s) \tag{4.53}$$

$$\tilde{Z}(s) = \frac{\sin(\pi s)}{\pi} \int_0^1 d\lambda\ \lambda^{-2s} \frac{d}{d\lambda} \log R_0(c;\imath\lambda)$$
$$+ \frac{\sin(\pi s)}{\pi} \int_1^{\infty} d\lambda\ \lambda^{-2s} \frac{d}{d\lambda} \left(\log R_0(c;\imath\lambda) - \log(A^+) - \sum_{i=-1}^{N-2} \lambda^{-i} \int_b^c dt \tilde{S}_i(t) \right) \tag{4.54}$$

$$\hat{Z}(s) = \frac{2\sin(\pi s)}{\pi} \sum_{k=1}^{\infty} \int_0^{\infty} duk\, (uk)^{-2s}$$

$$\times \frac{d}{duk}\left(\log R_k(c;\imath uk) - \log(A^+) - \sum_{i=-1}^{N-2} k^{-i}\int_b^c dt\hat{S}_i(t)\right), \quad (4.55)$$

and with the asymptotic terms given by

$$A_i(s) = \tilde{A}_i(s) + \hat{A}_i(s) \quad (4.56)$$

where $\tilde{A}_i(s)$ comes from the contributions of λ^{-i} coming from $\log A^+$ and $\tilde{S}(t)$, and $\hat{A}_i(s)$ can be obtained by considering the contributions of λ^{-i} that appear in $\log A^+$ and $\hat{(S)}(t)$. With this, the representation obtained in (4.52) is valid for $\Re(s) > 1 - N/2$.

4.2.8 Functional Determinant and Casimir Force

Considering the first 4 asymptotic terms will extend the convergence region of the integral representation of the zeta function for P to be $\Re(s) > -1$. Here we find

$$\tilde{A}_{-1}(s) = \frac{\sin(\pi s)}{\pi}\int_1^\infty d\lambda\,\lambda^{-2s}\frac{d}{d\lambda}\left(\int_b^c dt\,\tilde{S}_{-1}(t)\lambda\right) \quad (4.57)$$

$$\hat{A}_{-1}(s) = \frac{2\sin(\pi s)}{\pi}\sum_{k=1}^{\infty} k^{-2s+1}\int_0^\infty du\,u^{-2s}\frac{d}{du}\left(\int_b^c dt\,\hat{S}_{-1}(t)\right) \quad (4.58)$$

$$\tilde{A}_0(s) = \frac{\sin(\pi s)}{\pi}\int_1^\infty d\lambda\,\lambda^{-2s}\frac{d}{d\lambda}\left(-\log(2\lambda\tilde{S}_0(b)) + \lambda\int_b^c dt\,\tilde{S}_0(t)\right) \quad (4.59)$$

$$\hat{A}_0(s) = \frac{2\sin(\pi s)}{\pi}\sum_{k=1}^{\infty} k^{-2s}\int_0^\infty du\,u^{-2s}\frac{d}{du}\left(-\log(2k\hat{S}_0^+(b))) + \int_b^c dt\,\hat{S}_0(t)\right) \quad (4.60)$$

$$\tilde{A}_1(s) = \frac{\sin(\pi s)}{\pi}\int_1^\infty d\lambda\,\lambda^{-2s}\frac{d}{d\lambda}\left(\int_b^c dt\,\tilde{S}_1(t)\lambda^{-1}\right) \quad (4.61)$$

$$\hat{A}_1(s) = \frac{2\sin(\pi s)}{\pi}\sum_{k=1}^{\infty} k^{-2s-1}\int_0^\infty du\,u^{-2s}\frac{d}{du}\left(\int_b^c dt\,\hat{S}_1(t)\right) \quad (4.62)$$

$$\tilde{A}_2(s) = \frac{\sin(\pi s)}{\pi}\int_1^\infty d\lambda\,\lambda^{-2s}\frac{d}{d\lambda}\left(-\frac{\tilde{S}_1^+(b)}{\tilde{S}_{-1}^+(b)}\lambda^{-2} + \int_b^c dt\,\tilde{S}_2(t)\lambda^{-2}\right) \quad (4.63)$$

$$\hat{A}_2(s) = \frac{2\sin(\pi s)}{\pi} \sum_{k=1}^{\infty} k^{-2s-2} \int_0^{\infty} du\, u^{-2s} \frac{d}{du} \left(-\frac{\hat{S}_1^+(b)}{\hat{S}_{-1}^+(b)} + \int_b^c dt\, \hat{S}_2(t) \right) \tag{4.64}$$

which after performing the integration read

$$\tilde{A}_{-1}(s) = \frac{\sin(\pi s)}{2\pi(s-1/2)}(c-b) \tag{4.65}$$

$$\hat{A}_{-1}(s) = \frac{\Gamma(s-1/2)}{2s\pi^{1/2}\Gamma(s)} \left(c^{2s} - b^{2s}\right) \zeta_R(2s-1) \tag{4.66}$$

$$\tilde{A}_0(s) = -\frac{\sin(\pi s)}{2\pi s} \tag{4.67}$$

$$\hat{A}_0(s) = -\frac{1}{2}\left(c^{2s} + b^{2s}\right) \zeta_R(2s) \tag{4.68}$$

$$\tilde{A}_1(s) = -\frac{\sin(\pi s)}{2\pi(s+1/2)} \left(\frac{1}{8c} - \frac{1}{8b} + \frac{1}{2}\int_b^c dr\, V(r) \right) \tag{4.69}$$

$$\hat{A}_1(s) = \frac{\Gamma(s+1/2)(5s+1)}{6\pi^{1/2}\Gamma(s)} \left(c^{2s} - b^{2s}\right) \zeta_R(2s+1) - \frac{\Gamma(s+1/2)}{\pi^{1/2}\Gamma(s)} \int_b^c rdr\; r^{2s} V(r) \zeta_R(2s+1) \tag{4.70}$$

$$\tilde{A}_2(s) = -\frac{\sin(\pi s)}{2\pi(s+1)} \left(\frac{1}{8c^2} + \frac{1}{8b^2} \right) \tag{4.71}$$

$$\hat{A}_2(s) = -\frac{s^2(5s+3)}{16}(c^{2s} + b^{2s})\zeta_R(2s+2) + s(s+1)\int_b^c rdr\; r^{2s} V(r) \zeta_R(2s+2) + \frac{s}{2}\int_b^c rdr\; r^{2s+1} V'(r) \zeta_R(2s+2) \tag{4.72}$$

4.2.9 Functional Determinant

Taking the derivative with respect to s in the finite and the asymptotic parts of the zeta function and substituting $s = 0$ gives the functional determinant for P to be

$$
\begin{aligned}
\exp\left(-\zeta_P'(0)\right) = &-\log R_0(c;0) - \log A^+ - \sum_{i=-1}^{1}\int_b^c dt\tilde{S}_i(t) \\
&- 2\sum_{k=1}^{\infty}\left(\log R_k(c;0) - \log A^+\Big|_{u=0} - \sum_{i=-1}^{1} k^{-i}\int_b^c dt\ \ \hat{S}_i(t)\Big|_{u=0}\right) \\
&- (c-b) - \frac{1}{8}(c^{-1}-b^{-1}) - \frac{1}{2}\int_b^c dr\, V(r) - \frac{1}{16}(c^{-2}-b^{-2}) \\
&+ \frac{5}{6}\log c - \frac{1}{6}\log b + \log 2\pi - \frac{1}{2}\int_b^c r dr\, V(r)(2\log r + \psi(1/2) + 3\gamma) \\
&+ \frac{\pi^2}{12}\left((a+w)^2 V(a+w) - (a-w)^2 V(a-w)\right) \quad (4.73)
\end{aligned}
$$

where $\psi(s)$ is the digamma function.

4.2.10 Casimir Force

By taking $N = 2$ we can include the point $s = -1/2$ into the convergence region of the zeta function which enables us to find the Casimir force to be given by

$$
\begin{aligned}
F_{Cas} = \frac{\partial}{\partial c}\zeta_P(-1/2) = &-\frac{1}{\pi}\int_0^\infty d\lambda\ \lambda\frac{d}{d\lambda}\partial_c \log R_0(c;\imath\lambda) \\
&- \frac{2}{\pi}\sum_{k=1}^{\infty} k\int_0^\infty du\ u\frac{d}{du}\partial_c \log R_k(c;\imath uk)\,. \quad (4.74)
\end{aligned}
$$

As we are considering a general potential, a definite behavior for the force as being attractive or repulsive cannot be found. The analysis of the sign of the Casimir force will then depend on the specific characteristics of the potential considered. The expression derived in (4.74) can be used to numerically find the Casimir force and hence the behavior of the piston.

CHAPTER FIVE

Smooth Potentials in Spherical Shells

In this chapter we revisit the problem of having a piston modeled by a smooth potential. Here we consider higher dimensions where the piston configuration is a spherical shell. The analysis for this case is going to follow from the one made in the previous chapter, where we use the WKB method in order to find an asymptotic expansion for the characteristic function in order to be able to extend the convergence region of the integral representation of the zeta function.

5.1 Differential Equation

As in the one and two dimensional cases, we consider the eigenvalue problem

$$P\mu_\lambda = \lambda^2 \mu_\lambda \tag{5.1}$$

defined on the spherical shell $b \le \rho \le c$ in $\mathbb{R}^d$, where the second order differential operator P is given by

$$P = -\Delta + V(\rho)\,, \tag{5.2}$$

with $V(\rho)$ a smooth spherically symmetric potential compactly supported on $(a-w, a+w) \subset [b,c]$, and subject to Dirichlet boundary conditions at $\rho = b$ and $\rho = c$.

