DEDUCTION BY INDUCTION

Peter M. Maurer
Dept. of Computer Science
Baylor University
Waco, Texas 78799
Peter_Maurer@Baylor.edu

Introduction

I was amazed when I read the article "What Do You Get When You Cross a Power Sum with an Iraqi Bank Note" by Professor Boudreaux in the October 2009 Magazine[1]. Of course I also had to take a peek at a couple of the references[2,3], and I found them amazing as well. All these approaches to finding closed-form formulas for power sums are complex, wonderful, and truly impressive. However, I disagree with Professor Boudreaux about one thing. Not all approaches to power sums can be categorized as below (this is a direct quote from page 289 of [1]):

1. Intuitive and well-motivated, but not generalizable to all *k*.
2. Clever and often mystifying, yet generalizable.

In this note, I offer a technique that is: intuitive, well-motivated, generalizable to all *k*, and suitable for presentation to pre-calculus students.

Let's assume that I'm a student with a mathematical bent but very little mathematical knowledge. I want to know the formula for a summation of the form $\sum_{i=1}^{n} i^k$ for some specific k that I can't find in the book. It seems reasonable to me that $\sum_{i=1}^{n} i^k$ must equal $P(n)$ for some polynomial $P(n)$. And if it doesn't – well I suppose I'll find out soon enough. The first thing I notice is that if I replace i^k with n^k the summation is equal to n^{k+1}. If I knew something about asymptotic growth rates, I would note that because $\sum_{i=0}^{n} i^k \le \sum_{i=1}^{n} n^k = n^{k+1}$, $P(n)$ cannot have degree greater than $k+1$. Otherwise I would be forced to imitate the forms of known formulas. In either case, I make the assumption that:

$$\sum_{i=1}^{n} i^k = A_{k+1} n^{k+1} + A_k n^k + \cdots + A_1 n + A_0 \qquad (1)$$

I also note that if this formula were presented in class with actual values for k and $\{A_{k+1},\ldots,A_0\}$, the next thing my professor would say is, "As an exercise, prove this formula using induction!" So I decide that's exactly what I'll do. While I'm going through the "inductive proof" I'll replace $\{A_{k+1},\ldots,A_0\}$ with appropriate values to make the inductive proof work.

For the basis step, I'm a little puzzled about how to proceed because my professor suggested that I should start my inductive proofs with zero if at all possible, but I'm not really comfortable with things like $\sum_{i=1}^{0} i^k$. I decide to change the lower limit from one to zero, because $\sum_{i=1}^{n} i^k = \sum_{i=0}^{n} i^k$, and I can handle things like $\sum_{i=0}^{0} i^k$.

For the basis step of an inductive proof to work, this must be true:

$$\sum_{i=0}^{0} i^k = 0^k = 0 = A_{k+1} \cdot 0^{k+1} + A_k \cdot 0^k + \cdots + A_1 \cdot 0 + A_0 = A_0$$

Since A_0=0, I can forget about it.

For the inductive step, I substitute $n+1$ for n in Equation (1) and get the following.

$$\sum_{i=0}^{n+1} i^k = A_{k+1}(n+1)^{k+1} + A_k(n+1)^k + \cdots + A_1(n+1) \tag{2}$$

But on the other hand, this must also true:

$$\sum_{i=0}^{n+1} i^k = \sum_{i=0}^{n} i^k + (n+1)^k \tag{3}$$

Applying the inductive hypothesis to Equation (3), I get:

$$\sum_{i=0}^{n+1} i^k = A_{k+1}n^{k+1} + A_k n^k + \cdots + A_1 n + (n+1)^k \tag{4}$$

Combining Equations (2) and (4) I get a nice polynomial equation:

$$\begin{aligned} &A_{k+1}n^{k+1} + A_k n^k + \cdots + A_1 n + (n+1)^k = \\ &A_{k+1}(n+1)^{k+1} + A_k(n+1)^k + \cdots + A_1(n+1) \end{aligned} \tag{5}$$

For these polynomials to be equal, the coefficients of the corresponding terms must be equal. I notice that this will give me $k+2$ equations in $k+1$ unknowns. At first I'm worried that I will end up with an over-specified set of equations that has no solution. But I examine Equation (5) and I notice that on the left, the only term containing n^{k+1} is $A_{k+1}n^{k+1}$. On the right, the only term containing n^{k+1} is the first term of $A_{k+1}(n+1)^{k+1}$, which is also $A_{k+1}n^{k+1}$. This gives me the equation $A_{k+1} = A_{k+1}$. I don't get any information from this but I can eliminate it from further consideration. This leaves me with $k+1$ equations in $k+1$ unknowns, a far happier situation.

