ABSTRACT

Two-Dimensional Hořava-Lifshitz Theory of Gravity

Baofei Li, Ph.D.

Advisor: Anzhong Wang, Ph.D.

In this dissertation, two-dimensional Hařava theory of gravity has been studied on the classical and quantum mechanical levels. The classical solutions of the projectable and nonprojectable Hořava gravity have been found and their spacetime structures are investigated by Penrose diagrams. When quantizing the theory in the canonical approach, the integral Hamiltonian constraint in the projectable case will generate the so-called Wheeler-DeWitt equation which can be exactly solved if the invariant length and its conjugate momentum are used as the new variables. On the other hand, for the nonprojectable case, the lapse function is no longer a Lagrangian multiplier but one of the canonical variables. This results in a local and second-class Hamiltonian constraint which can be solved for the lapse function. The quantization of nonprojectable case is carried out by directly dropping the unphysical degrees of freedom, that is, replacing Poisson brackets with Dirac brackets. In the last part of the dissertation, the interactions between two-dimensional Hořava gravity and a non-relativistic scalar field are considered. In the projectable case, the minimal coupling is adopted and canonical quantization is implemented in the same way as we have done for the pure gravity case. In the nonprojectable case, we turn to the non-minimal couplings and find both Killing and universal horizons from the classical solutions.

Two-Dimensional Hořava-Lifshitz Theory of Gravity

by

Baofei Li, B.S., M.S.

A Dissertation

Approved by the Department of Physics

Dwight P. Russell, Ph.D., Interim Chairperson

Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Philosophy

Approved by the Dissertation Committee

Anzhong Wang, Ph.D., Chairperson

Gregory A. Benesh, Ph.D.

Gerald B. Cleaver, Ph.D.

Klaus Kirsten , Ph.D.

Walter Wilcox, Ph.D.

Accepted by the Graduate School
December 2017

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

TABLE OF CONTENTS

APPENDIX

LIST OF FIGURES

ACKNOWLEDGMENTS

First of all, I would like to thank my mentor Dr. Anzhong Wang, I am grateful for his constant guidance and encouragement.

I am also very grateful to Dr. Gregory A. Benesh, Dr. Gerald B. Cleaver, Dr. Walter Wilcox and Dr. Klaus Kirsten for serving on my committee.

Thanks to the faculty and staff in the physics department at Baylor to create such a friendly environment for the graduate students to pursue their PhD.

It is my pleasure to work with several visiting scholars from China in the past years: Dr. Jie Yang, Dr. Tao Zhu, Dr. Miao Tian, Dr. Chikun Ding, Dr. Xiuming Zhang.

Thanks to my friends: Xinwen Wang, Yanbin Deng, Suman Baral, Indra-Mani Ghimire, Zhiyue Ding, Yang Liu, V. H. Satheeshkumar, Kai Lin, Ahmad Borzou, Bahram Shakerin, Dhurba Sapkota, Meera Gurung, Otávio Goldoni, Jared Fier, Madhurima Bhattacharjee.

Special thanks to Yanbin Deng, Xinwen Wang and Dr. Tao Zhu. There were so many wonderful hours together with Yanbin and Xinwen, we always celebrated our traditional festivals in Yanbin's apartment, thanks for his generosity, friendship and assistance. Thanks to Xinwen for his company while more and more of our friends graduated and left Baylor. I really appreciate our casual talks in everyday life. I am also grateful to Dr. Zhu for kind discussions and comments when I have problems in my study.

To My Parents

CHAPTER ONE

Introduction to Hořava-Lifshitz Theory of Gravity

1.1 Problems in General Relativity

1.1.1 Somes Aspects of General Relativity

Since Albert Einstein wrote his paper on field equations of general relativity (GR) in 1915 [1], great achievements have been attained with the development of the theory in the last one hundred years. The first exact solution with spherical symmetry was discovered by Schwarzschild [2] in 1915 shortly after Einstein's paper was written. It describes the spacetime around an uncharged and non-rotating spherical body which collapses into a black hole when its radius is less than Schwarzschild radius. Later, the charged black hole solution was independently found between 1916 and 1918 by Reissner [3] and Nordstrom [4]. It was as late as in 1963 that the uncharged, rotating black hole solution was discovered by R. P. Kerr [5]. Soon, the no-hair theorem about black holes was proposed by Werner Israel in 1967 [6]. This theorem postulates that the black hole solutions can be completely characterized by only three classical parameters: mass, electric charge and angular momentum. This finally led to the discovery of four laws of black hole mechanics. In 1973, Bekenstein-Hawking black hole entropy was presented in its current form. The next year, S. Hawking published his work on the Hawking radiation. The black hole thermodynamics was thus constructed. On the other hand, together with cosmological principle and Weyl's postulate, general relativity has become one of the three bases in the big bang model of cosmology where the cosmic background radiation (CMB) [7] was discovered and the abundances of the light elements in the universe (BBN) were explained. Later in order to resolve the horizon and magnetic monopole problems in the big bang model, inflation was also proposed.

In the development of general relativity, one of the important modifications to the classical Einstein-Hilbert action is the inclusion of the surface term. The necessity of adding the boundary term was first realized by J. W. York [8], later it was redefined by Gary Gibbons and S. Hawking[1] [9] in their path-integral formulation of general relativity. They showed that in order to have a well-defined variational principle for a manifold $\mathcal{M}$ with a boundary $\partial\mathcal{M}$, the complete action of general relativity should be

$$S = \frac{1}{16\pi G}\left\{\int_{\mathcal{M}} d^4x\sqrt{-g^4}\,(R-2\Lambda) + 2\oint_{\partial\mathcal{M}} d^3x\sqrt{|h|}K\right\}, \tag{1.1}$$

where G is Newton constant, g^4 the determinant of four-dimensional metric, R the four-dimensional Ricci scalar, Λ the cosmological constant, h the determinant of the induced metric on the boundary and K the trace of the extrinsic curvature of the boundary. The surface term also plays an important role in the Hamiltonian formulation of the theory. One can derive the gravitational Hamiltonian from the covariant action, however, if the surface term in the action is ignored. The resultant Hamiltonian will only consist of constraints even for the asymptotically flat spacetime, this implies the energy for any gravitational system is zero which obviously does not make any sense. The solution to this question, as shown in [15], is that the Hamiltonian also requires a boundary term for the noncompact spacetimes. This boundary term does not vanish on the constraint surface so it gives the definition of so-called Arnowitt-Deser-Misner (ADM) energy of the system. Besides, the boundary term also makes the equations of motion (EOM) in the phase space well defined. It can been shown [10] that the surface terms in the Hamiltonian can be directly derived from the boundary terms in the action Eq. (1.1).[2]

[1] Therefore, this surface term is usually called Gibbons-Hawking-York boundary term. For a more recent discussion of variational principle and surface terms, see [10, 14].

[2] Unlike other field theories, the physical quantities in general relativity, like energy, momentum, and angular momentum, are usually given by integrals at the boundaries of the spacetime, this is due to the general covariance which makes the contribution from the bulk vanish, as well as the special asymptotic behavior of the metric at the boundaries: different from gravity, fields in other

1.1.2 Non-renormalizability of General Relativity

Despite all the successes achieved with general relativity, some intrinsic problems in the theory imply that the theory has its own limitations. Firstly, the presence of spacetime singularities in general relativity signifies its incompleteness. These singularities usually show up in two circumstances, one is in the black hole solutions while the other in the big bang cosmology. The singularities in the black hole solutions can be better understood in terms of geodesic incompleteness by Penrose-Hawking singularity theorem [11]. It states the geodesics can not be extended beyond a certain affine parameter and any physics processes simply terminate at the singularities where either the matter density or the curvature becomes infinite. The initial singularity in the big bang cosmology even predicts at the beginning of the time the universe is in a state of infinite density and energy. So both of these singularities are related with the infinite physical quantities which demand further treatment from a higher perspective. Just like the problem of the electron's infinite self-energy in the classical electromagnetic theory is finally resolved by its quantized version QED, one possible way to overcome the singularity problem is to solve it in the context of quantum gravity. However, the search of quantum gravity poses one of the biggest challenges in physics since it turns out general relativity defies a consistent quantum mechanical description: it is non-renormalizable in the covariant quantization approach.[3]

In simple words, the non-renormalizability of general relativity is deeply rooted in the fact that the only coupling constant of the theory, Newton constant G, has a negative dimension $[m]^{-2}$.[4] Thus, according to the simple argument in [13], the perturbation expansion of a physical quantity F in powers of the Newton constant

branches of physics are always assumed to approach zero at a rate a little bit faster than $1/r^{3/2}$ at infinity.

[3] See [12] for a brief history of quantum gravity before 2000.

[4] Of course, we use natural units $\hbar = 1$ and $c = 1$, so $[m]$ indicates the dimension of mass/energy.

must be in the form

$$F = \sum_{n=0}^{\infty} a_n \left(GE^2\right)^n , \tag{1.2}$$

which implies that the expansion breaks up when the energy of the system E becomes larger than $G^{-1/2}$. So the theory is inevitably UV divergent and perturbation approach fails. Another viewpoint to look at the same question is due to S. Weinberg. It is shown in [16] that the loop integral in nth order Feynman diagram for perturbation calculation behaves like $\int dpp^{A-nD}$ at large momentum, where A depends on the specific process and D denotes the dimension of coupling constant, thus if the coupling constant is of negative dimension, then the integral will become divergent for any process at any sufficiently large orders, so correspondingly in order to renormalize this type of theory, in principle, an infinite number of counter-terms is required to absorb these divergences at higher orders.[5] As a result, GR is perturbatively non-renormalizable: the divergences at each order of perturbation expansion can not be absorbed into a redefinition of fields and coupling constants.

1.2 Hořava-Lifshitz Theory of Gravity

1.2.1 High Derivative Theory

Since general relativity is non-renormalizable in the covariant quantization programme, in the 1970s, people began to study high derivative theory in the hope of finding a version of modified GR in the UV regime which would give a renormalizable perturbation expansion. In 1977, Stelle [18] showed that once terms quadratic in the curvature are added to the Einstein-Hilbert action, the new action[6]

$$S = \int d^4x \sqrt{-g} \left(\alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2 + \gamma R\right) , \tag{1.3}$$

[5] See [17] for an example of two loop divergences in Einstein's theory, there the divergent term is composed of high-order terms in Riemann tensor.

[6] It is understood that a common factor $\frac{1}{16\pi G}$ has already been absorbed into the coefficient of each term in the action.

becomes renormalizable with an appropriate choice of the coupling constants. However, due to the appearance of time derivatives with orders higher than one, the theory is plagued with the problem of not being stable since there exist negative energy modes which kinematically make the energy of the system unbounded from below. Actually, as early as in 1850, Ostrogradsky [19] found that any system with a nondegenerate higher time derivative Lagrangian is not stable. It can be shown clearly that the Hamiltonians of these systems will include terms linear in the canonical momentum so there exists no lowest energy state (ground sate) and the system itself is unstable. Therefore, any high derivative theory with Lorentz invariance (LI) appears a poor extension of general relativity.

On the other side, the possibility of breaking LI at high energy scales was studied by several groups of researchers [20–24]. From the viewpoint of field theory, one can treat Lorentz symmetry breaking as a regulator to regularize the divergences in the loop integrals of perturbation expansion. The renormalizability of these LI breaking theories is analyzed in the papers [25, 26]. In order to understand some preliminary concepts of these theories, one can simply start with an action of scalar field in d+1 dimensions

$$S_{\text{free}} = \int dt d^d x \left\{ \dot{\phi}^2 - \phi(\Delta)^z \phi \right\}, \tag{1.4}$$

here dot means time derivative of the scalar field, $\Delta = \vec{\nabla}^2$ is the spatial Laplacian and z is the dynamical critical exponent. Based on the fact that the kinetic and potential terms in the action should have the same dimension, one can conclude

$$[t] = [x]^z. \tag{1.5}$$

Besides, since the action is a dimensionless quantity in the units $\hbar = 1$, the dimension of the scalar field is

$$[\phi] = [x]^{(z-d)/2} = [m]^{(d-z)/(2z)}, \tag{1.6}$$

where $[m]$ denotes the dimension of energy. Thus if one introduces polynomial interactions

$$S_{\text{interaction}} = \int dtd^dx \left\{ \sum_{n=1}^{N} g_n \phi^n \right\}, \tag{1.7}$$

the dimensions of coupling constants g_n are

$$[g_n] = [m]^{[d+z-n(d-z)/2]/z}, \tag{1.8}$$

Now for the power-counting renormalizable theories, the dimensions of coupling constants can not be negative (as discussed in the last section) which will identically hold if the dynamical critical exponent z is no less than spatial dimensions d.[7]

The lessons we can learn from above analysis are twofold: firstly, to find a high derivative unitary theory, there should only be first-order time derivatives in the action. Besides, to make the theory power-counting renormalizable, the order of the spatial derivative operators in the action can not be lower than twice of the spatial dimensions. Actually, these are exactly two important features of the Hořava-Lifshitz gravity which will be introduced below.

1.2.2 3+1 Decomposition of General Relativity

One of the basic features of general relativity is that the theory is invariant under the diffeomorphism

$$t \to t'(t, x^k), \quad x^i \to x'^i \left(t, x^k\right). \tag{1.9}$$

This invariance, usually termed as general covariance, is also the origin of the problem of time in general relativity. In any theory with time reparametrization symmetry, we are forced to distinguish two different types of time: one is the coordinate time as that appears in the coordinate transformation, the other is the physical time which is much more subtle and elusive to identify. In the canonical formulation of general relativity, for the Lorentzian manifold, one has to choose a timelike direction and slice

[7] One can find the same conclusion from the analysis of superficial degrees of divergence of the loop integrals [25].

the entire manifold into a sequence of spacelike hypersurfaces. The ADM (Arnowitt-Deser-Misner) decomposition [27] is a particular foliation of spacetime where the spacelike hypersurfaces are given by $t =$ Constant, thus

$$\mathcal{M} = \mathbb{R} \times \Sigma_t, \tag{1.10}$$

here Σ_t represents the spacelike hypersurfaces $t =$ Constant and $t \in \mathbb{R}$. In this decomposition, the future-directed normal vector n^μ of the hypersurfaces[8] are given by the lapse function N and the shift vectors N^i

$$n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N} \right), \tag{1.11}$$

and the metric takes the well-known form

$$ds^2 = -N^2 dt^2 + g_{ij} \left(N^i dt + dx^i \right) \left(N^j dt + dx^j \right). \tag{1.12}$$

Now by applying the Gauss-Godazzi relations, the covariant action Eq. (1.1) also acquires its decomposition form [28]

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{g} N \left(K^{ij} K_{ij} - K^2 + {}^{(3)}R \right), \tag{1.13}$$

here g represents the determinant of 3-metric g_{ij}, ${}^{(3)}R$ is the intrinsic curvature of the hypersurface, K the trace of extrinsic curvature K_{ij} of the hypersurface which is defined as

$$K_{ij} = \frac{1}{2N} \left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i \right). \tag{1.14}$$

Here ∇_i represents the covariant derivative with respect to the 3-metric g_{ij} and $\dot{g}_{ij} \equiv \frac{\partial g_{ij}}{\partial t}$. The action (1.13) serves as the starting point to shift to the Hamiltonian formulation of general relativity. Furthermore, with the help of the DeWitt metric

$$G^{ijkl} = \frac{1}{2} \left(g^{ik} g^{jl} + g^{il} g^{jk} \right) - g^{ij} g^{kl}, \tag{1.15}$$

[8] From now on, the Greek letters are reserved for the indices of 4-dimensional tensors while the Latin letters for the indices of 3-dimensional tensors on the hypersurface.

the kinetic terms in the action can be put into a symmetric form

$$K^{ij}K_{ij} - K^2 = K_{ij}G^{ijkl}K_{kl}. \tag{1.16}$$

1.2.3 Foliation-Preserving Diffeomorphism

Hořava-Lifshitz gravity was first proposed in 2009 [29] with the purpose of modifying the UV behavior of general relativity while keeping the Ostrogradsky ghost out of the theory. This purpose was realized by assuming the anisotropic scaling between time and space

$$t \to b^{-z}t, \quad x^i \to b^{-1}x^i, \tag{1.17}$$

where z is the dynamical critical exponent. As discussed in Sec. 1.2.1, in order to make the theory power-counting renormalizable, one sufficient condition is $z \geq d$. Thus Lorentz symmetry is broken at high energy and later re-emerge when energy becomes low. In this theory, the coordinate time plays a more special role than in general relativity as Hořava assumed that the general covariance is broken to the so-called foliation-preserving diffeomorphism

$$t \to t'(t), \quad x^i \to x'^i\left(t, x^k\right), \tag{1.18}$$

which is usually denoted by Diff($\mathcal{M}$, $\mathcal{F}$). This modified symmetry indicates there exists a fixed foliation of spacetime, one can rescale the coordinate time in any arbitrary way but can not rotate the time direction. Therefore, the most natural coordinate system for this spacetime is that of the ADM variables (N, N^i, g_{ij}) introduced in the last subsection. The scaling dimensions of these variables are assumed to be [29]

$$[N] = [g_{ij}] = 0, \quad [N^i] = z - 1. \tag{1.19}$$

Besides, under the infinitesimal Diff($\mathcal{M}$, $\mathcal{F}$)

$$t \to \xi_0(t), \quad x^i \to \xi^i\left(t, x^k\right), \tag{1.20}$$

the change of the 4d metric $g_{\mu\nu}$ is just its Lie derivative along the direction (ξ_0, ξ^i). In terms of ADM variables, their variances can be shown as [30]

$$\begin{aligned}
\delta N &= \xi^k \nabla_k N + \dot{N}\xi_0 + N\dot{\xi}_0, \\
\delta N_i &= N_k \nabla_i \xi^k + \xi^k \nabla_k N_i + g_{ik}\dot{\xi}^k + \dot{N}_i \xi_0 + N_i \dot{\xi}_0, \\
\delta g_{ij} &= \nabla_i \xi_j + \nabla_j \xi_i + \xi_0 \dot{g}_{ij}.
\end{aligned} \tag{1.21}$$

With the transformation properties of the metric components, one can find that the basic building blocks of the action that is invariant under the Diff($\mathcal{M}$, $\mathcal{F}$) are the 3-dimensional Ricci tensor R_{ij}, the extrinsic curvature K_{ij}, the covariant derivative with respect to 3-metric ∇_i and the 3-vector $a_i \equiv \frac{d \ln N}{dx^i}$, so the general form of the action is[9]

$$S = \int d^4x \sqrt{g} N \left(\mathcal{L}_K - \mathcal{L}_V\right), \tag{1.22}$$

where the kinetic term $\mathcal{L}_K$ is a functional of K_{ij}. Since there appears to be only two independent scalars quadratic in the velocity $\dot{g}_{ij}$, which are $K_{ij}K^{ij}$ and K^2, we must have

$$\mathcal{L}_K = K_{ij}K^{ij} - \lambda K^2. \tag{1.23}$$

The parameter λ approaches one in the relativistic limit. On the other hand, the potential term $\mathcal{L}_V$ is a functional of R_{ij}, ∇_i and a_i, so any scalar composed of these three tensors are allowed in the action with only one restriction coming from the renormalizability condition, that is, for the d+1 dimensional Hořava-Lifshitz gravity, the potential $\mathcal{L}_V$ should include all the independent scalars with spatial derivatives up to the 2zth order. As one can expect the number of independent terms in $\mathcal{L}_V$ is huge in the 4-dimensional theory[10] . Therefore, there are too many coupling constants in the theory which greatly restrict the predictive power of the theory. In order to reduce the number of free parameters, in his seminal paper [29], Hořava introduced two more conditions: the projectable and detailed balance.

[9] Here we set $16\pi G$ to unity.

[10] One can refer to [13] for a list of independent terms up to the order $[k]^6$.

Firstly, the projectable condition is guaranteed by the Diff($\mathcal{M}$, $\mathcal{F}$) as it can be seen directly from the infinitesimal transformation (1.21) that if the lapse function only depends on time, it will remain so after the coordinate transformation (1.18). So once we assume the lapse funtion N stays constant on each spacelike hypersurface Σ_t, all the terms related with a_i in the potential will drop out. On the other hand, the condition in which N also changes with the locations is called the nonprojectable condition. There are various versions of Hořava-Lifshitz gravity based on whether the lapse funtion depends on the space coordinates. I will talk a little about each version in the next section.

In order to further reduce the number of the independent coupling constants, Hořava imposed detailed balance condition which requires the special form of the potential

$$\mathcal{L}_V = E_{ij}\mathcal{G}^{ijkl}E_{kl}, \tag{1.24}$$

$$\sqrt{g}E^{ij} = \frac{\delta W[g_{kl}]}{\delta g_{ij}}, \tag{1.25}$$

where $W[g_{kl}]$ is the superpotential[11] and $\mathcal{G}^{ijkl}$ the generalized DeWitt metric

$$\mathcal{G}^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk}\right) - \lambda g^{ij}g^{kl}. \tag{1.26}$$

Thus, the number of free parameters in the original version of Hořava-Lifshitz gravity are reduced to five: the Newton constant, cosmological constant, λ in the generalized DeWitt metric and the other two constants from the superpotential. It looks like a promising candidate for quantum gravity with UV completion and a decent amount of parameters. However, as it turns out, this version is also plagued by inconsistency due to the additional degree of freedom: the scalar mode.

1.2.3.1. The Scalar Mode There are at least two different viewpoints to understand why, other than the spin-two graviton, there is such a scalar mode in the original

[11] See [29] for more details on the expression of the superpotential.

Hořava-Lifshitz theory of gravity. Firstly, as is well known, the physical degrees of freedom for any system is the same as the dimension of the reduced phase space which in turn depends on the constraints in the theory[12] . Now since the projectable condition in which the lapse function is only a function of time is applied, there is no local Hamiltonian constraint in the canonical formalism of the theory. As a result, the number of local constraints are reduced by two which results in the addition of two degrees of freedom (i.e. the scalar mode) to the reduced phase space. Besides, the most straightforward way to verify that this scalar mode is also a propagating mode of the theory is to consider the linear scalar perturbations of the metric on the Minkowski background as we will discuss in the next section.