As the Laplacian is separable in this manifold, let $\mu(\rho,\theta) = R(\rho)\Theta(\theta)$ where $\theta \in \mathbb{S}^{d-1}$. Therefore the eigenvalue problem can be written as

$$P\mu = -\frac{\Theta}{\rho^{d-1}}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial R}{\partial\rho}\right) - \frac{R}{\rho^2}\Delta_{\mathbb{S}^{d-1}}\Theta + V(\rho)R\Theta = \lambda^2 R\Theta \tag{5.3}$$

which gives

$$-\frac{1}{\rho R}((d-1)R' + \rho R'') - \frac{1}{\rho^2\Theta}\Delta_{\mathbb{S}^{d-1}}\Theta + V(\rho) = \lambda^2\,. \tag{5.4}$$

The eigenvalue problem (5.4) can be restated as a system of ordinary differential equations

$$-\frac{\rho}{R}((d-1)R' + \rho R'') + \rho^2\left(V(\rho) - \lambda^2\right) = -\kappa^2 \tag{5.5}$$

$$\frac{1}{\Theta}\Delta_{\mathbb{S}^{d-1}}\Theta = -\kappa^2 \tag{5.6}$$

where from continuity conditions on (5.6) we find the eigenvalues for the Laplacian of $\mathbb{S}^{d-1}$ to be [11]

$$\kappa^2 = m(m+d-2) \tag{5.7}$$

with multiplicities

$$d(m) = \binom{m+d-2}{d-2} + \binom{m+d-3}{d-2}. \tag{5.8}$$

5.2 Associated Zeta Function

As for the lower dimensional cases, it is useful to rewrite the eigenvalue problem as an initial value problem in order to obtain a characteristic equation for the eigenvalues of P by means of imposing boundary conditions to the differential equation.

For this, consider the initial value problem for $R_m(\rho;\lambda)$,

$$-\rho^2\frac{R''}{R} - \rho(d-1)\frac{R'}{R} + \rho^2(V(r) - \lambda^2) + m(m+d-2) = 0\,, \tag{5.9}$$

with initial conditions $R_m(b;\lambda) = 0$ and $R'_m(b;\lambda) = 1$. Therefore, the eigen-values for P can be obtained from the characteristic equation

$$R_m(c;\lambda) = 0\,, \tag{5.10}$$

and the zeta function associated with P can then be written as

$$\zeta_P(s) = \sum_m d(m)\lambda_m^{-2s}\,, \tag{5.11}$$

where λ_m satisfies $R_m(c;\lambda)=0$, and admits the integral representation

$$\zeta_P(s)=\frac{\sin(\pi s)}{\pi}\sum_{m=0}^{\infty}d(m)\int_0^{\infty}d\lambda\ \lambda^{-2s}\frac{d}{d\lambda}\log R_m(c;\imath\lambda)\,. \tag{5.12}$$

5.3 Characteristic Function Asymptotic Expansion

For convenience we make the substitution

$$\nu^2=\left(m+\frac{d-2}{2}\right)^2 \tag{5.13}$$

and hence (5.9) becomes

$$-\rho^2\frac{R''}{R}-\rho(d-1)\frac{R'}{R}+\rho^2(V(\rho)-\lambda^2)+\nu^2-\left(\frac{d-2}{2}\right)^2=0\,. \tag{5.14}$$

With this change, there are two cases to consider for ν, when there is a zero angular momentum and when there is not. When the overall dimension is $d=2$, we have a zero angular momentum for ν that is not present for higher dimensions. This case has already been studied on the previous chapter, so we will focus in the case where $d>2$ only, in which case no zero angular momentum is present.

For $\nu\neq 0$, we need to control the behavior as both λ and ν grow to infinity. Using the substitution

$$\lambda=\xi\nu \tag{5.15}$$

we can proceed using WKB to get the asymptotic expansion for $R_m(c;\lambda)$. Let

$$S(\rho,\nu)=\frac{\partial}{\partial r}\log R_\nu(\rho;\imath\xi\nu) \tag{5.16}$$

be the auxiliary function we use to find the WKB asymptotic expansion as before. Then (5.14) is to have the form

$$S'=\left(V(\rho)-\left(\frac{d-2}{2\rho}\right)^2+\nu^2\left(\xi^2+\frac{1}{\rho^2}\right)\right)-S^2-\frac{d-1}{\rho}S\,, \tag{5.17}$$

from where we find a recursion equation for the asymptotic expansion

$$S(\rho,\nu)\sim\sum_{i=-1}^{\infty}S_i(\rho)\nu^{-i}\,, \tag{5.18}$$

given by

$$S_{i+1}(\rho) = -\frac{1}{2S_{-1}(\rho)}\left(S_i'(\rho) + \frac{d-1}{\rho}S_i(\rho) + \sum_{j=0}^{i} S_j(\rho)S_{i-j}(\rho) + f_{i+1}(\rho)\right) . \quad (5.19)$$

We have that the initial condition is given by

$$S_{-1}^{\pm}(\rho) = \pm\sqrt{\xi^2 + \frac{1}{\rho^2}}\,, \quad (5.20)$$

and the functions f_i are

$$f_{-1}(\rho) = \xi^2 + \frac{1}{\rho^2}, \qquad f_0 = 0, \quad f_1(\rho) = \left(\frac{d-1}{2\rho}\right)^2 - V(\rho), \quad (5.21)$$

and $f_i(\rho) = 0$ for $i > 1$.

With this, we find that the characteristic function $R_\nu(c; \imath\xi\nu)$ is a combination of exponentially growing and decaying terms

$$R_\nu(c;\imath\lambda) = A^+ \exp\left(\int_b^c dt\ S^+(t,\lambda)\right) + A^- \exp\left(\int_b^c dt\ S^-(t,\lambda)\right) , \quad (5.22)$$

and from the initial conditions (5.14) we have that

$$A^+ = -A^- = \frac{1}{S^+(b,\lambda) - S^-(b,\lambda)} . \quad (5.23)$$

Hence we obtain the asymptotic expansion for the characteristic function to have the form

$$\log R_\nu(c;\imath\lambda) \sim \log A^+ + \sum_{i=-1}^{\infty} \nu^{-i} \int_b^c d\rho\, S_i^+(\rho) \quad (5.24)$$

where A^+ can be found to have the asymptotic expansion

$$\log A^+ = -\log\left(2\nu S_{-1}^+(b)\right) - \log\left(1 + \sum_{j=1}^{\infty} \frac{S_{2j-1}^+(b)}{S_{-1}^+(b)}\nu^{-2j}\right) . \quad (5.25)$$

5.4 Analytic Continuation

With the asymptotic expansion of the characteristic function for the eigenvalues it is possible to find an analytic continuation for the zeta function associated with P as

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{N-2} A_i(s)\,, \tag{5.26}$$

where $Z(s)$ is the zeta function with N asymptotic terms subtracted and $A_i(s)$ is the asymptotic contribution of ν^{-i} coming from $\log A^+$ and from $S_i^+(t)$. Thus, we have that

$$Z(s) = \frac{\sin(\pi s)}{\pi}\zeta_{\mathbb{S}}(s)\int_0^\infty d\xi\,\xi^{-2s} \times \frac{d}{d\xi}\left(\log R_\nu(c;\imath\xi\nu) - \log A^+ - \sum_{i=-1}^{N-2}\nu^{-i}\int_b^c d\rho\, S_i^+(\rho)\right) \tag{5.27}$$

where $\zeta_{\mathbb{S}}(s)$ is the zeta function arising from the Laplacian on $\mathbb{S}^{d-1}$ and has the form

$$\zeta_{\mathbb{S}}(s) = \sum_\nu d(\nu)\nu^{-2s}\,, \tag{5.28}$$

which can be expressed as a sum of Barnes zeta functions [11] as

$$\zeta_{\mathbb{S}}(s) = \sum_\nu d(\nu)\nu^{-2s} = \zeta_{\mathcal{B}}\left(2s, \frac{d-2}{2}\right) + \zeta_{\mathcal{B}}\left(2s, \frac{d}{2}\right)\,, \tag{5.29}$$

where

$$\zeta_{\mathcal{B}}(s,r) = \sum_{m\in(\mathbb{Z}^+)^{d-1}} \frac{1}{(m_1+m_2+\cdots+m_{d-1}+r)^s}\,. \tag{5.30}$$

5.5 First Asymptotic Terms

The case when $d=1$ and $d=2$ was studied in the previous chapter, so we will perform the analysis when $d=3$. For this, we need to find the first 5 asymptotic terms of the characteristic function in order to be able to extend the convergence region of the zeta function associated with P to the left so we can include the points $s=0$ and $s=-1/2$ into the valid region of the continuation.