Examining Equation (5) further I observe that when I multiply through by a coefficient A_i on the right, I will do so *after* raising $n+1$ to the power i. This means I'm going to end up with a system of $k+1$ *linear* equations in $k+1$ unknowns, and I know how to solve that! The only thing I might still be worried about is whether the system of equations is singular. Intuition tells me that this cannot be so, otherwise the inductive proof wouldn't work. But I need a bit more reassurance. I play with the equations a little more and I discover something interesting. The equation I get from the n^k terms looks like this.

$$A_k + c_k = A_k + F_k(A_{k+1})$$

The term c_k is a constant and $F_k(A_{k+1})$ is a linear function of A_{k+1}. After cancelling the A_k terms, I'm left with this:

$$F_k(A_{k+1}) = c_k$$

Similarly the equation I get from the n^{k-1} term reduces to:

$$F_{k-1}(A_{k+1}, A_k) = c_{k-1}$$

Again, c_{k-1} is a constant and $F_{k-1}(A_{k+1}, A_k)$ is a linear function of A_{k+1} and A_k. As I continue, I realize that *every* equation is of this form. The equation I get from the n^i term is:

$$F_i(A_{k+1}, A_k, ..., A_{i+1}) = c_i$$

This takes care of everything but the constant terms. Since these are all 1 I get this equation:

$$A_{k+1} + A_k + \cdots + A_1 = 1$$

I also see that none of the linear functions F_i have constant terms, and none of their coefficients are equal to zero. This means that I can arrange the $k+1$ linear functions into an upper triangular matrix with no zeros on the main diagonal. This of course, implies that my system of equations must have a unique solution. I don't need Gaussian elimination because the matrix is already in upper triangular form. I can obtain that solution through division and back-substitution. In fact, in most cases, the solution will be so simple that I won't need to bother with the matrix.

To see how this works, let's set k=4, and proceed. For clarity we will drop the subscripts.

$$\sum_{i=1}^{n} i^4 = An^5 + Bn^4 + Cn^3 + Dn^2 + En$$

Referring back to Equation (5), the left-hand side reduces to:

$$An^5 + Bn^4 + Cn^3 + Dn^2 + En + (n+1)^4 =$$
$$An^5 + (B+1)n^4 + (C+4)n^3 + (D+6)n^2 + (E+4)n + 1$$

The right-hand side reduces to:

$$A(n+1)^5 + B(n+1)^4 + C(n+1)^3 + D(n+1)^2 + E(n+1) =$$
$$An^5 + (5A+B)n^4 + (10A+4B+C)n^3 + (10A+6B+3C+D)n^2 +$$
$$(5A+4B+3C+2D+E)n + (A+B+C+D+E)$$

Before cancelling we get this system of equations.

$$5A + B = B + 1$$
$$10A + 4B + C = C + 4$$
$$10A + 6B + 3C + D = D + 6$$
$$5A + 4B + 3C + 2D + E = E + 4$$
$$A + B + C + D + E = 1$$

This reduces to:

$$5A = 1$$
$$10A + 4B = 4$$
$$10A + 6B + 3C = 6$$
$$5A + 4B + 3C + 2D = 4$$
$$A + B + C + D + E = 1$$

A quick calculation gives the solution:

$$A = 1/5$$
$$B = 1/2$$
$$C = 1/3$$
$$D = 0$$
$$E = -1/30$$

Thus:

$$\sum_{i=1}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

As an exercise, prove this formula using induction!

REFERENCES

1. G. Boudreaux, What do you get when you cross a power sum with an Iraqi bank note, *Mathematics Magazine* **82** (2009) 289-293.

2. A. W. F. Edwards, A quick route to sums of powers, *Amer. Math. Monthly* **93** (1986) 451-455.

3. B Turner, Sums of powers of integers via the binomial theorem, *Mathematics Magazine* **53** (1980) 92-96.

Reviewer's Note:

I invented the technique used in this paper as a pedagogical tool, not as a means for discovering formulas for power sums. I wanted to dispel the notion that induction is a magic infinite staircase that somehow leads us where we want to go, and show that it is actually a rigorous and useful mathematical tool. I also wanted to dispel the incorrect notion that induction is inferior to other proof techniques in that it can only be used to prove things that you already know.

