ABSTRACT

An Explicit Description of Pieri Inclusions

John A. Miller II, Ph.D.

Mentor: Markus Hunziker, Ph.D.

By the Pieri rule, the tensor product of an exterior power and a finite-dimensional irreducible representation of a general linear group has a multiplicity-free decomposition. The embeddings of the constituents are called Pieri inclusions and were first studied by Weyman in his thesis and described explicitly by Olver. More recently, these maps have appeared in the work of Eisenbud, Fløstad, and Weyman and of Sam and Weyman to compute pure free resolutions for classical groups. We give a new closed form, non-recursive description of Pieri inclusions. For partitions with a bounded number of distinct parts, the resulting algorithm has polynomial time complexity whereas the previously known algorithm has exponential time complexity.

An Explicit Description of Pieri Inclusions

by

John A. Miller II, B.S.

A Dissertation

Approved by the Department of Mathematics

Dorina Mitrea, Ph.D., Chairperson

Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of

Doctor of Philosophy

Approved by the Dissertation Committee

Markus Hunziker, Ph.D., Chairperson

Mark Sepanski, Ph.D.

Daniel Herden, Ph.D.

David Kahle, Ph.D.

Accepted by the Graduate School

August 2020

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I am very thankful for the guidance and support given to me throughout my graduate career by the faculty of the Baylor University mathematics department. I am appreciative of Manfred Dugas, Jameson Graber, Paul Hagelstein, Jon Harrison, Connie Liaw, and Ron Stanke for teaching the courses in which I was able to build a foundation and maturity in mathematics. I am also thankful of my wonderful undergraduate teachers, especially Gordon Johnson for instilling in me a passion for advanced mathematics and Simon Bott for the many words of encouragement and inspiration.

Thank you to Markus Hunziker, Mark Sepanski, Daniel Herden, and David Kahle for taking the time to serve on my dissertation committee. I would like to especially recognize the frequent guidance, kindness, and open-door policy of both Markus Hunziker and Mark Sepanski over the past six years. This dissertation would certainly not exist without these two mentors.

I have been lucky to have the support of many friends and family. I owe a special thanks to Matt Jaynes for always receiving my frequent rambling and random questions with a smile. Thank you to Dale Frymark and Alan Mullenix for sitting through many of my presentations "about boxes" and to my parents, Connie and John A., for their belief in me. Finally, thank you to my wonderful wife, Sarah, for your unwavering support.

To Sarah, Tucker, and Sully

CHAPTER ONE

Preliminaries

In this chapter we construct the Schur-Weyl modules, first giving the necessary background, and then outline the description of Pieri Inclusions. We also give two examples computing the image of a Pieri inclusion removing a single box, one acting on a highest weight vector and one where the image must be straightened.

1.1 Partitions, Young Tableaux, and the Pieri Rule

1.1.1

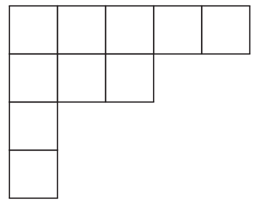
A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of a non-negative integer d is a sequence of non-negative integers in weakly decreasing order where the sum of the λ_i is d. We call the number of nonzero parts λ_i the length of λ, denoted by $l(\lambda)$, which is clearly finite. We will write partitions without trailing zeroes, e.g.

$$(5, 3, 1, 1, 0, 0, \ldots) = (5, 3, 1, 1).$$

To each partition we can associate an array of boxes called a Young diagram where the diagram associated to $\lambda = (\lambda_1, \ldots, \lambda_k)$ has k rows with λ_1 boxes in the first row from the top, λ_2 boxes in the second row from the top, etc. We will frequently denote by λ both a partition and its corresponding Young diagram.

A Young tableaux of shape λ is a filling of a Young diagram λ with positive integers. A Young tableaux is called semistandard if the entries are weakly increasing across the rows and strictly increasing down the columns.

Example. The Young diagram associated to $\lambda = (5, 3, 1, 1)$ is



and a semistandard tableaux on the shape λ is

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 5 & \multicolumn{2}{c}{} \\ \cline{1-3} 4 & 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-3} 5 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} .$$

1.1.2

If V is a complex vector space of dimension $n \leq l(\lambda)$, we can apply the Schur–Weyl functor $\mathbb{S}_\lambda$ as in (Fulton & Harris, 2013) to V to obtain an irreducible representation $\mathbb{S}_\lambda(V)$ of the general linear group $GL(V)$. It follows from Pieri's formula for the product of an elementary symmetric polynomial and a Schur polynomial that the tensor product representation $\wedge^m(V) \otimes \mathbb{S}_\lambda(V)$ decomposes multiplicity-free into a direct sum of irreducible representations

$$\wedge^m(V) \otimes \mathbb{S}_\lambda(V) \cong \bigoplus_\mu \mathbb{S}_\mu(V),$$

where the sum is over all partitions μ with $l(\mu) \leq n$ whose Young diagram is obtained from the Young diagram of λ by adding exactly m boxes, at most one to each row. Since the decomposition is multiplicity-free, it is natural to ask for explicit descriptions of the embeddings $\Phi_m : \mathbb{S}_\mu(V) \hookrightarrow \wedge^m(V) \otimes \mathbb{S}_\lambda(V)$. Following (Sam & Weyman, 2011), we call these embeddings (skew) Pieri inclusions. Similarly, Pieri's formula for the product of a complete symmetric function and a Schur polynomial yields a multiplicity-free decomposition of the tensor product representation $S^m(V) \otimes \mathbb{S}_\lambda(V)$ into a direct sum of irreducible representations

$$S^m(V) \otimes \mathbb{S}_\lambda(V) \cong \bigoplus_\mu \mathbb{S}_\mu(V),$$

where the sum is over all partitions μ with $l(\mu) \leq n$ whose Young diagram is obtained from the Young diagram of λ by adding exactly m boxes, at most one in each column. As the decomposition is again multiplicity-free, we can ask for explicit descriptions of the (symmetric) Pieri inclusions $\mathbb{S}_\mu(V) \hookrightarrow S^m(V) \otimes \mathbb{S}_\lambda(V)$.

1.1.3

Given a basis $\{e_1, e_2, \ldots, e_n\}$ of V, the representation $\mathbb{S}_\lambda(V)$ is equipped with a canonical basis indexed by the set of semistandard tableaux of shape λ with fillings from the set $\{1, \ldots, n\}$. In (Olver, 1982), Olver gave an explicit description of the Pieri inclusions with respect to these canonical bases in the special case when $m = 1$. We will denote this description by Φ_1. When $m > 1$, the Pieri inclusion Φ_m can be obtained by iteration of the special case (Sam & Weyman, 2011, Corollary 1.8). The main purpose of this paper is to give a new combinatorial description of Φ_m that (a) leads to a more efficient algorithm and (b) can be given in a general closed form (avoiding iteration) for $m \geq 1$. In regard to (a), we will show that our algorithm achieves an exponential speed-up over Olver's algorithm when it is restricted to partitions with a bounded number of distinct parts. More precisely, if we fix a positive integer N and consider partitions λ that can be written in exponential notation as $\lambda = (1^{h_1}, 2^{h_2}, 3^{h_3}, \ldots)$ with at most N nonzero exponents h_i, then our algorithm to compute the image of a highest weight vector under a Pieri inclusion $\Phi_1 : \mathbb{S}_\mu(V) \hookrightarrow V \otimes \mathbb{S}_\lambda(V)$ has a run-time complexity of $O(l(\lambda)^N)$, whereas Olver's algorithm has a run-time complexity of $\Omega(2^{l(\lambda)})$.

1.2 Constructing Schur–Weyl Modules

1.2.1

From now on, let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a fixed partition of d. Let $\mathcal{T}_{\lambda,n}$ be the set of all tableaux T of shape λ with filling from the alphabet $\{1, \ldots, n\}$. Fix the canonical tableau T_0 of shape λ labeled with $\{1, \ldots, d\}$, starting with the top left most box and filling across each row, so the first box of the first row is labeled 1, the first box of the second row is labeled $\lambda_1 + 1$, etc.

Example. If $\lambda = (6, 3, 3, 2, 1)$, then

$$T_0 = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \cline{1-3} 10 & 11 & 12 \\ \cline{1-3} 13 & 14 \\ \cline{1-2} 15 \\ \cline{1-1} \end{array} \quad .$$

1.2.2

Via this labeling, the symmetric group $\mathfrak{S}_d$ acts on the set of tableau with shape λ with respect to any given alphabet. Let

$$P = P_\lambda = \{\pi \in \mathfrak{S}_d : \pi \text{ preserves the rows of } T_0\}$$

and

$$Q = Q_\lambda = \{\sigma \in \mathfrak{S}_d : \sigma \text{ preserves the columns of } T_0\} .$$

As elements of the group algebra of $\mathfrak{S}_d$, $\mathbb{C}\mathfrak{S}_d$, define

$$A_\lambda = \sum_{\pi \in P} \pi \qquad \text{and} \qquad B_\lambda = \sum_{\sigma \in Q} (-1)^\sigma \sigma.$$

The Young Symmetrizer is then defined as $C_\lambda = A_\lambda B_\lambda$. Note this convention symmetrizes along rows first and antisymmetrizes along columns second (following, for example, (Sternberg, 1995)).

1.2.3

From now on, fix a complex vector space V of dimension n. Let $\mathfrak{S}_d$ also act on elements of $V^{\otimes d}$ by permuting the coordinates. In particular, the Young symmetrizer C_λ acts on $V^{\otimes d}$. The corresponding Schur–Weyl module is

$$\mathbb{S}_\lambda(V) = C_\lambda \cdot V^{\otimes d}.$$

Clearly, $\mathbb{S}_\lambda(V)$ is a $GL(V)$-module. When r, the number of non-zero parts or the number of rows of λ, is at most n it is known that $\mathbb{S}_\lambda(V)$ is an irreducible representation of $GL(V)$ and that all (in-equivalent) polynomial irreducible representations are constructed this way.

Write $\{e_i\}_{1 \le i \le n}$ for the standard basis of V. For $T \in \mathcal{T}_{\lambda,n}$, define $e_T \in \mathbb{S}_\lambda(V)$ by

$$e_T = C_\lambda \cdot ((e_{T_{11}} \otimes \cdots \otimes e_{T_{1\lambda_1}}) \otimes \cdots \otimes (e_{T_{r1}} \otimes \cdots \otimes e_{T_{r\lambda_r}}))$$

where T_{ij} is the entry in the ith row and jth column of T starting from the top left. Clearly $\mathbb{S}_\lambda(V)$ is spanned by such elements, and it is known that a basis is given by the semistandard ones.

1.2.4

Let $V_\bullet$ be the standard flag in V,

$$V_\bullet : \quad V_i = \operatorname{span}\{e_1, \dots, e_i\} \quad (1 \leq i \leq n).$$

Let $B \subset GL(V)$ be the Borel subgroup given by

$$B = \{g \in GL(V) : gV_i \subset V_i \text{ for } 1 \leq i \leq n\}.$$

Throughout this paper, all highest weights are with respect to B. The highest weight vector of $\mathbb{S}_\lambda(V)$ is

$$e_{T_\lambda} = C_\lambda \cdot ((\underbrace{e_1 \otimes \cdots \otimes e_1}_{\lambda_1}) \otimes \cdots \otimes (\underbrace{(e_r \otimes \cdots \otimes e_r}_{\lambda_r})).$$

That is, T_λ is the tableau of shape λ with all ones in the first row, all twos in the second row, etc.

Example. If $\lambda = (5, 3, 3, 1, 1)$, then

$$T_\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$$

and

$$e_{T_\lambda} = C_\lambda \cdot (e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_2 \otimes e_2 \otimes e_3 \otimes e_3 \otimes e_3 \otimes e_4 \otimes e_5).$$

1.2.5

For any subset A of boxes of T_0, let w_A be the maximum width of a row containing an element of A. Let

$$\mathfrak{S}_A = \{\sigma \in \mathfrak{S}_d : \sigma \text{ preserves } A \text{ and fixes } T_0 \setminus A\}.$$

When $|A| > w_A$, define a Garnir operator as an element of $\mathbb{C}\mathfrak{S}_d$ by

$$G_A = \sum_{\sigma \in \mathfrak{S}_A} \sigma.$$

1.2.6

Let $\mathcal{F}_{\lambda,n}$ be the formal $\mathbb{C}$-span of symbols $T \in \mathcal{T}_{\lambda,n}$ and let $\mathcal{R}_{\lambda,n}$ be the subspace of $\mathcal{F}_{\lambda,n}$ generated by all

$$T_1 - T_2, \text{ where } T_1 \text{ and } T_2 \text{ agree up to a row permutation,} \tag{1.2.6.1}$$

and

$$G_A(T), \text{ where } A \subset T_0 \text{ with } |A| > w_A. \tag{1.2.6.2}$$

Theorem. As $GL(V)$-modules, we have

$$\mathcal{F}_{\lambda,n}/\mathcal{R}_{\lambda,n} \cong \mathbb{S}_\lambda(V).$$

Proof. The map is induced by $T \mapsto e_T$. See, for example, (Fulton, 1997, §8), where the convention is transpose to ours. □

1.3 Outlining the General Formula for Pieri Inclusions

1.3.1

Our general formula for a Pieri inclusion $\Phi_m : \mathbb{S}_\mu(V) \hookrightarrow \wedge^m(V) \otimes \mathbb{S}_\lambda(V)$, where μ is obtained from λ by adding m boxes with no two in the same row, is as follows. If T is a semistandard tableau of shape μ with filling in $\{1, \ldots, n\}$ and $e_T \in \mathbb{S}_\mu(V)$ is the corresponding basis element, then

$$\Phi_m(e_T) = \sum_P \frac{(-1)^P}{H(P)} P(T)$$

where the sum is over a certain set of "m-paths" P which remove m boxes from the shape λ, $(-1)^P$ is a sign, and $H(P)$ is a positive integer that is a product of certain "hook lengths." We will write the path P acting on T as

$$P(T) = e_{Y_P} \otimes e_{T_P}$$

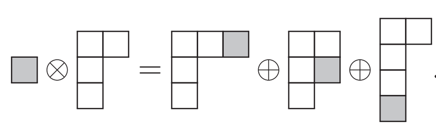
where $e_{Y_P} = e_{i_1} \wedge \cdots \wedge e_{i_m} \in \wedge^m(V)$ is given by the entries of the boxes removed by P, and T_P is a (not necessarily semistandard) tableau of shape λ with filings in $\{1, \ldots, n\}$ such that

$$\{\text{entries of } T\} = \{\text{entries of } Y_P\} \cup \{\text{entries of } T_P\}$$

as a multi-set. All of this will be defined rigorously in Chapter Two.

1.3.2

To illustrate how our formula works, we look at an example in the case when $n = 4$ and $m = 1$. Let $\lambda = (2, 1, 1, 1)$, and $\mu = (2, 1, 1)$. Then the Schur–Weyl module $\mathbb{S}_\lambda(V)$ appears as a summand in the decomposition of $\mathbb{S}_{(1)}(V) \otimes \mathbb{S}_{(2,1,1)}(V) = V \otimes \mathbb{S}_{(2,1,1)}$,



Consider the Pieri inclusion

$$\Phi_1 : \mathbb{S}_{(2,1,1,1)}(V) \hookrightarrow V \otimes \mathbb{S}_{(2,1,1)}(V).$$

By abuse of notation, we will identify semistandard tableaux and their corresponding basis vectors. We will compute

$$\Phi_1\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\right) = \sum_P \frac{(-1)^P}{H(P)} P\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\right).$$

The sum is over all "1-paths" P on λ. That is, certain maps on the boxes in λ that removes a single box. In Figure 1.1 we illustrate all such 1-paths with arrows, shading the boxes on which the path acts, and give the corresponding terms in the image. We will view the box removed by a 1-path as being moved to a "zeroth row" at the top of the diagram and we give the image up to a row permutation, so that it is semistandard.

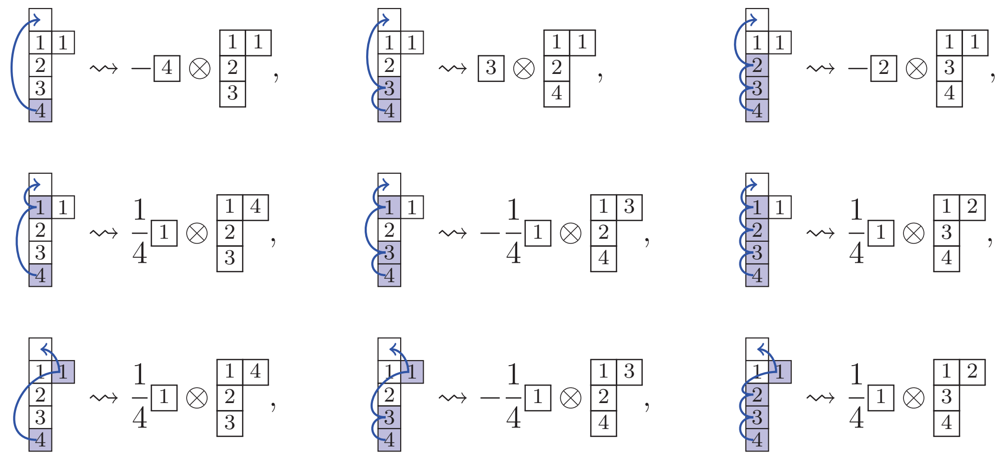


Figure 1.1. The 1-paths acting on the highest weight vector T_λ for $\lambda = (2,1,1,1)$ and the corresponding terms in the image $\Phi_1(T_\lambda)$.

Then up to row permutations we have

$$\Phi_1\left(\ytableaushort{11,2,3,4}\right) = -\,\ytableaushort{4}\otimes\ytableaushort{11,2,3} + \ytableaushort{3}\otimes\ytableaushort{11,2,4} - \ytableaushort{2}\otimes\ytableaushort{11,3,4}$$
$$+\frac{1}{2}\,\ytableaushort{1}\otimes\ytableaushort{14,2,3} - \frac{1}{2}\,\ytableaushort{1}\otimes\ytableaushort{13,2,4} + \frac{1}{2}\,\ytableaushort{1}\otimes\ytableaushort{12,3,4}.$$

1.3.3

In the computation from Section 1.3.2, all terms T_P that appeared were (after row permutations) semistandard. We now compute an example where some of the terms that appear in the image are not semistandard, and so must be straightened. Let Φ_1 be as in Section 1.3.2. We will compute

$$\Phi_1\left(\ytableaushort{12,2,3,4}\right) = \sum_P \frac{(-1)^P}{H(P)} P\left(\ytableaushort{12,2,3,4}\right),$$

where the terms of the sum in the image are again indexed by the 1-paths on λ removing a single box. In Figure 1.2 we illustrate all such 1-paths with arrows, shading the boxes on which the path acts, and give the corresponding terms in the image. As before, we give the image up to row permutations and we now star the terms that need to be straightened.

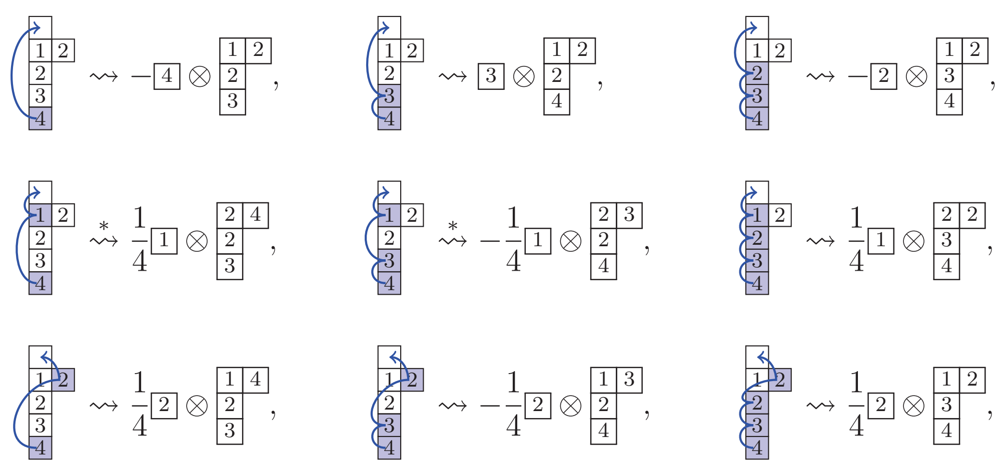


Figure 1.2. The 1-paths acting on the tableau $\ytableaushort{12,2,3,4}$ and the corresponding terms in the image $\Phi_1(T_\lambda)$. The terms that require straightening are starred.

In this case, we must straighten the image of two of the 1-paths (starred), which we show in Section 2.1.8. After straightening we have, up to row permutations,

$$\begin{aligned}\Phi_1\left(\begin{ytableau} 1 & 2 \\ 2 \\ 3 \\ 4 \end{ytableau}\right) = &- \begin{ytableau} 4 \end{ytableau} \otimes \begin{ytableau} 1 & 2 \\ 2 \\ 3 \end{ytableau} + \begin{ytableau} 3 \end{ytableau} \otimes \begin{ytableau} 1 & 2 \\ 2 \\ 4 \end{ytableau} - \frac{3}{4} \begin{ytableau} 2 \end{ytableau} \otimes \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau} \\ &+ \frac{1}{4} \begin{ytableau} 1 \end{ytableau} \otimes \begin{ytableau} 2 & 2 \\ 3 \\ 4 \end{ytableau} + \frac{1}{4} \begin{ytableau} 2 \end{ytableau} \otimes \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau} - \frac{1}{4} \begin{ytableau} 2 \end{ytableau} \otimes \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}.\end{aligned}$$

1.4 Motivation and Organization

1.4.1

Part of the motivation for giving explicit descriptions of Pieri inclusions comes from the frequent use of Pieri inclusions to construct differentials of complexes and resolutions. For example, in results of Eisenbud, Fløstad, and Weyman (Eisenbud et al., 2011) and of Sam and Weyman (Sam & Weyman, 2011), Pieri inclusions are used to compute pure free resolutions for classical groups and in (Pragacz & Weyman, 1985), Weyman and Pragacz use Pieri inclusions to describe Lascoux resolutions. Sam has also built a package for Macaulay2 (PieriMaps) (Sam, 2009) that computes Pieri inclusions explicitly using the algorithm from (Sam & Weyman, 2011).

In a forthcoming paper, (Hunziker et al., n.d.), we will show how our description of Pieri inclusions can be used to explicitly describe the differentials in minimal free resolutions of modules of covariants (in the context of Weyl's fundamental theorems). These resolutions will be obtained via Howe duality from Bernstein—Gelfand—Gelfand resolutions of unitary highest weight modules, the terms of which are direct sums of (parabolic) Verma modules.

1.4.2

The rest of the dissertation is organized in the following way. In Chapter Two we describe the construction of the Pieri inclusion in the one-box removal case (V) and the general m-box removal case ($\wedge^m(V)$), respectively. In Chapter Three we show that all Garnir relations are generated by Garnir relations of minimal size over hooks and give tools for collapsing the sum in the image of a Pieri Inclusion. In Chapters Four and Five we use the tools from Chapter Three to show that the Pieri inclusions are $GL(V)$-maps in the one-box removal and m-box removal cases, respectively. In Chapter Six we show that the one-box removal map is the negative of Olver's description via the uniqueness of an equivariant map, then use this and (Sam & Weyman, 2011) to show that this same description of Pieri inclusions gives a map in the case $\mathbb{S}_\mu(V) \hookrightarrow \mathrm{Sym}^m(V) \otimes \mathbb{S}_\lambda(V)$ and that iterating the the-box removal map is also a $GL(V)$ map. In this chapter we also show that when removing a column of boxes from a diagram the m-box removal map and iterating one-box removal m times differ by a multiple of $m!$. Finally, in Chapter Seven we compare the computational complexity of the one-box removal description to that of the description given by Olver and use the Pieri inclusion removing one box to optimally describe the image of a highest weight vector.

CHAPTER TWO

Constructing the Pieri Inclusions

In this chapter we construct the Pieri inclusions, first for removing a single box and then the general case removing many boxes.

2.1 Constructing the Pieri Inclusion for Removing One Box

2.1.1

We will write $\lambda = (\lambda_1, \ldots, \lambda_r)$ in block form as $\lambda = (w_1^{h_1}, \ldots, w_N^{h_N})$, where $w_i < w_{i+1}$ and exactly h_i parts of λ are equal to w_i. That is, N is the number of blocks in the shape λ, where block [1] is the lowest geometrically, w_b is the width of block $[b]$, and h_b is the height of block $[b]$. See Figure 2.1.

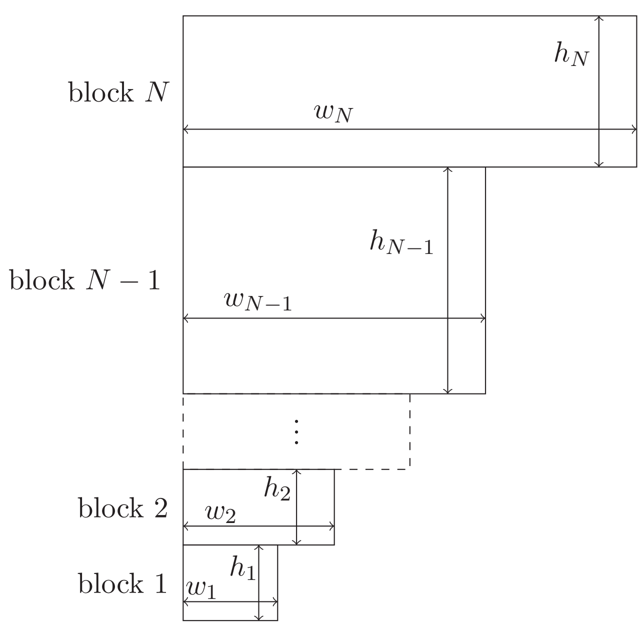
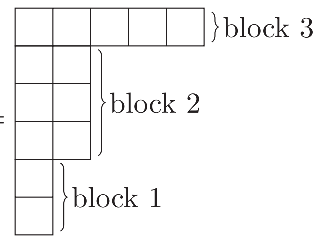


Figure 2.1. The shape λ with N blocks.

Example. We will write $(5,2,2,2,1,1)$ in block form as $(1^2, 2^3, 5)$,

$$(5,2,2,2,1,1) = (1^2, 2^3, 5) =$$



.

2.1.2

For any box $x \in T_0$ at the bottom right of some block k, let $\lambda \backslash \{x\}$ be the partition whose shape is obtained by removing the box x from T_0 as in Figure 2.2.

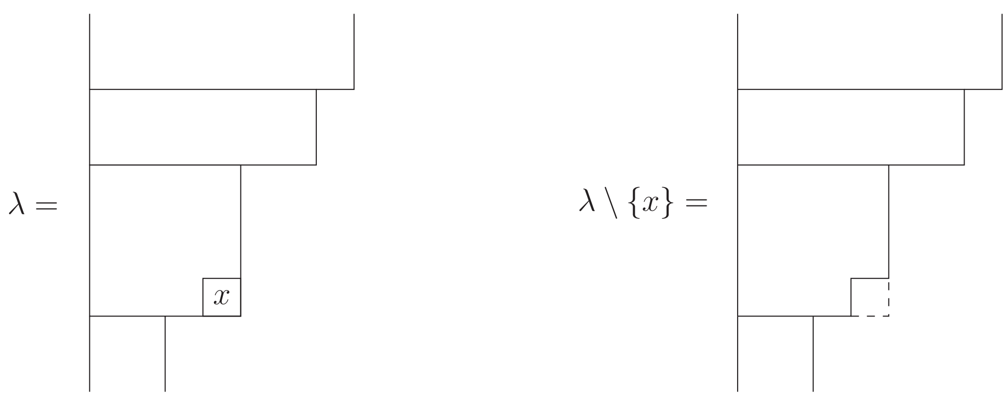


Figure 2.2. Obtaining $\lambda \setminus \{x\}$ from λ.

We will define the map

$$\Phi_1 : \mathcal{F}_{\lambda,n} \to V \otimes \mathcal{F}_{\lambda\setminus\{x\},n}$$

on a basis and then show that Φ_1 descends to a $GL(V)$-module map as in Figure 2.3.

2.1.3

We first introduce further notation. For a given shape λ, let $[b]$ denote the bth block, $[b](i)$ denote the ith row of the bth block, and $[b](i,j)$ denote the box in block

$$
\begin{tikzcd}
\mathcal{F}_{\lambda,n} \arrow[r, "\Phi_1"] \arrow[d] & V \otimes \mathcal{F}_{\lambda\setminus\{x\},n} \arrow[d] \\
\mathcal{F}_{\lambda,n}/\mathcal{R}_{\lambda,n} \cong \mathbb{S}_\lambda(V) \arrow[r, dashed, "\Phi_1"] & V \otimes \mathcal{F}_{\lambda\setminus\{x\},n}/\mathcal{R}_{\lambda,n} \cong V \otimes \mathbb{S}_{\lambda\setminus\{x\}}(V)
\end{tikzcd}
$$

Figure 2.3. Φ_1 descending to a $GL(V)$-module map.

$[b]$, row i, and column j, with block 1 and row 1 the lowest geometrically and column 1 the furthest left. We write

$$[b](i,j) \leq [c](k,l)$$

if $[b](i,j)$ is geometrically (weakly) lower than $[c](k,l)$, i.e. $b < c$ or $b = c$ and $i \leq k$. The strict inequality is defined in the natural way. We will extend this notation to compare boxes, rows, and blocks in the natural way. For a given $T \in \mathcal{F}_{\lambda,n}$, we denote the entry in box $[b](i,j)$ by $T_{[b](i,j)}$.

Example. If T is the tableau

$$T = \begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 3 & 3 & 4 \\
\hline
2 & 2 & \multicolumn{3}{c}{} \\
\cline{1-2}
3 & 4 & \multicolumn{3}{c}{} \\
\cline{1-2}
4 & 5 & \multicolumn{3}{c}{} \\
\cline{1-2}
6 & \multicolumn{4}{c}{} \\
\cline{1-1}
7 & \multicolumn{4}{c}{} \\
\cline{1-1}
\end{array},$$

then $T_{[1](2,1)} = 6$ and $T_{[3](1,5)} = 4$.

2.1.4

An evacuation route R is a selection of a string of boxes starting from the bottom of some block. An example of an evacuation route on $\lambda = (1^2, 3, 4^3, 7^2)$ is given by the shaded boxes in the diagram in Figure 2.4.

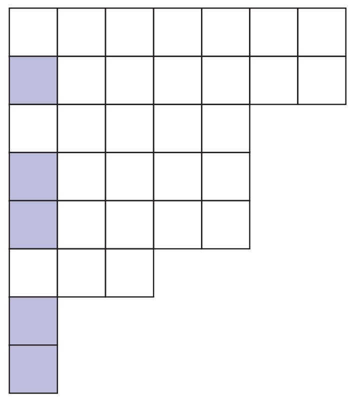


Figure 2.4. A simple example of an evacuation route on $\lambda = (1^2, 3, 4^3, 7^2)$.

An evacuation route has more freedom than the previous example shows, where each of the chosen boxes was in the first column of the row. The boxes in an evacuation route must be chosen starting from the bottom of some block, and the route can jump up to higher blocks, however the boxes need not be only in the first column. The chosen boxes can in fact be in any column in the diagram. Another example of an evacuation route on $\lambda = (1^2, 3, 4^3, 7^2)$ is given by the shaded boxes in the diagram in Figure 2.5.