1.2.4 Development of Hořava-Lifshitz Gravity Since 2009

In addition to the ghost and instability problems caused by the extra scalar mode in the original Hořava-Lifshitz gravity [32–34], the detailed balance condition will entail a non-zero cosmological constant of the wrong sign to be compatible with observation and consequently the Minkowski spacetime is not compatible with the detailed balance condition either. Therefore, in the so-called minimal theory, only the parity invariance and projectable condition are adopted [35]. A study of scalar perturbation on the Minkowski background shows that in order to evade the ghost problem in the theory, the parameter λ in the kinetic term (1.23) must take the values in the intervals $\lambda > 1$ or $\lambda < 1/3$ [36]. However, the instability problem still persists as it turns out the dispersion relation will carry a negative sign in front of the k^2 term in the infrared limit (IR). Now path diverges when people want to resolve the instability problem of the minimal theory. There are mainly two directions as one can follow to get out of this problem. The first one due to Hořava himself is to eliminate the scalar mode from the theory. Since the appearance of the scalar mode is a direct result from the Diff($\mathcal{M}$, $\mathcal{F}$) which can be viewed as a reduced symmetry group of general

[12] See Appendix A for details.

covariance, Hořava introduced into the theory an extra U(1) symmetry to eliminate this degree of freedom [37]. Therefore, this version is called the projectable Hořava-Lifshitz gravity with U(1) symmetry. In order to realize this extra U(1) symmetry, two auxiliary fields, the gauge field A and Newtonian pre-potential ϕ have to be introduced. Under the U(1) transformation, these fields, together with the ADM variables, are assumed to change like

$$\begin{aligned} \delta_\alpha A = \dot{\alpha} - N^i \nabla_i \alpha, \quad \delta_\alpha \phi = -\alpha, \\ \delta_\alpha N_i = N \nabla_i \alpha, \quad \delta_\alpha g_{ij} = \delta_\alpha N = 0, \end{aligned} \tag{1.27}$$

where α is an arbitrary function of spacetime. While under the foliation-preserving diffeomorphism (1.20), they transform as

$$\begin{aligned} \delta A &= \xi^i \nabla_i A + \dot{\xi}_0 A + \xi \dot{A}, \\ \delta \phi &= \xi_0 \dot{\phi} + \xi^i \nabla_i \phi. \end{aligned} \tag{1.28}$$

In Hořava's paper [37], it is asserted that the parameter λ is forced to be one by U(1) symmetry since this symmetry is passed over to the complete theory from its linearized version in which the global U(1) transformation leaves the action invariant only at $\lambda = 1$. However, the extension to the version with an arbitrary value of λ was soon proposed in [38] where the author used minimal substitution approach to construct the Lagrangian. Since the gauged shift vectors and extrinsic curvature

$$\begin{aligned} \tilde{N}_i &= N_i + N \nabla_i \phi, \\ \tilde{K}_{ij} &= K_{ij} - \nabla_i \nabla_j \phi, \end{aligned} \tag{1.29}$$

still act like a scalar under the U(1) transformation, one can simply replace K_{ij} in the minimal theory by its tilded counterpart. As for the gauge field A, one can also introduce a new field

$$a = -\dot{\phi} + N^i \nabla_i \phi + \frac{N}{2} \nabla^i \phi \nabla_i \phi. \tag{1.30}$$

Then, it can been shown that this field acts in the way as the gauge field A under the U(1) transformation, thus the combination $A - a$ is left invariant under the U(1) transformation. Therefore, the total Lagrangian takes the form

$$\tilde{S} = S[N, N_i + N\nabla_i\phi, g_{ij}] + \int dtd^3xN\sqrt{g}\,(R - 2\Omega)\,(A - a)\,, \tag{1.31}$$

where S represents the action from the minimal theory while the second term incorporates the couplings between the metric and the gauge field. The extra scalar mode is still eliminated by the Gauss constraint $R = 2\Omega$ which is generated by the variation of the total action $\tilde{S}$ with respect to the gauge field A [38, 39].

Another approach to alleviate the problems caused by the scalar mode is to live with it but introduce the nonprojectable condition, i.e. the lapse function also depends on the space coordinates. Therefore, terms related to the vector a_i should be added into the action in this case. In the IR, the only relevant term is a_ia^i, so the action will become

$$S_{\rm IR} = \int dtd^3xN\sqrt{g}\left(\mathcal{L}_K + {}^{(3)}R + \beta a_ia^i\right), \tag{1.32}$$

where β is a coupling constant to be fixed by stability conditions. The linear scalar perturbation of the metric on the Minkowski background will generate the dispersion relation [40, 41]

$$\omega^2 = \frac{\lambda - 1}{3\lambda - 1}\left(\frac{2}{\beta} - 1\right)k^2. \tag{1.33}$$

Thus, the scalar mode can be stabilized if $\beta \in (0, 2)$. The requirement $\lambda > 1$ or $\lambda < 1/3$ is still necessary to circumvent the ghost problem. As it turns out the lapse function in the nonprojectable Hořava-Lifshitz gravity becomes one of the canonical variables since the terms related to the vector a_i will make the local Hamiltonian constraint depend on the lapse function.

Therefore, the Hamiltonian constraint and the canonical momentum of the lapse function are both the second-class constraints of the theory which leads to the conclusion that nonprojectable condition alone will not eliminate the scalar mode in the minimal theory, that is why the fourth version, nonprojectable Hořava-Lifshitz gravity with U(1) symmetry was proposed in a series of papers [42–44]. This is actually the final version of the Hořava-Lifshitz gravity with a local Hamiltonian constraint and 4-dimensional symmetry at each point of the spacetime which makes it as one of the best candidates for quantization of gravity.

Apart from the four different versions of the Hořava-Lifshitz gravity mentioned above, $\lambda = 1/3$ is actually a special case when the generalized DeWitt metric (1.26) becomes degenerate. This degeneracy results in the emergence of two additional second-class constraints which remove the extra scalar mode in the theory. Besides, when $\lambda = 1/3$, the kinetic term $\mathcal{L}_K$ becomes conformal invariant, so this version is called Hořava-Lifshitz gravity at the kinetic-conformal point. One can refer to [45–47] for the discussion of this special version.

1.2.5 Organization of Dissertation

In this dissertation, I will mainly focus on the two dimensional Hořava-Lifshitz gravity. Unlike Einstein's theory, the two dimensional Hořava-Lifshitz gravity is nontrivial due to the special Diff($\mathcal{M}$, $\mathcal{F}$). So Chapter Two will discuss the projectable 2d Hořava-Lifshitz gravity, the classical solutions will be derived and the corresponding spacetime structure will be studied by using Penrose diagram. Then the metric will be quantized in the canonical approach. This chapter is a published paper co-authored by the author of this dissertation. Dr. A. Wang and Dr. Y. Wu are Baylor physics professors. Dr. Z.C. Wu is a professor from Zhejiang University of Technology. Dr. A. Wang supervised the whole project. Dr. Y. Wu and Dr. Z.C. Wu gave some advise. The author of this dissertation completed most of the calculations in this paper.

In Chapter Three, the nonprojectable 2d Hořava-Lifshitz gravity will be considered. In this case, the Hamiltonian constraint will be solved and the quantization is also implemented in the canonical approach. This chapter is a published paper co-authored by the author of this dissertation. Dr. A. Wang is Baylor physics professor who supervised the whole project. V. H. Satheeshkumar and Baofei Li are Baylor physics Ph.D. students who are research performers. They are approximately equal contributors to this paper.

Chapter Four will be devoted to the discussion of the coupled system between Hořava-Lifshitz gravity and a non-relativistic scalar field. The attention will be focused on the search of black hole solutions and finding the event and universal horizons and Hawking radiation temperature. This chapter is a published paper co-authored by the author of this dissertation. Dr. A. Wang is Baylor physics professor who supervised the whole project. Madhurima Bhattacharjee and Baofei Li are Baylor physics Ph.D. students who are research performers. They are approximately equal contributors to this paper.

In the last chapter, I will summarize my work and give a general overview on the status quo of the quantization of the Hořava-Lifshitz gravity.

CHAPTER TWO

Projectable Two-Dimensional Hořava-Lifshitz Theory of Gravity

This chapter published as [48]: Bao-Fei Li, Anzhong Wang, Yumei Wu, Z.C. Wu, "Quantization of (1+1)-Dimensional Hořava-Lifshitz Theory of Gravity," Phys. Rev. D 90, 124076 (2014).

2.1 Introduction

In this chapter, the spacetime structure and quantization of two-dimensional projectable Hořava-Lifshitz gravity will be addressed. I shall first provide a brief review on the 2d HL gravity from which it can be seen that, unlike the 2d GR, the 2d HL gravity is non-trivial even without coupling to matter. This point can be further confirmed by the fact that there exist non-trivial vacuum solutions of the theory with the projectability condition. Studying the local and global properties of these solutions will give us a general view of the structure, especially the singularities, of the spacetime. Then the quantization of the 2d HL gravity is carried out explicitly with the canonical quantization method. It turns out that the problem can be reduced to the quantization of a simple harmonic oscillator [49], for which the expectation value of the gauge-invariant length operator in the ground state provides a fundamental length scale.

2.2 Horava-Lifshitz Theory of Gravity in (1+1)-Dimensions

Einstein's theory of gravity in (1+1)-dimensional spacetimes is trivial, as the Riemann and Ricci tensors $\mathcal{R}_{\mu\nu\beta\gamma}$ and $\mathcal{R}_{\mu\nu}$ are uniquely determined by the Ricci scalar $\mathcal{R}$ via the relations [50]

$$\begin{aligned} \mathcal{R}_{\mu\nu\beta\gamma} &= \frac{1}{2}\left(g_{\mu\beta}g_{\nu\gamma} - g_{\mu\gamma}g_{\nu\beta}\right)\mathcal{R}, \\ \mathcal{R}_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}\mathcal{R}, \end{aligned} \tag{2.1}$$

where the Greek letters run from 0 to 1. Then, the Einstein tensor $E_{\mu\nu}[= \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}]$ always vanishes, and the Einstein-Hilbert action [1]

$$S_{EH} = \zeta^2 \int d^2x\sqrt{{}^{(2)}g}\left(\mathcal{R} - 2\Lambda + \zeta^{-2}\mathcal{L}_M\right), \tag{2.2}$$

leads to a set of non-dynamical field equations, in which the metric $g_{\mu\nu}$ is directly related to the energy-momentum tensor $T_{\mu\nu}$ via the relation

$$\Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.3}$$

where $\zeta^2 = 1/(16\pi G)$ [2] . Therefore, in order to have a non-trivial theory of gravity in 2-dimensions (2d), extra degrees are often introduced, such as a dilaton [51] or a Liouville field [52].

However, this is not the case for the HL gravity [29–31], as the latter has a different symmetry, the foliation-preserving diffeomorphisms (1.18). Then, the general gravitational action takes the form (1.22), that is

$$S_{HL} = \zeta^2 \int dtdxN\sqrt{g}\,(\mathcal{L}_K - \mathcal{L}_V), \tag{2.4}$$

where N denotes the lapse function in the ADM decompositions [27], and $g \equiv \det(g_{ij})$, here g_{ij} is the spatial metric defined on the leaves t = Constant. $\mathcal{L}_K$ is the kinetic part of the action in Eq. (1.23)

$$\mathcal{L}_K = K_{ij}K^{ij} - \lambda K^2, \tag{2.5}$$

where λ is a dimensionless constant, and K_{ij} denotes the extrinsic curvature tensor of the leaves t = Constant as given by Eq. (1.14)

$$K_{ij} = \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right), \tag{2.6}$$

[1] In 2d spacetimes, the integral $\int d^2x\sqrt{{}^{(2)}g}\ \mathcal{R}$ always gives a boundary term. So, normally one does not consider it. This can also be seen from the field equations (2.3).

[2] It should be noted that, unlike in the 4-dimensional case, now ζ is dimensionless (so is G).

and $K \equiv g^{ij}K_{ij}$. Here as mentioned previously $\dot{g}_{ij} \equiv \partial g_{ij}/\partial t$, ∇_i denotes the covariant derivative of the metric g_{ij}, and N^i the shift vector. In the (1+1)-dimensional case, since there is only one spatial dimension, we have $i, j = 1$, and

$$K \;=\; g^{11}K_{11} = -\frac{1}{N}\left(\frac{\dot{\gamma}}{\gamma} - \frac{N_1'}{\gamma^2} + \frac{N_1\gamma'}{\gamma^3}\right), \tag{2.7}$$

where $\gamma \equiv \sqrt{g_{11}}$, $\gamma' \equiv \partial\gamma/\partial x$, etc.

On the other hand, $\mathcal{L}_V$ denotes the potential part of the action, and is made of R, ∇_i and a_i, that is

$$\mathcal{L}_V = \mathcal{L}_V\left(R,\ \nabla_i,\ a_i\right), \tag{2.8}$$

where $a_i \equiv N_{,i}/N$ and R denotes the Ricci scalar of the leaves $t =$ Constant, which identically vanishes in one-dimension, i.e., $R = 0$. As pointed out in Sec. 1.2.1, power-counting renormalizibility condition requires that $\mathcal{L}_V$ should contain spatial operators with the highest dimensions that are not less than $2z$, where $z \geq d$ [25, 29], and d denotes the number of the spatial dimensions. Taking the minimal requirement, that is, $z = d$, we find that in the current case ($d = 1$) we have

$$\mathcal{L}_V = 2\Lambda - \beta a_i a^i, \tag{2.9}$$

where Λ denotes the cosmological constant, and β is another dimensionless coupling constant. Collecting all the above together, we find that the gravitational action of the HL gravity in $(1+1)$-dimensional spacetimes can be cast in the form

$$S_{HL} = \zeta^2 \int dtdxN\sqrt{g}\left[(1-\lambda)K^2 - 2\Lambda + \beta a_i a^i\right]. \tag{2.10}$$

2.3 Classical Solutions of the 2d HL Gravity with the Projectable Condition

Assuming the projectability condition, we have [29]

$$N = N(t), \tag{2.11}$$

from which we immediately find $a_i = 0$. In the rest of this section, we shall assume this condition. Then, the variations of the action S_{HL} with respect to N and N_1 yield the Hamiltonian and momentum constraints, and are given, respectively, by

$$\int dx\gamma(K^2 + 4\tilde{\Lambda}) = 0, \tag{2.12}$$

$$K' = 0, \tag{2.13}$$

where $\tilde{\Lambda} \equiv \Lambda/[2(1-\lambda)]$. The variation of the action S_{HL} with respect to γ, on the other hand, yields the dynamical equation

$$\begin{aligned} &\dot{K} + \frac{1}{2}N(K^2 - 4\tilde{\Lambda}) + \frac{K\dot{\gamma}}{\gamma} - \frac{2KN_1'}{\gamma^2} \\ &+ \left(\frac{N_1 K}{\gamma^2}\right)' + \frac{3KN_1\gamma'}{\gamma^3} = 0. \end{aligned} \tag{2.14}$$

Using the gauge freedom of Eq. (1.18), without loss of the generality, we can always set

$$N = 1, \quad N_1 = 0, \tag{2.15}$$

so that the 2d metric takes the form

$$ds^2 = -dt^2 + \gamma^2(t,x)dx^2. \tag{2.16}$$

It should be noted that Eq. (2.15) uniquely fixes the gauge only up to

$$t' = t + t_0, \quad x' = \zeta(x), \tag{2.17}$$

where t_0 is a constant, and $\zeta(x)$ is an arbitrary function of x only.

With the above gauge choice, Eq. (2.14) reduces to

$$K^2 - 2\dot{K} + 4\tilde{\Lambda} = 0. \tag{2.18}$$

On the other hand, from the momentum constraint (2.13) we can see that K is independent of x, so the Hamiltonian constraint Eq. (2.12) reduces to,

$$(K^2 + 4\tilde{\Lambda}) \int dx\gamma(t,x) = 0. \tag{2.19}$$

Therefore, there exist two possibilities

$$i)\ K^2 + 4\tilde{\Lambda} = 0, \quad ii) \int dx\gamma(t,x) = 0. \tag{2.20}$$

In the following, we consider them separately.

2.3.1 $K^2 + 4\tilde{\Lambda} = 0$

In this case, the extrinsic curvature K is just a constant given by

$$K = \pm 2\sqrt{-\tilde{\Lambda}}, \tag{2.21}$$

which makes sense only when $\tilde{\Lambda} < 0$. From Eq. (2.25), we can find

$$\gamma = e^{\pm 2\sqrt{-\tilde{\Lambda}}t + F(x)}, \tag{2.22}$$

here $F(x)$ is an arbitrary function of x only. Using the gauge residual (2.17), we can always set $F(x) = 0$, so the metric reduces to

$$ds^2 = -dt^2 + e^{4\sqrt{-\tilde{\Lambda}}t}dx^2. \tag{2.23}$$

This is nothing but the de Sitter spacetime.

2.3.2 $\int dx\gamma(t,x) = 0$

In this case, we can see that $\gamma(t,x)$ has to be an odd function of x, i.e., $\gamma(t,x) = -\gamma(t,-x)$. Then, from Eq. (2.18) we find that

$$\frac{dK}{K^2 + 4\tilde{\Lambda}} = \frac{1}{2}dt. \tag{2.24}$$

Since K is independent of x, we find

$$\frac{\dot{\gamma}}{\gamma} = -K(t). \tag{2.25}$$

To solve the above equations under the constraint $\int dx\gamma(t,x) = 0$, it is found convenient to consider the cases $\tilde{\Lambda} > 0$, $\tilde{\Lambda} < 0$, and $\tilde{\Lambda} = 0$, separately.

2.3.2.1. $\tilde{\Lambda} > 0$ Straightforward integration of Eq. (2.24) gives us

$$K = \beta \tan\left[\frac{\beta}{2}(t - t_0)\right], \tag{2.26}$$

where $\beta \equiv \sqrt{4|\tilde{\Lambda}|}$. Then, from Eq. (2.25) we find

$$\gamma = \cos^2\left(\frac{\beta(t - t_0)}{2}\right)\hat{\gamma}(x). \tag{2.27}$$

To satisfy the Hamiltonian constraint, $\hat{\gamma}(x)$ must be an odd function of x, so that

$$\int_{-L_\infty}^{L_\infty} \hat{\gamma}(x)dx = 0, \tag{2.28}$$

where $x = \pm L_\infty$ denote the boundaries of the spacetime in the spatial direction, which can be taken to infinity. With this in mind, we can introduce a new coordinate x' by $dx' = \hat{\gamma}(x)dx$, so the metric takes the form

$$ds^2 = -dt^2 + cos^4\left(\frac{\beta t}{2}\right)dx'^2. \tag{2.29}$$

Note that in writing the above expression, we had set $t_0 = 0$ by using another gauge freedom given in Eq. (2.17). Setting

$$T = \frac{2}{\beta}\tan\left(\frac{\beta t}{2}\right), \tag{2.30}$$

the above metric can be cast in the conformally-flat form

$$ds^2 = \left(1 + \frac{\beta^2}{4}T^2\right)^{-2}\left(-dT^2 + dx'^2\right), \tag{2.31}$$

for which we have

$$K = \frac{\beta^2}{2}T. \tag{2.32}$$

That is, the space-time is singular at $T = \pm\infty$. This is a real space-time singularity in the HL gravity [53], since it is a scalar one and cannot be removed by any coordinate transformations allowed by the symmetry of the theory. The corresponding Penrose diagram is given by Fig. 2.1.

2.3.2.2. $\tilde{\Lambda} < 0$ In this case, Eq. (2.24) has the solution

$$K = \begin{cases} -\beta\tanh\left[\frac{\beta}{2}(t-t_0)\right], & |K| < \beta \\ -\beta\coth\left[\frac{\beta}{2}(t-t_0)\right], & |K| > \beta \end{cases} \tag{2.33}$$

In the following, let us consider the two cases separately, as they will have different properties.

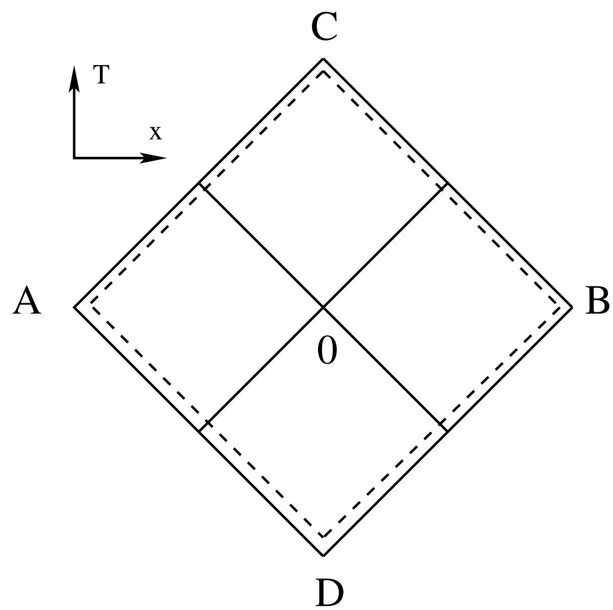

Figure 2.1: The Penrose diagram for the solution (2.31), in which the space-time is singular at both past and further null infinities ($T = \pm\infty$), denoted by the lines $\overline{AC}$, $\overline{AD}$, $\overline{BC}$ and $\overline{BD}$.

Case a) $|K| < \beta$**:** Then, from Eq. (2.25) we find that

$$\gamma = \cosh^2\left[\frac{\beta}{2}(t - t_0)\right]\hat{\gamma}(x). \tag{2.34}$$

Again, using the gauge residual (2.17), without loss of the generality, we can always set $t_0 = 0$ and $dx' = \hat{\gamma}(x)dx$, so the metric finally takes the form

$$ds^2 = -dt^2 + \cosh^4\left(\frac{\beta t}{2}\right)dx^2. \tag{2.35}$$

Note that we dropped the prime from x in writing down the above expression. Then, we can see that the metric is singular at $t = \pm\infty$. However, Eq. (2.33) shows that K is finite at these two limits. In addition, the corresponding 2d Ricci scalar $\mathcal{R}$ is given by

$$\mathcal{R} = \beta^2 \frac{\cosh(\beta t)}{\cosh^2\left(\frac{\beta t}{2}\right)}, \tag{2.36}$$

which is also finite as $t \to \pm\infty$. To further study the properties of these singularities, let us consider the tidal forces experienced by a free-falling observer, whose trajectory is given by the timelike geodesics, satisfying the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}_p}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}_p}{\partial \dot{x}^\mu}\right) = 0, \tag{2.37}$$

where τ denotes the affine parameter along the geodesics, and

$$\mathcal{L}_p \equiv \left(\frac{ds}{d\tau}\right)^2 = -\dot{t}^2 + \cosh^4\left(\frac{\beta t}{2}\right)\dot{x}^2, \tag{2.38}$$

but now with $\dot{t} \equiv dt/d\tau$, etc. For timelike geodesics we have $\mathcal{L}_p = -1$. Since the metric (2.35) does not depend on x explicitly, Eq. (2.37) yields the conservation law of momentum

$$2\cosh^4\left(\frac{\beta t}{2}\right)\dot{x} = p, \tag{2.39}$$

where p denotes the momentum of the observer. Inserting the above expression into Eq. (2.38), we find that

$$\dot{t} = \pm\frac{\sqrt{4\cosh^4\left(\frac{\beta t}{2}\right) + p^2}}{2\cosh^2\left(\frac{\beta t}{2}\right)}, \tag{2.40}$$

where "+" ("-") corresponds to the observer moving along the positive (negative) direction of the x-axis. Setting $e^\mu_{(0)} = dx^\mu/d\tau$, we can construct another space-like unit vector, $e^\mu_{(1)}$ as

$$e^\mu_{(1)} = \left(\pm\frac{p}{2\cosh^2\left(\frac{\beta t}{2}\right)}, \frac{\sqrt{4\cosh^4\left(\frac{\beta t}{2}\right) + p^2}}{2\cosh^4\left(\frac{\beta t}{2}\right)}\right), \tag{2.41}$$

which is orthogonal to $e^\mu_{(0)}$, and parallelly transported along the time-like geodesics

$$g_{\mu\nu}e^\mu_{(a)}e^\nu_{(b)} = \eta_{ab}, \quad e^\mu_{(1);\nu}e^\nu_{(0)} = 0, \tag{2.42}$$

where $\eta_{ab} = \text{diag.}(-1, 1)$, and a semicolon ";" denotes the covariant derivative with respect to the 2d metric $g_{\mu\nu}$. Projecting the 2d Ricci tensor onto the above orthogonal frame, we find that

$$\begin{aligned} R_{(0)(0)} &= -R_{(1)(1)} = -\frac{\beta^2\cosh(\beta t)}{2\cosh^2\left(\frac{\beta t}{2}\right)}, \\ R_{(0)(1)} &= 0, \end{aligned} \tag{2.43}$$

which are all finite as $t \to \pm\infty$. Therefore, the singularities at $t = \pm\infty$ must be coordinate ones. In fact, they represent the boundaries of the space-time. To see this, let us consider the proper time that the observer needs to travel from a given time t_0 to $t = \infty$, which is given by

$$\Delta\tau = \int_{t_0}^{\infty} \frac{2\cosh^2\left(\frac{\beta t}{2}\right)}{\sqrt{4\cosh^4\left(\frac{\beta t}{2}\right) + p^2}} = \infty, \tag{2.44}$$

for any finite t_0. That is, starting at any given finite moment, t_0, the observer always needs to spend infinite proper time to reach the time $t = \infty$. In other words, $t = \infty$ indeed represents the future timelike infinity of the space-time. Similarly, one can see that $t = -\infty$ represents the past timelike infinity.

To study its global structure, let us first introduce the new timelike coordinate T via the relation

$$T = \frac{2}{\beta}\tanh\left(\frac{\beta t}{2}\right), \tag{2.45}$$

we find that the metric takes the form

$$ds^2 = \left(1 - \frac{\beta^2}{4}T^2\right)^{-2}(-dT^2 + dx^2),\ (|T| \leq 2/\beta)\,. \tag{2.46}$$

It is interesting to note that the above metric is singular at $T = \pm 2/\beta$. But, as shown above, this corresponds to coordinate singularities. In fact, they are the space-time boundaries, and any observer will need infinite proper time to reach them starting from any finite time. The corresponding Penrose diagram is given by Fig. 2.2.

