ABSTRACT

Generalized Inverse Limits and the Intermediate Value Property

Tavish J. Dunn, Ph.D.

Mentor: David J. Ryden, Ph.D.

We introduce and discuss various notions of the intermediate value property applicable to upper-semicontinuous set-valued functions $f : [0, 1] \to 2^{[0,1]}$. In the first part, we present sufficient conditions such that an inverse limit of a sequence of bonding functions of this type is a continuum. In the second part, we examine the relationship between the dynamics of an upper-semicontinuous function with the intermediate value property and the topological structure of the corresponding inverse limit. In particular, we present conditions under which the existence of a cycle of period not a power of 2 implies indecomposability in the inverse limit and vice-versa. Lastly, we show that these conditions are sharp by constructing a family of upper-semicontinuous functions with the intermediate value property and cycles of all periods, yet admits a hereditarily decomposable inverse limit.

Generalized Inverse Limits and the Intermediate Value Property

by

Tavish J. Dunn, B.S.

A Dissertation

Approved by the Department of Mathematics

____

Dorina Mitrea, Ph.D., Chairperson

Submitted to the Graduate Faculty of
Baylor University in Partial Fulfillment of the
Requirements for the Degree
of

Doctor of Philosophy

Approved by the Dissertation Committee

____

David J. Ryden, Ph.D., Chairperson

____

Jonathan Meddaugh, Ph.D.

____

Brian Raines, Ph.D.

____

Mark Sepanski, Ph.D.

____

David Kahle, Ph.D.

Accepted by the Graduate School
August 2021

____

J. Larry Lyon, Ph.D., Dean

Page bearing signatures is kept on file in the Graduate School.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I would like to thank the faculty and staff of Baylor University and the Department of Mathematics for their support throughout my time in graduate school. I am grateful to Dr. David Ryden for the numerous hours he spent encouraging me and helping me in research, writing, and preparing for conferences. Thank you for helping me be clear in what I am trying to say and helping me learn how to investigate the literature. I am also grateful to Dr. Jonathan Meddaugh for his help proofreading and challenging my proofs to be rigorous, helping me develop as a researcher.

I also want to thank my family for their support and encouragement. Moving across the country is a large endeavor, and the help family members and the wonderful people I have come to know in Texas have helped me reach this point and become the person I am today.

To the reader.

CHAPTER ONE

Introduction

Inverse limits play an important role in continuum theory and dynamics. They were originally developed in conjunction with cohomology theory in the 1930s and 40s. At this time it was found that the inverse limit of continua is itself a continuum, making inverse limits relevant to continuum theory. A compilation of the fundamental properties of inverse limits along with new results of particular interest to continuum theorists was written by Capel in 1954 [14]. In the late 1950s and early 60s, the value of inverse limits for the study of continuum theory became apparent with their use in constructing exotic spaces from relatively simple spaces [1, 27, 45]. They are also useful for representing and studying properties of known spaces.

Inverse limits caught the interest of dynamicists in 1967 when R.F. Williams demonstrated a relationship between attractors and the shift map on an inverse limit [50]. He showed that for a given inverse limit X with shift map h, there is a diffeomorphism $f : S^4 \to S^4$ and indecomposable subset Ω_0 of its non-wandering set such that (X, h) is conjugate to $(\Omega_0, f|_{\Omega_0})$. Conversely, he also showed that, given a diffeomorphism of manifolds $f : M \to M$ and a one-deminsional set Ω_0 that is an irreducible subset of the non-wandering set with hyperbolic and associated stable structure, then there is an inverse limit X such that (Ω_0, f) is conjugate to (X, h).

In the 1980s, Barge and Martin showed that every inverse limit space of a mapping $f : [0, 1] \to [0, 1]$ can be realized as a global attractor A for a homeomorphism g of the plane such that $\left(\varprojlim\{[0,1],f\},h\right)$ is conjugate to $(A,g|_A)$ [6]. This provided an impetus for the study of topological dynamics of inverse limits of the unit interval. Accordingly, they also showed the existence of a periodic point of a continuous function $f:[0,1]\to[0,1]$ implies the existence of an indecomposable subcontinuum of its inverse limit, and proved a pseudo-converse [9].

In 2004, Mahavier introduced a notion of inverse limits involving set-valued bonding functions on $[0,1]$ instead of continuous bonding functions [37], and worked with Ingram in 2006 to extend this concept to inverse limits of upper-semicontinuous set-valued functions on compact Hausdorff spaces [31]. In these papers, Ingram and Mahavier gave examples of these generalized inverse limits for which many of the well-known properties of inverse limits of continuous function did not hold. This new context, in which little could be taken for granted, has seen a large body of research in recent years over a wide variety of topics, including connectedness [18, 24], indecomposability [34, 47], modeling spaces [2], the full-projection property [4], and specification [21].

The focus of this dissertation is on inverse limits of set-valued functions satisfying either of two notions of the intermediate value property for set-valued functions. We show these functions may be used to generate connected inverse limits, and we explore the relationship between the dynamics of a set-valued function f on $[0,1]$ and the topological structure of the inverse limit generated by f.

In Chapter Two, we give give preliminary definitions with some history of the field interspersed and a preview of the main results of this dissertation. In Chapter Three, we define the weak intermediate value property and intermediate value property for set-valued functions. We then prove theorems about the structure of functions with the intermediate value property and give sufficient conditions such that a sequence of set-valued functions with the weak intermediate value property admits a continuum as its inverse limit, leading to a generalization of work by Nall [10].

In Chapter Four, we look at inverse limits of a function with the intermediate value property and explore the relationship between the existence of periodic cycles of the bonding function whose period is not a power of 2 and indecomposability in the corresponding inverse limit. This generalizes work from Barge and Martin [5]. Key in these proofs is the full-projection property, which we prove holds for inverse limits of this type.

This leads immediately into Chapter Five, in which we show one of the main results of Chapter Four does not hold if we drop the assumption that the bonding function is almost nonfissile. We construct a family of set-valued functions with the intermediate value property that has periodic cycles of all periods, yet admits a hereditarily decomposable inverse limit. Lastly, in Chapter Six we explore some avenues for future work.

CHAPTER TWO

Preliminaries

We give preliminary definitions and related results from continuum theory, classical inverse limits, classical dynamical systems, set-valued inverse limits, and set-valued dynamical systems in Sections 2.1 through 2.5 respectively. This context allows us to provide in Sections 2.4 and 2.5 a preview of the main results of the dissertation.

2.1 Continuum Theory

We begin with some preliminary definitions from continuum theory. A more in-depth introduction to the subject can be found in [40]. For an introduction to topology, see [48].

Definition 2.1.1 . Let X and Y be topological spaces. A *mapping, or map, from X to Y* is a continuous function $f : X \to Y$.

Definition 2.1.2 . A *continuum* is a nonempty, compact, connected metric space. For a continuum X, a set $K \subseteq X$ is a *subcontinuum* if K is a continuum. We denote the collection of nonempty compact subsets of X by 2^X and denote the collection of nonempty subcontinua of X by $C(X)$.

It is well-known that if X is a compact metric space, both 2^X and C^X are compact.

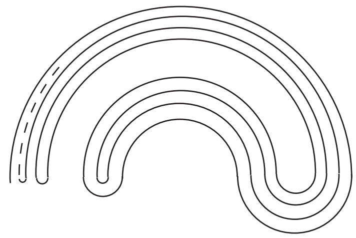

Figure 2.1: The BJK Continuum

Definition 2.1.3 . A nondegenerate continuum X is *decomposable* if it is the union of two proper subcontinua and *indecomposable* if it is not decomposable. X is *hereditarily decomposable* if each of its nondegenerate subcontinua is decomposable.

It is not intuitive that indecomposable continua exist. The topology of indecomposable continua is exotic; for example, an indecomopsable continuum is not locally connected at any point. A famous example of an indecomposable planar continuum is the Brouwer-Janiszewski-Knaster (BJK) continuum, or buckethandle continuum (See Figure 2.1).

Definition 2.1.4 . A continuum K is *irreducible* about a nonempty closed set $A \subseteq K$ if no proper subcontinuum of K contains A.

A famous characterization of indecomposability given by Stefan Mazurkiewicz is that a continuum X is indecomposable if and only if X contains three points such that X is irreducible about any two of them [39].

Definition 2.1.5 . A continuum X is *unicoherent* if, for every pair of subcontinua $A, B \subseteq X$ with $A \cup B = X$, $A \cap B$ is a continuum. X is *hereditarily unicoherent* if every subcontinuum is unicoherent, or equivalently, the intersection of any non-disjoint pair of subcontinua of X is itself a subcontinuum of X.

2.2 Classical Inverse Limits

For a general introduction to inverse limits, see [32].

The *graph* of a function $f : [a, b] \to 2^{[c,d]}$ is the set $G(f) = \{(x, y) \in [a, b] \times [c, d] : y \in f(x)\}$.

Definition 2.2.1 . Let $X_0, X_1, X_2, \dots$ be a sequence of continua and for all $i \in \mathbb{N}$ let $f_i : X_{i-1} \to X_i$ be continuous. The pair $\{X_i, f_i\}$ called an *inverse sequence*. The *inverse limit* of the inverse sequence is the subspace of $\prod_{i \in \omega} X_i$ given by

$$\varprojlim\{X_i, f_i\} = \left\{x = (x_0, x_1, \dots) \in \textstyle\prod_{i \in \omega} X_i : x_{i-1} = f_i(x_i) \ \forall i \geq 1\right\}.$$

The spaces X_i are called the *factor spaces* of the inverse limit, and the functions f_i are called the *bonding maps* For each $n \in \omega$, the map $\pi_n : \varprojlim\{X_i, f_i\} \to X_n$ defined by $\pi_n(x) = x_n$ is the *projection map* onto the nth factor space. For two consecutive integers n and $n + 1$, $\pi_{n+1,n} : \varprojlim\{X_i, f_i\} \to X_{n+1} \times X_n$ defined by $\pi_{n+1,n}(x) = (x_{n+1}, x_n)$ is the *projection map* into $X_{n+1} \times X_n$.

It is well-known that if $\{X_i, f_i\}$ is an inverse where each factor space is a continuum, then $X = \varprojlim\{X_i, f_i\}$ is a continuum.

Definition 2.2.2 . Let $X_0, X_1, X_2, \dots$ be a sequence of continua and for all $i \in \mathbb{N}$ let $f_i : X_i \to X_{i-1}$ be continuous. $X = \varprojlim\{X_i, f_i\}$ has the *closed-set property* if for every closed subset C of X, $C = \varprojlim\{\pi_i[C], f_i|_{\pi_i[C]}\}$.

Capel showed that inverse limits of continua with continuous bonding maps have the closed-set property [14]. Thus we are able to regard closed subsets of X as inverse limits in their own right. But under the generalized notion of an inverse limit that we will introduce in Section 2.4, this does not always hold.

Definition 2.2.3 . Let X be a continuum and $f : X \to X$ be continuous. The *shift map*, or *forgetful shift*, on $\varprojlim\{X, f\}$ is the map $h : \varprojlim\{X, f\} \to \varprojlim\{X, f\}$ defined by $h(x) = (x_1, x_2, \dots)$, where $x = (x_0, x_1, \dots)$.

The shift map is a homeomorphism of an inverse limit space with itself, with $h^{-1}(x) = (f(x_0, x_0, x_1, \dots))$, although it is not a homeomorphism in the generalized setting. Sometimes the literature refers to h^{-1} as the shift map instead.

Definition 2.2.4 . Let $\{X_i, f_i\}$ be an inverse sequence and $X = \varprojlim\{X_i, f_i\}$. We say X has the *full-projection property* if and only if $K = X$ for every subcontinuum K of X such that $\pi_i[K] = X_i$ for infinitely many $i \in \omega$.

The full-projection property holds for inverse limits of continua with continuous bonding maps. The full-projection property and closed-set property both hold for calssical inverse limits, but they become distinct concepts in the set-valued setting.

Definition 2.2.5 . Let $\{X_i\}_{i\in\omega}$ be a sequence of compact Hausdorff spaces and $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of mappings $f_i : X_i \to X_{i-1}$. We say $\varprojlim\{X_i, f_i\}$ has the *subsequence property* if for every increasing sequence $\{n_i\}_{i\in\omega}$ in ω, $\varprojlim\{X_{n_i}, g_i\}$ is homeomorphic to $\varprojlim\{X_i, f_i\}$, where $g_i = f_{n_i}^{n_{i+1}}$.

The subsequence property holds for all inverse limits of continuous functions and shows that the representation of a given space as an inverse limit is not unique.

Like the closed-set and full-projection properties, the subsequence property does not always hold for generalized inverse limits.

Definition 2.2.6 . A *tree* is a uniquely arcwise connected union of finitely many arcs. A continuum is *tree-like* if it is homeomorphic to an inverse limit on trees.

It is well-known that tree-like continua are hereditarily unicoherent.

2.3 Classical Dynamical Systems

Definition 2.3.1 . A *dynamical system* is a pair (X, f) consisting of a metric space X and a continuous function $f : X \to X$.

Definition 2.3.2 . Let $f : X \to X$ be a continuous function. The *orbit* of a point $x \in X$ is the sequence $\{f^i(x)\}_{i\in\omega}$, where $f^0(x) = x$. If there is some $n \in \mathbb{N}$ such that $f^n(x) = x$, then x is *periodic*. The *period* of x is the smallest such natural number n. A finite sequence $(x_0, x_1, \ldots, x_{n-1})$ is called a *cycle* if $(x_0, x_1, \ldots, x_{n-1}, x_0, x_1, \ldots)$ is a periodic orbit.

R.F. Williams's result that attractors can be realized as homeomorphic to an inverse limit where the dynamical system is conjugate to the shift map [50] provided a major impetus to the use of inverse limits to study dynamical systems.

The dynamical system $(\varprojlim\{X, f\}, h)$, where h is the shift map, provides a useful tool for studying the dynamics of (X, f) and vice-versa. For example, a periodic point $(x_0, x_1, x_2, \ldots) \in \varprojlim\{X, f\}$ of period n indicates $(x_n, x_{n-1}, \ldots, x_2, x_1)$ is a periodic cycle of (X, f). Conversely, if x_0 is a periodic point of (X, f) with period n, then $(x_0, f^{n-1}(x_0), f^{n-2}(x_0), \ldots, f(x_0), x_0, f^{n-1}(x_0, \ldots))$ is a periodic point of $(\varprojlim\{X, f\}, h)$.

In the 1980s, Marcy Barge and Joe Martin undertook a study to discern for a map $f : [0,1] \to [0,1]$ the relationship of its dynamics to the dynamics of the shift map on its inverse limit [6, 7] and to the topology of its inverse limit [5, 8]. The following two results are of particular interest for this dissertation and form pseudo-converses to one another:

Theorem 2.3.3 . (Barge and Martin [5, Theorem 1]) Suppose that k and n are integers, $k \geq 0$, $n \geq 1$, and that $f : [0,1] \to [0,1]$ has a point of period $2^k(2n+1)$, i.e., of period not a power of 2. Then $\varprojlim\{[0,1], f\}$ has an indecomposable subcontinuum that is invariant under $h^{2^{k+1}}$ where h is the shift homeomorphism.

Theorem 2.3.4 . (Barge and Martin, [5, Theorem 7]) If $f : [0,1] \to [0,1]$ is organic, and $\varprojlim\{[0,1], f\}$ is indecomposable, then f has a periodic point whose period is not a power of 2.

Since Theorem 2.3.3 only specifies the inverse limit contains an (possibly proper) indecomposable subcontinuum, Theorems 2.3.3 and 2.3.4 are not true converses, though both connect the existence of a period not a power of 2 for the bonding map to indecomposability in the inverse limit. The subsequence property is critical to the proof of Theorem 2.3.3.

Also critical to the proof of Theorem 2.3.3 is the relationship between periodic points. A.N. Sharkovskii introduced the following ordering of the positive integers, now known as the Sharkovskii ordering [46].

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 3 \cdot 2 \prec 5 \cdot 2 \prec 7 \cdot 2 \prec 9 \cdot 2 \prec \ldots$$

$$\prec 3 \cdot 2^2 \prec 5 \cdot 2^2 \prec 7 \cdot 2^2 \prec 9 \cdot 2^2 \prec \cdots \prec 3 \cdot 2^k \prec 5 \cdot 2^k \prec 7 \cdot 2^k \prec 9 \cdot 2^k \prec \ldots$$

$$\cdots \prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1$$

Theorem 2.3.5 . (Sharkovskii, [46]) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a mapping and f has a periodic point of period k. If $k \prec j$, then f has a point of period k.

We may apply this theorem to a map $f : [0, 1] \to [0, 1]$ by continuously extending f to be constant on $(-\infty, 0]$ and $[1, \infty)$. Note that if f has a point of period not a power of 2, as is assumed in Theorem 2.3.3, then f has periodic points of infinitely many periods. In particular, f has a periodic point whose period is a multiple of 3. Thus Theorem 2.3.3 demonstrates a connection between complicated dynamics and exotic topology.

2.4 Set-Valued Inverse Limits

In 2004 and 2006, Mahavier and Ingram introduced inverse limits with upper-semicontinuous set-valued functions [31, 37]. Since then, extensive work has been done to generalize and extend results from the classical setting. Many fundamental results of classical inverse limits do not carry into the set-valued setting.

Definition 2.4.1 . A function $f : X \to 2^Y$ is *upper-semicontinuous at* x if for every open set U containing $f(x)$ there is an open set V containing x such that $f(V) \subseteq U$. The function f is *upper-semicontinuous* if it is upper-semicontinuous at every point in its domain.

Ingram and Mahavier proved that f is upper-semicontinuous if and only if $G(f)$ is closed [31].

Definition 2.4.2 . Let $X_0, X_1, X_2, \dots$ be a sequence of continua and for all $i \in \mathbb{N}$ let $f_i : X_i \to 2^{X_{i-1}}$ be upper-semicontinuous. The pair $\{X_i, f_i\}$ is called an *inverse sequence*, and the *inverse limit* of $\{X_i, f_i\}$, sometimes called the *generalized inverse limit* of $\{X_i, f_i\}$, is the subspace of $\prod_{i\in\omega} X_i$ given by

$$\varprojlim\{X_i, f_i\} = \left\{x = (x_0, x_1, \dots) \in \textstyle\prod_{i\in\omega} X_i : x_{i-1} \in f_i(x_i)\ \forall i \geq 1\right\}.$$

The spaces X_i are called the *factor spaces* of the inverse limit, and the functions f_i are the *bonding functions.* For $n > i$, $f_i^n : X_n \to X_i$ denotes the composition $f_{i+1} \circ f_{i+2} \circ \ldots \circ f_n$. For each $n \in \omega$, the map $\pi_n : \varprojlim\{X_i, f_i\} \to X_n$ defined by $\pi_n(x) = x_n$ is the *projection map* onto the nth factor space. For two consecutive integers n and $n+1$, $\pi_{n+1,n} : \varprojlim\{X_i, f_i\} \to X_{n+1} \times X_n$ defined by $\pi_{n+1,n}(x) = (x_{n+1}, x_n)$ is the *projection map* into $X_{n+1} \times X_n$.

This new notion opened up the possibility for a wider variety of spaces to be modeled as inverse limits on a given sequence of factor spaces. Shortly after its introduction, a large body of research developed around inverse limits with set-valued functions, dealing with topics that include the kinds of spaces that can be realized as inverse limits [2, 11, 12, 28] and conditions under which results from classical inverse limits are extended [3, 16, 17, 24, 25, 30, 34, 47].

2.4.1 Connectedness

One property of classical inverse limits that does not hold in general for inverse limits of upper-semicontinuous functions is that, in the classical setting, the inverse limit of continua is a continuum. Ingram and Mahavier found in [31] that even inverse limits of upper-semicontinuous functions on $[0,1]$ are compact and nonempty but not necessarily connected, and gave examples of such functions. Ingram presents the following problems in [29]:

Problem. **6.1** Characterize connectedness of inverse limits on continua with upper-semicontinuous bonding functions.

Problem. **6.2** Characterize connectedness of inverse limits on continua with upper-semicontinuous bonding functions on $[0,1]$.

Problem. **6.3** Find sufficient conditions that an inverse limit on continua with upper-semicontinuous bonding functions be a continuum.

Problem. **6.4** Solve Problem 6.3 on $[0,1]$.

Theorem 2.4.3 . (Ingram and Mahavier, [31]) $\varprojlim\{X_n, f_n\}$ is connected if and only if the sets

$$\Gamma_n(f) = \{(x_0, x_1, x_2, \dots x_n) \in [0,1]^{n+1} : x_{i-1} \in f_i(x_i)\ \forall i \leq n\}$$

are connected for each n.

These sets $\Gamma_n(f)$, subsequently dubbed Mahavier products by Charatonik and Roe, have been the subject of research in their own right [18, 20, 26].

While this criterion characterizes connectedness, it is difficult to verify due to the need to check infinitely many conditions. Ingram and Mahavier also presented some sufficient conditions that are more readily verfied.

Theorem 2.4.4 . (Ingram and Mahavier, [31]) If for each $i \in \omega$, X_i is a continuum and $f_i : X_{i+1} \to C(X_i)$ is upper-semicontinuous, then $\varprojlim\{X_i, f_i\}$ is a continuum.

Nall also presented some conditions sufficient to ensure the connectedness of the inverse limit of a single set-valued function on $[0, 1]$.