The above examples show that an evacuation route does not need to contain a box from every row, however, it cannot skip rows within a block. The shaded selection of boxes in Figure 2.6 is not an evacuation route on $(2, 3^2, 5^4, 7^2)$ since a box in row

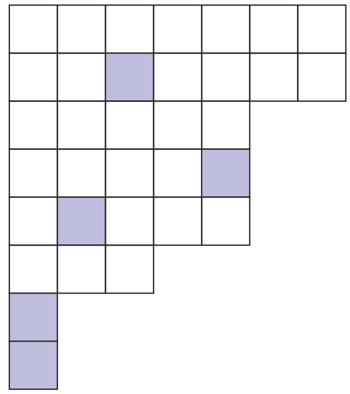


Figure 2.5. A more complex example of an evacuation route on $\lambda = (1^2, 3, 4^3, 7^2)$.

[3](3) (that is, the third row in the third block) is selected, while there is no box selected from row [3](2), which is below row [3](3) but still in block [3].

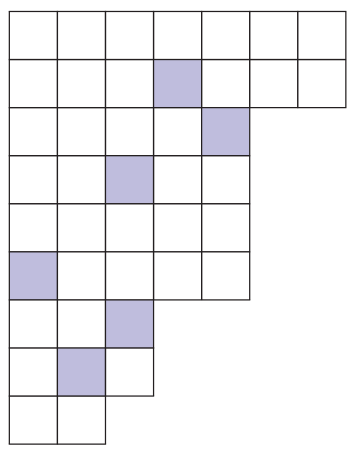


Figure 2.6. A non-example of an evacuation route on $(2, 3^2, 5^4, 7^2)$.

Formally, we have the following definition.

Definition (Evacuation Route). An evacuation route R starting at $[b_0]$ is a subset of boxes in T_0 such that R contains a box in row $[b_0](1)$, R contains at most one box per row, and if $[b](i, j) \in R$, then $[b](k, j_k) \in R$ for all $1 \leq k < i$ and some $1 \leq j_k \leq w_b$.

2.1.5

A 1-path P on λ moves boxes up the diagram via some associated evacuation route R^P. We will treat a 1-path as acting on general shapes, where the highest box in R^P is "removed" by the 1-path and viewed as being moved to the box $[N+1](1,1)$ attached to the top of T_0. In Figure 2.7 we illustrate a 1-path moving boxes up a diagram via an evacuation route, highlighting only the boxes in the evacuation route.

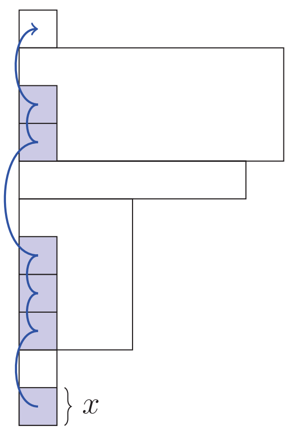


Figure 2.7. A simple example of a 1-path removing the box x.

As before, an evacuation route can contain boxes in any column in the diagram. Another 1-path moving boxes up a diagram is shown in Figure 2.8.

Formally, we have the following definition.

Definition (1-path). Let $X = \{x_1 := [b_1](1, w_{b_1})\}$ and $Y = \{y_1 := [N+1](1,1)\}$, where Y is viewed as block $[N+1]$ attached to the top of T_0. A 1-path P removing X is a map of boxes

$$P : \lambda \cup Y \to \lambda \cup Y$$

along with an evacuation route $R = R^P$ such that the following hold.

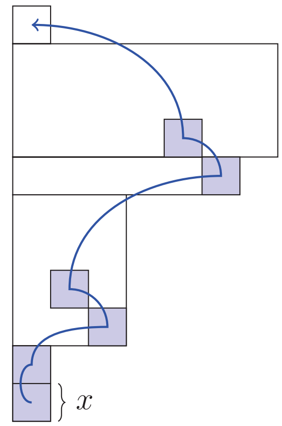


Figure 2.8. A more complex example of a 1-path removing the box x.

- R starts at $[b_1]$. Note that R can contain x_1, though this is not a requirement.
- P is geometrically increasing on rows, with P strictly increasing on R. That is, for all boxes $x \in \lambda \cup Y$, $x \leq P(x)$.
- If R_1 is the orbit of x_1 under $P^{\mathbb{N}}$, then $y_1 \in R_1$ and $R \cup X \cup Y = R_1$.
- P preserves row order in R within blocks. That is, if $[b](i,j),\ [b](k,l) \in R$ with $i < k$ and $P([b](i,j)), P([b](k,l)) \in [b]$, then $P([b](i,j)) < P([b](k,l))$.
- P fixes those boxes not in R or X, i.e. $P = \mathrm{id}_{\lambda \cup Y}$ except on $R \cup X$, and $P(R) = R \setminus X \cup Y$.

2.1.6

We now define the components of the formulation of the Pieri inclusion removing one box, Φ_1. For a 1-path P removing X with evacuation route R^P, let h^P be the number of rows in R^P and $(-1)^P := (-1)^{h^P}$. For $b = b_1, \ldots, N$, let h^P_b to be the number of rows in $R^P \cap [b]$. For $b \geq b_1 + 1$ define $h(b) = w_b - w_{b-1} + h_{b-1}$ to be the hook length of block $[b]$, and for $b = b_1 + 1, \ldots, N$ define

$$H(b) = \sum_{j=b_1+1}^{b} h(j).$$

Then for $b = b_1 + 1, \ldots, N$, let

$$H_b(P) = \begin{cases} 1 & \text{if } R^P \cap [b] = \emptyset \\ H(b) & \text{otherwise} \end{cases},$$

and let $H_{b_1}(P) = 1$. Define

$$H(P) = \prod_{b=b_1}^{n} H_b(P).$$

Example. For the partition $(1, 3^2, 4^3, 6^2)$, shown in Figure 2.9, and $X = \{[1](1,1)\}$ (shaded), we have $h(2) = 3 - 1 + 1 = 3$, $h(3) = 4 - 3 + 2 = 3$, $h(4) = 6 - 4 + 3 = 5$, $H(1) = 1$, $H(2) = 3$, $H(3) = 6$, and $H(4) = 11$.

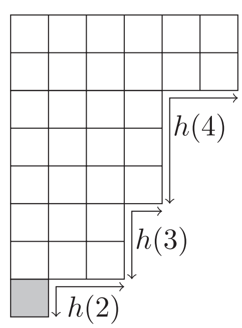


Figure 2.9. The hook length of a block, $h(b)$.

2.1.7

For $T \in \mathcal{F}_{\lambda,n}$, denote by α_1^P the entry in the box $P^{-1}(y_1) \in T$ and extend P to T by acting on the entries, with the image

$$P(T) = Y_P \otimes T_P \in V \otimes \mathcal{F}_{\lambda\backslash X,n},$$

where

$$Y_P = E_X \boxed{\alpha_1^P},$$

which is standard form notation is $e_{\alpha_1^P} \in V$, and $T_P \in \mathcal{F}_{\lambda\backslash X,n}$ is defined by

$$(T_P)_{[b](i,j)} = T_{P^{-1}([b](i,j))}.$$

We omit E_X and just write

$$\boxed{\alpha_1^P} \quad \text{in place of} \quad E_X\boxed{\alpha_1^P}$$

in the image of $P(T)$.

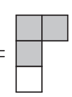
Definition (Φ_1). The map $\Phi_1 : \mathcal{F}_{\lambda,n} \to V \otimes \mathcal{F}_{\lambda\backslash\{x\},n}$ is given by

$$\Phi_1(T) = \sum_P \frac{(-1)^P}{H(P)} P(T)$$

where the sum is over all 1-paths P removing X.

2.1.8

We now compute the straightening example from Section 1.3.3. If

$$A_1 = \{[2](1,1), [2], (1,2), [1](2,1)\} =$$ 

then modulo $\mathcal{R}_{(2,1,1),4}$ we have

$$\frac{1}{2} G_{A_1}\left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}\right) = 2\,\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}.$$

Then if

$$A_2 = \{[1](1,1), [1](2,1)\} = \begin{ytableau} \; & \; \\ *(gray) \\ *(gray) \end{ytableau},$$

modulo $\mathcal{R}_{(2,1,1),4}$,

$$G_{A_2}\left(\begin{ytableau} 2 & 2 \\ 4 \\ 3 \end{ytableau}\right) = \begin{ytableau} 2 & 2 \\ 4 \\ 3 \end{ytableau} + \begin{ytableau} 2 & 2 \\ 3 \\ 4 \end{ytableau}.$$

Thus modulo $\mathcal{R}_{(2,1,1),4}$ we get

$$\frac{1}{4}\, \begin{ytableau} 1 \end{ytableau} \otimes \begin{ytableau} 2 & 4 \\ 2 \\ 3 \end{ytableau} = \frac{1}{8}\, \begin{ytableau} 1 \end{ytableau} \otimes \begin{ytableau} 2 & 2 \\ 3 \\ 4 \end{ytableau}.$$

Similarly, modulo $\mathcal{R}_{(2,1,1),4}$ we have

$$\frac{1}{4}\, \begin{ytableau} 1 \end{ytableau} \otimes \begin{ytableau} 2 & 3 \\ 2 \\ 4 \end{ytableau} = -\frac{1}{8}\, \begin{ytableau} 1 \end{ytableau} \otimes \begin{ytableau} 2 & 2 \\ 3 \\ 4 \end{ytableau}.$$

Note that in this example the terms that were straightened cancelled with each other and so did not appear in the image. This will not be the case in general.

2.2 Constructing the Pieri Inclusion for Removing Many Boxes

2.2.1

Let $X = \{x_1 = [b_1](1, w_{b_1}), \ldots, x_m = [b_m](i_m, w_{b_m})\}$ be a set of m boxes in λ with $x_i < x_{i+1}$ so that removing the boxes in X from T_0 gives a Young diagram and let $\lambda \setminus X$ be the associated partition. See Figure 2.10.

We call such a set X a removal set for T_0 (or for λ). As before, we will define the map $\Phi_m : \mathcal{F}_{\lambda,n} \to F_m \otimes \mathcal{F}_{\lambda\setminus X,n}$ on a basis, where $F_m = \bigwedge^m V$, and then show that Φ_m is a $GL(V)$-map. See Figure 2.11.

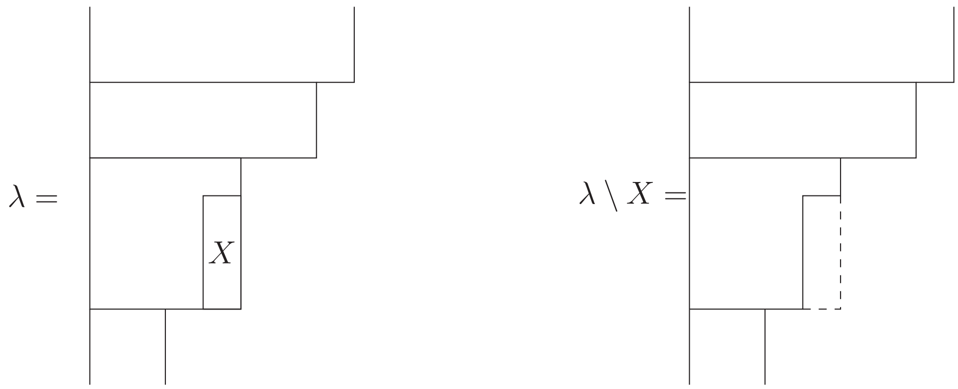


Figure 2.10. Obtaining $\lambda \setminus X$ from λ.

$$\begin{tikzcd}
\mathcal{F}_{\lambda,n} \arrow[rr, "\Phi_m"] \arrow[d] & & F_m \otimes \mathcal{F}_{\lambda\setminus X,n} \arrow[d] \\
\mathcal{F}_{\lambda,n}/\mathcal{R}_{\lambda,n} \cong \mathbb{S}_\lambda(V) \arrow[r, dashed, "\Phi_m"] & F_m \otimes \mathcal{F}_{\lambda\setminus X,n}/\mathcal{R}_{\lambda,n} \cong F_m \otimes \mathbb{S}_{\lambda\setminus X}(V) &
\end{tikzcd}$$

Figure 2.11. Φ_m descending to a $GL(V)$-module map.

2.2.2

Extending the notion of a 1-path, an m-path on λ is a map of boxes that moves boxes up the diagram via some associated evacuation route with m interlaced orbits. As with 1-paths, we treat m-paths as acting on general shapes, where the highest m boxes in R^P are "removed" by the m-path and viewed as being moved to the boxes $[N+1](1,1), \ldots, [N+1](m,1)$ attached to the top of T_0. An example of a 2-path removing X is given in Figure 2.12, where we highlight only the boxes in the evacuation route and distinguish the two distinct orbits.

The interlacing property for m-paths is not so strict as the above example suggests. We require that an m-path interlaces orbits only within blocks while multiple orbits are present. This is illustrated further in Figure 2.13, where we show a 2-

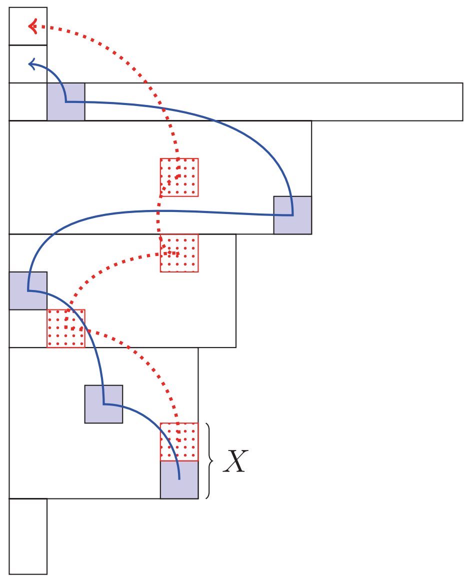


Figure 2.12. An example of a 2-path.

path removing X and again highlight only the boxes in the evacuation route and distinguish the two distinct orbits.

Formally, we have the following definition.

Definition (m-path). Let $X = \{x_1 = [b_1](1, w_{b_1}), \ldots, x_m = [b_m](i_m, w_{b_m})\}$ be a removal set for T_0 and $Y = \{y_1 := [N+1](1,1), \ldots, y_m := [N+1](m,1)\}$, where Y is viewed as block $N+1$ attached to the top of T_0. An m-path P removing X is a map of boxes

$$P : \lambda \cup Y \to \lambda \cup Y$$

along with an evacuation route $R = R^P$ such that the following hold.

- R starts at $[b_1]$. Note that R can intersect X, though this is not a requirement.

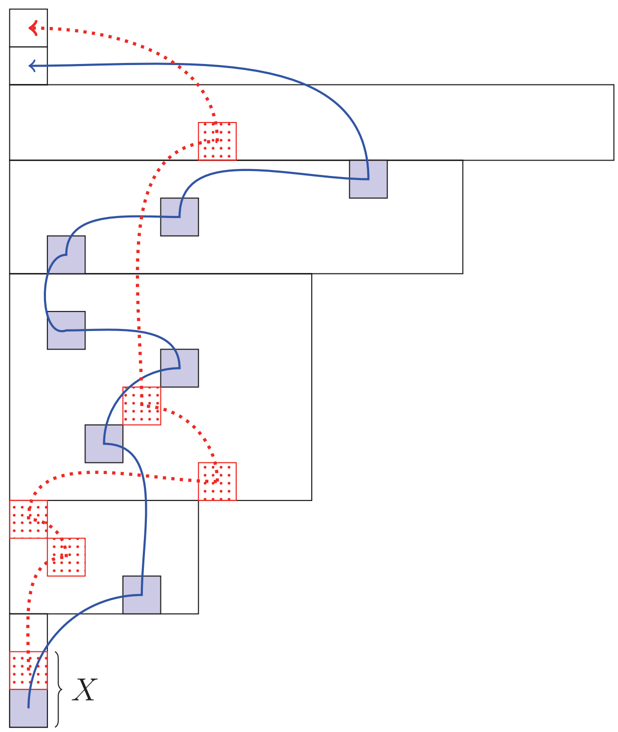


Figure 2.13. An example of a 2-path showing the interlacing property.

- P is geometrically increasing on rows, with P strictly increasing on R. That is, for all boxes $x \in \lambda \cup Y$, $x \leq P(x)$.

- If R_i is the orbit of x_i under $P^{\mathbb{N}}$, then $y_i \in R_i$ and $R \cup X \cup Y = \bigsqcup_{i=1}^{m} R_i$.

- If there are k distinct orbits in a block, then the first k rows of that block must be in different orbits. i.e., if $R_{i_1}^P, \ldots, R_{i_k}^P$ intersect some block $[b]$, then for $j = 1, \ldots k$, up to relabeling, $R_{i_j}^P \cap [b](j) \neq \emptyset$.

- P preserves row order in R within blocks, and so interlaces orbits. That is, if $[b](i,j)$, $[b](k,l) \in R$ with $i < k$ and $P([b](i,j)), P([b](k,l)) \in [b]$, then $P([b](i,j)) < P([b](k,l))$.

- P fixes those boxes not in R or X, i.e. $P = \mathrm{id}_{\lambda \cup Y}$ except on $R \cup X$, and $P(R) = R \setminus X \cup Y$.

2.2.3

For an m-path P with evacuation route R^P, let h^P, $(-1)^P$, h_b^P, and $H(b)$ be defined as in Section 2.1.6. For $b = b_1 + 1, \ldots, N$, let $H_b(P) = 1$ if $R^P \cap [b] = \emptyset$ and let $H_{b_1}(P) = 1$. If $b \geq b_1 + 1$ and $\left| R^P \cap [b] \right| = k_b \neq 0$, then let

$$H_b(P) = \prod_{i=1}^{k_b} (H(b) - (m - i))$$

and

$$H(P) = \prod_{b=b_1}^{n} H_b(P).$$

2.2.4

For $T \in \mathcal{F}_{\lambda,n}$, denote by α_i^P the entry in the box $P^{-1}(y_i) \in T$ and extend P to T by acting on the entries, with the image

$$P(T) = Y_P \otimes T_P \in F_m \otimes \mathcal{F}_{\lambda \backslash X,n} = \bigwedge^m V \otimes \mathcal{F}_{\lambda \backslash X,n}.$$

Here

$$Y_P = E_X \begin{array}{|c|}\hline \alpha_m^P \\ \hline \vdots \\ \hline \alpha_1^P \\ \hline \end{array},$$

which is standard form notation is $e_{\alpha_1^P} \wedge \cdots \wedge e_{\alpha_m^P} \in \bigwedge^m V$, and $T_P \in \mathcal{F}_{\lambda \backslash X,n}$ is defined by $(T_P)_{[b](i,j)} = T_{P^{-1}([b](i,j))}$. As before, we omit E_X and just write

$$\begin{array}{|c|}\hline \alpha_m^P \\ \hline \vdots \\ \hline \alpha_1^P \\ \hline \end{array} \quad \text{in place of} \quad E_X \begin{array}{|c|}\hline \alpha_m^P \\ \hline \vdots \\ \hline \alpha_1^P \\ \hline \end{array}$$

in the image of $P(T)$.

Definition (Φ_m). The map $\Phi_m : \mathcal{F}_{\lambda,n} \to F_m \otimes \mathcal{F}_{\lambda \backslash X,n}$ is given by

$$\Phi_m(T) = \sum_P \frac{(-1)^P}{H(P)} P(T),$$

where the sum is over all m-paths P removing X.

2.2.5

We now compute an example of the Pieri inclusion when $m = 2$. Let $n = 6$, $\lambda = (2, 2, 1, 1, 1, 1)$, and $\mu = (2, 2, 1, 1)$. Then the Schur–Weyl module $\mathbb{S}_\lambda(V)$ appears as a summand in the decomposition of $\mathbb{S}_{(1,1)}(V) \otimes \mathbb{S}_\mu(V)$, as seen in Figure 2.14.

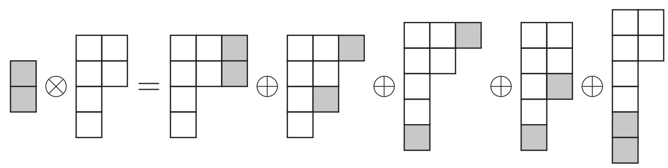


Figure 2.14. The Pieri rule for $\mathbb{S}_{(1,1)}(V) \otimes \mathbb{S}_{(2,2,1,1)}(V)$.

Consider the Pieri inclusion $\Phi_2 : \mathbb{S}_{(2,2,1,1,1,1)}(V) \longrightarrow \mathbb{S}_{(1,1)}(V) \otimes \mathbb{S}_{(2,2,1,1)}(V)$. We will show the image of the highest weight vector

$$T_{(2,2,1,1,1,1)} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

under this map,

$$\Phi_2 \left(T_{(2,2,1,1,1,1)} \right) = \sum_P \frac{(-1)^P}{H(P)} P \left(T_{(2,2,1,1,1,1)} \right),$$

where the sum is over all m-paths P on λ removing $X = \{x_1 = [1](1,1), x_2 = [1](2,1)\}$. In Figure 2.15 we illustrate all such paths with arrows, where we shade the boxes in the evacuation route, distinguishing the orbits of x_1 and x_2. For paths hitting rows [2](1) and [2](2), we only show the path that hits the first column of both rows, as the paths that hit the second column in either row will give the same result. As in the 1-box removal example, we give the images up to row permutations and we star the paths whose images require straightening.

If

$$A_1 = \{[2](1,1), [2], (1,2), [2](2,2)\} = \begin{ytableau} {} & *(gray) \\ *(gray) & *(gray) \\ {} \\ {} \end{ytableau}$$

then we have, modulo $\mathcal{R}_{(2,2,1,1),6}$,

$$\begin{ytableau} 1 & 5 \\ 2 & 4 \\ 3 \\ 6 \end{ytableau} = \frac{1}{2} G_{A_1} \left(\begin{ytableau} 1 & 5 \\ 2 & 4 \\ 3 \\ 6 \end{ytableau} \right) - \begin{ytableau} 1 & 2 \\ 4 & 5 \\ 3 \\ 6 \end{ytableau} - \begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \\ 6 \end{ytableau}.$$

Then if

$$A_2 = \{[2](1,1), [2], (1,2), [1](2,1)\} = \begin{ytableau} {} & {} \\ *(gray) & *(gray) \\ *(gray) \\ {} \end{ytableau},$$

modulo $\mathcal{R}_{(2,2,1,1),6}$,

$$\begin{ytableau} 1 & 2 \\ 4 & 5 \\ 3 \\ 6 \end{ytableau} = \frac{1}{2} G_{A_2} \left(\begin{ytableau} 1 & 2 \\ 4 & 5 \\ 3 \\ 6 \end{ytableau} \right) - \begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 \\ 6 \end{ytableau} - \begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 \\ 6 \end{ytableau}.$$

Thus, modulo $\mathcal{R}_{(2,2,1,1),6}$,

$$\begin{ytableau} 1 & 5 \\ 2 & 4 \\ 3 \\ 6 \end{ytableau} = \begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 \\ 6 \end{ytableau} + \begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 \\ 6 \end{ytableau} - \begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \\ 6 \end{ytableau}.$$

Similarly, via straightening we have, modulo $\mathcal{R}_{(2,2,1,1),6}$,

$$\ytableaushort{15,23,4,6} = -\ytableaushort{12,35,4,6} - \ytableaushort{13,25,4,6}, \qquad \ytableaushort{16,25,3,4} = -\ytableaushort{12,36,4,5} - \ytableaushort{12,35,4,6} - \ytableaushort{15,26,3,4},$$

$$\ytableaushort{16,24,3,5} = \ytableaushort{12,36,4,5} - \ytableaushort{12,34,5,6} - \ytableaushort{14,26,3,5}, \qquad \ytableaushort{16,23,4,5} = -\ytableaushort{12,36,4,6} - \ytableaushort{13,26,4,5},$$

and

$$\ytableaushort{14,23,5,6} = -\ytableaushort{12,34,5,6} - \ytableaushort{13,24,5,6}.$$

Recall that for all 2-paths P, $Y_P \in \bigwedge^2 V$, and so

$$\ytableaushort{\alpha,\beta} = -\ytableaushort{\beta,\alpha}.$$

Thus,

$$\begin{aligned}
\Phi_2\left(\ytableaushort{11,22,3,4,5,6}\right) = {} & \ytableaushort{5,6} \otimes \ytableaushort{11,22,3,4} - \ytableaushort{4,6} \otimes \ytableaushort{11,22,3,5} + \ytableaushort{3,6} \otimes \ytableaushort{11,22,4,5} - \frac{1}{2}\ytableaushort{2,6} \otimes \ytableaushort{12,25,3,4} + \frac{1}{2}\ytableaushort{2,6} \otimes \ytableaushort{11,24,3,5} \\
& -\frac{1}{2}\ytableaushort{2,6} \otimes \ytableaushort{11,23,4,5} + \ytableaushort{1,6} \otimes \ytableaushort{12,25,3,4} - \ytableaushort{1,6} \otimes \ytableaushort{12,24,3,5} + \ytableaushort{1,6} \otimes \ytableaushort{12,23,4,5} + \ytableaushort{4,5} \otimes \ytableaushort{11,22,3,6} \\
& -\ytableaushort{3,5} \otimes \ytableaushort{11,22,4,6} + \frac{1}{2}\ytableaushort{2,5} \otimes \ytableaushort{11,26,3,4} - \frac{1}{2}\ytableaushort{2,5} \otimes \ytableaushort{11,24,3,6} + \frac{1}{2}\ytableaushort{2,5} \otimes \ytableaushort{11,23,4,6} - \ytableaushort{1,5} \otimes \ytableaushort{12,26,3,4} \\
& +\ytableaushort{1,5} \otimes \ytableaushort{12,24,3,6} - \ytableaushort{1,5} \otimes \ytableaushort{12,23,4,6} + \ytableaushort{3,4} \otimes \ytableaushort{11,22,5,6} + \frac{1}{2}\ytableaushort{2,4} \otimes \ytableaushort{11,25,3,6} - \frac{1}{2}\ytableaushort{2,4} \otimes \ytableaushort{11,23,5,6} \\
& -\frac{1}{2}\ytableaushort{2,4} \otimes \ytableaushort{11,26,3,5} - \ytableaushort{1,4} \otimes \ytableaushort{12,25,3,6} + \ytableaushort{1,4} \otimes \ytableaushort{12,23,5,6} - \ytableaushort{1,4} \otimes \ytableaushort{12,26,3,5} + \frac{1}{2}\ytableaushort{2,3} \otimes \ytableaushort{11,24,5,6}
\end{aligned}$$

$$+\frac{1}{2}\ \ytableaushort{2,3} \otimes \ytableaushort{11,26,4,5} - \frac{1}{2}\ \ytableaushort{2,3} \otimes \ytableaushort{11,25,4,6} - \ytableaushort{1,3} \otimes \ytableaushort{12,24,5,6} - \ytableaushort{1,3} \otimes \ytableaushort{12,26,4,5} + \frac{1}{2}\ \ytableaushort{1,3} \otimes \ytableaushort{12,25,4,6}$$

$$+\frac{2}{5}\ \ytableaushort{1,2} \otimes \ytableaushort{15,26,3,4} - \frac{2}{5}\ \ytableaushort{1,2} \otimes \ytableaushort{14,26,3,5} + \frac{2}{5}\ \ytableaushort{1,2} \otimes \ytableaushort{13,26,4,5} - \frac{1}{5}\ \ytableaushort{1,2} \otimes \ytableaushort{12,35,4,6}$$

$$-\frac{1}{5}\ \ytableaushort{1,2} \otimes \ytableaushort{12,34,5,6} + \frac{2}{5}\ \ytableaushort{1,2} \otimes \ytableaushort{14,25,3,6} + \frac{2}{5}\ \ytableaushort{1,2} \otimes \ytableaushort{13,24,5,6} - \frac{2}{5}\ \ytableaushort{1,2} \otimes \ytableaushort{13,25,4,6}$$

$$+\frac{2}{5}\ \ytableaushort{1,2} \otimes \ytableaushort{12,36,4,5}.$$

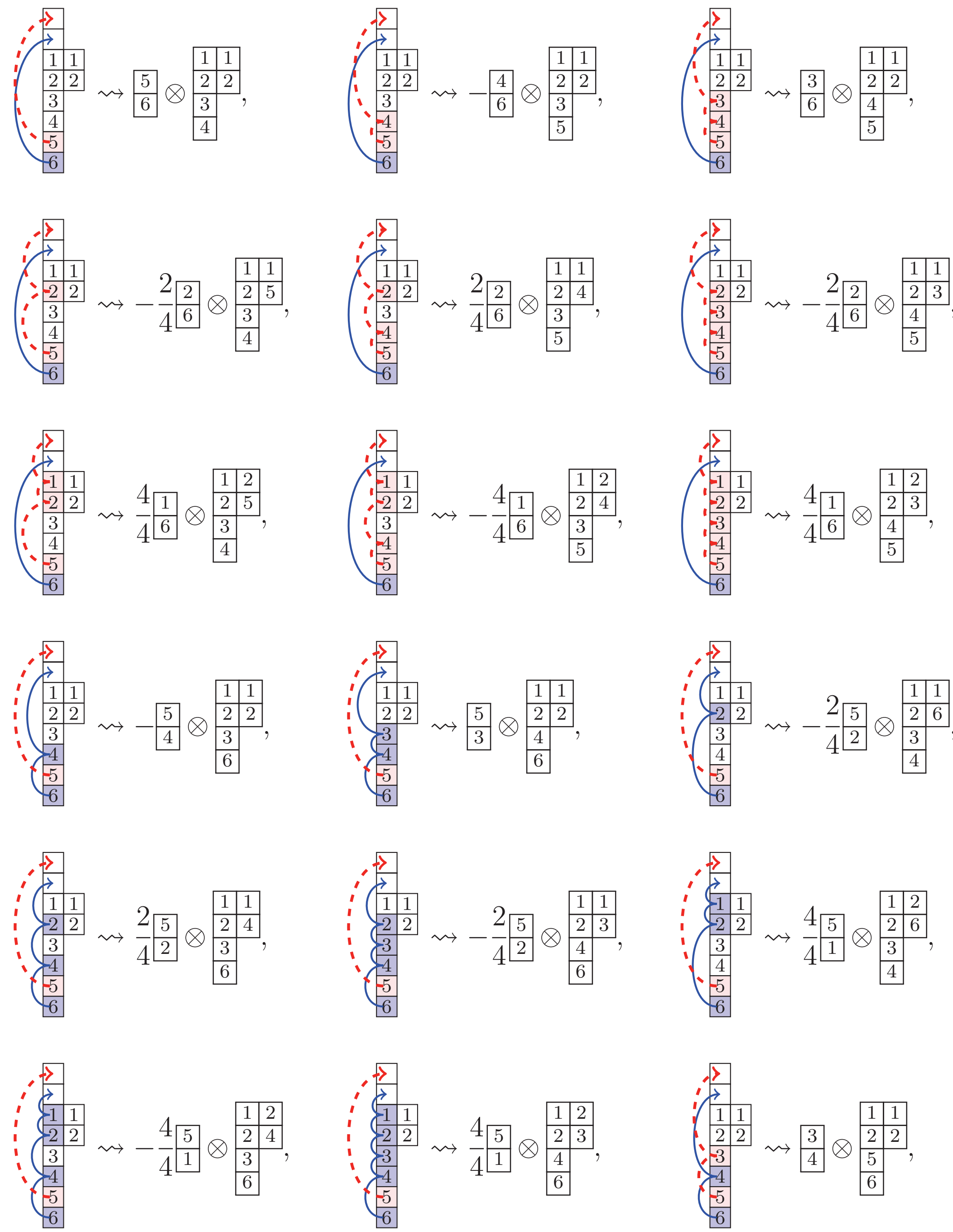


Figure 2.15. The 2-paths acting on $T_{(2,2,1,1,1,1)}$ with the distinct orbits distinguished and the corresponding terms in the image $\Phi_2\left(T_{(2,2,1,1,1,1)}\right)$. The terms that require straightening are starred.