Finally, we note that the similarity of the metric (2.35) with the dS_2 metric

$$ds^2_{dS_2} = -dt^2 + \cosh^2(\beta t)d\chi^2, \tag{2.47}$$

where $0 \leq \chi \leq \pi$ with the hypersurfaces $\chi = 0$ and $\chi = \pi$ identified, so the whole space-time has a $R^1 \times S^1$ topology. The space-time is complete in these coordinates. This can be seen clearly by embedding Eq. (2.47) into a 3-dimensional Minkowski

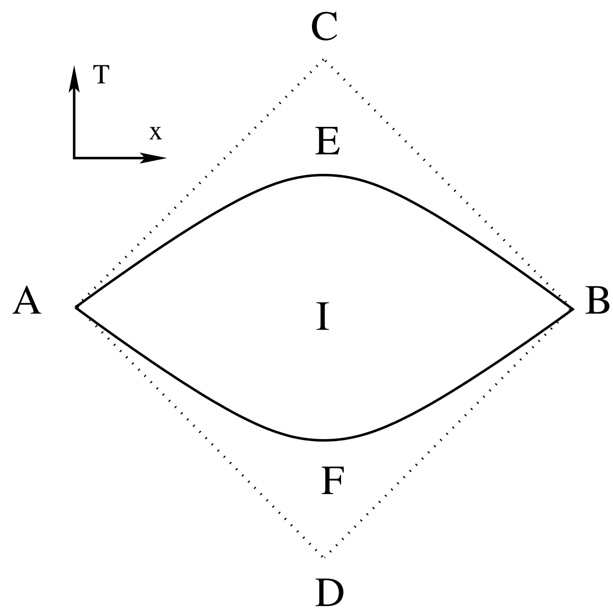

Figure 2.2: The Penrose diagram for the solution (2.35), in which the singularities at $t = \pm\infty$, denoted by the curves $\widehat{AEB}$ and $\widehat{AFB}$, are coordinate ones, and represent the physical boundaries of the space-time.

space-time $ds_3^2 = -dv^2 + dw^2 + dX^2$ with [11]

$$\begin{aligned} v &= \frac{1}{\beta}\sinh(\beta t), \quad w = \frac{1}{\beta}\cosh(\beta t)\cos\left(\frac{\chi}{\beta}\right), \\ X &= \frac{1}{\beta}\cosh(\beta t)\sin\left(\frac{\chi}{\beta}\right), \end{aligned} \tag{2.48}$$

which is a hyperboloid

$$-v^2 + w^2 + X^2 = \beta^{-2}, \tag{2.49}$$

in the 3-dimensional Minkowski space-time. The two metrics (2.35) and (2.47) becomes asymptotically identical when $|t| \gg \beta^{-1}$, provided that the coordinate χ is unrolled to $-\infty < \chi < \infty$.

Case b) $|K| > \beta$: In this case, following what was done in the last case, it can be shown that

$$K = -\beta\coth\left(\frac{\beta t}{2}\right), \quad \gamma = \sinh^2\left(\frac{\beta t}{2}\right)\hat{\gamma}(x), \tag{2.50}$$

and the corresponding line element takes the form

$$ds^2 = -dt^2 + \sinh^4\left(\frac{\beta}{2}t\right)dx'^2. \tag{2.51}$$

Similar to the last case, the metric is singular at $t = \pm\infty$. However, these are coordinate ones, as in the last case. In fact, following what we did there, we find that the following forms a freely-falling frame

$$
\begin{aligned}
e^{\mu}_{(0)} &= \left(\pm\sqrt{1+\frac{p^2}{4\sinh^4\left(\frac{\beta t}{2}\right)}}, \frac{p}{2\sinh^4\left(\frac{\beta t}{2}\right)}\right), \\
e^{\mu}_{(1)} &= \left(\pm\frac{p}{2\sinh^2\left(\frac{\beta t}{2}\right)}, \frac{\sqrt{1+\frac{p^2}{4\sinh^4\left(\frac{\beta t}{2}\right)}}}{\sinh^2\left(\frac{\beta t}{2}\right)}\right),
\end{aligned}
\tag{2.52}
$$

for which we have

$$
\begin{aligned}
R_{(0)(0)} &= -R_{(1)(1)} = -\frac{1}{2}\beta^2\cosh(\beta t)\cosh^{-2}\left(\frac{\beta t}{2}\right), \\
R_{(1)(0)} &= 0.
\end{aligned}
\tag{2.53}
$$

It is clear that all of these components, representing the tidal forces exerted on the observer, are finite. From Eq. (2.52) one can also show that

$$
\Delta\tau = \int_{t_0}^{\infty}\frac{2\sinh^2\left(\frac{\beta t}{2}\right)}{\sqrt{4\sinh^4\left(\frac{\beta t}{2}\right)+p^2}} = \infty,
\tag{2.54}
$$

for any finite t_0. That is, starting at any given finite moment, t_0, the observer will reach $t = \infty$ after spending infinite proper time, i.e., $t = \infty$ represents the space-time boundary. Similarly, one can show that $t = -\infty$ represents the past timelike infinity.

However, in contrast to the last case, the space-time now becomes singular at $t = 0$. This singularity is a scalar singularity, as one can see from Eq. (2.50) and the expression for the 2-dimensional Ricci scalar

$$
\mathcal{R} = \beta^2\left[1+\coth^2\left(\frac{\beta}{2}t\right)\right].
\tag{2.55}
$$

To study its global properties, we first introduce the new coordinate T via the relation

$$
T = -\frac{2}{\beta}\coth\left(\frac{\beta}{2}t\right),
\tag{2.56}
$$

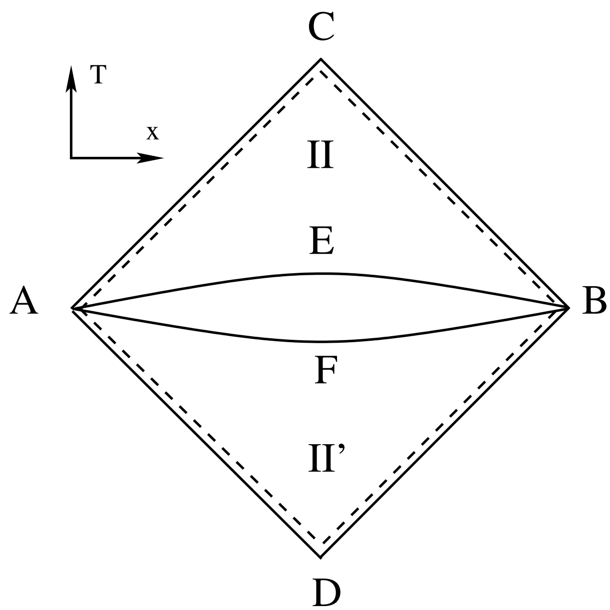

Figure 2.3: The Penrose diagram for the solution (2.51), in which the space-time is singular at both past and further null infinities ($T = \pm\infty$ or $t = 0$), denoted by the lines $\overline{AC}$, $\overline{AD}$, $\overline{BC}$ and $\overline{BD}$. The curved lines, $\widehat{AEB}$ and $\widehat{AFB}$, are free of space-time singularities, and represent the physical boundaries of the space-time.

which maps $t \in (-\infty, 0)$ into the region $T \in (2/\beta, \infty)$, and $t \in (0, \infty)$ into the region $T \in (-\infty, -2/\beta)$. In particular, the times $t = 0^{\pm}$ are mapped to $T = \mp\infty$, and $t = \pm\infty$ to $T = \mp 2/\beta$. In terms of T, we find that

$$ds^2 = \left[1 - \frac{\beta^2}{4}T^2\right]^{-2} \left(-dT^2 + dx'^2\right), \; (|T| \geq 2/\beta)\,. \tag{2.57}$$

The corresponding Penrose diagram is given by Fig. 2.3, from which we can see that the nature of the singularity at $t = 0$ is null.

It is remarkable to note that the metrics (2.46) and (2.57) take the same form, but with different covering ranges. In Eq. (2.46) we have $|T| \in (0, 2/\beta)$, while in Eq. (2.57) we have $|T| \in (2/\beta, \infty)$. The metrics are singular at $|T| = 2/\beta$, which represent the boundaries of the spacetimes, represented, respectively, by Eqs. (2.46) and (2.57).

2.3.2.3. $\tilde{\Lambda} = 0$ Following what we have done in the above, it can be shown that

$$K = -\frac{2}{t}, \qquad \gamma = t^2\hat{\gamma}(x), \tag{2.58}$$

and the line element takes the form

$$ds^2 = -dt^2 + t^4 dx'^2. \tag{2.59}$$

Setting

$$T = 1/t, \tag{2.60}$$

then in the new coordinates we find that the metric takes the form

$$ds^2 = \frac{1}{T^4}\big(- dT^2 + dx'^2\big), \tag{2.61}$$

for which we have

$$K = -2T. \tag{2.62}$$

That is, the space-time is singular at $T = \pm\infty$, and the corresponding Penrose diagram is similar to that given in Fig. 2.1.

2.4 Quantization of 2d Hořava-Lifshitz Gravity

In the projectable Hořava-Lifshitz gravity, the action (2.10) reduces to

$$S_{HL} = \zeta^2 \int dtdx N\gamma \left[(1-\lambda)K^2 - 2\Lambda\right], \tag{2.63}$$

where K is given by Eq. (2.7). In the following, we'll quantize the field by following Dirac's approach.

2.4.1 Hamiltonian Formulation and Dirac Quantization

Starting from the action Eq. (2.63), if we treat γ as a dynamical variable, its canonical momentum is found to be

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = 2\zeta^2(\lambda - 1)K. \tag{2.64}$$

After the Legendre transformation, the corresponding canonical Hamilton is given by

$$H_c(t) = \int dx \Big(N\mathcal{H}(x) + N_1(x)\mathcal{H}_1(x)\Big), \tag{2.65}$$

here the time variable is suppressed. With the projectability condition, the momentum constraint is local while the Hamiltonian constraint is global, that is

$$\mathcal{H}_1 = -\frac{\pi'}{\gamma} \approx 0, \tag{2.66}$$

$$\begin{aligned}\int dx \mathcal{H}(x) &= \int dx \Big(\frac{\pi^2 \gamma}{4\zeta^2(1-\lambda)} + 2\Lambda\zeta^2\gamma \Big) \\ &\approx 0.\end{aligned} \tag{2.67}$$

Straightforward calculations give us their Poisson brackets

$$\begin{aligned}\{\mathcal{H}(x), \mathcal{H}(x')\} &= 0, \\ \{\mathcal{H}(x), \mathcal{H}_1(x')\} &= \frac{\mathcal{H}(x')}{\gamma^2(x')}\delta_{x'}(x-x') \\ &\quad + \frac{\pi\mathcal{H}_1\delta(x-x')}{\zeta^2(1-\lambda)} \approx 0, \\ \{\mathcal{H}_1(x), \mathcal{H}_1(x')\} &= \frac{2\mathcal{H}_1(x')\delta_{x'}(x-x')}{\gamma^2(x')} \\ &\quad - \frac{2\gamma'\mathcal{H}_1}{\gamma^3}\delta(x-x') + \frac{\mathcal{H}_1'}{\gamma^2}\delta(x-x') \\ &\approx 0.\end{aligned} \tag{2.68}$$

Therefore, we've got all the constraints and the physical degrees of freedom of the theory per space-time point ($\mathcal{N}$) is given by

$$\begin{aligned}\mathcal{N} &= \frac{1}{2}\big(\dim\mathcal{P} - 2\mathcal{N}_1 - \mathcal{N}_2\big), \\ &= \frac{1}{2}\big(4 - 2*2 - 0\big) = 0.\end{aligned} \tag{2.69}$$

Here $\dim\mathcal{P}$ means the dimension of the phase space, $\mathcal{N}_1$ ($\mathcal{N}_2$) denotes the number of first-class (second-class) constraints. Meanwhile, the local momentum constraint indicates that π is a function of time only, i.e.

$$\pi(x,t) = \pi(t). \tag{2.70}$$

Note also that the canonical momentum $\pi(t)$ is invariant under the gauge transformation, as can been seen from the expression

$$\Big\{\pi(x), \int dx' \xi(x')\mathcal{H}_1(x')\Big\} = \frac{\xi(x)\mathcal{H}_1(x)}{\gamma(x)}, \tag{2.71}$$

which vanishes on the constraint surface. For completeness, we also give the variation of γ under the spatial diffeomorphism

$$\left\{\gamma(x), \int dx'\xi(x')\mathcal{H}_1(x')\right\} = \left(\frac{\xi}{\gamma}\right)'. \tag{2.72}$$

Since the momentum π is only a function of time, we can obtain an equivalent constraint by integrating Eq. (2.67) directly, and then we have

$$H(\pi, L) = \frac{\pi^2 L}{4\zeta^2(1-\lambda)} + 2\Lambda\zeta^2 L \approx 0, \tag{2.73}$$

with

$$L(t) = \int dx\gamma(t, x), \tag{2.74}$$

which is gauge-invariant owing to Eq. (2.72). It's worth noting that $\pi(t)$ can be regarded as conjugate momentum to the invariant length $L(t)$. Starting from the basic relation

$$\{\gamma(x), \pi(y)\} = \delta(x-y), \tag{2.75}$$

then integrating both sides with respect to x, since π is independent of spatial coordinate y, we directly get

$$\{L(t), \pi(t)\} = 1. \tag{2.76}$$

Now following Dirac's approach, by promoting Eq. (2.76) to the commutation relation $[\hat{L}, \hat{\pi}] = i$, we get the Wheeler-DeWitt equation in the coordinate representation,

$$\hat{H}\Psi = 0. \tag{2.77}$$

However, there is ordering ambiguity arising from the term $L\pi^2$ in Eq. (2.73) [49]. In the following we consider each of the possible orderings separately.

2.4.1.1. $:\pi^2 L: = \hat{L}\hat{\pi}^2$ In this case, the Hamiltonian constraint reads

$$L\left(\frac{\partial^2}{\partial L^2} - \epsilon_{\tilde{\Lambda}}\mu^2\right)\Psi = 0, \tag{2.78}$$

where $\mu \equiv 4\zeta^2|1-\lambda|\sqrt{|\tilde{\Lambda}|}$, and $\epsilon_{\tilde{\Lambda}}$ is a sign function which is one for $\tilde{\Lambda} > 0$, zero for $\tilde{\Lambda} = 0$ and negative one for $\tilde{\Lambda} < 0$. For $\tilde{\Lambda} > 0$, the general solution is

$$\Psi(L,t) = C_1 e^{\mu L} + C_2 e^{-\mu L}. \tag{2.79}$$

It can be shown that this solution is not normalizable even with $C_1 = 0$ with respect to the measure $L^{-1}dL$ in the interval $(0,+\infty)$. For $\tilde{\Lambda} = 0$, we have

$$\Psi(L,t) = A_1 L + A_2, \tag{2.80}$$

while when $\tilde{\Lambda} < 0$, we find

$$\Psi(L,t) = B_1 \sin\left(\mu L + B_2\right), \tag{2.81}$$

here A_1, A_2, B_1 and B_2 are some parameters independent of L. Again none of these wavefunctions are normalizable with respect to the measure $L^{-1}dL$.

2.4.1.2. $:\pi^2 L: = \hat{\pi}\hat{L}\hat{\pi}$ In this case, we have

$$\frac{\partial}{\partial L}\left(L\frac{\partial \Psi}{\partial L}\right) - \epsilon_{\tilde{\Lambda}}\mu^2 L\Psi = 0. \tag{2.82}$$

When $\tilde{\Lambda} > 0$, its general solution is given by the linear combination of the modified Bessel functions of the first and second kind which are denoted respectively by I and K, so

$$\Psi(L,t) = C_3 I_0(\mu L) + C_4 K_0(\mu L). \tag{2.83}$$

However, the normalizable condition with the flat measure dL in the interval $(0,+\infty)$ leads to

$$C_3 = 0, \quad C_4 = \frac{2}{\pi}\sqrt{\mu}. \tag{2.84}$$

For $\tilde{\Lambda} = 0$, we obtain

$$\Psi(L,t) = A_3 \ln L + A_4, \tag{2.85}$$

which cannot be normalized in the interval $(0, +\infty)$. When $\tilde{\Lambda} < 0$, the general solution is given by

$$\Psi(L, t) = B_3 J_0(\mu L) + B_4 Y_0(\mu L), \tag{2.86}$$

which is a linear combination of Bessel functions of the first and second kind. This wave function can't be normalized either.

2.4.1.3. $: \pi^2 L := \hat{\pi}^2 \hat{L}$ In this case, we have

$$\frac{\partial^2}{\partial L^2}\Big(L\Psi\Big) - \epsilon_{\tilde{\Lambda}} \mu^2 L \Psi = 0. \tag{2.87}$$

When $\tilde{\Lambda} > 0$, the general solution of the above equation is given by

$$\Psi(L, t) = \frac{1}{L}\left(C_5 e^{-\mu L} + C_6 e^{\mu L}\right), \tag{2.88}$$

where C_5 and C_6 are the integration constants. Similar to the first case, the wavefunction now is also not normalizable for any given C_5 and C_6 with respect to the measure LdL in the interval $(0, +\infty)$. When $\tilde{\Lambda} = 0$, the solution is

$$\Psi(L, t) = A_5 + \frac{A_6}{L}, \tag{2.89}$$

For $\tilde{\Lambda} < 0$, we find

$$\Psi(L, t) = \frac{1}{L}\left[B_5 \sin\left(\mu L + B_6\right)\right]. \tag{2.90}$$

None of these wavefunctions are normalizable with respect to the measure LdL in the interval $(0, +\infty)$.

2.4.2 Simple Harmonic Oscillator

In this subsection, we shall show that under canonical transformation the above system can be reduced to that of a simple harmonic oscillator. By using the gauge freedom, we can always set

$$N(t) = 1, \qquad N_1 = 0. \tag{2.91}$$

Then, applying the momentum constraint (2.70), the canonical Hamilton (2.65) reduces to

$$H(L,\pi) = L\left[\frac{\pi^2}{4\zeta^2(1-\lambda)} + 2\zeta^2\Lambda\right], \tag{2.92}$$

with L given by Eq. (2.74). After the canonical transformation,

$$L = x^2, \quad \pi = \frac{p}{2x}, \tag{2.93}$$

we find that Eq. (2.76) yields $\{x,p\}=1$, and Eq. (2.92) takes the form

$$H'(x,p) = \frac{p^2}{16\zeta^2(1-\lambda)} + 2\Lambda\zeta^2 x^2. \tag{2.94}$$

However, this new Hamilton constraint (2.94) can only be equivalent to the original one (2.92) on the classical level. One can immediately understand this point when trying to find the solution to the corresponding Wheeler-DeWitt equation

$$H'(\hat{x},\hat{p})\Psi = 0, \tag{2.95}$$

which yields no physical states due to non-vanishing of the energy of the ground state of the quantized oscillator. Hence, we employ the following ansatz for the quantum canonical transformation

$$\hat{L} = \hat{x}^2, \quad \hat{\pi} = \frac{1}{4}\left(\frac{1}{\hat{x}}\hat{p} + \hat{p}\frac{1}{\hat{x}}\right). \tag{2.96}$$

Correspondingly, some terms can be transformed into the forms

$$\hat{L}\hat{\pi}^2 \Longrightarrow \frac{\hat{p}^2}{4} + \frac{i}{2}\frac{1}{\hat{x}}\hat{p} - \frac{5}{16\hat{x}^2}, \tag{2.97}$$

$$\hat{\pi}\hat{L}\hat{\pi} \Longrightarrow \frac{\hat{p}^2}{4} - \frac{1}{16\hat{x}^2}, \tag{2.98}$$

$$\hat{\pi}^2\hat{L} \Longrightarrow \frac{\hat{p}^2}{4} - \frac{i}{2}\frac{1}{\hat{x}}\hat{p} + \frac{3}{16\hat{x}^2}. \tag{2.99}$$

Now setting

$$L\pi^2 \mapsto \frac{1}{3}\left(\hat{L}\hat{\pi}^2 + \hat{\pi}\hat{L}\hat{\pi} + \hat{\pi}^2\hat{L}\right), \tag{2.100}$$

we find that the new Hamilton under the canonical transformation (2.96) is given by

$$\tilde{H} = \frac{\hat{p}^2}{16\zeta^2(1-\lambda)} + 2\Lambda\zeta^2\hat{x}^2 - \frac{1}{64\zeta^2(1-\lambda)\hat{x}^2}. \tag{2.101}$$

Then, we can introduce the creation and annihilation operators

$$\begin{aligned} a &= c_0\left(x + \frac{ip}{8\zeta^2(1-\lambda)\sqrt{\tilde{\Lambda}}}\right), \\ a^\dagger &= c_0\left(x - \frac{ip}{8\zeta^2(1-\lambda)\sqrt{\tilde{\Lambda}}}\right), \end{aligned} \tag{2.102}$$

with $c_0 \equiv 2\zeta\sqrt{1-\lambda}\tilde{\Lambda}^{1/4}$, and

$$[a, a^\dagger] = 1. \tag{2.103}$$

In terms of a, $a^\dagger$ and $\hat{x}$, we find

$$\tilde{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) - \frac{1}{64\zeta^2(1-\lambda)\hat{x}^2}, \tag{2.104}$$

where $\hbar\omega \equiv \sqrt{\tilde{\Lambda}}$. Clearly, to have a well defined vacuum, we must require $\tilde{\Lambda} > 0$, that is

$$\frac{\Lambda}{1-\lambda} > 0. \tag{2.105}$$

Then, the Wheeler-DeWitt equation reads

$$\tilde{H}(\hat{x}, \hat{p})|\Psi\rangle = 0. \tag{2.106}$$

Expanding $|\Psi\rangle$ in terms of the complete set $\{|n\rangle\}$

$$|\Psi\rangle = \sum_{n=0}^{\infty} a_n|n\rangle, \tag{2.107}$$

we find that

$$a_0 + 10\sqrt{2}a_2 = 0, \tag{2.108}$$

$$17a_1 + 14\sqrt{6}a_3 = 0, \tag{2.109}$$

and for $n \geq 2$,

$$\begin{aligned} &(4n-6)\sqrt{n(n-1)}a_{n-2} + (8n^2+8n+1)a_n \\ &\quad + (4n+10)\sqrt{(n+1)(n+2)}a_{n+2} = 0. \end{aligned} \tag{2.110}$$

Therefore, the wavefunction is given by

$$\Psi(x) = \langle x|\Psi\rangle = \sum_{n=0}^{\infty} a_n \psi_n(x), \tag{2.111}$$

where $x = \sqrt{L}$, and

$$\begin{aligned}\psi_n(x) \quad &\equiv \quad \langle x|n\rangle = \frac{(2\mu)^{2n+1}}{\pi^{1/4}\sqrt{2^n n!}} \\ &\quad \times \left(x - \frac{1}{2\mu}\frac{d}{dx}\right)^n e^{-\mu x^2}.\end{aligned} \tag{2.112}$$

Thus, we find that $\Psi(L) \propto e^{-\mu L}$, which is similar to the ones obtained by the Dirac quantization, although they are not precisely equal, as we used two quite different approaches to obtain the corresponding Hamiltons of quantum mechanics, as one can see from Eqs. (2.73) and (2.104).

