ABSTRACT

On Certain Results of Fourier Analysis on Non-abelian Discrete Groups

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The topics of this thesis lie in the intersection of harmonic analysis, functional analysis and operator algebras. Recently, advancement made in the area of quantum information has ignited a wave of interest in the non-commutative aspects of analysis. This, in turn, helped to connect the tools in operator algebras with the techniques used in harmonic analysis to produce various deep results.

The first part of the thesis will discuss about the characterization of positive definite, radial functions on some non-commutative groups. The subject of positive definite functions is widely studied due to its link with the area of Fourier Multipliers. So, a characterization of positive, definite functions will help in identifying the class of Fourier Multipliers which is completely positive. There is also some surprising consequences which slightly improves the classical Schoenberg-Bochner theorem.

The second part of the thesis focus on the non-commutative Khintchine inequality and its relation with the Z_2 property. The classical Khintchine inequality has been studied extensively and greatly improved by many mathematicians. In our setting, we show that the existence of the Z_2 property in an orthonormal system will result in its matrix-valued coefficient function satisfying a Khintchine-type inequality. For the case of an abelian group, we also obtain some partial converse result on that, where

the existence of a Khtinchine-type inequality will imply that the group has finite Z_2 constants. In some examples, the optimal constants can be computed.

On Certain Results of Fourier Analysis on Non-abelian Discrete Groups

by

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To the teachers and mentors who guided me here

CHAPTER ONE

Introduction

1.1 Background

The main topics of this thesis lie in the intersection of areas of functional analysis and harmonic analysis. More precisely, we are interested in understanding the theories of harmonic analysis in the non-commutative settings. In recent times, the advances in areas of physics such as quantum mechanics, quantum field theory and quantum information theory have spurred a huge wave of interest in non-commutative analysis. As a simple example, the notion of observables in physics can be understood as a linear operator acting on a Hilbert space. Meanwhile, the quantum state of a particle refers to a positive, unital linear functionals acting on a C^* algebra.

On the side of harmonic analysis, I am interested in exploring the abstract generalization of the various beautiful results in the classical harmonic analysis to the non-commutative settings. Even in the level of basic definitions, there is a shift in paradigm from commutative analysis to the realm of non-commutativity. For example, the notion of measure theory and integration is now replaced by the notion of positive linear functionals on C^* algebras. The loss of commutativity does pose quite some challenges in the studies of this area. As an example, the maximal function is a basic tool in classical analysis. However, even in the 2 by 2 matrix case, the supremum of a pair of matrices might not be well-defined. This is due to the fact that the order imbued on matrices is only a partial order instead of a total order. Nevertheless, there are also some positive results regarding this area.

There will be two main topics discussed in this thesis, which focus on the various parts of the non-commutative harmonic analysis. They topics discussed are as follows:

1. Characterization of the positive definite, radial functions.

2. Non-commutative Khintchine inequalities.

1.2 Characterization of The Positive Definite, Radial Functions

The study of the relationship between positive definite functions and Fourier multipliers has always been a fascinating topic to study about. The deepest results concerning this area goes back to Bochner's celebrated theorem, which is as followed:

If ϕ is a positive definite function on a locally compact, abelian group G, then there exists a unique positive, Radon measure $\mu \in M(\widehat{G})$ such that

$$\phi(x) = \int_{\widehat{G}} \langle x, \xi \rangle \, d\mu(\xi)$$

Here, $\widehat{G}$ denotes the dual group of G, which is the group of group homomorphisms from G to the unit circle $\mathbb{T}$.

Since then, progress have been made in order to generalize Bochner's theorem, especially in non-commutative analysis. This, in turn, helps to establish some deep connections with the study of approximation properties of operator algebras. Given a group G, we consider the function $\varphi : G \to \mathbb{C}$. Let λ_s be the left regular representation of $s \in G$. The Fourier multiplier $M_\varphi : C_r^*(G) \to C_r^*(G)$ is defined by:

$$M_\varphi \left(\sum_{s \in G} c_s \lambda_s \right) = \sum_{s \in G} \varphi(s) c_s \lambda_s$$

It turns out that the function φ is positive definite if and only if M_φ is a completely positive operator on $C_r^*(G)$. Also, such multipliers are useful in understanding differential operators on the C^* algebra of a free group, which has deep connections to the field of non-commutative geometry pioneered by Alain Connes.

In [7] and [10], Haagerup and Knudby managed to characterize the positive definite, ℓ_1 radial functions on $\mathbb{F}_\infty$, the free group with infinitely many generators.

In a joint collaboration, we managed to obtain results for the generalization of the above problem which can be roughly be stated as follows.

An element $h \in \mathbb{F}_\infty$ can be expressed in the form of $h = g_{m_1}^{n_1} g_{m_2}^{n_2} \cdots g_{m_k}^{n_k}$, where $g_1, g_2, \cdots$ are the generators of $\mathbb{F}_\infty$. The ℓ^2 length of h is defined to be

$$\|h\|_2 := \left(\sum_{j=1}^{k} |n_j|^2 \right)^{\frac{1}{2}} \tag{1.1}$$

Then, we obtain the following result. An ℓ^2-radial function on $\mathbb{F}_\infty$ is positive definite if and only if there exists a unique Radon measure on $[-1, 1]$ such that

$$\varphi(g) = \int_{-1}^{1} s^{\|g\|_2^2} \, d\mu(s)$$

The above result is actually proved in a more general setting and also covers the case for the free real line.

In addition, we also manage to provide a slight generalization for the Schoenberg-Bochner theorem [23], which, in its original form is stated as follows:

Theorem (Schoenberg-Bochner): A function $f : (0, \infty) \to [0, \infty)$ is a Bernstein function if, and only if, for all $d \in \mathbb{N}$, the function $\xi \mapsto f(|\xi|^2)$, $\xi \in \mathbb{R}^d$, is continuous and negative definite.

We manage to generalize the above theorem to the case where the domain of the function f is $[0, \infty)$ instead of $(0, \infty)$ and this in turns, drops the requirement of f to be continuous. Also, the $|\cdot|_2$ norm can be generalized to an $|\cdot|_p$ norm for any $0 < p \leqslant 2$.

1.3 Non-commutative Khintchine Inequalities

The classical Khintchine inequalities states that given any $0 < p < \infty$ and any $(c_n)_{n=1}^{\infty} \subseteq \ell^2(\mathbb{C})$, we have that:

$$B_p\left(\sum_{n=1}^{\infty}|c_n|^2\right)^{\frac{1}{2}} \leqslant \left\|\sum_{n=1}^{\infty}c_nr_n(\cdot)\right\|_{L^p([0,1])} \leqslant A_p\left(\sum_{n=1}^{\infty}|c_n|^2\right)^{\frac{1}{2}}, \tag{1.2}$$

for some constant A_p, B_p, where r_n denotes the n-th Rademacher functions. I am interested to understand its extension to a more general non-commutative setting, especially for the case $p = 1$. Earlier progress has been made in this area by Lust-Piquard and Pisier ([16]) and many others. I am interested in the connection of this analytic property with the algebraic property of the so called Z_2 property of orthonormal systems. For simplicity, we explain the concept of the Z_2 property on a discrete group.

Given a discrete group Γ, the Z_2 constant of a subset $V \subseteq \Gamma$ is defined by:

$$Z_2(V) := \sup_{g\in\Gamma, g\neq e} \#\{(x,y) \in V \times V : x^{-1}y = g\} \tag{1.3}$$

The set V is said to have the Z_2 property if $Z_2(V) < \infty$.

It turns out that in the non-commutative setting, the optimal Khintchine inequality is deeply related to the properties of an orthonormal system having the Z_2 property. If an orthonormal system satisfies the Z_2 property, then the above Khintchine inequality holds true for the non-commutative L^1 functions with constants depending on the Z_2 constants of the othonormal system.