Theorem 2.4.5 . (Nall, [10]) Let X be a continuum and $f : X \to 2^X$ be a surjective upper-semicontinuous function with connected graph $G(f)$ such that $G(f) = \bigcup_\alpha G(f_\alpha)$, where each $f_\alpha : X \to C(X)$ is upper-semicontinuous. Then $\varprojlim\{X, f\}$ is a continuum.

While it is true that the inverse limit need not be connected, Banič and Kennedy showed there is a component of inverse limits with set-valued functions on $[0, 1]$ that is large in the sense that its projections encapsulate the entire graphs of the bonding functions.

Theorem 2.4.6 . (Banič and Kennedy, [3]) For all $n \in \omega$, let $f_n : [0, 1] \to 2^{[0,1]}$ be an upper-semicontinuous function such that the graph $G(f_n)$ is connected and surjective. Then there is a continuum $C \subseteq \varprojlim\{[0, 1], f_n\}$ such that $\pi_{n+1,n}[C] = G(f_n)$ for any n.

Greenwood and Kennedy developed a characterization for the connectedness of inverse limits of upper-semicontinuous functions on $[0, 1]$ that are surjective and have connected graphs in [24] and refined the result in [25]. Their result, stated in Theorem 3.3.3, requires preliminary technical definitions, which will be introduced in Chapter Three. We use this to show the following in Theorem 3.3.5:

Theorem. For each $n \in \mathbb{N}$, let $f_n : [0, 1] \to 2^{[0,1]}$ be a surjective, upper-semicontinuous function with the weak intermediate value property and a connected graph $G(f_n)$. Then $\varprojlim\{[0, 1], f_n\}$ is connected.

We show in Theorem 3.3.8 that in the special case of a single bonding function, the criteria in the previous theorem is equivalent to that of Nall's theorem (2.4.5). Like Nall's theorem, this provides sufficient conditions that are verified through checking each bonding function individually, providing an easily verifiable condition to check for connectedness in the inverse limit. Theorem 3.3.5 is the most general result that provides sufficient conditions for the connectedness of a generalized inverse limit using criteria restricted to individual bonding maps.

2.4.2 Indecomosability and Full-Projection Property

With the advent of generalized inverse limits, another area of particular interest has been to identify and analyze circumstances that give rise to indecomposable subcontinua. Ingram [29], James P. Kelly [33], Jonathan Meddaugh [34], and Scott Varagona [47] have all written on the subject, using the full-projection property as a crucial tool to demonstrate indecomposability.

Example 2.4.7 . (Varagona, [47]) Let $f : [0,1] \to 2^{[0,1]}$ be defined by the graph made by drawing straight line segments from $(0,0)$ to $\left(\frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{1}{3}, \frac{1}{3}\right)$ to $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\left(\frac{1}{3}, \frac{2}{3}\right)$ to $\left(\frac{1}{2}, 1\right)$, along with the reflection of this figure about the line $x = \frac{1}{2}$ (pictured on the left in Figure 2.2). Then $\varprojlim\{[0,1], f\}$ is indecomposable.

Example 2.4.8 . (Kelly, [33]) Let $f : [0,1] \to 2^{[0,1]}$ be the graph found on the right of Figure 2.2. Then $\varprojlim\{[0,1], f\}$ is an indecomposable continuum.

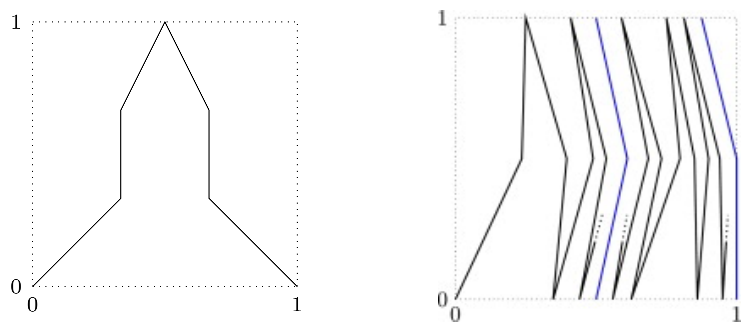

Figure 2.2: Set-valued functions that generate indecomposable inverse limits

While there is no proof that the full-projection property is a necessary condition for indecomposability, many if not all of the results related to the indecomposability of inverse limits with set-valued functions make use of it. Thus an important question related to the study of indecomposability is if there is a theorem that ensures an indecomposable inverse limit has the full-projection property or if there is an inverse limit that is an indecomposable continuum but does not have the full-projection property.

Since the bonding function is set-valued, the shift map on the generalized inverse limit $\varprojlim\{X, f\}$ is no longer injective, though it is still a continuous surjection. The corresopndence between periodic points of the inverse limit and periodic cycles of the dynamical system (X, f) is preserved, however, allowing us to model a more robust variety of dynamical systems as inverse limits.

In general, the subsequence property does not hold for inverse limits of upper-semicontinuous functions. Thus the proof for Theorem 2.3.3 cannot be lifted to the set-valued setting. However, we form an alternate proof making use of the following from Theorem 4.1.12:

Theorem. Suppose $\{[0,1], f_n\}$ is an inverse sequence where, for each $n \in \mathbb{N}$, $f_n : [0,1] \to 2^{[0,1]}$ is a surjective, light, almost nonfissile, upper-semicontinuous map with the intermediate value property. Then $\varprojlim\{[0,1], f_n\}$ has the full-projection property.

The added assumptions that f is light and almost nonfissile are necessary for the proof and will be introduced in Chapter Four.

2.5 Set-Valued Dynamical Systems

Set-valued dynamical systems have been used to model switched circuit networks [13], economics [19], and game theory [22, 38]. Recent years have seen the field grow as a robust area of study in its own right, with results involving the specification property [21], chaos [23], entropy [15, 35], and shadowing [42, 43]. In this dissertation, we are interested in the connection between complicated dynamics and exotic topology.

As with inverse limits, many fundamental properties of dynamics fail when moved to the generalized setting.

Definition 2.5.1 . Let X and Y be metric spaces and $f : X \to 2^Y$. An *orbit* of f is a sequence $\{x_i\}_{i\in\omega}$ where $x_{i+1} \in f(x_i)$ for all i. If $x \in X$, an orbit of x is an orbit of f where $x_0 = x$. The orbit is said to be *periodic* if there is some $n \in \mathbb{N}$ such that $x_{n+i} = x_i$ for all i. The smallest such n is called the *period* of the orbit. A finite sequence $(x_0, x_1, \ldots, x_{n-1})$ is called a *cycle* if $(x_0, x_1, \ldots, x_{n-1}, x_0, x_1, \ldots)$ is a periodic orbit.

As f is a set-valued function, a given point may not have a unique orbit. Because of this, for a given orbit $\{x_i\}_{i\in\omega}$ there may be some $i \in \mathbb{N}$ such that $x_i = x_0$, even if $\{x_i\}_{i\in\omega}$ is not periodic. Similarly if $\{x_i\}_{i\in\omega}$ is an orbit of period n, there may be some $0 < j < n$ such that $x_j = x_0$. For instance, the function $f : [0,1] \to 2^{[0,1]}$ defined by $f(0) = f(1) = [0,1]$ and $f(x) = \{0\}$ for $x \in (0,1)$ has $(0,0,1,0,1)$ as a periodic cycle. Although $x_0 = x_1 = 0$, the period of the orbit is 5.

In general, the Sharkovskii order does not hold for upper-semicontinuous set-valued functions.

Example 2.5.2 . Define $f : [0,1] \to 2^{[0,1]}$ by

$$f(x) = \begin{cases} 1/2 & x \in [0,1) \\ [1/2,1] & x = 1/2 \\ 1-x & x \in (1/2,1] \end{cases}$$

Then $\left(0, \frac{1}{2}, 1\right)$ is a period 3 cycle, but there is no period 2 cycle. So the Sharkovskii order fails for f. See Figure 2.3 for the graph of this function.

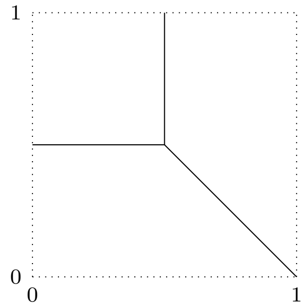

Figure 2.3: Upper-semicontinuous function where the Sharkovskii order fails.

However, the intermediate value property (defined in Chapter Three) allows us to recover the Sharkovskii order.

Theorem 2.5.3 . (Otey, Ryden, [41]) Let $f : [0,1] \to 2^{[0,1]}$ be upper-semicontinuous and have the intermediate value property. If f has a cycle of period n, then f has cycles of every period m such that $n \prec m$.

With the Sharkovskii order and the full-projection property in place, we provide the following generalizations of Theorems 2.3.3 and 2.3.4 in Theorems 4.2.2 and 4.2.8 respectively, connecting complicated dynamics of set-valued functions to the exotic topology of their corresponding inverse limits.

Theorem. Let $f : [0,1] \to 2^{[0,1]}$ be upper-semicontinuous, surjective, almost non-fissile, light, and have the intermediate value property, and $G(f)$ have empty interior. If f has an orbit of period not a power of 2, then $\varprojlim\{[0,1], f\}$ contains an indecomposable subcontinuum.

Theorem. If $f : [0,1] \to 2^{[0,1]}$ is upper-semicontinuous, organic and has the intermediate value property and $\varprojlim\{[0,1], f\}$ is indecomposable, then f has a periodic cycle with a period that is not a power of 2.

The intermediate value property will be key as it allows us to generalize Theorem 2.3.5 and use the Sharkovskii order with set-valued functions.

CHAPTER THREE

Connectedness of Inverse Limits of Set-Valued Functions on [0,1] with the (Weak) Intermediate Value Property

3.1 Introduction

In this chapter, we consider the connectedness of generalized inverse limits on $[0,1]$ whose bonding functions have the weak intermediate value property. In Section 3.2, we introduce two generalized notions of the intermediate value property that are applicable to set-valued functions. Then we prove some structure theorems to give the reader some intuition as to the nature of the weak intermediate value property and intermediate value property for upper-semicontinuous functions.

In Section 3.3, we introduce a characterization of connectedness of inverse limits of upper-semicontinuous functions on $[0,1]$ that are surjective and have connected graphs from [24, 25]. This allows us to prove the main theorem of the chapter, Theorem 3.3.5, in which we show a sequence of upper-semicontinuous functions with the weak intermediate value property generates a connected inverse limit under modest conditions. The sufficient conditions provided in Theorem 3.3.5 are easily verfiable, as they pertain to the bonding functions taken in isolation, rather than taken together in compositions of finite subsequences. We show in Theorem 3.3.8 that in the special case of a single bonding function, the criteria in Theorem 3.3.5 is equivalent to that of Nall's theorem (2.4.5).

In Section 3.4, we present examples demonstrating that the assumptions of Theorem 3.3.5 are sharp, and thus cannot be dropped. We also present an upper-semicontinuous function that does not have the weak intermediate value property yet admits a connected inverse limit, indicating the sufficient conditions of Theorem 3.3.5 are not necessary. This indicates room for further research on necessary or sufficient conditions on a sequence of upper-semicontinuous bonding functions to admit a connected inverse limit.

3.2 Intermediate Value Properties for Set-Valued Functions

Definition 3.2.1 . Let $f : [a,b] \to 2^{[c,d]}$ be an upper-semicontinuous function. We say f has the *intermediate value property* if, given distinct x_1, x_2 and distinct $y_1 \in f(x_1)$, $y_2 \in f(x_2)$, there is some x strictly between x_1 and x_2 such that $y \in f(x)$.

We say f has the *weak intermediate value property* if, given distinct x_1, x_2, and $y_1 \in f(x_1)$ there is some $y_2 \in f(x_2)$ such that if y is between y_1 and y_2, then there is x between x_1 and x_2 such that $y \in f(x)$.

Note that we do not specify if x_2 is larger than x_1. So for a function to have the weak intermediate value property, it is necessary for the condition to hold when $x_2 > x_1$ and $x_1 > x_2$. If f is upper-semicontinuous and has the intermediate value property, it follows that f is weakly continuous via Theorem 3.2.10.

Let $f : [a,b] \to 2^{[c,d]}$ and $g : [c,d] \to 2^{[i,j]}$ be upper-semicontinuous, I be a closed subinterval of $[a,b]$, and J be a closed subinterval of $[c,d]$ such that if $x \in I$, then $f(x) \cap J \neq \emptyset$. Let $f|_I : I \to 2^{[c,d]}$ denote the function $f|_I(x) = f(x)$. Let $f|_I^J : I \to J$ denote the function $f|_I^J(x) = f(x) \cap J$. Note that if f and g have the (weak) intermediate value property, then each of $f|_I$, $f|_I^J$, and $g \circ f$ has the (weak) intermediate value property as well.

Below we present some examples to demonstrate what it means for an upper-semicontinuous function to have the intermediate value property and weak intermediate value property. Examples 3.2.2, 3.2.3, and 3.2.4 are upper-semicontinuous functions that have the intermediate value property, the weak intermediate value property but not the intermediate value property, and neither property respectively.

Example 3.2.2 . Let $f : [0,1] \to 2^{[0,1]}$ be defined by

$$f(x) = \begin{cases} \left[0, \frac{1}{4}\right] & x = 0 \\ \frac{1}{4}\sin\left(\frac{1}{x}\right) + \frac{1}{4} & 0 \leq x \leq \frac{1}{\pi} \\ \frac{1}{4\pi - 1}(3\pi x + \pi - 1) & \frac{1}{\pi} \leq x \leq 1 \end{cases}$$

Then f has the intermediate value property, the graph of which can be seen in Figure 3.1.

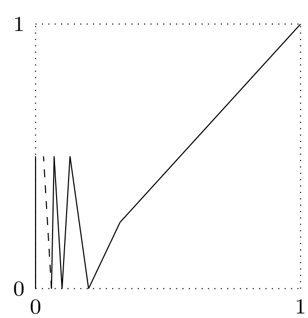

Figure 3.1: Upper-semicontinuous function with intermediate value property.

Example 3.2.3 . Let $f : [0, 1] \to 2^{[0,1]}$ be given by $f(x) = \{x, 1 - x\}$. Then f is upper-semicontinuous and has the weak intermediate value property but does not have the intermediate value property.

To see why f does not have the intermediate value property, consider $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = \left(\frac{1}{4}, \frac{1}{4}\right)$. There is no $x \in \left(0, \frac{1}{4}\right)$ such that $f(x)$ contains $\frac{1}{2}$.

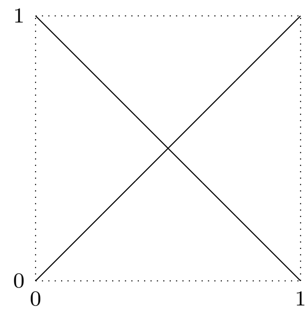

Figure 3.2: Upper-semicontinuous function with the weak intermediate value property.

Example 3.2.4 . Let $f : [0, 1] \to 2^{[0,1]}$ be given by

$$f(x) = \begin{cases} \frac{1}{3}x & 0 \leq x < \frac{1}{2} \\ \left\{\frac{1}{3}x, 2x - 1\right\} & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Then f has neither the intermediate value property nor the weak intermediate value property.

Let $(x_1, y_1) = \left(\frac{1}{2}, 0\right)$ and $x_2 = \frac{1}{4}$. Since $f\left(\frac{1}{4}\right) = \left\{\frac{1}{12}\right\}$, y_2 must be $\frac{1}{12}$. But $\frac{1}{24} \notin f(x)$ for any $x \in \left(\frac{1}{4}, \frac{1}{2}\right)$. So f does not have the weak intermediate value property. Note, from the graph of f in Figure 3.3 that the weak intermediate value property holds for $x_2 > x_1$ and only fails in a case where $x_1 > x_2$.

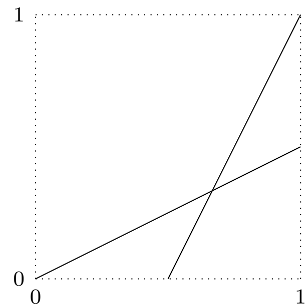

Figure 3.3: Upper-semcontinuous function w/o the weak intermeidate value property.

Definition 3.2.5 . The function $f : [a, b] \to 2^{[c,d]}$ is *weakly continuous from the left at* x if it is upper-semicontinuous and, for each $y \in f(x)$, there is a sequence $\{(x_n, y_n)\}_{n\in\omega}$ that converges to (x, y) such that $x_n < x$ and $y_n \in f(x_n)$ for each n.

The function $f : [a, b] \to 2^{[c,d]}$ is *weakly continuous from the right at* x if it is upper-semicontinuous and, for each $y \in f(x)$, there is a sequence $\{(x_n, y_n)\}_{n\in\omega}$ that converges to (x, y) such that $x_n > x$ and $y_n \in f(x_n)$ for each n.

The function $f : [a, b] \to 2^{[c,d]}$ is *weakly continuous at* x if f is weakly continuous from the left and from the right at x, and f is *weakly continuous* if it is weakly continuous for each $x \in (a, b)$.

Theorem 3.2.6 . Suppose $f : [0, 1] \to 2^{[0,1]}$ is upper-semicontinuous. Then f has the intermediate value property if and only if f is weakly continuous and $f(x)$ is connected for each x.

Proof. If f has the intermediate value property, then f is weakly continuous and $f(x)$ is connected for each x by Theorem 3.2.8. To see the converse, suppose f does not have the intermediate value property but that f is weakly continuous. We show that $f(x)$ fails to be connected for some $x \in [0,1]$. Since f does not have the intermediate value property, there are $(x_1, y_1), (x_2, y_2) \in G(f)$ and y strictly between y_1 and y_2 such that $y \notin f(x)$ for all x strictly between x_1 and x_2. There are four cases, all similar, corresponding to the orders of x_1 and x_2 and of y_1 and y_2. We consider only the case in which $x_1 < x_2$ and $y_1 < y_2$.

Since f is weakly continuous from the right at x_1, there is $(x_1', y_1') \in G(f)$ such that $x_1 < x_1' < x_2$ and $y_1' < y$. Since f is weakly continuous from the left at x_2, there is $(x_2', y_2') \in G(f)$ such that $x_1' < x_2' < x_2$ and $y < y_2'$. It follows that, for each $x \in [x_1', x_2']$, $y \notin f(x)$. Since $y_1' < y < y_2'$ it follows that $V_{[x_1', x_2']} \cap G(f)$ is the union of two disjoint compact sets K_1 and K_2. Then $\pi_1[K_1] \cup \pi_1[K_2] = [x_1', x_2']$. Consequently, there is $c \in \pi_1[K_1] \cap \pi_1[K_2]$. It follows that $\{c\} \times f(c)$ is a subset of $K_1 \cup K_2$ that intersects both K_1 and K_2. Hence $f(c)$ is not connected. $\square$

Notation. For $0 \leq a < b \leq 1$, let $V_{[a,b]} = [a,b] \times [0,1]$.

Lemma 3.2.7 . Let $f : [0,1] \to 2^{[0,1]}$ be an upper-semicontinuous function with $G(f)$ connected. Then for all $a \leq b$ there is some subcontinuum C of $G(f) \cap V_{[a,b]}$ such that $C \cap \{a\} \times [0,1] \neq \emptyset$ and $C \cap \{b\} \times [0,1] \neq \emptyset$.

Proof. Let

$$\mathcal{L} = \{H : H \text{ is a component of } G(f) \cap V_{[a,b]} \text{ and } H \cap \{a\} \times [0,1] \neq \emptyset\},$$

$$\mathcal{R} = \{K : K \text{ is a component of } G(f) \cap V_{[a,b]} \text{ and } K \cap \{b\} \times [0,1] \neq \emptyset\}.$$

Note for all $H \in \mathcal{L}$ and $K \in \mathcal{R}$, either $H \cap K = \emptyset$ or $H = K$ by the maximality of components. Also $G(f) \cap V_{[a,b]} = (\bigcup \mathcal{L}) \cup (\bigcup \mathcal{R})$; otherwise there would be some component D of $G(f) \cap V_{[a,b]}$ that does not intersect either $\{a\} \times [0,1]$ or $\{b\} \times [0,1]$. Then D would be a proper component of $G(f)$, contradicting the connectedness of $G(f)$.

Suppose that $\mathcal{L} \cap \mathcal{R} = \emptyset$. Then $f(a)$ and $f(b)$ are disjoint closed subsets of $G(f) \cap V_{[a,b]}$ such that no component of $G(f) \cap V_{[a,b]}$ intersects $f(a)$ and $f(b)$. Then by the Cut-Wire Theorem, there are two disjoint closed sets A and B such that $G(f) \cap V_{[a,b]} = A \cup B$, $f(a) \subseteq A$, and $f(b) \subseteq B$. As each $H \in \mathcal{L}$ and $K \in \mathcal{R}$ is connected, $H \subseteq A$ and $K \subseteq B$. Thus $A = \bigcup \mathcal{L}$ and $B = \bigcup \mathcal{R}$. Then $A \cup \big(G(f) \cap V_{[0,a]}\big)$ and $B \cup \big(G(f) \cap V_{[b,1]}\big)$ are nonempty disjoint closed sets whose union is $G(f)$, contradicting the connectedness of $G(f)$. $\square$

The following theorem provides a graphical characterization of the weak intermediate value property. A similar result characterizing the Intermediate Value Property is provided in Theorem 3.2.10 for the purpose of comparison.

Theorem 3.2.8 . Let $f : [0,1] \to 2^{[0,1]}$ be a upper-semicontinuous function such that $G(f)$ is connected. The following are equivalent:

1. f has the weak intermediate value property.

2. For all $a \leq b$, each component of $G(f) \cap V_{[a,b]}$ intersects both $\{a\} \times [0,1]$ and $\{b\} \times [0,1]$.