I think everything is clear until we get to the point of deriving the equations of the form : $F_i(A_{k+1}, A_k, ..., A_{i+1}) = c_i$. Proving that the equations have this form is boring, but straightforward. Consider following equation:

$$A_{k+1}n^{k+1} + A_k n^k + \cdots + A_1 n + (n+1)^k =$$
$$A_{k+1}(n+1)^{k+1} + A_k(n+1)^k + \cdots + A_1(n+1)$$

On the left, it is obvious that the coefficient of n^i is $A_i + c_i$, where $c_i = \binom{k}{i}$. On the right, we observe that if $j < i$ then $(n+1)^j$ has no n^i term, so we need only concern ourselves with the coefficients of the terms $(n+1)^m$, where $m \geq i$. The coefficient of the n^i term in $A_i(n+1)^i$ is A_i. This gives us the cancellable term. For $m \geq i$, the coefficient of the n^i term is $\binom{m}{i} A_m$. This justifies the observations that F_i is a linear function of the variables $A_{i+1}, A_{i+2}, ..., A_{k+1}$, that the constant term is zero, and that the coefficient of each A_m is non-zero.

The reason I changed the lower limit of the summation to zero even though it isn't necessary, is because that's what virtually all of the students in my discrete mathematics classes do when proving power-sum formulas.

Addendum for the Tech Report: This technique applied to the first 10 values of *k*.

$$\sum_{i=1}^{n} i^0 = An$$

$$An + 1 = A(n+1)$$

$$A = 1$$

$$\sum_{i=1}^{n} i^1 = An^2 + Bn$$

$$An^2 + Bn + n + 1 = A(n+1)^2 + B(n+1)$$
$$An^2 + (B+1)n + 1 = An^2 + (2A+B)n + (A+B)$$

$$2A = 1$$
$$A + B = 1$$

$$A = 1/2$$
$$B = 1/2$$

$$\sum_{i=1}^{n} i = \frac{n^2}{2} + \frac{n}{2} = \frac{n^2+n}{2}$$

$$\sum_{i=1}^{n} i^2 = An^3 + Bn^2 + Cn$$

$$An^3 + Bn^2 + Cn + (n+1)^2 = A(n+1)^3 + B(n+1)^2 + C(n+1)$$
$$An^3 + (B+1)n^2 + (C+2)n + 1 = An^3 + (3A+B)n^2 + (3A+2B+C)n + (A+B+C)$$

$$3A = 1$$
$$3A + 2B = 2$$
$$A + B + C = 1$$

$$A = 1/3$$
$$B = 1/2$$
$$C = 1/6$$

$$\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{2n^3 + 3n^2 + n}{6}$$

$$\sum_{i=1}^{n} i^3 = An^4 + Bn^3 + Cn^2 + Dn$$

$$An^4 + (B+1)n^3 + (C+3)n^2 + (D+3)n + 1 =$$
$$An^4 + (4A+B)n^3 + (6A+3B+C)n^2 + (4A+3B+2C+D)n + (A+B+C+D)$$

$$4A = 1$$
$$6A + 3B = 3$$
$$4A + 3B + 2C = 3$$
$$A + B + C + D = 1$$

$$A = 1/4$$
$$B = 1/2$$
$$C = 1/4$$
$$D = 0$$

$$\sum_{i=1}^{n} i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \frac{n^4 + 2n^3 + n^2}{4}$$

$$\sum_{i=1}^{n} i^4 = An^5 + Bn^4 + Cn^3 + Dn^2 + En$$

$$An^5 + Bn^4 + Cn^3 + Dn^2 + En + (n+1)^4 = A(n+1)^5 + B(n+1)^4 + C(n+1)^3 + D(n+1)^2 + E(n+1)$$

$$An^5 + (B+1)n^4 + (C+4)n^3 + (D+6)n^2 + (E+4)n + 1 =$$
$$An^5 + (5A+B)n^4 + (10A+4B+C)n^3 + (10A+6B+3C+D)n^2 +$$
$$(5A+4B+3C+2D+E)n + (A+B+C+D+E)$$