We proceed to compute the first 5 asymptotic terms for any dimension d and then we are going to analyze the special case when $d=3$

5.5.1 Calculation of the First S_i^+

Using the recursion expression derived in (5.19) we find the first 5 asymptotic terms to be

$$S_{-1}^+(\rho) = \sqrt{\frac{1}{\rho^2} + \xi^2}\,, \tag{5.31}$$

$$S_0^+(\rho) = -\frac{d - 2 + (d-1)\rho^2\xi^2}{2\left(\rho + \rho^3\xi^2\right)}\,, \tag{5.32}$$

$$S_1^+(\rho) = \frac{4\xi^2 - \rho^2\xi^4 + 4\left(1+\rho^2\xi^2\right)^2 V(\rho)}{8\sqrt{\frac{1}{\rho^2}+\xi^2}\left(1+\rho^2\xi^2\right)^2}\,, \tag{5.33}$$

$$S_2^+(\rho) = -\frac{\rho\left(\xi^2\left(4 - 10\rho^2\xi^2 + \rho^4\xi^4\right) + 4\left(1+\rho^2\xi^2\right)^2 V(\rho) + 2\rho\left(1+\rho^2\xi^2\right)^3 V'(\rho)\right)}{8\left(1+\rho^2\xi^2\right)^4}\,, \tag{5.34}$$

$$S_3^+(\rho) = \frac{64\xi^2 - 560\rho^2\xi^4 + 456\rho^4\xi^6 - 25\rho^6\xi^8}{128\sqrt{\frac{1}{\rho^2}+\xi^2}\left(1+\rho^2\xi^2\right)^5} + \frac{\left(8 - 16\rho^2\xi^2 + \rho^4\xi^4\right)V(\rho)}{16\sqrt{\frac{1}{\rho^2}+\xi^2}\left(1+\rho^2\xi^2\right)^3} \tag{5.35}$$

$$- \frac{\rho^4\sqrt{\frac{1}{\rho^2}+\xi^2}V(\rho)^2}{8\left(1+\rho^2\xi^2\right)^2} + \frac{5\rho V'(\rho)}{8\sqrt{\frac{1}{\rho^2}+\xi^2}\left(1+\rho^2\xi^2\right)^2} + \frac{\rho^4\sqrt{\frac{1}{\rho^2}+\xi^2}V''(\rho)}{8\left(1+\rho^2\xi^2\right)^2}\,. \tag{5.36}$$

5.5.2 Calculation of the First A_i

With the first S_i^+ computed as above, we find the first 5 asymptotic parts of the zeta function to be

$$A_{-1}(s) = \frac{\sin(\pi s)}{\pi}\zeta_\nu(s-1/2)\int_0^\infty d\xi\,\xi^{-2s}\frac{d}{d\xi}\int_b^c d\rho\,S_{-1}^+(\rho)\,, \tag{5.37}$$

$$A_0(s) = \frac{\sin(\pi s)}{\pi}\zeta_\nu(s)\int_0^\infty d\xi\,\xi^{-2s}\frac{d}{d\xi}\left(-\log\left(2\nu S_{-1}^+(b)\right) + \int_b^c d\rho\,S_0^+(\rho)\right)\,, \tag{5.38}$$

$$A_1(s) = \frac{\sin(\pi s)}{\pi}\zeta_\nu(s+1/2)\int_0^\infty d\xi\,\xi^{-2s}\frac{d}{d\xi}\int_b^c d\rho\,S_1^+(\rho)\,, \tag{5.39}$$

$$A_2(s) = \frac{\sin(\pi s)}{\pi}\zeta_\nu(s+1)\int_0^\infty d\xi\,\xi^{-2s}\frac{d}{d\xi}\left(-\frac{S_1^+(b)}{S_{-1}^+(b)} + \int_b^c d\rho\,S_2^+(\rho)\right)\,, \tag{5.40}$$

$$A_3(s) = \frac{\sin(\pi s)}{\pi}\zeta_\nu(s+3/2)\int_0^\infty d\xi\,\xi^{-2s}\frac{d}{d\xi}\int_b^c d\rho\,S_3^+(\rho)\,, \tag{5.41}$$

which after performing the integration read

$$A_{-1}(s) = \frac{\Gamma(s-1/2)}{4s\sqrt{\pi}\Gamma(s)}(c^{2s}-b^{2s})\zeta_\nu(s-1/2)\,, \tag{5.42}$$

$$A_0(s) = -\frac{1}{4}(c^{2s}+b^{2s})\zeta_\nu(s)\,, \tag{5.43}$$

$$A_1(s) = \frac{\Gamma(s+1/2)(5s+1)}{12\sqrt{\pi}\Gamma(s)}(c^{2s}-b^{2s})\zeta_\nu(s+1/2) \tag{5.44}$$

$$- \frac{\Gamma(s+1/2)}{2\sqrt{\pi}\Gamma(s)}\int_b^c d\rho\,r^{2s+1}V(\rho)\zeta_\nu(s+1/2)\,, \tag{5.45}$$

$$A_2(s) = -\frac{s^2(5s+3)}{32}(c^{2s}+b^{2s})\zeta_\nu(s+1) \tag{5.46}$$

$$+ \frac{s}{4}(c^{2s+2}V(c)+b^{2s+2}V(b))\zeta_\nu(s+1)\,, \tag{5.47}$$

$$A_3(s) = -\frac{\Gamma(s+3/2)(1105s^3+1326s^2+131s-42)}{7560\sqrt{\pi}\Gamma(s)}(c^{2s}-b^{2s})\zeta_\nu(s+3/2) \tag{5.48}$$

$$- \frac{\Gamma(s+3/2)(5s+6)(s+1)}{6\sqrt{\pi}\Gamma(s)}\int_b^c d\rho\,\rho^{2s+1}V(\rho)\zeta_\nu(s+3/2) \tag{5.49}$$

$$+ \frac{\Gamma(s+3/2)}{4\sqrt{\pi}\Gamma(s)}\int_b^c d\rho\,\rho^{2s+3}V(\rho)^2\zeta_\nu(s+3/2) \tag{5.50}$$

$$- \frac{5\Gamma(s+3/2)(2s+3)}{12\sqrt{\pi}\Gamma(s)}\int_b^c d\rho\,\rho^{2s+2}V'(\rho)\zeta_\nu(s+3/2) \tag{5.51}$$

$$- \frac{\Gamma(s+3/2)}{4\sqrt{\pi}\Gamma(s)}\int_b^c d\rho\,\rho^{2s+3}V''(\rho)\zeta_\nu(s+3/2)\,. \tag{5.52}$$

5.5.3 Residues

With the previous information, it is possible to find the first residues of the zeta function associated with P by analyzing the poles of the asymptotic parts. We calculate the first residues of ζ_P by finding the residues of the asymptotic terms A_i, which can be computed by knowing the location of the poles and the residues of the Barnes zeta function, which can be found in Appendix B.

Thus the first residues read,

$$\text{Res } \zeta_P\left(\frac{d}{2}\right) = \frac{1}{2^d\Gamma\left(\frac{d+2}{2}\right)\Gamma\left(\frac{d}{2}\right)}\left(c^d - b^d\right) \tag{5.53}$$

$$\text{Res } \zeta_P\left(\frac{d-1}{2}\right) = -\frac{1}{4\Gamma(d)}(c^{d-1} + b^{d-1}) \tag{5.54}$$

$$\text{Res } \zeta_P\left(\frac{d-2}{2}\right) = \frac{d-1}{3\cdot 2^{d-1}\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{d}{2}\right)}(c^{d-2} - b^{d-2}) \tag{5.55}$$

$$- \frac{1}{2^{d-1}\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{d}{2}\right)}\int_b^c d\rho\, \rho^{d-1}V(\rho) \tag{5.56}$$

$$\text{Res } \zeta_P\left(\frac{d-3}{2}\right) = \frac{(d-3)\left((d-1)^2 - 22(d-1) + 24\right)}{768\Gamma(d-1)}(c^{d-3} + b^{d-3}) + \frac{d-3}{8\Gamma(d)}(c^{d-1}V(c) + b^{d-1}V(b)) \tag{5.57}$$

$$\begin{aligned}\text{Res } \zeta_P\left(\frac{d-4}{2}\right) = &-\frac{d\left(348672 - 444368d + 204308d^2 - 40732d^3 + 2995d^4\right)}{120960\cdot 2^{d-1}\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{d-4}{2}\right)}(c^{d-4} - b^{d-4})\\ &+ \frac{1}{3\cdot 2^{d-1}\Gamma\left(\frac{d-4}{2}\right)^2}\int_b^c d\rho\, \rho^{d-3}V(\rho) + \frac{1}{2^d\Gamma\left(\frac{d-4}{2}\right)\Gamma\left(\frac{d}{2}\right)}\int_b^c d\rho\, \rho^{d-1}V^2(\rho)\\ &- \frac{1}{2^d\Gamma\left(\frac{d-4}{2}\right)\Gamma\left(\frac{d}{2}\right)}(c^{d-1}V'(c) - b^{d-1}V'(b))\\ &- \frac{d-1}{3\cdot 2^{d-1}\Gamma\left(\frac{d-4}{2}\right)\Gamma\left(\frac{d}{2}\right)}(c^{d-2}V(c) - b^{d-2}V(b))\end{aligned} \tag{5.58}$$

5.6 Three Dimensional Spherical Shell

In the previous chapter we talked about the case of one and two dimensional radially symmetric potentials. Here we explore the next natural case which is the three dimensional spherical shell. We calculate some special values for this configuration, namely the finite part value at $s = -1/2$, and the value of the derivative at $s = 0$. Since our formalism was made for a general potential $V(r)$, in order to obtain explicit results about these quantities, one has to do some numerical analysis on the expressions obtained once a potential is specified.

5.6.1 Functional Determinant

In order to include the point $s = 0$ into the convergence region, we need to subtract at least 4 asymptotic terms from the integral representation of the zeta function.

Thus, we use

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{2} A_i(s)\,, \tag{5.59}$$

which is valid for $\Re(s) > -1/2$.