It should be noted that, in the above studies, either in terms of the Dirac quantization or in terms of the harmonic oscillator, we implicitly assumed $L(t) \neq 0$. Classically, this corresponds to the case studied in Sec. III.A, in which solutions exist only when $\tilde{\Lambda} > 0$, and the resulted space-time is de Sitter. But, quantum mechanically the quantization can be carried out for any $\tilde{\Lambda}$. In addition, classical solutions exist even when $L(t) = 0$. In order to deal with this case, we first note in the classical solutions, we assume $\gamma(t,x) = \hat{\gamma}(x)\gamma(t)$, then $\hat{\gamma}(x)$ should be an odd function of x and $\gamma(t)$ satisfies the EOM (2.18). Of course, in this case constraints (2.12) and (2.13) are satisfied simultaneously. Now we only need to start from the EOM (2.18), in terms of γ it can be recast into the form

$$2\ddot{\gamma}\gamma - \dot{\gamma}^2 + 4\tilde{\Lambda}\gamma^2 = 0. \tag{2.113}$$

Then its corresponding Lagrange can be found as

$$L = \frac{\dot{\gamma}^2}{\gamma} - 4\tilde{\Lambda}\gamma. \tag{2.114}$$

After Legendre transformation, the Hamilton turns out to be

$$H = \frac{p}{4}\gamma^2 + 4\tilde{\Lambda}\gamma, \tag{2.115}$$

where p is conjugate momentum. This Hamilton happens to take the similar form as Eq. (2.73).

2.5 Summary

In this chapter, all the solutions in the projectable case of 2d Hořava gravity are found. These solutions can be divided into three different classes, and each of them have different local and global properties. Their corresponding Penrose diagrams are given, respectively, by Figs. 2.1, 2.2 and 2.3. After solving the momentum constraint explicitly for the projectable pure HL gravity, we have showed that the resulting Hamilton can be quantized by using the standard Dirac quantization. In addition, it can also be written in the form of a simple harmonic oscillator, with the expectation value of the gauge-invariant length operator $L(t)$ defined by Eq. (2.74) given by

$$\langle 0 \,| L(t) |\, 0 \rangle = \frac{1}{16\zeta^2}\sqrt{\frac{2}{(1-\lambda)\Lambda}} \equiv \ell_{HL}, \tag{2.116}$$

which defines a fundamental length of the theory. Here $(\lambda - 1)$ denotes the deviation of the kinetic part of the gravitational action from the relativistic one [cf. Eq. (2.5)], and Λ denotes the cosmological constant. In order for the oscillator to have a stable ground state, one has to assume that

$$\frac{\Lambda}{1-\lambda} > 0, \tag{2.117}$$

which also guarantees that ℓ_{HL} is real. A remarkable feature is that the space-time can be quantized, even it classically has various singularities [cf. Fig. 2.1]. In this sense, the singularities are indeed smoothed out by the quantum effects.

CHAPTER THREE

Nonprojectable Two-Dimensional Hořava-Lifshitz Theory of Gravity

This chapter published as [54]: Bao-Fei Li, V. H. Satheeshkumar, Anzhong Wang, "Quantization of 2d Hořava Gravity: Nonprojectable Case," Phys. Rev. D 93, 064043 (2016).

3.1 Introduction

In this chapter, we examine the two-dimensional version of Hořava-Lifshitz (HL) theory without projectability, where I shall extend the canonical quantization techniques employed in the projectable version of the theory in the last chapter to the nonprojectable case. I'll first give the action of the two-dimensional HL theory without projectability and discuss its classical solutions, then analyze its Hamiltonian structure and quantize the theory.

The general gravitational action of the HL gravity is given by Eq. (1.22)

$$S_{HL} = \zeta^2 \int dt\, dx\, N\sqrt{g}\,(\mathcal{L}_K - \mathcal{L}_V), \tag{3.1}$$

where N denotes the lapse function in the Arnowitt-Deser-Misner (ADM) decomposition [27], and $g \equiv \det(g_{ij})$. As discussed in the last chapter, in the 2-dimensions, the action can be reduced to a simple form (2.10), that is

$$S_{HL} = \zeta^2 \int dt\, dx\, N\gamma\left[(1-\lambda)K^2 - 2\Lambda + \beta a_i a^i\right], \tag{3.2}$$

where $\gamma \equiv \sqrt{g_{11}}$, $\gamma' \equiv \partial\gamma/\partial x$, and

$$K \;=\; g^{11}K_{11} = -\frac{1}{N}\left(\frac{\dot{\gamma}}{\gamma} - \frac{N_1'}{\gamma^2} + \frac{N_1\gamma'}{\gamma^3}\right), \tag{3.3}$$

with $N_1 \equiv g_{1i}N^i = \gamma^2 N^1$.

With regard to the above general action (3.2), it is interesting to note that, in a particular gauge, the so-called T-gauge [55, 56], in which the aether field u_a can be written as [57], $u_a = t_{,a}/\sqrt{-t_{,b}t^{,b}}$, where t is the global time introduced above in the HL gravity, the action of the 2d Einstein-aether theory [58] is identical to the action (3.2). It should be noted that this identification is only on the action level, as the two theories have different gauge symmetries, and the 2d HL theory is only a gauge-fixed form of the 2d Einstein-aether one. Contrary examples can be found in [53, 56]. Besides, one can also find the general classical solutions of the 2d Einstien-aether theory without the cosmological constant Λ in detail in [58].

3.2 Classical Solutions

The line element in terms of N, N^1 and γ, takes the form

$$ds^2 = -N^2(t,x)dt^2 + \gamma^2(t,x)\big(dx + N^1(t,x)dt\big)^2, \tag{3.4}$$

with the gauge freedom

$$t' = \xi^0(t), \quad x' = \xi^1(t,x), \tag{3.5}$$

where $\xi^0(t)$ and $\xi^1(t,x)$ are arbitrary functions of their indicated arguments. Variations of the action Eq. (3.2) with respect to γ, N, and N_1 yield, respectively, the following equations

$$\begin{aligned} 2(1-\lambda)\left[\dot{K} - \frac{NK^2}{2} - \frac{KN_1'}{\gamma^2} + \frac{2KN_1\gamma'}{\gamma^3} + \left(\frac{KN_1}{\gamma^2}\right)'\right]& \\ - \frac{\beta N'^2}{N\gamma^2} - 2\Lambda N = 0,& \end{aligned} \tag{3.6}$$

$$(1-\lambda)\gamma K^2 + 2\Lambda\gamma + 2\beta\left(\frac{N'}{N\gamma}\right)' + \beta\frac{N'^2}{N^2\gamma} = 0, \tag{3.7}$$

and $K' = 0$. Thus, we have $K = K(t)$. Using the gauge freedom (3.5), we can always set $N^1(t,x) = 0$ without loss of the generality. It should be noted that this gauge choice does not completely fix the gauge freedom, and the remaining one is

$$t' = \xi^0(t), \quad x' = \hat{\xi}^1(x). \tag{3.8}$$

With the gauge $N_1 = 0$, Eq. (3.3) reduces to $K(t) = -\dot{\gamma}/(N\gamma)$, while Eqs. (3.6) and (3.7) reduce to

$$\begin{aligned}(1-\lambda)K^2 - 2(1-\lambda)\frac{\dot{K}}{N} + \beta y^2 + 2\Lambda &= 0, &(3.9)\\ 2y' + \left(y^2 - g(t)\right)\gamma = 0, && (3.10)\end{aligned}$$

where $y \equiv N'/(N\gamma)$ and $g(t) \equiv -\beta^{-1}\left[(1-\lambda)K^2 + 2\Lambda\right]$. Equation (3.10) has the general solution,

$$\begin{aligned}y(t,x) &= -\sqrt{g(t)}\tanh\Delta(t,x),\\ \Delta(t,x) &\equiv -\sqrt{g(t)}\left[\frac{\int^x \gamma(t,x')dx'}{2} - c_1(t)\right], \end{aligned} \tag{3.11}$$

where $c_1(t)$ is an arbitrary function of t only. On the other hand, from Eqs. (3.9) and (3.10), we find that

$$(1-\lambda)\gamma\dot{K} + \beta N y' = 0, \tag{3.12}$$

from which, together with Eq.(3.11), we find $N(t,x) = N_0(t)\,\hat{N}(t,x)$, where $\hat{N}(t,x) = 2\cosh^2\Delta(t,x)$ and $N_0(t) = (\lambda-1)\dot{K}/[\beta g(t)]$. Using the remaining gauge freedom of Eq. (3.8), we can always absorb the factor $N_0(t)$ into t', so the lapse function finally takes the form

$$N(t,x) = 2\cosh^2\Delta(t,x). \tag{3.13}$$

Inserting it, together with y given by Eq. (3.11), into Eq. (3.12) we find that

$$\dot{K}(t) - K^2(t) + \eta = 0, \tag{3.14}$$

where $\eta \equiv 2\Lambda/(\lambda-1)$. When $\dot{K} = 0$, Eq. (3.14) has the solution $K = \pm\sqrt{\eta}$. Clearly, for K to be real, we must assume $\eta \geq 0$. Then, from (3.13), we find that $g(t) = 0$ and $N(t,x) = 2$. Redefining t, we can always set $N = 1$. Then, from Eq. (3.12), we find that $\gamma(t,x) = \gamma_0(x)e^{\mp 2\sqrt{\eta}(t-t_0)}$, where $\gamma_0(x)$ is an arbitrary function of x, and t_0 is a constant. Using the gauge residuals of Eq. (3.8), we can always set $\gamma_0(x) = 1$

and $t_0 = 0$, so the corresponding metric finally takes the form,

$$ds^2 = -dt^2 + e^{\mp 4\sqrt{\eta}t}dx^2, \ (\dot{K} = 0), \tag{3.15}$$

which is nothing but the de Sitter spacetime.

When $\dot{K} \neq 0$, Eq. (3.14) has a solution, $K(t) = -\sqrt{\eta}\tanh\left[\sqrt{\eta}(t-t_0)\right]$, from which we find that, $g(t) = -(2\Lambda/\beta)\cosh^{-2}\left[\sqrt{\eta}(t-t_0)\right]$. On the other hand, combining Eqs. (3.12) and (3.13) we find

$$\dot{\gamma} + 2K(t)\cosh^2\Delta\gamma = 0, \ (\dot{K} \neq 0), \tag{3.16}$$

where $\Delta(t,x)$ is given by Eq. (3.11).

3.3 Hamiltonian Structure and Canonical Quantization

Now, let us turn to the Hamiltonian structure and canonical quantization. For such a purpose, in this section, we shall not restrict ourselves to any gauge. Then, from the action (3.2), we find that the canonical momenta are given by,

$$\begin{aligned}
\pi_N &\equiv \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0, \quad \pi_{N_1} \equiv \frac{\partial \mathcal{L}}{\partial \dot{N}_1} = 0, \\
\pi &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = 2\zeta^2(\lambda - 1)K,
\end{aligned}$$

with K given by Eq. (3.3). After Legendre transformation, the Hamiltonian density is given by

$$\begin{aligned}
\mathcal{H} &= \frac{N\gamma\pi^2}{4\zeta^2(1-\lambda)} + 2\zeta^2\Lambda\gamma N - \frac{N_1\pi'}{\gamma} \\
&- \frac{\beta\zeta^2 N}{\gamma}\left(\frac{N'}{N}\right)^2 + \pi_N\sigma + \pi_{N_1}\sigma_1,
\end{aligned} \tag{3.17}$$

where σ and σ_1 are the Lagrangian multipliers. Then, the Hamiltonian takes the form,

$$H = \int dx\, \mathcal{H}(x).$$

Now, the preservation of the primary constraints, $\pi_N \approx 0$ and $\pi_{N_1} \approx 0$, gives us the secondary constraints. By evaluating the poisson brackets we find

$$\dot{\pi}_{N_1} = \left\{\pi_{N_1}, H\right\} = -\mathcal{H}_1 \approx 0,$$
$$\dot{\pi}_N = \left\{\pi_N, H\right\} = -\mathcal{H}_2 \approx 0.$$

Here

$$\begin{aligned} \mathcal{H}_1 &\equiv -\frac{\pi'}{\gamma}, \qquad (3.18)\\ \mathcal{H}_2 &\equiv \frac{\pi^2\gamma}{4\zeta^2(1-\lambda)} + 2\zeta^2\Lambda\gamma \\ &+ \beta\zeta^2\frac{N'^2}{N^2\gamma} + 2\beta\zeta^2\left(\frac{N'}{N\gamma}\right)'. \qquad (3.19)\end{aligned}$$

Rearranging the Hamiltonian in terms of the constraints, we end up with

$$\begin{aligned} \mathcal{H} &= N_1\mathcal{H}_1 + N\mathcal{H}_2 + \pi_N\sigma + \pi_{N_1}\sigma_1 \\ &-2\beta\zeta^2\left(\frac{N'}{\gamma}\right)'. \qquad (3.20)\end{aligned}$$

In the following analysis, we will drop the last surface term. By straightforward calculations, we can obtain the structure functions of the constraints, which are given by

$$\begin{aligned} \left\{\mathcal{H}_1(x), \mathcal{H}_1(x')\right\} &= \left(\frac{\mathcal{H}_1(x')}{\gamma^2(x')} + \frac{\mathcal{H}_1(x)}{\gamma^2(x)}\right)\partial_{x'}\delta(x-x'),\\ \left\{\mathcal{H}_1(x), \mathcal{H}_2(x')\right\} &= -\frac{\pi(x)\mathcal{H}_1(x)}{\zeta^2(1-\lambda)}\delta(x-x') \\ &+ \frac{\mathcal{H}_2(x)}{\gamma^2(x)}\partial_x\delta(x-x') \\ &+ \frac{2\beta\zeta^2 N'}{\gamma^3 N}\partial_{xx}\delta(x-x') \\ &- 2\beta\zeta^2\left(\frac{N'\gamma'}{N\gamma^4} + \frac{N'^2}{N^2\gamma^3}\right)\partial_x\delta(x-x') \\ &- \frac{\beta\zeta^2}{\gamma}\left(\frac{N'^2}{\gamma^2N^2}\right)'\delta(x-x'). \qquad (3.21)\end{aligned}$$

Clearly, $\mathcal{H}_1$ and $\mathcal{H}_2$ don't commute with each other on the constraint surface due to the last three terms on the right-hand side of Eq. (3.21) (all the functions on the right-hand side of this commutator are functions of x). In addition, we also have

$$\begin{aligned}
\left\{\mathcal{H}_2(x), \mathcal{H}_2(x')\right\} &= -\frac{2\beta\pi(x)N'(x)}{(1-\lambda)N(x)\gamma(x)}\partial_x\delta(x-x') \\
&- \frac{\beta}{1-\lambda}\left(\frac{N'\pi}{N\gamma}\right)'\delta(x-x'), \\
\left\{\pi_N(x), \mathcal{H}_2(x')\right\} &= -\frac{2\beta\zeta^2}{N(x')\gamma(x')}\partial_{x'x'}\delta(x-x') \\
&- 2\beta\zeta^2\partial_{x'}\left(\frac{1}{N(x')\gamma(x')}\right)\partial_{x'}\delta(x-x') \\
&+ \frac{2\beta\zeta^2}{N}\left(\frac{N'}{N\gamma}\right)'\delta(x-x').
\end{aligned}$$

So, π_N and $\mathcal{H}_2$ don't commute either. In this case, we need to define a new constraint via the relation

$$\tilde{\mathcal{H}}_1 = \mathcal{H}_1 + \frac{N'}{\gamma^2}\pi_N. \tag{3.22}$$

As it turns out that $\tilde{\mathcal{H}}_1$ commutes with both $\mathcal{H}_2$ and π_N on the constraint surface, and their structure functions are given by

$$\begin{aligned}
\left\{\tilde{\mathcal{H}}_1(x), \mathcal{H}_2(x')\right\} &= -\frac{\pi(x)\tilde{\mathcal{H}}_1(x)}{\zeta^2(1-\lambda)}\delta(x-x') \\
&+ \frac{\mathcal{H}_2(x)}{\gamma^2(x)}\partial_x\delta(x-x'), \\
\left\{\tilde{\mathcal{H}}_1(x), \pi_N(x')\right\} &= \frac{\pi_N(x)}{\gamma^2(x)}\partial_x\delta(x-x'), \\
\left\{\tilde{\mathcal{H}}_1(x), \tilde{\mathcal{H}}_1(x')\right\} &= \left(\frac{\tilde{\mathcal{H}}_1(x')}{\gamma^2(x')} + \frac{\tilde{\mathcal{H}}_1(x)}{\gamma^2(x)}\right)\partial_{x'}\delta(x-x').
\end{aligned}$$

Correspondingly, the Hamiltonian now takes the form [1]

$$\tilde{\mathcal{H}} = N_1\tilde{\mathcal{H}}_1 + \sigma_1\pi_{N_1} + N\mathcal{H}_2 + \sigma\pi_N. \tag{3.23}$$

Then, one can show that $\pi_{N_1} \approx 0$ and $\tilde{\mathcal{H}}_1 \approx 0$ are the first-class constraints, while $\pi_N \approx 0$ and $\mathcal{H}_2 \approx 0$ are the second-class constraints. These constraints are preserved

[1] Hamiltonian structure of four-dimensional HL theory without the projectability condition was studied in [61], and a similar structure was obtained (See also [62]). We thank T. Jacobson for pointing this out to us.

under time evolution. So, the physical degrees ($\mathcal{N}$) of freedom of the theory per spacetime point is given by the formula (2.69), that is

$$\begin{aligned}\mathcal{N} &= \frac{1}{2}\big(\text{dim}\mathcal{P} - 2\mathcal{N}_1 - \mathcal{N}_2\big),\\ &= \frac{1}{2}\big(6 - 2*2 - 2\big) = 0.\end{aligned}$$

Here dim$\mathcal{P}$ means the dimension of the phase space, and $\mathcal{N}_1$ ($\mathcal{N}_2$) denotes the number of first-class (second-class) constraints. It is interesting to note that $\mathcal{N}$ is not equal to -1, as in the usual 2d relativistic case [59], due to the new gauge symmetry (3.5) of the theory. It is also interesting to note that in the projectable case, the physical degrees of freedom is also zero.

Now we proceed to the canonical quantization of the system by following Dirac [60]. First, for the two second-class constraints $\pi_N \approx 0$ and $\mathcal{H}_2 \approx 0$, we can make them strongly equal to zero,

$$(i)\;\; \pi_N = 0, \quad (ii)\;\; \mathcal{H}_2 = 0, \tag{3.24}$$

by simply replacing the Poisson bracket with the Dirac bracket. The first condition is actually empty, while from the second condition, we can express N as a functional of γ and π by solving the equation $\mathcal{H}_2 = 0$, where $\mathcal{H}_2$ is given by Eq. (3.19). The general solution is given by

$$N(t,x) = N_0(t)\exp\left\{\int^x y(t,x')\gamma(t,x')dx'\right\}, \tag{3.25}$$

where $N_0(t)$ is an integration function of t only, and $y(t,x)$ is given by Eq. (3.11). As a result, we can drop N and π_N by going to the "reduced" phase space spanned by $(N_1, \pi_{N_1}; \gamma, \pi)$. However, the phase space can be further reduced by noting that the first-class constraint $\pi_{N_1} \approx 0$ simply yields

$$-i\hbar\frac{\delta\psi}{\delta N_1} = 0,$$

that is, the wave function ψ will not depend on N_1 and π_{N_1}. Then, the reduced phase space actually becomes two-dimensional, spanned by γ and π.

On the other hand, with the first condition (3.24), the first-class constraint $\tilde{\mathcal{H}}_1 \approx 0$ reduces to $\mathcal{H}_1 \approx 0$, as one can see from Eq. (3.22). This in turn implies

$$\pi - \alpha(t) \approx 0,$$

where $\alpha(t) \equiv 2\zeta^2(\lambda - 1)K(t)$. Then, the corresponding Wheeler-DeWitt equation takes the form

$$\left(-i\hbar\frac{\delta}{\delta\gamma} - \alpha(t)\right)\psi\left(\gamma; t\right) = 0, \tag{3.26}$$

which has the general (plane wave) solution

$$\psi\left(\gamma, t\right) = \psi_0 e^{\frac{i\alpha}{\hbar}L}. \tag{3.27}$$

Here, $L \equiv L(t)$ is the gauge-invariant length, defined in Eq. (2.74),

$$L(t) \equiv \int_{-L_\infty}^{L_\infty} \gamma(t, x) dx, \tag{3.28}$$

where $x = \pm L_\infty$ represent the boundaries of the one-dimensional spatial space. The integration "constant" ψ_0 in general is a function of t. But, the normalization condition, $\int_{-L_\infty}^{L_\infty} |\psi|^2\, dx$ requires $\psi_0 = e^{i\beta(t)}/(2L_\infty)$, where $\beta(t)$ is real and otherwise arbitrary function of t only. However, without loss of the generality, we can always set $\beta(t) = 0$.

3.4 Summary

In this chapter, we have studied the quantization of 2d Hořava theory of gravity without the projectability condition, that is, the lapse function N in general is a function of both time and space, $N = N(t, x)$. The classical solutions have been studied in some detail and shown that the extrinsic curvature of the leaves $t =$ constant is always independent of the spatial coordinates. In the case of a constant extrinsic curvature, the corresponding spacetime is de Sitter, while in the general case, the dynamical variable $\gamma(t, x)$ satisfies a master equation given by Eq. (3.16). Once γ is known, the rest of metric coefficients can be found algebraically.

Our investigation of the Hamiltonian structure of the theory shows that the system consists of two first-class and two second-class constraints. As a result, the number of total degrees of freedom is zero. Following Dirac [60], we have first turned the two second-class constraints into strong ones, by requiring that they be strongly equal to zero, from which we can express the lapse function N as a functional of the canonical variable γ and its momentum conjugate π, so the phase space is reduced from six to four dimensions, spanned by $(N_1, \pi_{N_1}; \gamma, \pi)$. But, one of the two first-class constraints further tells us that the actual dimension of the phase space is two, since the wave function of the system is independent of the shift vector N_1 and its momentum conjugate π_{N_1}. As a result, the corresponding Wheeler-DeWitt equation simply takes the form of Eq. (3.26) and has a plane wave solution (3.27), in terms of the gauge-invariant length $L(t)$ defined by Eq. (3.28). Therefore, similar to the projectable case , this system is also quantum mechanical in nature. This is understandable, as this system also has zero-degree of freedom. However, what is a bit surprising is that the corresponding Wheeler-DeWitt equation simply yields the plane wave solution.

In addition, the classical spacetimes do not play important role in the process of quantization. In particular, it does not matter whether the classical background is de Sitter or not, the wave function is always a plane wave solution. The only effects of the classical backgrounds are encoded in the phase of the plane wave, in terms of the extrinsic curvature $K(t)$ of the leaves $t =$ constant, where t is the time coordinate, with which the spacetime is foliated globally.

CHAPTER FOUR

Two-Dimensional Hořava-Lifshitz Theory of Gravity Coupled with Matter

This chapter published as [63]: Bao-Fei Li, Madhurima Bhattacharjee, Anzhong Wang, "Universal Horizons and Hawking Radiation in Nonprojectable 2d Hořava Gravity Coupled with A Non-relativistic Scalar Field," Phys. Rev. D 96, 084006(2017).

4.1 Introduction

In this chapter, we will first generalize our studies to the case where the projectable Hořava-Lifshitz gravity is minimally coupled to a scalar field, which shares the same gauge symmetry as the 2d Hořava-Lifshitz gravity. Unlike the vacuum case, we find that now the momentum constraint cannot be solved explicitly except for the case in which the fundamental variables depend only on time. Similar to the vacuum case, now the system can also be quantized by the standard Dirac quantization. When the self-interaction of the scalar field vanishes, the problem reduces to two independent simple harmonic oscillators, one has positive energy and the other has negative energy. In the second part of this chapter, we will study universal horizons and their thermodynamics in 2d nonprojectable Hořava gravity, coupled with a non-relativistic scalar field. The existence of universal horizons is closely related to the existence of a globally defined time-like khronon field φ [13]. Then, all the particles are assumed to move in the increasing direction of φ. At the beginning, universal horizons were studied in the framework of the Einstein-aether theory with spherical symmetry, in which the time-like aether naturally plays the role of the khronon field [64, 65]. To generalize such concepts to other theories, including Hořava-Lifshitz gravity, in which the aether field is not part of the theory, one can consider the khronon field as a test field [66], a role similar to a Killing vector field ξ_μ, which satisfies the Killing equations, $\nabla_{(\nu}\xi_{\mu)} = 0$, on a given spacetime background $g_{\mu\nu}$. In this chapter, we shall adopt this generalization, and assume that the test khronon field satisfies the same equations as the aether field, the most general second-order partial differential equations in terms of the aether four-velocity [67].