Proof. ($1 \Rightarrow 2$) As in Lemma 3.2.7, let

$$\mathcal{L} = \{H : H \text{ is a component of } G(f) \cap V_{[a,b]} \text{ and } H \cap \{a\} \times [0,1] \neq \emptyset\},$$

$$\mathcal{R} = \{K : K \text{ is a component of } G(f) \cap V_{[a,b]} \text{ and } K \cap \{b\} \times [0,1] \neq \emptyset\}.$$

Note every component D of $G(f) \cap V_{[a,b]}$ is a member of either $\mathcal{L}$ or $\mathcal{R}$; otherwise D would be a proper component of $G(f)$, contradicting the assumption that $G(f)$ is connected. By contradiction, suppose there is some component C of $G(f) \cap V_{[a,b]}$ that does not intersect both $\{a\} \times [0,1]$ and $\{b\} \times [0,1]$. Without loss of generality, suppose $C \cap (\{b\} \times [0,1]) = \emptyset$. Note this implies $C \in \mathcal{L}$. Then C and $f(b)$ are nonempty disjoint closed subsets of $G(f) \cap V_{[a,b]}$ such that no connected subset of $G(f) \cap V_{[a,b]}$ intersects both C and $\{b\} \times f(b)$. Then by the Cut-Wire Theorem, there are disjoint closed sets A and B in $G(f) \cap V_{[a,b]}$ such that $A \cup B = G(f) \cap V_{[a,b]}$, $C \subseteq A$, and $\{b\} \times f(b) \subseteq B$. Note that by the connectedness of each $H \in \mathcal{L}$ and $K \in \mathcal{R}$, $H \subseteq A$ and $K \subseteq B$. Thus $A = \bigcup_{H \in \mathcal{L} \setminus (\mathcal{L} \cap \mathcal{R})} H$ and $B = \bigcup_{K \in \mathcal{R}} K$.

Let $x_1 = \max\{x : (x,y) \in A\}$. By Lemma 3.2.7, there is some connected set $D \subseteq B$ such that $D \cap \{a\} \times [0,1] \neq \emptyset$ and $D \cap \{b\} \times [0,1] \neq \emptyset$. As D is connected, there is some point $z \in f(x_1)$ such that $(x_1, z) \in D$. Note there is some $y \in f(x_1)$ such that $(x_1, y) \in A$ and either $y > z$ or $z > y$. Without loss of generality, suppose $z > y$. Let $y_1 = \max\{y : (x_1, y) \in A \text{ and } y < z\}$. Define $\epsilon = \min\{d(A,B), b - x_1\} > 0$. Let $x_2 = x_1 + \frac{\epsilon}{2}$ and $y_2 \in f(x_2)$. By the construction of x_1 and x_2, $(x_2, y_2) \in B$. So $|y_2 - y_1| > \frac{\epsilon}{2}$; otherwise $d((x_1, y_1), (x_2, y_2)) < \epsilon$. Let $y \in (y_1 - \frac{\epsilon}{2}, y_1 + \frac{\epsilon}{2})$ be between y_1 and y_2 and x between x_1 and x_2. By definition of x_1, $(x, y) \notin A$. But

$d((x_1, y_1), (x, y)) < \epsilon$ so $(x, y) \notin B$. Thus for each x between x_1 and x_2, $y \notin f(x)$, a contradiction.

$(2 \Rightarrow 1)$ Choose $x_1, x_2 \in [0, 1]$ and let $y_1 \in f(x_1)$. Suppose $x_1 < x_2$. Let C be a component of $G(f) \cap V_{[x_1,x_2]}$ containing (x_1, y_1). Choose y_2 such that $(x_2, y_2) \in C$. As C is connected, $\pi_2(C)$ is connected where $\pi_2 : [0, 1]^2 \to [0, 1]$ is the projection map given by $\pi_2(x, y) = y$. Thus $\pi_2(C)$ is an interval containing y_1 and y_2. So for any y between y_1 and y_2, there is some $x \in [x_1, x_2]$ such that $(x, y) \in C$, i.e. $y \in f(x)$. The case where $x_1 > x_2$ follows by a similar argument. □

Theorem 3.2.9 . Let $f : [0, 1] \to 2^{[0,1]}$ be upper-semicontinuous function such that $G(f)$ is connected. If $f(x)$ is connected for every $x \in [0, 1]$, then f has the weak intermediate value property.

Proof. Let $0 \leq a \leq b \leq 1$. By Lemma 3.2.7, there is some component C of $G(f) \cap V_{[a,b]}$ that intersects $\{a\} \times [0, 1]$ and $\{b\} \times [0, 1]$. Since $f(a)$ and $f(b)$ are connected, $C \cap (\{a\} \times [0, 1]) = \{a\} \times f(a)$ and $C \cap (\{b\} \times [0, 1]) = \{b\} \times f(b)$. Let D be a component of $G(f) \cap V_{[a,b]}$. Since $G(f)$ is connected, either $D \cap (\{a\} \times [0, 1]) \neq \emptyset$ or $D \cap (\{b\} \times [0, 1]) \neq \emptyset$. In either case, $C \cap D \neq \emptyset$. Therefore $D = C$. Thus every component of $G(f) \cap V_{[a,b]}$ intersects $\{a\} \times [0, 1]$ and $\{b\} \times [0, 1]$, and f has the weak intermediate value property by Theorem 3.2.8. □

Theorem 3.2.10 . Let $f : [0, 1] \to 2^{[0,1]}$ be upper-semicontinuous. Then the following are equivalent.

1. f has the intermediate value property.

2. For all $a \leq b$, $G(f) \cap V_{[a,b]}$ is connected and $G(f) \cap V_{[a,b]} = \overline{G(f) \cap V_{(a,b)}}$.

Proof. ($1 \Rightarrow 2$) Suppose f has the intermediate value property. In order to show that for all $a \leq b$ $G(f) \cap V_{[a,b]}$ is connected, we first show that $f(x)$ is connected for each $x \in [0,1]$. By way of contradiction, suppose there is an x_1 such that $f(x_1)$ is disconnected. Let U_1 and U_2 be two intervals open in $[0,1]$ such that $f(x_1) \subseteq U_1 \cup U_2$, $f(x_1) \cap U_1 \neq \emptyset$, $f(x_1) \cap U_2 \neq \emptyset$ and $\overline{U}_1 \cap \overline{U}_2 = \emptyset$.

Since f is upper-semicontinuous, there is an open set $V \ni x_1$ such that if $x \in V$, then $f(x) \subseteq U_1 \cup U_2$. Let $x_2 \in V \setminus \{x_1\}$ and choose $y_2 \in f(x_2)$. Then $y_2 \in U_1$ or $y_2 \in U_2$. Without loss of generality, suppose $y_2 \in U_2$. Let $y_1 \in f(x_1) \cap U_1$. Then there is some $y \in [0,1] \setminus (U_1 \cup U_2)$ strictly between y_1 and y_2. Let x be between x_1 and x_2. Then since $x \in V$, $f(x) \subseteq U_1 \cup U_2$. So $y \notin f(x)$, contradicting the assumption that f has the Intermediate Value Property. Thus $f(x)$ is connected for all $x \in [0,1]$.

Let K_1 and K_2 be components of $G(f) \cap V_{[a,b]}$. Since f has the Intermediate Value Property and therefore the weak intermediate value property, $K_1 \cap (\{a\} \times [0,1]) \neq \emptyset$ and $K_2 \cap (\{a\} \times [0,1]) \neq \emptyset$ by Theorem 3.2.8. But because $f(a)$ is connected, $\{a\} \times f(a)$ is connected and intersects both K_1 and K_2. So $K_1 \cap K_2 \neq \emptyset$ as K_1 and K_2 are components. Hence $K_1 = K_2$, making $G(f) \cap V_{[a,b]}$ connected.

Next we show $G(f) \cap V_{[a,b]} = \overline{G(f) \cap V_{(a,b)}}$. If $f(a)$ is a singleton, then $(a, f(a)) \in \overline{G(f) \cap V_{(a,b)}}$ as f is upper-semicontinuous. Similarly if $f(b)$ is a singleton, then $(b, f(b)) \in \overline{G(f) \cap V_{(a,b)}}$. Suppose $f(a)$ is nondegenerate. Let $\{(a_n, z_n)\}_{n \in \omega}$ be a sequence in $G(f) \cap V_{(a,b)}$ such that $a_n \to a$. Taking subsequences if necessary, we may choose the sequence such that $\{z_n\}_{n\in\omega}$ converges to some $z \in f(a)$. So $(a,z) \in \overline{G(f) \cap V_{(a,b)}}$.

Let $y \in f(a)\setminus\{z\}$ and let $\epsilon > 0$ such that $|z-y| > \epsilon$. Since $z_n \to z$, there is $N \in \mathbb{N}$ such that if $n \geq N$, then y is not between z_n and z. Let y_ϵ be strictly between y and z_n for all $n \geq N$ such that $|y - y_\epsilon| < \epsilon$. Since f has the Intermediate Value Property, for each $n \geq N$ there is some $x_n \in (a, a_n)$ such that $y_\epsilon \in f(a_n)$. Then $a_n \to a$ and $(a, y_\epsilon) \in \overline{G(f) \cap V_{(a,b)}}$. By the same argument, for all $0 < \delta < \epsilon$ there is some y_δ such that $(a, y_\delta) \in \overline{G(f) \cap V_{(a,b)}}$. Since $d((a,y),(a,y_\delta)) = \delta$, $(a,y) \in \overline{G(f) \cap V_{(a,b)}}$. Therefore $\{a\} \times f(a) \subseteq \overline{G(f) \cap V_{(a,b)}}$. By a similar argument, $\{b\} \times f(b) \subseteq \overline{G(f) \cap V_{(a,b)}}$. Thus $\overline{G(f) \cap V_{(a,b)}} = G(f) \cap V_{[a,b]}$.

($2 \Rightarrow 1$) The converse follows by contradiction. Suppose for all $a \leq b$, $G(f) \cap V_{[a,b]}$ is connected and $G(f) \cap V_{[a,b]} = \overline{G(f) \cap V_{(a,b)}}$, but f does not have the Intermediate Value Property. Then there is some distinct x_1, x_2, $y_1 \in f(x_1)$, $y_2 \in f(x_2)$, and y strictly between y_1 and y_2 such that $y \notin f(x)$ for all x strictly between x_1 and x_2. There are four cases depending on which of x_1 and x_2 is larger and which of y_1 and y_2 is larger. Suppose $x_1 < x_2$ and $y_1 < y_2$. The proofs for the remaining cases are similar.

Since $G(f) \cap V_{[x_1,x_2]} = \overline{G(f) \cap V_{(x_1,x_2)}}$, there are $x_1 < a' < b' < x_2$, $y_{a'} \in f(a')$ with $y_{a'} < y$, and $y_{b'} \in f(b')$ with $y_{b'} > y$. So y is strictly between $y_{a'}$ and $y_{b'}$, but $y \notin f(x)$ for any $x \in [a', b']$. Thus $G(f) \cap V_{[a',b']}$ is disconnected, a contradiction. $\square$

3.3 Showing Connectedness of Generalized Inverse Limits with the Weak Intermediate Value Property

Notation. Let $i > 0$, $\epsilon > 0$, and $A_i = [a_i, b_i]$ be a subset of $[0, 1]$ for each i. For $j \in \{i, i-1\}$, define

$$J_j = [0, a_j), \qquad R_i = (K_i \times [0,1]) \cup Z_i,$$

$$K_j = (b_j, 1] \qquad TL_i = T_i \cup L_i,$$

$$Z_i = A_i \times A_{i-1}, \qquad TR_i = T_i \cup R_i,$$

$$T_i = ([0,1] \times K_{i-1}) \cup Z_i, \qquad BL_i = B_i \cup L_i$$

$$B_i = ([0,1] \times J_{i-1}) \cup Z_i, \qquad BR_i = B_i \cup R_i,$$

$$L_i = (J_i \times [0,1]) \cup Z_i,$$

$$Z_i(\epsilon) = ((a_i - \epsilon, b_i + \epsilon) \times (a_{i-1} - \epsilon, b_{i-1} + \epsilon)) \cap ([0,1] \times [0,1]).$$

Definition 3.3.1 . Suppose $i > 0$ and $f : [0, 1] \to 2^{[0,1]}$ is upper-semicontinuous. If for each $j \in \{i, i-1\}$, $A_i = [a_i, b_i] \subsetneq [0, 1]$, either

- $S \in \{BL, BR, TL, TR\}$, or
- $A_i \cap \{0, 1\} = \emptyset$ and $S \in \{L_i, R_i\}$, or
- $A_{i-1} \cap \{0, 1\} = \emptyset$ and $S \in \{B_i, T_i\}$,

and there exists $\epsilon > 0$ and a component C' of the set $G(f) \cap Z_i(\epsilon)$ such that $C' \subset S$, then any component C of $C' \cap Z_i$ is an S*-set in* $G(f)$ *framed by* $A_i \times A_{i-1}$, denoted $G(f) \sqsubset_C S$.

In the following definition, only condition (1) is used for the purpose of proving our main result. But for the sake of completeness and to properly introduce Greenwood's and Kennedy's criterion for connected inverse limits of surjective upper-semicontinuous functions on $[0,1]$ with connected graphs, we present the full definition of a CC-sequence.

Definition 3.3.2 . Suppose for each $i > 0$, $f_{i+1} : [0,1] \to 2^{[0,1]}$ is a surjective upper-semicontinuous function with a connected graph, denoted G_{i+1}, and $m, n \in \mathbb{N}$ are such that $m + 1 < n$. Suppose that there exist

- a closed interval $A_i \subsetneq [0,1]$ for each i, $m \leq i \leq n$, and
- a point

$$(p_k)_{k\in\omega} \in \varprojlim\{[0,1], f_i\} \cap \left(\prod_{i<m}[0,1] \times \prod_{m\leq i\leq n} A_i \times \prod_{i>n}[0,1]\right).$$

For each $i > 0$, let C_i be the component of $G_i \cap Z_i$ containing (p_i, p_{i-1}) and suppose the following properties hold:

1. $G_{m+1} \sqsubset_{C_{m+1}} R_{m+1}$ or $G_{m+1} \sqsubset_{C_{m+1}} L_{m+1}$;
2. - if $n = m + 2$, then $G_{m+2} \sqsubset_{C_{m+2}} T_{m+2}$ if $G_{m+1} \sqsubset_{C_{m+1}} L_{m+1}$, and $G_{m+2} \sqsubset_{C_{m+2}} B_{m+2}$ if $G_{m+1} \sqsubset_{C_{m+1}} R_{m+1}$;
   - if $n > m + 2$, then $G_{m+2} \sqsubset_{C_{m+2}} BR_{m+2}$ or $G_{m+2} \sqsubset_{C_{m+2}} BL_{m+2}$ if $G_{m+1} \sqsubset_{C_{m+1}} R_{m+1}$, and $G_{m+2} \sqsubset_{C_{m+2}} TL_{m+2}$ or $G_{m+2} \sqsubset_{C_{m+2}} TR_{m+2}$ if $G_{m+1} \sqsubset_{C_{m+1}} L_{m+1}$;

3. if $m+2 \le i < n-1$, then $G_{i+1} \sqsubset_{C_{i+1}} BL_{i+1}$ or $G_{i+1} \sqsubset_{C_{i+1}} BR_{i+1}$, if $G_i \sqsubset_{C_i} BR_i$ or $G_i \sqsubset_{C_i} TR_i$, and $G_{i+1} \sqsubset_{C_{i+1}} TL_{i+1}$ or $G_{i+1} \sqsubset_{C_{i+1}} TR_{i+1}$ if $G_i \sqsubset_{C_i} BL_i$ or $G_i \sqsubset_{C_i} TL_i$;

4. if $n > m+2$, then $G_n \sqsubset_{C_n} B_n$ if $G_{n-1} \sqsubset_{C_{n-1}} BR_{n-1}$ or $G_{n-1} \sqsubset_{C_{n-1}} TR_{n-1}$, and $G_n \sqsubset_{C_n} T_n$ if $G_{n-1} \sqsubset_{C_{n-1}} BL_{n-1}$ or $G_{n-1} \sqsubset_{C_{n-1}} TL_{n-1}$.

Then $\{f_i : i > 0\}$ admits a *component cropping sequence*, or *CC-sequence*,

$$\{A_i : m \le i \le n\},$$

over $[m, n]$ with *pivot point* $(p_k)_{k \in \omega}$. The collection $\{f_i : i > 0\}$ of functions admits a CC-sequence if there exist $m, n \in \mathbb{N}$ such that $\{f_i : i > 0\}$ admits a CC-sequence over $[m, n]$ with some pivot point.

Theorem 3.3.3 . (Greenwood, Kennedy [25]) Suppose that for each $i \ge 0$, I_i is an interval, $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper-semicontinuous function, and $G(f_{i+1})$ is connected. The system admits a CC-sequence if and only if $\varprojlim\{I_i, f_i\}$ is disconnected.

We show that if all bonding functions have the weak intermediate value property, then (1) from the above definition cannot be met, i.e. that the graph of such a function contains no L-sets or R-sets. This along with Theorem 3.3.3 is sufficient to establish the main result.

Theorem 3.3.4 . Let $f : [0, 1] \to 2^{[0,1]}$ be a function that is upper-semicontinuous and has the weak intermediate value property, and suppose $G(f)$ be connected. Then $G(f)$ contains no L-sets or R-sets.

Proof. We show that $G(f)$ contains no L-sets. That there are no R-sets follows by a symmetric argument. By way of contradiction, suppose there are some $A_1 = [a, b]$ and $A_0 = [c, d]$ such that $A_1 \times A_0$ frames an L-set of $G(f)$. Let $\epsilon > 0$ and $C' \subset L$ be a component of $G(f) \cap Z(\epsilon)$ that contains an L-set C. Then C' and C are connected sets that do not intersect any of $[a, b] \times [0, c)$, $[a, b] \times (d, 1]$, or $(b, 1] \times [0, 1]$. Thus C is a component of $V_{[a,b+\epsilon]}$. However by Theorem 3.2.8, $C \cap (\{b + \epsilon\} \times [0, 1]) \neq \emptyset$, a contradiction. □

Theorem 3.3.5 . For each $n \in \mathbb{N}$, let $f_n : [0, 1] \to 2^{[0,1]}$ be a surjective, upper-semicontinuous function with the weak intermediate value property and a connected graph $G(f_n)$. Then $\varprojlim\{[0, 1], f_n\}$ is connected.

Proof. By Theorem 3.3.4, $G(f)$ contains no L-sets or R-sets. Then condition (1) of the definition of a CC-sequence cannot be met. It follows that the system does not admit a CC-sequence, and therefore $\varprojlim\{[0, 1], f\}$ is connected by Theorem 3.3.3. □

Remark. Jonathan Medaugh has informed the author that Theorem 3.3.5 would hold if f_n^{-1}, rather than f_n, were assumed to have the weak intermediate value property for each n. One approach to the proof of such a result would involve finite Mahavier products $\Gamma_n(f)$ and the fact that $\Gamma_n(f)$ and $G_n(f^{-1})$ are homeomorphic where $f^{-1} = \{f_i^{-1}\}_{i\in\mathbb{N}}$ [18, Theorem 2.11].

We now establish a structure theorem regarding the graphs of functions with the weak intermediate value property. Theorem 3.3.8 reveals a kinship with Nall's theorem (Theorem 2.4.5). In particular, it shows that upper-semicontinuous functions with the weak intermediate value property are precisely those that satisfy Nall's criteria. Thus Theorems 3.3.5 and 3.3.8 provide a generalization of Nall's Theorem, which is stated and proved in the context of a single bonding function.

Before examining the structure of upper-semicontinuous functions with the weak intermediate value property as unions of their subgraphs, we must introduce the notion of convergence in the hyperspace 2^X with the Hausdorff topology, i.e. what it means for a sequence of subsets of a metric space to converge. The following definition and theorem can be found in [36].

Definition 3.3.6 . Let X be a space and $\{A_i\}_{i\in\omega}$ be a sequence of subsets of X. We define $\overline{\lim}\ A_i$ and $\underline{\lim}\ A_i$ as follows:

$$\overline{\lim}\ A_i = \{x \in X : \text{for every open set } U \ni x,\ U \cap A_i \neq \emptyset \text{ for infinitely many } i\},$$

$$\underline{\lim}\ A_i = \{x \in X : \text{for every open set } U \ni x,\ U \cap A_i \neq \emptyset \text{ for cofinitely many } i\}.$$

If $\overline{\lim}\ A_i = \underline{\lim}\ A_i$, we define $\lim\ A_i = \overline{\lim}\ A_i = \underline{\lim}\ A_i$.

Theorem 3.3.7 . If X is a compact metric space and if $\{E_i\}_{i\in\omega}$ is a sequence of connected subsets of X such that $\underline{\lim}\ E_i \neq \emptyset$, then $\overline{\lim}\ E_i$ is connected.

Theorem 3.3.8 . Let $f : [0,1] \to 2^{[0,1]}$ be an upper-semicontinuous function such that $G(f)$ is connected. Then f has the weak intermediate value property if and only if there is a collection $\mathcal{F}$ of upper-semicontinuous functions $g : [0,1] \to C([0,1])$ such that $G(f) = \bigcup_{g\in\mathcal{F}} G(g)$.