$$5A = 1$$
$$10A + 4B = 4$$
$$10A + 6B + 3C = 6$$
$$5A + 4B + 3C + 2D = 4$$
$$A + B + C + D + E = 1$$

$5A = 1/5$

$B = 1/2$

$3C = 1/3$

$D = 0$

$E = -1/30$

$$\sum_{i=1}^{n} i^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$$

$$\sum_{i=1}^{n} i^5 = An^6 + Bn^5 + Cn^4 + Dn^3 + En^2 + Fn$$

$$An^6 + Bn^5 + Cn^4 + Dn^3 + En^2 + Fn + (n+1)^5 =$$
$$A(n+1)^6 + B(n+1)^5 + C(n+1)^4 + D(n+1)^3 + E(n+1)^2 + F(n+1)$$

$$An^6 + (B+1)n^5 + (C+5)n^4 + (D+10)n^3 + (E+10)n^2 + (F+5)n + 1 =$$
$$An^6 + (6A+B)n^5 + (15A+5B+C)n^4 + (20A+10B+4C+D)n^3 + (15A+10B+6C+3D+E)n^2 +$$
$$(6A+5B+4C+3D+2E+F)n + (A+B+C+D+E+F)$$

$6A = 1$

$15A + 5B = 5$

$20A + 10B + 4C = 10$

$15A + 10B + 6C + 3D = 10$

$6A + 5B + 4C + 3D + 2E = 5$

$A + B + C + D + E + F = 1$

$A = 1/6$

$B = 1/2$

$C = 5/12$

$D = 0$

$E = -1/12$

$F = 0$

$$\sum_{i=1}^{n} i^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12} = \frac{2n^6 + 6n^5 + 5n^4 - n^2}{12}$$

$$\sum_{i=1}^{n} i^6 = An^7 + Bn^6 + Cn^5 + Dn^4 + En^3 + Fn^2 + Gn$$

$$An^7 + Bn^6 + Cn^5 + Dn^4 + En^3 + Fn^2 + Gn + (n+1)^6 =$$
$$A(n+1)^7 + B(n+1)^6 + C(n+1)^5 + D(n+1)^4 + E(n+1)^3 + F(n+1)^2 + G(n+1)$$

$$An^7 + (B+1)n^6 + (C+6)n^5 + (D+15)n^4 + (E+20)n^3 + (F+15)n^2 + (G+6)n + 1 =$$
$$(An^7 + (7A+B)n^6 + (21A+6B+C)n^5 + (35A+15B+5C+D)n^4 +$$
$$(35A+20B+10C+4D+E)n^3 + (21A+15B+10C+6D+3E+F)n^2 +$$
$$(7A+6B+5C+4D+3E+2F+G)n + (A+B+C+D+E+F+G)$$

$$7A = 1$$
$$21A + 6B = 6$$
$$35A + 15B + 5C = 15$$
$$35A + 20B + 10C + 4D = 20$$
$$21A + 15B + 10C + 6D + 3E = 15$$
$$7A + 6B + 5C + 4D + 3E + 2F = 6$$
$$A + B + C + D + E + F + G = 1$$

$$A = 1/7$$
$$B = 1/2$$
$$C = 1/2$$
$$D = 0$$
$$E = -1/6$$
$$F = 0$$
$$G = 1/42$$

$$\sum_{i=1}^{n} i^6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42} = \frac{6n^7 + 21n^6 + 21n^5 - 7n^3 + n}{42}$$

$$\sum_{i=1}^{n} i^7 = An^8 + Bn^7 + Cn^6 + Dn^5 + En^4 + Fn^3 + Gn^2 + Hn$$

$$An^8 + Bn^7 + Cn^6 + Dn^5 + En^4 + Fn^3 + Gn^2 + Hn + (n+1)^7 =$$
$$A(n+1)^8 + B(n+1)^7 + C(n+1)^6 + D(n+1)^5 + E(n+1)^4 + F(n+1)^3 + G(n+1)^2 + H(n+1)$$

$$An^8+(B+1)n^7+(C+7)n^6+(D+21)n^5+(E+35)n^4+(F+35)n^3+(G+21)n^2+(H+7)n+1=$$
$$An^8+(8A+B)n^7+(28A+7B+C)n^6+(56A+21B+6C+D)n^5+(70A+35B+15C+5D+E)n^4+$$
$$(56A+35B+20C+10D+4E+F)n^3+(28A+21B+15C+10D+6E+3F+G)n^2+$$
$$(8A+7B+6C+5D+4E+3F+2G+H)n+(A+B+C+D+E+F+G+H)$$