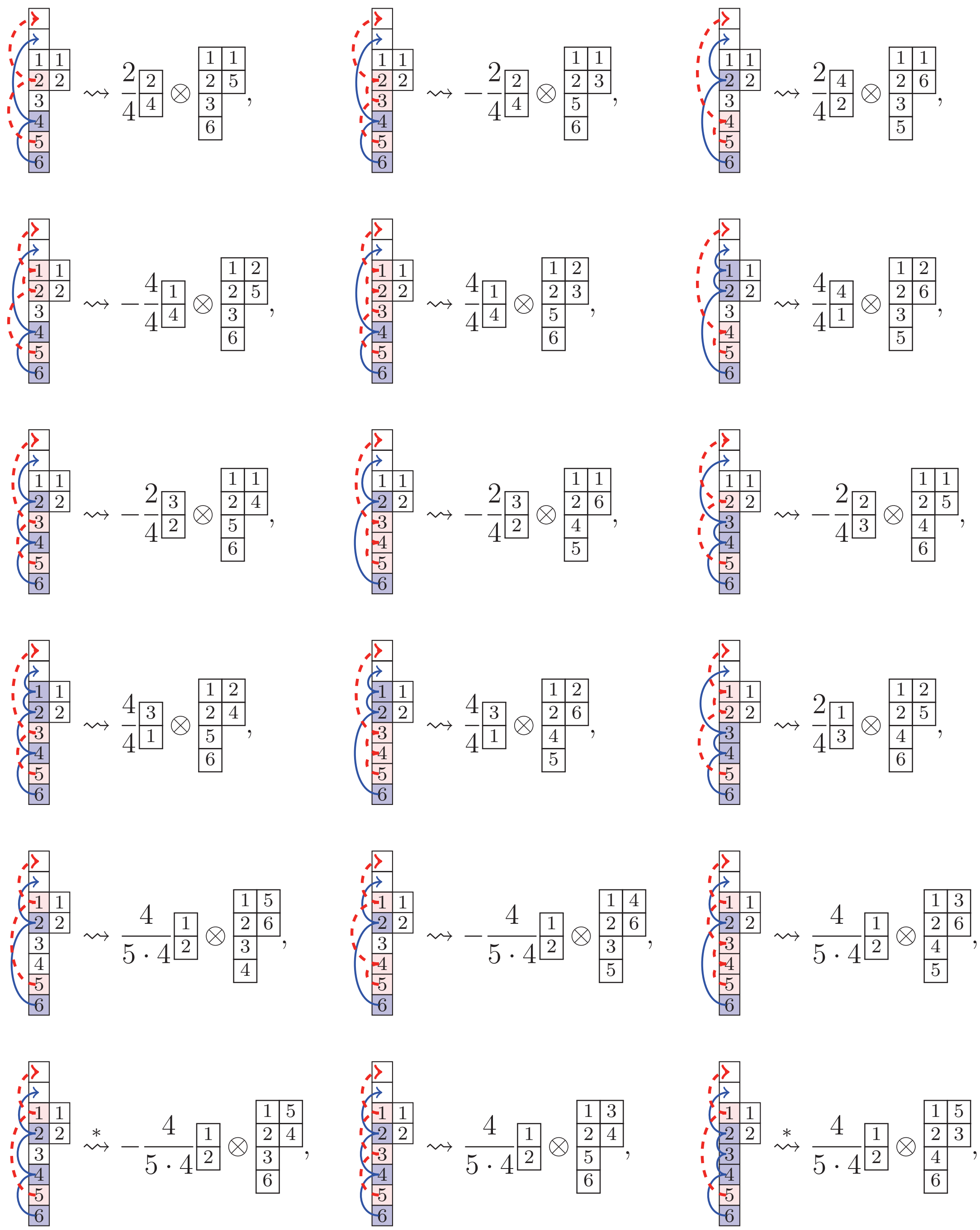


Figure 2.15 (Cont.). The 2-paths acting on $T_{(2,2,1,1,1,1)}$ with the distinct orbits distinguished and the corresponding terms in the image $\Phi_2\left(T_{(2,2,1,1,1,1)}\right)$. The terms that require straightening are starred.

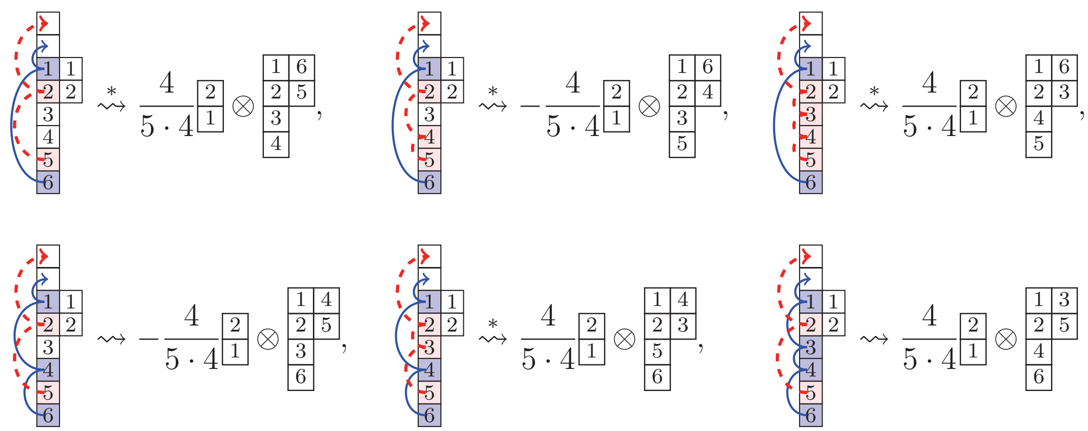


Figure 2.15 (Cont.). The 2-paths acting on $T_{(2,2,1,1,1,1)}$ with the distinct orbits distinguished and the corresponding terms in the image $\Phi_2\left(T_{(2,2,1,1,1,1)}\right)$. The terms that require straightening are starred.

CHAPTER THREE

Generating Garnir Relations and Tools for Collapsing Sums

In this chapter we will show that all Garnir relations are generated by Garnir relations of minimal size, i.e. those G_A with $|A| = w_A + 1$, where w_A is the maximum width of a row containing an element of A. We then show that all Garnir relations of minimal size are themselves generated by Garnir relations over "hooks", which are those G_A where A is of minimal size and consists of exactly a complete row and one other box. We then give the tools for collapsing the sum in the image $\Phi_m(T)$.

3.1 Generating All Garnir Relations by Those of Minimal Size

3.1.1

We start by formalizing the idea of a hook.

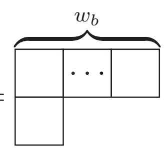
Definition (Hook). We say that $A \subset T_0$ is a hook if for some row $[b](r)$,

$$A = [b](r) \cup \{a_0\}$$

where

$$a_0 = \begin{cases} [b](r-1, 1) & \text{if } r \neq 1 \\ [b-1](h_{b-1}, 1) & \text{if } r = 1 \end{cases}.$$

That is,

$A =$ 

Theorem. Let $T \in \mathcal{F}_{\lambda,n}$ and $A \subset T_0$ such that $|A| > w_A$. Then

$$G_A(T) \in \langle G_{A'}(T') : T' \in \mathcal{F}_{\lambda,n},\ A' \text{ is a hook}\rangle.$$

3.1.2

To prove Theorem 3.1.1, we first show that $G_A(T)$ is generated by Garnir operators of minimal size for any $T \in \mathcal{F}_{\lambda,n}$ and any $A \subset T_0$ with $|A| > w_A$. We will also show that if $|A| > w_A + 1$, then $G_A(T)$ is generated by Garnir operators over $A \setminus \{y\}$ for any $y \in A$.

Lemma. Let $T \in \mathcal{F}_{\lambda,n}$. If $A \subset T_0$ with $|A| > w_A$, then for any $x \in T_0$ such that $|A \cup \{x\}| > w_{A \cup \{x_0\}}$,

$$G_{A\cup\{x\}}(T) \in \langle G_A(T') : T' \in \mathcal{F}_{\lambda,n}\rangle.$$

Proof. Let $A \subset T_0$ with $|A| > w_A$ and $x \in T_0 \setminus A$. For all $y \in A \cup \{x\}$, let $\tau_{x,y}$ be the permutation that switches x and y and fixes the rest of $A \cup \{x\}$. Then for any $\sigma \in \mathfrak{S}_{A\cup\{x\}}$,

$$\sigma(y) = x \iff (\sigma\tau_{x,y})(x) = x \iff \sigma\tau_{x,y} \in \mathfrak{S}_A.$$

Then,

$$\begin{aligned} G_{A\cup\{x\}}(T) &= \sum_{\sigma \in \mathfrak{S}_{A\cup\{x\}}} \sigma T \\ &= \sum_{y \in A\cup\{x\}} \sum_{\substack{\sigma \in \mathfrak{S}_{A\cup\{x\}} \\ \text{s.t. } \sigma(y)=x}} \sigma T \\ &= \sum_{y \in A\cup\{x\}} \sum_{\substack{\sigma \in \mathfrak{S}_{A\cup\{x\}} \\ \text{s.t. } \sigma(y)=x}} (\sigma\tau_{x,y})(\tau_{x,y}T) \end{aligned}$$

$$= \sum_{y \in A \cup \{x\}} \sum_{\tilde{\sigma} \in \mathfrak{S}_A} \tilde{\sigma}\left(\tau_{x,y} T\right)$$

$$= \sum_{y \in A \cup \{x\}} G_A(\tau_{x,y} T) \in \langle G_A(T') : T' \in \mathcal{F}_{\lambda,n} \rangle$$

□

3.1.3

We now show that all Garnir operators of minimal size are generated by Garnir operators over a set consisting of a full row and a box below that row.

Lemma. Let $T \in \mathcal{F}_{\lambda,n}$. If $A \subset T_0$ of minimal size, then

$G_A(T) \in \langle G_{A' \cup \{b_0\}}(T') : T' \in \mathcal{F}_{\lambda,n}, A' = [b](r)$ for some $[b](r) \subset T_0, b_0 < [b](r)\rangle$.

Proof. Let $T \in \mathcal{F}_{\lambda,n}$ and $A \subset T_0$ such that $A \subset [b](r)$. Assume, without loss of generality, that $r \neq 1$ (else, replace $[b](r-1)$ in the following argument with $[b-1](h_{b-1})$). Let $B \subset T_0$ such that $B < [b](r)$, $B \not\subset [b](r-1)$, and $|A \cup B| = w_b + 1$. Assume $A \neq [b](r)$, i.e. $|B| \neq 1$. We will show that

$$G_{A \cup B}(T) \in \langle G_{A' \cup B'}(T') : T' \in \mathcal{F}_{\lambda,n},\ A' = [b](r),\ |B'| = 1, B < A\rangle.$$

Pick $x_0 \in [b](r) \setminus A$ and $b_0 \in B$, as in Figure 3.1.

For all $x \in A \cup \{x_0\} \cup B$, define $\tau_{x_0,x}$ as before. Then for all $\sigma \in \mathfrak{S}_{A \cup \{x_0\} \cup B}$,

$$\sigma(x_0) = x \iff (\tau_{x_0,x}\sigma)\, x_0 = x_0 \iff \tau_{x_0,x}\sigma \in \mathfrak{S}_{A \cup B}$$

and

$$\sigma(x_0) = x \iff (\sigma\tau_{x_0,x})\, x = x \iff \sigma\tau_{x_0,x} \in \mathfrak{S}_{A \cup B \cup \{x_0\} \setminus \{x\}}.$$

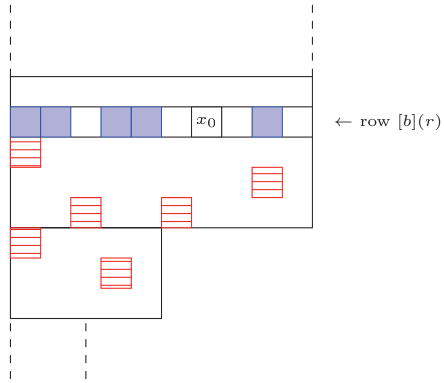


Figure 3.1. A set of boxes in T_0 with a distinguished top row. The blue solid boxes represent the set A and the red striped boxes represent the set B where $|A \cup B| = w_b + 1$.

Then,

$$\begin{aligned}
G_{A\cup\{x_0\}\cup B}(T) &= \sum_{\sigma\in\mathfrak{S}_{A\cup\{x_0\}\cup B}} \sigma T \\
&= \sum_{a\in A\cup\{x_0\}} \sum_{\substack{\sigma\in\mathfrak{S}_{A\cup\{x_0\}\cup B},\\ \sigma(x_0)=a}} \sigma T + \sum_{b\in B} \sum_{\substack{\sigma\in\mathfrak{S}_{A\cup\{x_0\}\cup B},\\ \sigma(x_0)=b}} \sigma T \\
&= \sum_{a\in A\cup\{x_0\}} \sum_{\substack{\sigma\in\mathfrak{S}_{A\cup\{x_0\}\cup B},\\ \sigma(x_0)=a}} \tau_{x_o,a}\left(\tau_{x_0,a}\sigma\right) T + \sum_{b\in B} \sum_{\substack{\sigma\in\mathfrak{S}_{A\cup\{x_0\}\cup B},\\ \sigma(x_0)=b}} \left(\sigma\tau_{x_0,b}\right)\tau_{x_0,b}T \\
&= \sum_{a\in A\cup\{x_0\}} \sum_{\tilde\sigma\in\mathfrak{S}_{A\cup B}} \tau_{x_o,a}\tilde\sigma T + \sum_{b\in B} \sum_{\tilde\sigma\in\mathfrak{S}_{A\cup\{x_0\}\cup B\setminus\{b\}}} \tilde\sigma\tau_{x_0,b}T \\
&= \sum_{a\in A\cup\{x_0\}} \tau_{x_0,a} G_{A\cup B}(T) + \sum_{b\in B} G_{A\cup\{x_0\}\cup B\setminus\{b\}}(\tau_{x_0,b}T)
\end{aligned}$$

and as $\tau_{x_0,a}$ is a row permutation for all $a \in A \cup \{x_0\}$, up to row permutations we have

$$G_{A\cup\{x_0\}\cup B}(T) = |A\cup\{x_0\}| G_{A\cup B}(T) + \sum_{b\in B} G_{A\cup\{x_0\}\cup B\setminus\{b\}}(\tau_{x_0,b}T). \tag{3.1.3.1}$$

Solving for $G_{A\cup B}(T)$ in equation 3.1.3.1 we get

$$G_{A\cup B}(T) = \frac{1}{|A\cup\{x_0\}|}\left(G_{A\cup\{x_0\}\cup B}(T) - \sum_{b\in B} G_{A\cup\{x_0\}\cup B\setminus\{b\}}(\tau_{x_0,b}T)\right).$$

By Lemma 3.1.2, $G_{A\cup B\cup\{x_0\}}(T)$ is generated by Garnir relations over $A\cup\{x_0\}\cup B\setminus\{b_0\}$. Thus $G_{A\cup B}(T)$ is generated by Garnir relations over $A'\cup B'$, where $A' = A\cup\{x_0\}$, so that $|A'\cap[b](r)| = |A\cap[b](r)|+1$, and $B' = B\setminus\{b\}$ for some $b\in B$, so that $|B'| = |B|-1$. By induction, we get that

$$G_{A\cup B}(T) \in \langle G_{A'\cup B'}(T') : T'\in\mathcal{F}_{\lambda,n},\ A' = [b](r),\ |B'| = 1\rangle.$$

□

3.1.4

We now give a way to to write $G_{A\cup B}(T)$ as above as a sum of 2-cycles, which will make our calculations easier throughout.

Lemma. Let $A = [b](r)$ and $b_0 \in T_0\setminus A$. Then for all $T\in\mathcal{F}_{\lambda,n}$, modulo $\mathcal{R}_{\lambda,n}$ we have

$$G_A(T) = w_b!\sum_{a\in A}\tau_{a,b_0}T.$$

Proof. As all $\tilde\sigma\in\mathfrak{S}_A$ are row permutations and $|A| = w_b$ we have, modulo $\mathcal{R}_{\lambda,n}$,

$$\begin{aligned} G_A(T) &= \sum_{\sigma\in\mathfrak{S}_{A\cup\{b_0\}}}\sigma T \\ &= \sum_{\tilde\sigma\in\mathfrak{S}_A}\sum_{a\in A\cup\{b_0\}}\tilde\sigma\tau_{a,b_0}T \\ &= w_b!\sum_{a\in A\cup\{b_0\}}\tau_{a,b_0}T. \end{aligned}$$

□

3.1.5

To prove Theorem 3.1.1, it remains to show that all Garnir relations of the form $G_{A\cup B}(T)$ where $A = [b](r)$ and $|B| = 1$, with $B < A$, are generated by Garnir relations over hooks. We show that for any such A and B, $G_{A\cup B}(T)$ is generated by Garnir relations over $A' \cup B'$ where A' is a full row and $|B| = 1$ with $B' < A'$, and where the distance between A' and B' is less than the distance between A and B. Theorem 3.1.1 is then proved by iterating this until we get that $G_{A\cup B}(T)$ is generated (up to row permutation) by Garnir relations over hooks.

Lemma. Let $T \in \mathcal{F}_{\lambda,n}$, $A = [b](r)$, $B \subset T_0$ with $|B| = 1$ and $B < A$. Then

$$G_{A\cup B}(T) \in \langle G_{A'}(T') : T' \in \mathcal{F}_{\lambda,n}, A' \text{ is a hook}\rangle.$$

Proof. Let $A = [b](r)$ and $B = \{b_0\}$ with $b_0 \in [c](s)$ and $[c](s) < [b](r)$. Let j be the number of rows between $[b](r)$ and $[c](s)$. Without loss of generality we will assume that $r > j+1$ and $b_0 = [b](r-j-1, 1)$. Then

$$\begin{aligned} G_{A\cup B}(T) &= \sum_{\sigma \in A\cup B} \sigma T \\ &= \sum_{\tilde{\sigma} \in \mathfrak{S}_A} \sum_{a \in A\cup B} \tilde{\sigma}\tau_{a,b_0} T \\ &= w_b! \sum_{a \in A\cup B} \tau_{a,b_0} T. \end{aligned}$$

We also have that for all $a \in A \cup B$,

$$G_{[b](r-j)\cup B}(\tau_{a,b_0}T) = w_b! \left(\tau_{a,b_0}T + \sum_{x \in [b](r-j)} \tau_{x,b_0}\tau_{a,b_0}T \right)$$

and hence

$$\tau_{a,b_0}T = \frac{1}{w_b!}G_{[b](r-j)\cup B}(\tau_{a,b_0}T) - \sum_{x\in[b](r-j)} \tau_{x,b_0}\tau_{a,b_0}T.$$

Now observe that for all $a \in A \cup B$ and all $x \in [b](r-j)$,

$$\tau_{x,b_0}\tau_{a,b_0}T = \tau_{a,x}\tau_{x,b_0}T.$$

Then we have

$$\begin{aligned}
G_{A\cup B}(T) &= w_b! \sum_{a\in A\cup B} \tau_{a,b_0}T \\
&= w_b! \left(\sum_{a\in A\cup B} \frac{1}{w_b!} G_{[b](r-j)\cup B}(\tau_{a,b_0}T) - \sum_{x\in[b](r-j)} \tau_{x,b_0}\tau_{a,b_0}T \right) \\
&= \sum_{a\in A\cup B} G_{[b](r-j)\cup B}(\tau_{a,b_0}T) - w_b! \sum_{a\in A\cup B}\sum_{x\in[b](r-j)} \tau_{x,b_0}\tau_{a,b_0}T \\
&= \sum_{a\in A\cup B} G_{[b](r-j)\cup B}(\tau_{a,b_0}T) - w_b! \sum_{x\in[b](r-j)} \left(\tau_{x,b_0}T + \sum_{a\in A} \tau_{x,b_0}\tau_{a,b_0}T \right) \\
&= \sum_{a\in A\cup B} G_{[b](r-j)\cup B}(\tau_{a,b_0}T) - w_b! \sum_{x\in[b](r-j)} \left(\tau_{x,b_0}T + \sum_{a\in A} \tau_{a,x}\tau_{x,b_0}T \right) \\
&= \sum_{a\in A\cup B} G_{[b](r-j)\cup B}(\tau_{a,b_0}T) - w_b! \sum_{x\in[b](r-j)} G_{A\cup\{x\}}(\tau_{x,b_0}T).
\end{aligned}$$

□

So for any $T \in \mathcal{F}_{\lambda,n}$ and any $A \subset T_0$ with $|A| > w_A$, $G_A(T)$ is generated by Garnir relations over hooks.

3.2 Collapsing the Sum in the Image of a Pieri Inclusion

3.2.1

The rest of this chapter is devoted to collapsing the sum in the image $\Phi_m(T)$. We first consider the 1-path case, where the idea is that the sum over all possible paths between two boxes can be collapsed to a single tableau, modulo $\mathcal{R}_{\lambda\setminus X,n}$, with parity depending only on the number of rows between the two boxes. See Figure 3.2. We then generalize the result to 2-paths, before considering the m-path case.

Definition (σ_k^A). Let $A \subset T_0$ be a hook with $[b](r)$ the top row of A. Label the boxes in $[b](r)$ as $a_1, \ldots, a_{w_b}$ and let a_0 be the box in A below $[b](r)$. For $k = 0, \ldots, w_b$, define σ_k^A to be the permutation of A that switches a_0 and a_k and is the identity otherwise. For $T \in \mathcal{F}_{\lambda,n}$ and $0 \leq k \leq w_b$, let $A_k = T_{a_k}$ and extend σ_k^A to act on the entries of T, so that $\sigma_k^A A_k = A_0$ and σ_k^A is the identity on T otherwise. Then, by Lemma 3.1.4, modulo $\mathcal{R}_{\lambda,n}$

$$G_A(T) = w_b! \sum_{k=0}^{w_b} \sigma_k^A T.$$

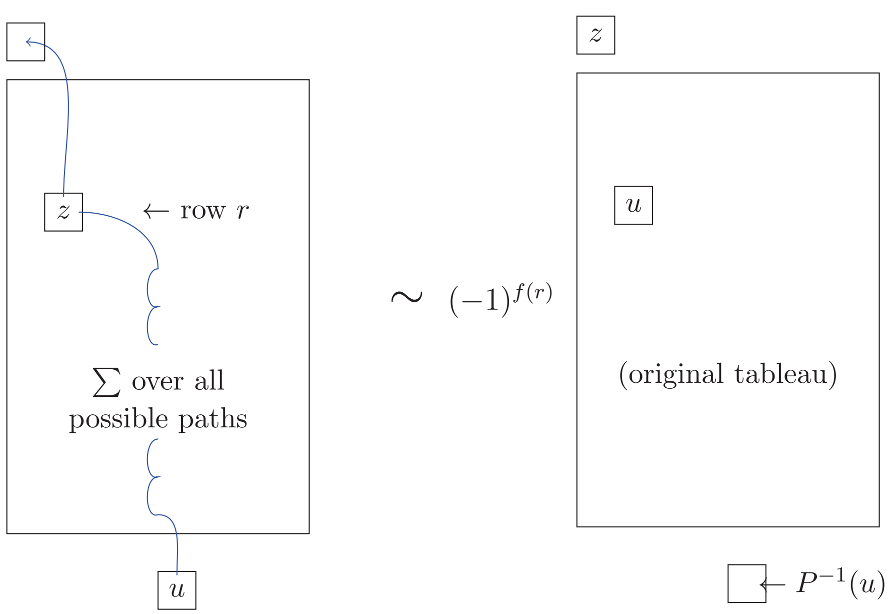


Figure 3.2. Collapsing a sum of paths.

It will be useful to be able to identify those paths that are similar to a given m-path. Given an m-path P and two rows $[b](r)$ and $[c](s)$, a $([b](r),[c](s))$-path extension of P is an m-path Q that is identical to P except on the interval of rows $([b](r),[c](s))$ and on any boxes whose image under P is in the interval of rows $([b](r),[c](s))$. In the row interval $([b](r),[c](s))$, Q can differ from P, and in fact can even act on different boxes.

Example. Let the 1-path P be as in Figure 3.3. For any $([2](2),[4](1))$-path extension Q of P, it must be that $\{[1](1,1),\{[1](2,1),[2](1,3)\} \subset R^Q$ as these are the boxes in R^P outside of the interval of rows $([2](2),[4](1))$. As $[2](1,3) \in P^{-1}(([2](2),[4](1)))$, Q must be identical to P on $\{[1](1,1),\{[1](2,1)\}$, but it can be the case that $Q([2](1,3)) \neq P([2](1,3))$. Two such examples of $([2](2),[4](1))$-path extensions of P are given in Figure 3.4.

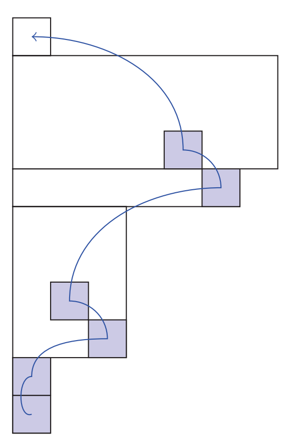


Figure 3.3. An example of a 1-path.

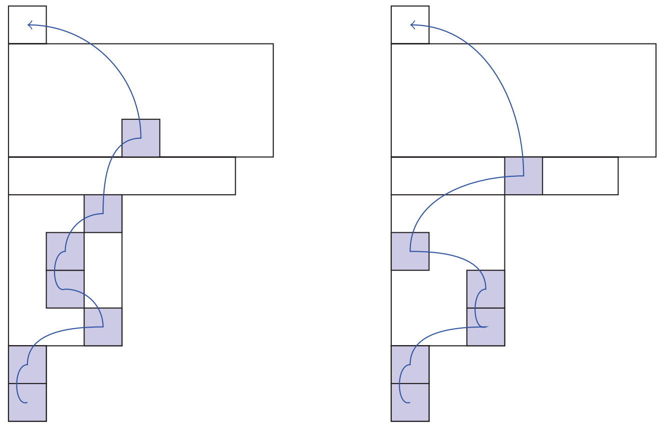


Figure 3.4. Two examples of $([2](2),[4](1))$-path extensions of the 1-path in Figure 3.3.

Example. Let the 2-path P be as in Figure 3.5. For any $([3](1),[4](1))$-path extension Q of P, it must be that $R^Q \setminus ([3](1),[4](1)) = R^P \setminus ([3](1),[4](1))$. As

$$\{[2](2,2),[2](3,1)\} \subset P^{-1}(([3](1),[4](1))),$$

Q must be identical to P on

$$R^Q \setminus (([3](1),[4](1)) \cup \{[2](2,2),[2](3,1)\}),$$

but Q can differ from P otherwise. An example of a $([3](1),[4](1))$-path extension of P is given in Figure 3.6.

Given an evacuation route R and a row $[b](r)$, define

$$R_{<[b](r)} := \{x \in R : x < [b](r)\} \qquad \text{and} \qquad R_{>[b](r)} := \{x \in R : [b](r) < x\}.$$

We formalize the notion of path extensions with the following definitions.

Definition 3.2.2.1 (Route Extension). Given an evacuation route R and two rows $[b](r)$ and $[c](s)$ with $[b](r) \leq [c](s)$, an evacuation route B is a $([b](r),[c](s))$-route extension of R if $R_{<[b](r)} = B_{<[b](r)}$ and $R_{>[c](s)} = B_{>[c](s)}$.

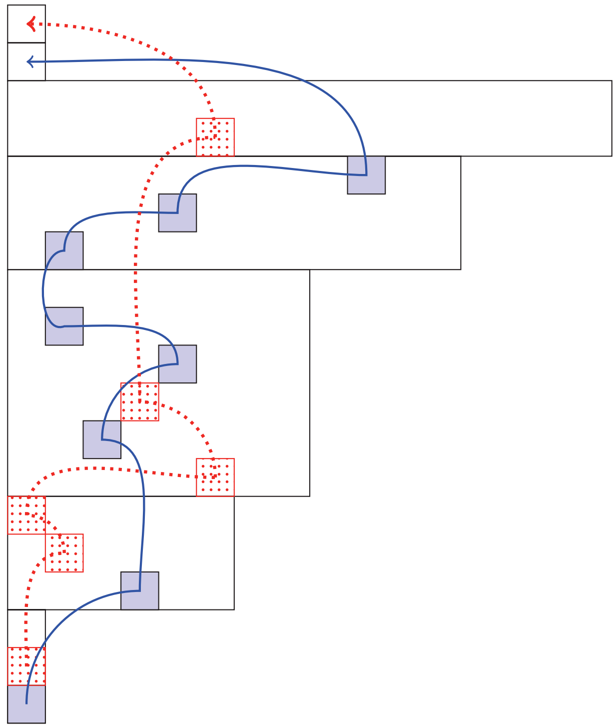


Figure 3.5. An example of a 2-path.

Definition 3.2.2.2 (Path Extension). Given an m-path P and two rows $[b](r)$ and $[c](s)$ with $[b](r) \leq [c](s)$, an m-path Q is a $([b](r), [c](s))$-path extension of P if:

- R^Q is a $([b](r), [c](s))$-route extension of R^P,
- $P|_{R^P_{>[c](s)}} = Q|_{R^P_{>[c](s)}}$
- $P|_{R^P_{<[b](r)} \setminus I} = Q|_{R^P_{<[b](r)} \setminus I}$, where $I = \{x \in R^P_{<[b](r)} : P(x) \in ([b](r), [c](s))\}$.