Proof. ($\Leftarrow$) Suppose there is a collection $\mathcal{F}$ of upper-semicontinuous functions g : $[0,1] \to C([0,1])$ such that $G(f) = \bigcup_{g\in\mathcal{F}} G(g)$. Let $(x_1, y_1) \in G(f)$ and $x_2 \in [0,1]$ be distinct from x_1. There is some $g \in \mathcal{F}$ such that $(x_1, y_1) \in G(g)$. By Theorem 3.2.9, g has the weak intermediate value property. Then there is some y_2 such that if y is between y_1 and y_2, there is some x between x_1 and x_2 such that $y \in g(x) \subseteq f(x)$. Thus f has the weak intermediate value property.

($\Rightarrow$) It suffices to show that for each $(x, y) \in G(f)$, there is an upper-semicontinuous function $g : [0, 1] \to C([0, 1])$ such that $(x, y) \in G(g) \subseteq G(f)$. To that end, suppose $(x, y) \in G(f)$. For each $n \in \mathbb{N}$, define $\mathcal{G}_n$ to be the collection of set-valued functions g_n that satisfy the following: let $0 \leq i < 2^n$ such that $\frac{i}{2^n} \leq x \leq \frac{i+1}{2^n}$ and $C_{n,i}$ be the component of $G(f) \cap V_{\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]}$ containing (x, y). By Theorem 3.2.8, $C_{n,i} \cap \left(\left\{\frac{i+1}{2^n}\right\} \times [0, 1]\right) \neq \emptyset$. Choose a component $C_{n,i+1}$ of $G(f) \cap V_{\left[\frac{i+1}{2^n}, \frac{i+2}{2^n}\right]}$ such that $C_{n,i} \cap C_{n,i+1} \neq \emptyset$. Then by Theorem 3.2.8, $C_{n,i+1} \cap \left(\left\{\frac{i+2}{2^n}\right\} \times [0, 1]\right) \neq \emptyset$. Continuing inductively, for each $i < j < 2^n$, we may choose a component $C_{n,j}$ of $G(f) \cap V_{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]}$ such that $C_{n,j-1} \cap C_{n,j} \neq \emptyset$ and $C_{n,j} \cap \left(\left\{\frac{j+1}{2^n}\right\} \times [0, 1]\right) \neq \emptyset$. By a similar argument, for $0 \leq j < i$, we may choose a component $C_{n,j}$ of $G(f) \cap V_{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]}$ such that $C_{n,j} \cap C_{n,j+1} \neq \emptyset$ and $C_{n,j} \cap \left(\left\{\frac{j}{2^n}\right\} \times [0, 1]\right) \neq \emptyset$. So for $0 \leq j < 2^n$ there are components $C_{n,j}$ of $G(f) \cap V_{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right]}$ such that $C_{n,j} \cap C_{n,k} \neq \emptyset$ if and only if $|j - k| \leq 1$. Define $G(g_n) = \bigcup_{j<2^n} C_{n,j}$. Then g_n is upper-semicontinuous and has a connected graph. Since for any $i < 2^n$ and component $C_{n,i}$ of $G(f) \cap V_{\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]}$ there is some $g_n \in \mathcal{G}_n$ with $C_{n,i} \subseteq G(g_n)$, $\bigcup_{g_n \in \mathcal{G}_n} G(g_n) = G(f)$.

Let $\{G(g_n)\}_{n\in\mathbb{N}}$ be a sequence of subcontinua of $G(f)$ where $g_n \in \mathcal{G}_n$ for each n. Since 2^X is compact, there is a convergent subsequence $\{G(g_{n_k})\}_{k\in\mathbb{N}}$ such that $\lim G(g_{n_k})$ is a subcontinuum of $G(f)$. Define g by $G(g) = \lim G(g_{n_k})$. Then g is upper-semicontinuous and has a connected graph. Furthermore $(x, y) \in G(g)$, as

$(x, y) \in G(g_{n_k})$ for every k. Since $(\{0\} \times g_{n_k}(0)) \in 2^{[0,1]^2}$ for each k, there is a convergent subsequence of $\{\{0\} \times \{g_{n_k}(0)\}\}$. Then $\overline{\lim}\,(\{0\} \times g_{n_k}(0))$ is a nonempty subset of $\overline{\lim}\ G(g_{n_k}) = \lim G(g_{n_k}) = G(g)$. Thus $g(0) \neq \emptyset$. Similarly $g(1) \neq \emptyset$. As $G(g)$ is connected, g is a upper-semicontinuous function on $[0, 1]$. Let $\mathcal{F}$ consist of all such functions g for any $(x, y) \in G(f)$ and any sequence $\{G(g_n)\}_{n\in\mathbb{N}}$ where $g_n \in \mathcal{G}_n$ for each n.

In order to show $g(x)$ is connected for each $x \in [0, 1]$, let $(x, y), (x, y') \in G(g_\alpha)$. Then for each k there are points $(x_{n_k}, y_{n_k}), (x'_{n_k}, y'_{n_k}) \in G(g_{n_k})$ such that $(x_{n_k}, y_{n_k}) \to (x, y)$ and $(x'_{n_k}, y'_{n_k}) \to (x, y')$. Let i_{n_k} and j_{n_k} be the largest and smallest integers respectively such that $x_{n_k}, x'_{n_k} \in \left[\frac{i_{n_k}}{2^{n_k}}, \frac{j_{n_k}}{2^{n_k}}\right]$. Note that $\frac{i_{n_k}}{2^{n_k}}, \frac{j_{n_k}}{2^{n_k}} \to x$ because $x_{n_k}, x'_{n_k} \to x$. By the construction of the $g'_{n_k}s$, $A_{n_k} = G(g_{n_k}) \cap V_{\left[\frac{i_{n_k}}{2^{n_k}}, \frac{j_{n_k}}{2^{n_k}}\right]}$ is a subcontinuum containing (x_{n_k}, y_{n_k}) and (x'_{n_k}, y'_{n_k}). So $(x, y), (x, y') \in \underline{\lim}\ A_{n_k}$. Then by Theorem 3.3.7, $\overline{\lim}\ A_{n_k}$ is a connected subset of $\lim G(g_{n_k}) = G(g)$ containing (x, y) and (x, y'). Since $\frac{i_{n_k}}{2^{n_k}}, \frac{j_{n_k}}{2^{n_k}} \to x$, $\overline{\lim} A_{n_k} \subseteq (\{x\} \times f(x))$. As y and y' are arbitrary elements of $g(x)$, $g(x)$ is connected. Note that by Theorem 3.2.9, g has the weak intermediate value property.

Next we show that $G(f) = \bigcup_{\alpha\in\Lambda} G(g)$. That $\bigcup_{\alpha\in\Lambda} G(g_\alpha) \subseteq G(f)$ follows from the fact that $G(g_\alpha) \subseteq G(f)$ for each α. To show $G(f) \subseteq \bigcup_{\alpha\in\Lambda} G(g_\alpha)$, let $(x, y) \in G(f)$. Then for each $n \in \mathbb{N}$, there is a upper-semicontinuous function $g_n \in \mathcal{G}_n$ such that $(x, y) \in G(g_n)$. Then there is a convergent subsequence $\{G(g_{n_k})\}$. Let $g_\alpha \in \mathcal{F}$ be such that $G(g_\alpha) = \lim\ G(g_{n_k})$. Then $(x, y) \in G(g_\alpha)$. $\square$

3.4 Examples

The first three examples demonstrate that Theorem 3.3.5 is sharp in that the conditions on the bonding functions cannot be dropped. Example 3.4.4 demonstrates that the inverse limit may be connected even if the bonding functions do not have the weak intermediate value property.

Example 3.4.1 (Nall). Let $f : [0,1] \to 2^{[0,1]}$ be given by

$$f(x) = \begin{cases} \frac{1}{3}x & 0 \le x < \frac{1}{2} \\ \left\{\frac{1}{3}x, 2x-1\right\} & \frac{1}{2} \le x \le 1 \end{cases}$$

Then f is an upper-semicontinuous surjective function such that $G(f)$ is connected that does not have the weak intermediate value property. Nall shows in [10] that $G(f^2)$ has an isolated point at $(1,0)$, so $\varprojlim\{[0,1], f\}$ is not connected. Notice that f satisfies the weak intermediate value property in the case of $x_2 \ge x_1$. However, the definition of the weak intermediate value property does not allow us to restrict ourselves to considering only the case where $x_2 \ge x_1$. For example, let $(x_1, y_1) = \left(\frac{1}{2}, 0\right)$ and $x_2 = \frac{1}{4}$. Since $f\left(\frac{1}{4}\right) = \left\{\frac{1}{12}\right\}$, y_2 must be $\frac{1}{12}$. But $\frac{1}{24} \notin f(x)$ for any $x \in \left[\frac{1}{4}, \frac{1}{2}\right]$. So f does not have the weak intermediate value property.

Example 3.4.2 . The function $f : [0,1] \to 2^{[0,1]}$ given by $f(x) = \left\{\frac{2}{3}x, \frac{2}{3}x + \frac{1}{3}\right\}$ is an upper-semicontinuous function that is surjective and satisfies the weak intermediate value property. But $G(f)$ is not connected, therefore $\varprojlim\{[0,1], f\}$ is not connected.

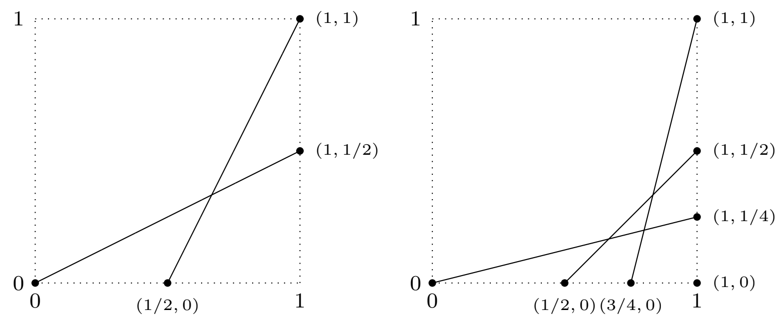

Figure 3.4: $G(f)$ (left) and $G(f^2)$ (right) from Example 3.4.1

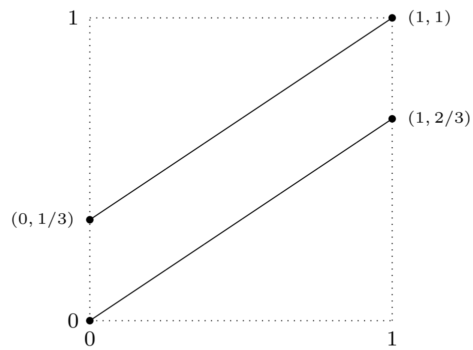

Figure 3.5: $G(f)$ from Example 3.4.2

Example 3.4.3 . The function $f : [0, 1] \to 2^{[0,1]}$ given by $f(x) = \left\{\frac{1}{4}, \frac{3}{4}x + \frac{1}{4}\right\}$ is upper-semicontinuous, satisfies the weak intermediate value property, and has a connected graph. But f is not surjective. Specifically if $y \in \left[0, \frac{1}{4}\right)$, there is no x with $y \in f(x)$. Thus $\varprojlim\{[0, 1], f\} = \varprojlim \left\{ \left[\frac{1}{4}, 1\right], f|_{\left[\frac{1}{4},1\right]} \right\}$ which is not connected.

Example 3.4.4 . (Ingram) Let $f : [0, 1] \to 2^{[0,1]}$ be given by

$$f(x) = \begin{cases} \{0, x\} & 0 \le x \le \frac{1}{4} \\ 0 & \frac{1}{4} \le x \le 1 \\ [0, 1] & x = 1 \end{cases}$$

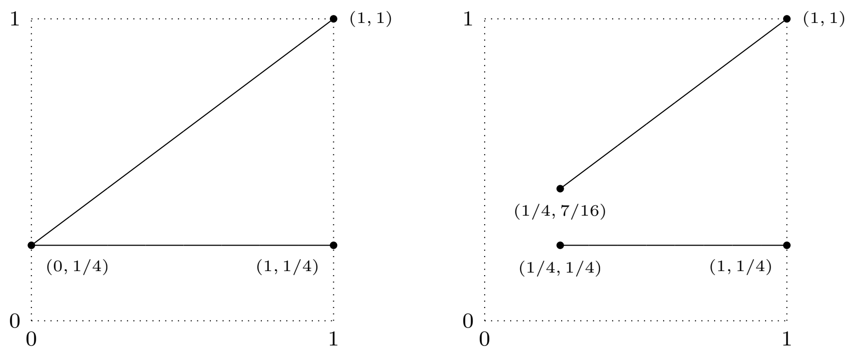

Figure 3.6: $G(f)$ (left) and $G\left(f|_{\left[\frac{1}{4},1\right]}\right)$ (right) for Example 3.4.3

Then f does not have the weak intermediate value property but $\varprojlim\{[0,1], f\}$ is connected. If $(x_1, y_1) = \left(\frac{1}{4}, \frac{1}{4}\right)$ and $x_2 = \frac{1}{2}$, then $y_2 = 0$. But $\frac{1}{8} \notin f(x)$ for any $x \in \left[\frac{1}{4}, \frac{1}{2}\right]$. Ingram gives a proof that $\varprojlim\{[0,1], f\}$ is connected, which can be found in Example 2.9 pg. 24 of [29].

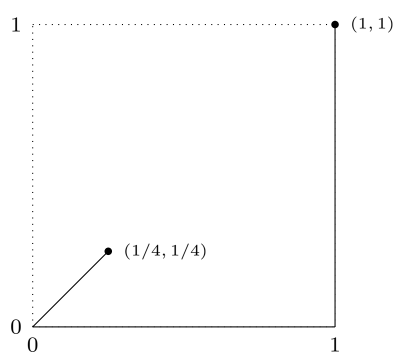

Figure 3.7: $G(f)$ for Example 3.4.4

CHAPTER FOUR

Connecting the Dynamics of the Bonding Function with the Structure of the Inverse Limit

Now that we have proven this class of bonding functions may be used to construct connected inverse limits, it is natural to further study the properties of their inverse limits. In this chapter, we explore the relationship between the topological structure of an inverse limit generated by a single bonding function on $[0,1]$ with the intermediate value property and the dynamics of the shift map on the inverse limit. Barge and Martin examined the relationship between the dynamics of a continuous function $f : [0,1] \to [0,1]$ and the dynamics of the shift map on the corresponding inverse limit [6, 7]. They also examined the relationship between the dynamics of f and the topological structure of the corresponding inverse limit [5], [8] . Two of their results are of primary concern:

Theorem 4.0.1 . [5, Theorem 1] If f has a periodic point whose period is not a power of two, then the inverse limit has an indecomposable subcontinuum that is invariant under the shift.

Theorem 4.0.2 . [5, Theorem 7] If f is organic and the inverse limit is indecomposable, then f has a periodic cycle whose period is not a power of two.

A particular area of interest in the study of generalized inverse limits has been to identify and analyze circumstances that give rise to indecomposable subcontinua in the inverse limit. Ingram, James P. Kelly, Jonathan Meddaugh, and Scott Varagona have all written on the subject, using the full-projection property as a crucial tool to demonstrate indecomposability [29], [33], [34], [47].

In Section 4.1, we establish the full-projection property for inverse limits of surjective, light, almost nonfissile, upper-semicontinuous functions with the intermediate value property (Theorem 4.1.12) and for all subcontinua whose projections are nondegenerate (Theorem 4.1.13). A key aspect of that role is to establish the full-projection property. The main result, stated here, follows from Theorems 4.2.3 and 4.2.8.

Then in Section 4.2, we generalize Theorems 4.0.1 and 4.0.2 as Theorems 4.2.3 and 4.2.8 respectively, in which the existence of periodic cycles with period not a power of two in the bonding function gives rise to indecomposability in the inverse limit and vice versa, respectively.

4.1 Full-Projection Property

In this section we consider the full-projection property for inverse limits of upper-semicontinuous functions with the intermediate value property. It is shown elsewhere [44] that an inverse limit with upper-semicontinuous bonding functions has the full-projection property if and only if its nonfissile points constitute a dense G_δ subset of the inverse limit. In light of this, it is reasonable to wonder whether an equivalent or even sufficient condition might be to require that the bonding functions of the inverse limit be almost nonfissile. Alone, almost nonfissile does not suffice; in tandem with surjectivity, lightness, and the intermediate value property, it does. Theorem 4.1.12 establishes this, and Theorem 4.1.13 provides a generalization, that any subcontinuum with nondegenerate projections in all coordinates may also be written as an inverse limit with the full-projection property by restricting the bonding functions appropriately. These are the main results of Section 4.1.2

In Section 4.1.1, we present results intended to provide intuition regarding the structure of almost nonfissile functions and their graphs. In Proposition 4.1.4, it is shown that an upper-semicontinuous interval function is almost nonfissile if and only if it is irreducible with respect to domain. B.R. Williams [49] defined "irreducible with respect to domain" to study the full-projection property. Iztok Banič, Matevž Črepnjak, Matej Merhar, and Uroš Milutinović [4] studied the property further and introduced variations to Williams's definition.

4.1.1 Equivalence of Almost Nonfissile to Irreducibility with Respect to Domain

Definition 4.1.1 . Let X and Y be metric spaces and $f : X \to 2^Y$. A point $x \in X$ is a *fissile* point of f if $|f(x)| > 1$ and a *nonfissile* point otherwise, i.e. $f(x) = \{y\}$.

A point $(x, y) \in G(f)$ is a *fissile* point of $G(f)$ if x is a fissile point of f and a *nonfissile* point of $G(f)$ otherwise.

The function f is *almost nonfissile* if the set of nonfissile points of $G(f)$ is a dense G_δ subset of $G(f)$.

Let $\{X_i, f_i\}$ be an inverse sequence. A point $(x_0, x_1, x_2, ...) \in \varprojlim\{X_i, f_i\}$ is a *fissile* point of $\varprojlim\{X_i, f_i\}$ if x_i is a fissile point of f_i for some i and a *nonfissile* point of $\varprojlim\{X_i, f_i\}$ otherwise.

The requirement that a function $f : X \to 2^Y$ be almost nonfissile is not equivalent to the requirement that the set of nonfissile points of f be a dense G_δ subset of X. For example, consider the function $f : [0, 1] \to 2^{[0,1]}$ defined by $f(x) = 0$ if $x \neq 1$ and

$f(1) = [0,1]$. The set of nonfissile points of f is the interval $[0,1)$ which is a dense G_δ subset of $[0,1]$, but the set of nonfissile points of $G(f)$ is $[0,1) \times \{0\}$ which is not dense in $G(f)$.

However, it is true that if f is almost nonfissile, then the set of nonfissile points of f is a dense G_δ set in X. It is straight forward to show density, and it is shown in Lemma 4.1.2 that the set of nonfissile points of f is a G_δ set.

The set of fissile points of $f : X \to 2^Y$, the set of fissile points of $G(f)$, and the set of fissile points of an inverse limit are all F_σ sets. The first and last of these is proved in [44]. The first is also a consequence of Lemma 4.1.2.

Lemma 4.1.2 . Let $f : [0,1] \to 2^{[0,1]}$ be an upper-semicontinuous function. Then the set of nonfissile points of f is a G_δ subset of $[0,1]$. If int $G(f) = \emptyset$, then it is a dense G_δ subset of $[0,1]$.

Proof. Define $A = \{x \in [0,1] : |f(x)| > 1\}$, and, for each $n \in \mathbb{N}$, define

$$D_n = \left\{ x \in [0,1] : \text{diam } f(x) \geq \frac{1}{n} \right\}.$$

As f is upper-semicontinuous, D_n is closed for each n. Note that $A = \bigcup_{n \in \mathbb{N}} D_n$, making A an F_σ set. It follows that the set of nonfissile points of f is a G_δ set.

We prove the second statement by contraposition. To that end, suppose the set of nonfissile points of f is not dense or, equivlaently, that A is nonmeager. Then there is some fixed n such that D_n is not nowhere dense, i.e. int $D_n \neq \emptyset$. So there is some nondegenerate interval $[a,b] \subseteq D_n$.

Let $\epsilon = \inf_{x\in[a,b]} \text{diam } f(x) \geq \frac{1}{n}$. Then for any $\eta > 0$, there exists $z \in [a,b]$ such that $\epsilon \leq \text{diam } f(z) < \epsilon + \eta$. In particular, for $\eta = \frac{\epsilon}{8}$, there is a $z \in [a,b]$ such that $\text{diam } f(z) < \frac{9\epsilon}{8}$. We assume the case $z \in [a,b)$, as the argument for $z = b$ follows a similar argument. Let $c = \min f(z)$ and $d = \max f(z)$. Since f is upper-semicontinuous, there is some $\delta > 0$ such that if $x \in (z, z+\delta)$, then $f(x) \subseteq \left(c - \frac{\epsilon}{8}, d + \frac{\epsilon}{8}\right)$.

Let $x \in (z, z+\delta)$. As $\text{diam } f(x) \geq \epsilon$, $f(x) \subseteq \left(c - \frac{\epsilon}{8}, d + \frac{\epsilon}{8}\right)$, and $\text{diam}\left(c - \frac{\epsilon}{8}, d + \frac{\epsilon}{8}\right) < \frac{11\epsilon}{8}$, $f(x) \supseteq \left[c + \frac{\epsilon}{4}, d - \frac{\epsilon}{4}\right]$, an interval with nonempty interior. As x was arbitrary, the set

$$U = \left\{(x,y) : z < x < z+\delta \text{ and } y \in \left(c + \frac{\epsilon}{4}, d - \frac{\epsilon}{4}\right)\right\}$$

is an open subset of $G(f)$, so $G(f)$ has nonempty interior. Therefore, by contraposition, if $\text{int } G(f) = \emptyset$, then A is meager. So the set of points in $[0,1]$ on which f is single-valued is a dense G_δ. $\square$

Definition 4.1.3 . A function $f : [0,1] \to 2^{[0,1]}$ is *irreducible with respect to domain* if no closed subgraph of $G(f)$ has full domain, that is, $\pi_1[H] \neq [0,1]$ for every closed set $H \subsetneq G(f)$.