More precisely, given n, $d \in \mathbb{N}$ and $C_1, \cdots C_d \in M_n(\mathbb{C})$, we consider the $S^1(l^2_{rc})$ norm of the matrix sequence $(C_k)_{k=1}^d$ given by:

$$\|(C_k)_{k=1}^d\|_{S^1(\ell^2_{rc})} := \inf\left\{\operatorname{Tr}\left[\left(\sum_{k=1}^d X_k^*X_k\right)^{\frac{1}{2}} + \left(\sum_{k=1}^d Y_kY_k^*\right)^{\frac{1}{2}}\right] : C_k = X_k + Y_k \in M_n(\mathbb{C})\right\} \tag{1.4}$$

Then, we have the following result:

Let n, $d \in \mathbb{N}$. Suppose that $W \subseteq \Gamma$ is a Z_2 set. Let $x_1, \cdots, x_d \in W$ and $C_{x_1}, \cdots, C_{x_d} \in M_n(\mathbb{C})$. Then,

$$\frac{\left\|(C_{x_i})_{i=1}^d\right\|_{S_n^1(\ell^2_{rc})}}{\sqrt{Z_2(W)+1}} \leqslant \left\|\sum_{i=1}^d C_{x_i}\otimes\lambda_{x_i}\right\|_{L^1[M_n(\mathbb{C})\otimes\mathfrak{L}(\Gamma)]} \leqslant \left\|(C_{x_i})_{i=1}^d\right\|_{S^1(\ell^2_{rc})}.$$

For the case of the commutative group, we also obtained a converse result. If the non-commutative L^1 functions on an orthonormal system satisfy the Khintchine inequality with appropriate constants, then the orthonormal system must possess the Z_2 property. Optimal constants which relate the Z_2 property and the Khintchine property has also been obtained.

CHAPTER TWO

Positive Definite Function

2.1 Introduction

Let G be a group. A function $\varphi : G \to \mathbb{C}$ is called *positive definite* if the associated Toeplitz-type matrix $[\varphi(x_i^{-1}x_j)]_{1\leqslant i,j\leqslant n}$ is positive definite for any $n \in \mathbb{N}$ and any $(x_i)_{i=1}^n \in G$. i.e. $\sum_{i,j=1}^n \bar{c}_i c_j \varphi(x_i^{-1}x_j) \geqslant 0$ for any complex numbers $(c_i)_{i=1}^n$. The classical Bochner-Herglotz theorem ([2, 5.5.2] says that a function on the integer group is positive definite if and only if it is the Fourier transform of a finite nonnegative Borel measure on the torus.

There is a similar concept of *positive definiteness* on semigroups. Let G_+ be a semigroup. A function $\dot{\varphi} : G_+ \to \mathbb{C}$ is called *positive definite* in the semigroup sense if the associated Hankel-type matrix $[\dot{\varphi}(x_i x_j)]_{1\leqslant i,j\leqslant n}$ is positive definite for any $n \in \mathbb{N}$ and any n elements $x_i \in G_+$. This is equivalent to saying that $\sum_{i,j=1}^n \bar{c}_i c_j \dot{\varphi}(x_i x_j) \geqslant 0$ for any complex numbers c_i. The Hamburger theorem ([19, Theorem 7.1] says that a bounded function $\dot{\varphi}$ is positive definite on the semigroup $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ if and only if $\dot{\varphi}$ is the moment of a nonnegative Borel measure μ on $[-1, 1]$, i.e. there exists μ such that

$$\dot{\varphi}(k) = \int_{-1}^{1} t^k d\mu(t). \tag{2.1}$$

Note that the support of μ is a subset of $[-1, 1]$ as $\dot{\varphi}$ is bounded. Given any bounded positive definite function $\dot{\varphi}$ on $\mathbb{Z}_+$, the formula $\varphi(k) = \dot{\varphi}(|k|)$ defines a symmetric positive definite function on $\mathbb{Z}$. This can be seen by (2.1) and the well-known fact that $k \mapsto t^{|k|}$ is positive definite on $\mathbb{Z}$ for any $-1 \leqslant t \leqslant 1$. However, not every symmetric positive definite function φ on $\mathbb{Z}$ is of the form of $\varphi(k) = \dot{\varphi}(|k|)$. In fact, Haagerup and Knudby proved in [10, Theorem 3.3] that there is a one to one correspondence between the class of bounded positive definite functions on $\mathbb{Z}_+$ and the class of radial positive definite functions on the infinite free product of $\mathbb{Z}$. We restate it as follows.

Theorem (Haagerup-Knudby, [10, Theorem 3.3]): Let $\mathbb{F}_\infty$ be the free group of countable many infinite generators. Let $\|g\|_1$ be the reduced word length of an element $g \in \mathbb{F}_\infty$. Given a bounded function $\dot{\varphi}$ on $\mathbb{Z}_+$, the following are equivalent.

1. The function $\varphi(g) = \dot{\varphi}(\|g\|_1)$ is positive definite on $\mathbb{F}_\infty$.

2. There is a finite positive Borel measure μ on $[-1, 1]$ such that

$$\dot{\varphi}(n) = \int_{-1}^{1} s^n \, d\mu(s), \ n \in \mathbb{N}$$

Together with the Hamberger theorem (2.1), Haagerup-Knudby's theorem gives a one to one correspondence between the class of bounded positive definite functions on $\mathbb{Z}_+$ and the class of radial positive definite functions on the infinite free product of $\mathbb{Z}$, though the article does not provide a direct proof of this correspondence. Also, in the commutative case, a function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is positive definite in the semi-group sense if and only if $\varphi \circ |\cdot|_2^2$ is positive definite on $\mathbb{R}^d$ for all $d \in \mathbb{N}$.

In this article, we give a direct argument for this correspondence, which works for a more general setting (see Lemma 3.2). As a consequence, we obtain the characterization of ℓ_p-length radial, positive definite functions on the infinitely generated free real line and the ℓ_2-length radial, positive definite functions on the infinitely generated free group.

Definition 2.1. Fix a set of generators $\{g_i, i \in \mathbb{N}\}$ of the free group $\mathbb{F}_\infty$. Let $0 < p \leqslant 2$. For a reduced word $g = g_{i_1}^{k_1} g_{i_2}^{k_2} ... g_{i_n}^{k_n} \in \mathbb{F}_\infty$, define the ℓ^p length of g, denoted by $\|g\|_p$ as:

$$\|g\|_p = \left(\sum_{j=1}^{n} |k_j|^p \right)^{\frac{1}{p}}$$

The maps $g \mapsto \|g\|_p$ are still conditionally negative (see Definition 2.6) on the free group (see Proposition 2.31). We say a function φ on $\mathbb{F}_\infty$ is $\|\cdot\|_p$-radial if the value of $\varphi(g)$ only depends on $\|g\|_p$. Our first main result is stated as follows.

Theorem 2.2. *Suppose φ is a $\|\cdot\|_2$-radial function on $\mathbb{F}_\infty$ with $\varphi(e) = 1$, the following are equivalent.*

1. *$\varphi(g)$ defines a positive definite function on $\mathbb{F}_\infty$.*

2. *There is a probability measure μ on $[-1, 1]$ such that*

$$\varphi(g) = \int_{-1}^{1} s^{\|g\|_2^2} \, d\mu(s)$$

Moreover, if (2) holds, then μ is uniquely determined by φ.