$8A=1$

$28A+7B=7$

$56A+21B+6C=21$

$70A+35B+15C+5D=35$

$56A+35B+20C+10D+4E=35$

$28A+21B+15C+10D+6E+3F=21$

$8A+7B+6C+5D+4E+3F+2G=7$

$A+B+C+D+E+F+G+H=1$

$A=1/8$

$B=1/2$

$C=7/12$

$D=0$

$E=-7/24$

$F=0$

$G=1/12$

$H=0$

$$\sum_{i=1}^{n} i^7 = \frac{n^8}{8}+\frac{n^7}{2}+\frac{7n^6}{12}-\frac{7n^4}{24}+\frac{n^2}{12}=\frac{3n^8+12n^7+14n^6-7n^4+2n^2}{24}$$

$$\sum_{i=1}^{n} i^8 = An^9+Bn^8+Cn^7+Dn^6+En^5+Fn^4+Gn^3+Hn^2+Kn$$

$$An^9 + Bn^8 + Cn^7 + Dn^6 + En^5 + Fn^4 + Gn^3 + Hn^2 + Kn + (n+1)^8 =$$
$$A(n^9 + 9n^8 + 36n^7 + 84n^6 + 126n^5 + 126n^4 + 84n^3 + 36n^2 + 9n + 1)^9 +$$
$$B(n^8 + 8n^7 + 28n^6 + 56n^5 + 70n^4 + 56n^3 + 28n^2 + 8n + 1)^8 +$$
$$C(n^7 + 7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n + 1) +$$
$$D(n^6 + 6n^5 + 15n^4 + 20n^3 + 15n^2 + 6n + 1) +$$
$$E(n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1)^5 +$$
$$F(n^4 + 4n^3 + 6n^2 + 4n + 1) +$$
$$G(n^3 + 3n^2 + 3n + 1) +$$
$$H(n^2 + 2n + 1) +$$
$$K(n+1)$$

$$An^9 + (B+1)n^8 + (C+8)n^7 + (D+28)n^6 + (E+56)n^5 + (F+70)n^4 + (G+56)n^3 + (H+28)n^2 + (K+8)n + 1$$
$$An^9 +$$
$$(9A + B)n^8 +$$
$$(36A + 8B + C)n^7 +$$
$$(84A + 28B + 7C + D)n^6 +$$
$$(126A + 56B + 21C + 6D + E)n^5 +$$
$$(126A + 70B + 35C + 15D + 5E + F)n^4 +$$
$$(84A + 56B + 35C + 20D + 10E + 4F + G)n^3 +$$
$$(36A + 28B + 21C + 15D + 10E + 6F + 3G + H)n^2 +$$
$$(9A + 8B + 7C + 6D + 5E + 4F + 3G + 2H + K)n +$$
$$(A + B + C + D + E + F + G + H + K)$$

$$9A = 1$$
$$36A + 8B = 8$$
$$84A + 28B + 7C = 28$$
$$126A + 56B + 21C + 6D = 56$$
$$126A + 70B + 35C + 15D + 5E = 70$$
$$84A + 56B + 35C + 20D + 10E + 4F = 56$$
$$36A + 28B + 21C + 15D + 10E + 6F + 3G = 28$$
$$9A + 8B + 7C + 6D + 5E + 4F + 3G + 2H = 8$$
$$A + B + C + D + E + F + G + H + K = 1$$

$A = 1/9$
$B = 1/2$
$C = 2/3$
$D = 0$
$E = -7/15$
$F = 0$
$G = 2/9$
$H = 0$
$K = -1/30$

$$\sum_{i=1}^{n} i^8 = \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^5}{15} + \frac{2n^3}{9} - \frac{n}{30} = \frac{10n^9 + 45n^8 + 60n^7 - 42n^5 + 20n^3 - 3n}{90}$$

$$\sum_{i=1}^{n} i^9 = An^{10} + Bn^9 + Cn^8 + Dn^7 + En^6 + Fn^5 + Gn^4 + Hn^3 + Kn^2 + Ln$$