Proposition 4.1.4 . Let $f : [0,1] \to 2^{[0,1]}$ be an upper-semicontinuous function. Then f is almost nonfissile if and only if f is irreducible with respect to domain.

Proof. First note that if $\text{int } G(f) \neq \emptyset$, then f is neither almost nonfissile nor irreducible with respect to domain. Suppose $\text{int } G(f) = \emptyset$. Let $\text{Fi}(f)$ be the set of fissile points of $G(f)$ and $A = G(f) \setminus \text{Fi}(f)$, i.e. the set of nonfissile points of $G(f)$. By

Lemma 4.1.2, $\pi_1[A]$ is a dense G_δ subset of $[0,1]$. Then $\overline{A}$ is a closed subgraph of $G(f)$ with full domain. So if f is irreducible with respect to domain, $\overline{A} = G(f)$, making f almost nonfissile. Conversely, if f is almost nonfissile, then as A is composed of nonfissile points, any closed subgraph with full domain must contain A and hence contains $\overline{A}$. Thus if f is almost nonfissile and H is a closed subgraph of $G(f)$ with full domain, $H \supseteq \overline{A} = G(f)$, making f irreducible with respect to domain. □

4.1.2 The Full-projection Property in Inverse Limits of maps with the Intermediate Value Property

Theorem 4.1.5 . (Ryden, [44]) Suppose $\{X_n, f_n\}$ is an inverse sequence and $X = \varprojlim\{X_n, f_n\}$. Then X has the full-projection property if and only if the set of fissile points of X is a meager F_σ set.

Lemma 4.1.6 . Suppose $f : [0,1] \to 2^{[0,1]}$ is a surjective, almost nonfissile, upper-semicontinuous map with the intermediate value property. If $f(x)$ is nondegenerate for some interior point x of $[0,1]$, then there are sequences L_1, L_2, ..., and R_1, R_2, ... of nondegenerate closed subintervals of $[0,1]$ such that

1. $z < x$ for all $z \in \bigcup L_n$ and $z > x$ for all $z \in \bigcup R_n$,
2. $\lim L_n = \{x\}$ and $\lim R_n = \{x\}$,
3. $\lim f[L_n] = f(x)$ and $\lim f[R_n] = f(x)$.

Proof. We construct the sequence L_1, L_2,... only and note that the construction of R_1, R_2,... is similar. Let a and b denote the points such that $f(x) = [a,b]$. Since upper-semicontinuous maps with the intermediate value property are weakly continuous by Theorem 3.2.6, there are sequences α_1, α_2,...; β_1, β_2,...; a_1, a_2,...; and b_1, b_2, ...such that each of the following is true:

- $a_n \in f(\alpha_n)$ for all n,
- $b_n \in f(\beta_n)$ for all n,
- $\{\alpha_n\}$ and $\{\beta_n\}$ converge to x from the left,
- $\{a_n\}$ and $\{b_n\}$ converge to a and b respectively.

Furthermore, since f is almost nonfissile, the sequences may be chosen so that $f(\alpha_n) = \{a_n\}$ and $f(\beta_n) = \{b_n\}$. It follows that, for sufficiently large n, a_n and b_n are distinct. Finally, taking subsequences if necessary, the sequences may be chosen so that $a_n < \frac{a+b}{2} < b_n$ for each $n \in \mathbb{N}$ and $\alpha_n, \beta_n < \alpha_{n+1}, \beta_{n+1}$ for each $n \in \mathbb{N}$.

Since $a_n \neq b_n$ for each $n \in \mathbb{N}$, it follows that $\alpha_n \neq \beta_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define L_n to be the nondegenerate closed interval with endpoints α_n and β_n. Then L_1, L_2,...satisfies (1) and (2). To see that it satisfies (3), note that $\lim\ \inf f[L_n]$ contains both a and b, and hence $f(x)$, since $a_n \to a$ and $b_n \to b$ as $n \to \infty$. On the other hand, $\lim\ \sup f[L_n] \subseteq f(x)$ since the graph of f is closed. Hence $\lim f[L_n] = f(x)$, and $\{L_n\}$ satisfies (3). $\square$

Lemma 4.1.7 . Suppose $f : [0,1] \to 2^{[0,1]}$ is an almost nonfissile upper-semicontinuous function with the intermediate value property. If $y \in f(x)$, and D_x and D_y are open sets such that $y \in D_y$ and $x \in \overline{D_x}$, then there is an open subset D of D_x such that $f[D] \subset D_y$.

Proof. Since f is weakly continuous from both the left and the right by Theorem 3.2.6, there is a point $x_1 \in D_x$ such that $f(x_1)$ intersects D_y. Since f is almost nonfissile, there is a nonfissile point $x_2 \in D_x$, i.e. that $f(x_2) = \{y_2\} \subseteq D_y$. Put $D = \{x \in [0,1] : f(x) \subset D_y\} \cap D_x$. Then D is a nonempty open subset of D_x that contains x_2. Furthermore, $f[D] \subset D_y$. □

Lemma 4.1.8 . Suppose $f : [0,1] \to 2^{[0,1]}$ is a light, almost nonfissile, upper-semicontinuous map with the intermediate value property. If G is a G_δ subset of D for some open subset D of $[0,1]$ then $\{x \in [0,1] : f(x) \subseteq G\}$ is a G_δ subset of $[0,1]$. Furthermore, if G is dense in D, then $\{x \in [0,1] : f(x) \subseteq G\}$ is dense in $\{x \in [0,1] : f(x) \subseteq D\}$.

Proof. There are open sets G_1, G_2, …such that $\bigcap G_n = G$. Since f is upper-semicontinuous, $\{x \in [0,1] : f(x) \subset G_n\}$ is open in $[0,1]$. Note that $\bigcap\{x \in [0,1] : f(x) \subset G_n\} = \{x \in [0,1] : f(x) \subset \bigcap G_n\} = \{x \in [0,1] : f(x) \subset G\}$. It follows that $\{x \in [0,1] : f(x) \subset G\}$ is a G_δ set.

Suppose further that G is dense in some open set D. Replacing G_n with $G_n \cap D$ for each $n \in \mathbb{N}$ if necessary, the open sets G_n may be taken to be open subsets of D for which $\bigcap G_n = G$. Note that G_n is dense in D for each $n \in \mathbb{N}$. Suppose U is an open interval in $\{x \in [0,1] : f(x) \subset D\}$. Since f is light and has the intermediate value property, $f[U]$ is a nondegenerate interval in D. Then int $f[U]$ contains a point of G_n. It follows that there is a point u of U and a point w of int $f[U] \cap G_n$ such that $w \in f(u)$. Since f is almost nonfissile, u and w may be chosen so that $f(u) = \{w\}$. It follows that G_n contains $f(u)$ and U contains a point of $\{x \in [0,1] : f(x) \subset G_n\}$.

Hence $\{x \in [0,1] : f(x) \subset G_n\}$ is a dense open subset of $\{x \in [0,1] : f(x) \subset D\}$. As this is true for each $n \in \mathbb{N}$, $\{x \in [0,1] : f(x) \subset G\}$ is a dense G_δ subset of $\{x \in [0,1] : f(x) \subset D\}$. $\square$

Definition 4.1.9 . A function $f : X \to 2^Y$ is *light* if for every $y \in [0,1]$, the set $\{x \in [0,1] : y \in f(x)\}$ has no interior.

Lemma 4.1.10 . Suppose $f : [0,1] \to 2^{[0,1]}$ is a surjective, light, almost nonfissile, upper-semicontinuous map with the intermediate value property. If $y \in f(x)$ and D_y is an open set such that $y \in \overline{D_y \cap (-\infty, y)} \cap \overline{D_y \cap (y, \infty)}$, then there is an open set D_x such that $x \in \overline{D_x \cap (-\infty, x)} \cap \overline{D_x \cap (x, \infty)}$ and such that $f[D_x] \subset D_y$.

Proof. First suppose $f(x)$ is nondegenerate. Then, by the Lemma 4.1.6, there are sequences L_1, L_2,... and R_1, R_2,... of nondegenerate closed subintervals of $[0,1]$ such that

1. $z < x$ for all $z \in \bigcup L_n$, and $z > x$ for all $z \in \bigcup R_n$,
2. $\lim L_n = \{x\}$, and $\lim R_n = \{x\}$, and
3. $\lim f[L_n] = f(x)$ and $\lim f[R_n] = f(x)$.

Since $f(x)$ is a nondegenerate interval containing y, at least one of $f(x) \cap (-\infty, y)$ and $f(x) \cap (y, \infty)$ is a nondegenerate interval with one endpoint equal to y, say $f(x) \cap (y, \infty)$. Since $y \in \overline{D_y \cap (y, \infty)}$, every open interval whose left endpoint is y contains a point of $D_y \cap (y, \infty)$. It follows that $\mathrm{int} f(x) \cap (y, \infty)$ contains a point of $D_y \cap (y, \infty)$. Hence $\mathrm{int} f(x) \cap D_y \cap (y, \infty)$ contains an open interval (y_1, y_2); furthermore, y_1 and

y_2 may be chosen so that neither of them is an endpoint of $f(x)$. Since $f[L_n]$ and $f[R_n]$ are connected for each $n \in \mathbb{N}$ by dint of the intermediate value property and since $\lim f[L_n] = \lim f[R_n] = f(x)$, it follows that there is $N \in \mathbb{N}$ such that $f[L_n]$ and $f[R_n]$ both contain (y_1, y_2) for each $n \geq N$. Hence, for each $n \geq N$, some point of L_n has an image that intersects (y_1, y_2). As f is weakly continuous from both the left and the right by Theorem 3.2.6, there are, for each $n \geq N$, points $l_n \in \text{int } L_n$ and $\tilde{l}_n \in (y_1, y_2)$ such that $\tilde{l}_n \in f(l_n)$. Furthermore, since f is almost nonfissile, l_n and $\tilde{l}_n$ may be chosen so that l_n is a nonfissile point of f. Then (y_1, y_2) is an open set containing $f(l_n)$. Hence $\{x \in [0,1] : f(x) \subset (y_1, y_2)\}$ is an open set containing l_n. For each $n \geq N$, put $U_n = \text{int } L_n \cap \{x \in [0,1] : f(x) \subset (y_1, y_2)\}$. Then $U_n \subset L_n$, and $f[U_n] \subset (y_1, y_2) \subset D_y$. Similarly, for $n \geq N$, there are open sets $V_n \subset R_n$ such that $f[V_n] \subset D_y$. Finally, put $D_x = (\bigcup_{n \geq N} U_n) \cup (\bigcup_{n \geq N} V_n)$. Note that $f[D_x] \subset D_y$. Thus it remains only to show that $x \in \overline{D_x \cap (-\infty, x)} \cap \overline{D_x \cap (x, \infty)}$.

To that end note that, by (1) and the fact that $U_n \subset L_n$ and $V_n \subset R_n$ for each $n \geq N$, we have $D_x \cap (-\infty, x) = \bigcup_{n \geq N} U_n$ and $D_x \cap (x, \infty) = \bigcup_{n \geq N} V_n$. It follows from (2) that $x \in \overline{\bigcup_{n \geq N} U_n}$ and $x \in \overline{\bigcup_{n \geq N} V_n}$. Consequently, $x \in \overline{D_x \cap (-\infty, x)} \cap \overline{D_x \cap (x, \infty)}$.

Now suppose $f(x)$ is degenerate, that is, suppose $f(x) = \{y\}$. Suppose $n \in \mathbb{N}$ is given, and consider the interval $\left(x, x + \frac{1}{n}\right)$. Since f is light and uppersemicontinuous, $f\left(x, x + \frac{1}{n}\right)$ is a nondegenerate interval. Since the graph of f is closed, $y \in \overline{f\left(x, x + \frac{1}{n}\right)}$. Since $f\left(x, x + \frac{1}{n}\right)$ is an interval, this is equivalent to $y \in \overline{\text{int } f\left(x, x + \frac{1}{n}\right)}$. It follows that $\text{int } f\left(x, x + \frac{1}{n}\right) \cap D_y$ is nonempty. By Lemma 4.1.7, there is an open subset V_n of $\left(x, x + \frac{1}{n}\right)$ such that $f[V_n] \subset D_y$. Similarly, there is an open subset U_n of $\left(x - \frac{1}{n}, x\right)$ such that $f[U_n] \subset D_y$. Thus U_n and V_n

are defined for $n \in \mathbb{N}$. Put $D_x = (\bigcup_{n\in\mathbb{N}} U_n) \cup (\bigcup_{n\in\mathbb{N}} V_n)$. Then $f[D_x] \subset D_y$ and $x \in \overline{(\bigcup_{n\in\mathbb{N}} U_n)} \cap \overline{(\bigcup_{n\in\mathbb{N}} V_n)} = \overline{D_x \cap (-\infty, x)} \cap \overline{D_x \cap (x, \infty)}$. $\square$

Lemma 4.1.11 . Suppose $\{[0,1], f_n\}$ is an inverse sequence where, for each $n \in \mathbb{N}$, f_n is a surjective almost nonfissile, light, upper-semicontinuous map with the intermediate value property. For each $N \in \mathbb{N}$, if $x \in \varprojlim\{[0,1], f_n\}$ and U_0, U_1, $\ldots$, U_N are open sets containing x_0, x_1, $\ldots$, x_N respectively, then there are open subsets D_0, D_1, $\ldots$, D_N of U_0, U_1, $\ldots$, U_N respectively such that

1. $x_n \in \overline{D_n \cap (-\infty, x_n)} \cap \overline{D_n \cap (x_n, \infty)}$ for $n = 0, 1, \ldots, N$,

2. $f_n[D_n] \subset D_{n-1}$ for $n = 1, 2, \ldots, N$, and

3. z_N, $f^N_{N-1}(z_N)$, $\ldots$, $f^N_1(z_N)$ are nonfissile points of f_N, f_{N-1}, $\ldots$, f_1 respectively for all z_N in some comeager subset of D_N.

Proof. The proof is by induction. First consider $N = 1$. Suppose $x \in \varprojlim\{[0,1], f_n\}$, and suppose U_0 and U_1 are open sets containing x_0 and x_1 respectively. Put $D_0 = U_0$. Note that D_0 satisfies the requirement in (1). By the Lemma 4.1.10, there is an open set $\tilde{D}_1$ such that $x_1 \in \overline{\tilde{D}_1 \cap (-\infty, x_1)} \cap \overline{\tilde{D}_1 \cap (x_1, \infty)}$ and $f[\tilde{D}_1] \subset D_0$. Put $D_1 = U_1 \cap \tilde{D}_1$. Then D_0 and D_1 satisfy (1) and (2). The set of nonfissile points of f_1 is a G_δ subset of $[0,1]$ by Lemma 4.1.2 and dense in $[0,1]$ since f_1 is almost nonfissile. Since D_1 is open, the set of nonfissile points of f_1 that lie in D_1 is a comeager subset of D_1. Hence (3) holds, and the result is true for $N = 1$.

Suppose that the result is true for $N = k$ for some $k \geq 1$, and consider $n = k + 1$. Suppose $x \in \varprojlim\{[0,1], f_n\}$, and suppose U_0, U_1, $\ldots$, U_{k+1} are open sets containing

x_1, x_2, …, x_{k+1} respectively. Since the result holds for $N = k$, there are open subsets D_0, D_1, …, D_k of U_0, U_1, …, U_k that satisfy (1), (2), and (3). By Lemma 4.1.10, there is an open set D_{k+1} such that $x_{k+1} \in \overline{D_{k+1} \cap (-\infty, x_{k+1})} \cap \overline{D_{k+1} \cap (x_{k+1}, \infty)}$ and $f_{k+1}[D_{k+1}] \subset D_k$. Replacing D_{k+1} with $D_{k+1} \cap U_{k+1}$ if necessary, we may assume $D_{k+1} \subset U_{k+1}$. Note that D_{k+1} satisfies (1) and (2). Thus it remains to show that D_{k+1} satisfies (3).

For each $n \in \mathbb{N}$, denote the set of fissile points of f_n by $\mathrm{Fi}(f_n)$. By Lemma 4.1.2, $\mathrm{Fi}(f_n)$ is an F_σ set for each $n = 1, 2, \ldots, k+1$. Since f_n^{k+1} is upper-semicontinuous for each n, $(f_n^{k+1})^{-1}(\mathrm{Fi}(f_n))$ is an F_σ set for each $n = 1, 2, \ldots, k+1$. Hence $\bigcup_{n=1}^{k+1} (f_n^{k+1})^{-1}(\mathrm{Fi}(f_n))$ is an F_σ set. Equivalently, $\{z \in D_{k+1} : z, f_k^{k+1}(z), \ldots, f_1^{k+1}(z)$ are nonfissile points of f_k, f_{k-1}, …, f_1 respectively$\}$ is a G_δ set. Denote it by A, and note that $A \cap D_{k+1}$ is a G_δ subset of D_{k+1}. To see that $A \cap D_{k+1}$ is dense in D_{k+1}, suppose D is an open interval in D_{k+1}. Since f_{k+1} is light and has the intermediate value property, $f_{k+1}[D]$ is a nondegenerate interval in D_k. Since D_k satisfies (3), $\{z \in D_k : z, f_{k-1}^k(z), \ldots, f_1^k(z)$ are nonfissile points of f_k, f_{k-1}, …, f_1 respectively$\}$ is a dense G_δ set in D_k. Denote this set by G. Then, by Lemma 4.1.8, $\{x \in [0,1] : f_{k+1}(x) \subset G\}$ is a dense G_δ set in $\{x \in [0,1] : f_{k+1}(x) \subset D_k\}$. Since $D_{k+1} \subset \{x \in [0,1] : f_{k+1}(x) \subset D_k\}$, it follows that $D_{k+1} \cap \{x \in [0,1] : f_{k+1}(x) \subset G\}$ is a dense G_δ subset of D_{k+1}. The set of nonfissile points of f_{k+1} in D_{k+1} is also a dense G_δ subset of D_{k+1} by Lemma 4.1.2 and the fact that f_{k+1} is almost nonfissile. Put $A = ([0,1] - \mathrm{Fi}(f_{k+1})) \cap D_{k+1} \cap \{x \in [0,1] : f_{k+1}(x) \subset G\}$. Then A is a dense G_δ subset of D_{k+1}, and, for each $z \in A$, z, $f_k^{k+1}(z)$, $f_{k-1}^{k+1}(z)$, …, $f_1^{k+1}(z)$ are nonfissile points of f_{k+1}, f_k, $\dots$, f_1 respectively. Hence D_{k+1} satisfies (3), and the inductive step is complete. $\square$

Theorem 4.1.12 . Suppose $\{[0,1], f_n\}$ is an inverse sequence where, for each $n \in \mathbb{N}$, $f_n : [0,1] \to 2^{[0,1]}$ is a surjective, light, almost nonfissile, upper-semicontinuous map with the intermediate value property. Then $\varprojlim\{[0,1], f_n\}$ has the full-projection property.

Proof. Denote $\varprojlim\{[0,1], f_n\}$ by X. By Theorem 4.1.5, it suffices to show that the set of nonfissile points of X is dense in X. For each $n \in \mathbb{N}$, denote $\{x \in X : |f_n(x_n)| = 1\}$ by $\sim \mathrm{Fi}_n(X)$, and note that the set of nonfissile points of X is $\sim \mathrm{Fi}_1(X) \cap \sim \mathrm{Fi}_2(X) \cap \sim \mathrm{Fi}_3(X) \cap ....$ Since $\sim \mathrm{Fi}_n(X)$ is a G_δ subset of X for each n, it suffices to show that $\sim \mathrm{Fi}_1(X) \cap \sim \mathrm{Fi}_2(X) \cap \cdots \cap \sim \mathrm{Fi}_n(X)$ is dense in X for each $n \geq 1$.

To that end, suppose n is given and D is a nonempty basic open set in X. Then D has the form $D = D_1 \times D_2 \times \cdots \times D_m \times [0,1] \times [0,1] \times \dots$ where D_i is an open subset of $[0,1]$ for $i = 1, 2, \dots, m$, and where $m \geq n$. We must show that D contains a point of $\sim \mathrm{Fi}_1(X) \cap \sim \mathrm{Fi}_2(X) \cap \cdots \cap \sim \mathrm{Fi}_n(X)$, to which end it suffices to show that D contains a point of $\sim \mathrm{Fi}_1(X) \cap \sim \mathrm{Fi}_2(X) \cap \cdots \cap \sim \mathrm{Fi}_m(X)$. This is a consequence of Lemma 4.1.11. $\square$

Theorem 4.1.13 . Suppose $\{[0,1], f_n\}$ is an inverse sequence where, for each $n \in \omega$, $f_n : [0,1] \to 2^{[0,1]}$ is a surjective, light, almost nonfissile, upper-semicontinuous map with the intermediate value property. Let K be a subcontinuum of $\varprojlim\{[0,1], f\}$ such that $\pi_n[K]$ is nondegenerate for each n. Then K can be written as the inverse limit of its projections and has the full-projection property.

Proof. For each $n \in \omega$, let $K_n = \pi_n[K]$. Then f_n maps $\pi_{n+1}[K]$ onto $\pi_n[K]$. Denote by f'_n the restriction of f_n, $f_n|_{\pi_{n+1}[K]}^{\pi_n[K]} : \pi_{n+1}[K] \to 2^{\pi_n[K]}$. Note f'_n inherits the properties of f_n given in the hypothesis.