Therefore, the functional determinant can be found by computing $\zeta'_P(0)$, which is found to be

$$\zeta'_P(0) = Z'(0) + \sum_{i=-1}^{2} A'_i(0)\,, \tag{5.60}$$

where

$$Z'(0) = -\sum_{\nu} d(\nu)\left(\log R_m(c;0) - \log A^{+} - \sum_{i=-1}^{2} \nu^{-i}\int_b^c d\rho\, S_i^{+}(\rho)\right)\,, \tag{5.61}$$

and

$$A'_{-1}(0) = -\sum_{\nu} d(\nu)\int_b^c d\rho\, S_{-1}^{+}(\rho)\,, \tag{5.62}$$

$$A'_0(0) = -\sum_{\nu} d(\nu)\left(-\log\left(2\nu S_{-1}^{+}(b)\right) + \int_b^c d\rho\, S_0^{+}(\rho)\right)\,, \tag{5.63}$$

$$A_1'(0) = -\sum_\nu d(\nu) \int_b^c d\rho\, S_1^+(\rho)\,, \tag{5.64}$$

$$A_2'(0) = -\sum_\nu d(\nu) \left(-\frac{S_1^+(b)}{S_{-1}^+(b)} + \int_b^c d\rho\, S_2^+(\rho) \right) . \tag{5.65}$$

5.6.2 Casimir Force

In order to evaluate the zeta function at $s = -1/2$ we need to subtract at least 5 asymptotic terms. Hence, we work with

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{3} A_i(s)\,, \tag{5.66}$$

which is valid for $\Re(s) > -1$.

Since the first asymptotic terms do not depend on the position of the potential for $s = -1/2$, the only contribution is made by the finite part

$$F_{Cas} = \frac{\partial}{\partial a}\zeta_P(-1/2) = -\frac{1}{\pi}\zeta_{\mathbb{S}}(-1/2) \int_0^\infty d\xi\, \xi \frac{\partial}{\partial a}\frac{d}{d\xi} \left(\log R_\nu(c; \imath\xi\nu)\right) . \tag{5.67}$$

CHAPTER SIX

Smooth Potentials in Cylindrical Shells

Another way of generalizing the configuration treated in Chapter 4 to higher dimensions is by considering the 2 dimensional configuration and adding extra euclidean dimensions. In this chapter we deal with the 2 dimensional circular ring configuration cross a $d-2$ dimensional Euclidean space which constitutes a d dimensional cylindrical shell.

6.1 Differential Equation

For this discussion we consider the eigenvalue problem

$$P\mu_\lambda = \lambda^2 \mu_\lambda \tag{6.1}$$

defined on the cylindrical shell $R_b^c \times [0, L]$, where R_b^c is the region enclosed by the circle of radius b and the circle of radius c around the origin in $\mathbb{R}^2$, $L = (L_1, L_2, \ldots, L_{d-2})$ is a multi-vector in $(\mathbb{R}^+)^{d-2}$ and the second order differential operator P is given by

$$P = -\Delta + V(r)\,, \tag{6.2}$$

where $V(r)$ is a smooth cylindrical symmetric potential with support in $(a-w, a+w) \subset [b, c]$, and where we consider Dirichlet boundary conditions at $r = b$, $r = c$, $x_i = 0$, and $x_i = L_i$, for all the Euclidean coordinates $x_1, \ldots, x_{d-2}$.

Using separation of variables we write $\mu(r, \theta, z) = R(r)\Theta(\theta)Z(z)$, where $r \in \mathbb{R}^+$, $z \in \mathbb{R}^{d-2}$, and $\theta \in [0, 2\pi]$. Hence we have that the eigenvalue problem reads

$$-\frac{\Theta Z}{r}(R' + rR'') - \frac{RZ}{r^2}\Theta'' - R\Theta Z'' + V(r)R\Theta Z = \lambda^2 R\Theta Z \tag{6.3}$$

which can be written as

$$-\frac{1}{rR}(R' + rR'') - \frac{1}{r^2\Theta}\Theta'' - \frac{1}{Z}Z'' + V(r) = \lambda^2\,. \tag{6.4}$$

Therefore, we can rewrite the eigenvalue problem as the system

$$-\frac{1}{rR}(R' + rR'') - \frac{1}{r^2\Theta}\Theta'' + V(r) = \eta^2 \tag{6.5}$$

$$-\frac{1}{Z}Z'' = \nu^2 \tag{6.6}$$

where $\eta^2 + \nu^2 = \lambda^2$, which gives the system of ordinary differential equations

$$-\frac{r}{R}(R' + rR'') + r^2\left(V(r) - \eta^2\right) = -\kappa^2 \tag{6.7}$$

$$\frac{1}{\Theta}\Theta'' = -\kappa^2\,. \tag{6.8}$$

Solving the eigenvalue equation for the Laplacian in $\mathbb{R}^{d-2}$ from (6.6) and imposing periodic boundary conditions on the angular part in (6.8) gives

$$\nu^2 = \sum_{i=1}^{d-2}\left(\frac{n_i\pi}{L_i}\right)^2, \quad n_i \in \mathbb{Z}^+ \tag{6.9}$$

$$\kappa \in \mathbb{Z}\,. \tag{6.10}$$

6.2 Zeta Function

We have that the zeta function for the differential operator P is given by

$$\zeta_P(s) = \sum_{m,n,\kappa}\lambda_{mn\kappa}^{-2s} = \sum_{m,n,\kappa}\left(\eta_{m\kappa}^2 + \sum_{i=1}^{d-2}\left(\frac{n_i\pi}{L_i}\right)^2\right)^{-s}, \tag{6.11}$$

where $\eta_{m\kappa}$ satisfies the equation $R_\kappa(c;\eta) = 0$, where R is a solution to the initial value problem

$$-r^2\frac{R''}{R} - r\frac{R'}{R} + r^2(V(r) - \eta^2) + \kappa^2 = 0 \tag{6.12}$$

with the initial conditions $R_\kappa(b;\eta) = 0$ and $R'_\kappa(b;\eta) = 1$.

Thus, the zeta functions associated with P admits the integral representation

$$\zeta_P(s) = \frac{\sin(\pi s)}{\pi}\sum_{n\in(\mathbb{Z}^+)^{d-2}}\sum_{\kappa=-\infty}^{\infty}\int_{l(n)}^{\infty} d\eta\,\left(\eta^2 - l(n)^2\right)^{-s}\frac{d}{d\eta}\log R_\kappa(c;\imath\eta) \tag{6.13}$$

where

$$l(n) = \left(\sum_{i=1}^{d-2}\left(\frac{n_i\pi}{L_i}\right)^2\right)^{1/2}. \tag{6.14}$$

6.3 Characteristic Function Asymptotic Expansion

As this configuration is a product of the two dimensional circular ring configuration with an Euclidean space, we will have a zero case in the angular part and hence a similar analysis as the approach made is Chapter 4 is needed. We are going to find an asymptotic expansion for the characteristic function in terms of the eigenvalues η for the zero case $\kappa = 0$ and for the case of $\kappa \neq 0$ an uniform asymptotic expansion in η and κ is needed.

6.3.1 Zero Case

Since $\kappa = 0$, it suffices to get the asymptotic expansion for large η. Let

$$\tilde{S}(r,\eta) = \frac{\partial}{\partial r} \log R_0(r; \imath\eta)\,, \tag{6.15}$$

which by (6.12) will satisfy

$$\tilde{S}' = \big(V(r) + \eta^2\big) - \tilde{S}^2 - \frac{1}{r}\tilde{S}\,. \tag{6.16}$$

Considering the asymptotic series for $\tilde{S}$

$$\tilde{S}(r,\eta) \sim \sum_{i=-1}^{\infty} \tilde{S}_i(r)\eta^{-i}\,, \tag{6.17}$$

where we can find $\tilde{S}_i$ recursively by

$$\tilde{S}_{i+1}(r) = -\frac{1}{2\tilde{S}_{-1}(r)}\left(\tilde{S}_i'(r) + \frac{1}{r}\tilde{S}_i(r) + \sum_{j=0}^{i} \tilde{S}_j(r)\tilde{S}_{i-j}(r) + \tilde{f}_{i+1}(r)\right) \tag{6.18}$$

with the initial condition

$$\tilde{S}_{-1}^{\pm}(r) = \pm 1 \tag{6.19}$$

and where the functions f_i are given by

$$\tilde{f}_{-1}(r) = 1, \quad \tilde{f}_0(r) = 0, \quad \tilde{f}_1(r) = -V(r)\,, \tag{6.20}$$

and $\tilde{f}_i(r) = 0$ for $i > 1$.

6.3.2 Non-zero Case

In the case where $\kappa \neq 0$, we need to control the behavior as both η and κ grow to infinity. Using the substitution

$$\eta = \xi\kappa \tag{6.21}$$

we can proceed as before. Let

$$\hat{S}(r,\kappa) = \frac{\partial}{\partial r} \log R_\kappa(r; \imath\xi\kappa) \tag{6.22}$$

which by (6.12) satisfies

$$\hat{S}' = \left(V(r) + \kappa^2\left(\xi^2 + \frac{1}{r^2}\right)\right) - \hat{S}^2 - \frac{1}{r}\hat{S}\,. \tag{6.23}$$

Consider the asymptotic expansion for $\hat{S}$

$$\hat{S}(r,\kappa) \sim \sum_{i=-1}^{\infty} \hat{S}_i(r)\kappa^{-i} \tag{6.24}$$

where we can find $\hat{S}_i$ recursively by

$$\hat{S}_{i+1}(r) = -\frac{1}{2\hat{S}_{-1}(r)}\left(\hat{S}_i'(r) + \frac{1}{r}\hat{S}_i(r) + \sum_{j=0}^{i} \hat{S}_j(r)\hat{S}_{i-j}(r) + \hat{f}_{i+1}(r)\right) \tag{6.25}$$

with the initial condition

$$\hat{S}_{-1}^{\pm}(r) = \pm\sqrt{\xi^2 + \frac{1}{r^2}} \tag{6.26}$$

and where the functions $\hat{f}_i$ are given by

$$\hat{f}_{-1}(r) = \xi^2 + \frac{1}{r^2}, \quad \hat{f}_0(r) = 0, \quad \hat{f}_1(r) = -V(r)\,, \tag{6.27}$$

and $\hat{f}_i(r) = 0$ for $i > 1$.