3.2.3

For any $T \in \mathcal{F}_{\lambda,n}$, let

$$X = \{x_1 := [b_1](1, w_{b_1})\} \text{ and } Y = \{y_1 := [N+1](1,1)\}$$

and let

$$z_1 := T_{[b_z](i_1,j_1)}$$

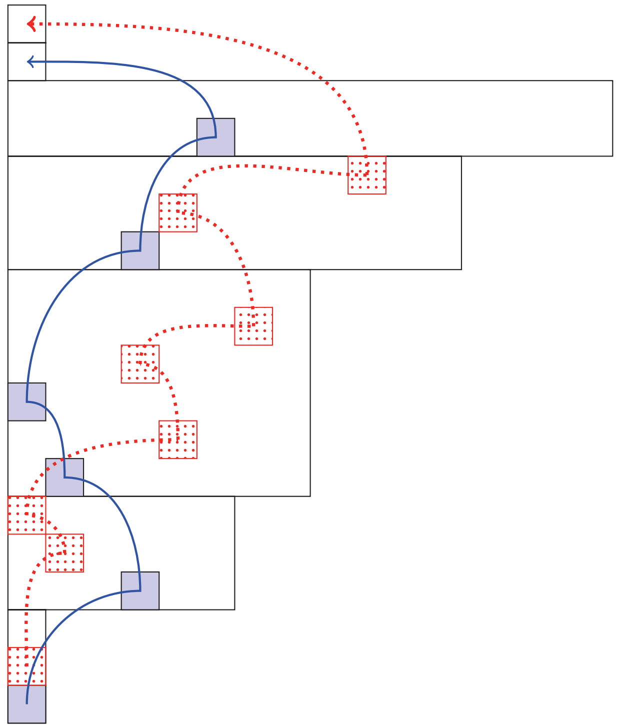


Figure 3.6. An example of a $([3](1),[4](1))$-path extension of the 2-path in Figure 3.5.

for some $1 \leq b_1 \leq b_z \leq N$, $1 \leq i_1 \leq h_{b_z}$, and $1 \leq j_1 \leq w_{b_z}$. Let $u := T_{[b_u](i_u,j_u)})$ for some $b_1 \leq b_u \leq b_z$, $1 \leq i_u \leq h_{b_u}$, and $1 \leq j_u \leq w_{b_u}$, and let P be any 1-path on λ removing X such that $P([b_u](i_u,j_u)) \in [b_z](1)$ and $P([b_z](i_1,j_1)) > [b_z](i_1)$, including the case $P([b_z](i_1,j_1)) = y_1$. Let

$$[P] = \{\text{1-paths } Q \text{ on } \lambda : Q \text{ is a } ([b_z](1),[b_z](i_1))\text{-path extension of } P$$
$$\text{with } Q([b_z](i_1,j_1)) = P([b_z](i_1,j_1))\}$$

and $T' \in \mathcal{F}_{\lambda \setminus X,n}$ be the unique tableau such that $T' = T_P$ on $(\lambda \setminus X)$, except on the interval of rows $([b_z](1),[b_z](i_1))$, where $T' = T$, except $T'_{[b_z](i_1,j_1)} = u$. We then have the following.

Lemma. For $[P]$ as above, modulo $F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}$,

$$\sum_{Q\in[P]} Q(T) = (-1)^{i_1-1} \boxed{\alpha_1^P} \otimes T'.$$

Proof. Assume, without loss of generality, that $j_1 = 1$. We will show the case $b_u < b_z$, the case $b_u = b_z$ is similar. If $i_1 = 1$, then $[P] = \{P\}$, and so modulo $F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}$ we get

$$\begin{aligned}\sum_{Q\in[P]} Q(T) &= P(T)\\ &= \boxed{\alpha_1^P} \otimes T',\end{aligned}$$

as desired. Let $i_1 = 2$ and

$$A = \{a_1 := [b_z](1,k), \ldots, a_{w_{b_z}} := [b_z](1, w_{b_z})\} \cup \{a_0 := [b_z](2,1)\}.$$

Then by Lemma 3.1.4 we have the following, modulo $F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}$. See Figure 3.7.

$$\begin{aligned}\sum_{Q\in[P]} Q(T) &= \sum_{k=1}^{w_{b_z}} \boxed{\alpha_1^P} \otimes \sigma_k^A T'\\ &= \boxed{\alpha_1^P} \otimes \left(\frac{1}{w_{b_z}!} G_A\left(T'\right) - T'\right)\\ &= -\boxed{\alpha_1^P} \otimes T'.\end{aligned}$$

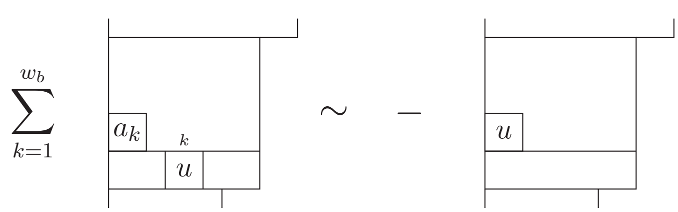


Figure 3.7. Collapsing a sum via Garnir relations on the bottom row of a block.

Now let $i_1 > 2$ and

$$B = \{b_1 := [b_z](i_1 - 1, k), \ldots, b_{w_{b_z}} := [b_z](i_1 - 1, w_{b_z})\} \cup \{b_0 := [b_z](i_1, 1)\}.$$

By Lemma 3.1.4 and induction applied to each entry in $(i_1 - 1)^{b_z}$, see Figure 3.8, modulo $F_1 \otimes \mathcal{R}_{\lambda \setminus X, n}$ we have

$$\begin{aligned}
\sum_{Q \in [P]} Q(T) &= \sum_{k=1}^{w_{b_z}} (-1)^{i_1 - 2} \boxed{\alpha_1^P} \otimes \sigma_k^B T' \\
&= (-1)^{i_1 - 2} \boxed{\alpha_1^P} \otimes \left(\frac{1}{w_{b_z}!} G_B(T') - T' \right) \\
&= (-1)^{i_1 - 1} \boxed{\alpha_1^P} \otimes T'.
\end{aligned}$$

Thus the claim holds for $1 \leq i_1 \leq h_{b_z}$. $\square$

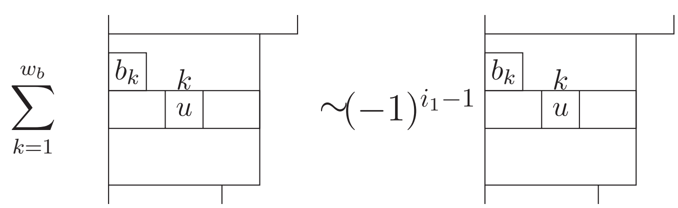


Figure 3.8. Collapsing a sum via Garnir relations in the middle of a block.

3.2.4

Lemma 3.2.3 also allows for calculations of sums of 2-paths, by applying the technique of the proof twice and "skipping" certain rows each time. That is, for any $T \in \mathcal{F}_{\lambda,n}$, let

$$X := \{x_1 := [b_1](1, w_{b_1}), x_2 := [b_2](i_2, w_{b_2})\},$$

$$Y := \{y_1 := [N+1](1,1), y_2 := [N+1](2,1)\}$$

and let

$$z_1 := T_{[b_z](i_1,j_1)}, \;\; z_2 := T_{[b_z](i_2,j_2)}$$

for some $1 \le b_1 \le b_z \le N$, $1 \le i_2 < i_1 \le h_{b_z}$, and $1 \le j_1, j_2 \le w_{b_z}$. Let

$$u_1 := T_{[b_{u_1}](i_{u_1},j_{u_1})}, \;\; u_2 := T_{[b_{u_2}](i_{u_2},j_{u_2})}$$

for some $b_1 \le b_{u_1}, b_{u_2} \le b_z$, $1 \le i_{u_1} \le h_{b_{u_1}}$, $1 \le j_{u_1} \le w_{b_{u_1}}$, $1 \le i_{u_2} \le h_{b_{u_2}}$, and $1 \le j_{u_2} \le w_{b_{u_2}}$. If $b_{u_1} = b_{u_2}$, then we also assume that $i_{u_1} \ne i_{u_2}$. Let P be any 2-path on λ such that

$$P([b_{u_1}](i_{u_1}, j_{u_1})) \in [b_z](1), P([b_{u_2}](i_{u_2}, j_{u_2})) \in [b_z](2)$$

and

$$P([b_z](i_1, j_1)), P([b_z](i_2, j_2)) > [b_z](i_1).$$

Assume, without loss of generality, that $P([b_z](i_1, j_1)), P([b_z](i_2, j_2)) \notin Y$. Let

$$\begin{aligned}[P] = \{&\text{2-paths } Q \text{ on } \lambda : Q \text{ is a } ([b_z](1), [b_z](i_1))\text{-path extension of } P \\ &\text{with } Q([b_z](i_1, j_1)) = P([b_z](i_1, j_1)), Q([b_z](i_2, j_2)) = P([b_z](i_2, j_2))\}\end{aligned}$$

and $T' \in \mathcal{F}_{\lambda \setminus X,n}$ be the unique tableau such that $T' = T_P$ on $(\lambda \setminus X)$, except on the interval of rows $([b_z](1), [b_z](i_1))$, where $T' = T$, except $T'_{[b_z](i_1,j_1)} = u$, $T'_{[b_z](i_2,j_2)} = v$. See Figure 3.9.

Corollary. For $[P]$ as above, modulo $F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}$ we have

$$\sum_{Q \in [P]} Q(T) = (-1)^{i_1-2+i_2-2} \begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T'.$$

Proof. Apply the techniques from the proof of Lemma 3.2.3 to get a sum of tableaux with u_1 in the box $[b_z](i_1, j_1)$, skipping row $[b_z](i_2)$, then apply the techniques again to get u_2 in the box $[b_z](i_2, j_2)$, skipping row $[b_z](i_2 - 1)$. $\square$

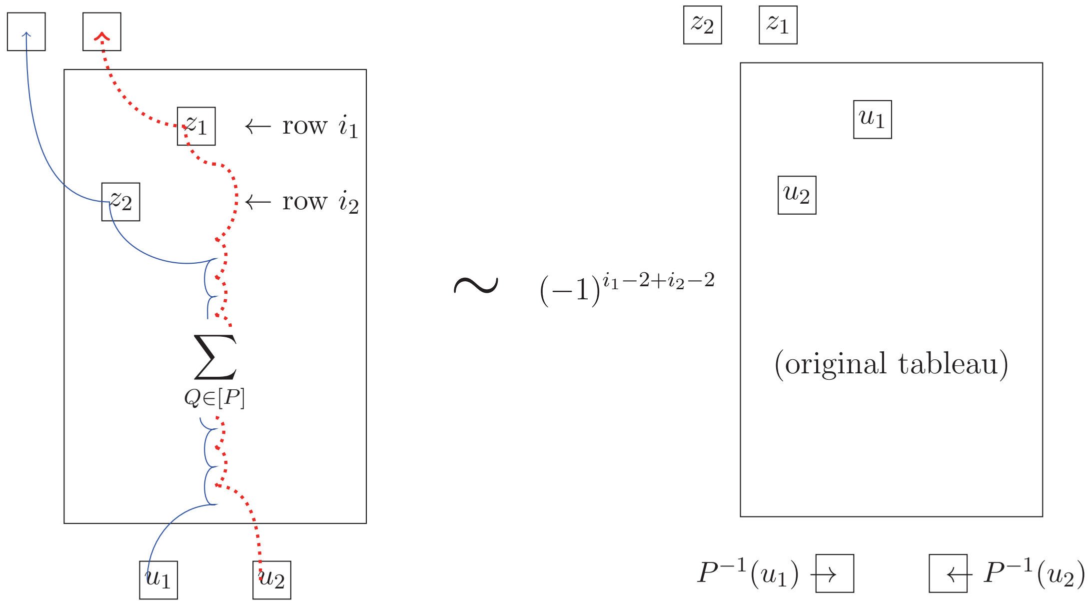


Figure 3.9. Collapsing a sum of 2-paths.

3.2.5

The same technique used in 3.2.4 immediately generalizes to sums of m-path extensions. Fix $m > 2$. For any $T \in \mathcal{F}_{\lambda,n}$, let

$$X = \{x_1 = [b_1](1, w_{b_1}), \ldots, x_m = [b_m](i_m, w_{b_m})\}$$

be a removal set and

$$z_k = T_{[b_z](i_k,j_k)} \quad \text{for } 1 \leq k \leq m$$

for some $1 \leq b_1 \leq b_z \leq N$, $1 \leq i_m < \cdots < i_1 \leq h_{b_z}$, and $1 \leq j_k \leq w_{b_z}$ for $1 \leq k \leq m$. Let

$$u_k = T_{[b_{u_k}](i_{u_k},j_{u_k})} \quad \text{for } 1 \leq k \leq m$$

for some $b_1 \leq b_{u_k} \leq b_z$, $1 \leq i_{u_k} \leq h_{b_{u_k}}$, $1 \leq j_{u_k} \leq w_{b_{u_k}}$. If $b_{u_k} = b_{u_l}$ for $k \neq l$, then we also assume that $i_{u_k} \neq i_{u_l}$. Let P be any m-path on λ such that, for $1 \leq k \leq m$,

$$P([b_{u_k}](i_{u_k}, j_{u_k})) \in [b_z](k)$$

and

$$P([b_z](i_k, j_k)) > [b_z](i_1).$$

Assume, without loss of generality, that $P([b_z](i_k, j_k)) \notin Y$ for $1 \leq k \leq m$. Let

$$[P] = \{m\text{-paths } Q \text{ on } \lambda : Q \text{ is a } ([b_z](1), [b_z](i_1))\text{-path extension of } P$$

$$\text{such that } Q([b_z](i_k, j_k)) = P([b_z](i_k, j_k)) \text{ for } 1 \leq k \leq m\}$$

and $T' \in \mathcal{F}_{\lambda \setminus X, n}$ be the unique tableau such that $T' = T_P$ on $(\lambda \setminus X)$ except on $([b_z](1), [b_z](i_2))$, where $T' = T$ except $T'_{[b_z](i_k, j_k)} = u_k$ for $1 \leq k \leq m$.

Corollary. For $[P]$ and T' as above, modulo $F_m \otimes \mathcal{R}_{\lambda \setminus X, n}$,

$$\sum_{Q \in [P]} Q(T) = (-1)^{i_1 - m + \cdots + i_m - m} \begin{array}{|c|} \hline \alpha_m^P \\ \hline \vdots \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T'$$

Proof. Assume, without loss of generality, that

$$i_1 > i_2,$$

$$i_2 > i_3 + 1,$$

$$\vdots$$

$$i_{m-1} > i_m + (m-1) - 1.$$

Otherwise, the following goes through by skipping the appropriate rows. Apply the techniques from the proof of Lemma 3.2.3 to get a sum of tableaux with u_1 in the box $[b_z](i_1, j_1)$, skipping rows $i_m^z, \ldots, i_2^z$. Then iterate the techniques again to get u_k in the box $[b_z](i_k, j_k)$, skipping rows $[b_z](i_m), \ldots, [b_z](i_{k+1})$ and rows $[b_z](i_k - 1), \ldots, [b_z](i_k - (k-1))$. □

CHAPTER FOUR

The Pieri Inclusion Removing One Box is a $GL(V)$-map

In this chapter we show that the Pieri inclusion removing one box is a $GL(V)$-map. We start by stating this as a theorem and then prove it in the two possible cases.

4.1 The Theorem Statement and Set-Up

4.1.1

For all of Chapter 4, fix a removal set $X = \{x_1 := [b_1](1, w_{b_1})\} \subset T_0$. Let

$$\Phi_1 : \mathcal{F}_{\lambda,n} \to V \otimes \mathcal{F}_{\lambda\setminus X,n}$$

be as in Section 2.1.

Theorem. Φ_1 is a $GL(V)$-map, i.e. Φ_1 descends to

$$\Phi_1 : \mathbb{S}_\lambda(V) \to F_1 \otimes \mathbb{S}_{\lambda\setminus X}(V)$$

and Φ_1 is $GL(V)$-equivaraint.

4.1.2

For each simple root vector α_i with respect the standard Cartan subalgebra, the action of e_{α_i} on a tableau T generates a sum of tableau $\widetilde{T}$ where each entry i in T is replaced by an $i+1$. Similarly, for each $e_{-\alpha_i}$, where each entry i in T is replaced by an $i-1$. As Φ_1 is a sum over 1-paths that move entries up the diagram, acting with e_{α_i} and applying Φ_1 to the sum is the same as the opposite order. Then as the simple root vectors generate $\mathfrak{gl}(V)$, Φ_1 is $\mathfrak{gl}(V)$-equivariant.

4.1.3

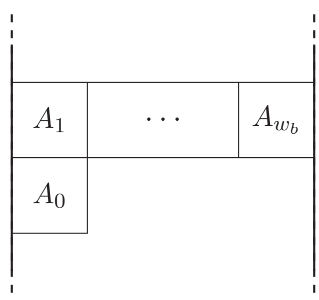
To prove Theorem 4.1.1, it remains to show that

$$\Phi_1(\mathcal{R}_{\lambda,n}) \subset F_1 \otimes \mathcal{R}_{\lambda\setminus X,n}.$$

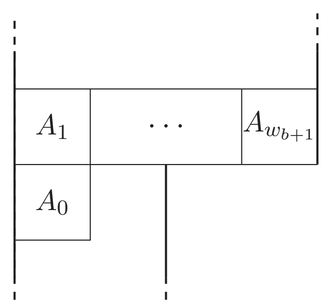
It is clear that Φ_1 preserves property 1.2.6.1 as it is a sum over all 1-paths. It remains to show that property 1.2.6.2 holds, i.e. for all $T \in \mathcal{F}_{\lambda,n}$ and all $A \subset T_0$ with $|A| > w_A$,

$$\Phi_1(G_A(T)) \in F_1 \otimes \mathcal{R}_{\lambda\setminus X,n}. \tag{4.1.3.1}$$

By Theorem 3.1.1, it is enough to show that equation 4.1.3.1 holds for all hooks A. If A is a hook, either A is completely contained in a block $[b]$, with $1 \leq b \leq N$, or A is contained in two blocks, $[b]$ and $[b+1]$, with $1 \leq b \leq N-1$ as in Figure 4.1. We consider these two options separately.



(a) A hook $A \subset [b]$.



(b) A hook $A \subset [b] \cup [b+1]$.

Figure 4.1. Hooks contained in a single block or two blocks.

4.2.1

We first show that Equation 4.1.3.1 holds for all hooks $A \subset [b]$, for some $1 \leq b \leq N$. For the rest of Section 4.2, fix $T \in \mathcal{F}_{\lambda,n}$ and

$$A = \{a_0 := [b](i_0, 1), a_1 := [b](i_0 + 1, 1), \ldots, a_{w_b} := [b](i_0 + 1, w_b)\} \subset T_0$$

with $1 \leq i_0 < h_b$, so that $A \subset [b]$. Denote the entries of A in T by $A_k = T_{a_k}$ for $k = 0, 1, \ldots, w_b$. Then by Lemma 3.1.4, modulo $F_1 \otimes \mathcal{R}_{\lambda \setminus X,n}$ we have

$$\Phi_1 \left(G_A(T) \right) = \sum_P \frac{(-1)^P}{H(P)} P \left(\sum_{\sigma \in \mathfrak{S}_A} \sigma T \right) = C \sum_P \sum_{k=0}^{w_b} \frac{(-1)^P}{H(P)} P \left(\sigma_k^A T \right),$$

where the sum is over all 1-paths P on λ removing X. The set of all $P_k := P(\sigma_k^A T)$, which we will generally call “paths,” appearing in the image $\Phi_1 \left(G_A(T) \right)$ above is the union of the following disjoint sets.

The P_ks that miss A,

$$\mathfrak{T}_1 = \{P_k : R^P \cap A = \emptyset\}. \tag{4.2.1.1}$$

The P_ks that hit A and keep A in block $[b]$,

$$\mathfrak{T}_2 = \{P_k : R^P \cap A \neq \emptyset, \ P(A) \leq [b]\}. \tag{4.2.1.2}$$

The P_ks that hit A and move the entry A_i above block $[b]$, including $P(\sigma_k^A A_i) \in Y$,

$$\mathfrak{T}_3 = \bigsqcup_{i=0}^{w_b} \mathfrak{T}_3^i, \tag{4.2.1.3}$$

where

$$\mathfrak{T}_3^i = \{P_k : R^P \cap A \neq \emptyset, \ P(\sigma_k^A A_i) > [b]\}.$$

We then have, modulo $F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}$,

$$\Phi_1(G_A(T)) = C \sum_P \sum_{k=0}^{w_b} \frac{(-1)^P}{H(P)} P\left(\sigma_k^A T\right) = C \sum_{j=1}^{3} \sum_{P_k \in \mathfrak{T}_j} \frac{(-1)^P}{H(P)} P_k.$$

4.2.2

We show that for each of the cases (4.2.1.1) - (4.2.1.3),

$$\sum_{P_k \in \mathfrak{T}_j} \frac{(-1)^P}{H(P)} P_k \in F_1 \otimes \mathcal{R}_{\lambda\backslash X,n},$$

and hence equation 4.1.3.1 holds for all hooks A with $A \subset [b]$.

Case (4.2.1.1). In this case we show that the sum over all paths that miss A is in $F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}$, i.e.

$$\sum_{P_k \in \mathfrak{T}_1} \frac{(-1)^P}{H(P)} P_k \in F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}.$$

Proof. As P misses A for all $P_k \in \mathfrak{T}_1$, $P|_A = \mathrm{id}_A$, and thus

$$P\left(\sigma_k^A T\right) = Y_P \otimes \sigma_k^A T_P \text{ for all } 0 \leq k \leq w_b.$$

Then modulo $F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}$ we have

$$\begin{aligned}
\sum_{P_k \in \mathfrak{T}_1} \frac{(-1)^P}{H(P)} P_k &= \sum_{P_0 \in \mathfrak{T}_1} \sum_{k=0}^{w_b} \frac{(-1)^P}{H(P)} P\left(\sigma_k^A T\right) \\
&= \sum_{P_0 \in \mathfrak{T}_1} \sum_{k=0}^{w_b} \frac{(-1)^P}{H(P)} \boxed{\alpha_1^P} \otimes \sigma_k^A T_P \\
&= \sum_{P_0 \in \mathfrak{T}_1} \frac{1}{C_A} \frac{(-1)^P}{H(P)} \boxed{\alpha_1^P} \otimes G_A(T_P) \\
&= 0.
\end{aligned}$$

$\square$

Case (4.2.1.2). In this case we show that the sum over all paths that hit A and keep A in block $[b]$ is in $F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}$, i.e.

$$\sum_{P_k \in \mathfrak{T}_2} \frac{(-1)^P}{H(P)} P_k \in F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}.$$

Proof. For a 1-path P, let P^{-1} be the unique map of boxes

$$P^{-1} : \lambda \cup Y \to \lambda \cup Y$$

such that for all $x \in \lambda \cup Y$, $P^{-1}(P(x)) = x$. For all $k = 0, 1, \ldots, w_b$, let

$$\tau_k^A := P \sigma_k^A P^{-1} \in S_{P(A)},$$

so that τ_k^A permutes $P(a_0)$ and $P(a_k)$ and is the identity otherwise. Extend τ_k^A to act on the entries of T_P.

Then, modulo $F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}$, we have

$$\begin{aligned}
\sum_{P_k \in \mathfrak{T}_2} \frac{(-1)^P}{H(P)} P_k &= \sum_{P_0 \in \mathfrak{T}_2} \sum_{k=0}^{w_b} \frac{(-1)^P}{H(P)} P\left(\sigma_k^A T\right) \\
&= \sum_{P_0 \in \mathfrak{T}_2} \sum_{k=0}^{w_b} \frac{(-1)^P}{H(P)} P\left(\sigma_k^A P^{-1}\left(P(T)\right)\right) \\
&= \sum_{P_0 \in \mathfrak{T}_2} \sum_{k=0}^{w_b} \frac{(-1)^P}{H(P)} \boxed{\alpha_1^P} \otimes \tau_k^A T_P.
\end{aligned}$$

As $P(A) \subset [b]$ and $|P(A)| = w_b + 1$, by the proof of Lemma 3.1.4 we have, modulo $F_1 \otimes \mathcal{R}_{\lambda\backslash X,n}$,

$$\sum_{P_0 \in \mathfrak{T}_2} \sum_{k=0}^{w_b} \frac{(-1)^P}{H(P)} \boxed{\alpha_1^P} \otimes \tau_k^A T_P = \sum_{P_0 \in \mathfrak{T}_2} \frac{1}{C_A} \frac{(-1)^P}{H(P)} \boxed{\alpha_1^P} \otimes G_{P(A)}\left(T_P\right) = 0.$$

$\square$

Remark. Notice that the proofs of Case (4.2.1.1) and Case (4.2.1.2) did not depend on removing a single box nor on A being contained in a single block, and so this will generalize to $m \geq 1$ for both options of a hook A.

Case (4.2.1.3). In this case we show that the sum over all paths that hit A and move the entry A_i above block $[b]$ is in $F_1 \otimes \mathcal{R}_{\lambda \setminus X,n}$. We will assume that $b > b_1$, as the case $b = b_1$ can be treated similarly. It is enough to show that for each $i = 0, \ldots, w_b$,

$$\sum_{P_k \in \mathcal{T}_3^i} \frac{(-1)^P}{H(P)} P_k \in F_1 \otimes \mathcal{R}_{\lambda \setminus X,n}.$$

We will show the case $i = 0$, with the cases $i = 1, \ldots, w_b$ being similar.

Proof. For the rest of Case (4.2.1.3) let $\mathcal{T} := \mathcal{T}_3^0$ and, for any 1-path P, let $\tilde{h^P} := h^P - h_b^P$. Define the relation $\sim$ on $\mathcal{T}$ by

$$P_k \sim Q_j \iff Q \text{ is a } ([b](1), [b](i_0+1))\text{-path extension of } P.$$

It is clear that this defines an equivalence relation on $\mathcal{T}$, so that

$$\sum_{P_k \in \mathcal{T}} \frac{(-1)^P}{H(P)} P_k = \sum_{[P_k] \in \mathcal{T}/\sim} \; \sum_{Q_k \in [P_k]} \frac{(-1)^Q}{H(Q)} Q_k.$$

Pick $P_0 \in \mathcal{T}$ with $[b](i,1) \in R^P$ for all $i = 1, \ldots, i_0$, and let $[b_u](i_u, j_u) = P^{-1}([b](1,1))$, with $u := T_{[b_u](i_u,j_u)}$ as in Figure 4.2.

It is then enough to show that

$$\sum_{Q_k \in [P_0]} \frac{(-1)^Q}{H(Q)} Q_k \in F_1 \otimes \mathcal{R}_{\lambda \setminus X,n}.$$

In fact, as $\tilde{h^Q} = \tilde{h^P}$ and $H(Q) = H(P)$ for all $Q_k \in [P_0]$, it is enough to show that

$$\sum_{Q_k \in [P_0]} (-1)^{h_b^Q} Q_k \in F_1 \otimes \mathcal{R}_{\lambda \setminus X,n}.$$

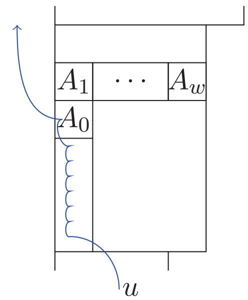


Figure 4.2. A path P_0 with $[b](i,1) \in R^P$ for all $i = 1, \cdots, i_0$.

Observe that $[P_0]$ can be written as the disjoint union

$$[P_0] = \bigsqcup_{i=1}^{3} [P_0]_i,$$

where the $[P_0]_i$ are defined as follows.

$[P_0]_1$ is the set of all paths acting on $\sigma_0^A T$ as in Figure 4.3,

$$[P_0]_1 = \{Q_0 \in [P_0]\}.$$

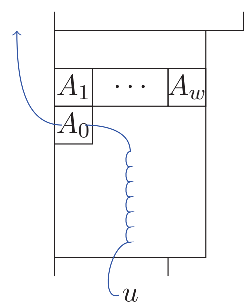


Figure 4.3. The paths in $[P_0]_1$ acting on $\sigma_0^A T$.

$[P_0]_2$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that hit $a_0 = \sigma_k^A a_k$ as in Figure 4.4,

$$[P_0]_2 = \{Q_k \in [P_0] : k \neq 0, a_0 \in R^Q\}.$$

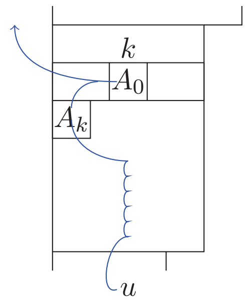


Figure 4.4. The paths in $[P_0]_2$ acting on $\sigma_k^A T$ for $k \neq 0$ that hit $a_0 = \sigma_k^A a_k$.

$[P_0]_3$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that miss $a_0 = \sigma_k^A a_k$ as in Figure 4.5,

$$[P_0]_3 = \{Q_k \in [P_0] : k \neq 0, a_0 \notin R^Q\}.$$

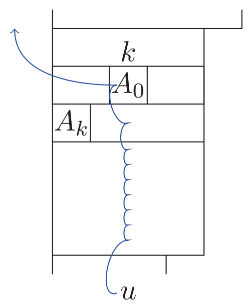


Figure 4.5. The paths in $[P_0]_3$ acting on $\sigma_k^A T$ for $k \neq 0$ that miss $a_0 = \sigma_k^A a_k$.

Let $T' \in \mathcal{F}_{\lambda \setminus X}$ be the unique tableau with $T' = T_P$ on $(\lambda \setminus X) \setminus [b]$ and $T' = T$ on $[b]$ except $T'_{a_0} = u$, as in Figure 4.6.

Then by Lemma 3.2.3 and applications of G_A, modulo $F_1 \otimes \mathcal{R}_{\lambda\setminus X,n}$, we have

$$\begin{aligned} \sum_{Q_0 \in [P_0]_1} (-1)^{h_b^Q} Q_0 &= (-1)^{i_0+i_0-1} \boxed{\alpha_1^P} \otimes T' \\ &= - \boxed{\alpha_1^P} \otimes T', \end{aligned}$$

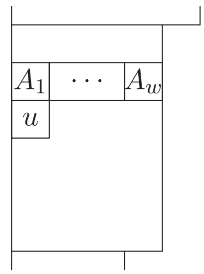


Figure 4.6. The unique tableau T' with $T' = T_P$ on $(\lambda \setminus X) \setminus [b]$ and $T' = T$ on $[b]$ except $T'_{a_0} = u$.

$$\sum_{Q_k \in [P_0]_2} (-1)^{h_b^Q} Q_k = (-1)^{i_0+1+i_0-1} w_b \boxed{\alpha_1^P} \otimes T'$$

$$= w_b \boxed{\alpha_1^P} \otimes T',$$

and

$$\sum_{Q_k \in [P_0]_3} (-1)^{h_b^Q} Q_k = (-1)^{i_0+1+1+i_0-1} (w_b - 1) \boxed{\alpha_1^P} \otimes T'$$

$$= -(w_b - 1) \boxed{\alpha_1^P} \otimes T'.$$

As

$$\sum_{Q_k \in [P_0]} (-1)^{h_b^Q} Q_k = \sum_{i=1}^{3} \sum_{Q_k \in [P_0]_i} (-1)^{h_b^Q} Q_k,$$

modulo $F_1 \otimes \mathcal{R}_{\lambda \setminus X, n}$, we have

$$\sum_{Q_k \in [P_0]} (-1)^{h_b^Q} Q_k = (-1 + w_b - w_b + 1) \boxed{\alpha_1^P} \otimes T'$$

$$= 0.$$

$\square$

4.3.1

We now show that Equation 4.1.3.1 holds for all hooks $A \subset [b] \cup [b+1]$ for some $1 \leq b \leq N-1$. For the rest of Section 4.3, fix $T \in \mathcal{T}_{\lambda,n}$ and

$$A = \{a_0 := [b](h_b, 1), a_1 := [b+1](1,1), \ldots, a_{w_{b+1}} := [b+1](1, w_{b+1})\} \subset T_0,$$

so that $A \subset [b] \cup [b+1]$. Denote the entries of A in T by $A_k = T_{a_k}$ for $k = 0, 1, \ldots, w_{b+1}$.