$$An^{10} + Bn^9 + Cn^8 + Dn^7 + En^6 + Fn^5 + Gn^4 + Hn^3 + Kn^2 + Ln + (n+1)^9 =$$
$$A(n+1)^{10} + B(n+1)^9 + C(n+1)^8 + D(n+1)^7 + E(n+1)^6 +$$
$$F(n+1)^5 + G(n+1)^4 + H(n+1)^3 + K(n+1)^2 + L(n+1)$$

$$An^{10} + (B+1)n^9 + (C+9)n^8 + (D+36)n^7 + (E+84)n^6 + (F+126)n^5 + (G+126)n^4 +$$
$$(H+84)n^3 + (K+36)n^2 + (L+9)n + 1 =$$
$$(An^{10} + 10An^9 + 45An^8 + 120An^7 + 210An^6 + 252An^5 + 210An^4 + 120An^3 + 45An^2 + 10An + A) +$$
$$(Bn^9 + 9Bn^8 + 36Bn^7 + 84Bn^6 + 126Bn^5 + 126Bn^4 + 84Bn^3 + 36Bn^2 + 9Bn + B) +$$
$$(Cn^8 + 8Cn^7 + 28Cn^6 + 56Cn^5 + 70Cn^4 + 56Cn^3 + 28Cn^2 + 8Cn + C) +$$
$$(Dn^7 + 7Dn^6 + 21Dn^5 + 35Dn^4 + 35Dn^3 + 21Dn^2 + 7Dn + D) +$$
$$(En^6 + 6En^5 + 15En^4 + 20En^3 + 15En^2 + 6En + E) +$$
$$(Fn^5 + 5Fn^4 + 10Fn^3 + 10Fn^2 + 5Fn + F) +$$
$$(Gn^4 + 4Gn^3 + 6Gn^2 + 4Gn + G) +$$
$$(Hn^3 + 3Hn^2 + 3Hn + H) +$$
$$(Kn^2 + 2Kn + K) +$$
$$(Ln + L)$$

$$An^{10}+(B+1)n^{9}+(C+9)n^{8}+(D+36)n^{7}+(E+84)n^{6}+(F+126)n^{5}+(G+126)n^{4}+$$
$$(H+84)n^{3}+(K+36)n^{2}+(L+9)n+1=$$
$$An^{10}$$
$$(10A+B)n^{9}+$$
$$(45A+9B+C)n^{8}+$$
$$(120A+36B+8C+D)n^{7}+$$
$$(210A+84B+28C+7D+E)n^{6}+$$
$$(252A+126B+56C+21D+6E+F)n^{5}+$$
$$(210A+126B+70C+35D+15E+5F+G)n^{4}+$$
$$(120A+84B+56C+35D+20E+10F+4G+H)n^{3}+$$
$$(45A+36B+28C+21D+15E+10F+6G+3H+K)n^{2}+$$
$$(10A+9B+8C+7D+6E+5F+4G+3H+2K+L)n+$$
$$(A+B+C+D+E+F+G+H+K+L)$$

$$10A=1$$
$$45A+9B=9$$
$$120A+36B+8C=36$$
$$210A+84B+28C+7D=84$$
$$252A+126B+56C+21D+6E=126$$
$$210A+126B+70C+35D+15E+5F=126$$
$$120A+84B+56C+35D+20E+10F+4G=84$$
$$45A+36B+28C+21D+15E+10F+6G+3H=36$$
$$10A+9B+8C+7D+6E+5F+4G+3H+2K=9$$
$$A+B+C+D+E+F+G+H+K+L=1$$

$$A=1/10$$
$$B=1/2$$
$$C=3/4$$
$$D=0$$
$$E=-7/10$$
$$F=0$$
$$G=1/2$$
$$H=0$$
$$K=-3/20$$
$$L=0$$

$$\sum_{i=1}^{n} i^9 = \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{3n^8}{4} - \frac{7n^6}{10} + \frac{n^4}{2} - \frac{3n^2}{20} = \frac{2n^{10} + 10n^9 + 15n^8 - 14n^6 + 10n^4 - 3n^2}{20}$$

$$\sum_{i=1}^{n} i^{10} = An^{11} + Bn^{10} + Cn^9 + Dn^8 + En^7 + Fn^6 + Gn^5 + Hn^4 + Kn^3 + Ln^2 + Mn$$