6.3.3 Asymptotics

Following the notation used in Chapter 4, let

$$S(r,\eta) = \begin{cases} \tilde{S}(r,\eta) & \kappa = 0 \\ \hat{S}(r,\kappa) & \kappa \neq 0 \end{cases} \tag{6.28}$$

and let $S^{\pm}$ be the corresponding series obtained when $S^{\pm}_{-1}$ is chosen.

Therefore we have that $R_\kappa(c;\imath\eta)$ is a combination of exponentially growing and decaying terms

$$R_\kappa(c;\imath\eta) = A^+ \exp\left(\int_b^c dt\ S^+(t,\eta)\right) + A^- \exp\left(\int_b^c dt\ S^-(t,\eta)\right) \tag{6.29}$$

and from the initial conditions (6.12) we have that

$$A^+ = -A^- = \frac{1}{S^+(b,\eta) - S^-(b,\eta)}\,. \tag{6.30}$$

As before, we have the asymptotic series for $\log A^+$ be given by

$$\log A^+ = \begin{cases} -\log\left(2\eta\tilde{S}^+_{-1}(b)\right) - \log\left(1 + \sum_{j=1}^{\infty} \frac{\tilde{S}^+_{2j-1}(b)}{\tilde{S}^+_{-1}(b)}\eta^{-2j}\right) & \text{for } \kappa = 0 \\ -\log\left(2\kappa\hat{S}^+_{-1}(b)\right) - \log\left(1 + \sum_{j=1}^{\infty} \frac{\hat{S}^+_{2j-1}(b)}{\hat{S}^+_{-1}(a)}\kappa^{-2j}\right) & \text{for } \kappa \neq 0 \end{cases}. \tag{6.31}$$

With this we have the asymptotic expansion for the logarithm of the characteristic function

$$\log R_\kappa(c;\imath\eta) \sim \begin{cases} \log A^+ + \sum_{i=-1}^{\infty} \eta^{-i} \int_b^c dt\tilde{S}^+_i(t) & \text{for } \kappa = 0 \\ \log A^+ + \sum_{i=-1}^{\infty} \kappa^{-i} \int_b^c dt\hat{S}^+_i(t) & \text{for } \kappa \neq 0 \end{cases} \tag{6.32}$$

6.4 Analytic Continuation

With the asymptotic expression for $R_\kappa(c;\imath\eta)$, we have that the spectral zeta function for P can be written as

$$\zeta(s) = Z(s) + \sum_{i=-1}^{N-2} A_i(s) \tag{6.33}$$

where $Z(s)$ is the zeta function with N asymptotic terms subtracted and the $A_i(s)$ are the asymptotic contribution of η^{-i} coming from $\log A^+$ and S_i^+. Thus, we have that

$$Z(s) = \frac{\sin(\pi s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \int_{l(n)}^{\infty} d\eta \, \left(\eta^2 - l(n)^2\right)^{-s}$$
$$\times \frac{d}{d\eta}\left(\log R_0(c;\imath\eta) - \log A^+ - \int_b^c dt \sum_{i=-1}^{N-2} \tilde{S}_i^+(t)\eta^{-i}\right)$$
$$+ \frac{2\sin(\pi s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \sum_{\kappa=1}^{\infty} \kappa^{-2s} \int_{\frac{l(n)}{\kappa}}^{\infty} d\xi \, \left(\xi^2 - \left(\frac{l(n)}{\kappa}\right)^2\right)^{-s}$$
$$\times \frac{d}{d\xi}\left(\log R_\kappa(c;\imath\eta) - \log A^+ - \int_b^c dt \sum_{i=-1}^{N-2} \hat{S}_i^+(t)\kappa^{-i}\right) . \quad (6.34)$$

In order to find the analytic continuation for the asymptotic terms $A_i(s)$, we are going to exploit the structure of the Epstein zeta function that arises from the Euclidean space,

$$\zeta_E(s) = \sum_{n\in(\mathbb{Z}^+)^{d-2}} \frac{1}{\left(\left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2 + \cdots + \left(\frac{n_d\pi}{L_d}\right)^2\right)^s} \quad (6.35)$$

$$= \sum_{n\in(\mathbb{Z}^+)^{d}} l(n)^{-2s} \quad (6.36)$$

whose meromorphic structure is discussed in the Appendix B, as well as the evaluation of special integrals that come from the form of the asymptotic terms S_i.

The integrals obtained from the zero case $\kappa = 0$ are straightforward, as the asymptotic terms $\tilde{S}_i$ are the same as the ones obtained in the two dimensional case and the Euclidean space contribution shows up just as a multiplication by the Epstein zeta function. In the non-zero case, a different approach is needed. Here, a convolution type of the zeta function coming from the circular ring and the Euclidean space is more evident, giving terms of the form

$$\sum_{n\in(\mathbb{Z}^+)^{d-2}} \sum_{k=1}^{\infty} \kappa^{-2s-i} \int_{\frac{l(n)}{\kappa}}^{\infty} d\xi \, \left(\xi^2 - \frac{l(n)^2}{\kappa^2}\right)^{-s} \int_b^c dr \, r^\alpha \xi^\beta (1 + r^2\xi^2)^\delta f(r) , \quad (6.37)$$

where $f(r)$ is a function of r only being mainly integrals over the potential $V(r)$ and its derivatives.

In order to evaluate and find the analytic continuation for these expressions, we rewrite the above integral as a contour integral in the complex plane and then use properties of hypergeometric functions, so we can separate the r terms from the ξ terms to be able to perform the r and ξ integrals independently and find the analytic continuation of the series.

For this, we have the following

Lemma 6.1.

$$\sum_{n\in(\mathbb{Z}^+)^{d-2}}\sum_{k=1}^{\infty}\kappa^{-2s-i}\int_{\frac{l(n)}{\kappa}}^{\infty}d\xi\,\left(\xi^2-\frac{l(n)^2}{\kappa^2}\right)^{-s}\int_b^c dr\, r^\alpha\xi^\beta(1+r^2\xi^2)^\delta f(r) \qquad (6.38)$$

admits an analytic continuation in s *as a finite sum of meromorphic functions for* $\alpha,\beta/2,2\delta\in\mathbb{Z}$ *and* $i+2\delta$*, and even integer.*

Proof. Suppose that $\Re(s)$ is big enough so we have convergence of the ξ integral. Using change of variables, we can rewrite the summand as

$$\begin{aligned}
&\kappa^{-2s-i}\int_{\frac{l(n)}{\kappa}}^{\infty}d\xi\,\left(\xi^2-\frac{l(n)^2}{\kappa^2}\right)^{-s}\int_b^c dr\, r^\alpha\xi^\beta(1+r^2\xi^2)^\delta f(r)\\
&\quad=\kappa^{-\beta-i-1}l(n)^{1-2s+\beta}\int_b^c dr\int_1^\infty du\,\left(u^2-1\right)^{-s}r^\alpha u^\beta\left(1+\frac{r^2l(n)^2}{\kappa^2}u^2\right)^\delta f(r)\\
&\quad=\kappa^{-\beta-i-1-2\delta}l(n)^{1-2s+\beta+2\delta}\\
&\quad\times\int_b^c dr\int_0^1 dv\,\left(1-v^2\right)^{-s}r^{\alpha+2\delta}v^{2s-\beta-2\delta-2}\left(1+\frac{\kappa^2}{r^2l(n)^2}v^2\right)^\delta f(r)\\
&\quad=\frac{1}{2}\kappa^{-\beta-i-1-2\delta}l(n)^{1-2s+\beta+2\delta}\\
&\quad\times\int_b^c dr\int_0^1 dt\,(1-t)^{-s}r^{\alpha+2\delta}t^{s-\beta/2-\delta-3/2}\left(1+\frac{\kappa^2}{r^2l(n)^2}t\right)^\delta f(r)\,, \qquad (6.39)
\end{aligned}$$

and using the integral representation of the hypergeometric function we have that the integral equals

$$\frac{1}{2}\kappa^{-\beta-i-1-2\delta} l(n)^{1-2s+\beta+2\delta} \frac{\Gamma(s-\beta/2-\delta-1/2)\Gamma(1-s)}{\Gamma(1/2-\beta/2-\delta)}$$
$$\times \int_b^c dr\, r^{\alpha+2\delta}\, {}_2F_1\left(-\delta, s-\beta/2-\delta-1/2, 1/2-\beta/2-\delta; -\frac{\kappa^2}{r^2 l(n)^2}\right) f(r)\,. \quad (6.40)$$

The Barnes-Mellin integral representation for the hypergeometric function provides a way to rewrite the above expression as

$$\frac{1}{2}\frac{\Gamma(1-s)}{\Gamma(-\delta)}\frac{1}{2\pi\imath}\int_b^c dr \int_\gamma dt\, \frac{\Gamma(t-\delta)\Gamma(t+s-\beta/2-\delta-1/2)}{\Gamma(t+1/2-\beta/2-\delta)}$$
$$\times \Gamma(-t)\kappa^{-\beta-i-1-2\delta+2t} l(n)^{1-2s+\beta+2\delta-2t} r^{\alpha+2\delta-2t} f(r)\,, \quad (6.41)$$

where γ is a contour such that the poles of $\frac{\Gamma(t-\delta)\Gamma(t+s-\beta/2-\delta-1/2)}{\Gamma(t+1/2-\beta/2-\delta)}$ lie to the left of γ and the poles of $\Gamma(-t)$ to the right.