Then by Lemma 3.1.4, modulo $F_1 \otimes \mathcal{R}_{\lambda \setminus X, n}$, we have

$$\Phi_1\left(G_A(T)\right) = \sum_P \frac{(-1)^P}{H(P)} P\left(\sum_{\sigma \in \mathfrak{S}_A} \sigma T\right) = C \sum_P \sum_{k=0}^{w_{b+1}} \frac{(-1)^P}{H(P)} P\left(\sigma_k^A T\right),$$

where the sum is over all 1-paths P on λ removing X. The set of all $P_k := P(\sigma_k^A T)$ appearing in the image $\Phi_1\left(G_A(T)\right)$ above is the union of the following disjoint sets.

The P_ks that miss A,

$$\mathcal{T}_1 = \{P_k : R^P \cap A = \emptyset\}. \tag{4.3.1.1}$$

The P_ks that hit A and keep A in blocks $[b]$ and $[b+1]$,

$$\mathcal{T}_2 = \{P_k : R^P \cap A \neq \emptyset,\ P(A) \leq [b+1]\}. \tag{4.3.1.2}$$

The P_ks that hit A and move the entry A_i above block $[b+1]$, including the case $P(\sigma_k^A A_i) \in Y$,

$$\mathcal{T}_3 = \bigsqcup_{i=0}^{w_{b+1}} \mathcal{T}_3^i \tag{4.3.1.3}$$

where

$$\mathcal{T}_3^i = \{P_k \in \mathcal{T}_3 : R^P \cap A \neq \emptyset,\ P(\sigma_k^A A_i) > [b+1]\}.$$

Then we have, modulo $F_1 \otimes \mathcal{R}_{\lambda \backslash X,n}$,

$$\Phi_1(G_A(T)) = C \sum_{P} \sum_{k=0}^{w_{b+1}} \frac{(-1)^P}{H(P)} P\left(\sigma_k^A T\right) = C \sum_{j=1}^{3} \sum_{P_k \in \mathfrak{T}_j} \frac{(-1)^P}{H(P)} P_k.$$

4.3.2

We show that for each of the cases (4.3.1.1) - (4.3.1.3),

$$\sum_{P_k \in \mathfrak{T}_j} \frac{(-1)^P}{H(P)} P_k \in F_1 \otimes \mathcal{R}_{\lambda \backslash X,n},$$

and hence Equation 4.1.3.1 holds for all hooks $A \subset [b] \cup [b+1]$. Case (4.3.1.1) follows from the proof of Case (4.2.1.1) and Case (4.3.1.2) follows from the proof of Case (4.2.1.2). It remains to show Case (4.3.1.3).

Case (4.3.1.3). In this case we show that the sum over all paths that hit A and move the entry A_i above block $[b+1]$ is in $F_1 \otimes \mathcal{R}_{\lambda \backslash X,n}$. It is enough to show that for $i = 0, \ldots, w_{b+1}$,

$$\sum_{P_k \in \mathfrak{T}_3^i} \frac{(-1)^P}{H(P)} P_k \in F_1 \otimes \mathcal{R}_{\lambda \backslash X,n}.$$

We will show the case $i = 0$, with the cases $i = 1, \ldots, w_{b+1}$ being similar.

Proof. Note that as we are considering paths that hit A in this case, we must have that $b \geq b_1 - 1$. We will consider the case $b = b_1 - 1$ (and hence $a_{w_{b+1}} = x_1$) and the case $b > b_1 - 1$ separately.

Subcase (4.3.1.3.1). We first show the case where $b = b_1 - 1$. We want to show that

$$\sum_{P_k \in \mathcal{T}_3^0} \frac{(-1)^P}{H(P)} P_k \in F_1 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

Proof. As $a_0 < x_1$, for all $P_k \in \mathcal{T}_3^0$ we must have $k \neq 0$. We can then write $\mathcal{T}_3^0$ as

$$\mathcal{T}_3^0 = \bigsqcup_{P_1 \in \mathcal{T}_3^0} \mathcal{T}_{P_1}$$

where

$$\mathcal{T}_{P_1} = \{Q_k =\in \mathcal{T}_3^0 : Q \text{ is a } ([b+1](1), [b+1](1))\text{-path extension of } P\}.$$

It is then enough to show that for each $P_1 \in \mathcal{T}_3^0$,

$$\sum_{k=1}^{w_{b+1}} \frac{(-1)^P}{H(P)} P_1(\sigma_k^A T) \in F_1 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

Pick a $P_1 \in \mathcal{T}_3^0$ and let $A' = A \setminus \{a_{w_b+1}\} \subset \lambda \setminus X$. As $|A'| = w_{b+1} > w_{b+1} - 1$, by the proof of Lemma 3.1.4 we have

$$\begin{aligned}
\sum_{k=1}^{w_{b+1}} P_1(\sigma_k^A T) &= \sum_{k=1}^{w_{b+1}} \boxed{\alpha_1^P} \otimes \sigma_k^A (T_P) \\
&= \frac{1}{C_A} \boxed{\alpha_1^P} \otimes G_{A'} (T_P) \in F_1 \otimes \mathcal{R}_{\lambda \setminus X, n}.
\end{aligned}$$

□

Subcase (4.3.1.3.2). We now show the case $b > b_1$. For the rest of Subcase 4.3.1.3.2, let $\mathcal{T} := \mathcal{T}_3^0$ and for any 1-path P define $\tilde{h^P} := h^P - h_b^P$ and

$$\tilde{H(P)} = \frac{H(P)}{H_b(P) H_{b+1}(P)}.$$

Define the relation $\sim$ on $\mathcal{T}$ by

$$P_k \sim Q_j \iff Q_j \text{ is a } ([b](1),[b+1](1))\text{-path extension of } P.$$

It is clear that this defines an equivalence relation on $\mathcal{T}$, so that

$$\sum_{P_k \in \mathcal{T}} \frac{(-1)^P}{H(P)} P_k = \sum_{[P_k] \in \mathcal{T}/\sim} \; \sum_{Q_k \in [P_k]} \frac{(-1)^Q}{H(Q)} Q_k.$$

Pick $P_0 \in \mathcal{T}$ with $[b](i,1) \in R^P$ for all $i = 1, \ldots, h_b$, and let $[b_u](i_u, j_u) = P^{-1}([b](1,1))$ with $u := T_{[b_u](i_u,j_u)}$ as in Figure 4.7.

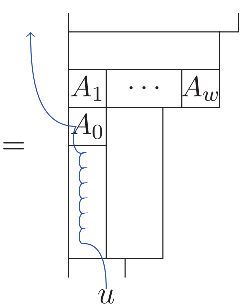
$P_0 =$ 

Figure 4.7. A path P_0 with $[b](i,1) \in R^P$ for all $i = 1, \ldots, h_b$.

It is then enough to show that

$$\sum_{Q_k \in [P_0]} \frac{(-1)^Q}{H(Q)} Q_k \in F_1 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

In fact, as $\widetilde{h^Q} = \widetilde{h^P}$ and $\widetilde{H(Q)} = \widetilde{H(P)}$ for all $Q_k \in [P_0]$, it is enough to show that

$$\sum_{Q_k \in [P_0]} \frac{(-1)^{h_b^Q}}{H_b(Q) H_{b+1}(Q)} Q_k \in F_1 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

Observe that $[P_0]$ can be written as the disjoint union

$$\mathcal{T} = \bigsqcup_{i=1}^{6} [P_0]_i,$$

where the $[P_0]_i$ are defined as follows.

$[P_0]_1$ is the set of all paths acting on $\sigma_0^A T$ as in Figure 4.8,

$$[P_0]_1 = \{Q_0 \in [P_0]\}.$$

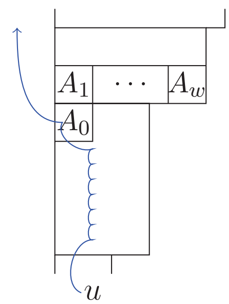


Figure 4.8. The paths in $[P_0]_1$ acting on $\sigma_0^A T$.

$[P_0]_2$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that miss block $[b]$ as in Figure 4.9,

$$[P_0]_2 = \{Q_k \in [P_0] : k \neq 0, R^Q \cap [b] = \emptyset\}.$$

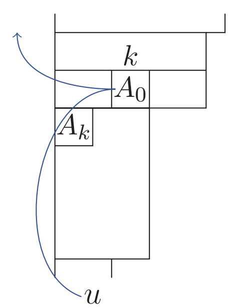


Figure 4.9. The paths in $[P_0]_2$ acting on $\sigma_k^A T$ for $k \neq 0$ that miss block $[b]$.

$[P_0]_3$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that hit $a_0 = \sigma_k^a a_k$ as in Figure 4.10,

$$[P_0]_3 = \{Q_k \in [P_0] : k \neq 0, a_0 \in R^Q\}.$$

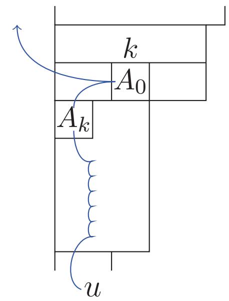


Figure 4.10. The paths in $[P_0]_3$ acting on $\sigma_k^A T$ for $k \neq 0$ that hit $a_0 = \sigma_k^a a_k$.

$[P_0]_4$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that miss $a_0 = \sigma_k^a a_k$ but hit row $[b](h_b)$ as in Figure 4.11,

$$[P_0]_4 = \{Q_k \in [P_0] : k \neq 0, [b](h_b, j) \in R^Q \text{ for some } 2 \leq j \leq w_b\}.$$

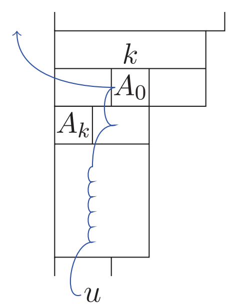


Figure 4.11. The paths in $[P_0]_4$ acting on $\sigma_k^A T$ for $k \neq 0$ that miss $a_0 = \sigma_k^a a_k$ but hit row $[b](h_b)$.

$[P_0]_5$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that miss row $[b](h_b)$ and leave block $[b]$ from an odd row and $[P_0]_6$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that miss row $[b](h_b)$ and leave block $[b]$ from an even row as in Figure 4.12,

$$[P_0]_5 = \{Q_k \in E^P : k \neq 0, Q([b](i,j)) = a_k \text{ for some } 1 \leq j \leq w_b, 1 \leq i < h_b, i \text{ odd}\},$$

and

$$[P_0]_6 = \{Q_k \in E^P : k \neq 0, Q([b](i,j)) = a_k \text{ for some } 1 \leq j \leq w_b, 1 \leq i < h_b, i \text{ even}\}.$$

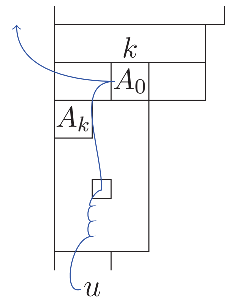


Figure 4.12. The paths in $[P_0]_5$ and $[P_0]_5$ acting on $\sigma_k^A T$ for $k \neq 0$ that miss row $[b](h_b)$ and leave block $[b]$ from an odd or even row, respectively.

Let $T' \in \mathcal{F}_{\lambda\setminus X,n}$ be the unique tableau with $T' = T_P$ on $(\lambda \setminus X) \setminus ([b] \cup [b+1])$ and $T' = T$ on $[b] \cup [b+1]$ except $T'_{a_0} = u$, as in Figure 4.13.

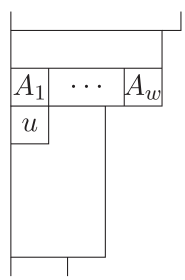


Figure 4.13. The unique tableau T' with $T' = T_P$ on $(\lambda \setminus X) \setminus ([b] \cup [b+1])$ and $T' = T$ on $[b] \cup [b+1]$ except $T'_{a_0} = u$.

By Lemma 3.2.3 and applications of G_A we have, modulo $F_1 \otimes \mathcal{R}_{\lambda\setminus X,n}$,

$$\begin{aligned}\sum_{Q_0 \in [P_0]_1} (-1)^{h_b^Q} Q_0 &= \frac{(-1)^{h_b+h_b-1}}{H(b)} \boxed{\alpha_1^P} \otimes T' \\ &= \frac{-H(b+1)}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T',\end{aligned}$$

$$\begin{aligned}\sum_{Q_k \in [P_0]_2} (-1)^{h_b^Q} Q_k &= \frac{(-1)^{1+1}}{H(b+1)} \boxed{\alpha_1^P} \otimes T' \\ &= \frac{H(b)}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T',\end{aligned}$$

$$\sum_{Q_k \in [P_0]_3} (-1)^{h_b^Q} Q_k = \frac{(-1)^{h_b+1+h_b-1} w_{b+1}}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T'$$

$$= \frac{w_{b+1}}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T',$$

$$\sum_{Q_k \in [P_0]_4} (-1)^{h_b^Q} Q_k = \frac{(-1)^{h_b+1+1+h_b-1} (w_b - 1)}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T'$$

$$= \frac{1 - w_b}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T',$$

$$\sum_{Q_k \in [P_0]_5} (-1)^{h_b^Q} Q_k = \sum_{\substack{1 \le i < h_b \\ i \text{ odd}}} \frac{(-1)^{i+1+1+i-1+1}}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T'$$

$$= \sum_{\substack{1 \le i < h_b \\ i \text{ odd}}} \frac{1}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T',$$

and

$$\sum_{Q_k \in [P_0]_5} (-1)^{h_b^Q} Q_k = \sum_{\substack{1 \le i < h_b \\ i \text{ even}}} \frac{(-1)^{i+1+1+i-1+1}}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T'$$

$$= \sum_{\substack{1 \le i < h_b \\ i \text{ even}}} \frac{1}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T'.$$

Then as $H(b+1) = H(b) + w_{b+1} - w_b + h_b$ and

$$\sum_{\substack{1 \le i < h_b \\ i \text{ odd}}} \frac{1}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T' + \sum_{\substack{1 \le i < h_b \\ i \text{ even}}} \frac{1}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T'$$

$$= \frac{h_b - 1}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T'$$

we get, modulo $F_1 \otimes \mathcal{R}_{\lambda \setminus X, n}$,

$$\sum_{Q_k \in [P_0]} (-1)^{h_b^Q} Q_k = \sum_{i=1,\ldots,6} \sum_{Q_k \in [P_0]_i} (-1)^{h_b^Q} Q_k$$

$$= \frac{-H(b+1) + H(b) + w_{b+1} + 1 - w_b + h_b - 1}{H(b)H(b+1)} \boxed{\alpha_1^P} \otimes T'$$

$$= 0.$$

$\square$

Thus Equation 4.1.3.1 holds for all hooks $A \subset [b] \cup [b+1]$, which proves Theorem 4.1.1.

CHAPTER FIVE

The Pieri Inclusion Removing Many Boxes is a $GL(V)$-map

In this chapter we show that the Pieri inclusion removing many boxes is a $GL(V)$-map. As in Chapter Four, we start by stating this as a theorem and then prove it in the two possible cases.

5.1 The Theorem Statement and Set-Up

5.1.1

Let $X = \{x_1 = [b_1](1, w_{b_1}), \ldots, x_m = [b_m](i_m, w_{b_m})\} \subset \lambda$ be a removal set and

$$\Phi_m : \mathcal{F}_{\lambda,n} \to F_m \otimes \mathcal{F}_{\lambda\setminus X,n}$$

be as in Section 2.2.4.

Theorem. Φ_m is a $GL(V)$-map, i.e. Φ_m descends to

$$\Phi_m : \mathbb{S}_\lambda(V) \to F_m \otimes \mathbb{S}_{\lambda\setminus X}(V)$$

and Φ_m is $GL(V)$-equivariant.

5.1.2

As before, it is clear that Φ_m is $\mathfrak{gl}(V)$-equivariant by construction. To prove Theorem 5.1.1, it remains to show that

$$\Phi_m(\mathcal{R}_{\lambda,n}) \subset F_m \otimes \mathcal{R}_{\lambda\setminus X,n}.$$

It is clear that Φ_m preserves Property 1.2.6.1 as it is a sum over all m-paths, and hence we must show that Property 1.2.6.2 holds, i.e. for all $T \in \mathcal{F}_{\lambda,n}$ and all $A \subset T_0$

with $|A| > w_A$,

$$\Phi_m\left(G_A(T)\right) \in F_m \otimes \mathcal{R}_{\lambda\setminus X,n}. \tag{5.1.2.1}$$

Recall that by Theorem 3.1.1, it is enough to show that 5.1.2.1 holds for all hooks A. As any hook consists of exactly two rows at most two orbits of any m-path can intersect A, and so it is enough to show that Equation 5.1.2.1 holds for $m = 2$. As before, there are two options for hooks in T_0, which we consider separately. For the rest of Chapter 5, fix the removal set

$$X = \{x_1 = [b_1](1, w_{b_1}), x_2 = [b_2](i_2, w_{b_2})\}.$$

5.2 Preserving Garnir Relations for Hooks Contained in a Single Block

5.2.1

We first show that Equation 5.1.2.1 holds when $m = 2$ for all hooks $A \subset [b]$, for some $1 \leq b \leq N$. For the rest of Section 5.2, fix $T \in \mathcal{F}_{\lambda,n}$ and let

$$A = \{a_0 := [b](i_0, 1), a_1 = [b](i_0 + 1, 1), \ldots, a_{w_b} = [b](i_0 + 1, w_b)\} \subset T_0$$

with $1 \leq i_0 < h_b$, so that $A \subset [b]$. Denote the entries of A in T by $A_k = T_{a_k}$ for $k = 0, 1, \ldots, w_b$. Then by Lemma 3.1.4, modulo $F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}$,

$$\begin{aligned}
\Phi_2\left(G_A(T)\right) &= \sum_P \frac{(-1)^P}{H(P)} P\left(\sum_{\sigma\in\mathfrak{S}_A} \sigma T\right) \\
&= C \sum_P \sum_{k=0}^{w_b} \frac{(-1)^P}{H(P)} P\left(\sigma_k^A T\right),
\end{aligned}$$

where the sum is over all 2-paths P on λ removing X. The set of all $P_k := P(\sigma_k^A T)$ appearing in the image $\Phi_2\left(G_A(T)\right)$ above is the union of the following disjoint sets.

The P_ks that miss A,

$$\mathcal{T}_1 = \{P_k : R^P \cap A = \emptyset\}. \tag{5.2.1.1}$$

The P_ks that hit A and keep A in block $[b]$,

$$\mathcal{T}_2 = \{P_k : R^P \cap A \neq \emptyset,\ P(A) \subset [b]\}. \tag{5.2.1.2}$$

The P_ks that have exactly one orbit in $[b]$ and move A_i above $[b]$,

$$\mathcal{T}_3 = \bigsqcup_{0 \leq i \leq w_b} \mathcal{T}_3^i \tag{5.2.1.3}$$

where

$$\mathcal{T}_3^i = \{P_k :\ \text{exactly one of } R_1, R_2 \text{ intersects } [b] \text{ and } P(\sigma_k A_i) > [b]\}.$$

The P_ks that move A_i and A_j above $[b]$,

$$\mathcal{T}_4 = \bigsqcup_{0 \leq i < j \leq w_b} \mathcal{T}_4^{i,j} \tag{5.2.1.4}$$

where

$$\mathcal{T}_4^{i,j} = \{P_k \in \mathcal{T}_4 : P(\sigma_k^A a_i) > [b] \text{ and } P(\sigma_k^A a_j) > [b]\}.$$

The P_ks that move A_i and a box $z \in [b]$, with $z < A$, above $[b]$,

$$\mathcal{T}_5 = \bigsqcup_{\substack{0 \leq i \leq w_b,\ z = [b](i_z, j_z), \\ 1 \leq i_z \leq i_0 - 1, 1 \leq j_z \leq w_b}} \mathcal{T}_5^{i,z}, \tag{5.2.1.5}$$

where

$$\mathcal{T}_5^{i,z} = \{P_k \in \mathcal{T}_5 : P(\sigma_k^A a_i) > [b], P(z) > [b]\}.$$

The P_ks that move A_i and a box $z \notin A$ in row i_0 above $[b]$,

$$\mathcal{T}_6 = \bigsqcup_{0 \leq i \leq w_b,\ 2 \leq j \leq w_b} \mathcal{T}_6^{i,j} \tag{5.2.1.6}$$

where

$$\mathcal{T}_6^{i,j} = \{P_k \in \mathcal{T}_6 : P(\sigma_k^A a_i) > [b], P([b](i_0, j)) > [b]\}.$$

The P_ks that move A_i and a box in $[b]$ above A above $[b]$,

$$\mathcal{T}_7 = \bigsqcup_{\substack{0 \le i \le w_b \\ 1 \le j \le w_b}} \mathcal{T}_7^{i,j} \tag{5.2.1.7}$$

where

$$\mathcal{T}_7^{i,j} = \{P_k \in \mathcal{T}_7 : P(\sigma_k^A A_i) > [b], [b](i_0 + 2, j) \in R^P\}.$$

Then, modulo $F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}$, we have

$$\begin{aligned}\Phi_2\left(G_A(T)\right) &= C \sum_P \sum_{k=0}^{w_b} \frac{(-1)^P}{H(P)} P\left(\sigma_k^A T\right) \\ &= C \sum_{j=1,\ldots,7} \sum_{P_k \in \mathcal{T}_j} \frac{(-1)^P}{H(P)} P_k.\end{aligned}$$

5.2.2

We show that for $j = 1, \ldots, 7$,

$$\sum_{P_k \in \mathcal{T}_j} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda \setminus X, n},$$

and hence Equation 5.1.2.1 holds when $m = 2$ for all blocks $A \subset [b]$. The proofs of Cases (5.2.1.1), (5.2.1.2), and (5.2.1.3) are similar to the proofs of Cases (4.2.1.1), (4.2.1.2), and (4.2.1.3), respectively. It remains to show the proofs of Cases (5.2.1.4), (5.2.1.5), (5.2.1.6), and (5.2.1.7). In each case we will assume $b > b_1$, with the case $b = b_1$ being similar. We will also assume in each case that $A \cap X = \emptyset$, as if $A \cap X \neq \emptyset$ we may follow the proof of Subcase (4.3.1.3.1).

Case (5.2.1.4). In this case we show that the sum over all paths that move A_i and A_j above $[b]$ is in $F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}$. Recall that

$$\mathcal{T}_4 = \bigsqcup_{0 \le i < j \le w_b} \mathcal{T}_4^{i,j},$$

where

$$\mathcal{T}_4^{i,j} = \{P_k \in \mathcal{T}_4 : P(\sigma_k^A a_i) > [b] \text{ and } P(\sigma_k^A a_j) > [b]\}.$$

It is enough to show that for $0 \le i < j \le w_b$,

$$\sum_{P_k \in \mathcal{T}_4^{i,j}} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}.$$

We will show the case $i = 0$ and $j = 1$, with rest being similar.

Proof. For the rest of Case (5.2.1.4) let $\mathcal{T} := \mathcal{T}_4^{0,1}$. Observe that for all $P_k \in \mathcal{T}$, either $k = 0$ or $k = 1$, as otherwise $\sigma_k^P A_0$ and $\sigma_k^A A_1$ are in the same row.

Next we will define for each $P_0 \in \mathcal{T}$ a unique $P_1' \in \mathcal{T}$ that agrees with P except on $\{a_0, a_1\}$. See Figure 5.1. The conditions on P_1' will depend on whether or not P "removes" (i.e. maps to Y) either or both of a_0, a_1. We want to construct P_1' so that it sends A_0 and A_1 to the same place P does, but with the freedom to do so with either the orbit of x_1 or x_2. For each $P_0 \in \mathcal{T}$, let $P_1' \in \mathcal{T}$ such that $P' \equiv P$ except on $\{a_0, a_1\}$, and

- if $\{P(a_0), P(a_1)\} \cap Y = \emptyset$,

$$P'(\sigma_1^A a_0) = P(a_0) \text{ and } P'(\sigma_1^A(a_1)) = P(a_1).$$

- if $P(a_0) \in Y$ and $P(a_1) \notin Y$,

$$P'(\sigma_1^A a_0) \in Y \text{ and } P'(\sigma_1^A(a_1)) = P(a_1).$$

- if $P(a_0) \notin Y$ and $P(a_1) \in Y$,

$$P'(\sigma_1^A a_0) = P(a_0) \text{ and } P'(\sigma_1^A(a_1)) \in Y.$$

- if $\{P(a_0), P(a_1)\} = Y$,

$$\{P'(\sigma_1^A a_0), P'(\sigma_1^A(a_1))\} = Y.$$

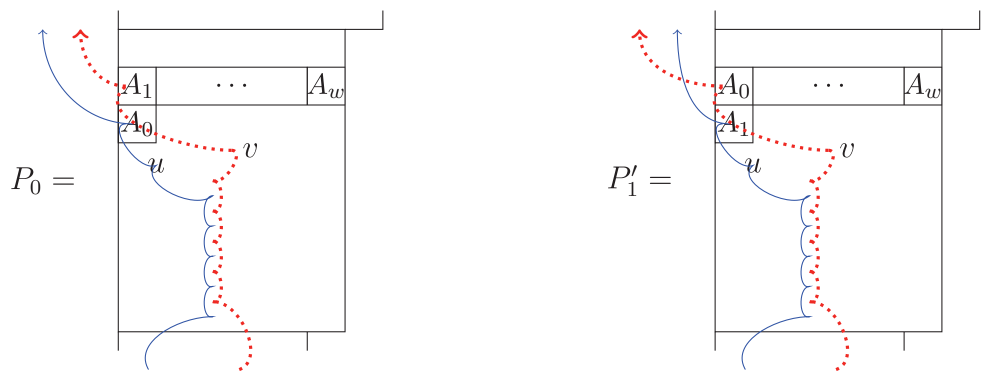


Figure 5.1. A path P_0 removing the entries A_0 and A_1 from block $[b]$ and its dual path P_1'.

It is clear that for for each $P_0 \in \mathfrak{T}$ the choice of P_1' is unique, and that all $Q_1 \in \mathfrak{T}$ arise in such a way. Thus

$$\sum_{P_k \in \mathfrak{T}} \frac{(-1)^P}{H(P)} P_k = \sum_{P_0 \in \mathfrak{T}} \left(\frac{(-1)^P}{H(P)} P_0 + \frac{(-1)^{P'}}{H(P')} P_1' \right)$$

As $(-1)^P = (-1)^{P'}$ and $H(P) = H(P')$, it is then enough to show that

$$\sum_{P_0 \in \mathfrak{T}} P_0 + P_1' \in F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

We will in fact show that for each $P_0 \in \mathfrak{T}$, $P_0 + P_1' \in F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}$. Pick P_0 and the corresponding $P_1' \in \mathfrak{T}$ and let $u = P^{-1}(a_0)$ and $v = P^{-1}(A_1)$. Let $T' \in \mathcal{F}_{\lambda \setminus X, n}$ be the unique tableau with $T' = T_P$ on $(\lambda \setminus X) \setminus \{[b](i_0), [b](i_0+1)\}$ and $T' = T$ on $\{[b](i_0), [b](i_0+1)\}$ except $T'_{a_0} = u$ and $T'_{a_1} = v$, as in Figure 5.2.

Then, modulo $F_2 \otimes \mathcal{R}_{\lambda\backslash X,n}$,

$$P_0 + P_1' = \left(\begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} + \begin{array}{|c|}\hline \alpha_1^P \\ \hline \alpha_2^P \\ \hline \end{array} \right) \otimes T'$$

$$= 0.$$

□

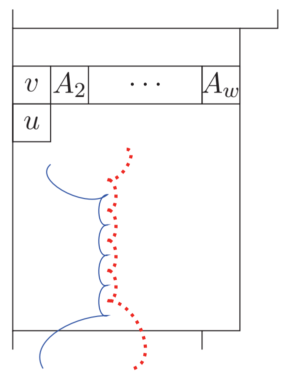


Figure 5.2. The unique tableau T' corresponding to the path P_0.

Case (5.2.1.5). In this case we show that the sum over all paths that move A_i and a box $z \in [b]$, with $z < A$, above $[b]$ is in $F_2 \otimes \mathcal{R}_{\lambda\backslash X,n}$. Recall that

$$\mathcal{T}_5 = \bigsqcup_{\substack{0 \le i \le w_b,\ z=[b](i_z,j_z), \\ 1 \le i_z \le i_0 - 1, 1 \le j_z \le w_b}} \mathcal{T}_5^{i,z},$$

where

$$\mathcal{T}_5^{i,z} = \{P_k \in \mathcal{T}_5 : P(\sigma_k^A a_i) > [b], P(z) > [b]\}.$$

It is enough to show that

$$\sum_{P_k \in \mathcal{T}_5^{0,z}} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda\backslash X,n},$$

for some $z = [b](i_z, j_z)$ a fixed box with $Z = T_z$, $1 \le i_z \le i_0 - 1$ odd, and $1 \le j_z \le w_b$, with the other cases being similar.

Proof. For the rest of Case (5.2.1.5) let $\mathcal{T} := \mathcal{T}_5^{0,z}$ and, for any 2-path P, let $\tilde{h^P} := h^P - h_b^P$. Define the relation $\sim$ on $\mathcal{T}$ by

$$P_k \sim Q_j \iff Q \text{ is a } ([b](1), [b](i_0+1))\text{-path extension of } P$$
$$\text{and if } P(u) \in [b](1) \text{ for some box } u < [b], \text{ then } Q(u) \in [b](1).$$

It is clear that this defines an equivalence relation on $\mathcal{T}$, so that

$$\sum_{P_k \in \mathcal{T}_5^{0,z}} \frac{(-1)^P}{H(P)} P_k = \sum_{[P_k] \in \mathcal{T}/\sim} \sum_{Q_k \in [P_k]} \frac{(-1)^Q}{H(Q)} Q_k.$$

Pick $P_0 \in \mathcal{T}$ with $[b](i,1) \in R^P$ for all $1 \le i \neq i_z \le i_0$, and let $[b_u](i_u, j_u) = P^{-1}([b](1,1))$ and $[b_v](i_v, j_v) = P^{-1}([b](2,1))$ with $u = T_{[b_u](i_u,j_u)}$ and $v = T_{[b_v](i_v,j_v)}$. See Figure 5.3.

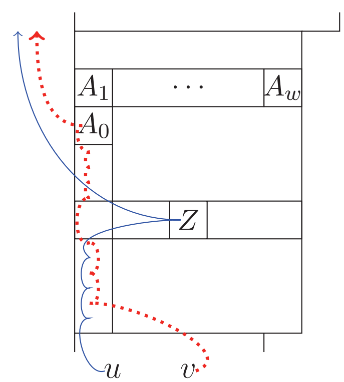


Figure 5.3. A path P_0 removing the entries A_0 and Z from block $[b]$.

It is then enough to show that

$$\sum_{Q_k \in [P_0]} \frac{(-1)^Q}{H(Q)} Q_k \in F_2 \otimes \mathcal{R}_{\lambda \backslash X, n}.$$

In fact, as $\tilde{h^Q} = \tilde{h^P}$ and $H(Q) = H(P)$ for all $Q_k \in [P_0]$, it is enough to show that

$$\sum_{Q_k \in [P_0]} (-1)^{h_b^Q} Q_k \in F_2 \otimes \mathcal{R}_{\lambda \backslash X, n}.$$

Observe that $[P_0]$ can be written as the disjoint union

$$[P_0] = \bigsqcup_{i=1}^{3} [P_0]_i,$$

where the $[P_0]_i$ are defined as follows.