Theorem 2.3. *Suppose $\psi : \mathbb{F}_\infty \to \mathbb{C}$ is an ℓ^2-radial function with $\psi(e) = 0$. Then, the following are equivalent:*

1. *ψ is conditionally negative definite on $\mathbb{F}_\infty$*

2. *There is a probability measure ν on $[-1, 1]$ such that*

$$\psi(g) = \int_{-1}^{1} \frac{1 - s^{\|g\|_2^2}}{1 - s} \, d\nu(s)$$

Moreover, if (2) holds, then ν is uniquely determined by ψ.

We obtain characterizations for $\|\cdot\|_p$-radial, positive definite and $\|\cdot\|_p$-radial conditionally negative definite functions on the infinite free product of the group of real numbers as well. See Corollary 2.36 and Corollary 2.37. Similar results for the commutative case is also generalized in Corollary 2.38.

Positive definite functions are closely connected to completely positive maps of the Fourier multiplier type. Let G be a group and $\varphi : G \to \mathbb{C}$ a bounded function.

Let λ_s be the left regular representation of $s \in G$. Consider the associated multiplier M_φ on $Span[\lambda(G)]$ defined as:

$$M_\varphi \left(\sum c_s \lambda_s \right) = \sum \varphi(s) c_s \lambda_s \tag{2.2}$$

Then M_φ extends to a completely positive map on $C_r^*(G)$ if and only if φ is positive definite. In this case, M_φ is also completely bounded on $C_r^*(G)$ with norm $\varphi(e)$.

Following Haagerup's pioneer work ([7]), the completely positivity and the completely boundedness of the map M_φ, with φ being a radial function with respect to the ℓ_1-length, are fully characterized ([12, 10]). These works significantly improve the understanding of the approximation properties of the free groups and the associated noncommutative L^p-spaces. Nevertheless, our understanding is still incomplete. For instance, the existence of a Schauder basis for the reduced free group C^* algebra is still a mystery. A better understanding of positive definite functions beyond ℓ_1-radial type would help. The main results of this article (Theorem 1.2 and 1.3) complement Haagerup and Knudby's work ([10]) and provide characterizations of the complete positivity of the corresponding multipliers M_φ defined as in (2.2) with φ being ℓ_2-radial. The classical ℓ_2-radial Fourier multipliers are those associated with the Laplacian operators. We hope the results obtained in this article will shed light on determining an appropriate Laplace type operator on the free group C^*-algebras.

2.2 Preliminaries

First, we recall the general definition of a positive definite function and a conditionally negative definite function.

Definition 2.4. Let G be a group. A function $\varphi : G \to \mathbb{C}$ is Hermitian if $\varphi(g^{-1}) = \overline{\varphi(g)}$ for all $g \in G$.

Definition 2.5. Let G be a group. A function $\varphi : G \to \mathbb{C}$ is positive definite if for each $n \in \mathbb{N}$, $\{x_1, ..., x_n\} \subseteq G$ and $\{c_1, ..., c_n\} \subseteq \mathbb{C}$,

$$\sum_{j,k=1}^{n} c_j \; \overline{c_k} \; \varphi(x_j^{-1} x_k) \geqslant 0.$$

Definition 2.6. Let G be a group. A function $\psi : G \to \mathbb{C}$ is conditionally negative definite if

1. ψ is Hermitian.

2. For each $n \in \mathbb{N}$, $\{x_1, ..., x_n\} \subseteq G$ and $\{c_1, ..., c_n\} \subseteq \mathbb{C}$ satisfying

$$\sum_{j=1}^{n} c_j = 0, \text{ we have that } \sum_{j,k=1}^{n} c_j \; \overline{c_k} \; \varphi(x_j^{-1} x_k) \leqslant 0.$$

Bochner's Theorem[page 19, [22]]: Let G be a locally compact, abelian group and let $\phi : G \to \mathbb{C}$ be a continuous function. Then, ϕ is positive definite on G if and only if there exists a non-negative, finite Radon measure μ on the dual group of G, Γ such that

$$\phi(x) = \int_{\Gamma} \gamma(x) \; d\mu(\gamma) \qquad (x \in G).$$

Now, we recall Schoenberg's theorem which characterizes the conditionally negative definite functions on G.

Schoenberg Theorem: Let G be a group. Let $\psi : G \to \mathbb{C}$ be a Hermitian function. Then, the following are equivalent.

1. ψ is conditionally negative definite on G.

2. For each $t > 0$, the function $\varphi_t : G \to \mathbb{C}$ defined by $\varphi_t(g) := e^{-t\psi(g)}$ is positive definite.

Besides Schoenberg's theorem, there is also another classical result which relates a conditionally negative definite kernel with a function on a Hilbert space. However, this requires some additional assumption that the kernel is real valued and is zero on the diagonal.

Lemma 2.7. *Let G be a group. Let $\psi : G \to \mathbb{R}$ be a real-valued function where $\psi(e) = 0$. Then, the following are equivalent.*

1. *ψ is conditionally negative definite on G.*

2. *There exists a Hilbert space H and a function $f : G \to H$ such that $\psi(x^{-1}y) = \|f(y) - f(x)\|^2$ for all $x,\, y \in G$.*

Definition 2.8. Let G be a group and $\theta : G \to \mathbb{R}_+$ be a function. A function $\varphi : G \to \mathbb{C}$ is said to be radial with respect to θ if there exists $\dot{\varphi} : \mathrm{ran}(\theta) \to \mathbb{R}_+$ such that for all $g \in G$, $\varphi(g) = \dot{\varphi}[\theta(g)]$.

Apart from the case for groups, there is also an analogous definition of positive definite functions in the setting of an abelian semi-group. However, care must be taken since in general, a semi-group does not have an inverse. As a remark, one can prove the theory for a general involution semigroup. However, for our purpose, we always assume the involution operator to be the identity operator.

Definition 2.9. Let S be an abelian semigroup. A function $\varphi : S \to \mathbb{C}$ is positive definite in the semi-group sense if for each $n \in \mathbb{N}$, $\{s_1, ..., s_n\} \subseteq \mathbb{N}$ and $\{c_1, ..., c_n\} \subseteq \mathbb{C}$,

$$\sum_{j,k=1}^{n} c_j \, \overline{c_k} \, \varphi(s_j + s_k) \geqslant 0.$$

Definition 2.10. Let S be an abelian semigroup. A function $\psi : S \to \mathbb{C}$ is conditionally negative definite in the semi-group sense if

1. ψ is real-valued.

2. For each $n \in \mathbb{N}$, $\{s_1, ..., s_n\} \subseteq G$ and $\{c_1, ..., c_n\} \subseteq \mathbb{C}$ satisfying

$$\sum_{j=1}^{n} c_j = 0, \text{ we have that } \sum_{j,k=1}^{n} c_j \, \overline{c_k} \, \varphi(s_j + s_k) \leqslant 0.$$

Definition 2.11. A function $\rho : S \to \mathbb{R}$ is called a semicharacter if

1. $\rho(0) = 1$.
2. $\rho(s + t) = \rho(s)\rho(t)$ for $s, t \in S$.

Definition 2.12. Let S be an abelian semigroup. The set $S^* := \{\rho : \rho \text{ is a semicharacter}\}$ equipped with the topology of pointwise convergence is called the dual semigroup of S.

It turns out that being equipped with the topology of pointwise convergence, S^* becomes a completely regular space, in particular it is a Hausdorff space. Moreover, it forms a topological semigroup, with the multiplication defined via pointwise multiplication and the constant function 1 being the identity.

Definition 2.13. : Let S be an abelian semigroup. The set $\widehat{S} := \{\rho \in S^* : |\rho(s)| \leqslant 1$ for $s \in S\}$ is called the restricted dual semigroup.

By inheriting the subspace topology of S^*, $\widehat{S}$ becomes a compact subsemigroup of S^*.

We need the following result from [Theorem 4.2.8, page 96, [1]].