Provided that $\Re(s) > (d-2+i-i)/2$, the above expression can be summed in the region where $\beta/2+i/2+\delta > \Re(t) > d/2-3/2+\beta/2+\delta-\Re(s)$. Thus using the Riemann zeta function and the Epstein zeta function we obtain that the series (6.38) can be written as the contour integral

$$\frac{1}{2}\frac{\Gamma(1-s)}{\Gamma(-\delta)}\frac{1}{2\pi\imath}\int_b^c dr \int_\gamma dt\, \frac{\Gamma(t-\delta)\Gamma(t+s-\beta/2-\delta-1/2)}{\Gamma(t+1/2-\beta/2-\delta)}$$
$$\times \Gamma(-t)\zeta_R(\beta+i+1+2\delta-2t)\zeta_E(t+s-\beta/2-\delta-1/2) r^{\alpha+2\delta-2t} f(r)\,. \quad (6.42)$$

Notice that closing the contour γ to the right will capture the poles located at $t=n$, $n\in\mathbb{N}$ and $t=\beta/2+i/2+\delta$, therefore we find the contour integral using Cauchy's residue theorem to be

$$\frac{1}{2}\frac{\Gamma(1-s)}{\Gamma(-\delta)}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\Gamma(n-\delta)\Gamma(s-\beta/2-\delta-1/2+n)}{n!\Gamma(1/2-\beta/2-\delta+n)}$$
$$\times \zeta_R(\beta+i+1+2\delta-2n)\zeta_E(s-\beta/2-\delta-1/2+n)\int_b^c dr\, r^{\alpha+2\delta-2n} f(r)$$
$$-\frac{1}{4}\frac{\Gamma(1-s)}{\Gamma(-\delta)}\frac{\Gamma(\beta/2+i/2)\Gamma(s+i/2-1/2)\Gamma(-\beta/2-i/2-\delta)}{\Gamma(i/2+1/2)}$$
$$\times \zeta_E(s+i/2-1/2)\int_b^c dr\, r^{\alpha-\beta-i} f(r)\,. \quad (6.43)$$

Since $\beta, 2\delta \in \mathbb{Z}$ and $i + 2\delta$ is even, we have that the n series only contributes with a finite number of summands

$$\frac{1}{2}\frac{\Gamma(1-s)}{\Gamma(-\delta)}\sum_{0\le n<\beta/2+i/2+3/2+\delta}\frac{(-1)^{n+1}\Gamma(n-\delta)\Gamma(s-\beta/2-\delta-1/2+n)}{n!\Gamma(1/2-\beta/2-\delta+n)}$$
$$\times\,\zeta_R(\beta+i+1+2\delta-2n)\zeta_E(s-\beta/2-\delta-1/2+n)\int_b^c dr\, r^{\alpha+2\delta-2n}f(r)$$
$$-\frac{1}{4}\frac{\Gamma(1-s)}{\Gamma(-\delta)}\frac{\Gamma(\beta/2+i/2)\Gamma(s+i/2-1/2)\Gamma(-\beta/2-i/2-\delta)}{\Gamma(i/2+1/2)}$$
$$\times\,\zeta_E(s+i/2-1/2)\int_b^c dr\, r^{\alpha-\beta-i}f(r) \quad (6.44)$$

which provides the analytic continuation needed. □

6.4.1 First Asymptotic Terms

We are going to find the first asymptotic terms for any dimension d and then we are going to consider the case of a three dimensional cylindrical shell, as the one and two dimensional configurations were already treated in Chapter 4.

For this, we calculate the first five asymptotic terms coming from the WKB expansion for both zero and non-zero modes

6.4.2 Zero Case Asymptotic Terms

Using the recursion (6.18) we find the first five asymptotic terms of the zero case to be

$$\tilde{S}_{-1}(r) = 1 \quad (6.45)$$

$$\tilde{S}_0(r) = -\frac{1}{2r} \quad (6.46)$$

$$\tilde{S}_1(r) = -\frac{1}{8r^2} + \frac{1}{2}V(r) \quad (6.47)$$

$$\tilde{S}_2(r) = -\frac{1}{8r^3} - \frac{1}{4}V'(r) \tag{6.48}$$

$$\tilde{S}_3(r) = -\frac{25}{128r^4} + \frac{V(r)}{16r^2} - \frac{V(r)^2}{8} + \frac{V''(r)}{8}\,. \tag{6.49}$$

6.4.3 Non-zero Case Asymptotic Terms

Similarly, using the WKB recursion (6.25) we calculate the first five asymptotic terms in the non-zero case WKB asymptotic expansion to be

$$\hat{S}_{-1}(r) = \frac{(r^2\xi^2+1)^{1/2}}{r} \tag{6.50}$$

$$\hat{S}_0(r) = -\frac{r\xi^2}{2+2r^2\xi^2} \tag{6.51}$$

$$\hat{S}_1(r) = \frac{4r\xi^2 - r^3\xi^4 + 4r\left(1+r^2\xi^2\right)^2 V(r)}{8\left(1+r^2\xi^2\right)^{5/2}} \tag{6.52}$$

$$\hat{S}_2(r) = -\frac{r\xi^2\left(4-10r^2\xi^2+r^4\xi^4\right) + 4r\left(1+r^2\xi^2\right)^2 V(r) + 2r^2\left(1+r^2\xi^2\right)^3 V'(r)}{8\left(1+r^2\xi^2\right)^4}\,. \tag{6.53}$$

$$\begin{aligned}\hat{S}_3(r) = {} & \frac{64r\xi^2 - 560r^3\xi^4 + 456r^5\xi^6 - 25r^7\xi^8}{128\left(1+r^2\xi^2\right)^{11/2}} \\ & + \frac{8r\left(8-16r^2\xi^2+r^4\xi^4\right)V(r)}{128\left(1+r^2\xi^2\right)^{7/2}} - \frac{16r^3\left(1+r^2\xi^2\right)^2 V(r)^2}{128\left(1+r^2\xi^2\right)^{7/2}} \\ & + \frac{80r^2V'(r)}{128\left(1+r^2\xi^2\right)^{5/2}} + \frac{16r^3V''(r)}{128\left(1+r^2\xi^2\right)^{3/2}}\end{aligned} \tag{6.54}$$

6.4.4 Integral Representation for A_i

The first five asymptotic terms in the meromorphic structure of the zeta function associated with the operator P in the cylindrical shell piston configuration are found to be given by

$$A_{-1}(s) = \frac{\sin(\pi s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \int_{l(n)}^{\infty} d\eta \left(\eta^2 - l(n)^2\right)^{-s} \frac{d}{d\eta}\left(\int_b^c dt\ \tilde{S}^+_{-1}(t)\eta\right)$$
$$+ \frac{2\sin(\pi+s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \sum_{\kappa=1}^{\infty} \kappa^{-2s+1} \int_{\frac{l(n)}{\kappa}}^{\infty} d\xi\ \left(\xi^2 - \left(\frac{l(n)}{\kappa}\right)^2\right)^{-s} \frac{d}{d\xi}\left(\int_b^c dt\ \hat{S}^+_{-1}(t)\right) \tag{6.55}$$

$$A_{0}(s) = \frac{\sin(\pi s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \int_{l(n)}^{\infty} d\eta \left(\eta^2 - l(n)^2\right)^{-s} \frac{d}{d\eta}\left(-\log(2\eta\tilde{S}_{-1}(b)) + \int_b^c dt\ \tilde{S}^+_{0}(t)\right)$$
$$+ \frac{2\sin(\pi+s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \sum_{\kappa=1}^{\infty} \kappa^{-2s} \int_{\frac{l(n)}{\kappa}}^{\infty} d-2\xi\ \left(\xi^2 - \left(\frac{l(n)}{\kappa}\right)^2\right)^{-s}$$
$$\times \frac{d}{d\xi}\left(-\log(2\kappa\hat{S}_{-1}(b)) + \int_b^c dt\ \ + \hat{S}^+_{0}(t)\right) \tag{6.56}$$

$$A_{1}(s) = \frac{\sin(\pi s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \int_{l(n)}^{\infty} d\eta \left(\eta^2 - l(n)^2\right)^{-s} \frac{d}{d\eta}\left(\int_b^c dt\ \tilde{S}^+_{1}(t)\eta^{-1}\right)$$
$$+ \frac{2\sin(\pi+s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \sum_{\kappa=1}^{\infty} \kappa^{-2s-1} \int_{\frac{l(n)}{\kappa}}^{\infty} d\xi\ \left(\xi^2 - \left(\frac{l(n)}{\kappa}\right)^2\right)^{-s} \frac{d}{d\xi}\left(\int_b^c dt\ \hat{S}^+_{1}(t)\right) \tag{6.57}$$

$$A_{2}(s) = \frac{\sin(\pi s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \int_{l(n)}^{\infty} d\eta \left(\eta^2 - l(n)^2\right)^{-s} \frac{d}{d\eta}\left(-\frac{\tilde{S}_1(b)}{\tilde{S}_{-1}(b)}\eta^{-2} + \int_b^c dt\ \tilde{S}^+_{2}(t)\eta^{-2}\right)$$
$$+ \frac{2\sin(\pi+s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \sum_{\kappa=1}^{\infty} \kappa^{-2s-2} \int_{\frac{l(n)}{\kappa}}^{\infty} d\xi\ \left(\xi^2 - \left(\frac{l(n)}{\kappa}\right)^2\right)^{-s}$$

$$\times \frac{d}{d\xi}\left(-\frac{\hat{S}_1(b)}{\hat{S}_{-1}(b)} + \int_b^c dt\ \hat{S}_2^+(t)\right) \quad (6.58)$$

$$A_3(s) = \frac{\sin(\pi s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \int_{l(n)}^{\infty} d\eta \left(\eta^2 - l(n)^2\right)^{-s} \frac{d}{d\eta}\left(\int_b^c dt\ \tilde{S}_3^+(t)\eta^{-3}\right)$$
$$+ \frac{2\sin(\pi+s)}{\pi} \sum_{n\in(\mathbb{Z}^+)^{d-2}} \sum_{\kappa=1}^{\infty} \kappa^{-2s-3} \int_{\frac{l(n)}{\kappa}}^{\infty} d\xi \left(\xi^2 - \left(\frac{l(n)}{\kappa}\right)^2\right)^{-s} \frac{d}{d\xi}\left(\int_b^c dt\,\hat{S}_3^+(t)\right)$$
$$(6.59)$$