Define $K' = \varprojlim\{\pi_n[K], f'_n\}$. Then K' is a subcontinuum of $\varprojlim\{[0,1], f_n\}$ by Theorem 3.3.5 and has the full-projection property by Theorem 4.1.12.

To show $K' = K$, let $x \in K$. Then for all n, $\pi_n(x) \in \pi_n(K)$ and $\pi_n(x) \in f(\pi_{n+1}(x))$ for all n. So $\pi_n(x) \in f'_n(\pi_{n+1}(x))$, i.e. $x \in K'$. Therefore $K \subseteq K'$. But $\pi_n(K) = \pi_n(K')$ for all n. Then as K' has the full-projection property, $K' = K$. $\square$

4.2 Relationship Between Periodicity and Indecomposability

We now turn to the connection between periodicity in an upper-semicontinuous function $f : [0,1] \to 2^{[0,1]}$ with the intermediate value property and indecomposability in the corresponding inverse limit. In particular, we generalize a connection established in the classical setting by Barge and Martin [5, Theorems 1 & 7].

In Section 4.2.1, we examine how a periodic cycle of f with period not a power of two gives rise to an indecomposable subcontinuum of the inverse limit. The primary result is Theorem 4.2.3. The proof leans heavily on the intermediate value property, appealing to both the Sharkovskii order and the full-projection property, each of which holds in a context involving the intermediate value property (Theorems 4.1.12 and 2.5.3).

We then explore a pseudo converse in Section 4.2.2, that is, how the indecomposability of $\varprojlim\{[0,1], f\}$ gives rise to a periodic cycle of f with period not a power of two. This subsection focuses on organic maps and has Theorem 4.2.8 as its main result.

4.2.1 Periodicity Giving Rise to Indecomposability

Lemma 4.2.1 . Let $f : [0,1] \to 2^{[0,1]}$ be upper-semicontinuous, surjective, and almost nonfissile and $G(f)$ be connected and have empty interior. If there is some $y \in [0,1]$ and a nondegenerate interval $I \subseteq [0,1]$ such that $y \in f(x)$ for every $x \in I$, then f is constant and single valued on I and $f[I] = \{y\}$.

Proof. Let $x \in I$ and $y' \in f(x)$. Then either $x > \inf I$ or $x < \sup I$. The two cases proceed similarly, so we shall prove the result for $x > \inf I$. As f is weakly continuous, there is a sequence $\{(x_n, y_n)\}_{n\in\omega}$ in $G(f)$ converging to (x, y') such that $x_n \in I$ and $x_n < x$. Since f is almost nonfissile, we may choose each (x_n, y_n) so that x_n is a nonfissile point of f. Thus $f(x_n) = \{y_n\}$ for all n. But $y \in f(x_n)$, so $y_n = y$ for all n. As $y_n \to y'$, this implies $y' = y$. As x and y' were arbitrary, $f[I] = \{y\}$. □

Theorem 4.2.2 . Let $f : [0,1] \to 2^{[0,1]}$ be upper-semicontinuous, surjective, almost nonfissile, light, and have the intermediate value property, and $G(f)$ have empty interior. If f has an orbit of period not a power of 2, then $\varprojlim\{[0,1], f\}$ contains an indecomposable subcontinuum.

A natural question arising from this theorem is whether it is necessary to assume f is almost nonfissile and light in order to guarantee the inverse limit contains an indecomposable subcontinuum. In Chapter Five, we construct an upper-semicontinuous, surjective function f with the intermediate value property that is not almost nonfissile and has a hereditarily decomposable inverse limit even though it has periodic points of all periods. Thus the additional assumption that f is almost nonfissile cannot be dropped from Theorem 4.2.2.

Proof. Suppose f has an orbit of period $n \cdot 2^k$. By the Theorem 2.5.3, there is a periodic orbit of f with period $3 \cdot 2^{k+1}$. Let $x \in \varprojlim\{[0,1], f\}$ be the point that models this orbit. Then $(x_{3\cdot 2^{k+1}-1}, \ldots, x_1, x_0)$ is a cycle f of period $3 \cdot 2^{k+1}$. Let $h : \varprojlim\{[0,1], f\} \to \varprojlim\{[0,1], f\}$ be the forgetful shift. Then x has a period 3 orbit under $h^{2^{k+1}}$, namely $\left(x, h^{2^{k+1}}(x), h^{2^{k+2}}(x)\right)$. To show this, suppose to the contrary that x does not have a period 3 orbit. By the construction of x, $h^{3\cdot 2^{k+1}}(x) = x$. So either $h^{2^{k+1}}(x) = x$ or $h^{2^{k+2}}(x) = x$. If $h^{2^{k+1}}(x) = x$, then for all n, $x_{n+2^{k+1}} = x_n$, contradicting the fact $(x_0, x_1, \ldots, x_{3\cdot 2^{k+1}-1})$ is an orbit of period $3 \cdot 2^{k+1}$. By a similar argument, $h^{2^{k+2}}(x) \neq x$. Thus x has an orbit of period 3 under $h^{2^{k+1}}$.

Let S be a subcontinuum of $\varprojlim\{[0,1], f\}$ that is irreducible about x, $h^{2^{k+1}}(x)$, and $h^{2^{k+2}}(x)$. By Theorem 4.1.13, there are restrictions f'_n of f such that each f'_n inherits the properties of f listed in the hypothesis, $S = \varprojlim\{\pi_n(S), f'_n\}$, and S has the full-projection property. We show that S is indecomposable by showing it is irreducible about any two points of x, $h^{2^{k+1}}(x)$, and $h^{2^{k+2}}(x)$.

By way of contradiction, suppose S is not irreducible between two points of $\{x, h^{2^{k+1}}(x), h^{2^{k+2}}(x)\}$, say x and $h^{2^{k+1}}(x)$. Then there is a proper subcontinuum

$H \subsetneq S$ containing x and $h^{2^{k+1}}(x)$. So $h^{2^{k+2}}(x) \notin H$ as S is irreducible about x, $h^{2^{k+1}}(x)$, and $h^{2^{k+2}}(x)$.

Since $(x_{3\cdot 2^{k+1}-1}, \ldots, x_1, x_0)$ is a cycle f of period $3 \cdot 2^{k+1}$, there is some $i \in \{0, 1, \ldots, 2^{k+1}-1\}$ such that for all $n \in \mathbb{N}$, $\pi_{3n\cdot 2^{k+1}+i}(x) \neq \pi_{3n\cdot 2^{k+1}+i}(h^{2^{k+1}}(x))$. As $h^{2^{k+1}}$ permutes x, $h^{2^{k+1}}(x)$, and $h^{2^{k+2}}(x)$, there is some $j \in \{0, 1, 2\}$ such that for all $n \in \mathbb{N}$, $\pi_{(3n+j)2^{k+1}+i}(h^{2^{k+2}}(x))$ is between $\pi_{(3n+j)2^{k+1}+i}(x)$ and $\pi_{(3n+j)2^{k+1}+i}(h^{2^{k+1}}(x))$. Furthermore, $\pi_{(3n+j)2^{k+1}+i}(h^{2^{k+2}}(x))$ is distinct from at least one of $\pi_{(3n+j)2^{k+1}+i}(x)$ or $\pi_{(3n+j)2^{k+1}+i}(h^{2^{k+1}}(x))$. So $\pi_{(3n+j)2^{k+1}+i}[H]$ is nondegenerate for each n, and $\pi_{(3n+j)2^{k+1}+i}(h^{2^{k+2}}(x)) \in \pi_{(3n+j)2^{k+1}+i}[H]$. As f is weakly continuous and almost nonfissile, there is a sequence of nonfissile points $\{(x_k, y_k)\}_{k\in\mathbb{N}}$ of $G(f)$ such that $x_k \in \pi_{(3n+j)2^{k+1}+i}[H]$ and

$$(x_k, y_k) \to (\pi_{(3n+j)2^{k+1}+i}(h^{2^{k+2}}(x)), \pi_{(3n+j)2^{k+1}+i-1}(h^{2^{k+2}}(x))).$$

Since $f(x_k) = \{y_k\}$, $y_k \in \pi_{(3n+j)2^{k+1}+i-1}[H]$. Then $\pi_{(3n+j)2^{k+1}+i-1}(h^{2^{k+2}}(x)) \in \pi_{(3n+j)2^{k+1}+i-1}[H]$ because $y_k \to \pi_{(3n+j)2^{k+1}+i-1}(h^{2^{k+2}}(x))$ and $\pi_{(3n+j)2^{k+1}+i-1}[H]$ is closed. Furthermore, since f is almost nonfissile and light and $\pi_{(3n+j)2^{k+1}+i}[H]$ is nondegenerate, $\pi_{(3n+j)2^{k+1}+i-1}[H]$ is nondegerate by Lemma 4.2.1.

Proceeding inductively, we see that $\pi_l[H]$ is nondegenerate and $\pi_l(h^{2^{k+2}}(x)) \in \pi_l[H]$ for all $l \leq (3n+j)2^{k+1}+i$. As this holds for any $n \in \mathbb{N}$, $\pi_l(h^{2^{k+2}}(x)) \in \pi_l[H]$ for all $l \in \mathbb{N}$. Since H is the inverse limit of its own projections by Theorem 4.1.13, $h^{2^{k+2}}(x) \in H$, a contradiction. Therefore S is irreductible about any two points of $\{x, h^{2^{k+1}}(x), h^{2^{k+2}}(x)\}$ and is indecomposable. $\square$

Theorem 4.2.3 . Suppose $f : [0,1] \to 2^{[0,1]}$ is upper-semicontinuous, surjective, has the intermediate value property, and has an orbit of period not a power of 2. If $f|_{[0,1]\setminus\pi_1(int(G(f)))}$ is almost nonfissile and light, then $\varprojlim\{[0,1], f\}$ contains an indecomposable subcontinuum.

Proof. If int $G(f) = \emptyset$, then the conclusion follows form Theorem 4.2.2. Suppose int $G(f) \neq \emptyset$. Let $(x_0, x_1, \ldots, x_{p-1})$ be a cycle of f where p is not a power of 2. It is sufficient to show there is a map $g : [0,1] \to 2^{[0,1]}$ that is upper-semicontinuous, almost nonfissile, and light, that has the intermediate value property and retains $(x_0, x_1, \ldots, x_{p-1})$ as a periodic cycle, and such that $G(g)$ has empty interior and $G(g) \subseteq G(f)$. Then $\varprojlim\{[0,1], g\}$ is a subcontinuum of $\varprojlim\{[0,1], f\}$ that contains an indecomposable subcontinuum by Theorem 4.2.2.

Note $\pi_1[\text{int } G(f)]$ is an open subset of $[0,1]$. Let $\{O_n\}_{n\in\mathbb{N}}$ be an enumeration of the components of $\pi_1[\text{int } G(f)]$. We construct g as follows: if $x \in [0,1] \setminus \pi_1[\text{int } G(f)]$, let $g(x) = f(x)$. For each n, we construct $G(g)$ on $\overline{O}_n$ to contain any of (x_0, x_1), (x_1, x_2),...,(x_{p-1}, x_0) for which $x_i \in O_n$, and some $(a_n, \max f(\overline{O}_n))$, $(b_n, \min f(\overline{O}_n))$ where $a_n, b_n \in \overline{O}_n$. To that end, let $C = \{x_0, x_1, \ldots, x_{p-1}\} \cup \bigcup_{n\in\mathbb{N}}\{a_n, b_n\}$. Define $g(x)$ to be $f(x)$ if $x \in C$.

Note that for each n, $C \cap O_n$ is finite. Define g on $O_n \setminus C$ to be single-valued and continuous according to the following conditions:

1. $g(x) \subseteq f(x)$,

2. g is light on $O_n \setminus C$ and

3. if x is in C or $\operatorname{bd} O_n$, then for any component U of $O_n \setminus C$ with $x \in \overline{U}$, $\overline{G(g|_U)} \cap (\{x\} \times [0,1]) = \{x\} \times g(x)$.

That g may be light on $O_n \setminus C$ while maintaining $G(g) \subseteq G(f)$ follows from the fact that $O_n \subseteq \pi_1[\text{int } G(f)]$. Regarding (3), since $C \cap O_n$ is finite and f is weakly continuous, g may also be constructed such that as y approaches x from within U, the graph of g is a ray with remainder $g(x)$. Therefore such a map g exists. Note that by (1) and the fact that $g(x) = f(x)$ on C, $g[\overline{O}_n] = f[\overline{O}_n]$.

Note that $(x_0, x_1, \ldots, x_{p-1})$ is a periodic cycle of g. By this construction, g is light and almost nonfissile on each O_n and $G(g) \cap V_{O_n}$ is connected. Note that if $x \in \operatorname{bd} O_n$, condition (3) becomes $\overline{G(g|_U)} \cap (\{x\} \times [0,1]) = \{x\} \times g(x) = \{x\} \times f(x)$. Then since $g|_{[0,1] \setminus \pi_1[\text{int } G(f)]} = f|_{[0,1] \setminus \pi_1[\text{int } G(f)]}$, g is almost nonfissile and light on $[0,1]$, and $G(g)$ is connected. It remains to show g has the intermediate value property. Since $g(x)$ is connected for each $x \in [0,1]$, it is sufficient to show that g is weakly continuous.

We show that g is weakly continuous from the left. The proof that g is weakly continuous from the right is similar. Let $(x, y) \in G(g)$ with $x > 0$. Suppose first $x \in O_n$ for some n. If $x \in C$, then by (3) there is a sequence $\{(x_i, y_i)\}_{i \in \omega}$ in $G(g)$ such that $x_i \in O_n \cap (0, x)$ for all i and $(x_i, y_i) \to (x, y)$. Thus g is weakly continuous at x from the left. If $x \notin C$, then since g is single-valued and continuous on $O_n \setminus C$, it follows that g is weakly continuous at x from the left.

Next suppose $x \notin O_n$ for any n. Then either $x \in [0,1] \setminus \overline{\pi_1[\text{int } G(f)]}$, $x = \sup O_n$ for some n, or there is a subsequence O_{n_k} such that $x > \sup O_{n_k}$ for all k but $x = \sup \bigcup_k O_{n_k}$.

Case 1: Suppose $x \in [0,1] \setminus \overline{\pi_1[\text{int } G(f)]}$. Since f is weakly continuous, there is a sequence $\{(x_i, y_i)\}_{i \in \omega}$ in $G(f)$ such that $x_i < x$ for all i and $(x_i, y_i) \to (x, y)$. Then there is some $N \in \mathbb{N}$ such that for $i \geq N$, $x_i \in [0,1] \setminus \pi_1[\text{int } G(f)]$. Since g agrees with f on $[0,1] \setminus \overline{\pi_1[\text{int } G(f)]}$, $\{(x_i, y_i)\}_{i \geq N}$ is a sequence in $G(g)$ converging to (x, y).

Case 2: Suppose $x = \sup O_n$ for some n. Then by (2), there is a sequence $\{(x_i, y_i)\}_{i \in \omega}$ in $G(g)$ such that $x_i \in O_n$ for all i and $(x_i, y_i) \to (x, y)$.

Case 3: Suppose there is a sequence $\{O_{n_k}\}_{k \in \omega}$ such that $\sup O_{n_k} < x$ and $x = \sup \bigcup_k O_{n_k}$. Note that any such sequence may be ordered so that $O_{n_k} = (c_k, d_k)$ where $d_k < c_{k+1}$, $c_k \to x$, and $d_k \to x$. Then $d_k - c_k \to 0$, i.e. diam $O_{n_k} \to 0$. Let $\{(x_i, y_i)\}_{i \in \omega}$ be a sequence in $G(f)$ such that $x_i < x$ for all i and $(x_i, y_i) \to (x, y)$. Recall that f Define a sequence $\{(x_i', y_i)\}_{i \in \omega}$ in $G(g)$ where x_i' is a point of some O_{n_i} with $y_i \in g(x_i')$ if $x_i \in O_{n_i}$ and $x_i' = x_i$ if $x_i \in [0,1] \setminus \pi_1[\text{int } G(f)]$. Note $d((x_i', y_i), (x_i, y_i)) = |x_i' - x_i| <$ diam O_{n_i} if $x_i \in O_{n_i}$. Let $\epsilon > 0$ and N_1 such that if $i \geq N_1$, $d((x_i, y_i), (x, y)) < \frac{\epsilon}{2}$. Since diam $O_{n_i} \to 0$, there is some N_2 such that if $i \geq N_2$, then diam $O_{n_i} < \frac{\epsilon}{2}$. Then for $i \geq \max\{N_1, N_2\}$,

$$d((x_i', y_i), (x, y)) \leq d((x', y_i), (x_i, y_i)) + d((x_i, y_i), (x, y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Then $(x_i', y_i) \to (x, y)$. Therefore g is weakly continuous from the left. By a similar argument, g is weakly continuous from the right. Thus g is weakly continuous. Then g has the intermediate value property. $\square$

Notation. If $x_1, x_2 \in [a,b]$, let $\overline{x_1 x_2}$ denote the closed interval with endpoints x_1 and x_2.

Definition 4.2.4 . If $f : [a,b] \to 2^{[a,b]}$ is upper-semicontinuous, we say f is *organic* if for every $x, y \in \varprojlim\{[a,b], f\}$ such that $\varprojlim\{[a,b], f\}$ is irreducible between x and y, then there exists $n \in \mathbb{N}$ such that $f^n(\overline{x_n y_n}) = [a,b]$.

Lemma 4.2.5 . Let $f : [0,1] \to 2^{[0,1]}$ be such that f is upper-semicontinuous, surjective, and has the intermediate value property. Further suppose that $\varprojlim\{[0,1], f\}$ is irreducible between x and y. For $k \geq 0$, let $J_k = \overline{\bigcup_{n \geq k} f^{n-k}(\overline{x_n y_n})}$. Then for each k, J_k is a closed subinterval of $[0,1]$ with $f(J_{k+1}) = J_k$.

Proof. Since f has intermediate value property, $x_i \in f(x_{i+1})$, and $y_i \in f(y_{i+1})$ for all i, $\overline{x_i y_i} \subseteq f(\overline{x_{i+1} y_{i+1}})$. Thus if $n_2 > n_1$, $f^{n_1}(\overline{x_{n_1} y_{n_1}}) \subseteq f^{n_2}(\overline{x_{n_2} y_{n_2}})$. So

$$\overline{x_0 y_0} \subseteq f(\overline{x_1 y_1}) \subseteq f^2(\overline{x_2 y_2}) \subseteq \dots .$$

Since f has the intermediate value property, for each k $f^k(\overline{x_n y_n})$ is an interval. Thus J_k is a closed subinterval of $[0,1]$. Note for $n \geq k+1$,

$$f(J_{k+1}) \supseteq f(f^{n-(k+1)}(\overline{x_n y_n})) = f^{n-k}(\overline{x_n y_n}).$$

So $f(J_{k+1}) \supseteq \bigcup_{n \geq k+1} f^{n-k}(\overline{x_n y_n})$. But because $\overline{x_k y_k} \subseteq f(\overline{x_{k+1} y_{k+1}})$, we have

$$f(J_{k+1}) \supseteq \bigcup_{n \geq k} f^{n-k}(\overline{x_n y_n}).$$

As $f(J_{k+1})$ is closed,

$$f(J_{k+1}) \supseteq \overline{\bigcup_{n\geq k} f^{n-k}(\overline{x_n y_n})} = J_k.$$

Similarly for $n \geq k+1$,

$$J_k \supseteq f^{n-k}(\overline{x_n y_n}) = f(f^{n-(k+1)}(\overline{x_n y_n})).$$

Thus $J_k \supseteq f\left(\bigcup_{n\geq k+1} f^{n-(k+1)}(\overline{x_n y_n})\right)$. Since J_k is closed and f has the intermediate value property and is therefore weakly continuous, by Theorem 3.2.10,

$$\overline{f\left(\bigcup_{n\geq k+1} f^{n-(k+1)}(\overline{x_n y_n})\right)} = f\left(\overline{\bigcup_{n\geq k+1} f^{n-(k+1)}(\overline{x_n y_n})}\right).$$

Thus

$$J_k = \overline{f\left(\bigcup_{n\geq k+1} f^{n-(k+1)}(\overline{x_n y_n})\right)} = f\left(\overline{\bigcup_{n\geq k+1} f^{n-(k+1)}(\overline{x_n y_n})}\right) = f(J_{k+1}).$$

□

Lemma 4.2.6 . Let $f : [0,1] \to 2^{[0,1]}$ be such that $G(f)$ is connected and f is upper-semicontinuous, surjective, and has the intermediate value property. Further suppose that $\varprojlim\{[0,1], f\}$ is irreducible between x and y. If $0 < c < d < 1$, then there is some $N \in \omega$ such that $n > N$ implies $[c,d] \subseteq f^n([x_n, y_n])$.