Lemma 2.14. *Let S be an abelian semigroup. A function $\varphi : S \to \mathbb{C}$ is positive definite and bounded on S if and only if*

$$\varphi(s) = \int_{\widehat{S}} \rho(s) \, d\mu(\rho) \qquad (s \in S),$$

where μ is a Radon measure on $\widehat{S}$. Moreover, if we assume that $\varphi(e) = 1$, then μ is a probability measure.

Remark 2.15. Note that there is a slight difference between the formulation of the results for case of groups and semigroups.

2.3 Key Lemma

Definition 2.16. Let G be a group and $\theta : G \to \mathbb{R}_+$ be a function. We denote the abelian semigroup S_θ induced by θ to be:

$$S_\theta := \left\{ s \in \mathbb{R}_+ : s = \sum_{k=1}^{N} c_k \theta(g_k) \text{ where } c_k \in \mathbb{N}, g_k \in G \right\}$$

Definition 2.17. Let G be a group. A function $\theta : G \to \mathbb{R}_+$ is said to be a partial morphism on G if it satisfies the following property:

Given $N \in \mathbb{N}$, $M \in \mathbb{N}$ and $s_1, \cdots, s_M \in S$, there exist elements $\{g_{n,s_k} \in G : 1 \leqslant n \leqslant N \text{ and } 1 \leqslant k \leqslant M\}$ such that $\theta(g_{n,s_j}^{-1} g_{m,s_k}) = s_k + s_j$ for $n \neq m$.

Remark 2.18. If θ is a partial morphism, then the range of θ is a semi-group.

Lemma 2.19. *Let $\theta : G \to \mathbb{R}_+$ be a partial morphism. Let $\varphi : G \to \mathbb{R}$ be a positive definite, radial function with respect to θ. Suppose that θ is a partial morphism on G. Then, the corresponding function $\dot{\varphi} : S_\theta \to \mathbb{R}$ satisfying $\dot{\varphi}[\theta(g)] = \varphi(g)$ is positive definite in the semi-group sense and bounded.*

Proof. Fix $N \in \mathbb{N}$, $K \in \mathbb{N}$. Consider $\{s_1, \cdots, s_K\} \subseteq S_\theta$, $\{c_1, \cdots, c_K\} \subseteq \mathbb{C}$.

Next, we set $d_{1,1} = d_{2,1} = \cdots = d_{N,1} :\equiv c_1$, $d_{1,2} = d_{2,2} = \cdots = d_{N,2} :\equiv c_2$, $\cdots, d_{1,K} = d_{2,K} = \cdots = d_{N,K} :\equiv c_K$.

Since θ is a partial morphism on G, there exist distinct elements $\{g_{n,s_k} \in G : 1 \leqslant n \leqslant N \text{ and } 1 \leqslant k \leqslant K\}$ such that $\theta(g_{n,s_j}^{-1} g_{m,s_k}) = s_k + s_j$ for $n \neq m$.

Since φ is positive definite on G, $\sum_{n,m=1}^{N} \sum_{k,j=1}^{K} d_{m,k} \overline{d_{n,j}} \varphi(g_{n,s_j}^{-1} g_{m,s_k}) \geqslant 0$. Next, we perform some calculation.

For each $1 \leqslant n, m \leqslant N$, we have that:

$$\sum_{j,k=1}^{K} d_{m,k}\overline{d_{n,j}}\varphi(g_{n,s_j}^{-1}g_{m,s_k}) = \sum_{j,k=1}^{K} d_{m,k}\overline{d_{n,j}}\dot{\varphi}[\theta(g_{n,s_j}^{-1}g_{m,s_k})].$$

If $m \neq n$, then

$$\sum_{j,k=1}^{K} d_{m,k}\overline{d_{n,j}}\varphi(g_{n,s_j}^{-1}g_{m,s_k}) = \sum_{j,k=1}^{K} d_{m,k}\overline{d_{n,j}}\dot{\varphi}[\theta(g_{n,s_j}^{-1}g_{m,s_k})] = \sum_{j,k=1}^{K} d_{m,k}\overline{d_{n,j}}\dot{\varphi}(s_j + s_k).$$

Taking the sum of m, n from 1 to N, we obtain:

$$\begin{aligned}
&\sum_{m,n=1}^{N}\sum_{j,k=1}^{K} d_{m,k}\overline{d_{n,j}}\varphi(g_{n,s_j}^{-1}g_{m,s_k}) \\
&= \sum_{n=1}^{N}\sum_{j,k=1}^{K} d_{n,k}\overline{d_{n,j}}\varphi(g_{n,s_j}^{-1}g_{m,s_k}) + \sum_{1\leqslant m\neq n\leqslant N}\sum_{k,j=1}^{K} d_{m,k}\overline{d_{n,j}}\dot{\varphi}(s_j + s_k).
\end{aligned}$$

Using the relationship between $d_{n,k}$ and c_k, we obtain:

$$\begin{aligned}
0 \leqslant & \sum_{m,n=1}^{N}\sum_{j,k=1}^{K} d_{m,k}\overline{d_{n,j}}\varphi(g_{n,s_j}^{-1}g_{m,s_k}) = \sum_{m,n=1}^{N}\sum_{j,k=1}^{K} c_k\overline{c_j}\varphi(g_{n,s_j}^{-1}g_{m,s_k}) \\
=& \sum_{n=1}^{N}\sum_{j,k=1}^{K} c_k\overline{c_j}\varphi(g_{n,s_j}^{-1}g_{n,s_k}) + (N^2 - N)\sum_{k,j=1}^{M} c_k\overline{c_j}\dot{\varphi}(s_j + s_k) \\
\leqslant& \sum_{n=1}^{N}\sum_{j,k=1}^{K} |c_k||c_j||\varphi(g_{n,s_j}^{-1}g_{n,s_k})| + (N^2 - N)\sum_{k,j=1}^{M} c_k\overline{c_j}\dot{\varphi}(s_j + s_k) \\
\leqslant& \sum_{n=1}^{N}\sum_{j,k=1}^{K} |c_k||c_j|\varphi(0) + (N^2 - N)\sum_{k,j=1}^{M} c_k\overline{c_j}\dot{\varphi}(s_j + s_k) \\
=& N\sum_{j,k=1}^{K} |c_k||c_j|\varphi(0) + N(N-1)\sum_{k,j=1}^{M} c_k\overline{c_j}\dot{\varphi}(s_j + s_k).
\end{aligned}$$

Thus, $-\dfrac{1}{N-1}\left[\sum_{j,k=1}^{K} |c_k||c_j|\varphi(0)\right] \leqslant \sum_{k,j=1}^{M} c_k\overline{c_j}\dot{\varphi}(s_j + s_k).$

Since the above inequality holds true for all $N \in \mathbb{N}$, we have $\sum_{k,j=1}^{M} c_k \overline{c}_j \dot{\varphi}(s_j + s_k) \geqslant 0$. Since the above inequality holds true for any $\{s_1, \cdots, s_K\} \subseteq G$ and any $\{c_1, \cdots, c_K\} \subseteq \mathbb{C}$, $\dot{\varphi}$ is positive definite in the semi-group sense on S.

Finally, $\dot{\varphi}$ is bounded because φ is bounded.

□

Remark 2.20. It turns out that Lemma 2.19 still holds true under a slightly weaker assumption. We will explain the settings and state the theorem.