4.2 2D Projectable Hořava-Lifshitz Gravity Coupled with a Scalar Field

When the 2d HL gravity couples to a scalar field ϕ, the total action becomes

$$S = S_{HL} + S_{\phi}, \tag{4.1}$$

where S_{HL} is the action of 2d projectable Hořava-Lifshitz gravity given by Eq. (2.4) while S_{ϕ} denotes the action of the scalar field. To be power-counting renormalizable, the marginal terms of S_{ϕ} must be at least of dimension $2z$ with $z \geq d$. Since ϕ is dimensionless, one can see that the marginal terms are $\nabla_i \phi \nabla^i \phi$ and $a_i \nabla^i \phi$. Then, S_{ϕ} must take the form

$$\begin{aligned} S_{\phi} \;\; = \;\; & \int dt dx N \sqrt{g} \Big[\frac{1}{2} \left(\partial_{\perp}\phi\right)^2 - \alpha_0 \left(\nabla_i \phi\right)^2 - V(\phi) \\ & - \alpha_1 \phi \nabla^i a_i - \alpha_2 \phi a^i \nabla_i \phi \Big]. \end{aligned} \tag{4.2}$$

Here $\partial_{\perp} \equiv N^{-1}\left(\partial_t - N^i \nabla_i\right)$, $V(\phi)$ denotes the potential of the scalar field, and α_n are dimensionless coupling constants. Since the scalar field ϕ is dimensionless, these coefficients in principle can be arbitrary functions of ϕ. In this chapter, we consider only the case where they are constants. Besides, in the relativistic limit, we have $(\alpha_0, \alpha_1, \alpha_2)_{GR} = (1/2, 0, 0)$.

4.2.1 Classical Field Equations

In the projectable case, we have $a_i = 0$ and the last two terms in Eq. (4.2) vanish. Then, the variations of the total action with respect to N, γ, N_1 and ϕ yield, respectively,

$$\int dx \Bigg\{ \left[\frac{\dot{\gamma}^2}{\kappa\gamma} + 8\zeta^2\Lambda\gamma\right] + \frac{2c_\phi^2}{\gamma}\phi'^2 + \left[2\gamma\dot{\phi}^2 + 4\gamma V(\phi)\right] \Bigg\} = 0, \tag{4.3}$$

$$\left(\frac{\dot{\gamma}}{\gamma}\right)^{\cdot} + \frac{1}{2}\left(\frac{\dot{\gamma}}{\gamma}\right)^2 + 2\tilde{\Lambda} = \kappa\left(\dot{\phi}^2 + \frac{c_\phi^2}{\gamma^2}\phi'^2 - 2V(\phi)\right), \tag{4.4}$$

$$\left(\frac{\dot{\gamma}}{\gamma}\right)' = 2\kappa\dot{\phi}\phi', \tag{4.5}$$

$$\left(\gamma\dot{\phi}\right)^{\cdot} - c_\phi^2\left(\frac{\phi'}{\gamma}\right)' + \gamma\frac{dV(\phi)}{d\phi} = 0, \tag{4.6}$$

where $c_\phi^2 \equiv 2\alpha_0$ must be non-negative in order for the scalar field to be stable, and

$$\kappa = \frac{1}{4\zeta^2(1-\lambda)}. \tag{4.7}$$

Note that in the vacuum case γ is a function of t only, as shown previously. However, because of the presence of the scalar field, now it is in general a function of both t and x. To compare it with the vacuum case, in the following let us consider the case $\gamma = \gamma_0(x)\gamma(t)$ only. In fact, as to be shown below, this is also the case where the corresponding Hamiltonian constraint becomes local, while the momentum constraint can be solved explicitly.

Setting $\gamma = \gamma_0(x)\gamma(t)$, from Eq. (4.5) we can choose that $\phi = \phi(t)$. Then, Eqs. (4.3), (4.4) and (4.6) reduce, respectively, to

$$\int dx \left\{\frac{\dot{\gamma}^2}{\kappa\gamma} + 8\zeta^2\Lambda\gamma + 2\gamma\dot{\phi}^2 + 4\gamma V(\phi)\right\} = 0, \tag{4.8}$$

$$\left(\frac{\dot{\gamma}}{\gamma}\right)^{\cdot} + \frac{1}{2}\left(\frac{\dot{\gamma}}{\gamma}\right)^2 + 2\tilde{\Lambda} = \kappa\left(\dot{\phi}^2 - 2V(\phi)\right), \tag{4.9}$$

$$\left(\gamma\dot{\phi}\right)^{\cdot} + \gamma\frac{dV(\phi)}{d\phi} = 0. \tag{4.10}$$

To solve the above equations, we further assume that $V(\phi) = \tilde{\Lambda} = 0$. Then from Eq. (4.10), we know

$$\dot{\phi} = \frac{\phi_0}{\gamma(t)}, \tag{4.11}$$

here ϕ_0 is a constant. Combining with Eq. (4.9), we derive an equation for $\gamma(t)$

$$\ddot{\gamma}(t)\gamma(t) - \frac{1}{2}\dot{\gamma}(t)^2 = \kappa\phi_0^2. \tag{4.12}$$

One of the solutions can be easily obtained, and is given by

$$\begin{aligned}
\gamma(t) &= (c_0 + c_1 t)^2 + \frac{\kappa\phi_0^2}{2c_1^2}, && (4.13)\\
\phi(t) &= \sqrt{\frac{2}{\kappa}}\arctan\left(\sqrt{\frac{2}{\kappa}}\frac{c_1(c_0 + c_1 t)}{\phi_0}\right) + \phi_1, && (4.14)
\end{aligned}$$

where c_0, c_1 and ϕ_1 are constants. In order to make our solution consistent with the integral constraint (4.8), we require $\gamma(t, x)$ to be an odd function of x, so that the integration of γ over the whole interval $x \in (-\infty, \infty)$ vanishes. Keeping this in mind and then using the residual gauge freedom, we find the metric takes the form,

$$ds^2 = -dt^2 + \left(t^2 + \epsilon_\kappa t_s^2\right)^2 dx^2, \tag{4.15}$$

here $\epsilon_\kappa \equiv \text{sign}(\kappa)$, and

$$t_s^2 \equiv \frac{|\kappa|\phi_0^2}{2c_1^4}. \tag{4.16}$$

Following what we did in Chapter I, we can derive the extrinsic curvature K, Ricci scalar R, and the components of the tidal forces, given respectively by

$$\begin{aligned}
K &= -\frac{t}{2}R = -\frac{2c_1^2 t}{t^2 + \epsilon_\kappa t_s^2}, && (4.17)\\
R_{(1)(1)} &= -R_{(0)(0)} = \frac{2c_1^2}{t^2 + \epsilon_\kappa t_s^2}. && (4.18)
\end{aligned}$$

Therefore, the singularities of the spacetime are determined directly by the signs of κ. In particular, if $\lambda \leq 1$, the spacetime is free of space-time singularities. For $\lambda > 1$, on the other hand, there is a curvature singularity located at

$$t = \pm t_s. \tag{4.19}$$

The corresponding Penrose diagrams are given in Fig. 4.1.

It should be noted that, instead of imposing the condition that γ is an odd function of x, we can set the integrand of the Hamiltonian constraint to zero. But, this will require $c_1 = 0$, and the corresponding space-time is flat.

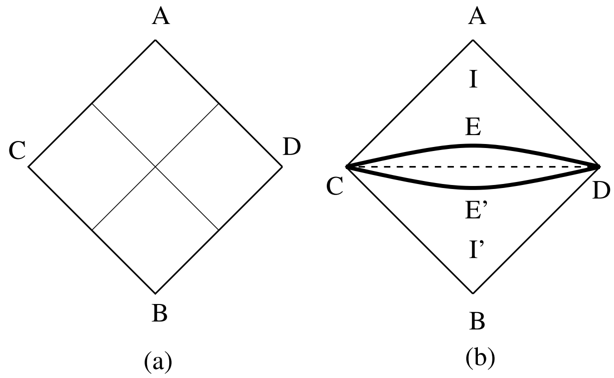

Figure 4.1: (a) The Penrose diagram for the solution (4.15) with $\lambda \leq 1$ (or $\kappa \geq 0$), in which the whole space-time is free of space-time singularities. (b) The Penrose diagram for the solution (4.15) with $\lambda > 1$ (or $\kappa < 0$), in which the space-time is singular on $t = \pm t_s$, denoted by the thick solid curves $\widehat{CED}$ and $\widehat{CE'D}$. Thus, in this case the two regions I and I' are causally disconnected.

4.2.2 Hamiltonian Structure and Canonical Quantization

When coupling with the scalar field, the Hamiltonian and momentum constraints become

$$\begin{aligned} \int dx \mathcal{H}(x) &= \int dx \left[\frac{\pi^2\gamma}{4\zeta^2(1-\lambda)} + 2\Lambda\zeta^2\gamma + \frac{\pi_\phi^2}{2\gamma}\right. \\ &\qquad \left. + \frac{\alpha_0\phi'^2}{\gamma} + \gamma V(\phi)\right], \end{aligned} \tag{4.20}$$

$$\mathcal{H}_1 = -\frac{\pi'}{\gamma} + \frac{\pi_\phi\phi'}{\gamma^2}, \tag{4.21}$$

here π_ϕ denotes the canonical moment conjugate to the scalar field ϕ. Similarly, the Poisson brackets of the two constraints are given by

$$\begin{aligned} \{\mathcal{H}(x), \mathcal{H}_1(x')\} &= \frac{\mathcal{H}(x')\delta_x(x-x')}{\gamma^2(x')} + \frac{\pi\mathcal{H}_1\delta(x-x')}{\zeta^2(1-\lambda)}, \\ \{\mathcal{H}_1(x), \mathcal{H}_1(x')\} &= \frac{2\mathcal{H}_1(x)\delta_{x'}(x-x')}{\gamma^2(x)} \\ &\quad + \frac{2\gamma'\mathcal{H}_1}{\gamma^3}\delta(x-x') - \frac{\mathcal{H}_1'}{\gamma^2}\delta(x-x'). \end{aligned} \tag{4.22}$$

For the non-local Hamiltonian constraint we also find

$$\left\{\int dx\mathcal{H}(x), \int dx'\mathcal{H}(x')\right\} = 0, \tag{4.23}$$

as long as $\pi_\phi \phi'/\gamma^2$ vanishes on boundaries.

In the rest of this section, we only consider the quantization of the system for the case

$$\phi' = 0 = \pi', \tag{4.24}$$

in order to compare with what we obtained in the pure gravity case. As a matter of fact, this also makes the problem considerably simplified and become tractable. Under the above assumption, the Hamiltonian constraint reads

$$H(t) = \frac{\pi^2 L}{4\zeta^2(1-\lambda)} + 2\Lambda\zeta^2 L + \frac{L\dot{\phi}^2}{2} + LV(\phi) \simeq 0. \tag{4.25}$$

It must be noted that in writing down the above expression, we performed the spatial integration and used the fact that

$$\pi_\phi = \gamma\dot{\phi}, \tag{4.26}$$

with the gauge choice $N = 1$ and $N_1 = 0$. On the other hand, from the canonical relation

$$\{\phi(x), \pi_\phi(y)\} = \delta(x-y), \tag{4.27}$$

we can integrating both sides with respect to the spatial coordinates x and y, and then use Eq. (4.26) and the constraint $\phi = \phi(t)$, to obtain

$$\{\phi(t), L(t)\dot{\phi}(t)\} = 1, \tag{4.28}$$

which enables us to identify π_ϕ as $\pi_\phi = L\dot{\phi}$. Now making this substitution in the Hamiltonian constraint (4.25), we find the Hamilton with two discrete physical degrees of freedom, L and ϕ, takes the form

$$H(t) = \frac{\pi^2 L}{4\zeta^2(1-\lambda)} + 2\Lambda\zeta^2 L + \frac{\pi_\phi^2}{2L} + LV(\phi). \tag{4.29}$$

Thus, the Wheeler-Dewitt equation now reads

$$\hat{H}(t)\Psi(L, \phi; t) = 0. \tag{4.30}$$

If we further assume that the potential of the scalar field can be ignored, $V(\phi) \simeq 0$, we are able to find solutions to Eq. (4.30) by separation of variables. In this case, assuming

$$\Psi(L, \phi) = X(L)Y(\phi), \tag{4.31}$$

we obtain two independent equations

$$Y''(\phi) + mY(\phi) = 0, \tag{4.32}$$

$$\left[L\pi^2\right] X(L) + \left(\mu^2 L + \epsilon_\lambda \frac{m\mu}{2\sqrt{\tilde{\Lambda}}L}\right) X(L) = 0. \tag{4.33}$$

here $[L\pi^2]$ means some specific ordering of L and π, m is an undetermined parameter, μ is given as in the pure gravity case, and ϵ_λ is one for $\lambda < 1$ and negative one for $\lambda > 1$. Just as in the pure gravity case, there are three different orderings, which will be considered below, separately.

4.2.2.1. $: \pi^2 L :=\hat{L}\hat{\pi}^2$ In this case, the Hamiltonian constraint reads

$$L^2 X'' - \left(\epsilon_{\tilde{\Lambda}} \mu^2 L^2 + k^2\right) X = 0, \tag{4.34}$$

where $k^2 = 2\epsilon_\lambda m\zeta^2|1-\lambda|$ and $\epsilon_{\tilde{\Lambda}}$ is defined in Sec. 2.4.1.1. For $\tilde{\Lambda} > 0$, the general solution is given by the linear combination of the modified Bessel functions of the first and second kind, denoted by I_ν and K_ν respectively, that is

$$X = \sqrt{L}\left\{C_1 I_\nu(L\mu) + C_2 K_\nu(L\mu)\right\}, \tag{4.35}$$

Here $\nu \equiv \sqrt{1+4k^2}/2$. Generally, this wave-function is not normalizable with respect to the measure dL/L in the interval $(0, +\infty)$. However, if $|\,\mathrm{Re}(\nu)| < 1/2$, we have normalized function as

$$X_{\text{norm}} = \frac{1}{\pi}\sqrt{\frac{4\mu}{\sec(\pi\nu)}} K_\nu(L\mu). \tag{4.36}$$

In this particular case for $-1/4 \leq k^2 < 0$, depending on the value of λ, m can be either positive or negative. In both cases, in order to have a normalizable wave function (4.31), we need to restrict the domain of ϕ to some finite region, for example $(0, 2\pi)$, then it would be straightforward to normalize $Y(\phi)$ from Eq. (4.32) in that finite region.

For $\tilde{\Lambda} = 0$, the solution is given by

$$X = \sqrt{L}\left(A_1 L^{+\nu} + A_2 L^{-\nu}\right), \tag{4.37}$$

while for $\tilde{\Lambda} < 0$, we find

$$X = \sqrt{L}\left(B_1 J_\nu(\mu L) + B_2 Y_\nu(\mu L)\right). \tag{4.38}$$

Here ν is defined as in the case $\tilde{\Lambda} > 0$. None of these two wave functions are normalizable with respect to the measure $L^{-1}dL$ in the interval $(0, +\infty)$.

4.2.2.2. $: \pi^2 L : = \hat{\pi}\hat{L}\hat{\pi}$ In this case, we have

$$L^2 X'' + LX' - \left(\epsilon_{\tilde{\Lambda}}\mu^2 L^2 + k^2\right) X = 0. \tag{4.39}$$

Thus, for $\tilde{\Lambda} > 0$, the general solution is given by

$$X = C_1 I_k(L\mu) + C_2 K_k(L\mu). \tag{4.40}$$

Again, for $0 \leq k^2 < 1/4$, we have the normalized function $X(L)$ given by

$$X_{\text{norm}} = \frac{1}{\pi}\sqrt{\frac{4\mu}{\sec(\pi k)}} K_k(L\mu). \tag{4.41}$$

The same discussion from above applies to $Y(\phi)$.

When $\tilde{\Lambda} = 0$, its general solution is

$$X = A_1 L^k + A_2 L^{-k}, \tag{4.42}$$

while for $\tilde{\Lambda} < 0$, it is given by

$$X = B_1 J_k(\mu L) + B_2 Y_k(\mu L). \tag{4.43}$$

It can be shown that none of these two wavefunctions are normalizable in the interval $(0, +\infty)$.

4.2.2.3. $:\pi^2 L: = \hat{\pi^2}\hat{L}$ In this case, we have

$$L^2 X'' + 2LX' - \left(\epsilon_{\tilde{\Lambda}}\mu^2 L^2 + k^2\right) X = 0. \tag{4.44}$$

Then, for $\tilde{\Lambda} > 0$, we find

$$X = C_1 j_{-\nu-1/2}(-iL\mu) + C_2 y_{-\nu-1/2}(-iL\mu), \tag{4.45}$$

here j_ν, y_ν denote the spherical Bessel functions of the first and second kind. When $\tilde{\Lambda} = 0$, we find that

$$X = L^{-1/2}\left(A_1 L^\nu + A_2 L^{-\nu}\right), \tag{4.46}$$

while for $\tilde{\Lambda} < 0$, we have

$$X = C_1 j_{\nu-1/2}(\mu L) + C_2 y_{\nu-1/2}(\mu L). \tag{4.47}$$

It can be shown that in this case none of these wave functions are normalizable with respect to the measure LdL in the interval $(0, +\infty)$.

Just like in the pure gravity part, in the $L(t) = 0$ case, we can again assume $\gamma(t,x) = \hat{\gamma}(x)\gamma(t)$, then requiring $\hat{\gamma}(x)$ to be an odd function, in this condition, the constraints (4.3) and (4.5) are automatically satisfied, then from the equations of motion (4.4) and (4.6), we are able to deduce the corresponding Lagrange

$$L = \frac{\dot{\gamma}^2}{\gamma} + 2\kappa\gamma\dot{\phi}^2 - 4\tilde{\Lambda}\gamma - 4\kappa\gamma V(\phi), \tag{4.48}$$

Here p is the canonical momentum conjugate to γ, P is conjugate momentum to ϕ. After Legendre transformation, the Hamilton reads

$$H = \frac{p^2}{4}\gamma + \frac{P^2}{8\kappa\gamma} + 4\tilde{\Lambda}\gamma + 4\kappa\gamma V(\phi), \tag{4.49}$$

which has the similar form as the Hamilton constraint (4.29).

4.2.3 Two Interacting Simple Harmonic Oscillators

Similar to what we have done in the pure gravity case, we can also treat the Hamilton given by Eq. (4.29) as consisting of harmonic oscillators. To this goal, let us first make the transformations

$$\begin{aligned} L(t) &= y_1^2(t) - y_2^2(t), \\ \phi(t) &= \sqrt{2\zeta^2(\lambda-1)}\ln\left(\frac{y_1(t)+y_2(t)}{y_1(t)-y_2(t)}\right), \end{aligned} \tag{4.50}$$

for which we are able to convert Eq. (4.26) into the form

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}m\Big[\left(\dot{y}_1^2-\omega^2y_1^2\right)-\left(\dot{y}_2^2-\omega^2y_2^2\right)\Big] \\ &\quad -V_e(y_1,y_2), \end{aligned} \tag{4.51}$$

but now with

$$\begin{aligned} &m \equiv 8(1-\lambda)\zeta^2, \quad \omega^2 \equiv \frac{\Lambda}{2(1-\lambda)}, \\ &V_e(y_1,y_2) \equiv \left(y_1^2-y_2^2\right)V\Big(\phi(y_1,y_2)\Big). \end{aligned} \tag{4.52}$$

Clearly, Eq. (4.51) describes the interaction between two simple harmonic oscillators, one with positive energy and the other with negative energy. Thus, in order for the system to have a total positive energy, the interaction between them is important.

4.3 2D Nonprojectable Hořava-Lifshitz Gravity Coupled with a Scalar Field

For the nonprojectable case, the gravitational action of Hořava-Lifshitz gravity is given in the last chapter by Eq. (3.2) which is

$$S_{HL} = \zeta^2\int dt\,dx\,N\gamma\left[(1-\lambda)K^2-2\Lambda+\beta a_1a^1\right], \tag{4.53}$$

where $a_1 = (\ln N)'$, and

$$K = -\frac{1}{N}\left(\frac{\dot{\gamma}}{\gamma}-\frac{N_1'}{\gamma^2}+\frac{N_1\gamma'}{\gamma^3}\right), \tag{4.54}$$

with $\gamma' \equiv \partial\gamma/\partial x$, etc. On the other hand, the action for a non-relativistic scalar field takes the form,

$$\begin{aligned} S_{\phi} &= \int dt\, dx\, N\sqrt{g}\Bigg\{\frac{1}{2}\left(\partial_{\perp}\phi\right)^2 - \alpha_0\left(\nabla_i\phi\right)^2 \\ &\qquad\qquad -V(\phi) - f(\phi)R\Bigg\}, \end{aligned} \tag{4.55}$$

where $\partial_{\perp} \equiv N^{-1}(\partial_t - N^i\nabla_i)$, α_0 is a dimensionless coupling constant. In the relativistic case, it is equal to $1/2$. The function $f(\phi)$ is arbitrary and depends on ϕ only, and R denotes the Ricci scalar of the 2d spacetimes. The total action is

$$S = S_{HL} + S_{\phi} = \zeta^2 \int dt\, dx\, N\sqrt{g}\mathcal{L}. \tag{4.56}$$

4.3.1 Hamiltonian Structure

The 2d spacetimes are described by the general metric

$$ds^2 = -N^2dt^2 + \gamma^2\big(dx + N^1dt\big)^2, \tag{4.57}$$

subjected to the gauge freedom (1.18), where N, N^1 and γ are in general functions of t and x. To be as much general as possible, we shall not impose any gauge conditions in this section. Then, in terms of N, N_1 and γ, the matter action takes the form

$$\begin{aligned} S_{\phi} &= \int dtdxN\gamma\left\{\frac{1}{2N^2}\left(\dot{\phi} - \frac{N_1\phi'}{\gamma^2}\right)^2 - \frac{\alpha_0}{\gamma^2}\phi'^2 \right. \\ &\qquad \left. -V(\phi) - f(\phi)R\ \right\}, \end{aligned} \tag{4.58}$$

where

$$R = \frac{2}{N\gamma}\left[\partial_{\mu}(N\gamma n^{\mu}K) - \left(\frac{N'}{\gamma}\right)'\right]. \tag{4.59}$$

Here $n^{\mu} \equiv N^{-1}(1, -N^1)$ denotes the normal vector to the hypersurfaces $t =$ Constant. Then, we find

$$\begin{aligned} \pi_N &\equiv \frac{\partial\mathcal{L}}{\partial\dot{N}} = 0, \quad \pi_{N_1} \equiv \frac{\partial\mathcal{L}}{\partial\dot{N}_1} = 0, \\ \pi &= \frac{\partial\mathcal{L}}{\partial\dot{\gamma}} = 2K(\lambda - 1) - 2f'\frac{\dot{\phi}}{N} + 2f'\frac{\phi' N_1}{N\gamma^2}, \\ \pi_{\phi} &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\gamma}{N}(\dot{\phi} - N_1\frac{\phi'}{\gamma^2}) + 2f'\gamma K. \end{aligned} \tag{4.60}$$

After a Legendre transformation, it can be shown that the Hamiltonian can be cast into the form

$$\mathcal{H}_0 = N\mathcal{H} + N_1\mathcal{H}^1 - 2\beta\left(\frac{N'}{\gamma}\right)', \tag{4.61}$$

where

$$\begin{aligned}
\mathcal{H}^1 &= -\frac{\pi'}{\gamma} + \frac{\pi_\phi\phi'}{\gamma^2}, &(4.62)\\
\mathcal{H} &= -\frac{\pi_\phi\pi}{2f'} + \frac{(\lambda-1)\pi_\phi}{f'}K + (1-\lambda)K^2\gamma \\
&\quad +2\Lambda\gamma + \alpha_0\frac{\phi'^2}{\gamma} - \frac{\gamma}{2}\left(\frac{\pi_\phi}{\gamma} - 2f'K\right)^2 \\
&\quad +\gamma V(\phi) - 2\left(\frac{f'\phi'}{\gamma}\right)' \\
&\quad +\beta\frac{N'^2}{N\gamma} + 2\beta\left(\frac{N'}{N\gamma}\right)'. &(4.63)
\end{aligned}$$

Here K can be expressed in terms of the canonical fields and their momenta

$$K = \frac{\pi\gamma + 2f'\pi_\phi}{4\gamma f'^2 - 2\gamma(1-\lambda)}. \tag{4.64}$$

A straightforward evaluation of poisson brackets between momentum constraints shows

$$\left\{\mathcal{H}^1(x), \mathcal{H}^1(x')\right\} = \left(\frac{\mathcal{H}^1(x')}{\gamma^2(x')} + \frac{\mathcal{H}^1(x)}{\gamma^2(x)}\right)\partial_{x'}\delta(x-x'), \tag{4.65}$$

which is the same as in the pure gravity Eq. (3.21). The poisson bracket between $\mathcal{H}$ and $\mathcal{H}^1$ will not vanish on the constraint surface because of the terms related to the lapse function N in the Hamiltonian constraint $\mathcal{H}$. Therefore, we need to redefine the momentum constraint by adding a term proportional to the primary constraint π_N, which generates the diffeomorphisms of N,

$$\tilde{\mathcal{H}}^1 = \mathcal{H}^1 + \frac{N'}{\gamma^2}\pi_N. \tag{4.66}$$

In principle, one can also add a term generating diffeomorphisms of N_1. However, in the present case, since the Hamiltonian constraint doesn't depend on N_1, this term is not mandatory. In terms of $\tilde{\mathcal{H}}^1$, the structure of Eq. (4.65) will not change, while one can show that $\tilde{\mathcal{H}}^1$ now commutes with $\mathcal{H}$ on the constraint surface,

$$\begin{aligned}\left\{\tilde{\mathcal{H}}^1(x), \mathcal{H}(x')\right\} &= -\left(4\mathrm{c}\pi + \frac{2\mathrm{b}\pi_\phi}{\gamma}\right)\tilde{\mathcal{H}}^1(x)\delta(x-x') \\ &\quad + \frac{\mathcal{H}(x)}{\gamma^2(x)}\partial_x\delta(x-x'). \end{aligned} \tag{4.67}$$

Here $\mathrm{c} \equiv -\alpha/2 - 2\xi^2\alpha^2$ and $\mathrm{b} \equiv \alpha\xi(2\beta-1) - \frac{1}{2\xi}[1+2\alpha(1-\lambda)]$, where $\alpha^{-1} \equiv 4\xi^2 + 2(\lambda - 1)$. Note that in writing down the above expression, we had set $f(\phi) = \xi\phi$ for the sake of simplicity. Thus, the total Hamiltonian of the coupled system can be written as

$$\mathcal{H}_t = N\mathcal{H} + N_1\tilde{\mathcal{H}}^1 + \sigma\pi_N + \sigma_1\pi_{N_1}. \tag{4.68}$$

For this coupled system, there are two first-class constraints $\tilde{\mathcal{H}}^1$ and π_{N_1}, and two second-class constraints $\mathcal{H}$ and π_N.