$[P_0]_1$ is the set of all paths acting on $\sigma_0^A T$ as in Figure 5.4,

$$[P_0]_1 = \{Q_0 \in [P_0]\}.$$

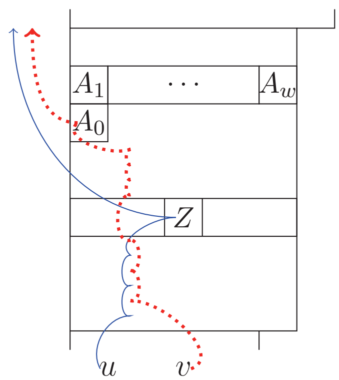


Figure 5.4. The paths in $[P_0]_1$ acting on $\sigma_0^A T$.

$[P_0]_2$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that hit $a_0 = \sigma_k^A a_k$ as in Figure 5.5,

$$[P_0]_2 = \{Q_k \in [P_0] : k \neq 0, a_0 \in R^Q\}.$$

$[P_0]_3$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that miss $a_0 = \sigma_k^A a_k$ as in Figure 5.6,

$$[P_0]_3 = \{Q_k \in [P_0] : k \neq 0, a_0 \notin R^Q\}.$$

Let $T' \in \mathcal{F}_{\lambda \setminus X}$ be the unique tableau with $T' = T_P$ on $(\lambda \setminus X) \setminus [b]$ and $T' = T$ on $[b]$ except $T'_z = v$ and $T'_{a_0} = u$ as in Figure 5.7.

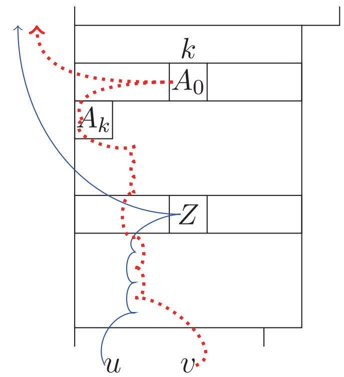


Figure 5.5. The paths in $[P_0]_2$ acting on $\sigma_k^A T$ for $k \neq 0$ that hit $a_0 = \sigma_k^A a_k$.

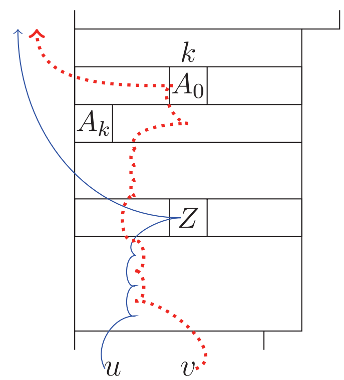


Figure 5.6. The paths in $[P_0]_3$ acting on $\sigma_k^A T$ for $k \neq 0$ that miss $a_0 = \sigma_k^A a_k$.

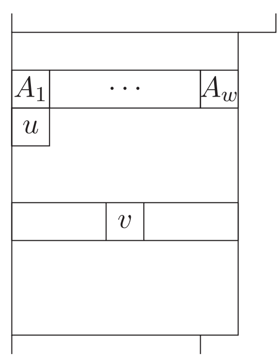


Figure 5.7. The unique tableau T' with $T' = T_P$ on $(\lambda \setminus X) \setminus [b]$ and $T' = T$ on $[b]$ except $T'_z = v$ and $T'_{a_0} = u$.

Then by Corollary 3.2.4 and applications of G_A, modulo $F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}$ we have

$$\begin{aligned}\sum_{Q_k \in [P_0]_1} (-1)^{h_b^Q} Q_k &= (-1)^{i_0+i_0-2+i_z-2} \begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T' \\ &= - \begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T',\end{aligned}$$

$$\begin{aligned}\sum_{Q_k \in [P_0]_2} (-1)^{h_b^Q} Q_k &= (-1)^{i_0+1+i_0-2+i_z-2} (w_b) \begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T' \\ &= w_b \begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T',\end{aligned}$$

and

$$\begin{aligned}\sum_{Q_k \in [P_0]_3} (-1)^{h_b^Q} Q_k &= (-1)^{i_0+1+1+i_0-2+i_z-2} (w_b - 1) \begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T' \\ &= -w_b + 1 \begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T'.\end{aligned}$$

Thus modulo $F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}$ we have

$$\begin{aligned}\sum_{Q_k \in [P_0]} (-1)^{h_b^Q} Q_k &= (-1 + w_b - w_b + 1) \begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T' \\ &= 0.\end{aligned}$$

$\square$

Case (5.2.1.6). In this case we show that the sum over all paths that that move A_i and a box $z \notin A$ in row i_0 above $[b]$ is in $F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}$. Recall that

$$\mathcal{T}_6 = \bigsqcup_{0 \le i \le w_b,\ 2 \le j \le w_b} \mathcal{T}_6^{i,j},$$

where

$$\mathcal{T}_6^{i,j} = \{P_k \in \mathcal{T}_6 : P(\sigma_k^A a_i) > [b], P([b](i_0, j)) > [b]\}.$$

It is enough to show that

$$\sum_{P_k \in \mathcal{T}_6^{0,2}} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda \backslash X, n},$$

with the other cases being similar.

Proof. Let $z := [b](i_0, 2)$ and $Z := T_z$, and observe that $\mathcal{T}_6^{0,2}$ is the union of the following disjoint sets:

$$\mathcal{T}_6^{0,2,1} = \{P_k \in \mathcal{T}_6^{0,2} : P(\sigma_k^A a_0), P(z) \notin Y\},$$

$$\mathcal{T}_6^{0,2,2} = \{P_k \in \mathcal{T}_6^{0,2} : P(\sigma_k^A a_0) \in Y, P(z) \notin Y\},$$

$$\mathcal{T}_6^{0,2,3} = \{P_k \in \mathcal{T}_6^{0,2} : P(z) \in Y, P(\sigma_k^A a_0) \notin Y\}, \text{ and}$$

$$\mathcal{T}_6^{0,2,4} = \{P_k \in \mathcal{T}_6^{0,2} : P(\sigma_k^A a_0), P(z) \in Y\}.$$

It is enough to show that for $1 \leq i \leq 4$,

$$\sum_{P_k \in \mathcal{T}_6^{0,2,i}} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda \backslash X, n}.$$

Define the relation $\sim^i$ on $\mathcal{T}_6^{0,2,i}$, for $1 \leq i \leq 4$, as follows. Define $\sim^1$ on $\mathcal{T}_6^{0,2,1}$ by

$$P_k \sim^1 Q_j \iff Q \text{ is a } ([b](1), [b](i_0+1))\text{-path extension of } P,$$

$$Q(\sigma_j^A A_0) = P(\sigma_k^A A_0), \text{ and } Q(Z) = P(Z).$$

Define $\sim^2$ on $\mathcal{T}_6^{0,2,2}$ by

$$P_k \sim^2 Q_j \iff Q \text{ is a } ([b](1), [b](i_0+1))\text{-path extension of } P,$$

$$\text{and } Q(Z) = P(Z).$$

Define $\sim^3$ on $\mathcal{T}_6^{0,2,3}$ by

$$P_k \sim^3 Q_j \iff Q \text{ is a } ([b](1),[b](i_0+1))\text{-path extension of } P,$$

$$\text{and } Q(\sigma_j^A A_0) = P(\sigma_k^A A_0).$$

Define $\sim^4$ on $\mathcal{T}_6^{0,2,4}$ by

$$P_k \sim^4 Q_j \iff Q \text{ is a } ([b](1),[b](i_0+1))\text{-path extension of } P.$$

It is clear that for $1 \le i \le 4$ $\sim^i$ an equivalence relation on $\mathcal{T}_6^{0,2,i}$, so that

$$\sum_{P_k \in \mathcal{T}_6^{0,2,i}} \frac{(-1)^P}{H(P)} P_k = \sum_{[P_k] \in \mathcal{T}_6^{0,2,i}/\sim^i} \sum_{Q_k \in [P_k]} \frac{(-1)^Q}{H(Q)} Q_k.$$

Thus it is enough to show that for $1 \le i \le 4$,

$$\sum_{[P_k] \in \mathcal{T}_6^{0,2,i}/\sim^i} \sum_{Q_k \in [P_k]} \frac{(-1)^Q}{H(Q)} Q_k \in F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}.$$

We will show the case $i = 1$, with the rest being similar. For the rest of Case (5.2.1.6), let $\mathcal{T} := \mathcal{T}_6^{0,2,1}$. For $l = 1,2$, let $\mathcal{T}_{x_l}$ be the set of all Q_k in $\mathcal{T}$ such that the orbit of x_l intersects the first row in $[b]$,

$$\mathcal{T}_{x_l} = \{P_k \in \mathcal{T} : R_l^P \cap [b](1) \neq \emptyset\}.$$

As $a_0 = [b](i_0,1)$ and $z = [b](i_0,2)$ are in the same row, it must be that $1 \le k \le w_b$ for all $P_k \in \mathcal{T}$. Pick $P_1 \in \mathcal{T}$ with $[b](i,1) \in R^P$ for all $i = 1,\ldots,i_0-1$, and let $[b_u](i_u,j_u) = P^{-1}([b](1,1))$ and $[b_v](i_v,j_v) = P^{-1}([b](2,1))$ with $u = T_{[b_u](i_u,j_u)}$ and $v = T_{[b_v](i_v,j_v)}$ as in Figure 5.8.

It is then enough to show that

$$\sum_{Q_k \in [P_1]} \frac{(-1)^Q}{H(Q)} Q_k \in F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}.$$

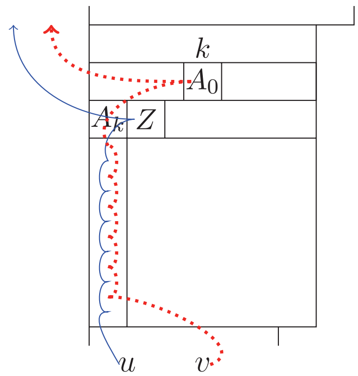


Figure 5.8. A path P_1 with $[b](i,1) \in R^P$ for all $i = 1, \ldots, i_0 - 1$.

In fact, as $(-1)^P = (-1)^Q$ and $H(Q) = H(P)$ for all $Q_k \in [P_1]$, it is enough to show that

$$\sum_{Q_k \in [P_1]} Q_k \in F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

Assume, without loss of generality, that $P_1 \in \mathcal{T}_{x_1}$, and let $[P_1]_1 = [P_1] \cap \mathcal{T}_{x_1}$ and $[P_1]_2 = [P_1] \cap \mathcal{T}_{x_2}$, so that

$$[P_1] = [P_1]_{x_1} \bigsqcup [P_1]_{x_2}.$$

See Figure 5.9.

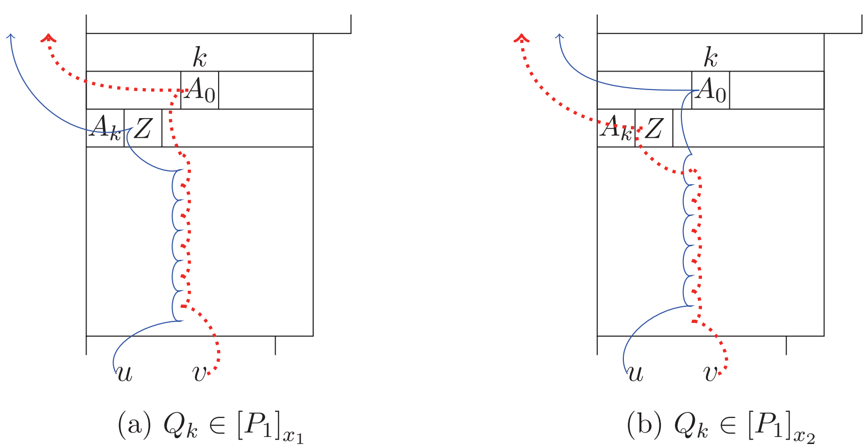


Figure 5.9. Paths in $[P_1]_{x_1}$ and $[P_1]_{x_2}$ removing the entries A_0 and Z from block $[b]$.

Let $T' \in \mathcal{F}_{\lambda\setminus X}$ be the unique tableau with $T' = T_P$ on $(\lambda \setminus X) \setminus [b]$ and $T' = T$ on $[b]$ except $T'_{a_0} = u$ and $T'_z = v$. See Figure 5.10

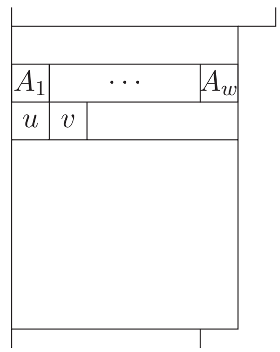


Figure 5.10. The unique tableau T' with $T' = T_P$ on $(\lambda \setminus X) \setminus [b]$ and $T' = T$ on $[b]$ except $T'_{a_0} = u$ and $T'_z = v$

By the proof of Lemma 3.2.3, the result of Corollary 3.2.4 still holds when moving u and v to boxes in the same row, which we have here after applying G_A. This gives, modulo $F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}$,

$$\begin{aligned}\sum_{Q_k\in[P_1]} Q_k &= \sum_{Q_k\in[P_1]_{x_1}} Q_k + \sum_{Q_k\in[P_1]_{x_2}} Q_k \\ &= (-1)^{i_0+1+1+2(i_0-1)} \begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T' + (-1)^{i_0+1+1+2(i_0-1)} \begin{array}{|c|}\hline \alpha_1^P \\ \hline \alpha_2^P \\ \hline \end{array} \otimes T' \\ &= 0.\end{aligned}$$

$\square$

Case (5.2.1.7). In this section we show that the sum over all paths that move A_i and a box in $[b]$ above A above $[b]$ is in $F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}$. Note that for any such path, there must be a box $[b](i_0+2, j) \in R^P$. Recall that

$$\mathcal{T}_7 = \bigsqcup_{\substack{0\le i\le w_b \\ 1\le j\le w_b}} \mathcal{T}_7^{i,j},$$

where

$$\mathcal{T}_7^{i,j} = \{P_k \in \mathcal{T}_7 : P(\sigma_k^A A_i) > [b], [b](i_0+2, j) \in R^P\}.$$

For $l = 1, 2$, let $\mathcal{T}_{7,x_l}$ be the set of all P_k in $\mathcal{T}_7$ such that the orbit of x_l intersects the first row in $[b]$,

$$\mathcal{T}_{7,x_l} = \{P_k \in \mathcal{T}_7 : R_l^P \cap [b](1) \neq \emptyset\}.$$

Then

$$\mathcal{T}_7 = \mathcal{T}_{7,x_1} \bigsqcup \mathcal{T}_{7,x_2},$$

and letting

$$\mathcal{T}_{7,x_l}^{i,j} = \mathcal{T}_{7,x_l} \bigcap \mathcal{T}_7^{i,j},$$

we have

$$\mathcal{T}_7 = \bigsqcup_{\substack{l=1,2,\ 0 \le i \le w_b \\ 1 \le j \le w_b}} \mathcal{T}_{7,x_l}^{i,j}.$$

It is then enough to show that

$$\sum_{P_k \in \mathcal{T}_{7,x_1}^{0,1}} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda \setminus X,n},$$

with $z = [b](i_0+2, 1)$ and $Z = T_z$, with the other cases being similar.

Proof. For the rest of Case (5.2.1.7) let $\mathcal{S}$ be the set of all $P_k \in \mathcal{T}_{7,x_1}^{0,1}$ that hit a box other than a_0 in row $[b](i_0)$,

$$\mathcal{S} := \{P_k \in \mathcal{T}_{7,x_1}^{0,1} : [b](i_0, j) \in R^P \text{ for some } 2 \le j \le w_b\},$$

and let $\mathcal{T} := T_{7,x_1}^{0,1} \setminus S$. One can show

$$\sum_{P_k \in \mathcal{S}} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda \setminus X,n}$$

by following the proof of Case (5.2.1.6). It remains to show

$$\sum_{P_k \in \mathfrak{T}} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

Observe that $\mathfrak{T}$ is the union of the following disjoint sets:

$$\mathfrak{T}^1 = \{P_k \in \mathfrak{T} : P(\sigma_k^A A_0), P(Z) \notin Y\},$$

$$\mathfrak{T}^2 = \{P_k \in \mathfrak{T} : P(\sigma_k^A A_0) \in Y, P(Z) \notin Y\},$$

$$\mathfrak{T}^3 = \{P_k \in \mathfrak{T} : P(Z) \in Y, P(\sigma_k^A A_0) \notin Y\}, \text{ and}$$

$$\mathfrak{T}^4 = \{P_k \in \mathfrak{T} : P(\sigma_k^A A_0), P(Z) \in Y\}.$$

So, it is enough to show that for $1 \leq i \leq 4$,

$$\sum_{P_k \in \mathfrak{T}^i} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

Define the relation $\sim^i$ on $\mathfrak{T}^i$, for $1 \leq i \leq 4$, as follows. Define $\sim^1$ on $\mathfrak{T}^1$ by

$$P_k \sim^1 Q_j \iff Q \text{ is a } ([b](i_0), [b](i_0+1))\text{-path extension of } P,$$
$$Q(\sigma_j^A A_0) = P(\sigma_k^A A_0), \text{ and } Q(Z) = P(Z).$$

Define $\sim^2$ on $\mathfrak{T}^2$ by

$$P_k \sim^2 Q_j \iff Q \text{ is a } ([b](i_0), [b](i_0+1))\text{-path extension of } P,$$
$$\text{and } Q(Z) = P(Z).$$

Define $\sim^3$ on $\mathfrak{T}^3$ by

$$P_k \sim^3 Q_j \iff Q \text{ is a } ([b](i_0), [b](i_0+1))\text{-path extension of } P,$$
$$\text{and } Q(\sigma_j^A A_0) = P(\sigma_k^A A_0).$$

Define $\sim^4$ on $\mathfrak{T}^4$ by

$$P_k \sim^4 Q_j \iff Q \text{ is a } ([b](i_0), [b](i_0+1))\text{-path extension of } P.$$

It is clear that for $1 \le i \le 4$, $\sim^i$ an equivalence relation on $\mathcal{T}^i$, so that

$$\sum_{P_k \in \mathcal{T}^i} \frac{(-1)^P}{H(P)} P_k = \sum_{[P_k] \in \mathcal{T}^i/\sim^i} \sum_{Q_k \in [P_k]} \frac{(-1)^Q}{H(Q)} Q_k.$$

Thus it is enough to show that for $1 \le i \le 4$,

$$\sum_{[P_k] \in \mathcal{T}^i/\sim^i} \sum_{Q_k \in [P_k]} \frac{(-1)^Q}{H(Q)} Q_k \in F_2 \otimes \mathcal{R}_{\lambda \backslash X, n}.$$

We will show the case $i = 1$, with the other cases being similar.

Pick $P_0 \in \mathcal{T}^1$ with $A_1 \in R^P$ and let $u := P^{-1}(A_0)$ and $v := P^{-1}(A_1)$. See Figure 5.11 and note that the image of Z can be in block $[b]$. It is then enough to show that

$$\sum_{Q_k \in [P_0]} \frac{(-1)^Q}{H(Q)} Q_k \in F_2 \otimes \mathcal{R}_{\lambda \backslash X, n},$$

with the other cases being similar.

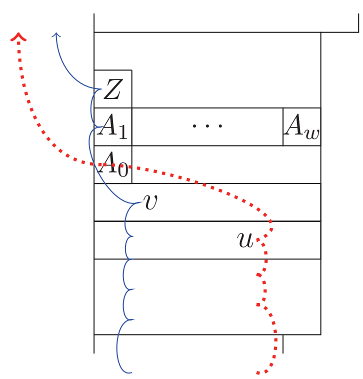


Figure 5.11. A path $P_0 \in \mathcal{T}^1$ with $A_1 \in R^P$.

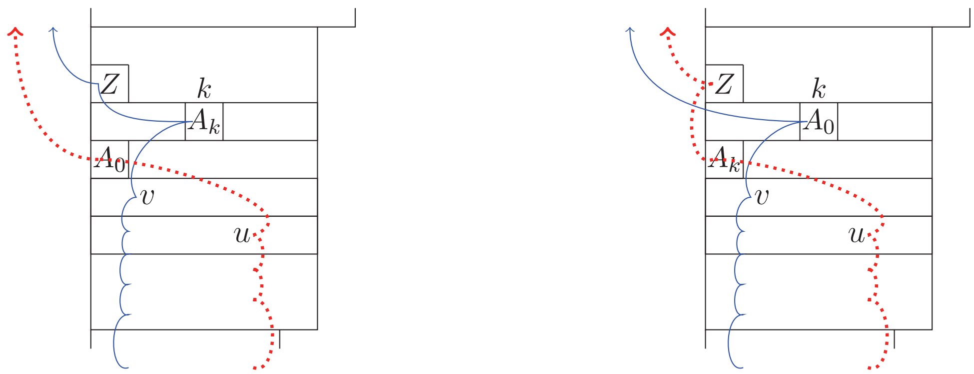
In fact, as $(-1)^P = (-1)^Q$ and $H(Q) = H(P)$ for all $Q_k \in [P_0]$, it is enough to show that

$$\sum_{Q_k \in [P_0]} Q_k \in F_2 \otimes \mathcal{R}_{\lambda \backslash X, n}.$$

Let $[P_0]_1 = \{Q_0 \in [P_0]\}$ and $[P_0]_2 = \{Q_k \in [P_0] : 1 \leq k \leq w_b\}$, so that

$$[P_0] = [P_0]_1 \bigsqcup [P_0]_2 \,.$$

See Figure 5.12, where the image of Z can be in $[b]$.



(a) A path $Q_0 \in [P_0]_1$. (b) A path $Q_k \in [P_0]_2$.

Figure 5.12. The paths in $[P_0]_1$ and $[P_0]_2$ removing the entry A_k from block $[b]$ and acting on a box in $[b]$ above A.

Let $T' \in \mathcal{F}_{\lambda \setminus X}$ be the unique tableau with $T' = T_P$ on $(\lambda \setminus X)\{([b](i_0), [b](i_0+2))\}$ and $T' = T$ on $([b](i_0), [b](i_0+2))$ except $T'_z = v$ and $T'_{a_0} = u$ as in Figure 5.13.

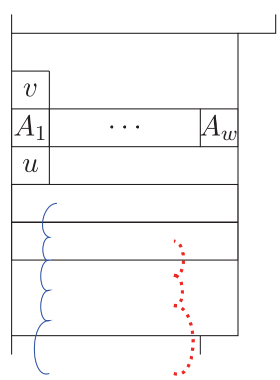


Figure 5.13. The unique tableau T' with $T' = T_P$ on $(\lambda \setminus X)\{([b](i_0), [b](i_0 + 2))\}$ and $T' = T$ on $([b](i_0), [b](i_0 + 2))$ except $T'_z = v$ and $T'_{a_0} = u$.

Then by Corollary 3.2.4 and applications of G_A we have, modulo $F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}$,

$$\sum_{Q_k \in E^P} Q_K = \sum_{Q_k \in [P_0]_1} Q_K + \sum_{Q_k \in [P_0]_2} Q_K = -\boxed{\begin{matrix}\alpha_2^P \\ \hline \alpha_1^P\end{matrix}} \otimes T' - \boxed{\begin{matrix}\alpha_1^P \\ \hline \alpha_2^P\end{matrix}} \otimes T' = 0.$$

□

5.3 Preserving Garnir Relations for Hooks Contained in Two Blocks

5.3.1

We now show that Equation 5.1.2.1 holds when $m = 2$ for all hooks $A \subset [b] \cup [b+1]$ for some $1 \leq b \leq N-1$. For the rest of Section 5.3, fix $T \in \mathcal{F}_{\lambda,n}$ and let

$$A = \{a_0 := [b](h_b, 1), a_1 := [b+1](1,1), \ldots, a_{w_{b+1}} := [b+1](1, w_{b+1})\} \subset T_0$$

so that $A \subset [b] \cup [b+1]$. Denote the entries of A in T by $A_k = T_{a_k}$ for $k = 0, 1, \ldots, w_{b+1}$. Then by Lemma 3.1.4, modulo $F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}$ we have

$$\begin{aligned}\Phi_2\left(G_A(T)\right) &= \sum_P \frac{(-1)^P}{H(P)} P \left(\sum_{\sigma \in \mathfrak{S}_A} \sigma T \right) \\ &= C \sum_P \sum_{k=0}^{w_{b+1}} \frac{(-1)^P}{H(P)} P\left(\sigma_k^A T\right),\end{aligned}$$

where the sum is over all 2-paths P on λ removing X. The set of all $P_k := P(\sigma_k^A T)$ appearing in the image $\Phi_2\left(G_A(T)\right)$ above is the union of the following disjoint sets.

The P_ks that miss A,

$$\mathcal{T}_1 = \{P_k : R^P \cap A = \emptyset\}. \tag{5.3.1.1}$$

The P_ks that hit A and keep A in $[b] \cup [b+1]$,

$$\mathcal{T}_2 = \{P_k : R^P \cap A \neq \emptyset,\ P(A) \leq [b+1]\}. \tag{5.3.1.2}$$

The P_ks that have exactly one orbit in $[b] \cup [b+1]$ and move A_i above $[b+1]$,

$$\mathcal{T}_3 = \bigsqcup_{i=0}^{w_{b+1}} \mathcal{T}_3^i, \tag{5.3.1.3}$$

where

$$\mathcal{T}_3^i = \{P_k \in \mathcal{T}_3 : \text{exactly one of } R_{x_1}^P, R_{x_2}^P, \text{ intersect } [b] \cup [b+1]$$
$$\text{and } P(\sigma_k^A A_i) > [b+1]\}.$$

The P_ks that move A_i and A_j above $[b+1]$,

$$\mathcal{T}_4 = \bigsqcup_{0 \le i < j \le w_{b+1}} \mathcal{T}_4^{i,j}, \tag{5.3.1.4}$$

where

$$\mathcal{T}_4^{i,j} = \{P_k : P(\sigma_k^A A_i) > [b+1], \text{ and } P(\sigma_k^A A_j) > [b+1]\}.$$

The P_ks that move A_i and a box Z in $[b]$ below A above $[b+1]$,

$$\mathcal{T}_5 = \bigsqcup_{\substack{0 \le i \le w_{b+1} \\ z = [b](j,k),\ 1 \le j < h_b \text{ and } 1 \le k \le w_b}} \mathcal{T}_5^{i,z}, \tag{5.3.1.5}$$

where

$$\mathcal{T}_5^{i,z} = \{P_k : P(\sigma_k^A A_i) > [b+1], P(z) > [b+1]\}.$$

The P_ks that move A_i and a box other than a_0 in row $[b](h_b)$ above $[b+1]$,

$$\mathcal{T}_6 = \bigsqcup_{0 \le i \le w_{b+1},\ 2 \le j \le w_b} \mathcal{T}_6^{i,j}, \tag{5.3.1.6}$$

where

$$\mathcal{T}_6^{i,j} = \{P_k : P(\sigma_k^A A_i) > [b+1], P([b](h_b, j)) > [b+1]\}.$$

The P_ks that move A_i and a box above A above $[b+1]$,

$$\mathcal{T}_7 = \bigsqcup_{\substack{0 \le i \le w_{b+1} \\ 1 \le j \le w_{b+1}}} \mathcal{T}_7^{i,j}, \tag{5.3.1.7}$$

where

$$\mathcal{T}_7^{i,z} = \{P_k : P(\sigma_k^A A_i) > [b+1], P([b+1](2,j)) \in R^P\}.$$

Then we have, modulo $F_2 \otimes \mathcal{R}_{\lambda \backslash X, n}$,

$$\begin{aligned}\Phi_2\left(G_A(T)\right) &= C \sum_P \sum_{k=0}^{w_{b+1}} \frac{(-1)^P}{H(P)} P\left(\sigma_k^A T\right) \\ &= C \sum_{j=1,\ldots,7} \sum_{P_k \in \mathcal{T}_j} \frac{(-1)^P}{H(P)} P_k.\end{aligned}$$

5.3.2

We show that for $1 \leq j \leq 7$,

$$\sum_{P_k \in \mathcal{T}_j} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda \backslash X, n},$$

and hence Equation 5.1.2.1 holds when $m = 2$ for all blocks $A \subset [b] \cup [b+1]$.

The proofs of Case (5.3.1.1) and Case (5.3.1.2) are similar to the proofs of Case (4.2.1.1) and Case (4.2.1.2), respectively. The proof of Case (5.3.1.3) is similar to the proof of Case (4.3.1.3), and goes through by observing that using the definition of $H(P)$ for a 2-path only adds and subtracts 1 in some of the terms. The proofs of Case (5.3.1.4) and Case (5.3.1.7) are similar to the proofs of Case (5.2.1.4) and Case (5.2.1.7), respectively, as these proofs did not depend on $H(P)$. It remains to show Case (5.3.1.5) and Case (5.3.1.6). In both cases we assume $b > b_1$ and $A \cap X = \emptyset$, as if $b = b_1$ or if $A \cap X \neq \emptyset$ we may follow the proof of Subcase (4.3.1.3.1).

Case (5.3.1.5). In this case we show that the sum over all paths that move A_i and a box Z in $[b]$ below A above $[b+1]$ is in $F_2 \otimes \mathcal{R}_{\lambda\backslash X,n}$. Recall that

$$\mathcal{T}_5 = \bigsqcup_{\substack{0\le i\le w_{b+1} \\ z=[b](j,k),\ 1\le j<h_b \text{ and } 1\le k\le w_b}} \mathcal{T}_5^{i,z},$$

where

$$\mathcal{T}_5^{i,z} = \{P_k : P(\sigma_k^A A_i) > [b+1], P(z) > [b+1]\}.$$

It is enough to show that

$$\sum_{P_k\in\mathcal{T}_5^{0,z}} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda\backslash X,n},$$

where $z = [b](i_z, j_z)$ is a fixed box with $1 \le i_z \le h_b - 1$ odd, $1 \le j_z \le w_{b+1}$, and $Z = T_z$, with the other cases being similar.