Theorem 2.21. *Given a group G, let $\theta : G \to \mathbb{R}_+$ be a function that satisfies the following property:*

For every finite subset $U := \{s_1, \cdots, s_K\} \subseteq S_\theta$, there exists a sequence of finite subsets of G, $(V_N)_{N=1}^{\infty}$, where $V_{U,N} := \{g_{n,s_k} \in G : 1 \leqslant n \leqslant N, 1 \leqslant k \leqslant K\}$ such that

$$\lim_{N\to\infty} \frac{\#[\{(m,n) \in \mathbb{N}^2 : 1 \leqslant m,n \leqslant N \text{ and for all } 1 \leqslant j,k \leqslant K, \theta(g_{m,j}^{-1} g_{n,k}) = s_j + s_k\}]}{N^2}$$

$$= 1.$$

Then, for each positive definite function $\varphi : G \to \mathbb{R}$ which is radial with respect to θ, the corresponding function $\dot{\varphi}$: range$(\theta) \to \mathbb{R}$ satisfying $\dot{\varphi}[\theta(g)] = \varphi(g)$ is positive definite in the semi-group sense and bounded.

Corollary 2.22. *Let $\theta : G \to \mathbb{C}$ be a function. Let $\psi : G \to \mathbb{R}$ be a conditionally negative definite, radial function with respect to θ. Suppose that θ is a partial morphism on G. Then, the corresponding function $\dot{\psi} : S_\theta \to \mathbb{R}$ satisfying $\dot{\psi}[\theta(g)] = \psi(g)$ is conditionally negative definite in the semi-group sense and bounded below.*

Proof. Since ψ is a conditionally negative definite function on the group G, by Schoenberg's theorem, for all $t > 0$, the function $\varphi_t : G \to \mathbb{C}$ defined by $\varphi_t(g) := e^{-t\psi(g)}$ is positive definite on G. By Lemma 2.19, the function $\dot{\varphi}_t : S_\theta \to \mathbb{R}$ defined by $\dot{\varphi}_t[\theta(g)] := e^{-t\dot{\psi}[\theta(g)]}$ is positive definite on S_θ. Again, by Schoenberg's theorem, the function $\dot{\psi}$ is conditionally negative definite in the semi-group sense on S_θ.

Finally, $\dot{\psi}$ is bounded below because ψ is bounded below. □

Theorem 2.23. *Let G be a group and $\theta : G \to \mathbb{R}_+$ be a partial morphism on G. Let $\varphi : G \to \mathbb{R}$ be a radial function with respect to θ where $\varphi(e) = 1$, If φ is positive definite on G, then there exists a unique probability measure μ on $\widehat{S}_\theta$ such that*

$$\varphi(g) = \dot{\varphi}[\theta(g)] = \int_{\widehat{S}} \rho[\theta(g)] \; d\mu(\rho).$$

Proof. Since φ is positive definite on G, $\dot{\varphi}$ is positive definite in the semi-group sense on S_θ by Lemma 2.19. Then, by Lemma 2.14, there exists a unique probability measure μ on $\widehat{S}$ such that

$$\varphi(g) = \dot{\varphi}[\theta(g)] = \int_{\widehat{S}} \rho[\theta(g)] \; d\mu(\rho).$$

□

2.4 Proof of Main Theorems

2.4.1 Case of ℓ^2 Length for $\mathbb{F}_\infty$

We consider the case where the group $G = \mathbb{F}_\infty$, the free group with infinite generators and the function $\theta = \|\cdot\|_2^2$ is the ℓ^2 length of an element in $\mathbb{F}_\infty$.

In this case, $S_\theta = \mathbb{N}$ and hence, $\widehat{S_\theta} \cong [-1, 1]$. More precisely, given $\rho \in \widehat{\mathbb{N}}$, there exists a unique $x \in [-1, 1]$ such that $\rho(n) = x^n$. First, we show that the ℓ^2 length function is indeed a partial morphism on $\mathbb{F}_\infty$.

Proposition 2.24. *The function $\| \cdot \|_2^2 : \mathbb{F}_\infty \to \mathbb{N}$ is a partial morphism on $\mathbb{F}_\infty$.*

Proof: We enumerate the infinite generators of $\mathbb{F}_\infty$ as follows:

$\{g_{1,1}, g_{1,2}, \cdots, g_{2,1}, g_{2,2}, \cdots g_{n,1}, g_{n,2} \cdots ; n \in \mathbb{N}^+\}.$

For each $n \in \mathbb{N}$, define $q_n : \mathbb{N}^+ \to \mathbb{F}_\infty$ by

$$q_n(j) := g_{n,j} g_{n,j-1} \cdots g_{n,1}.$$

So, for each m, n, j, $k \in \mathbb{N}$, we have:

$$\begin{aligned} \|[q_n(j)]^{-1} q_m(k)\|_2^2 &= \|g_{n,1}^{-1} g_{n,2}^{-1} \cdots g_{n,j}^{-1} g_{m,k} \cdots g_{m,1}\|_2^2 \\ &= \begin{cases} 0, & \text{if } \ m = n \ and \ k = j, \\ j + k, & \text{otherwise.} \end{cases} \end{aligned}$$

□

To proceed, we show that the ℓ^2 length function is a conditionally negative definite function on $\mathbb{F}_\infty$ by an explicit proof.

Proposition 2.25. *Let $\mathbb{F}_r$ be a free group with generators $g_1, g_2, \cdots, g_r$, where $r \in \mathbb{N} \cup \{\infty\}$.*

1. *Let $s \in [0, 1]$. Then, the function $\psi : \mathbb{F}_r \to \mathbb{R}$ defined by $\psi(g) := s^{\|g\|_2^2}$ is positive definite on $\mathbb{F}_r$, i.e. the function $\varphi : \mathbb{F}_r \to \mathbb{R}$ defined by $\varphi(g) := \|g\|_2^2$ is conditionally negative definite on $\mathbb{F}_r$.*

2. *The function $\varphi : \mathbb{F}_r \to \mathbb{R}$ defined by $\varphi(g) := (-1)^{\|g\|_2^2}$ is positive definite on $\mathbb{F}_r$.*

Moreover, $g \to s^{\|g\|_2^2}, s \in [-1, 1]$ is a positive definite function on $\mathbb{F}_r$.

Proof. Let $\{g_1, g_2, \cdots, g_r\}$ be the generators of $\mathbb{F}_r$ and let $g = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}, f = b_1^{l_1} b_2^{l_2} \cdots a_m^{l_m}$ be the reduced words of $g, f \in \mathbb{F}_r$, respectively.

(1):By Schoenberg's Theorem, it is enough to prove that the function $g \mapsto \|g\|_p^p$, $g \in \mathbb{F}_r$ is a conditionally negative definite function on $\mathbb{F}_r$. Now, let $\ell^2(\mathbb{C})$ be the space of square summable sequence. For any $g = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$, define $T : \mathbb{F}_r \to \ell^2(\mathbb{F}_r)$ by

$$T(g) = k_1 \delta_{a_1} + k_2 \delta_{a_1^{k_1} a_2} + \cdots + k_n \delta_{a_1^{k_1} a_2^{k_2} \cdots a_{n-1}^{k_{n-1}} a_n}.$$

Suppose that $g = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$, $f = b_1^{l_1} b_2^{l_2} \cdots b_m^{l_m} \in \mathbb{F}_r$. Without loss of generality, we also suppose that $n \geqslant m$ and $\sum_{i=1}^n |k_i| \geqslant \sum_{i=1}^m |l_i|$. Let $g = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$.

Case 1: f is a prefix subword of g.