$$An^{11} + Bn^{10} + Cn^9 + Dn^8 + En^7 + Fn^6 + Gn^5 + Hn^4 + Kn^3 + Ln^2 + Mn + (n+1)^{10} =$$
$$A(n+1)^{11} + B(n+1)^{10} + C(n+1)^9 + D(n+1)^8 + E(n+1)^7 + F(n+1)^6 +$$
$$G(n+1)^5 + H(n+1)^4 + K(n+1)^3 + L(n+1)^2 + M(n+1)$$

$$An^{11} + (B+1)n^{10} + (C+10)n^9 + (D+45)n^8 + (E+120)n^7 + (F+210)n^6 + (G+252)n^5 +$$
$$(H+210)n^4 + (K+120)n^3 + (L+45)n^2 + (M+10)n + 1 =$$
$$A(n^{11} + 11n^{10} + 55n^9 + 165n^8 + 330n^7 + 462n^6 + 462n^5 + 330n^4 + 165n^3 + 55n^2 + 11n + 1)^{11} +$$
$$B(n^{10} + 10n^9 + 45n^8 + 120n^7 + 210n^6 + 252n^5 + 210n^4 + 120n^3 + 45n^2 + 10n + 1)^{10} +$$
$$C(n^9 + 9n^8 + 36n^7 + 84n^6 + 126n^5 + 126n^4 + 84n^3 + 36n^2 + 9n + 1)^9 +$$
$$D(n^8 + 8n^7 + 28n^6 + 56n^5 + 70n^4 + 56n^3 + 28n^2 + 8n + 1)^8 +$$
$$E(n^7 + 7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n + 1)^7 +$$
$$F(n^6 + 6n^5 + 15n^4 + 20n^3 + 15n^2 + 6n + 1)^6 +$$
$$G(n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1)^5 +$$
$$H(n^4 + 4n^3 + 6n^2 + 4n + 1)^4 +$$
$$K(n^3 + 3n^2 + 3n + 1)^3 +$$
$$L(n^2 + 2n + 1)^2 +$$
$$M(n+1)$$

$$An^{11}+(B+1)n^{10}+(C+10)n^{9}+(D+45)n^{8}+(E+120)n^{7}+(F+210)n^{6}+(G+252)n^{5}+$$
$$(H+210)n^{4}+(K+120)n^{3}+(L+45)n^{2}+(M+10)n+1=$$
$$An^{11}+$$
$$(11A+B)n^{10}+$$
$$(55A+10B+C)n^{9}+$$
$$(165A+45B+9C+D)n^{8}+$$
$$(330A+120B+36C+8D+E)n^{7}+$$
$$(462A+210B+84C+28D+7E+F)n^{6}+$$
$$(462A+252B+126C+56D+21E+6F+G)n^{5}+$$
$$(330A+210B+126C+70D+35E+15F+5G+H)n^{4}+$$
$$(165A+120B+84C+56D+35E+20F+10G+4H+K)n^{3}+$$
$$(55A+45B+36C+28D+21E+15F+10G+6H+3K+L)n^{2}+$$
$$(11A+10B+9C+8D+7E+6F+5G+4H+3K+2L+M)n+$$
$$(A+B+C+D+E+F+G+H+K+L+M)$$

$$11A=1$$
$$55A+10B=10$$
$$165A+45B+9C=45$$
$$330A+120B+36C+8D=120$$
$$462A+210B+84C+28D+7E=210$$
$$462A+252B+126C+56D+21E+6F=252$$
$$330A+210B+126C+70D+35E+15F+5G=210$$
$$165A+120B+84C+56D+35E+20F+10G+4H=120$$
$$55A+45B+36C+28D+21E+15F+10G+6H+3K=45$$
$$11A+10B+9C+8D+7E+6F+5G+4H+3K+2L=10$$
$$A+B+C+D+E+F+G+H+K+L+M=1$$

$A = 1/11$

$B = 1/2$

$C = 5/6$

$D = 0$

$E = -1$

$F = 0$

$G = 1$

$H = 0$

$K = -1/2$

$L = 0$

$M = 5/66$

$$\sum_{i=1}^{n} i^{10} = \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66} = \frac{6n^{11} + 33n^{10} + 55n^9 - 66n^7 + 66n^5 - 33n^3 + 5n}{66}$$