Proof. For the rest of Case (5.3.1.5), let $\mathcal{T} := \mathcal{T}_5^{0,z}$ and, for any 2-path P on λ removing X, let $\tilde{h^P} = h^P - h_b^P - h_{b+1}^P$ and $H(\tilde{P}) = \dfrac{H(P)}{H_b(P)H_{b+1}(P)}$. Observe that $\mathcal{T}$ is the union of the following disjoint sets:

$$\mathcal{T}^1 = \{P_k \in \mathcal{T} : P(\sigma_k^A a_0), P(z) \notin Y\},$$

$$\mathcal{T}^2 = \{P_k \in \mathcal{T} : P(\sigma_k^A a_0) \in Y, P(z) \notin Y\},$$

$$\mathcal{T}^3 = \{P_k \in \mathcal{T} : P(z) \in Y, P(\sigma_k^A a_0) \notin Y\}, \text{ and}$$

$$\mathcal{T}^4 = \{P_k \in \mathcal{T} : P(\sigma_k^A a_0), P(z) \in Y\}.$$

So it is enough to show that for $1 \le i \le 4$,

$$\sum_{P_k\in\mathcal{T}^i} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda\backslash X,n}.$$

Define the relation $\sim^i$ on $\mathcal{T}^i$, for $1 \le i \le 4$, as follows. Define $\sim^1$ on $\mathcal{T}^1$ by

$$P_k \sim^1 Q_j \iff Q \text{ is a } ([b](1),[b+1](1))\text{-path extension of } P,$$

$$Q(\sigma_j^A a_0) = P(\sigma_k^A a_0), \text{ and } Q(z) = P(z).$$

Define $\sim^2$ on $\mathcal{T}^2$ by

$$P_k \sim Q_j \iff Q \text{ is a } ([b](1),[b+1](1))\text{-path extension of } P,$$

$$\text{and } Q(z) = P(z).$$

Define $\sim^3$ on $\mathcal{T}^3$ by

$$P_k \sim^3 Q_j \iff Q \text{ is a } ([b](1),[b+1](1))\text{-path extension of } P,$$

$$\text{and } Q(\sigma_j^A a_0) = P(\sigma_k^A a_0).$$

Define $\sim^4$ on $\mathcal{T}^4$ by

$$P_k \sim^4 Q_j \iff Q \text{ is a } ([b](1),[b+1](1))\text{-path extension of } P.$$

It is clear that for $1 \le i \le 4$, $\sim^i$ an equivalence relation on $\mathcal{T}^i$, so that

$$\sum_{P_k \in \mathcal{T}^i} \frac{(-1)^P}{H(P)} P_k = \sum_{[P_k] \in \mathcal{T}^i/\sim^i} \sum_{Q_k \in [P_k]} \frac{(-1)^Q}{H(Q)} Q_k.$$

Thus it is enough to show that, for $1 \le i \le 4$,

$$\sum_{[P_k] \in \mathcal{T}^i/\sim^i} \sum_{Q_k \in [P_k]} \frac{(-1)^Q}{H(Q)} Q_k \in F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}.$$

We will show the case $i = 1$, with the rest being similar. Pick $P_0 \in \mathcal{T}^1$ with $[b](i,1) \in R^P$ for all $1 \le i \ne i_z \le h_b$, and let $[b_u](i_u, j_u) = P^{-1}([b](1,1))$ and $[b_v](i_v, j_v) = P^{-1}([b](2,1))$ with $u = T_{[b_u](i_u,j_u)}$ and $v = T_{[b_v](i_v,j_v)}$ as in Figure 5.14.

It is then enough to show that

$$\sum_{Q_k \in [P_0]} \frac{(-1)^Q}{H(Q)} Q_k \in F_2 \otimes \mathcal{R}_{\lambda\setminus X,n}.$$

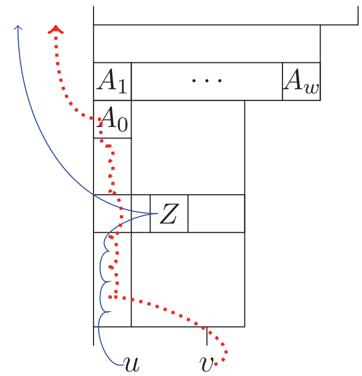


Figure 5.14. A path P_0 with $[b](i,1) \in R^P$ for all $1 \leq i \neq i_z \leq h_b$.

In fact, as $\tilde{h^Q} = \tilde{h^P}$ and $H(\tilde{Q}) = H(\tilde{P})$ for all $Q_k \in [P_0]$, it is enough to show that

$$\sum_{Q_k \in [P_0]} \frac{(-1)^{h_b^Q + h_{b+1}^Q}}{H_b^Q H_{b+1}^Q} Q_k \in F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

Observe that $[P_0]$ can be written as the disjoint union

$$[P_0] = \bigsqcup_{i=1}^{7} [P_0]_i \,,$$

where the $[P_0]_i$ are defined as follows.

$[P_0]_1$ is the set of all paths acting on $\sigma_0 T$ as in Figure 5.15,

$$[P_0]_1 = \{Q_0 \in [P_0]\}.$$

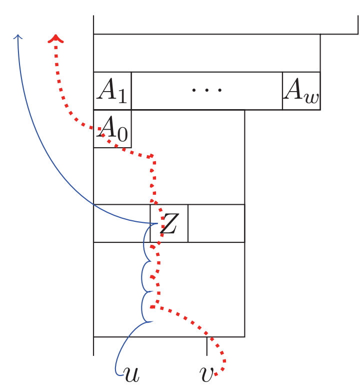


Figure 5.15. The paths in $[P_0]_1$ acting on $\sigma_0 T$.

$[P_0]_2$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that hit $\sigma_k^A a_k$ as in Figure 5.16,

$$[P_0]_2 = \{Q_k \in [P_0] : k \neq 0, a_0 \in R^Q\},$$

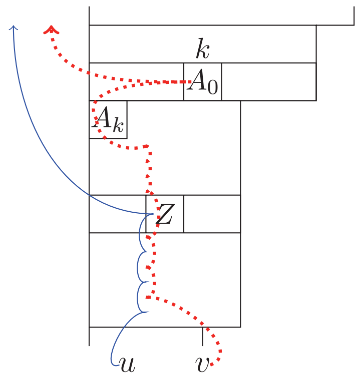


Figure 5.16. The paths in $[P_0]_2$ acting on $\sigma_k^A T$ for $k \neq 0$ that hit $\sigma_k^A a_k$.

$[P_0]_3$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that miss $\sigma_k^A a_k$ but hit row $[b](h_b)$ as in Figure 5.17,

$$[P_0]_3 = \{Q_k \in [P_0] : k \neq 0, [b](h_b, j) \in R^Q \text{ for some } 2 \leq j \leq w_b\},$$

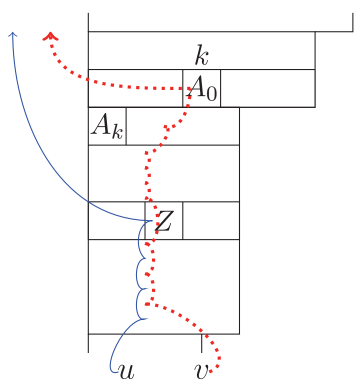


Figure 5.17. The paths in $[P_0]_3$ acting on $\sigma_k^A T$ for $k \neq 0$ that miss $\sigma_k^A a_k$ but hit row $[b](h_b)$.

$[P_0]_4$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that miss row $[b](h_b)$ with R_2^P leaving $[b]$ in a row above Z as in Figure 5.18,

$$[P_0]_4 = \{Q_k \in [P_0] : k \neq 0, Q([b](i,j)) = a_k \text{ for some } i_z < i < h_b, 1 \leq j \leq w_b\},$$

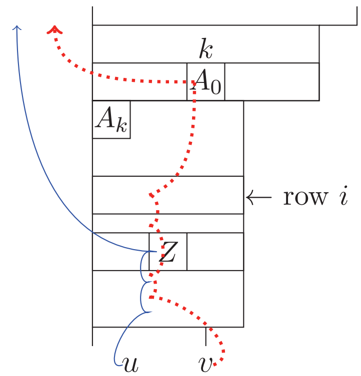


Figure 5.18. The paths in $[P_0]_4$ acting on $\sigma_k^A T$ for $k \neq 0$ that miss row $[b](h_b)$ with R_2^P leaving $[b]$ in a row above Z.

$[P_0]_5$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that miss row $[b](h_b)$ with R_2^P leaving $[b]$ from a row $[b](i)$ below Z with i even as in Figure 5.19,

$$[P_0]_5 = \{Q_k \in [P_0] : k \neq 0, Q([b](i,j)) = \sigma_k^A a_0 \text{ for some } 1 \leq i < i_z \text{ even,}$$
$$\text{and } 1 \leq j \leq w_b\},$$

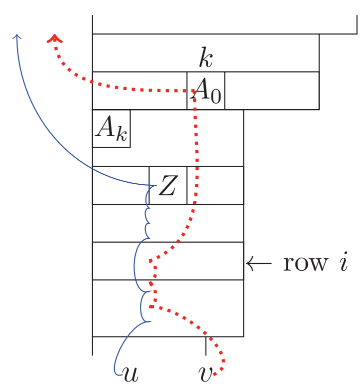


Figure 5.19. The paths in $[P_0]_5$ acting on $\sigma_k^A T$ for $k \neq 0$ that miss row $[b](h_b)$ with R_2^P leaving $[b]$ in an even row $[b](i)$ below Z.

$[P_0]_6$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ that miss row $[b](h_b)$ with R_2^P leaving $[b]$ from a row $[b](i)$ below Z with i odd as in Figure 5.20,

$$[P_0]_6 = \{Q_k \in [P_0] : k \neq 0, Q([b](i,j)) = \sigma_k^A a_0 \text{ for some } 1 \leq i < i_z \text{ odd,}$$
$$\text{and } 1 \leq j \leq w_b\},$$

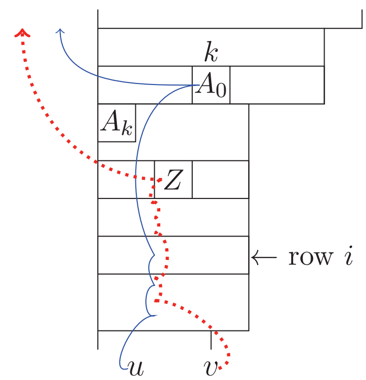


Figure 5.20. The paths in $[P_0]_6$ acting on $\sigma_k^A T$ for $k \neq 0$ that miss row $[b](h_b)$ with R_2^P leaving $[b]$ in an odd row $[b](i)$ below Z.

$[P_0]_7$ is the set of all paths acting on $\sigma_k^A T$ for $k \neq 0$ with R_2^P missing block $[b]$ as in Figure 5.21,

$$[P_0]_7 = \{Q_k \in [P_0] : k \neq 0, R_2^P \cap [b] = \emptyset\}.$$

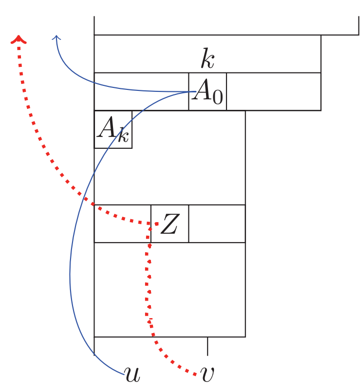


Figure 5.21. The paths in $[P_0]_7$ acting on $\sigma_k^A T$ for $k \neq 0$ with R_2^P missing block $[b]$.

Let $T' \in \mathcal{F}_{\lambda \setminus X}$ be the unique tableau with $T' = T_P$ on $(\lambda \setminus X) \setminus [b] \cup [b+1]$ and $T' = T$ on $[b] \cup [b+1]$ except $T'_z = u$ and $T'_{a_0} = v$ as in Figure 5.22.

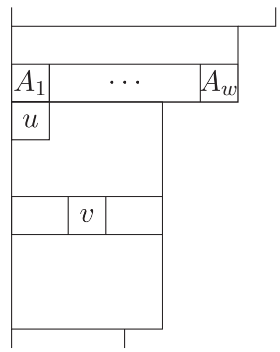


Figure 5.22. The unique tableau T' with $T' = T_P$ on $(\lambda \setminus X) \setminus [b] \cup [b+1]$ and $T' = T$ on $[b] \cup [b+1]$ except $T'_z = u$ and $T'_{a_0} = v$.

Then by Corollary 3.2.4 and applications of G_A we have, modulo $F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}$,

$$\begin{aligned}
\sum_{Q_k \in [P_0]_1} \frac{(-1)^{h_b^Q + h_{b+1}^Q}}{H_b^Q H_{b+1}^Q} Q_k =& \frac{(-1)^{h_b + h_b - 2 + i_z - 2}}{H(b)(H(b)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline \end{array} \otimes T' \\
=& \frac{-H(b+1)+1}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline \end{array} \otimes T',
\end{aligned}$$

$$\begin{aligned}
\sum_{Q_k \in [P_0]_2} \frac{(-1)^{h_b^Q + h_{b+1}^Q}}{H_b^Q H_{b+1}^Q} Q_k =& \frac{(-1)^{h_b + 1 + h_b - 2 + i_z - 2}(w_{b+1})}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline \end{array} \otimes T' \\
=& \frac{w_{b+1}}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline \end{array} \otimes T'
\end{aligned}$$

$$\begin{aligned}
\sum_{Q_k \in [P_0]_3} \frac{(-1)^{h_b^Q + h_{b+1}^Q}}{H_b^Q H_{b+1}^Q} Q_k =& \frac{(-1)^{h_b + 1 + 1 + h_b - 2 + i_z - 2}(w_b - 1)}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline \end{array} \otimes T' \\
=& \frac{-w_b + 1}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline \end{array} \otimes T'
\end{aligned}$$

$$\sum_{Q_k \in [P_0]_4} \frac{(-1)^{h_b^Q + h_{b+1}^Q}}{H_b^Q H_{b+1}^Q} Q_k = \sum_{i=i_z+1}^{h_b-1} \frac{(-1)^{i+1+1+i-2+i_z-2+1}}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline\end{array} \otimes T'$$

$$= \frac{h_b - 1 - i_z}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline\end{array} \otimes T'$$

$$\sum_{Q_k \in [P_0]_5} \frac{(-1)^{h_b^Q + h_{b+1}^Q}}{H_b^Q H_{b+1}^Q} Q_k = \sum_{\substack{1 \le i < i_z, \\ i \text{ even}}} \frac{(-1)^{i_z+1+1+i_z-2+i-2+1}}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline\end{array} \otimes T'$$

$$= \sum_{\substack{1 \le i < i_z, \\ i \text{ even}}} \frac{(-1)^{i+1}}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline\end{array} \otimes T'$$

$$\sum_{Q_k \in [P_0]_6} \frac{(-1)^{h_b^Q + h_{b+1}^Q}}{H_b^Q H_{b+1}^Q} Q_k = \sum_{\substack{1 \le i < i_z, \\ i \text{ odd}}} \frac{(-1)^{i_z+1+1+i_z-2+i-2+1}}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline Z \\ \hline A_0 \\ \hline\end{array} \otimes T'$$

$$= \sum_{\substack{1 \le i < i_z, \\ i \text{ odd}}} \frac{(-1)^{i+2}}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline\end{array} \otimes T'$$

and

$$\sum_{Q_k \in [P_0]_7} \frac{(-1)^{h_b^Q + h_{b+1}^Q}}{H_b^Q H_{b+1}^Q} Q_k = \frac{(-1)^{i_z+1+1+i_z-1}}{(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline Z \\ \hline A_0 \\ \hline\end{array} \otimes T'$$

$$= \frac{H(b)}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline\end{array} \otimes T'.$$

Then as $H(b+1) = H(b) + w_{b+1} - w_b + h_b$ and

$$\sum_{\substack{1 \le i < i_z, \\ i \text{ even}}} \frac{(-1)^{i}}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline\end{array} \otimes T'$$

$$+ \sum_{\substack{1 \le i < i_z, \\ i \text{ odd}}} \frac{(-1)^{i+1}}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline\end{array} \otimes T'$$

$$= \frac{i_z - 1}{H(b)(H(b)-1)(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline\end{array} \otimes T'$$

we get, modulo $F_2 \otimes \mathcal{R}_{\lambda\backslash X,n}$

$$\sum_{Q_k\in[P_0]} \frac{(-1)^{h_b^Q+h_{b+1}^Q}}{H_b^Q H_{b+1}^Q} Q_k = \sum_{i=1,\dots,7} \sum_{Q_k\in[P_0]_i} \frac{(-1)^{h_b^Q+h_{b+1}^Q}}{H_b^Q H_{b+1}^Q} Q_k$$

$$= \frac{-H(b+1)+1+w_{b+1}-w_b+1+h_b-1-i_z-+i_z-1+H(b)}{H(b)\,(H(b)-1)\,(H(b+1)-1)} \begin{array}{|c|}\hline A_0 \\ \hline Z \\ \hline \end{array} \otimes T'$$

$$= 0$$

$\square$

Case (5.3.1.6). In this case we show that the sum over all paths that move A_i and a box other than a_0 in row $[b](h_b)$ above $[b+1]$ is in $F_2 \otimes \mathcal{R}_{\lambda\backslash X,n}$. Recall that

$$\mathcal{T}_6 = \bigsqcup_{0\le i\le w_{b+1},\ 2\le j\le w_b} \mathcal{T}_6^{i,j},$$

where

$$\mathcal{T}_6^{i,j} = \{P_k : P(\sigma_k^A A_i) > [b+1], P([b](h_b,j)) > [b+1]\}.$$

It is enough to show that

$$\sum_{P_k\in\mathcal{T}_6^{0,2}} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda\backslash X,n},$$

with the other cases being similar.

Proof. For the rest of Case (5.3.1.6), let $z = [b](h_b,2)$ and $Z = T_z$, and observe that $\mathcal{T}_6^{0,2}$ is the union of the following disjoint sets.

$$\mathcal{T}_6^{0,2,1} = \{P_k \in \mathcal{T}_6^{0,2} : P(\sigma_k^A a_0), P(z) \notin Y\},$$

$$\mathcal{T}_6^{0,2,2} = \{P_k \in \mathcal{T}_6^{0,2} : P(\sigma_k^A a_0) \in Y, P(z) \notin Y\},$$

$$\mathcal{T}_6^{0,2,3} = \{P_k \in \mathcal{T}_6^{0,2} : P(z) \in Y, P(\sigma_k^A a_0) \notin Y\},\ \text{and}$$

$$\mathcal{T}_6^{0,2,4} = \{P_k \in \mathcal{T}_6^{0,2} : P(\sigma_k^A a_0), P(z) \in Y\}.$$

So it is enough to show that for $1 \leq i \leq 4$,

$$\sum_{P_k \in \mathcal{T}_6^{0,2,i}} \frac{(-1)^P}{H(P)} P_k \in F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

Now define the relation $\sim^i$ on $\mathcal{T}_6^{0,2,i}$, for $1 \leq i \leq 4$, as follows. Define $\sim^1$ on $\mathcal{T}_6^{0,2,1}$ by

$$\begin{aligned} P_k \sim^1 Q_j \iff & Q \text{ is a } ([b](1), [b+1](1))\text{-path extension of } P, \\ & Q(\sigma_j^A A_0) = P(\sigma_k^A A_0), \text{ and } Q(Z) = P(Z), \text{ and} \\ & Q^{-1}(\sigma_j^A A_0) \text{ and } P^{-1}(\sigma_k^A A_0) \text{ are in the same row.} \end{aligned}$$

Define $\sim^2$ on $\mathcal{T}_6^{0,2,2}$ by

$$\begin{aligned} P_k \sim^2 Q_j \iff & Q \text{ is a } ([b](1), [b+1](1))\text{-path extension of } P, \\ & Q(Z) = P(Z), \text{ and} \\ & Q^{-1}(\sigma_j^A A_0) \text{ and } P^{-1}(\sigma_k^A A_0) \text{ are in the same row.} \end{aligned}$$

Define $\sim^3$ on $\mathcal{T}_6^{0,2,3}$ by

$$\begin{aligned} P_k \sim^3 Q_j \iff & Q \text{ is a } ([b](1), [b+1](1))\text{-path extension of } P, \\ & Q(\sigma_j^A A_0) = P(\sigma_k^A A_0), \text{ and} \\ & Q^{-1}(\sigma_j^A A_0) \text{ and } P^{-1}(\sigma_k^A A_0) \text{ are in the same row.} \end{aligned}$$

Define $\sim^4$ on $\mathcal{T}_6^{0,2,4}$ by

$$\begin{aligned} P_k \sim^4 Q_j \iff & Q \text{ is a } ([b](1), [b+1](1))\text{-path extension of } P, \text{ and} \\ & Q^{-1}(\sigma_j^A A_0) \text{ and } P^{-1}(\sigma_k^A A_0) \text{ are in the same row.} \end{aligned}$$

It is clear that for $1 \leq i \leq 4$, $\sim^i$ an equivalence relation on $\mathcal{T}_6^{0,2,i}$, so that

$$\sum_{P_k \in \mathcal{T}_6^{0,2,i}} \frac{(-1)^P}{H(P)} P_k = \sum_{[P_k] \in \mathcal{T}_6^{0,2,i}/\sim^i} \sum_{Q_k \in [P_k]} \frac{(-1)^Q}{H(Q)} Q_k.$$

Thus it is enough to show that for $1 \le i \le 4$,

$$\sum_{[P_k]\in \mathcal{T}_6^{0,2,i}/\sim^i} \sum_{Q_k\in[P_k]} \frac{(-1)^Q}{H(Q)} Q_k \in F_2 \otimes \mathcal{R}_{\lambda\backslash X,n}.$$

We will show the case $i = 1$, with the rest being similar. For the rest of Case (5.3.1.6), let $\mathcal{T} := \mathcal{T}_6^{0,2,1}$.

For $l = 1, 2$, let $\mathcal{T}_{x_l}$ be the set of all Q_k in $\mathcal{T}$ such that the orbit of x_l intersects the first row in $[b]$,

$$\mathcal{T}_{x_l} = \{P_k \in \mathcal{T} : R_l^P \cap [b](1) \neq \emptyset\}.$$

Note that as $a_0 = [b](h_b, 1)$ is in the same row as $z = [b](h_b, 2)$, it must be that $1 \le k \le w_b$ for all $P_k \in \mathcal{T}$. Pick $P_1 \in \mathcal{T}$ with $[b](i, 1) \in R^P$ for all $i = 1, \ldots, h_b - 1$, and $P^{-1}(\sigma_1 A_0) \in [b](i)$ with i odd, and let $[b_u](i_u, j_u) = P^{-1}([b](1, 1))$ and $[b_v](i_v, j_v) = P^{-1}([b](2, 1))$ with $u = T_{[b_u](i_u,j_u)}$ and $v = T_{[b_v](i_v,j_v)}$, as in Figure 5.23.

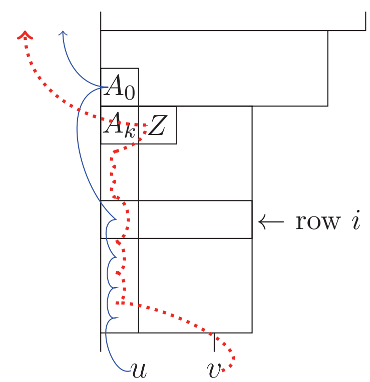


Figure 5.23. A path P_1 with $[b](i, 1) \in R^P$ for all $i = 1, \ldots, h_b - 1$ and $P^{-1}(\sigma_1 A_0) \in [b](i)$ with i odd.

It is then enough to show that

$$\sum_{Q_k\in[P_1]} \frac{(-1)^Q}{H(Q)} Q_k \in F_2 \otimes \mathcal{R}_{\lambda\backslash X,n},$$

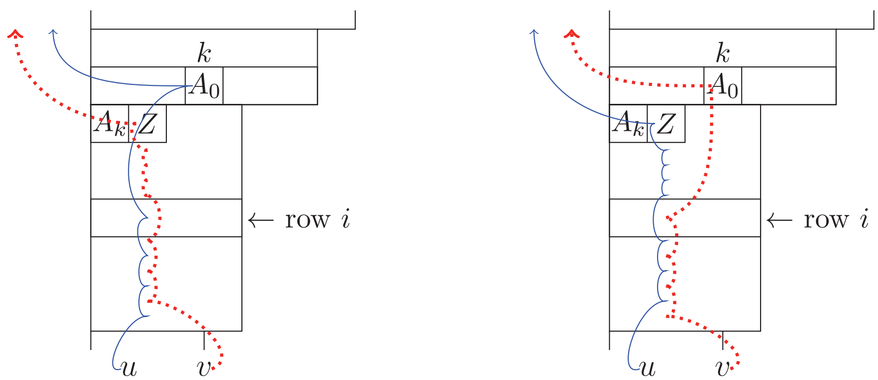
with the other cases being similar. In fact, as $(-1)^P = (-1)^Q$ and $H(Q) = H(P)$ for all $Q_k \in [P_1]$, it is enough to show that

$$\sum_{Q_k \in [P_1]} Q_k \in F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}.$$

Without loss of generality, assume $P_1 \in \mathfrak{T}_{x_1}$ and let $[P_1]_{x_1} = [P_1] \cap \mathfrak{T}_{x_1}$ and $[P_1]_{x_2} = [P_1] \cap \mathfrak{T}_{x_2}$, so that

$$[P_1] = [P_1]_{x_1} \bigsqcup [P_1]_{x_2}.$$

See Figure 5.24.



(a) $Q_k \in [P_1]_{x_1}$ (b) $Q_k \in [P_1]_{x_2}$

Figure 5.24. The paths in $[P_1]_{x_1}$ and $[P_1]_{x_2}$ removing the entry Z from block $[b]$ and the enrty A_0 from block $[b+1]$.

Let $T' \in \mathcal{F}_{\lambda \setminus X}$ be the unique tableau with $T' = T_P$ on $(\lambda \setminus X) \setminus ([b] \cup [b+1])$ and $T' = T$ on $[b] \cup [b+1]$ except $T'_z = v$ and $T'_{a_0} = u$ as in Figure 5.25.

As in the calculations for the proof of Case 5.2.1.6, by the proof of Lemma 3.2.3, the result of Corollary 3.2.4 still holds when moving u and v to boxes in the same row, which we have here after applying G_A. This gives, modulo $F_2 \otimes \mathcal{R}_{\lambda \setminus X, n}$,

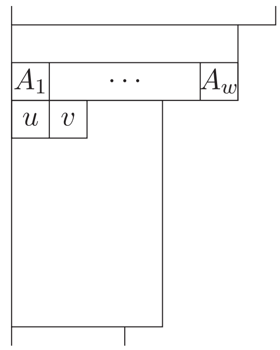


Figure 5.25. The unique tableau T' with $T' = T_P$ on $(\lambda \setminus X) \setminus ([b] \cup [b+1])$ and $T' = T$ on $[b] \cup [b+1]$ except $T'_z = v$ and $T'_{a_0} = u$.

$$
\begin{aligned}
\sum_{Q_k \in [P_1]} Q_k &= \sum_{Q_k \in [P_1]_{x_1}} Q_k + \sum_{Q_k \in [P_1]_{x_2}} Q_k \\
&= (-1)^{h_b+1+1h_b-2+i-2+1} \begin{array}{|c|}\hline \alpha_2^P \\ \hline \alpha_1^P \\ \hline \end{array} \otimes T' + (-1)^{h_b+1+1h_b-2+i-2+1} \begin{array}{|c|}\hline \alpha_1^P \\ \hline \alpha_2^P \\ \hline \end{array} \otimes T' \\
&= 0.
\end{aligned}
$$

$\square$

Thus Equation 5.1.2.1 holds for all hooks $A \subset [b] \cup [b+1]$, and so Theorem 5.1.1 holds.

CHAPTER SIX

Relating Pieri Inclusion Descriptions

In this chapter we show that our description of the Pieri inclusion removing one box, Φ_1, is the negative of Pieri inclusion description removing one box given in (Olver, 1982, §6). We then show that iterating Φ_1 is still a $GL(V)$ map and that our description of Pieri inclusions also describes the symmetric case. Finally, in the special case where the removal set is a column of boxes in the diagram, we show that iterating Φ_1 and the Pieri inclusion removing many boxes, Φ_m, differ my $m!$.

6.1 Comparing the One Box Removal Description to Olver's Description

Let $\widetilde{\Phi}_1$ be the Pieri inclusion removing one box described in (Olver, 1982, §6) and (Sam & Weyman, 2011) and Φ_1 be the Pieri inclusion removing one box described in 2.1.7.

Theorem. For Φ_1 and $\widetilde{\Phi}_1$ as above,

$$\Phi_1 = -\widetilde{\Phi}_1.$$

Proof. Let $T_\lambda \in \mathcal{T}_{\lambda,n}$ and $T_{\lambda\backslash X} \in \mathcal{T}_{\lambda\backslash X,n}$ be the diagrams corresponding to highest weight vectors as in 1.2.4. Then, in the image of T_λ, the coefficient of

$$\boxed{\alpha} \otimes T_{\lambda\backslash X}$$

where

$$\alpha = \sum_{i=b_1}^{N} h_i$$

is readily seen to be $-w_{b_1}$ in the image of Φ and w_{b_1} in the image of $\widetilde{\Phi}_1$. By uniqueness of the Pieri inclusion up to scalar multiple (Schur's Lemma), the result holds. $\square$

6.2 Iterating the One Box Removal Description and the Symmetric Case

6.2.1

Given a removal set $X = \{x_1 = [b_1](1, w_{b_1}), \ldots, x_m = [b_m](i_m, w_{b_m})\} \subset \lambda$, let

$$\Xi = X_1 \subset X_2 \subset \cdots \subset X_m = X$$

be a filtration of X where each $|X_k| = k$ so that the corresponding shapes $\lambda \setminus X_k$ are Young diagrams. For each such filtration, define Φ_1^m to be the map given by iterating Φ_1 where the box in X_1 is removed first, then the box in $X_2 \setminus X_1$, etc. That is, the first iteration is

$$\Phi_1^1(T) = \Phi_1(T) = \sum_P \frac{(-1)^P}{H(P)} P\,(T)$$

where the sum is over all 1-paths P on λ removing the box in X_1 and, for $k = 2, \ldots, m$, the kth iteration is

$$\Phi_1^k\,(T) = \sum_P \frac{(-1)^P}{H(P)} P\left(\Phi_1^{k-1}(T)\right)$$

where the sum is over all 1-paths P on $\lambda \setminus X_{k-1}$ removing the box in $X_k \setminus X_{k-1}$ and

$$P\,(Y_Q \otimes T_Q) = {}^{Y_P}_{Y_Q} \otimes P\,(T_Q)$$

where Y_P is the box removed by P from T_Q.

6.2.2

Lemma. Φ_1^m is a $GL(V)$-map.

Proof. By Theorem 6.1, this follows from the proof in (Sam & Weyman, 2011, Corollary 1.8), where it is shown for the iteration of Olver's map. □

6.2.3

Define the map

$$\Phi'_m : \mathbb{S}_\lambda(V) \to S^m(V) \otimes \mathbb{S}_{\lambda\setminus X}(V)$$

just as we have defined Φ_m in 2.2.4 except for redefining for all m-paths P on λ removing X

$$Y_P = E_X \boxed{\alpha_m^P}\boxed{\cdots}\boxed{\alpha_1^P},$$

which is standard form notation is $e_{\alpha_1^P} \cdots e_{\alpha_m^P} \in S^m V$.

Theorem. The map

$$\Phi'_m : \mathbb{S}_\lambda(V) \to S^m(V) \otimes \mathbb{S}_{\lambda\setminus X}(V)$$

is a $GL(V)$-map.

Proof. As Φ_m is a $GL(V)$-map, similar to (Sam & Weyman, 2011, Corollary 1.8), this follows by the results of Chapters Four and Five by keeping track of a sign. □

6.3 Relating The Pieri Inclusion Removing Many Boxes and the Iteration of the Pieri Inclusion Removing One Box in the Case of Removing a Column

Let Φ_m be the Pieri inclusion removing m boxes constructed in 2.2.4 and let Φ_1^m be the Pieri inclusion given by iterating one box removal constructed in 6.2.1.