Note that no other constraints will be generated by the EOM of the said four constraints because the secondary constraint $\tilde{\mathcal{H}}^1$ will not give rise to any tertiary constraints due to Eqs. (4.65) and (4.67), while on the other hand the preservation of $\mathcal{H}$ will only produce two differential equations for the lapse function N and Lagrange multiplier σ since $\mathcal{H}$ is a second-class constraint. Thus, the Dirac procedure of finding all the constraints in the Hamiltonian formulation terminates at the level of secondary constraints, and the physical degrees of freedom in the configuration space is one which is due to the introduction of the scalar field into the whole system, while in the pure gravity case it is zero as shown in Eq. (3.24).

4.3.2 Field Equations

The variations of the total action S with respect to N, N_1, γ and ϕ, yield, respectively,

$$
\begin{aligned}
&(1-\lambda)\gamma K^2 + 2\beta\left(\frac{N'}{N\gamma}\right)' + \frac{\beta N'^2}{N^2\gamma} + \gamma\left(2\Lambda + V\right) \\
&\quad + \frac{\gamma}{2N^2}\left(\dot{\phi} - \frac{N_1\phi'}{\gamma^2}\right)^2 + \frac{\alpha_0\phi'^2}{\gamma} \\
&\quad + \frac{2K}{N}\left(f'\dot{\phi}\gamma - \frac{f'\phi' N_1}{\gamma}\right) - \left(\frac{2f'\phi'}{\gamma}\right)' = 0,
\end{aligned}
\tag{4.69}
$$

$$
\begin{aligned}
&\frac{2(1-\lambda)K'}{\gamma} + \frac{\phi'}{N\gamma}\left(\dot{\phi} - \frac{N_1\phi'}{\gamma^2}\right) \\
&\quad + \frac{2f'\phi' K}{\gamma} + \left(\frac{2f'\dot{\phi}}{N\gamma} - \frac{2f'\phi' N_1}{N\gamma^3}\right)' \\
&\quad + \frac{2\gamma'}{N\gamma^3}\left(f'\dot{\phi}\gamma - \frac{f'\phi' N_1}{\gamma}\right) = 0,
\end{aligned}
\tag{4.70}
$$

$$
\begin{aligned}
&2(1-\lambda)\left(\dot{K} + \frac{N_1 K'}{\gamma^2} - \frac{NK^2}{2}\right) - \frac{\beta N'^2}{N\gamma^2} \\
&\quad + \frac{1}{2N}\left(\dot{\phi} - \frac{N_1\phi'}{\gamma^2}\right)^2 + \frac{2N_1\phi'}{N\gamma^2}\left(\dot{\phi} - \frac{N_1\phi'}{\gamma^2}\right) \\
&\quad - N(2\Lambda + V) + 2f'\dot{\phi}K + 2f'\phi'\frac{N_1 K}{\gamma^2} \\
&\quad + 2f'\phi'\frac{N'}{\gamma^2} + \alpha_0\phi'^2\frac{N}{\gamma^2} - 2K\left(f'\dot{\phi} - \frac{f'\phi' N_1}{\gamma^2}\right) \\
&\quad + \left(\frac{2f'\dot{\phi}\gamma}{N\gamma} - \frac{2f'\phi' N_1}{N\gamma^2}\right)_{,t} \\
&\quad - \frac{2N_1'}{\gamma^2}\left(f'\dot{\phi}\gamma - f'\phi'\frac{N_1}{\gamma}\right) \\
&\quad + \frac{4N_1\gamma'}{N\gamma^4}\left(f'\dot{\phi}\gamma - f'\phi'\frac{N_1}{\gamma}\right) \\
&\quad + \left(\frac{2N_1 f'\dot{\phi}}{N\gamma^2} - \frac{4f'\phi' N_1^2}{N\gamma^4}\right)' = 0,
\end{aligned}
\tag{4.71}
$$

$$\left(\frac{\gamma\dot{\phi}}{N} - \frac{N_1\phi'}{N\gamma}\right)_{,t} - \left(\frac{N_1\dot{\phi}}{N\gamma} - \frac{N_1^2\phi'}{N\gamma^3}\right)' - 2\alpha_0\left(\frac{N\phi'}{\gamma}\right)'$$
$$+N\gamma V' - 2f''\dot{\phi}\gamma K + 2(f'\gamma K)^{\cdot} + 2f''\phi'\frac{N_1 K}{\gamma}$$
$$-2\left(\frac{f' N_1 K}{\gamma}\right)' + 2f''\phi'\frac{N'}{\gamma} - 2\left(\frac{f'N'}{\gamma}\right)' = 0. \tag{4.72}$$

Here $f'(\phi) \equiv df(\phi)/d\phi$, etc. Note Eqs. (4.69)-(4.72) hold for any function $f(\phi)$.

4.4 Stationary Spacetimes

In this section, we will study stationary spacetimes of the 2d Hořava gravity coupled with a non-relativistic scalar field, presented in the last section. Setting all the time derivative terms to zero in Eqs. (4.69)-(4.72), and

$$f(\phi) = \xi\phi, \tag{4.73}$$

where ξ is a constant, we find that

$$(1-\lambda)\gamma K^2 + 2\beta\left(\frac{N'}{N\gamma}\right)' + \frac{\beta N'^2}{N^2\gamma} + \frac{N_1^2\phi'^2}{2N^2\gamma^3}$$
$$+ \frac{\alpha_0\phi'^2}{\gamma} + \gamma(2\Lambda + V) - \frac{2K\xi\phi' N_1}{N\gamma}$$
$$- \left(\frac{2\xi\phi'}{\gamma}\right)' = 0, \tag{4.74}$$

$$\frac{2(1-\lambda)K'}{\gamma} - \frac{N_1\phi'^2}{N\gamma^3} + \frac{2\xi\phi' K}{\gamma}$$
$$- \left(\frac{2\xi\phi' N_1}{N\gamma^3}\right)' - \frac{2\xi\phi'\gamma' N_1}{N\gamma^4} = 0, \tag{4.75}$$

$$2(1-\lambda)\left(\frac{N_1 K'}{\gamma^2} - \frac{NK^2}{2}\right) - \frac{\beta N'^2}{N\gamma^2} - \frac{3N_1^2\phi'^2}{2N\gamma^4}$$
$$+ \alpha_0\phi'^2\frac{N}{\gamma^2} + 4\xi\phi'\frac{N_1 K}{\gamma^2} + 2\xi\phi'\frac{N'}{\gamma^2}$$
$$- N(2\Lambda + V) + \frac{2\xi N_1'\phi' N_1}{\gamma^3}$$
$$- \frac{4\xi\gamma'\phi' N_1^2}{N\gamma^5} - \left(\frac{4\xi\phi' N_1^2}{N\gamma^4}\right)' = 0, \tag{4.76}$$

$$\left(\frac{N_1^2\phi'}{N\gamma^3}\right)' - 2\alpha_0\left(\frac{N\phi'}{\gamma}\right)' + N\gamma V'$$
$$- 2\xi\left(\frac{N_1 K}{\gamma}\right)' - 2\xi\left(\frac{N'}{\gamma}\right)' = 0. \tag{4.77}$$

4.4.1 Diagonal Solutions

When the metric is diagonal, we have

$$N_1 = 0, \tag{4.78}$$

so the extrinsic curvature K vanishes and Eq. (4.75) holds identically, while Eqs. (4.74), (4.76) and (4.77) reduce, respectively, to

$$2\beta\left(\nu'' - \nu'\mu'\right) + \beta\nu'^2 - 2\xi\left(\phi'' - \phi'\mu'\right) + \alpha_0\phi'^2$$
$$= -(V + 2\Lambda)e^{2\mu}, \tag{4.79}$$

$$\beta\nu'^2 - 2\xi\phi'\nu' - \alpha_0\phi'^2 = -(V + 2\Lambda)e^{2\mu}, \tag{4.80}$$

$$2\xi\left(\nu'' + \nu'^2 - \nu'\mu'\right) + 2\alpha_0\left(\phi'' - \phi'\mu' + \nu'\phi'\right)$$
$$= e^{2\mu}V', \tag{4.81}$$

where $\nu \equiv \ln N$ and $\mu \equiv \ln\gamma$.

It should be noted that static diagonal solutions were studied recently in [68] with $\Lambda = 0 = \xi$. However, comparing the above equation (4.79) with Eq. (12) given in [68], it can be seen that the second-order derivative term ν'' (or N'') is missing there. This is because, when taking the variation of the total action with respect to N, the authors of [68] incorrectly assumed that a_1 is independent of N. Unfortunately, as a result, all the solutions resulted from Eq. (12) given in [68] in general are not solutions of the field equations of the 2d Hořava gravity coupled with a non-relativistic scalar field.

Using the gauge freedom given by Eq. (1.18), without loss of the generality, we can always set $\mu = -\nu$, that is

$$N = \frac{1}{\gamma} = e^{\nu}. \tag{4.82}$$

To solve Eqs. (4.79)-(4.81), let us further consider the case where $V = -2\Lambda$, so that Eqs. (4.79) - (4.81) reduce to

$$\begin{aligned} 2\beta\left(\nu'' + \nu'^2\right) + \beta\nu'^2 - 2\xi\left(\phi'' + \phi'\nu'\right) \\ + \alpha_0\phi'^2 = 0, \end{aligned} \tag{4.83}$$

$$\beta\nu'^2 - 2\xi\phi'\nu' - \alpha_0\phi'^2 = 0, \tag{4.84}$$

$$\nu'' + 2\nu'^2 + \frac{\alpha_0}{\xi}\left(\phi'' + 2\nu'\phi'\right) = 0. \tag{4.85}$$

Then, from Eqs. (4.83) and (4.84) we find that

$$\nu'' + 2\nu'^2 - \frac{\xi}{\beta}\left(\phi'' + 2\nu'\phi'\right) = 0. \tag{4.86}$$

Thus, Eqs. (4.85) and (4.86) show that there are two possibilities,

$$(i)\ \alpha_0\beta + \xi^2 \neq 0; \quad (ii)\ \alpha_0\beta + \xi^2 = 0. \tag{4.87}$$

4.4.1.1. $\alpha_0\beta + \xi^2 \neq 0$ In this case we must have

$$\nu'' + 2\nu'^2 = 0, \tag{4.88}$$

$$\phi'' + 2\nu'\phi' = 0, \tag{4.89}$$

which have the solutions

$$\begin{aligned} N &= \sqrt{C_0 x + C_1}, \\ \phi &= \phi_0 \ln\left(C_0 x + C_1\right) + \phi_1, \end{aligned} \tag{4.90}$$

where C_i and ϕ_i are the integration constants. Without loss of the generality, we can always set $C_0 = 1$, so the metric and scalar field finally take the form

$$\begin{aligned} ds^2 &= -\left(x - x_0\right) dt^2 + \frac{dx^2}{x - x_0}, \\ \phi &= \phi_0 \ln\left(x - x_0\right) + \phi_1, \end{aligned} \tag{4.91}$$

where $x_0 \equiv -C_1$. Clearly, the scalar field is singular at $x = x_0$, so is the corresponding spacetime.

4.4.1.2. $\alpha_0\beta + \xi^2 = 0$ In this case, there are only two independent equations which are Eqs. (4.84) and (4.85). Now if substituting the relation $\alpha_0 = -\xi^2/\beta$ into these equations and defining a new constant $\kappa = \xi/\beta$, one can easily arrive at

$$\nu'^2 - 2\kappa\phi'\nu' + \kappa^2\phi'^2 = 0, \tag{4.92}$$

$$\nu'' + \nu'^2 - \kappa\phi'' - \kappa^2\phi'^2 = 0. \tag{4.93}$$

The first equation tells us that ν' and ϕ' are linearly dependent, that is

$$\nu = \frac{\xi}{\beta}\left(\phi - \phi_0\right), \tag{4.94}$$

which also makes the second equation hold identically, where ϕ_0 is a constant. Therefore, in the current case *for any chosen* ϕ*, the solution (4.94) will satisfy the field equations (4.83)-(4.85). The corresponding metric takes the form*

$$ds^2 = -e^{\frac{2\xi(\phi-\phi_0)}{\beta}}dt^2 + e^{-\frac{2\xi(\phi-\phi_0)}{\beta}}dx^2, \tag{4.95}$$

for $\alpha_0 = -\xi^2/\beta$*.*

4.4.2 Non-Diagonal Solutions

In this case, using the gauge transformations (1.18), without loss of generality, we can always set

$$\gamma = 1, \tag{4.96}$$

so the metric takes the form

$$ds^2 = -N^2(x)dt^2 + \big(dx + h(x)dt\big)^2. \tag{4.97}$$

Then, Eqs. (4.74)-(4.77) reduce to

$$
\begin{aligned}
&(1-\lambda)K^2 + 2\beta\left(\frac{N'}{N}\right)' + \frac{\beta N'^2}{N^2} + 2\Lambda + V(\phi) + \frac{h^2\phi'^2}{2N^2} \\
&\qquad + \alpha_0\phi'^2 - \frac{2K\xi\phi' h}{N} - 2\xi\phi'' = 0, && (4.98)\\
&2(1-\lambda)K' - \frac{h\phi'^2}{N} + 2\xi\phi' K - \left(\frac{2\xi\phi' h}{N}\right)' = 0, && (4.99)\\
&2(1-\lambda)\left(hK' - \frac{NK^2}{2}\right) - \frac{\beta N'^2}{N} - \frac{3h^2\phi'^2}{2N} \\
&\qquad - N(2\Lambda + V) + \alpha_0\phi'^2 N + 4\xi\phi' hK \\
&\qquad + 2\xi\phi' N' + 2\xi h'\phi' h - \left(\frac{4\xi\phi' h^2}{N}\right)' = 0, && (4.100)\\
&\left(\frac{h^2\phi'}{N}\right)' - 2\alpha_0(N\phi')' + NV' - 2\xi(hK)' - 2\xi N'' = 0, \\
& && (4.101)
\end{aligned}
$$

where

$$
K = \frac{h'}{N}. \tag{4.102}
$$

To solve the above equations, in the following we shall consider some particular cases.

4.4.2.1. $N(x) = 1$ In this case, let us first consider the solution with $\phi = \phi_0$, where ϕ_0 is a constant. Then, from Eq. (4.98) we find that

$$
h'^2 = \frac{2\hat{\Lambda}}{\lambda - 1}, \tag{4.103}
$$

where $\hat{\Lambda} \equiv \Lambda + V(\phi_0)/2$. The above equation has the solution

$$
h(x) = \pm\sqrt{\frac{2\hat{\Lambda}}{\lambda - 1}}\, x = \pm\eta x. \tag{4.104}
$$

It can be shown that in this case a killing horizon exists, located at $x_{KH} = \pm\eta^{-1}$.

4.4.2.2. $\xi = 0$ When $\xi = 0$, Eqs. (4.98)-(4.101) reduces to

$$(1-\lambda)\left(\frac{h'}{N}\right)^2 + 2\beta\left(\frac{N'}{N}\right)' + \frac{\beta N'^2}{N^2} + \hat{V}$$
$$+ \alpha_0\phi'^2 + \frac{h^2\phi'^2}{2N^2} = 0, \tag{4.105}$$

$$2(1-\lambda)\left(\frac{h''}{N} - \frac{h'N'}{N^2}\right) - \frac{h\phi'^2}{N} = 0, \tag{4.106}$$

$$2(1-\lambda)\left(\frac{h''}{N} - \frac{h'N'}{N^2} - \frac{h'^2}{2hN}\right) - \frac{\beta N'^2}{hN} - \frac{3h\phi'^2}{2N}$$
$$+ \frac{N}{h}\left(\alpha_0\phi'^2 - \hat{V}\right) = 0, \tag{4.107}$$

$$\left(\frac{h^2\phi'}{N}\right)' - 2\alpha_0(N\phi')' + N\hat{V}' = 0, \tag{4.108}$$

where $\hat{V} \equiv V + 2\Lambda$. To solve the above equations, let us consider the case

$$N = h, \quad \hat{V} = 0, \tag{4.109}$$

for which the above equations reduce to

$$2\beta\nu'' + (1-\lambda+\beta)\nu'^2 = -\frac{1+2\alpha_0}{2}\phi'^2, \tag{4.110}$$

$$2(1-\lambda)\nu'' = \phi'^2, \tag{4.111}$$

$$2(1-\lambda)\nu'' - (1-\lambda+\beta)\nu'^2 = \frac{3-2\alpha_0}{2}\phi'^2, \tag{4.112}$$

$$(1-2\alpha_0)\left(e^{\nu}\phi'\right)' = 0, \tag{4.113}$$

where $\nu = \ln N$. To solve the above equations, let us consider the cases $\alpha_0 = 1/2$ and $\alpha_0 \neq 1/2$ separately.

Case B.2.1) $\alpha_0 = 1/2$: This is the relativistic case, and Eq. (4.113) is satisfied identically, while from Eqs. (4.110) and (4.112), we find

$$(1-\lambda+\beta)\nu'' = 0. \tag{4.114}$$

If $\lambda \neq \beta + 1$, it can be shown that the above equations have only the trivial solution in which ν and ϕ are all constants. On the other hand, when $\lambda = \beta + 1$, Eqs. (4.110)-(4.112) reduce to a single equation

$$2\beta\nu'' = -\phi'^2, \ (\beta = \lambda - 1). \tag{4.115}$$

for the two arbitrary functions ν and ϕ. Again, similar to Case A.2 considered in the last subsection, the solutions are not uniquely determined. In fact, for any given ϕ, the solution

$$\nu(x) = -\frac{1}{2\beta}\int^{x} dx' \int^{x'} \phi'^2(x'')dx'' + C_1 x + C_0, \tag{4.116}$$

will satisfy the field equations (4.110) and (4.112), where C_1 and C_0 are two integration constants.

Case B.2.2) $\alpha_0 \neq 1/2$: In this case, from Eq. (4.113) we find

$$\phi' = C_0 e^{-\nu}, \tag{4.117}$$

where C_0 is another constant. Substituting it into Eq. (4.111), we obtain

$$NN'' - N'^2 + \mathcal{D} = 0, \tag{4.118}$$

where $\mathcal{D} \equiv C_0^2/(2(\lambda-1))$. The above equation has two particular solutions

$$\begin{aligned} N_A(x) &= \frac{1}{2C_1^2}e^{C_1(x+C_2)} - \frac{\mathcal{D}}{2}e^{-C_1(x+C_2)}, & (4.119)\\ N_B(x) &= \frac{1}{2C_1^2}e^{-C_1(x+C_2)} - \frac{\mathcal{D}}{2}e^{C_1(x+C_2)}, & (4.120)\end{aligned}$$

where C_1 and C_2 are two integration constants. Correspondingly, the scalar field ϕ is given, respectively, by

$$\begin{aligned} \phi_A(x) &= -\frac{2}{\sqrt{\mathcal{D}}}\tanh^{-1}\left(\frac{e^{C_1(C_2+x)}}{\sqrt{\mathcal{D}}C_1}\right), & (4.121)\\ \phi_B(x) &= \frac{2}{\sqrt{\mathcal{D}}}\tanh^{-1}\left(C_1\sqrt{\mathcal{D}}e^{C_1(C_2+x)}\right). & (4.122)\end{aligned}$$

4.5 Universal Horizons and Hawking Radiation

In this section, we shall consider two issues, universal horizons and the corresponding Hawking radiations. As a representative case, we shall focus on the solution given by Eqs. (4.97) and (4.104) with $N = 1$. Without loss of the generality, we consider only the case with "-" sign, that is

$$\begin{aligned} ds^2 &= -dt^2 + (dx - \eta x dt)^2 \\ &= -\left(1-\eta^2x^2\right)dt^2 - 2\eta x dt dx + dx^2, & (4.123)\end{aligned}$$

where $-\infty < t, x < \infty$. The corresponding inverse metric is given by

$$g^{tt} = -1, \quad g^{tx} = -\eta x, \quad g^{xx} = 1 - \eta^2 x^2, \tag{4.124}$$

which is non-singular, except at the infinities $x = \pm\infty$. The latter are coordinate singularities, similar to the 4d de Sitter space. In fact, the extrinsic curvature and 2d Ricci scalar are all finite, and given by $-\eta$ and $2\eta^2$, respectively. However, there exist two cosmological Killing horizons located, respectively, at $x_{KH} = \pm\eta^{-1}$. Similar to the 4d de Sitter space, the time-translation Killing vector, $\xi^\mu = \delta^\mu_t$, is time-like only in the region $x^2 < x^2_{KH}$. In the regions $x^2 > x^2_{KH}$, the Killing vector becomes spacelike, and only in these regions can the universal horizon exist, as the latter is defined by [13],

$$(\xi \cdot u) = 0. \tag{4.125}$$

Since the four-velocity u of the khronon field is always time-like, Eq. (4.125) has solutions only when ξ becomes spacelike, which are the regions in which $x^2 > x^2_{KH}$ holds.