6.4.5 *Analytic Continuation for A_i*

Applying Lemma 6.1 we find the first five asymptotic terms A_i to be

$$A_{-1}(s) = \frac{1}{4(s-1)}(c^2 - b^2)\zeta_E(s-1) \quad (6.60)$$

$$A_0(s) = -\frac{1}{2}\zeta_E(s) - \frac{\sqrt{\pi}\Gamma(s-1/2)}{4\Gamma(s)}(c+b)\zeta_E(s-1/2) \quad (6.61)$$

$$A_1(s) = -\frac{3\Gamma(s+1/2)}{8\sqrt{\pi}\Gamma(s)}(c^{-1} - b^{-1})\zeta_E(s+1/2)$$
$$- \frac{\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s)}\zeta_E(s+1/2)\int_b^c dr\, V(r) + \frac{1}{2}\zeta_E(s)\int_b^c dr\, rV(r) \quad (6.62)$$

$$A_2(s) = -\frac{s}{8}(c^{-2} + b^{-2})\zeta_E(s+1) + \frac{s}{2}(V(c) + V(b))\zeta_E(s+1)$$
$$- \frac{\sqrt{\pi}\Gamma(s+1/2)}{4\Gamma(s)}(cV(c) + bV(b))\zeta_E(s+1/2)$$
$$- \frac{\sqrt{\pi}\Gamma(s+1/2)}{128\Gamma(s)}(c^{-1} + b^{-1})\zeta_E(s+1/2) \quad (6.63)$$

$$
\begin{aligned}
A_3(s) = \frac{2458s}{3465}(c^{-2} - b^{-2})\zeta_E(s+1) - \frac{25\Gamma(s+3/2)}{384\sqrt{\pi}\Gamma(s)}(c^{-3} - b^{-3})\zeta_E(s+3/2) \\
- \frac{\Gamma(s+3/2))}{8\sqrt{\pi}\Gamma(s)}\zeta_E(s+3/2)\int_b^c dr\, r^{-2}V(r) + \frac{\Gamma(s+3/2)}{4\sqrt{\pi}\Gamma(s)}\zeta_E(s+3/2)\int_b^c dr\, V^2(r) \\
- \frac{s}{4}\zeta_E(s+1)\int_b^c dr\, rV^2(r) + \frac{5s}{12}(V(c) - V(b))\zeta_E(s+1) \\
- \frac{\Gamma(s+3/2)}{4\sqrt{\pi}\Gamma(s)}(V'(c) - V'(b))\zeta_E(s+3/2) + \frac{s}{4}\zeta_E(s+1)\int_b^c dr\, rV''(r) \\
- \frac{25\Gamma(s+3/2)}{192\sqrt{\pi}\Gamma(s)}(c^{-3} - b^{-3})\zeta_E(s+3/2) - \frac{\Gamma(s+3/2)}{8\sqrt{\pi}\Gamma(s)}\zeta_E(s+3/2)\int_b^c dr\, r^{-2}V(r) \\
+ \frac{\Gamma(s+3/2)}{4\sqrt{\pi}\Gamma(s)}\zeta_E(s+3/2)\int_b^c dr\, V^2(r) - \frac{\Gamma(s+3/2)}{4\sqrt{\pi}\Gamma(s)}(V'(c) - V'(b))\zeta_E(s+3/2)\,.
\end{aligned}
\tag{6.64}
$$

6.5 Three Dimensional Cylindrical Shell

In the case of a three dimensional spherical shell, in order to find the functional determinant and the Casimir force, we can subtract $N = 4$ and $N = 5$ asymptotic terms respectively. Here we can only obtain general expression for these quantities, as more detailed analysis requires an explicit knowledge of the potential $V(r)$.

6.5.1 Functional Determinant

For the functional determinant, we subtract the first 4 asymptotic terms from the integral representation of the zeta function, obtaining

$$
\zeta_P(s) = Z(s) + \sum_{i=-1}^{2} A_i(s)\,, \tag{6.65}
$$

which is valid for $\Re(s) > -1/2$. From here we have that

$$
\zeta_P(0) = Z'(0) + \sum_{i=-1}^{2} A'_{-1}(0)\,, \tag{6.66}
$$

where

$$Z'(0) = \sum_{n\in(\mathbb{Z}^+)^{d-2}} \left(\log R_0(c;\imath l(n)) - \log A^+ - \sum_{i=-1}^{2}\int_b^c dt\,\tilde{S}_i^+(t)l(n)^{-i}\right)$$

$$+ \sum_{n\in(\mathbb{Z}^+)^{d-2}}\sum_{\kappa=1}^{\infty}\left(\log R_\kappa\left(c;\frac{\imath l(n)}{\kappa}\right) - \log A^+ - \sum_{i=-1}^{2}\int_b^c dt\,\hat{S}_i^+(t)\kappa^{-i}\right), \quad (6.67)$$

$$A'_{-1}(0) = \sum_{n\in(\mathbb{Z}^+)^{d-2}}\int_b^c dt\ \tilde{S}_{-1}^+(t)l(n) + 2\sum_{n\in(\mathbb{Z}^+)^{d-2}}\sum_{\kappa=1}^{\infty}\kappa\int_b^c dt\ \hat{S}_{-1}^+(t) \quad (6.68)$$

$$A'_0(0) = \sum_{n\in(\mathbb{Z}^+)^{d-2}}\left(-\log(2l(n)\tilde{S}_{-1}(b)) + \int_b^c dt\ \tilde{S}_0^+(t)\right)$$

$$+ 2\sum_{n\in(\mathbb{Z}^+)^{d-2}}\sum_{\kappa=1}^{\infty}\left(-\log(2\kappa\hat{S}_{-1}(b)) + \int_b^c dt\ \hat{S}_0^+(t)\right) \quad (6.69)$$

$$A'_1(0) = \sum_{n\in(\mathbb{Z}^+)^{d-2}}\left(\int_b^c dt\ \tilde{S}_1^+(t)\eta^{-1}\right) + 2\sum_{n\in(\mathbb{Z}^+)^{d-2}}\sum_{\kappa=1}^{\infty}\kappa^{-1}\int_b^c dt\ \hat{S}_1^+(t) \quad (6.70)$$

$$A'_2(0) = \sum_{n\in(\mathbb{Z}^+)^{d-2}}\left(-\frac{\tilde{S}_1(b)}{\tilde{S}_{-1}(b)}l(n)^{-2} + \int_b^c dt\ \tilde{S}_2^+(t)l(n)^{-2}\right)$$

$$+ 2\sum_{n\in(\mathbb{Z}^+)^{d-2}}\sum_{\kappa=1}^{\infty}\kappa^{-2}\left(-\frac{\hat{S}_1(b)}{\hat{S}_{-1}(b)} + \int_b^c dt\ \hat{S}_2^+(t)\right). \quad (6.71)$$

6.5.2 Casimir Force

In order to calculate the Casimir force we need to consider 5 asymptotic terms

$$\zeta_P(s) = Z(s) + \sum_{i=-1}^{3} A_i(s)\,, \quad (6.72)$$

where the region of convergence is $\Re(s) > -1$. As the first asymptotic terms do not depend on the position of the piston when $s = -1/2$, the Casimir force can be calculated as

$$F_{Cas} = -\frac{1}{2}\frac{\partial}{\partial a}\zeta_P(-1/2) = -\frac{1}{\pi}\sum_{n\in(\mathbb{Z}^+)^{d-2}}\int_{l(n)}^{\infty} d\eta\, \left(\eta^2 - l(n)^2\right)^{1/2}\frac{\partial}{\partial a}\frac{d}{d\eta}\left(\log R_0(c;\imath\eta)\right)$$

$$-\frac{2}{\pi}\sum_{n\in(\mathbb{Z}^+)^{d-2}}\sum_{\kappa=1}^{\infty}\kappa\int_{\frac{l(n)}{\kappa}}^{\infty} d\xi\,\left(\xi^2 - \left(\frac{l(n)}{\kappa}\right)^2\right)^{1/2}\frac{\partial}{\partial a}\frac{d}{d\xi}\left(\log R_\kappa(c;\imath\eta)\right)\ . \quad (6.73)$$

APPENDICES

APPENDIX A

Basic Inequalities

1.1 Ratio of Hyperbolic Sines

We are going to analyze the behavior of a ratio of hyperbolic sines. As the hyperbolic sine is an exponentially increasing function, one can expect the ratio of the hyperbolic sines to be dominated by the one with larger argument.

Lemma A.1. *If* $m > n > 0$ *then*

$$f(x) = \frac{\sinh(mx)}{\sinh(nx)} \tag{A.1}$$

is increasing in $\mathbb{R}^+$.