Theorem. For Φ_1^m and Φ_m as above and for the removal set $X = \{x_1 = [b](1, w_b), x_2 = [b](2, w_b), \dots, x_m = [b](m, w_b)\}$,

$$\Phi_1^m = m! \cdot \Phi_m.$$

Proof. We will show that $\Phi_1^2 = 2 \cdot \Phi_2$ and then proceed via induction. Let $X_0 = \{x_1 = [b_0](1, w_{b_0}), x_2 = [b_0](2, w_{b_0})\}$ be a removal set in λ and let $T_\lambda \in \mathcal{T}_{\lambda,n}$ and $T_{\lambda\setminus X_0} \in \mathcal{T}_{\lambda\setminus X_0,n}$ be the diagrams corresponding to highest weight vectors as in Section 1.2.4. In the image of T_λ under Φ_2, the coefficient of

$$\begin{array}{|c|}\hline \alpha_2 \\ \hline \alpha_1 \\ \hline \end{array} \otimes T_{\lambda\setminus X_0}$$

where

$$\alpha_k = (T_\lambda)_{x_k} \qquad \text{for } k = 1, 2$$

is readily seen to be $w_{b_0}^2$ as the only 2-paths that remove the entries α_1 and α_2 are the ones with

$$R_1 \subset [b_0](1) \qquad \text{and} \qquad R_2 \subset [b_0](2),$$

where R_1 and R_2 are the orbits of x_1 and x_2, respectively. See figure 6.1.

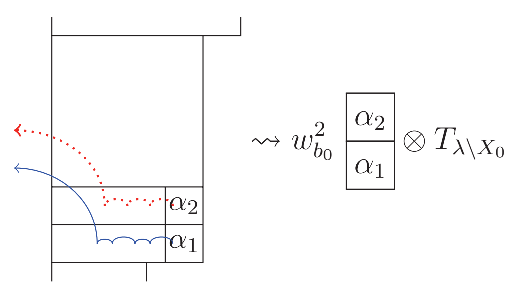


Figure 6.1. The 2-paths removing the entries α_1 and α_2 from T_λ.

In the image of T_λ under Φ_1^2, the only compositions of 1-paths that remove α_1 and α_2 are the ones where the orbits of x_1 and x_2 are contained in the rows $[b_0](1)$ and

$[b_0](2)$. See Figure 6.2. From this it is easy to see that the coefficient of

$$\begin{array}{|c|}\hline \alpha_2 \\ \hline \alpha_1 \\ \hline \end{array} \otimes T_{\lambda \backslash X_0}$$

in the image of T_λ under Φ_1^2 is $2 \cdot w_{b_0}^2$. Then by uniqueness of the Pieri inclusion up to scalar multiple we have that $\Phi_1^2 = 2 \cdot \Phi_2$.

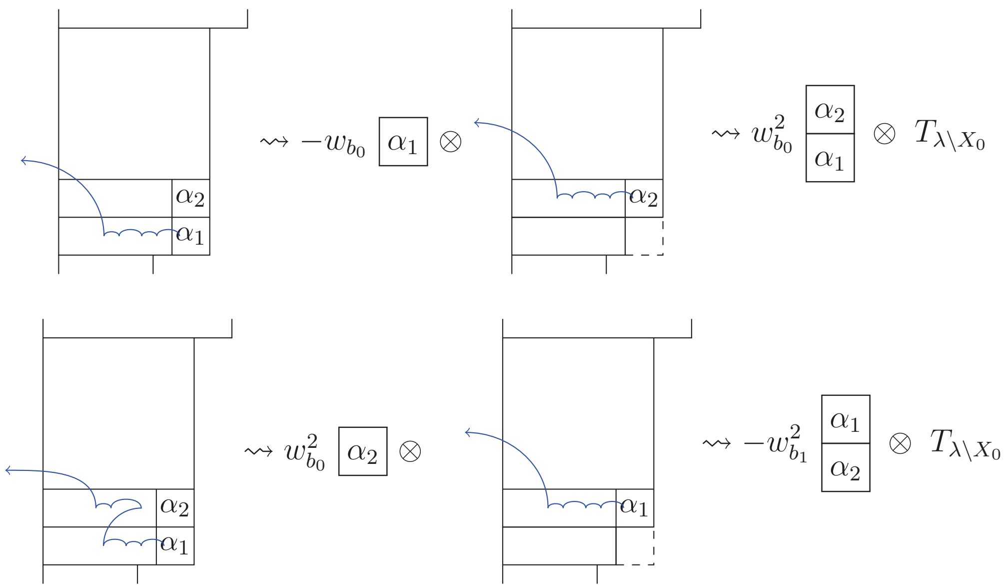


Figure 6.2. The compositions of 1-paths removing the entries α_1 and α_2 from T_λ.

We now show that $\Phi_1(\Phi_{m-1}) = m \cdot \Phi_m$, which proves the theorem. Let $X = \{x_1 = [b](1, w_b), x_2 = [b](2, w_b), \ldots, x_m = [b](m, w_b)\}$ be a removal set in λ and let $T_{\lambda \backslash X}$ be the diagram corresponding to the highest weight vector as in Section 1.2.4. In the image of T_λ under Φ_m, the coefficient of

$$\begin{array}{c}\boxed{\alpha_m} \\ \vdots \\ \boxed{\alpha_1} \end{array} \otimes T_{\lambda \backslash X}$$

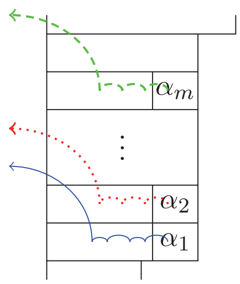
where

$$\alpha_k = (T_\lambda)_{x_k} \qquad \text{for } k = 1, \ldots, m$$

is readily seen to be $(-1)^m w_b^m$ as the only m-paths that remove the entries $\alpha_1, \ldots, \alpha_m$ are the ones with

$$R_k \subset [b](k) \qquad \text{for } k = 2, \ldots, m$$

where R_k is the orbit of x_k for $k = 1, \ldots, m$. See Figure 6.3.



$$\rightsquigarrow (-1)^m w_b^m \begin{array}{c} \boxed{\alpha_m} \\ \vdots \\ \boxed{\alpha_1} \end{array} \otimes T_{\lambda \setminus X}$$

Figure 6.3. The m-paths removing the entries $\alpha_1, \ldots, \alpha_m$ from T_λ.

We now show that in the image $\Phi_1 \left(\Phi_{m-1}\left(T_\lambda\right)\right)$ the coefficient of the term

$$\begin{array}{c} \boxed{\alpha_m} \\ \vdots \\ \boxed{\alpha_1} \end{array} \otimes T_{\lambda \setminus X}$$

is $m \cdot (-1)^m w_b^m$. As above, in the image of T_λ under Φ_{m-1} the coefficient of the term

$$\begin{array}{c} \boxed{\alpha_{m-1}} \\ \vdots \\ \boxed{\alpha_1} \end{array} \otimes T_{\lambda \setminus \{x_1, \ldots, x_{m-1}\}}$$

is $(-1)^{m-1}w_b^{m-1}$. Then in the image of this term under Φ_1 the coefficient of the term

$$\begin{array}{c}\boxed{\alpha_m}\\ \vdots\\ \boxed{\alpha_1}\end{array} \otimes T_{\lambda\setminus X}$$

is $(-1)^m w_b^m$ as the only 1-paths acting on $T_{\lambda\setminus\{x_1,\ldots,x_{m-1}\}}$ that remove α_m are the ones where the orbit $R_m \subset [b](m)$, see Figure 6.4.

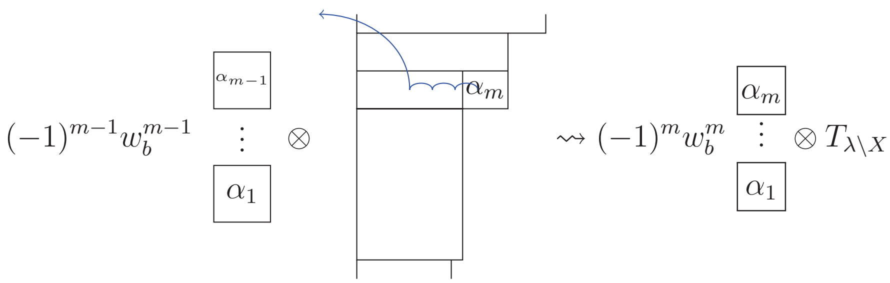
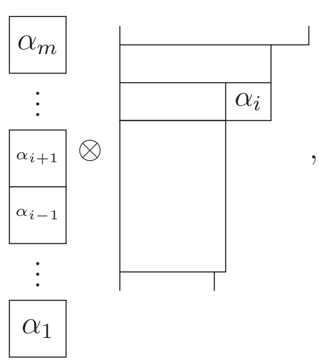


Figure 6.4. The 1-paths removing the entry α_m from $T_{\lambda\setminus\{x_1,\ldots,x_{m-1}\}}$.

Now fix an $i = 1, \ldots, m-1$ and consider the $(m-1)$-paths acting on T_λ that remove the entries α_k for $k = 1, \ldots, i-1, i+1, \ldots, m$. Such $(m-1)$-paths must have that the orbits $R_k \subset [b](k)$ for $k = 1, \ldots, i-1, i+1, \ldots, m-1$ and the orbit $R_i \subset [b](i) \cup [b](m)$. See Figure 6.5.

Then in the image of T_λ under Φ_{m-1} the coefficient of the term



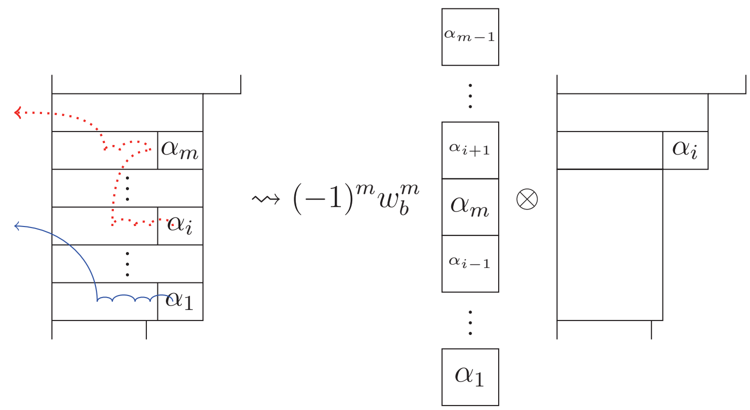


Figure 6.5. The $(m-1)$-paths removing the entries $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_m$ from T_λ for some $i = 1, \ldots, m-1$.

after ordering the term in $\bigwedge^m V$, is $(-1)^{2-1-i} w_b^m$. In the image of this term under Φ_1 the coefficient of the term

$$\begin{array}{c}\boxed{\alpha_m}\\ \vdots \\ \boxed{\alpha_1}\end{array} \otimes T_{\lambda\setminus X},$$

after again ordering the term in $\bigwedge^m V$, is $(-1)^{3m-2i} w_b^m = (-1)^m w_b^m$ as the only 1-path acting on it is the one the evacuation route $\{x_m\}$. See Figure 6.6.

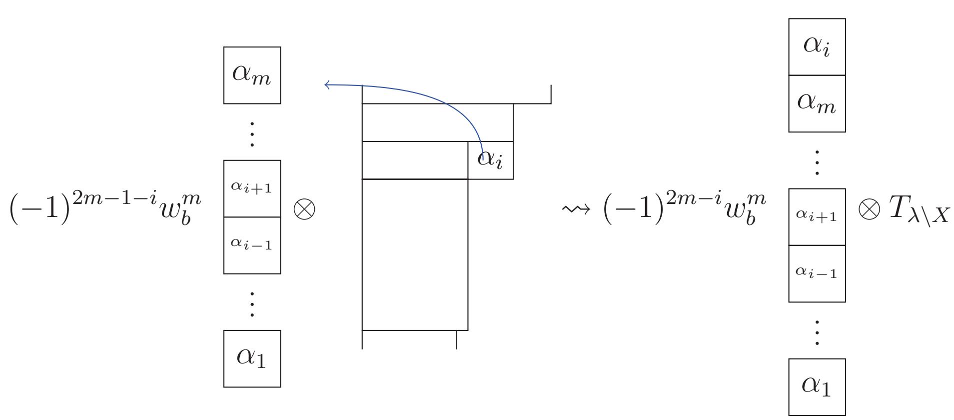


Figure 6.6. The 1-path removing the entry α_i from the box x_m.

Thus, in the image $\Phi_1 (\Phi_{m-1} (T_\lambda))$ the coefficient of the term

$$\begin{array}{c} \boxed{\alpha_m} \\ \vdots \\ \boxed{\alpha_1} \end{array} \otimes T_{\lambda \backslash X}$$

is $m \cdot (-1)^m w_b^m$. So, by the uniqueness of the Pieri inclusion up to scalar multiple, we have that $\Phi_1 (\Phi_{m-1}) = m \cdot \Phi_m$, which proves the claim. $\square$

CHAPTER SEVEN

Computational Complexity and the Image of a Highest Weight Vector

In this chapter we compute an example that illustrates the difference in the descriptions of Pieri inclusions removing one box given in Section 2.1 to that given in (Olver, 1982, §6). We then describe the image of a highest weight vector under our Pieri inclusion removing one box and show that this description is optimal and then compare the computational complexities of the descriptions of Pieri inclusions.

7.1 Computing the Image of a Highest Weight Vector Under the Different Descriptions of Pieri Inclusions Removing One Box

7.1.1

For a removal set $X = \{x_1 = [b_1](1, w_{b_1})\}$, let

$$\Phi_1 : \mathbb{S}_\lambda(V) \to V \otimes \mathbb{S}_{\lambda \setminus X}(V)$$

be the Pieri inclusion removing one box described in Section 2.1 and let

$$\widetilde{\Phi}_1 : \mathbb{S}_\lambda(V) \to V \otimes \mathbb{S}_{\lambda \setminus X}(V)$$

be the Pieri inclusion given in (Olver, 1982, §6) (see (Sam & Weyman, 2011, §1.2) and (Sam, 2009, §4) for an updated description).

7.1.2

The smallest example that illustrates the difference in the complexity of Φ_1 and $\widetilde{\Phi}_1$ is

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|}\hline \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array},$$

i.e.

$$\Phi_1, \widehat{\Phi}_1 : \mathbb{S}_{(1,1,1)}(V) \to V \otimes \mathbb{S}_{(1,1)}(V).$$

We will compute the image of the highest weight vector

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

under these maps.

Following the notation in (Sam, 2009),

$$\widetilde{\Phi}_1 = \sum_{J \in B_3} \frac{(-1)^{\#J} \tau_J}{c_J}$$

where B_3 is the set consisting of all "paths" that take the box in the bottom row up and out of the diagram $(1,1,1)$, with each path indexed only by the rows in which it acts,

$$B_3 = \{(0,3), \quad (0,1,3), \quad (0,2,3), \quad (0,1,2,3)\}.$$

Here row 0 is "removal," row 1 is the top row in the shape, etc. (Note that this convention is opposite ours, where we start counting from the bottom row of the shape.) The τ_J is the action of the path on the tableau, the $(-1)^{\#J}$ is a sign, and the c_J is a constant depending on the rows on which J acts.

Each of the following paths is pictured in Figure 7.1. The path $(0,3)$ results in

$$\frac{(-1)^{\#(0,3)}\tau_{(0,3)}}{c_{(0,3)}}\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = \tau_{0,3}\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = \begin{array}{|c|}\hline 3\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}.$$

The path $(0,1,3)$ results in

$$\frac{(-1)^{\#(0,1,3)}\tau_{(0,1,3)}}{c_{(0,1,3)}}\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = \frac{-\tau_{1,3}\circ\tau_{0,1}}{3-1}\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = -\frac{1}{2}\left(\begin{array}{|c|}\hline 1\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 3\\ \hline 2\\ \hline\end{array}\right).$$

The path $(0,2,3)$ results in

$$\frac{(-1)^{\#(0,2,3)}\tau_{(0,2,3)}}{c_{(0,2,3)}}\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = \frac{-\tau_{2,3}\circ\tau_{0,2}}{3-2}\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = -\left(\begin{array}{|c|}\hline 2\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\right).$$

The path $(0,1,2,3)$ results in

$$\frac{(-1)^{\#(0,1,2,3)}\tau_{(0,1,2,3)}}{c_{(0,1,2,3)}}\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = \frac{\tau_{2,3}\circ\tau_{1,2}\circ\tau_{0,1}}{(3-1)(3-2)}\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = \frac{1}{2}\left(\begin{array}{|c|}\hline 1\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\right).$$

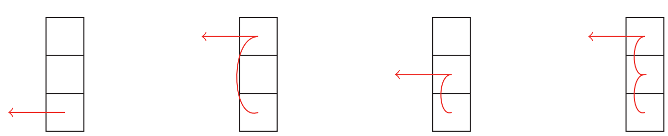


Figure 7.1. From left to right, the paths $(0,3)$, $(0,1,3)$, $(0,2,3)$, and $(0,1,2,3)$ acting on $(1,1,1)$.

So via straightening we have

$$\widetilde{\Phi}_1\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = \begin{array}{|c|}\hline 3\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} - \begin{array}{|c|}\hline 2\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array} - \frac{1}{2}\left(\begin{array}{|c|}\hline 1\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 3\\ \hline 2\\ \hline\end{array}\right) + \frac{1}{2}\left(\begin{array}{|c|}\hline 1\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\right)$$

$$= \begin{array}{|c|}\hline 3\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} - \begin{array}{|c|}\hline 2\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array} + \begin{array}{|c|}\hline 1\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}.$$

We now compute

$$\Phi_1\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = \sum_P \frac{(-1)^P}{H(P)} P\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right),$$

where the sum is over all 1-paths P on $(1,1,1)$ removing $[1](1,1)$. Note that as $(1,1,1)$ has only one block, for each such 1-path we have $H(P) = 1$. All such 1-paths are in fact the same as the paths $(0,3)$, $(0,2,3)$, and $(0,1,2,3)$ pictured in Figure 7.1. Thus without any straightening and without combining any like terms we get

$$\Phi\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\right) = \begin{array}{|c|}\hline 3\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} - \begin{array}{|c|}\hline 2\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array} + \begin{array}{|c|}\hline 1\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}.$$

In particular, note that the path $(0,1,3)$ pictured in Figure 7.1 is not a 1-path as rows $[1](1)$ and $[1](3)$ are included, but row $[1](2)$ is skipped. Further notice that the coefficients in the definitions of $\widetilde{\Phi}_1$ and Φ_1 are similar, however the c_J depend on each row on which a path acts while the $H(P)$ depend only on the blocks on which a path acts.

7.2.1

Given a removal set $X = \{x_1 = [b_1](1, w_{b_1})\} \subset \lambda$, it is clear by the construction of 1-paths that for all 1-paths on λ removing X, $(T_\lambda)_P$ is semi-standard. Define the relation $\sim$ on the set of all 1-paths on λ removing X by

$$P \sim Q \iff R^Q \text{and } R^P \text{ intersect the same set of rows.}$$

This clearly defines an equivalence relation. Let

$$[P] = \{Q : Q \sim P\}.$$

Then for all $Q \in [P]$ we have $(-1)^Q = (-1)^P$ and $H(Q) = H(P)$, and, when considering the image of a highest weight vector where each entry in a given row is the same,

$$Y_Q \otimes (T_\lambda)_Q = Y_P \otimes (T_\lambda)_P.$$

For distinct $[P]$ and $[P']$ we have (by construction) that $Y_P \otimes (T_\lambda)_P$ and $Y_{P'} \otimes (T_\lambda)_{P'}$ are linearly independent. Thus, $\Phi_1(T_\lambda)$ can be written as

$$\Phi_1(T_\lambda) = \sum_{[P_0]} \frac{(-1)^{P_0} |[P_0]|}{H(P_0)} P_0(T_\lambda)$$

where the sum is over all 1-paths P_0 on λ removing X which only hit boxes in the first column of λ. From the above, the terms in the image of $\Phi_1(T_\lambda)$ written as above are linearly independent and do not require straightening, and so this description is optimal. Two such examples are computed in Sections 1.3.2 and 7.1.2. To see the optimal description from the example in Section 1.3.2, take only the first six terms shown in Figure 1.1.

7.2.2

For a given 1-path P_0 as in 7.2.1, we now describe the corresponding term in the image of T_λ. Let $\{r_i\}_{1\leq i\leq |R^{P_0}|}$ be the rows in λ that P_0 hits, so that $\lambda_i > \lambda_{i+1}$ and $r_{|R^{P_0}|} = [b_1](1)$. Then

$$||[P_0]|| = \prod_{i=1}^{|P|} \lambda_{r_i}$$

and $(T_\lambda)_{P_0} \in \mathbb{S}_{\lambda\setminus X}(V)$ has λ_1 ones in the first row, λ_2 twos in the first row, etc. except for each row r_i, $1 \leq i \leq |R^{P_0}|$, where the last entry in row r_i of $(T_\lambda)_{P_0}$ is

$$((T_\lambda)_{P_0})_{(r_i,\lambda_{r_i})} = r_{i+1}.$$

7.2.3

We have built an algorithm computing this optimal description of the image of a highest weight vector using Macaulay2, with the output given as a hash table. With this one can quickly compute the image of the highest weight for very large examples. Figures 7.2 and 7.3 show the timed computation for the image of a highest weight vector, where the partition is given as the first input of the function oneboxremovalHW and the second input of the function is the row (from the top of the tableau) of the box to be removed.

```
i56 : time oneboxremovalHW({10,10,10,10}, 4)
     -- used 0.000466903 seconds

o56 = HashTable{{{0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1},
                 {{1}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
                 {{2}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
                 {{3}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
```

Figure 7.2. Computing the image of the highest weight vector for the inclusion $\mathbb{S}_{(10,10,10,10)}(V) \to V \otimes \mathbb{S}_{(10,10,10,9)}(V)$. Only the first four terms in the hash table are shown.

```
i57 : time oneboxremovalHW({10,10,10,10,10,10,10,7,7,7,7,7,7,3,3,3,3,3}, 18)
     -- used 0.0424879 seconds

o57 = HashTable{{{0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 2},
                {{0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 2},

                {{0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 1}, {1, 1, 1, 1, 1, 1, 1, 1, 1, 2},
```

Figure 7.3. Computing the image of the highest weight vector for the inclusion $\mathbb{S}_{(10,10,10,10,10,10,10,7,7,7,7,7,7,3,3,3,3,3)}(V) \to V \otimes \mathbb{S}_{(10,10,10,10,10,10,10,7,7,7,7,7,7,3,3,3,3,2)}(V)$. Only the first three terms in the hash table are shown.

7.3 Comparing the Computational Complexity of the Descriptions

7.3.1

We now formalize the difference in the computational complexity of the descriptions for $\widetilde{\Phi}_1$ and Φ_1.

Theorem. Fix a positive integer N and consider partitions λ that have at most N blocks. Then the algorithm to compute the image of a highest weight vector under a Pieri inclusion $\Phi_1 : \mathbb{S}_\lambda(V) \hookrightarrow V \otimes \mathbb{S}_{\lambda\backslash X}(V)$ has a worst-case time complexity of $O(l(\lambda)^N)$. On the other hand, the algorithm to compute the image of a highest weight vector under a Pieri inclusion $\widetilde{\Phi}_1 : \mathbb{S}_\lambda(V) \hookrightarrow V \otimes \mathbb{S}_{\lambda\backslash X}(V)$ has a worst-case time complexity of $\Omega(2^{l(\lambda)})$.

Proof. Let $\lambda = (w_1^{h_1}, \ldots, w_N^{h_N})$. We first consider the time complexity of the algorithm as given by Olver's construction. As in Section 7.2, when considering the image of a highest weight vector we only need to select paths on λ removing X that act on the first column of λ. From the description of the map $\widetilde{\Phi}_1$ removing X, the number of such paths in the computation of $\widetilde{\Phi}_1$ is equal to the number of choices of rows in λ above row $[b_1](1)$. Thus the complexity of the map $\widetilde{\Phi}_1$ acting on a highest weight vector is

$$2^{h_{b_1}-1} \cdot \prod_{i=b_1+1}^{N} 2^{h_i} \leq \tfrac{1}{2} \cdot \prod_{i=1}^{N} 2^{h_i} = \tfrac{1}{2} \cdot 2^{\sum_{i=1}^{N} h_i} = \tfrac{1}{2} \cdot 2^{\,l(\lambda)}.$$

In the worst-case when $b_1 = 1$, the inequality is in fact an equality. Furthermore, the paths that act on the first column of λ using Olver's algorithm can result in tableaux which are not semi-standard, and so must be straightened. Hence the worst-case complexity of Olver's algorithm is $\Omega(2^{\,l(\lambda)})$.

The map Φ_1 removing X restricts the choices of paths to those that act on a set of rows which describes an evacuation route, and hence the number of 1-paths acting on the first column of λ in the computation of Φ_1 is equal to the number of choices of rows in λ above row $[b_1](1)$ made without skipping rows within blocks. It is also clear from the definition of 1-paths that the image of a highest weight vector under a 1-path is semi-standard. Thus the complexity of the map Φ_1 acting on a highest weight vector is

$$h_{b_1} \cdot \prod_{i=b_1+1}^{N} (h_i + 1) < \prod_{i=1}^{N} (h_i + 1) \leq (l(\lambda) + 1)^N = \Theta(l(\lambda)^N).$$

$\square$

Remark. Similar to the Theorem 7.3.1, by restricting the maximum possible width of a block in λ we get that Φ_1 is an exponential speed up of $\widetilde{\Phi}_1$ on the image of basis vectors (semi-standard tableaux) in $\mathbb{S}_\lambda(V)$.

7.3.2

This exponential to polynomial speed up can be seen in the computation time for computing Pieri maps in Macaulay2 by replacing the description of $\widetilde{\Phi}_1$ within Sam's PieriMaps package (Sam, 2009) with the description of Φ_1. This comes down to restricting all possible paths to 1-paths and redifining the coefficient, which we have done via editing the pieriHelper function.

The computation time difference can be seen for even small examples. For example, computing the map

$$\mathbb{S}_{(6,6,6)} \to \mathbb{S}_{(1)} \otimes \mathbb{S}_{(6,6,5)}$$

was an order of magnitude faster, see Figure 7.4.

```
i3 : time pieri({6,6,6}, {3}, CC[a,b,c])
     -- used 0.723831 seconds

o3 = | 6c   |
     | -36b |
     | 216a |
```

(a) Using the algorithm for $\widetilde{\Phi}_1$.

```
i31 : time pieri({6,6,6}, {3}, CC[a,b,c])
      -- used 0.0219184 seconds

o31 = | 6c   |
      | -36b |
      | 216a |
```

(b) Using the algorithm for Φ_1.

Figure 7.4. Computing the inclusion $\mathbb{S}_{(6,6,6)}(V) \to V \otimes \mathbb{S}_{(6,6,5)}(V)$.

Computing the map

$$\mathbb{S}_{(7,7,7)}(V) \to V \otimes \mathbb{S}_{(7,7,6)(V)}$$

was four orders of magnitude faster, see Figure 7.5.

In Figure 7.6 we show the timed computations for computing the map

$$\mathbb{S}_{(8,8,8)}(V) \to V \otimes \mathbb{S}_{(8,8,7)}(V).$$

```
i4 : time pieri({7,7,7}, {3}, CC[a,b,c])
     -- used 88.0308 seconds

o4 = | 7c   |
     | -49b |
     | 343a |
```

(a) Using the algorithm for $\widetilde{\Phi}_1$.

```
i32 : time pieri({7,7,7}, {3}, CC[a,b,c])
     -- used 0.0350425 seconds

o32 = | 7c   |
      | -49b |
      | 343a |
```

(b) Using the algorithm for Φ_1.

Figure 7.5. Computing the inclusion $\mathbb{S}_{(7,7,7)}(V) \to V \otimes \mathbb{S}_{(7,7,6)(V)}$.

Using the algorithm for $\widetilde{\Phi}_1$ (as built in to PieriMaps), the process was interrupted after an hour with no output. Using the algorithm for Φ_1 computing this map takes only 0.07 seconds.

```
i3 : time pieri({8,8,8}, {3}, CC[a,b,c])
  ^C ^Cstdio:3:6:(3): error: interrupted
     -- used 3538.41 seconds
```

(a) Using the algorithm for $\widetilde{\Phi}_1$.

```
i33 : time pieri({8,8,8}, {3}, CC[a,b,c])
     -- used 0.0688948 seconds

o33 = | 8c   |
      | -64b |
      | 512a |
```

(b) Using the algorithm for Φ_1.

Figure 7.6. Computing the inclusion $\mathbb{S}_{(8,8,8)}(V) \to \mathbb{S}_{(1)}(V) \otimes \mathbb{S}_{(8,8,7)}(V)$.

We can also see this exponential speed up for examples with more than one block. In Figures 7.7 and 7.7 we show the computation times for the Pieri inclusion

$$\mathbb{S}_{(3,1,1,1,1,1,1,1,1,1)}(V) \to V \otimes \mathbb{S}_{(3,1,1,1,1,1,1,1,1)}(V)$$

using the algorithms for $\widetilde{\Phi}_1$ and Φ_1, respectively. Using the algorithm for $\widetilde{\Phi}_1$ this computation takes over eleven seconds, while using the algorithm for Φ_1 this computation takes less than two seconds.

```
i2 : time pieri({3,1,1,1,1,1,1,1,1,1}, {10}, CC[a,b,c,d,e,f,g,h,i,j]);
     -- used 11.6579 seconds

                                           540                                         55
o2 : Matrix (CC  [a, b, c, d, e, f, g, h, i, j])    <--- (CC  [a, b, c, d, e, f, g, h, i, j])
               53                                          53
```

Figure 7.7. Computing the inclusion $\mathbb{S}_{(3,1,1,1,1,1,1,1,1,1)}(V) \to V \otimes \mathbb{S}_{(3,1,1,1,1,1,1,1,1)}(V)$ using the algorithm for $\widetilde{\Phi}_1$.

```
i31 : time pieri({3,1,1,1,1,1,1,1,1,1}, {10}, CC[a,b,c,d,e,f,g,h,i,j]);
      -- used 1.3184 seconds

                                            540                                        55
o31 : Matrix (CC  [a, b, c, d, e, f, g, h, i, j])   <--- (CC  [a, b, c, d, e, f, g, h, i, j])
                53                                         53
```

Figure 7.8. Computing the inclusion $\mathbb{S}_{(3,1,1,1,1,1,1,1,1,1)}(V) \to V \otimes \mathbb{S}_{(3,1,1,1,1,1,1,1,1)}(V)$ using the algorithm for Φ_1.

BIBLIOGRAPHY

Eisenbud, D., Fløystad, G., & Weyman, J. (2011). The Existence of Equivariant Pure Free Resolutions [https://arxiv.org/abs/0709.1529v5]. *Annales de l'Institut Fourier*, *61*(3), 905–926.

Fulton, W. (1997). *Young Tableaux: With applications to Representation Theory and Geometry* (Vol. 35). Cambridge University Press.

Fulton, W., & Harris, J. (2013). *Representation Theory: A First Course* (Vol. 129). Springer Science & Business Media.

Hunziker, M., Miller, J. A., & Sepanski, M. (n.d.). Minimal Graded Free Resolutions of Modules of Covariants for Classical Groups.

Olver, P. J. (1982). Differential Hyperforms I [available at http://www.math.umn.edu/~olver/]. *University of Minnesota Mathematics Report*, 82–101.

Pragacz, P., & Weyman, J. (1985). Complexes Associated with Trace and Evaluation. Another Approach to Lascoux's Resolution. *Advances in Mathematics*, *57*(2), 163–207.

Sam, S. V. (2009). Computing Inclusions of Schur Modules. *Journal of Software for Algebra and Geometry*, *1*(1), 5–10.

Sam, S. V., & Weyman, J. (2011). Pieri Resolutions for Classical Groups [https://arxiv.org/abs/0907.4505v5]. *Journal of Algebra*, *329*(1), 222–259.

Sternberg, S. (1995). *Group Theory and Physics*. Cambridge University Press.