To see the difference between the physics at Killing horizons and that at universal horizons, let us first consider Hawking radiation at the Killing horizon.

4.5.1 Hawking Radiation at the Killing Horizon

As shown in [69], at a Killing horizon only relativistic particles are radiated quantum mechanically. So, in this subsection we consider only the relativistic limit in which the dispersion relation of radiated massless scalar particles satisfies $k^2 \equiv k_\lambda k^\lambda = 0$. Considering only the positive outgoing particles, $k_t = -\omega < 0$, we find

$$k_x^\pm = \frac{\omega(h \pm 1)}{1 - h^2}\ , \tag{4.126}$$

which is singular for k_x^+ at the Killing horizon at which we have $h\left(x_{KH}\right) = 1$. Then, from the following formula [69],

$$2\mathrm{ImS} = \mathrm{Im}\oint \mathrm{k_x^+ dx} = \frac{\omega}{\mathrm{T_{KH}}}, \tag{4.127}$$

we find that

$$T_{KH} = -\frac{h'(x_{KH})}{2\pi} = \frac{\eta}{2\pi}, \tag{4.128}$$

where $x_{KH} = -\eta^{-1}$. On the other hand, the surface gravity at the Killing horizon is given by [11]

$$\begin{aligned} \kappa_{KH} &\equiv \sqrt{-\frac{1}{2}\left(D_\mu \chi_\nu\right)\left(D^\mu \chi^\nu\right)} \\ &= \eta, \end{aligned} \tag{4.129}$$

where D_μ denotes the covariant derivative with respect to the 2d metric $g_{\mu\nu}$, and $\chi^\mu = \delta^\mu_t$ is the timelike Killing vector. Therefore, the standard form

$$T_{KH} = \frac{\kappa_{KH}}{2\pi}, \tag{4.130}$$

holds.

4.5.2 Universal Horizons and Hawking Radiation

The existence of a universal horizon is closely related to the existence of a globally defined timelike scalar field φ [13,66],

$$u_\mu = \frac{\partial_\mu \varphi}{\sqrt{-g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi}}, \quad u_\lambda u^\lambda = -1, \tag{4.131}$$

where the equation of φ is given by the action [58]

$$\begin{aligned} S_u &= \int dt dx N\gamma\Big[\frac{\kappa_1}{2}F^{\alpha\beta}F_{\alpha\beta} + \kappa_2(D_\alpha u^\alpha)^2 \\ &\qquad + \sigma(u^\alpha u_\alpha + 1)\Big], \end{aligned} \tag{4.132}$$

where $F_{\alpha\beta} \equiv D_\alpha u_\beta - D_\beta u_\alpha$, σ is a Lagrange multiplier, and $\kappa_{1,2}$ are two coupling constants. It should be noted that the action (4.132) remains unchanged under the transformations

$$\varphi = \mathcal{F}(\tilde{\varphi}), \tag{4.133}$$

where $\mathcal{F}(\tilde{\varphi})$ is a monotonically increasing or decreasing function of $\tilde{\varphi}$ only. In the following, we shall use this property to choose $\mathcal{F}(\tilde{\varphi})$ so that $d\varphi$ is along the same direction as dt in the regions we are interested in.

Under the background (4.123), we find that the equations of motion are given by

$$\kappa_1(1-\eta^2x^2)u_0'' - \sigma u_0 = 0, \tag{4.134}$$

$$\kappa_1\eta x u_0'' + \kappa_2(u^1)'' - \sigma u_1 = 0, \tag{4.135}$$

$$u_0^2 + 2\eta x u_0 u_1 - (1-\eta^2x^2)u_1^2 - 1 = 0. \tag{4.136}$$

Generally, these coupled non-linear equations are difficult to solve. One simple solution can be obtained when $\kappa_1 = 0$, in which we find $\sigma u_0 = 0$. Since $u_0 \neq 0$ we must have $\sigma = 0$, and Eqs. (4.134)-(4.136) have the solution [1]

$$\begin{aligned} u^0 &= \frac{\eta x u^1 - \sqrt{G(x)}}{\eta^2x^2-1}, \quad u^1 = cx + d, \\ G(x) &\equiv \left(c^2-\eta^2\right)x^2 + 2cdx + \left(d^2+1\right), \end{aligned} \tag{4.137}$$

or inversely

$$\begin{aligned} u_0 &= -\sqrt{G(x)}, \\ u_1 &= \frac{-(cx+d) + \eta x\sqrt{G(x)}}{\eta^2x^2-1}, \end{aligned} \tag{4.138}$$

where c and d are two integration constants. In asymptotically flat spacetimes, these two constants can be determined by requiring that [64, 66]: (a) it be aligned asymptotically with the time translation Killing vector; and (b) the khronon have a regular future sound horizon. However, the spacetime we are studying is asymptotically de Sitter, and these conditions cannot be applied to the present case. Instead, we shall leave this possibility open, as long as it allows a globally defined khronon field φ. Since only the latter is essential for the existence of the universal horizon, as explained previously in the Introduction Sec. 4.1. Then, one may ask what is their physical meanings. To see these, let us first calculate the quantity

$$\nabla_\alpha u_\beta = c s_\alpha s_\beta + \hat{c} u_\alpha s_\beta, \tag{4.139}$$

[1] Eq. (4.136) is a quadratic equation for u_0, so in general it has two solutions. In the following we shall consider only the one with the minus sign, as the one with the plus sign will give the same results.

where

$$\hat{c} \equiv \frac{x\eta^2 - c(cx+d)}{\sqrt{1+(cx+d)^2 - x^2\eta^2}}. \tag{4.140}$$

Thus, c is directly related to the expansion of the aether. In fact, we have $\theta \equiv g^{\alpha\beta}\nabla_\alpha u_\beta = c$. On the other hand, assuming that the aether is moving alone the trajectory $x^\mu = x^\mu(\tau)$, where τ is the proper time measured by aether, from Eq. (4.137) we find

$$u^1 \equiv \left.\frac{dx(\tau)}{d\tau}\right|_{c=0} = d, \tag{4.141}$$

that is, the parameter d is directly related to the constant part of the velocity of the aether.

In order to have the solution (4.137) well-defined for all the values of $x \in (-\infty, \infty)$, we must assume that $G(x) \geq 0$, which yields

$$c^2 \geq \left(1+d^2\right)\eta^2. \tag{4.142}$$

On the other hand, the universal horizon is located at [13], $(u \cdot \xi) = -\sqrt{G(x)} = 0$. Since $G(x) \geq 0$ for $x \in (-\infty, \infty)$, we must have [70],

$$G\left(x_{UH}\right) = 0, \qquad \left.\frac{dG(x)}{dx}\right|_{x=x_{UH}} = 0, \tag{4.143}$$

at the universal horizon $x = x_{UH}$. Inserting Eq. (4.137) into the above equations, we find that

$$c = \epsilon_c \eta\sqrt{1+d^2}, \quad x_{UH} = -\epsilon_c \frac{\sqrt{1+d^2}}{\eta d}, \tag{4.144}$$

where $\epsilon_c = \text{Sign}(c)$. It is interesting to note that the above solution for c saturates the bound of Eq. (4.142). We also note that

$$x_{UH}^2 - x_{KH}^2 = \frac{1}{(\eta d)^2} > 0, \tag{4.145}$$

as expected.

On the other hand, from Eqs. (4.131) and (4.133), we find that the khronon field takes the form

$$\varphi = t + f(x), \tag{4.146}$$

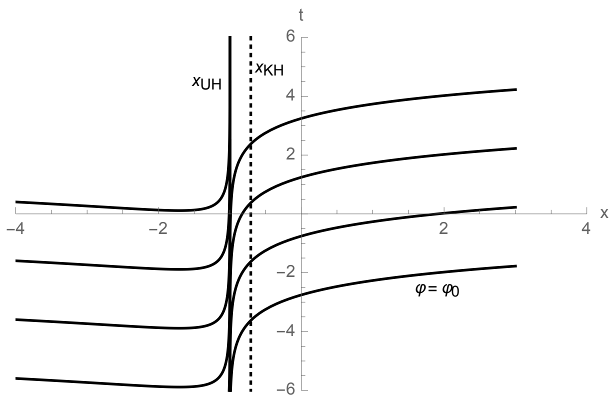

Figure 4.2: The curves of $\varphi =$ Constant. In this figure, we choose $\epsilon_c = 1,\ d = 1,\ \eta = \sqrt{2}$. The universal horizon (dotted vertical line) is located at $x_{UH} = -1$, and the black vertical line denotes the location of the cosmological Killing horizon located at $x_{KH} = -\frac{1}{\sqrt{2}}$.

where we had chosen $\mathcal{F} = -\tilde{\varphi}$, and dropped the tilde from $\tilde{\varphi}$ for the sake of simplicity, without causing any confusions. The function f satisfies the differential equation

$$f'(x) = \frac{u^1 - \eta x\sqrt{G(x)}}{(\eta^2 x^2 - 1)\sqrt{G(x)}}. \tag{4.147}$$

In Fig. 4.2, we show the curves of Constant φ, from which it can be seen clearly the peeling behavior of the curves of constant φ at the universal horizon, while these curves are well-behaved across the Killing horizon.

From Eq. (4.137), we can construct a spacelike unit vector $s_\mu = s_0\delta^t_\mu + s_1\delta^x_\mu$, which is orthogonal to u^μ. It can be shown that s_μ has the non-vanishing components

$$\begin{aligned} s_0 &= -(cx+d), \\ s_1 &= \frac{\eta x u^1 - \sqrt{G(x)}}{\eta^2 x^2 - 1}. \end{aligned} \tag{4.148}$$

Then, we can project k^μ onto u^α and s^α, and obtain

$$\begin{aligned} k_u &\equiv (k\cdot u) = -\omega u^0 + k_x u^1, \\ k_s &\equiv (k\cdot s) = -\omega u_1 - k_x u_0. \end{aligned} \tag{4.149}$$

To proceed further, we need to consider the aether four-velocity u_μ in the regions $x > x_{UH}$ and $x < x_{UH}$, separately. In particular, ϵ_c is set to unity in Eq. (4.144) which leads to the solution

$$
\begin{aligned}
u_0 &= -|d\eta x + \sqrt{d^2+1}|, \\
u_1 &= d, \\
u^0 &= \sqrt{d^2+1}, \\
u^1 &= \eta\sqrt{d^2+1}x + d, \\
f' &= -\frac{d}{x\eta d + \sqrt{d^2+1}}, \\
f &= -\frac{1}{\eta}\ln\left(\eta x d + \sqrt{d^2+1}\right),
\end{aligned}
\tag{4.150}
$$

for $x>x_{UH}$. When $x<x_{UH}$, we find that u_0 and u^1 remain the same while u^0, u_1, f' and f are changed to

$$
\begin{aligned}
u^0 &= \frac{\eta^2x^2\sqrt{d^2+1} + 2d\eta x + \sqrt{d^2+1}}{\eta^2x^2-1}, \\
u_1 &= -\frac{2\eta x\sqrt{d^2+1} + d\eta^2x^2 + d}{\eta^2x^2-1}, \\
f' &= \frac{d}{x\eta d + \sqrt{d^2+1}} + \frac{2\eta x}{1-x^2\eta^2}, \\
f &= \frac{1}{\eta}\ln\left(\frac{dx\eta + \sqrt{d^2+1}}{1-x^2\eta^2}\right), \ (x<x_{UH}).
\end{aligned}
\tag{4.151}
$$

At the universal horizon, similar to the (3+1)-dimensional case [69], relativistic particles cannot be emitted in the form of Hawking radiation. Thus, in the following we consider only the particles with the following non-relativistic dispersion relation [69]

$$
k_u^2 = k_s^2 + a_2\frac{k_s^4}{k_0^2}, \tag{4.152}
$$

where a_2 is a dimensionless constant of order one, and k_0 is the cutoff energy scale. For $k \ll k_0$, the particles become relativistic. Then, from Eq. (4.149) we find

$$
\begin{aligned}
k_u &= -\frac{1}{u_0}(k_s u^1 - \omega), \\
k_x &= -\frac{1}{u_0}(\omega u_1 + k_s).
\end{aligned}
\tag{4.153}
$$

Combined with the dispersion relation (4.152), we find that k_s has a simple pole at the universal horizon $x = x_{UH}$ with $u_0(x_{UH}) = 0$. Thus, we assume that near the universal horizon we have

$$k_s = -\frac{b(x)}{u_0}, \tag{4.154}$$

where $b(x = x_{UH}) \neq 0$. To calculate the temperature given by Eq. (4.127) but now at the universal horizon, in principle we only need the Laurent expansion of k_x in the neighborhood of the universal horizon. Setting $\epsilon = x - x_{UH}$, for the special case given by Eq. (4.150), we find

$$\begin{aligned}
u_0 &= -d\eta\epsilon, \\
u^1 &= -\frac{1}{d} + \epsilon\eta\sqrt{d^2+1}, \\
b(x) &= b_0 + b_1\epsilon + \mathcal{O}\left(\epsilon^2\right), \\
k_x &= \frac{b_0}{\eta^2 d^2 \epsilon^2} + \frac{1}{\epsilon}\left(\frac{\omega}{\eta} + \frac{b_1}{\eta^2 d^2}\right) + \mathcal{O}\left(1\right),
\end{aligned} \tag{4.155}$$

for $x > x_{UH}$, where

$$\begin{aligned}
b_0 &= \pm\frac{k_0}{\sqrt{a_2}d}, \\
b_1 &= \eta d^2\omega - \eta \mathrm{d}\mathrm{b}_0\sqrt{\mathrm{d}^2+1}.
\end{aligned} \tag{4.156}$$

When $x < x_{UH}$, the Taylor expansions of u^1 and $b(x)$ remain the same as in Eq. (4.155) while u_0 and k_x are changed to

$$\begin{aligned}
u_0 &= d\eta\epsilon, \\
k_x &= \frac{b_0}{\eta^2 d^2 \epsilon^2} + \frac{1}{\epsilon}\left(-\frac{\omega}{\eta} + \frac{b_1}{\eta^2 d^2}\right) + \mathcal{O}\left(1\right),
\end{aligned} \tag{4.157}$$

Correspondingly, with the help of dispersion relation Eq. (4.152), one can show

$$\begin{aligned}
b_0 &= \pm\frac{k_0}{\sqrt{a_2}d}, \\
b_1 &= -\eta d^2\omega - \eta \mathrm{d}\mathrm{b}_0\sqrt{\mathrm{d}^2+1}.
\end{aligned} \tag{4.158}$$

In order to figure out the temperature at the universal horizon, one needs to analytically continue the radial momentum k_x to the complex plane, combining Eqs. (4.155) and (4.157), it's easy to conclude that, by setting $x = x_{UH} + \epsilon e^{i\theta}$, for $\theta \in (0, 2\pi)$

$$k_x = \frac{b_0}{\eta^2 d^2 \epsilon^2 e^{2i\theta}} + \frac{2\omega}{\eta \epsilon e^{i\theta}} - \frac{b_0\sqrt{d^2+1}}{\eta d \epsilon}, \tag{4.159}$$

Then, using Eq. (4.127)

$$\frac{\omega}{T_{KH}} = \mathrm{Im} \oint \mathrm{k_x^+ dx} = \frac{4\pi\omega}{\eta}, \tag{4.160}$$

from which we find that

$$T_{UH} = \frac{\eta}{4\pi}. \tag{4.161}$$

The surface gravity at the universal horizon is given by [13] [2]

$$\kappa_{UH} = \frac{1}{2} D_u (u \cdot \zeta) = \frac{\eta}{2}, \tag{4.162}$$

from which we find that the standard relation

$$T_{UH} = \frac{\kappa_{UH}}{2\pi}, \tag{4.163}$$

is satisfied at the universal horizon. This is similar to the (3+1)-dimensional case [69, 71, 72]. For more general case with the dispersion relation

$$k_u^2 = k_s^2 \sum_{n=0}^{2z} a_n \left(\frac{k_s}{k_0}\right)^n, \tag{4.164}$$

it can be shown that the (3+1)-dimensional results [69]

$$T_{UH}^{z\geq 2} = \frac{\kappa_{UH}^{z\geq 2}}{2\pi} = \left(\frac{2(z-1)}{z}\right)\left(\frac{\kappa_{UH}}{2\pi}\right), \tag{4.165}$$

can be also obtained.

4.6 Summary

In this chapter, 2d projectable Hořava-Lifshitz gravity is coulped minimally to a scalar field, the momentum constraint of this coupled system can be solved only in

[2] It should be noted that κ_{UH} given by Eq. (4.162) can also be obtained by considering the peeling behavior of the khronon field φ given by Eq. (4.146), as it was done in [71].

the case where the fundamental variables are functions of time only. In this particular case, the quantization of the coupled system can also be carried out by the standard Dirac process. However, when the system is written in terms of two simple harmonic oscillators, one of them has positive energy, while the other has negative energy, whenever their interactions are ignored. The total energy of the system is always non-negative, provided that the expectation values of the gauge-invariant length operator L for any given physical state $|n_1, n_2\rangle$ must be non-negative. For the case when 2d non-projectable Hořava-Lifshitz gravity is coulped non-minimally with a non-relativistic scalar field, the Hamiltonian structure of this coupled system is very similar to that of pure gravity case. There exist two first-class constraints and two second-class constraints (The combinations of two second-class constraints will generate two global first-class constraints which account for global time reparametrization symmetry of Hořava gravity as first pointed out in [61]). Therefore, the local degrees of freedom is one due to the presence of the scalar field.

We also found diagonal static solutions for the couplings $f(\phi) = \xi\phi$, and showed that Killing horizons exist in such solutions, but the scalar field turns out to be singular at these Killing horizons. For the non-diagonal stationary solutions, when the lapse function and the spatial metric component g_{11} are set to one, we found that the solutions represent black holes, in which both Killing and universal horizons exist. At the Killing horizon, the temperature of Hawking radiation is proportional to its surface gravity defined as in the relativistic case [cf. Eq. (4.129)] [11].

To study locations of the universal horizons, we first considered a test timelike scalar field in such a fixed background [66], and found solutions of the test field, whereby the universal horizons located at $\chi{\cdot}u = 0$ were found. By using the Hamilton-Jacobi method [69], we calculated the temperature at the universal horizon, and found that it is proportional to the modified surface gravity defined by Eq. (4.162). For $z = 2$ of the dispersion relation (4.164), the modified surface gravity given by

Eq. (4.66) satisfies the standard relation with its temperature, $T_{UH} = \kappa_{UH}/(2\pi)$, similar to the (3+1)-dimensional case [71, 72]. But, in more general cases, both of them will depend on z, as shown by Eq. (4.165), although the standard relation, $T_{UH}^{z\geq 2} = \kappa_{UH}^{z\geq 2}/(2\pi)$, is still expected to hold [73, 74].

The results presented in this chapter show clearly that the existence of universal horizons and their thermodynamics are independent of dimensions of spacetimes concerned. Therefore, the 2d Hořava gravity provides an ideal place to address these important issues, which often technically become very complicated in higher dimensional spacetimes.

CHAPTER FIVE

Conclusions and Outlook

This dissertation is mostly centered around the topic of low dimensional Hořava-Lifshitz theory of gravity. I have studied both the projectable and nonprojectable versions of the theory. Although Hořava-Lifshitz gravity is not trivial in 1+1 dimension, from the Hamiltonian formulation of the theory, I find there is actually no local degree of freedom which makes the quantization of the theory easy to handle. The difference between two versions of the theory mainly lies in the behavior of the constraints. In the projectable case, since the lapse function only depends on time, there is only integral Hamiltonian constraint which can be directly solved to yield Wheeler-Dewitt equation. While for the nonprojectable case, the Hamiltonian constraint become both local and second-class, so one has to find the solution of the Hamiltonian constraint and its accompanied Lagrangian multiplier in order to show the theory is self-consistent. Later, I have considered the interactions between matter sector and the low dimensional gravity. It turns out that the constraint algebra will not be altered except the contributions from the scalar field should also be added into the constraints. In the special case when the nonminimal coupling is taken into account, universal horizon has been found from the classical solutions.

As a matter of fact, there are at least three approaches for the quantization of a theory, the covariant approach which turns to the perturbation expansion of the fields on the background of classical solutions (usually Minkowski spacetime), the canonical approach which is based on the Hamiltonian formulation of the theory and try to quantize it by solving the Wheeler-Dewitt equation, the path-integral approach which deals with the generating functional of the correlation functions of the theory. What I have followed in this dissertation is the canonical approach since for the pure gravity in two dimensions, there is no propagating degrees of freedom. The covariant approach is more approporiate for the higher dimensional theories where there are local degrees of freedom that can be considered as the fluctuations on the Minkowski background. In this direction, the renormalization of projectable Hořava-Lifshitz gravity in 2+1 dimensions has been studied in [75], recently they have also calculated the renormalization group flow and found the theory is asymptotically free [76]. As for the path integral approach, the typical example is the lattice theory called causal dynamical triangulations [77]. In this approach, the spacetime is discretized so one can apply numerical methods to compute the generating functional.

There is lots of work that is worth my efforts in the future. The first thing I would like to do is to study the coupled system of 2d Hořava gravity and the scalar field since the degree of freedom of this coupled system is one which makes it a good situation to apply the perturbation expansions and I want to see how the gravity sector would affect the matter part. Secondly, the 2+1 nonprojectable Hořava-Lifshitz gravity is still imposing challenges due to the second-class constraints, the closure of the constraint algebra after quantization is an interesting topic to study.

APPENDICES

APPENDIX A

Dirac's Algorithm for Constrained Systems

A.1 General Aspects of Dirac's Algorithm for Singular Lagrangian

In this Appendix, I will give a brief description of Dirac's Algorithm for discovering all the constraints of the degenerate systems. Here I shall focus on the systems with discrete degress of freedom for simplicity of notations. It can be easily extended to the systems with continuous degrees of freedom. Important procedures and results will be presented in a concise way, as for the details, since there exists a voluminous collection of references on this subject, I'd like to refer those interested readers to [78–84] as they cover a variety of topics as well as applications to concrete examples.

For the system described by a Lagrangian which is quadratic in velocities and free of high derivative operators [1]

$$S = \int dt\mathcal{L}(q^i, \dot{q}^i), \tag{A.1}$$

where superscript i denotes different degrees of freedom, the momentum in this case is defined by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}. \tag{A.2}$$

Now there exits an operator called Hessian matrix denoted by $W_{ij} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j}$, whose determinant $W = det(W_{ij})$ can be used to distinguish between degenerate and non-degenerate systems. If W is nonzero, there is one-to-one correspondence between the coordinate-velocity space (spannd by q^i and $\dot{q}^i$) and the phase space (spannd by q^i and p_i), therefore, all the velocities $\dot{q}_i$'s can be uniquely determined as some functions

[1] If Lagrangian contains high-order time derivative terms, the system is unstable since the energy is not bounded from below due to the presence of Ostrogradsky's ghosts [19, 85], on the other hand, if Lagrangian contains a polynomial of velocities which usually appears in some modified gravity theories [86], different techniques should be applied in order to obtain a unique Hamiltonian of the system as discussed in [87].

of q^i and p_i

$$\dot{q}^i = f_i(q^i, p_i). \tag{A.3}$$

Then after applying Legendre transformation

$$\mathcal{H} = p_i \dot{q}^i - \mathcal{L}, \tag{A.4}$$

the Hamiltonian can thus be derived so are the first-order EOM in the phase space which are equivalent to Euler-Lagrange equations directly obtained from the action by variational method.