Proof. Let $J_k = \overline{\bigcup_{n\geq k} f^{n-k}(\overline{x_n y_n})}$, as in Lemma 4.2.5. Let $J = \varprojlim\{J_k, f|_{J_{k+1}}\}$. As $f|_{J_{k+1}}$ also has the intermediate value property, is surjective, is upper-semicontinuous, and $G(f|_{J_k})$ is connected by Theorem 3.2.10, J is a subcontinuum of $\varprojlim\{[0,1], f\}$. Note $x, y \in J$; hence $J = \varprojlim\{[0,1], f\}$. As f is surjective, $J_0 = [0,1]$ and $[0,1] = \overline{\bigcup_{n\geq 0} f^n(\overline{x_n y_n})}$. Then because $f^n(\overline{x_n y_n}) \subseteq f^{n+1}(\overline{x_{n+1} y_{n+1}})$ for each n, there is some $N \in \omega$ such that if $n \geq N$, $[c,d] \subseteq f^n(\overline{x_n y_n})$. □

Lemma 4.2.7 . Suppose $f : [0,1] \to 2^{[0,1]}$ is upper-semicontinuous, surjective, has the intermediate value property. If there are $p, q \in (0,1)$ and $r, s \in \omega$ with $0 \in f^r(p)$ and $1 \in f^s(q)$, then f is organic.

Proof. Suppose $\varprojlim\{[0,1], f\}$ is irreducible between x and y. Then by Lemma 4.2.6, there are positive integers N_r and N_s such that if $n > N_r$, $p \in f^{n-r}(\overline{x_n y_n})$ and if $n > N_s$, $q \in f^{n-s}(\overline{x_n y_n})$. So if $n > N_r + N_s$, $f^n(\overline{x_n y_n}) = [0,1]$. □

Theorem 4.2.8 . If $f : [0,1] \to 2^{[0,1]}$ is upper-semicontinuous, organic and has the intermediate value property and $\varprojlim\{[0,1], f\}$ is indecomposable, then f has a periodic cycle with a period that is not a power of 2.

Proof. Since $\varprojlim\{[0,1], f\}$ is indecomposable, there are three points x, y, and z such that $\varprojlim\{[0,1], f\}$ is irreducible between any two of them. Because f is organic, there exists some n such that $f^n(\overline{x_n y_n}) = f^n(\overline{y_n z_n}) = f^n(\overline{x_n z_n}) = [0,1]$. Without loss of generality, suppose $x_n < y_n < z_n$. As f is upper-semicontinuous and has the intermediate value property, $f^n(y_n)$ is a closed interval. Thus either $y_n \in \mathrm{int}(f^n(y_n))$, $f^n(y_n) \subseteq [0, y_n]$, or $f^n(y_n) \subseteq [y_n, 1]$.

Case 1: Suppose $y_n \in \mathrm{int}(f^n(y_n))$. Then there are numbers c and d such that $c < y_n < d$ and $f(y_n) = [c,d]$. As f has the intermediate value property, f is weakly continuous. Thus, there exist sequences $\{(a_i, c_i)\}_{i\in\omega}$ and $\{(b_i, d_i)\}_{i\in\omega}$ in $G(f^n)$ such that for all i $a_i, b_i < y_n$, $a_i, b_i \to y_n$, $c_i \to c$, and $d_i \to d$. Furthermore these sequences may be chosen such that $a_i \leq b_i \leq a_{i+1}$ for all i. Then because $c < y_n < d$, there is some $N \in \mathbb{N}$ such that $i \geq N$ implies $c_i < y_n < d_i$. Since f^n has the intermediate value property, for $i \geq N$, there is a point $p_i \in [a_i, b_i]$ with $y_n \in f^n(p_i)$. Furthermore $p_i \to y_n$ since $a_i, b_i \to y_n$.

Note that as $y_n \in \mathrm{int} f^n(y_n)$ and $p_i \to y_n$, $p_i \in f^n(y_n)$ for cofinitely many i. Then $y_n \in f^n(p_i)$ and $p_i \in f^n(y_n)$ for cofinitely many i. Thus, for any $k \in \mathbb{N}$, there is a periodic orbit of the form $(q_0, \ldots, q_{2kn})$ where for $j = 0 \ldots, k$, $q_{2jn} = y_n$ and for $j = 0, \ldots, k-1$, $q_{(2j+1)n}$ is a distinct member of the p_i's. In particular, $k = 3$ gives a periodic cycle with a period that is not a power of 2, satisfying the conclusion of the theorem.

Case 2: Suppose $f^n(y_n) \subseteq [0, y_n]$. Then either $f^n(y_n) = \{y_n\}$ or there is some value $b \in f^n(y_n)$ with $b < y_n$. If $f^n(y_n) = y_n$, then there are values $a, b \in [x_n, y_n)$ such that $0 \in f^n(a)$ and $1 \in f^n(b)$. Thus there is a closed interval $J_1 \subseteq \overline{ab} \subseteq [x_n, y_n)$ and a restriction $f^n|_{J_1}^{[y_n,z_n]}$ of f^n such that $f^n|_{J_1}^{[y_n,z_n]}(J_1) = [y_n, z_n]$ [41].

We show that such a J_1 also exists if there is some $b \in f^n(y_n)$ with $b < y_n$. By the weak continuity of f^n there is a sequence $\{(a_i, b_i)\}_{i \in \omega}$ such that for all i, $x_n \leq a_i < y_n$, $b_i \in f^n(a_i)$, $a_i \to y_n$, and $b_i \to b$. Thus there is some $b_N < y_n$. Let $q \in [x_n, y_n)$ be a point such that $1 \in f^n(q)$. Then $f^n(\overline{b_N q}) \supseteq [b, 1] \supseteq [y_n, z_n]$, so there is a closed interval $J_1 \subseteq \overline{b_N q} \subseteq [x_n, y_n)$ and a restriction $f^n|_{J_1}^{[y_n,z_n]}$ of f^n such that $f^n|_{J_1}^{[y_n,z_n]}(J_1) = [y_n, z_n]$.

As $f^n([x_n, y_n]) \supseteq J_1$, there is a closed subinterval J_2 of $[x_n, y_n]$ and a restriction $f^n|_{J_2}^{J_1}$ of f^n such that $f^n|_{J_2}^{J_1}(J_2) = J_1$. Similarly there is a closed subinterval J_3 of $[y_n, z_n]$ and a restriction $f^n|_{J_3}^{J_2}$ of f^n such that $f^n|_{J_3}^{J_2}(J_3) = J_2$.

Thus $J_3 \subseteq f^n|_{J_1}^{[y_n,z_n]}(f^n|_{J_2}^{J_1}(f^n|_{J_3}^{J_2}(J_3))) \subseteq f^{3n}(J_3)$. Then there is a periodic orbit $(q_0, \ldots, q_{3n})$ with $q_0 = q_{3n} = q \in J_3$, $q_n \in J_2$, and $q_{2n} \in J_1$. Suppose $q_{2n} = q$.

Then $q \in J_1 \cap J_3 \subseteq [x_n, y_n] \cap [y_n, z_n] = \{y_n\}$. But then we would have $y_n \in J_1$, a contradiction. So $q \neq q_{2n}$.

Let s be the period of $(q_0, \ldots, q_{3n})$. Then $s \mid 3n$. As $q_{2n} \neq q_0 = q$, $s \nmid 2n$. If $s \mid n$, then $s \mid 2n$, a contradiction. It follows that $s \nmid n$. Therefore $3 \mid s$, and s is not a power of 2 as desired. The case for $f^n(y_n) \subseteq [y_n, 1]$ follows from a similar argument. $\square$

CHAPTER FIVE

A Hereditarily Decomposable Inverse Limit of a Map with Periodic Orbits of all Periods

We demonstrate the assumption in Theorem 4.2.2 that f is almost nonfissile is sharp. In Example 5.2.1, we construct a family of functions $F : [0,1] \to 2^{[0,1]}$ that satisfy the hypothesis of Theorem 4.2.2 other than being almost nonfissile, with the possible exception of being light, yet admit a hereditarily decomposable inverse limit. We note in Example 5.2.1 that many members of this family are light and all have the desired properties.

The construction of this family of functions relies heavily on collections of nested Cantor sets, in which no point of a Cantor set in the collection is an endpoint of any larger Cantor set in the collection. We establish the existence of such collections in Section 5.1 that are necessary to define the family of functions in Example 5.2.1.

Then in Section 5.2, we show that each function $F : [0,1] \to 2^{[0,1]}$ in this family is upper-semicontinuous, surjective, has the intermediate value property, has a graph with empty interior, has periodic cycles of all periods, and is not almost nonfissile. We then examine the structure of the inverse limit generated by any one of these functions and show that it is a tree-like, hereditarily decomposable continuum.

5.1 Cantor Sets

Definition 5.1.1 . A set $C \subseteq [0,1]$ is a *Cantor Set* if C is a closed, perfect, nowhere dense set. A point $x \in C$ is called a *left endpoint* of C if there is some number a

such that $(a, x) \subseteq [0, 1] \setminus C$ and a *right endpoint* if there is some number a such that $(x, a) \subseteq [0, 1] \setminus C$. These endpoints form a countable dense subset of C.

Definition 5.1.2 . Let K be a continuum and $p \in K$. The *composant of* p in K is the union of all proper subcontinua of K that contain p.

It is well-known that a metric continuum has three composants if it is decomposable and irreducible, one composant if it is decomposable but not irreducible, and uncountably many composants if it is indecomposable.

Lemma 5.1.3 . Let C_1 be the middle thirds Cantor set on $[1/4, 3/4]$. There is a Cantor set C_0 such that $C_1 \subsetneq C_0$ and no point of C_1 is an endpoint of C_0.

Proof. Construct C_0 as follows: Note that every left endpoint of C_1 corresponds to some point b for some maximal interval $(a, b) \subseteq [1/8, 7/8] \setminus C_1$, and every right endpoint of C_1 corresponds to some point a for some maximal interval $(a, b) \subseteq [1/8, 7/8] \setminus C_1$. Let K_a and K_b be the middle thirds Cantor sets on $[a, a + 1/3(b - a)]$ and $[b - 1/3(b - a), b]$ respectively.

Define $C_0 = C_1 \cup (\bigcup_a K_a) \cup (\bigcup_b K_b)$. Since C_0 is a countable union of Cantor sets, it is perfect and nowhere dense. To show C_0 is closed, let x be a limit point of $C_0 \setminus C_1$ and $\{x_n\}_{n\in\omega}$ be a sequence in C_0 converging to x. Since $x \notin C_1$, there is some maximal interval $(a, b) \subseteq [1/8, 7/8] \setminus C_1$ with $x \in (a, b)$, hence $x_n \in (a, b)$ for cofinitely many n. Thus $x_n \in K_a \cup K_b$ cofinitely often and $x \in K_a \cup K_b$ as $K_a \cup K_b$ is closed. Thus C_0 is a Cantor set containing C_1 such that no point of C_1 is an endpoint of C_0. $\square$

Lemma 5.1.4 . Let $C, D \subseteq [0,1]$ be Cantor sets where $C \subsetneq D$ and no point of C is an endpoint of D. Then there is a Cantor set E such that $C \subsetneq E \subsetneq D$, no point of C is an endpoint of E, and no point of E is an endpoint of D.

Proof. Let $\{p_n\}$ be an enumeration of the endpoints of D. Since C is closed and $p_1 \notin C$, there are points α_1 and β_1 of C such that $p_1 \in (\alpha_1, \beta_1) \subset [0,1] \setminus C$. As points of C, α_1 and β_1 are not endpoints of D. Consequently, there are points a_1 and b_1 of $D \setminus C$ that are not endpoints of D such that $\alpha_1 < a_1 < p_1 < b_1 < \beta_1$.

Proceeding inductively, suppose (a_i, b_i) has been defined for $i \leq n$ so that the following hold.

- $a_i, b_i \in D \setminus C$
- a_i and b_i are not endpoints of D
- $p_i \in (a_i, b_i)$
- if $p_i \in (a_j, b_j)$ for some $j < i$, then $(a_i, b_i) = (a_j, b_j)$
- if $p_i \notin (a_j, b_j)$ for each $j < i$, then $[a_i, b_i] \cap (\cup_{j<i}[a_j, b_j]) = \emptyset$

If $p_{n+1} \in [a_i, b_i]$ for some $i \leq n$ (and hence $p_{n+1} \in (a_i, b_i)$), let $(a_{n+1}, b_{n+1}) = (a_i, b_i)$. Suppose $p_{n+1} \notin \bigcup_{i\leq n}[a_i, b_i]$. Since $C \cup \left(\bigcup_{i\leq n}[a_i, b_i]\right)$ is closed and $p_{n+1} \notin C \cup \left(\bigcup_{i\leq n}[a_i, b_i]\right)$, there are points α_{n+1} and β_{n+1} of $C \cup \left(\bigcup_{i\leq n}[a_i, b_i]\right)$ such that $p_{n+1} \in (\alpha_{n+1}, \beta_{n+1}) \subset [0,1] \setminus \left(C \cup \left(\bigcup_{i\leq n}[a_i, b_i]\right)\right)$. As points of $C \cup \left(\bigcup_{i\leq n}\{a_i, b_i\}\right)$, α_{n+1} and β_{n+1} are not endpoints of D. Consequently, there are points a_{n+1} and b_{n+1} of $D \setminus \left(C \cup \left(\bigcup_{i\leq n}[a_i, b_i]\right)\right)$ that are not endpoints of D such that $\alpha_{n+1} < a_{n+1} <$

$p_{n+1} < b_{n+1} < \beta_{n+1}$. Then a_{n+1} and b_{n+1} satisfy the criteria above and, by induction, (a_i, b_i) is defined for each positive integer i.

Define $E = D \setminus \left(\bigcup_{n\in\mathbb{N}}(a_n, b_n)\right)$. Then $C \subsetneq E \subsetneq D$, E is closed, and no point of E is an endpoint of D. Note also that since $a_n, b_n \in D$ for all $n \in \mathbb{N}$, and since any two intervals of the form $[a_n, b_n]$ are identical or disjoint, it follows that $a_n, b_n \in E$ for each $n \in \mathbb{N}$. It remains to show that no point of C is an endpoint of E and E is perfect.

Let $x \in E \setminus \{b_n : n \in \mathbb{N}\}$. As x is not an endpoint of D, there is a subsequence $\{p_{n_k}\}_{k\in\mathbb{N}}$ such that $p_{n_k} \to x$ and $p_{n_k} < x$ for all k. Since $x \neq b_n$ for each n, then we may choose $\{p_{n_k}\}$ such that $[a_{n_k}, b_{n_k}]$ and $[a_{n_j}, b_{n_j}]$ are disjoint for $j \neq k$. Then, choosing a subsequence of p_{n_k} if necessary to have monotone convergence, we have

$$a_{n_k} < p_{n_k} < b_{n_k} < a_{n_{k+1}} < p_{n_{k+1}} < b_{n_{k+1}} < x.$$

So $a_{n_k} \to x$, making x a limit point of E from the left.

Similarly, let $x \in E \setminus \{a_n : n \in \mathbb{N}\}$. As x is not an endpoint of D, there is a subsequence $\{p_{n_k}\}_{k\in\mathbb{N}}$ such that $p_{n_k} \to x$ and $p_{n_k} > x$. Furthermore, since $x \neq a_n$ for each n, we may choose $\{p_{n_k}\}_{k\in\mathbb{N}}$ such that

$$x < a_{n_{k+1}} < p_{n_{k+1}} < b_{n_{k+1}} < a_{n_k} < p_{n_k} < b_{n_k}.$$

So $b_{n_k} \to x$, making x a limit point of E from the right.

Thus E is a perfect set and therefore a Cantor set. Furthermore, each point of $E \setminus \{a_n, b_n : n \in \mathbb{N}\}$ is not an endpoint of E. Thus every point of C is not an endpoint of E. $\square$

Proposition 5.1.5 . There is a collection of Cantor sets $\{C_r : r \in \mathbb{Q} \cap [0,1]\}$ such that when $r > s$, $C_r \subsetneq C_s$ and no point of C_r is an endpoint of C_s.

Proof. Let C_0 and C_1 be as in Lemma 5.1.3. By Lemma 5.1.4, there is a Cantor set $C_{1/2}$ such that $C_1 \subsetneq C_{1/2} \subsetneq C_0$, no point of C_1 is an endpoint of $C_{1/2}$, and no point of $C_{1/2}$ is an endpoint of C_0. By the same argument, there are Cantor sets $C_{1/4}$ and $C_{3/4}$ such that $C_1 \subsetneq C_{3/4} \subsetneq C_{1/2} \subsetneq C_{1/4} \subsetneq C_0$ and if $s > r$, no point of C_r is an endpoint of C_s. Continuing inductively, we may define a Cantor set C_r for each dyadic rational r in $[0,1]$ such that if $s > r$, then $C_r \subsetneq C_s$ and no point of C_r is an endpoint of C_s. By reindexing the subscripts according to an order-preserving bijection between the dyadic rationals of $[0,1]$ and $\mathbb{Q} \cap [0,1]$, we achieve the desired result. □

5.2 A Hereditarily Decomposable Inverse Limit

Example 5.2.1 . The following notation will be assumed for the remainder of the chapter.

- Let $\{C_r : r \in \mathbb{Q} \cap [0,1]\}$ denote a collection of Cantor sets in $(0,1]$ such that, for $r > s$, $C_r \subsetneq C_s$ and no point of C_r is an endpoint of C_s.

- Let $f : [0,1] \to [0,1]$ be a continuous function such that

  $f(t) = 0$ for all $t \in \{0\} \cup C_0$

  $f(t) < \min C_0$ for all $t \in [0,1]$

  $f(t) < t$ for all $t \in (0,1]$

  Note in particular the following possibilities: (1) f could be light and (2) f could be identically zero.

- Let $F : [0,1] \to C([0,1])$ be defined as follows:

  $$F(t) = \begin{cases} \{f(t)\} & \text{if } t \notin C_0 \\ \\ [0, \sup\{r : t \in C_r\}] & \text{if } t \in C_0 \end{cases}.$$

  Note that $G(f) \subsetneq G(F)$ and that F is light if and only if f is light.

- Let $X = \varprojlim\{[0,1], F\}$. Note that X is a continuum since $F(t)$ is connected for each $t \in [0,1]$.

- For each $x \in X$ and each $n \in \mathbb{N}$, let $L_{x_n} \subseteq [0,1]^2$ be the union of $\{(t, f(t)) : 0 \leq t \leq x_n\}$ and the (possibly degenerate) vertical line segment from $(x_n, f(x_n))$ to (x_n, x_{n-1}).

- For $n \geq 1$, define $G_{x,n} : [0, x_n] \to C([0, x_{n-1}])$ by $G(G_{x,n}) = L_{x_n}$. Note that $G(G_{x,n}) \subsetneq G(F)$.

- Let $L_x = \varprojlim\{[0, x_n], G_{x,n}\}$. Note that L_x is a continuum for each $x \in X$ since $G_{x,n}(t)$ is connected for each $n \in \mathbb{N}$ and $t \in [0, x_n]$.

Theorem 5.2.2 . F is upper semicontinuous, surjective, has the intermediate value property, and has periodic cycles of period n for every $n \in \mathbb{N}$. F is not almost nonfissile, and F is light if and only if f is light. $G(F)$ has empty interior.

Proof. As $F(t) = [0,1]$ for $t \in C_1$, F is surjective. To show F is upper semicontinuous, let $\{(t_n, y_n)\}_{n \in \omega}$ be a sequence in $G(F)$ that converges to some point (t, y). Then either $t_n \in C_0$ cofinitely often or $t_n \notin C_0$ for infinitely many n. First suppose $t_n \notin C_0$ for infinitely many n. Then there is a subsequence $\{(t_{n_k}, y_{n_k})\}_{k \in \omega}$ converging to (t, y)

such that $t_{n_k} \notin C_0$ for each k. Thus $(t_{n_k}, y_{n_k}) \in G(f)$; hence $(t, y) \in G(f) \subseteq G(F)$ by the continuity of f.

Next, suppose $t_n \in C_0$ for cofinitely many n. Since C_0 is closed, it follows that $t \in C_0$. If if $y = 0$, then $(t, y) \in G(F)$. Suppose $y \neq 0$ and $s \in (0, y) \cap \mathbb{Q}$. As $y_n \to y$, there is some $N_s \in \mathbb{N}$ such that $n \geq N_s$ implies $y_n \in (s, 1]$. Since $t_n \in C_0$ for cofinitely many n, we may choose N_s such that for $n \geq N_s$, $t_n \in C_0$. Then $t_n \in C_s$ for $n \geq N_s$. So $F(t) \supseteq [0, s]$. Then $F(t) \supseteq \bigcup_{s \in (0,y) \cap \mathbb{Q}} [0, s] = [0, y)$. As $F(t)$ is closed, $y \in F(t)$, hence $(t, y) \in G(F)$, making F upper semicontinuous.

To show F is weakly continuous, let $(t, y) \in G(F)$. If $t \notin C_r$ for any $r \neq 0$, $(t, y) \in G(f)$ and weak continuity at t is clear. If there is some $r \neq 0$ such that $t \in C_r$, then it is sufficient to show that weak continuity holds for $y = \max F(t) > 0$. Let $\{s_n\}_{n \in \omega}$ be a sequence in $\mathbb{Q} \cap [0, 1]$ such that $s_n \to y$ and $s_n < y$ for all n. By the construction of the Cantor sets, for each n, $t \in C_{s_n}$, and t is not an endpoint of any of the C_{s_n}'s. Thus there is some $t_n \in C_{s_n}$ such that $|t - t_n| < 1/n$ and $t_n < t$ for all n (or $t_n > t$ for all n). As $t_n \in C_{s_n}$, $s_n \in F(t_n)$. Thus $\{(t_n, s_n)\}_{n \in \omega}$ is a sequence in $G(F)$ converging to (t, y), making F weakly continuous at t. Since the image of each point is connected and F is weakly continuous, F has the intermediate value property by Theorem 2.15 of 3.2.6.