Proof. Consider the function

$$g(x) = (m-n)\sinh((m+n)x) - (m+n)\sinh((m-n)x) \tag{A.2}$$

on the interval $[0,\infty)$. Since cosh is increasing on $[0,\infty)$ and $g(0) = 0$, we have that the derivative

$$g'(x) = (m^2 - n^2)(\cosh((m+n)x) - \cosh((m-n)x)) \tag{A.3}$$

is positive on $[0,\infty)$ and thus $g(x) > 0$ for all $x \in (0,\infty)$. Therefore

$$\frac{g(x)}{2} = m\cosh(mx)\sinh(nx) - n\cosh(nx)\sinh(mx) > 0 \tag{A.4}$$

and hence we have that

$$\frac{d}{dx}f(x) = \frac{m\cosh(mx)\sinh(nx) - n\cosh(nx)\sinh(mx)}{\sinh^2(nx)} > 0 \tag{A.5}$$

which completes the proof. □

1.2 Product of positive functions

Here we study the behavior of the product of two competing terms. If we have the product of an increasing function and a decreasing function, the behavior of the product can be determined by the ratio of their derivatives,

Lemma A.2. *If $f(x), g(x) > 0$ are positive functions such that $f'(x) > 0$, $g'(x) < 0$ and*

$$\frac{|f'(x)|}{|g'(x)|} > \frac{f(x)}{g(x)}$$

for all x, then $f(x)g(x)$ is an increasing function.

Proof. Since

$$\frac{|f'(x)|}{|g'(x)|} > \frac{f(x)}{g(x)},$$

we have that $f'(x)g(x) > -f(x)g'(x)$ and hence $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x) > 0$. Therefore $f(x)g(x)$ is increasing as desired. □

APPENDIX B

Basic Zeta Functions

2.1 Barnes Zeta Function

The Barnes zeta function is a multi dimensional generalization of the usual zeta function of Riemann. It can be defined as

$$\zeta_B(s;a,r) = \sum_{n\in\mathbb{N}^d} (a_1 n_1 + a_2 n_2 + \cdots + a_d n_d + r)^{-s} \tag{B.1}$$

where $a = (a_1, a_2, \ldots, a_d) \in (\mathbb{R}^+)^d$ and $r \in \mathbb{R}^+$, which will be convergent for $\Re(s) > d$. In order to obtain an analytical continuation, we can make use of the integral representation

$$\zeta_B(s;a,r) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{e^{-ct}}{\prod_{j=1}^d (1 - e^{-a_j t})}\,, \tag{B.2}$$

which will give the meromorphic structure of the Barnes zeta function in terms of the generalized Bernoulli polynomials $B_n^{(d)}(x;a)$ that can be defined by the Taylor series

$$\frac{e^{-xt}}{\prod_{j=1}^d (1 - e^{-a_j t})} = \frac{(-1)^d}{\prod_{j=1}^d a_j} \sum_{n=0}^\infty \frac{(-1)^{n-d} t^{n-d}}{n!} B_n^{(d)}(c,a)\,. \tag{B.3}$$

With this we can find the residues of the Barnes zeta function and special values in terms of the generalized Bernoulli polynomials.

Theorem B.1. *The Barnes zeta function defined above has residues at the points* $s = d, d-1, \ldots, 1$ *with residues given by*

$$Res\ \zeta_B(k;a,r) = \frac{(-1)^{d+k}}{(k-1)!(d-k)!\prod_{j=1}^d a_j} B_{d-k}^{(d)}(c;a)\,, \tag{B.4}$$

and has the values at the special points $s = -n$, $n \in \mathbb{N}$

$$\zeta_B(-n;a,r) = \frac{(-1)^d n!}{(d+n)!\prod_{j=1}^d a_j} B_{d+n}^{(d)}(c;r)\,. \tag{B.5}$$

2.2 Epstein zeta function

Define the d dimensional Epstein Zeta function as

$$\zeta_E(s; a_1, a_2, \ldots, a_d) = \sum_{n \in (\mathbb{Z}^+)^d} (a_1 n_1^2 + a_2 n_2 + \cdots + a_d n_d^2)^{-s} \tag{B.6}$$

which will be convergent for $\Re(s) > d/2$. We are going to find an analytical continuation for the d dimensional Epstein zeta function in terms of the $d-1$ dimensional Epstein zeta function. Hence the poles and trivial zeros can be found by a recursive argument noticing that the case when $d = 1$ corresponds to the Riemann zeta function

$$\zeta_E(s; a) = a^{-s} \zeta_R(2s) \tag{B.7}$$

which has one simple pole at $s = 1/2$ with residue $a^{-1/2}/2$ and has trivial zeros at the negative integers.

To consider the case when $d > 1$, apply the Mellin transform to $(B.6)$ to obtain

$$\zeta_E(s; a_1, \ldots, a_d) = \sum_{n \in (\mathbb{Z}^+)^d} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \exp(-(a_1 n_1^2 + a_2 n_2 + \cdots + a_d n_d^2) t)\,. \tag{B.8}$$

The Poisson resumation formula enables us to rewrite a sum by

$$\sum_{l=-\infty}^{\infty} \exp(-rtl^2) = \sqrt{\frac{\pi}{rt}} \sum_{l=-\infty}^{\infty} \exp\left(-\frac{\pi^2}{rt} l^2\right)\,, \tag{B.9}$$

and performing the resumation to the n_d sum in the Mellin transform, we obtain

$$\begin{aligned}\zeta_E(s; a_1, \ldots, a_d) = &\sum_{n \in (\mathbb{Z}^+)^{d-1}} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \exp\left(-(a_1 n_1^2 + \cdots + a_{d-1} n_{d-1}^2) t\right) \\ &\times \left(\sqrt{\frac{\pi}{a_d t}} \sum_{n_d=1}^{\infty} \exp\left(-\frac{\pi^2}{a_d t} n_d^2\right) + \sqrt{\frac{\pi}{4 a_d t}} - \frac{1}{2} \right)\,. \end{aligned} \tag{B.10}$$

which gives

$$\zeta_E(s; a_1, \ldots, a_d) = \frac{1}{\Gamma(s)} \sqrt{\frac{\pi}{a_d}} \sum_{n \in (\mathbb{Z}^+)^{d-1}} \int_0^\infty dt\, t^{s-3/2}$$

$$
\times \exp\left(-\frac{\pi^2 n_d^2}{a_d t} - (a_1 n_1^2 + \cdots + a_{d-1} n_{d-1}^2)t\right)
$$
$$
+ \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{a_d}} \sum_{n\in(\mathbb{Z}^+)^{d-1}} \int_0^\infty dt\, t^{s-3/2} \exp\left(-(a_1 n_1^2 + \cdots + a_{d-1} n_{d-1}^2)t\right)
$$
$$
- \frac{1}{2\Gamma(s)} \sum_{n\in(\mathbb{Z}^+)^{d-1}} \int_0^\infty dt\, t^{s-1} \exp\left(-(a_1 n_1^2 + \cdots + a_{d-1} n_{d-1}^2)t\right) . \quad \text{(B.11)}
$$

using the integral representation for the modified Bessel function of the second kind

$$
K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty dt\, t^{-\nu-1} \exp\left(-t - \frac{z^2}{4t}\right) , \quad \text{(B.12)}
$$

for $|\arg(z)| < \pi/2$ and $\Re(z^2) > 0$, we can rewrite $(B.11)$ as

$$
\zeta_E(s; a_1, \ldots, a_d) = \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{a_d}} \sum_{n\in(\mathbb{Z}^+)^{d-1}} (a_1 n_1^2 + \cdots + a_{d-1} n_{d-1}^2)^{3/2-s}
$$
$$
\times \left(2\pi n_d \sqrt{\frac{a_1 n_1^2 + \cdots + a_{d-1} n_{d-1}^2}{a_d}}\right)^{1/2-s} K_{1/2-s}\left(2\pi n_d \sqrt{\frac{a_1 n_1^2 + \cdots + a_{d-1} n_{d-1}^2}{a_d}}\right)
$$
$$
+ \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{a_d}} \zeta_E(s - 1/2; a_1, \ldots, a_{d-1})
$$
$$
- \frac{1}{2\Gamma(s)}\sqrt{\frac{\pi}{a_d}} \zeta_E(s; a_1, \ldots, a_{d-1}) . \quad \text{(B.13)}
$$

As the function $K_\nu(z)$ is exponentially damped for large $z \in \mathbb{R}$, the only terms that contribute to the poles of the d dimensional Epstein zeta function are the second and third summand. Therefore we have that it is possible to find the poles of the d dimensional Epstein zeta function recursively in terms of the lower dimensional ones. Also because of the $\Gamma(s)$ factor in all three summands, it can be proved by induction that ζ_E has trivial zeros at the negative integers. This can be summarized as

Theorem B.2 (Epstein Zeta function). *The Epstein zeta function defined in* $(B.6)$ *admits an analytic continuation to the entire complex plane except for the simple poles located at* $s = \frac{d}{2}, \frac{d-1}{2}, \ldots, \frac{1}{2}$ *and has trivial zeros at* $s = -n$, $n \in \mathbb{Z}^+$. *Moreover,* *the residues can be found by*

$$Res\ \zeta_E(d/2; a_1, \ldots, a_d) = \frac{1}{\Gamma(d/2)}\sqrt{\frac{\pi}{a_d}} Res\ \zeta_E\left(\frac{d-1}{2}; a_1, \ldots, a_{d-1}\right) \tag{B.14}$$

and

$$\begin{aligned} Res\ \zeta_E(k; a_1, \ldots, a_d) = \frac{1}{\Gamma(k)}\sqrt{\frac{\pi}{a_d}} Res\ \zeta_E\left(k - 1/2; a_1, \ldots, a_{d-1}\right) \\ - \frac{1}{2\Gamma(k)}\sqrt{\frac{\pi}{a_a}} Res\ \zeta_E(k; a_1, \ldots, a_{d-1})\,. \end{aligned} \tag{B.15}$$

for $k = \frac{d-1}{2}, \ldots, \frac{1}{2}$.

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