Now comes the nondegenerate systems with constraints[2] which are featured by vanishing Hessian determinant $W = 0$. In this case, one can't solve for all the velocities from the definition of momentum (A.2). Assuming the rank of Hessian matrix W_{ij} is m and there are n degrees of freedom (i, j range from one to n), thus it can be shown that only m velocities $\dot{q}^\alpha$ (α takes the values from one to m) can be solved as functions of q^i, p_α and $\dot{q}^s$, where $s = m+1, ..., n$. The remaining $n - m$ definitions of p_s's will be reduced to $n - m$ relations between the phase space variables which are called *primary constraints* $\phi_s(q^i, p_i)$. Therefore, after Legendre transformation Eq. (A.4) is performed, the resulting canonical Hamiltonian reads

$$H = H_0(q^i, p_\alpha) + u^s \phi_s, \tag{A.5}$$

where summation over s from $m+1$ to n is understood.[3] As a result, the EOM in the phase space take the form of Poisson bracket

$$\dot{\zeta}^i \approx \Big\{\zeta^i, H\Big\}, \tag{A.6}$$

[2] The original purpose of studying nondegenerate systems is to quantize Einstein's theory of gravity. Although Bergmann and his students [88] first set out to formulate the difficulties encountered in the canoncial formulation of degenerate systems and partially solve them, it was Dirac [89] who proposed his method, known as Dirac's algorithm nowadays, to systematically unveil the Hamiltonian and all the constraints of degenerate systems.

[3] Actually u^s equals undetermined velocities $\dot{q}^s$, generally H_0 is a nonzero quantity except when the action is invariant under time reparametrization transformation [82], see [90] for a simple proof.

where $\zeta^i = (q^i, p_i)$ and " $\approx$ " is called "weakly equals" which implies that the equations should be evaluated on the *primary constraint surface* which is carved out of the whole phase space by $\phi_s = 0$ and therefore all the trajectories of the motion will be confined to it. However, this is not the end of the story since the primary constraints are identities due to the definition of momenta and degeneracy of the singular Lagrangian, they should be satisfied as the system evolves over the time, this equally states

$$\dot{\phi}_s \approx \Big\{\phi_s, H\Big\} \approx 0. \tag{A.7}$$

There are three possibilities resulting from the above evolution equation,

Case (a) Eq. (A.7) can be simply reduced to the trivial identity on the constraint surface, i.e. "$0 \approx 0$" which signifies the end of Dirac's procedures.

Case (b) Eq. (A.7) is now equivalent to an algebraic (for discrete case) or partial differential equation (for continuum case) of Lagrangian multiplies u^s which indicates the u^s must be restricted by the evolution of the system. In this case, the existence and uniqueness of the solutions of u^s is a vital test for the viability and consistency of the theory.[4]

Case (c) Generally, Eq. (A.7) will fall into this category in which the secondary constraints will be generated. In this case, the evaluation of Eq. (A.7) can give us relations between canonical variables

$$\chi_r(q^i, p_i) \approx 0, \tag{A.8}$$

which are independent of the primary constraints. These relations are called secondary constraints since they impose further restrictions on the possible surface where trajectories of the motion can exist. Now the particles can only travel on the subspace spanned by $\phi_s = \chi_r = 0$. Once all the secondary constraints are derived, it's necessary that they should be preserved over the time, so one simply evaluates the

[4] see [45] for one example on this topic.

time evolution of secondary constraints until case (a) or (b) is reached which terminates the whole process. In this way, all the constraints of the theory can be found systematically. In the rest part of this appendix, the secondary constraints will be denoted collectively by χ_r.[5]

A.2 First- and Second-Class Constraints

Based on the analysis in the last section, one can obtain canonical Hamiltonian H in Eq. (A.5) of the constrained system with primary and secondary constraints, i.e. $\phi_{\bar{s}}$ and $\chi_{\bar{r}}$ (or $\psi_{\bar{n}}$ [6] collectively). The motion of the particles is confined to the constraint surface spanned by $\psi_{\bar{n}} = 0$. In order to find the dimension of the physical phase space, the knowledge of first- and second-class constraints is required.

The first-class constraints $\sigma_{\bar{\mu}}$ are those which commute with all the constraints on the constraint surface, that is,

$$\Big\{\sigma_{\bar{\mu}}, \psi_{\bar{n}}\Big\} \approx 0, \tag{A.9}$$

here $\bar{\mu} = 1, ..., \mu$. For the second-class constraints $\tau_{\bar{\nu}}$ ($\bar{\nu} = 1, ..., \nu$), there at least exists one constraint $\tau_{\bar{\nu}'}$ which fails to commute with $\tau_{\bar{\nu}}$,

$$\mathcal{M}_{\bar{\nu}\bar{\nu}'} = \Big\{\tau_{\bar{\nu}}, \tau_{\bar{\nu}'}\Big\} \not\approx 0. \tag{A.10}$$

One notable feature of matrix $\mathcal{M}_{\bar{\nu}\bar{\nu}'}$ is that its determinant $\mathcal{M}$ is nonzero.[7] Therefore, the inverse matrix $\tilde{\mathcal{M}}^{\bar{\nu}\bar{\nu}'}$ exists. In order to reveal the intrinsic difference between first- and second-class constraints, one can consider the extended Hamiltonian defined by

$$H_{\text{ext}} = H_0 + \alpha^{\bar{\mu}}\sigma_{\bar{\mu}} + \beta^{\bar{\nu}}\tau_{\bar{\nu}}. \tag{A.11}$$

---

[5] No difference will be made between secondary and so-called tertiary constraints which are generated by time preservation of secondary constraints. All the constraints other than primary constraints will be called secondary constraints.

[6] Here the subscripts $\bar{s}$, $\bar{r}$ and $\bar{n}$ indicate the numbers of constraints are s, r, n respectively, so there is relation $n = s + r$.

[7] If $\mathcal{M}$ is zero, then there must be a linear combination of $\tau_{\bar{\nu}}$ which commutes with all the $\tau_{\bar{\nu}'}$ which goes against the hypothesis that $\tau_{\bar{\nu}}$s constitute an irreducible set of second-class constraints.

Correspondingly, the evolution of any functions of canonical variables is given by the EOM

$$\dot{f}(q^i, p_i) \approx \Big\{ f(q^i, p_i), H_{\text{ext}} \Big\}. \tag{A.12}$$

As a result, time invariance of the constraints immediately leads us to the conclusion that all the $\alpha_{\bar{\mu}}$ are truly arbitrary while the coefficients of the second class constraints must be fixed as

$$\beta^{\bar{\nu}} = -\tilde{\mathcal{M}}^{\bar{\nu}\bar{\nu}'} \Big\{ \tau_{\bar{\nu}'}, H_0 \Big\}. \tag{A.13}$$

Substituting the solution of $\beta^{\bar{\nu}}$ back into the evolution equation Eq. (A.12), one finally arrives at

$$\dot{f}(q^i, p_i) \approx \Big\{ f, H_0 + \alpha^{\bar{\mu}} \sigma_{\bar{\mu}} \Big\} - \Big\{ f, \tau_{\bar{\nu}''} \Big\} \tilde{\mathcal{M}}^{\bar{\nu}''\bar{\nu}'} \Big\{ \tau_{\bar{\nu}'}, H_0 + \alpha^{\bar{\mu}} \sigma_{\bar{\mu}} \Big\}. \tag{A.14}$$

This form of evolution equation provides us a way to find the right definition of Dirac bracket which is

$$\Big\{ f, g \Big\}_D = \Big\{ f, g \Big\} - \Big\{ f, \tau_{\bar{\nu}''} \Big\} \tilde{\mathcal{M}}^{\bar{\nu}''\bar{\nu}'} \Big\{ \tau_{\bar{\nu}'}, g \Big\}, \tag{A.15}$$

here f and g are two arbitrary functions of canonical variables. Dirac bracket inherits most of algebraic properties of Poisson bracket [82] while, on the other hand, has its own advantages: the second-class constraints are rendered strongly equal to zero once Poisson bracket is replaced with Dirac bracket in the EOM, since

$$\Big\{ \tau_{\bar{\nu}}, f \Big\}_D = 0, \tag{A.16}$$

for any function f in the phase space. As a result, the EOM in terms of Dirac bracket can be cast into the form

$$\dot{f} \approx \Big\{ f, H_0 + \alpha^{\bar{\mu}} \sigma_{\bar{\mu}} \Big\}_D. \tag{A.17}$$

This clearly shows that given the initial conditions which satisfy the first-class constraints at the initial time, the solutions of the motion can not be uniquely determined at a later time, and the degrees of freedom caused by the arbitrariness of $\alpha^{\bar{\mu}}$ are called gauge degrees of freedom, a reflection of gauge invariance of the degenerate theory.

A.3 Gauge Conditions and Reduced Phase Space

As pointed out at the end of the last section, the gauge degrees of freedom are closely related to the first-class constraints of the theory. Actually, it can be shown explicitly that the first-class constraints are the generators of gauge transformations which leave the EOM in the Hamiltonian formulation form-invariant. A generic form of these generators is of the type

$$G_i(\epsilon, \dot{\epsilon}, ..., \epsilon^{(k)}) = \sum_{l=0}^{k} \epsilon_i^{(l)} G_n, \tag{A.18}$$

where l stands for lth time derivative of gauge parameters ϵ_i, and G_n are some linear combinations of first-class constraints which can be fixed in the Dirac's procedure as discussed in Sec. A. 1 [81]. The number of independent gauge parameters ϵ_i coincides with the number of primary first-class constraints.[8]

In order to uniquely fix the trajectories of the motion, one needs to fix the coefficients $\alpha^{\bar{\mu}}$ in Eq. (A.17). The basic idea is to add the same number of additional conditions $\bar{\sigma}_{\bar{\mu}}$ [9] as that of first-class constraints so that all the first-class constraints together with the gauge conditions become second-class. So the EOM in terms of Possion bracket now becomes

$$\dot{f} \approx \Big\{ f, H_0 + \alpha^{\bar{\mu}} \sigma_{\bar{\mu}} + \beta^{\bar{\nu}} \tau_{\bar{\nu}} + \alpha^{\bar{\mu}'} \bar{\sigma}_{\bar{\mu}'} \Big\}. \tag{A.19}$$

The preservation of all the constraints $\sigma_{\bar{\mu}}$, $\tau_{\bar{\nu}}$ and $\bar{\sigma}_{\bar{\mu}'}$ in time will generate $(2\mu + \nu)$ equations of their coefficients. Now an important question arises: whether these $(2\mu + \nu)$ equations have unique solutions? The fact is that only with a discreet choice of gauge conditions, these equations can be solved for the coefficients $\alpha^{\bar{\mu}}$, $\beta^{\bar{\nu}}$, and $\alpha^{\bar{\mu}'}$.[10] Once the solutions of these coefficients are substituted back into Eq. (A.19),

---

[8] For example, in canonical formalism of general relativity, there are four primary first-class constraints which are canonical momenta of the lapse function and shift vectors, so there are also four independent gauge parameters which are the arbitrary functions in the general coordinate transformation.

[9] These conditions, called gauge conditions, are relations of canonical variables which are regarded as additional constraints added to the theory so that gauge degrees of freedom can be eliminated.

[10] See [91, 92] for a discussion of admissible gauge conditions for constrained systems.

the gauge degrees of freedom are eliminated and unique solutions of motion can be found as long as initial conditions are provided. The motion of particles are now confined to the subspace spanned by the constraints $\sigma_{\bar{\mu}}$, $\tau_{\bar{\nu}}$ and $\bar{\sigma}_{\bar{\mu}'}$. This subspace is called *reduced phase space* whose dimension can be calculated by the formula

$$\mathcal{D} = 2N - 2\mu - \nu, \tag{A.20}$$

with the understanding that half of $\mathcal{D}$ amounts to the physical degrees of freedom in the coordinate space. The reduced phase space is physical in the sense that the actual motion of particles can be parameterized by the canonical variables in that space. Once the reduced phase space is found, the quantization of the system can be implemented by promoting canonical variables to operators and replacing Poisson brackets with commutators. However, this reduced phase space is generally difficult to find. So when it comes to the quantization of systems with gauge symmetry, instead of diminishing the dimensions of the phase space, people usually expand the phase space by adding more degrees of freedom with opposite Grassman parity. The study of this subject is outside the scope of this dissertation.

[1] Einstein, Albert (November 25, 1915), Die Feldgleichungen der Gravitation. Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin: 844847. Retrieved 2017-08-21.

[2] Schwarzschild, Karl (1916a), ber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie, Sitzungsber. Preuss. Akad. D. Wiss.: 189196.

[3] Reissner, H., ber die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie. Annalen der Physik (in German). 50: 106120, (1916).

[4] Nordstrm, G., On the Energy of the Gravitational Field in Einstein's Theory. Verhandl. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk., Amsterdam. 26: 12011208, (1918).

[5] Kerr, Roy P., Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics. Physical Review Letters. 11 (5): 237238, (1963).

[6] Israel, Werner, Event Horizons in Static Vacuum Space-Times. Phys. Rev. 164 (5): 17761779, (1967).

[7] A. A. Penzias and R. W. Wilson, A measure of excess antenna temperature at 4090 Mc/s, Astrophys. J. **142** (1965) 419.

[8] J. W. York, Role of conformal three geometry in the dynamics of gravitation, Phys. Rev. Lett. 28 (1972) 1082.

[9] G. W. Gibbons and S. W. Hawking, Action Integrals And Partition Functions In Quantum Gravity, Phys. Rev. D15, 2752 (1977).

[10] S. W. Hawking and G. T. Horowitz, The gravitational Hamiltonian, action, entropy and surface terms, Classical Quantum Gravity 13, 1487 (1996).

[11] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge Monographs on Mathematical Physics, (Cambridge University Press, Cambridge, 1973).

[12] C. Rovelli, Notes for a Brief History of Quantum Gravity, [arXiv: gr-qc/0006061].

[13] A. Wang, *Hořava Gravity at a Lifshitz Point: A Progress Report*, Inter. J. Mod. Phys. D**26** (2017) 1730014 [arXiv:1701.06087].

[14] Ethan Dyer and Kurt Hinterbichler, Phys. Rev. D79, 024028 (2009).

[15] T. Regge and C. Teitelboim, Ann. Phys. 88 (1974) 286.

[16] S. Weinberg, in General Relativity. An Einstein Centenary Survey, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1980).

[17] M. H. Goroff and A. Sagnotti, The ultraviolet behavior of Einstein gravity, Nucl. Phys, B266 (1986) 709.

[18] K. S. Stelle, Renormalization of higher-derivative quantum gravity, Phys. Rev. D16 (1977) 953.

[19] M. Ostrogradsky, Mem. Ac. St. Petersbourg, VI**4** (1850) 385.

[20] J. Collins, A. Perez, D. Sudarsky, L. Urrutia, and H.Vucetich, Lorentz invariance and quantum gravity : an additional fine-tuning problem?, Phys. Rev. Lett. 93 (2004) 191301.

[21] M. Pospelov and C. Tamarit, Lifshitz-sector mediated SUSY breaking, J.High Energy Phys. (01) (2014) 048.

[22] G. Bednik, O. Pujolas and S. Sibiryakov, Emergent Lorentz invaraince from strong dynamics: Holographic examples, J. High Energy Phys. 11 (2013) 064.

[23] I. Kharuk and S. Sibiryakov, Emergent Lorentz invariance with chiral fermions, arXiv: 1505.04130.

[24] N. Afshordi, Why is high energy physics Lorentz invariant? arXiv: 1511.07879.

[25] M.Visser, Lorentz symmetry breaking as a quantum field theory regulator, Phys. Rev. D**80** (2009) 025011 [arXiv: 0912.4757].

[26] D. Anselmi and M. Halat, Renormalizationof Lorentz violating thoeries, Phys. Rev. D76 (2007) 125011.

[27] R. Arnowitt, S. Deser, and C.W. Misner, Gen. Relativ. Gravit. **40**, 1997 (2008); C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation (W.H. Freeman and Company, San Francisco, 973), pp.484-528.

[28] Gourgoulhon, E., 3+1 formalism and bases of numerical relativity, [arXiv:gr-qc/0703035].

[29] P. Hořava, Phys. Rev. D**79**, 084008 (2009).

[30] P. Hořava, J. High Energy Phys. **0903**, 020 (2009).

[31] P. Hořava, Phys. Rev. Lett. **102**, 161301 (2009).

[32] D. Blas, O. Pujolas, and S. Sibiyakov, On the extra mode and inconsistency of Horava Gravity, JHEP 0910 (2009) 029.

[33] M. Li and Y. Pang, A trouble with Horava-Lifshitz Gravity, JHEP 0908 (2009) 015.

[34] M. Henneaux, A. Kleinschmidt and G. Lucena Gomez, A dynamical inconsistency of Horava gravity, Phys. Rev. D81 (2010) 064002.

[35] T. P. Sotiriou, M. Visseer and S. Weinfurtner, Phys. Rev. Lett. 102 (2009) 25601.

[36] A. Wang and R. Maartens, Linear perturbations of cosmological models in the Horava-Lifshitz theory of gravity without detailed balance, Phys. Rev. D**81** (2010) 024009.

[37] P. Horava and C. M. Melby-Thompson, General covariance in quantum gravity at a Lifshitz point, Phys. Rev. D**82** (2010) 064027.

[38] A. M. da Silva, An alternative approach for general covariant Horva-Lifshitz gravity and matter coupling, Class. Quantum Grav. **28** (2011) 055011.

[39] Y.-Q. Huang and A. Wang, Stability, ghost, and strong coupling in the non-relativistic general covariant theory of graivty with $\lambda \neq 1$, Phys. Rev. D**83** (2011) 104012.

[40] D. Blas, O. Pujolas, and S. Sibiyakov, Phys. Rev. Lett. 104 (2010) 181302.

[41] D. Blas, O. Pujolas, and S. Sibiyakov, JHEP 04 (2011) 018.

[42] T. Zhu, F.-W. Shu, Q. Wu, and A. Wang, Phys. Rev. D**85** (2012) 044053.

[43] T. Zhu, F.-W. Shu, Q. Wu, and A. Wang, Phys. Rev. D**84** (2012) 101502.

[44] K. Lin, S. Mukohyama, A. Wang, and T. Zhu, Phys.Rev. D**89** (2014) 084022.

[45] J. Bellorin, A. Restuccia, A. Sotomayor, Phys. Rev. D**85** (2012) 124060.

[46] J. Bellorin, A. Restuccia, A. Sotomayor, Phys. Rev. D**87** (2013) 084020.

[47] J. Bellorin, A. Restuccia, Phys. Rev. D**94** (2016) 064041.

[48] B.-F. Li, A. Wang, Y. Wu, and Z.C. Wu, Phys. Rev. D**90**, 124076 (2014).

[49] J. Ambjorn, L. Glaser, Y. Sato, and Y. Watabiki, Phys. Lett. B**722**, 172 (2013).

[50] S. Carlip, *Quantum Gravity in 2+1 Dimensions*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2003).

[51] D. Grumiller, W. Kummer, and D.V. Vassilevich, [arXiv:hep-th/0204253].

[52] P. di Francesco, P. Ginsparg, and J. Zinn-Justin, Phys. Rept. **254**, 1 (1995) [arXiv:hep-th/0204253].

[53] R.-G. Cai and A. Wang, Phys. Lett. B**686**, 166 (2010).

[54] B.-F. Li, V. H. Satheeshkumar, A. Wang, Phys. Rev. D**93**, 064043 (2016).

[55] T. Jacobson, Phys. Rev. D**81**, 101502 (2010).

[56] A. Wang, arXiv:1212.1040; Phys. Rev. Lett. **110**, 091101 (2013).

[57] T.P. Sotiriou, M. Visser, and S. Weinfurtner, Phys. Rev. D**83**, 124021 (2011).

[58] C. Eling and T. Jacobson, Phys. Rev. D**74**, 084027 (2006).

[59] M. Henneaux, Phys. Rev. Lett. **54**, 959 (1985).

[60] P.A.M. Dirac, Phys. Rev. **114**, 924 (1959).

[61] W. Donnelly and T. Jacobson, Phys. Rev. D**84**, 104019 (2011).

[62] J. Kluson, J. High Energy Phys. **07**, (2010) 038.

[63] B.-F. Li, Madhurima Bhattacharjee, A. Wang, Phys. Rev. D**96**, 084006 (2017).

[64] D. Blas and S. Sibiryakov, *Horava gravity vs. thermodynamics: the black hole case*, Phys. Rev. D**84**, 124043 (2011) [arXiv:1110.2195].

[65] E. Barausse, T. Jacobson, and T. Sotiriou, *Black holes in Einstein-aether and Horava-Lifshitz gravity*, Phys. Rev. D**83** (2011) 124043 [arXiv:1104.2889].

[66] K. Lin, E. Abdalla, R.-G. Cai, and A. Wang, Inter. J. Mod. Phys. D**23**, 1443004 (2014).

[67] T. Jacobson and D. Mattingly, Phys. Rev. D**63**, 041502 (2001); T. Jacobson, Proc. Sci. QG-PH, **020** (2007).

[68] D. Bazeia, F.A. Brito, and F.G. Costa, *Two-dimensional Horava-Lifshitz black hole solutions*, Phys. Rev. D**91** (2015) 044026 [arXiv:1409.0490].

[69] C. Ding, A. Wang, X. Wang and T. Zhu, *Hawking radiation of charged Einstein-aether black holes at both Killing and universal horizons*, Nucl. Phys. B**913** (2016) 694 [arXiv:1512.01900].

[70] K. Lin, O. Goldoni, M. F. da Silva, and A. Wang, *New look at black holes: Existence of universal horizons*, Phys. Rev. D**91** (2015) 024047 [arXiv:1410.6678].

[71] B. Cropp, S. Liberati, A. Mohd and M. Visser, *Ray tracing Einstein-aether black holes: Universal versus Killing horizons*, Phys. Rev. D**89** (2014) 064061 [arXiv:1312.0405].

[72] P. Berglund, J. Bhattacharyya, and D. Mattingly, *Towards Thermodynamics of Universal Horizons in Einstein-Eather Theory* Phys. Rev. Lett. **110** (2013) 071301 [arXiv:1210.4940].

[73] C.-K. Ding and C.-Q. Liu, *Dispersion relation and surface gravity of universal horizons*, arXiv:1611.03153.

[74] B. Cropp, *Strange Horizons: Understanding Causal Barriers Beyond General Relativity*, arXiv:1611.00208.

[75] Andrei O. Barvinsky, Diego Blas, Mario Herrero-Valea, Sergey M. Sibiryakov, Christian F. Steinwachs, Phys. Rev. D**93** (2016) 064022.

[76] Andrei O. Barvinsky, Diego Blas, Mario Herrero-Valea, Sergey M. Sibiryakov, Christian F. Steinwachs, Hoava gravity is asymptotically free (in 2+1 dimensions) (2017), [arXiv:1706.06809].

[77] C. Anderson, S. Carlip, J. H. Cooperman, P. Horava, R. Kommu, and P. R. Zulkowski, Phys. Rev. D**85**, 044027 (2012).

[78] Paul A.M. Dirac, *Lectures on Quantum Mechanics*, Yeshiva, New York, 1964.

[79] Sundermeyer, K. (1982): *Constrained Dynamics*, Springer Lecture Notes 169.

[80] Paul Steinhardt, Ann. Phys., 128, 425 (1980).

[81] Leonardo Castellani, Ann. Phys., 143, 357 (1982).

[82] Henneaux, M. and Teitelboim, C.(1992): *Quanzation of Gauge Systems*, Princeton Univ. Press.

[83] Andreas W. Wipf, *Hamilton's Formalism for Systems with Constraints*, (1993) [arXiv:hep-th/9312078].

[84] Hans-Jurgen Matschull, *Dirac's Canonical Quantization Programme*, (1996) [arXiv:quant-ph/9606031].

[85] R.P.Woodard, *The Theorem of Ostrogradsky* , (2015) [arXiv:1506.02210].

[86] D.Lovelock, J. Math. Phys. (N.Y.) **12**,498 (1971).

[87] Eran Avraham and Ram Brustein, Phys. Rev. D**90** (2014) 024003.

[88] Bryce S. Dewitt, Phys. Rev. **160** (1967) 113.

[89] P.A.M. Dirac, Can. J. Math. **2**,129 (1950).

[90] Edmund Bertschinger (2002): *Symmetry Transformations, the Einstein-Hilbert Action, and Gauge Invaraince*, lecture notes.

[91] R. Ferraro and C. Simeone, J. Math. Phys. 38, 599 1997.

[92] S. A. Gogilidze, A. M. Khvedelidze, and V. N. Pervushin, Phys. Rev. D**53** (1996) 2160.