There are no nondegenerate intervals on which F is nondegenerate, so $G(F)$ has empty interior. F is not almost nonfissile as all nonfissile points are contained in $G(f)$, which is a closed proper subset of $G(F)$. Since C_0 is nowhere dense, F can fail to be light only on a subinterval of $[0, 1] \setminus C_0$, on which F agrees with f. Thus F is light if and only if f is light.

To see that F has cycles of all periods, let $n \in \mathbb{N}$. Choose distinct points $t^1, t^2, \ldots, t^n$ in C_1. Then for $1 \leq i \leq n$, $F(t^i) = [0,1]$. Thus $(t^1, \ldots, t^n)$ is a periodic cycle of period n. □

Lemma 5.2.3 . Let $x \in X \setminus \{\overline{0}\}$. Then there is some $N \in \omega$ such that $x_n \in C_0$ if and only if $n \geq N$.

Proof. Since $F(t) = f(t) < \min C_0$ for $t \notin C_0$, it follows that, if $x_n \notin C_0$ for some n, then $x_k \notin C_0$ for each $k \leq n$. Equivalently, if $x_n \in C_0$ for some n, then $x_k \in C_0$ for each $k \geq n$. But it is not the case that $x_n \notin C_0$ for every $n \in \omega$; otherwise $x_1, x_2, x_3, ...$ would be a nondecreasing sequence bounded above by $\min C_0$ that would converge to a fixed point of f lying in $(0, \min C_0]$, contrary to the definition of f. Thus there is $N \in \omega$ such that $x_n \in C_0$ if and only if $n \geq N$. □

Proposition 5.2.4 . For each $x \in X \setminus \{\overline{0}\}$, L_x is an arc with endpoints at x and $\overline{0}$.

Proof. By Lemma 5.2.3, there is some $N \in \omega$ such that $x_n \in C_0$ if and only if $n \geq N$. For $n \geq N$, let

$$K_n = \{(f^n(t), \ldots, f(t), t, x_{n+1}, x_{n+2}, \ldots) : 0 \leq t \leq x_n\}.$$

Then K_n is an arc from y^n to y^{n+1}, where $y^i = (0, \ldots, 0, x_i, x_{i+1}, \ldots)$ for each $i \geq N$. Furthermore, for each n, $K_{n+1} \cap \left(\bigcup_{i \leq n} K_i\right) = \{y^{n+1}\}$. Therefore, $\bigcup_{i \leq n} K_i$ is an arc from y^N to y^{n+1}. So $\bigcup_{n \geq N} K_n$ is a ray in L_x with endpoint y^N that does not contain $\overline{0}$.

We consider two cases. First suppose $N \neq 0$. Note that $y^N \neq x$. Define K_{N-1} by

$$K_{N-1} = \{(f^{N-1}(t), \dots, f(t), t, x_N, x_{N+1}, \dots) : 0 \le t \le x_{N-1}\}.$$

Then K_{N-1} is an arc from x to y^N that intersects $\bigcup_{n\geq N} K_n$ only at the point y^N. Hence $\bigcup_{n\geq N-1} K_n$ is a ray in L_x with endpoint x that does not contain $\overline{0}$. To complete the proof in the case where $N \neq 0$, it suffices to show $\overline{\bigcup_{n\geq N-1} K_n} = L_x$ and $\overline{\bigcup_{n\geq N-1} K_n} \setminus \left(\bigcup_{n\geq N-1} K_n\right) = \{\overline{0}\}$.

To show $L_x \subseteq \left(\bigcup_{n\geq N-1} K_n\right) \cup \{\overline{0}\}$, let $z \in L_x$. The claim trivially holds for $z = \overline{0}$. If $z \neq \overline{0}$, then by Lemma 5.2.3, $z_i \in C_0$ for cofinitely many i. Since $G_{x,i}(z_i) \cap C_0 \neq \emptyset$ only if $z_i = x_i$, it follows that $z_i = x_i$ for cofinitely many i. Thus there is an $M \in \omega$ such that for $z_i = x_i$ if and only if $i \geq M$. If $M \leq N$, then $z = (f^{N-1}(t) \dots, f(t), t, x_N, x_{N+1}, x_{N+2}, \dots) \in K_{N-1}$. Suppose $M > N$. Then for $i < M$, $z_i < x_i$. So for $1 \leq j \leq M-1$, $z_{j-1} = G_{x,j}(z_j) = f(z_j)$. Thus for $0 \leq j \leq M-1$, $z_{M-1-j} = f^j(z_{M-1})$. So $z = (f^{M-1}(t) \dots, f(t), t, x_M, x_{M+1}, x_{M+2}, \dots) \in K_{M-1}$ and $L_x \subseteq \left(\bigcup_{n\geq N} K_n\right) \cup \{\overline{0}\}$.

Since L_x is closed and $K_n \subseteq L_x$ for each $n \geq N$, $\overline{\bigcup_{n\geq N-1} K_x} \subseteq L_x$. As $y^n \to \overline{0}$, $\overline{0} \in \overline{\bigcup_{n\geq N-1} K_n} \setminus \left(\bigcup_{n\geq N-1} K_n\right)$. Since $L_x \subseteq \left(\bigcup_{n\geq N-1} K_n\right) \cup \{\overline{0}\}$, it follows that $\overline{\bigcup_{n\geq N-1} K_n} \setminus \left(\bigcup_{n\geq N-1} K_n\right) = \{\overline{0}\}$ and $L_x = \overline{\bigcup_{n\geq N-1} K_n}$, making L_x an arc from x to $\overline{0}$.

For the case in which $N = 0$, $y^N = x$. Hence $\bigcup_{n\geq N} K_n$ is a ray with endpoint x for which it can be shown by similar arguments that $\overline{\bigcup_{n\geq N} K_n} \setminus \left(\bigcup_{n\geq N} K_n\right) = \{\overline{0}\}$ and $L_x = \overline{\bigcup_{n\geq N} K_n}$, making L_x an arc from x to $\overline{0}$. $\square$

Corollary 5.2.5 . X is arcwise connected.

Theorem 5.2.6 . (Theorem 4.2 [30]) Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions such that $f_n : [0,1] \to C([0,1])$ is a surjective upper semicontinuous function for each positive integer n. If, for each $n > 1$, Z_n is a closed totally disconnected subset of $[0,1]$ such that if $f_n(t)$ is nondegenerate then $t \in Z_n$ and $(f_i^n)^{-1}(Z_i)$ is totally disconnected for each i, $1 \leq i \leq n$, then $\varprojlim\{[0,1], f_n\}$ is a tree-like continuum.

Proposition 5.2.7 . X is tree-like and therefore hereditarily unicoherent.

Proof. For each n, let $Z_n = C_0$. Then Z_n is a closed totally disconnected set. If $F(t)$ is nondegenerate then there is some $r \in \mathbb{Q} \cap [0,1]$ such that $t \in C_r \subset C_0$. Since $C_0 \subseteq (0,1]$, $F^{-1}(C_0) \subseteq C_0$. Thus $F^{-n}(C_0)$ is totally disconnected for every n. So by Theorem 5.2.6 X is tree-like. □

Proposition 5.2.8 . Let K be a subcontinuum of X.

1. If $\overline{0} \in K$, then $K = \bigcup_{x\in K} L_x$.

2. If $\overline{0} \notin K$, then $\pi_n[K]$ is degenerate for cofinitely many n.

Proof. First, suppose $\overline{0} \in K$. As X is hereditarily unicoherent, for each $x \in K$, $L_x \cap K$ is a subcontinuum containing both x and $\overline{0}$. As L_x is an arc irreducible between x and $\overline{0}$, $L_x \cap K = L_x$. Then $L_x \subseteq K$. So $K = \bigcup_{x\in K} L_x$.

Next, suppose $\overline{0} \notin K$. As $F(0) = \{0\}$ and K is closed, $0 \notin \pi_n[K]$ for cofinitely many n. Then $\pi_{n,n-1}[K]$ is a subcontinuum of $G(F)$ that does not touch $[0,1] \times \{0\}$ for cofinitely many n. For each such n, $\pi_{n,n-1}[K]$ is a (possibly degenerate) vertical line segment and $\pi_n[K]$ contains a single point. □

Theorem 5.2.9 . X is a hereditarily decomposable tree-like continuum.

Proof. Let K be a nondegenerate subcontinuum of X. If $\overline{0} \in K$, then $K = \bigcup_{x \in K} L_x$ by the above proposition. If there is some y such that $K = L_y$, then K is an arc and thus decomposable. Otherwise, each L_x is a proper subcontinuum. Then the composant of $\overline{0}$ in K is K itself, making K decomposable.

Now suppose $\overline{0} \notin K$. By Proposition 5.2.8, for cofinitely many n, $\pi_n[K]$ contains a single point, which we denote k_n. Since K is nondegenerate, $\pi_n[K]$ is nondegenerate for some $n \in \mathbb{N}$. Denote the largest such n by N. Then $\pi_N[K]$ contains a point c in its interior such that $c \notin C_0$. Let $x \in \pi_N^{-1}(c) \cap K$. Then $x_n = k_n$ for $n > N$. Since $c \notin C_0$, $F(c) = \{f(c)\}$. Since $f(t) < \min C_0$ for all $t \in [0,1]$, it follows that $x_n = f^{N-n}(c)$ for $n < N$. So $x = (f^N(c), \ldots, f(c), c, k_{N+1}, k_{N+2}, \ldots)$ is the unique point of $\pi_N^{-1}(c) \cap K$. As c separates $\pi_N(K)$, x is a separating point of K, and K is decomposable. $\square$

CHAPTER SIX

Future Work

Inverse limits with set-valued functions remains a field with many avenues for future study. One source of possible research is in attempting to find generalizations of the connections between topological and dynamic properties of classical inverse limits found by Barge and Martin [5, 6, 7, 8]. Since the weak intermediate value property is a weaker assumption than requiring that the image of each point be connected and solves the immediate issue of whether the generalized inverse limit is connected, it is worth applying to see what other results can be generalized. In [9], Barge and Martin showed that the classical inverse limit of a single function on the interval may be realized as a global attractor for a planar homeomorphism. This raises the following questions: What conditions are sufficient to make generalized inverse limits planar? What conditions make them global attractors?

An answer to the former question would provide a template for constructing more exotic planar continua, while an answer to the latter would further illuminate the connections between topology and dynamics in the setting of generalized inverse limits.

One topic of interest for set-valued dynamics is the relationship between the entropy of these types of functions and periodicity. The existence of a periodic cycle of a function on $[0, 1]$ with period not a power of 2 and the entropy of the function being positive are equivalent in the classical case, but Kelly and Tennant showed that this is not the case in general for set-valued functions on $[0, 1]$ [35]. This raises the question of whether the intermediate value property is sufficient to restore this equivalence. This would enhance our understanding chaos in dynamical systems with set-valued functions.

BIBLIOGRAPHY

[1] R. D. Anderson and Gustave Choquet. "A Plane Continuum no Two of Whose Non-Degenerate Subcontinua are Homeomorphic: An Application of Inverse Limits". In: *Proceedings of the American Mathematical Society* 10.3 (June 1959), p. 347. DOI: 10.2307/2032845.

[2] I. Banič and Veronica Martinez-de-la-Vega. "Universal dendrite D3 as a generalized inverse limit". In: *Houston Journal of Mathematics* 41 (Jan. 2015), pp. 669–682.

[3] Iztok Banič and Judy Kennedy. "Inverse limits with bonding functions whose graphs are arcs". In: *Topology and its Applications* 190 (Aug. 2015), pp. 9–21. DOI: 10.1016/j.topol.2015.04.009.

[4] Iztok Banič et al. "The (Weak) Full Projection Property for Inverse Limits with Upper Semicontinuous Bonding Functions". In: *Mediterranean Journal of Mathematics* 15.4 (June 2018). DOI: 10.1007/s00009-018-1209-6.

[5] Marcy Barge and Joe Martin. "Chaos, periodicity, and snakelike continua". In: *Transactions of the American Mathematical Society* 289.1 (Jan. 1985), pp. 355–355. DOI: 10.1090/s0002-9947-1985-0779069-7.

[6] Marcy Barge and Joe Martin. "Dense orbits on the interval." In: *Michigan Mathematical Journal* 34.1 (Jan. 1987). DOI: 10.1307/mmj/1029003477.

[7] Marcy Barge and Joe Martin. "Dense periodicity on the interval". In: *Proceedings of the American Mathematical Society* 94.4 (Apr. 1985), pp. 731–731. DOI: 10.1090/s0002-9939-1985-0792293-8.

[8] Marcy Barge and Joe Martin. "Endpoints of inverse limit spaces and dynamics". In: *Lecture Notes in Pure and Appl. Math* 289.1 (Jan. 1994), pp. 355–355.

[9] Marcy Barge and Joe Martin. "The construction of global attractors". In: *Proceedings of the American Mathematical Society* 110.2 (Feb. 1990), pp. 523–523. DOI: 10.1090/s0002-9939-1990-1023342-1. URL: https://doi.org/10.1090/s0002-9939-1990-1023342-1.

[10] Van C. Nall. "Connected Inverse Limits with a Set-Valued Function". In: *Topology Proceedings* 40 (2012), pp. 167–177.

[11] Van C. Nall. “Finite graphs that are inverse limits with a set valued function on $[0, 1]$”. In: *Topology and its Applications* 158.10 (June 2011), pp. 1226–1233. DOI: 10.1016/j.topol.2011.04.011.

[12] Van C. Nall. “The only finite graph that is an inverse limit with a set valued function on $[0, 1]$ is an arc”. In: *Topology and its Applications* 159.3 (Feb. 2012), pp. 733–736. DOI: 10.1016/j.topol.2011.11.029.

[13] M.K. Camlibel et al. “Switched networks and complementarity”. In: *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* 50.8 (Aug. 2003), pp. 1036–1046. DOI: 10.1109/tcsi.2003.815195.

[14] C. E. Capel. “Inverse limit spaces”. In: *Duke Mathematical Journal* 21.2 (June 1954). DOI: 10.1215/s0012-7094-54-02124-9.

[15] Dante Carrasco-Olivera et al. “Topological entropy for set-valued maps”. In: *Discrete & Continuous Dynamical Systems - B* 20.10 (2015), pp. 3461–3474. DOI: 10.3934/dcdsb.2015.20.3461.

[16] Włodzimierz J. Charatonik and Robert P. Roe. “Inverse Limits of Continua Having Trivial Shape”. In: *Houston Journal of Mathematics* 38.4 (Jan. 2012), pp. 1307–1312.

[17] Włodzimierz J. Charatonik and Robert P. Roe. “Mappings between inverse limits of continua with multivalued bonding functions”. In: *Topology and its Applications* 159.1 (Jan. 2012), pp. 233–235. DOI: 10.1016/j.topol.2011.09.008.

[18] Włodzimierz J. Charatonik and Robert P. Roe. “On Mahavier products”. In: *Topology and its Applications* 166 (Apr. 2014), pp. 92–97. DOI: 10.1016/j.topol.2014.02.008.

[19] Louis J. Cherene. *Set Valued Dynamical Systems and Economic Flow*. Springer Berlin Heidelberg, 1978. DOI: 10.1007/978-3-642-45504-9.

[20] Steven Clontz and Scott Varagona. *Mahavier Products, Idempotent Relations, and Condition Γ*. 2018. eprint: arXiv:1805.06827.

[21] Brian E. Raines and Tim Tennant. “The Specification property on a Set-valued Map and its Inverse Limit”. In: *Houston Journal of Mathematics* 44.2 (2018), pp. 665–677.

[22] Mathieu Faure and Gregory Roth. “Stochastic Approximations of Set-Valued Dynamical Systems: Convergence with Positive Probability to an Attractor”. In: *Mathematics of Operations Research* 35.3 (Aug. 2010), pp. 624–640. DOI: 10.1287/moor.1100.0455.

[23] A. Fedeli. “On chaotic set-valued discrete dynamical systems”. In: *Chaos, Solitons & Fractals* 23.4 (Feb. 2005), pp. 1381–1384. DOI: 10.1016/s0960-0779(04)00394-7.

[24] Sina Greenwood and Judy Kennedy. “Connected generalized inverse limits”. In: *Topology and its Applications* 159.1 (Jan. 2012), pp. 57–68. DOI: 10.1016/j.topol.2011.07.019.

[25] Sina Greenwood and Judy Kennedy. “Connected generalized inverse limits over intervals”. In: *Fundamenta Mathematicae* 236 (2017), pp. 1–43.

[26] Sina Greenwood and Judy Kennedy. “Connectedness and Ingram–Mahavier products”. In: *Topology and its Applications* 166 (Apr. 2014), pp. 1–9. DOI: 10.1016/j.topol.2014.01.016.

[27] George W. Henderson. “The pseudo-arc as an inverse limit with one binding map”. In: *Duke Mathematical Journal* 31.3 (Sept. 1964). DOI: 10.1215/s0012-7094-64-03140-0.

[28] Alejandro Illanes. “A circle is not the generalized inverse limit of a subset of $[0,1]$”. In: *Proceedings of the American Mathematical Society* 139.08 (Aug. 2011), pp. 2987–2987. DOI: 10.1090/s0002-9939-2011-10876-1.

[29] W. T. Ingram. *An Introduction to Inverse Limits with Set-valued Functions*. Springer New York, 2012. ISBN: 978-1-4614-4487-9. DOI: 10.1007/978-1-4614-4487-9.

[30] W. T. Ingram. “Concerning dimension and tree-likeness of inverse limits with set-valued functions”. In: *Houston Journal of Mathematics* 40.2 (2014), pp. 621–631.

[31] W. T. Ingram and William S. Mahavier. “Inverse limits of upper semi-continuous set-valued functions”. In: *Houston Journal of Mathematics* 32 (Jan. 2006), pp. 119–130.

[32] W.T. Ingram and William S. Mahavier. *Inverse Limits*. Springer New York, 2012. DOI: 10.1007/978-1-4614-1797-2.

[33] James P. Kelly. “Inverse limits with irreducible set-valued functions”. In: *Topology and its Applications* 166 (Apr. 2014), pp. 15–31. DOI: 10.1016/j.topol.2014.02.001.

[34] James P. Kelly and Jonathan Meddaugh. “Indecomposability in inverse limits with set-valued functions”. In: *Topology and its Applications* 160.13 (Aug. 2013), pp. 1720–1731. DOI: 10.1016/j.topol.2013.07.002.

[35] James P. Kelly and Tim Tennant. “Topological Entropy of Set-valued Functions”. In: *Houston Journal of Mathematics* 43.1 (2017), pp. 263–282.

[36] Sergio Macías. *Topics on Continua*. Chapman and Hall CRC, 2005. ISBN: 978-3-319-90902-8.

[37] William S. Mahavier. “Inverse limits with subsets of [0, 1]×[0, 1]”. In: *Topology and its Applications* 141.1-3 (June 2004), pp. 225–231. DOI: 10.1016/j.topol.2003.12.008.

[38] Michael Maschler and Bezalel Peleg. “Stable Sets and Stable Points of Set-Valued Dynamic Systems with Applications to Game Theory”. In: *SIAM Journal on Control and Optimization* 14.6 (Nov. 1976), pp. 985–995. DOI: 10.1137/0314062.

[39] Stefan Mazurkiewicz. “Un théorème sur les continus indécomposables”. In: *Fundamenta Mathematicae* 1.1 (1920), pp. 35–39. DOI: 10.4064/fm-1-1-35-39.

[40] Sam B. Nadler Jr. *Continuum Theory: An Introduction*. Monographs, Textbooks in Pure, and Applied Mathematics, vol. 158, Marcel Dekker Inc., New York, 1992. ISBN: 978-3-319-90902-8.

[41] Drew Otey and David J. Ryden. “Sarkovskii order for upper semicontinuous functions on $[0, 1]$ with intermediate value property”. In: *In Progress* ().

[42] Sergei Yu. Pilyugin and Janosch Rieger. “Shadowing and inverse shadowing in set-valued dynamical systems. Contractive case”. In: *Topological Methods in Nonlinear Analysis* 32.1 (2008), pp. 139–149. DOI: tmna/1463150468.

[43] Sergei Yu. Pilyugin and Janosch Rieger. “Shadowing and inverse shadowing in set-valued dynamical systems. Hyperbolic case”. In: *Topological Methods in Nonlinear Analysis* 32.1 (2008), pp. 151–164. DOI: tmna/1463150469.

[44] David J. Ryden. “The full-projection and closed-set properties”. In: *In Progress* ().

[45] Richard M. Schori. “A universal snake-like continuum”. In: *Proceedings of the American Mathematical Society* 16.6 (June 1965), pp. 1313–1313. DOI: 10.1090/s0002-9939-1965-0184209-x.

[46] A. N. Sharkovskiĭ. “Coexistence of cycles of a continuous map of the line into itself”. In: *International Journal of Bifurcation and Chaos* 05.05 (Oct. 1995), pp. 1263–1273. DOI: 10.1142/s0218127495000934.

[47] Scott Varagona. “Inverse limits with upper semi-continuous bonding functions and indecomposability”. In: *Houston Journal of Mathematics* 37 (Jan. 2011), pp. 1017–1034.

[48] S. Willard. *General Topology*. Addison Wesley series in mathematics/Lynn H.Loomis. Addison-Wesley Publishing Company, 1970. ISBN: 9780201087079. URL: https://books.google.com/books?id=e8IPAQAAMAAJ.

[49] B.R. Williams. “Indecomposability in inverse limits”. PhD dissertation. Baylor University, 2010.

[50] R.F. Williams. “One-dimensional non-wandering sets”. In: *Topology* 6.4 (Nov. 1967), pp. 473–487. DOI: 10.1016/0040-9383(67)90005-5.