Then, the reduced word of f has the form $f = a_1^{k_1} a_2^{k_2} \cdots a_m^{l_m}$, where $|l_m| \leqslant |k_m|$, $0 \leqslant m \leqslant n$ and l_m, k_m have the same signs. Note that $f = e$ when $m = 0$. It follows that $f^{-1} g = a_m^{k_m - l_m} a_{m+1}^{k_{m+1}} \cdots a_n^{k_n}$ and $\|f^{-1} g\|_2^2 = |k_m - l_m|^2 + |k_{m+1}|^2 + \cdots + |k_n|^2$. Moreover,

$$\begin{aligned} \|T(g) - T(f)\|^2 =& \|(k_m - l_m)\delta_{a_1^{k_1} a_2^{k_2} \cdots a_{m-1}^{k_{m-1}} a_m} + k_{m+1} \delta_{a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m} a_{m+1}} \\ &+ \cdots + k_n \delta_{a_1^{k_1} a_2^{k_2} \cdots a_{n-1}^{k_{n-1}} a_n} \|^2 \\ =& |k_m - l_m|^2 + |k_{m+1}|^2 + \cdots + |k_n|^2 = \|f^{-1} g\|_2^2. \end{aligned}$$

Case 2: f and g share a same sub-word prefix block but f travel in an opposite direction from g at the block where they differ.

More precisely, there exists $1 \leqslant i \leqslant m$ such that $f = a_1^{k_1} \cdots a_{i-1}^{k_{i-1}} a_i^{j_i} b_{i+1}^{l_{i+1}} \cdots b_m^{l_m}$, where k_i and j_i have opposite signs. This implies that $f^{-1} g = b_m^{-l_m} \cdots b_{i+1}^{-l_{i+1}} a_i^{k_i - j_i} a_{i+1}^{k_{i+1}} \cdots a_n^{k_n}$ and

$$\begin{aligned} \|f^{-1} g\|_2^2 &= |l_m|^2 + \cdots + |l_{i+1}|^2 + |k_i - j_i|^2 + |k_{i+1}|^2 + \cdots + |k_n|^2 \\ &= |l_m|^2 + \cdots + |l_{i+1}|^2 + (|k_i| + |j_i|)^2 + |k_{i+1}|^2 + \cdots + |k_n|^2. \end{aligned}$$

So, $\|T(g)-T(f)\|^2$

$$= \|k_n\delta_{a_1^{k_1}a_2^{k_2}\cdots a_{n-1}^{k_n-1}a_n} + \cdots + (k_i - j_i)\delta_{a_1^{k_1}a_2^{k_2}\cdots a_{i-1}^{k_i-1}a_i} - \cdots - l_m\delta_{a_1^{k_1}\cdots b_{m-1}^{l_{m-1}}b_m}\|^2$$

$$= |l_m|^2 + \cdots + |l_{i+1}|^2 + (|k_i| + |j_i|)^2 + |k_{i+1}|^2 + \cdots + |k_n|^2 = \|f^{-1}g\|_2^2.$$

Case 3: f and g share a same sub-word prefix block but f travel in a different direction which is not directly opposite from g at the block where they differ.

More precisely, there exist $1 \leqslant i \leqslant m$ and $b_i \neq a_i$ such that $f = a_1^{k_1}\cdots a_{i-1}^{k_{i-1}}b_i^{j_i}b_{i+1}^{j_{i+1}}\cdots b_m^{j_m}$.

So, $f^{-1}g = b_m^{-j_m}\cdots b_{i+1}^{-j_{i+1}}b_i^{-j_i}a_i^{k_i}a_{i+1}^{k_{i+1}}\cdots a_n^{k_n}$ and $\|f^{-1}g\|_2^2 = |j_m|^2 + \cdots + |j_{i+1}|^2 + |j_i|^2 + |k_i|^2 + |k_{i+1}|^2 \cdots + |k_n|^2$.

Next, $\|T(g)-T(f)\|_2^2$

$$= \|k_1\delta_{a_1} + k_2\delta_{a_1^{k_1}a_2} + \cdots + k_{i-1}\delta_{a_1^{k_1}\cdots a_{i-2}^{k_{i-2}}a_{i-1}} + k_i\delta_{a_1^{k_1}\cdots a_{i-1}^{k_{i-1}}a_i} + \cdots + k_n\delta_{a_1^{k_1}a_2^{k_2}\cdots a_{n-1}^{k_{n-1}}a_n}$$

$$- k_1\delta_{a_1} - k_2\delta_{a_1^{k_1}a_2} - \cdots - k_{i-1}\delta_{a_1^{k_1}\cdots a_{i-2}^{k_{i-2}}a_{i-1}} - j_i\delta_{a_1^{k_1}\cdots a_{i-1}^{k_{i-1}}b_i} - \cdots - j_m\delta_{a_1^{k_1}\cdots b_{m-1}^{j_{m-1}}b_m}\|$$

$$= |k_i|^2 + \cdots |k_n|^2 + |j_i|^2 + \cdots + |j_m|^2 = \left\|f^{-1}g\right\|_2^2.$$

Case 4: f and g share a same sub-word prefix block and the point of deviation occurs in the interior of a block.

More precisely, there exist $1 \leqslant i \leqslant m$ such that $f = a_1^{k_1}\cdots a_{i-1}^{k_{i-1}}a_i^{j_i}b_{i+1}^{j_{i+1}}\cdots b_m^{j_m}$, $|j_i| < |k_i|$ and j_i, k_i have the same sign. Thus, we obtain:

$$f^{-1}g = b_m^{-j_m}\cdots b_{i+1}^{-j_{i+1}}a_i^{k_i-j_i}a_{i+1}^{k_{i+1}}\cdots a_m^{k_m} \text{ and}$$

$$\left\|f^{-1}g\right\|_2^2 = |j_m|^2 + \cdots + |j_{i+1}|^2 + |k_i - j_i|^2 + |k_{i+1}|^2 \cdots + |k_m|^2.$$

Next, $\|T(g)-T(f)\|_2^2$

$$= \|k_1\delta_{a_1} + k_2\delta_{a_1^{k_1}a_2} + \cdots + k_{i-1}\delta_{a_1^{k_1}\cdots a_{i-2}^{k_{i-2}}a_{i-1}} + k_i\delta_{a_1^{k_1}\cdots a_{i-1}^{k_{i-1}}a_i} + \cdots + k_n\delta_{a_1^{k_1}a_2^{k_2}\cdots a_{n-1}^{k_{n-1}}a_n}$$

$$- k_1\delta_{a_1} - k_2\delta_{a_1^{k_1}a_2} - \cdots - k_{i-1}\delta_{a_1^{k_1}\cdots a_{i-2}^{k_{i-2}}a_{i-1}} - j_i\delta_{a_1^{k_1}\cdots a_{i-1}^{k_{i-1}}a_i} - \cdots - j_m\delta_{a_1^{k_1}\cdots b_{m-1}^{j_{m-1}}b_m}\|_2^2$$

$$= |k_m|^2 + \cdots + |k_{i+1}|^2 + |k_i - j_i|^2 + |j_{i+1}|^2 \cdots + |j_m|^2 = \left\|f^{-1}g\right\|_2^2.$$

So, the function $g \mapsto \|g\|_2^2, g \in \mathbb{F}_r$ is conditionally negative definite on $\mathbb{F}_r$.

(2):

$$\text{Let } \varphi(g) := \dot{\varphi}(\|g\|_2^2) \triangleq \begin{cases} 1, & if\ \|g\|_2^2\ is\ even, \\ -1, & if\ \|g\|_2^2\ is\ odd. \end{cases} \tag{2.3}$$

For any $n \in \mathbb{N}$, any $g_1, g_2, \cdots \in \mathbb{F}_r$ and any $c_1, \cdots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$,

$$\sum_{i,j=1}^n c_i\overline{c_j}\varphi(g_ig_j^{-1}) = \sum_{i,j=1}^n c_i\overline{c_j}\dot{\varphi}(\|g_ig_j^{-1}\|_2^2) = \sum_{\left\{\left\|g_ig_j^{-1}\right\|_2^2 \text{ even}\right\}} c_i\overline{c_j} - \sum_{\left\{\left\|g_ig_j^{-1}\right\|_2^2 \text{ odd}\right\}} c_i\overline{c_j}$$

$$= \sum_{\left\{\left|g_ig_j^{-1}\right| \text{ even}\right\}} c_i\overline{c_j} - \sum_{\left\{\left|g_ig_j^{-1}\right| \text{ odd}\right\}} c_i\overline{c_j} = \sum_{\{|g_i|-|g_j| \text{ even}\}} c_i\overline{c_j} - \sum_{\{|g_i|-|g_j| \text{ odd}\}} c_i\overline{c_j}$$

$$= \sum_{\{|g_i|,|g_j| \text{ even}\}} c_i\overline{c_j} + \sum_{\{|g_i|,|g_j| \text{ odd}\}} c_i\overline{c_j} - \sum_{\{|g_i| \text{ odd, } |g_j| \text{ even}\}} c_i\overline{c_j} - \sum_{\{|g_i| \text{even, } |g_j| \text{ odd}\}} c_i\overline{c_j}$$

$$= \left|\sum_{\{|g_i| \text{ is even}\}} c_i - \sum_{\{j\in\mathbb{N}:|g_j| \text{ is odd}\}} c_j\right|^2 \geqslant 0.$$

It follows that $\varphi(g) = \dot{\varphi}(\|g\|_2^2) = (-1)^{\|g\|_2^2}$ is positive definite.

(1) implies that for $0 \leqslant s \leqslant 1$, the function $g \mapsto s^{\|g\|_2^2}$ is positive definite. For $-1 \leqslant s < 0$, $s^{\|g\|_2^2} = (-1)^{\|g\|_2^2}(-s)^{\|g\|_2^2}$. Since a product of two positive definite functions is also positive definite, for $-1 \leqslant s \leqslant 1$, the function $g \to s^{\|g\|_2^2}$ is positive definite on $\mathbb{F}_r$.

□

Corollary 2.26. *Given $\varphi : \mathbb{F}_\infty \to \mathbb{R}$ a radial function where $\varphi(e) = 1$, the following are equivalent.*

1. *φ is positive definite on $\mathbb{F}_\infty$.*

2. *There is a probability measure μ on $[-1, 1]$ such that*

$$\varphi(g) = \dot{\varphi}(\|g\|_2^2) = \int_{-1}^{1} s^{\|g\|_2^2} \, d\mu(s)$$

Moreover, if (2) holds, then μ is uniquely determined by φ.

Proof. (1) $\implies$ (2) follows from Theorem 2.23. To prove (2) $\implies$ (1), let μ be a probability measure on $[-1, 1]$. For each $s \in [-1, 1]$, the function $\psi_s(g) := s^{\|g\|_2^2}$ is positive definite on $\mathbb{F}_\infty$ by Proposition 2.25. Taking finite sums and limits, we deduce that the function $\varphi : \mathbb{F}_\infty \to \mathbb{R}$ defined by: $\varphi(g) := \int_{-1}^{1} s^{\|g\|_2^2} \, d\mu(s)$ is positive definite. □

Theorem 2.27. *Let $\psi : \mathbb{F}_\infty \to \mathbb{C}$ be an ℓ^2-radial function where $\psi(e) = 0$. Then, the following are equivalent:*

1. *ψ is conditionally negative definite on $\mathbb{F}_\infty$.*

2. *There is a probability measure ν on $[-1, 1]$ such that*

$$\psi(g) = \int_{-1}^{1} \frac{1 - s^{\|g\|_2^2}}{1 - s} \, d\nu(s).$$

Moreover, if (2) holds, then ν is uniquely determined by ψ.

Proof: ($\implies$) By Schoenberg's theorem, since ψ is conditionally negative definite and $\psi(e) = 0$, the function $\varphi_t : \mathbb{F}_\infty \to \mathbb{C}$ defined by $\varphi_t(g) := e^{-t\psi(g)}$ is positive definite for each $t > 0$. Also, $\varphi_t(g) = 1$. By Corollary 2.26, there exists a unique probability measure μ_t on $[-1, 1]$ such that

$$e^{-t\psi(g)} = \varphi_t(g) = \int_{-1}^{1} s^{\|g\|_2^2} \, d\mu_t(s).$$

Now, let $t > 0$ and define a new measure on the Borel σ-algebra of $[-1, 1]$, ν_t by $\nu_t(E) := \int_{-1}^{1} \chi_E(s) \frac{1-s}{t} \, d\mu_t(s)$. Note that

$$\begin{aligned}
\frac{1 - e^{-t\psi(g)}}{t} &= \int_{[-1,1)} \frac{1 - s^{\|g\|_2^2}}{t} \, d\mu_t(s) + \int_{\{1\}} \frac{1 - s^{\|g\|_2^2}}{t} \, d\mu_t(s) \\
&= \int_{[-1,1)} \frac{1 - s^{\|g\|_2^2}}{t} \, d\mu_t(s) \left(\text{since } \frac{1 - s^{\|g\|_2^2}}{t} = 0 \text{ at } x = 1 \right) \\
&= \int_{[-1,1)} \frac{1 - s^{\|g\|_2^2}}{1 - s} \frac{1 - s}{t} \, d\mu_t(s) = \int_{[-1,1)} \frac{1 - s^{\|g\|_2^2}}{1 - s} \, d\nu_t(s) \\
&= \int_{[-1,1)} \frac{1 - s^{\|g\|_2^2}}{1 - s} \, d\nu_t(s) + \int_{\{1\}} \frac{1 - s^{\|g\|_2^2}}{1 - s} \, d\nu_t(s) \text{ (since } \nu_t(\{1\}) = 0) \\
&= \int_{[-1,1]} \frac{1 - s^{\|g\|_2^2}}{1 - s} \, d\nu_t(s).
\end{aligned}$$

Applying the identity for $h \in \mathbb{F}_\infty$ where $\|h\|_2^2 = 1$, we obtain:

$$\nu_t([-1, 1]) = \frac{1 - e^{-t\psi(h)}}{t}$$

Taking the supremum over all $t > 0$,

$$\sup_{t>0} \|\nu_t\|_{\text{var}} = \sup_{t>0} \frac{1 - e^{-t\psi(h)}}{t} = \psi(h).$$

So, the set $\{\nu_t : t > 0\}$ is uniformly bounded in the space of Radon measures on $[-1, 1]$. Note that for each $g \in \mathbb{F}_\infty$, $\frac{1-e^{-t\psi(g)}}{t} \to \psi(g)$ as $t \to 0$. Next, we focus on the terms ν_t.

Consider $\Lambda = \{t \in \mathbb{R} : 0 < t \leqslant 1\}$ as a directed set with partial order, $\dot{\leqslant}$ defined as follows: $s \dot{\leqslant} t$ means that $s > t$. So, ν_t is a net in $M^+([-1,1])$, the space of positive Radon measure on $[-1,1]$.

Since $(\nu_t)_{t\in\Lambda}$ is a bounded set in $M([-1,1])$, the space of Radon measures on $[-1,1]$, by the Banach-Alaoglu theorem, there exists a subnet $(\nu_{t_\alpha})_{\alpha\in E}$ and $\nu \in M([-1,1])$ such that $\nu_{t_\alpha} \to \nu$ in the weak-* topology of $M([-1,1])$.

$$\int_{-1}^{1} \frac{1-s^{\|g\|_2^2}}{1-s}\, d\nu(s) = \lim_{\alpha} \int_{-1}^{1} \frac{1-s^{\|g\|_2^2}}{1-s}\, d\nu_{t_\alpha}(s)$$
$$= \lim_{\alpha} \frac{1-e^{-t_\alpha \psi(g)}}{t_\alpha} = \lim_{t\to 0} \frac{1-e^{-t\psi(g)}}{t} = \psi(g)$$

The equality $\lim_{\alpha} \frac{1-e^{-t_\alpha \psi(g)}}{t_\alpha} = \lim_{t\to 0} \frac{1-e^{-t\psi(g)}}{t}$ holds true due to the following:

1. $t_\alpha \dot{\leqslant} t_\beta$ whenever $\alpha < \beta$ (in E)

2. For each $r \in \Lambda$, there exits $\alpha \in E$ such that $r \dot{\leqslant} t_\alpha$.

For the direction (2) $\implies$ (1), let ν be a probability measure on $[-1,1]$. Note that for each $s \in [-1,1]$, the function $g \mapsto \frac{1-s^{\|g\|_2^2}}{1-s}$ is conditionally negative definite. Taking finite sums and limits, we deduce that $g \mapsto \int_{-1}^{1} \frac{1-s^{\|g\|_2^2}}{1-s}\, d\nu(s)$ is conditionally negative definite. □

2.4.2 Case of ℓ^p Length of The Free Real Line with Infinite Generators for $0 < p \leqslant 2$

Now, we focus on the case where the group $G = \mathbb{R}_\infty$, the free real line with infinite generators and the function $\theta = \|\cdot\|_p^p$ is the ℓ^p length of an element in $\mathbb{R}_\infty$, where $0 < p \leqslant 2$.

In this case, $S_\theta = \mathbb{R}_+$ and hence, $\widehat{S_\theta} \cong [0,\infty]$. More precisely, given $\psi \in \widehat{\mathbb{R}_+}$, either there exists a unique $a \in [0,\infty)$ such that $\psi(s) := \rho_a(s) = e^{-as}$ or $\psi(s) = \rho_\infty(s) := \chi_{\{0\}}(s)$. Consequently, we have the following characterization of the positive definite functions on $\mathbb{R}_\infty$ and the conditionally negative definite functions on $\mathbb{R}_\infty$, as given in [1, Proposition 4.4.2 and Proposition 4.4.3].

Theorem 2.28. *A function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is positive definite and bounded if and only if it has the form*

$$\varphi(s) = \int_0^\infty e^{-as} d\mu(a) + b\chi_{\{0\}}(s), s \geqslant 0,$$

where $\mu \in M_+^b(\mathbb{R}_+)$ is a bounded positive Radon measure and $b \geqslant 0$. The pair (μ, b) is uniquely determined by φ.

Theorem 2.29. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be a function. Then, ψ is conditionally negative definite and bounded below if and only if it has the form*

$$\psi(s) = \psi(0) + cs + b\chi_{(0,\infty)}(s) + \int_0^\infty 1 - e^{-as}\ d\mu(a),\ \ s \geqslant 0,$$

where b, $c \geqslant 0$ and μ a positive Radon measure on $(0,\infty)$ (possibly infinite) are uniquely determined by ψ.

Next, we show that the ℓ^p length function is indeed a partial morphism on $\mathbb{R}_\infty$.

Proposition 2.30. *The function $\|\cdot\|_p^p : \mathbb{R}_\infty \to \mathbb{R}_+$ is a partial morphism on $\mathbb{R}_\infty$.*

Proof: Let $M \in \mathbb{N}$, $r_1, \cdots, r_M \in S$. First, we enumerate the generators as $\{g_{1,1}, g_{1,2}, \cdots, g_{2,1}, g_{2,2}, \cdots, g_{M,1}, g_{M,2}, \cdots\}$.

Now, let $1 \leqslant j \leqslant M$. There exists $\lambda_j \in \mathbb{R}$ such that $(\lambda_j)^p = r_j$. Then, define $q_n : \{r_1, \cdots, r_M\} \to R_\infty$ by $q_n(r_j) := g_{j,n}^{\lambda_j}$. We note the following observation: For each $n, m \in \mathbb{N}$ and $1 \leqslant j, k \leqslant M$,

$$\|[q_n(r_j)]^{-1}q_m(r_k)\|_p^p = \begin{cases} 0 & \text{if } \ m = n \textit{ and } r_j = r_k, \\ r_j + r_k & \text{otherwise.} \end{cases}$$

□

Now, we provide a proof that for each $t > 0$, the function $r \mapsto e^{-t\|r\|_p^p}$ is positive definite on $\mathbb{R}_\infty$.

Proposition 2.31. *(Bożejko [3, Corollary 1]) Let* $\mathbb{R}_q$ *be a free real line with generators* $r_1, r_2, \cdots, r_q$, *where* $q \in \mathbb{N} \cup \{\infty\}$. *Let* $0 < p \leqslant 2$. *Then, for all* $t > 0$, *the function* $\varphi : \mathbb{R}_q \to \mathbb{R}$ *defined by* $\varphi(r) := e^{-t\|r\|_p^p}$ *is positive definite on* $\mathbb{R}_q$.

Proof: First, let $t > 0$. Observe that the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(s) := e^{-t|s|^p}$ is positive definite for each $0 < p \leqslant 2$. The $p = 2$ case is well known. The $p < 2$ cases follow from the fact that $e^{-|s|^p}$ is an average of $e^{-t|s|^2}$ in t. Since $\mathbb{R}_q = *_{i=1}^q \mathbb{R}$ (where $*$ denotes the free product of groups) and $\varphi(r) = e^{-t\|r\|_p^p} = e^{-t|r_{j_1}|^p} \cdots e^{-t|r_{j_n}|^p} = \left(*_{i=1}^n \phi\right)(r)$, we deduce that φ is a positive definite function. Here we use the fact that a free product of unital positive definite functions is positive definite (see [3, Corollary 1]). □

Next, we can verify that the function $\psi_\infty : \mathbb{R}_\infty \to \mathbb{R}_+$ defined by: $\psi_\infty(r) := \chi_{\{e\}}(r)$ is positive definite. With this, we provide the following characterization:

Corollary 2.32. *Let* $0 < p \leqslant 2$. *Given* $\varphi : \mathbb{R}_\infty \to \mathbb{R}$ *an* ℓ^p *radial function, the following are equivalent.*

1. φ *is positive definite on* $\mathbb{R}_\infty$.

2. *There exist a bounded, positive, Radon measure* μ *on* $[0, \infty)$ *and* $b \geqslant 0$ *such that*

$$\varphi(r) = \dot{\varphi}(\|r\|_p^p) = \int_0^\infty e^{-t\|r\|_p^p} \, d\mu(t) + b\chi_{\{e\}}(r).$$

Moreover, if (2) holds, then μ is uniquely determined by φ.

Proof. (1) $\implies$ (2) follows from Theorem 2.23, Theorem 2.28 and Proposition 2.30. To prove (2) $\implies$ (1), let μ be a bounded, positive, Borel measure on $[0,\infty)$. By Corollary 2.32, for each $t \in [0,\infty)$, the function $\psi_t(r) := e^{-t\|r\|_p^p}$ is positive definite on $\mathbb{R}_\infty$. Also, the function defined by $\psi_\infty(r) := \chi_e(r)$ is positive definite on $\mathbb{R}_\infty$. Taking finite sums and limits, we deduce that the function $\varphi : \mathbb{R}_\infty \to \mathbb{R}$ defined by: $\varphi(r) := \int_0^\infty e^{-t\|r\|_p^p}\, d\mu(t) + b\chi_e(r)$ is positive definite on $\mathbb{R}_\infty$. □

Corollary 2.33. *Let $0 < p \leqslant 2$. Let $\psi : \mathbb{R}_\infty \to \mathbb{R}$ be an ℓ^p-radial function. Then, the following are equivalent:*

1. *ψ is conditionally negative definite and bounded below on $\mathbb{R}_\infty$*
2. *There exist unique b, $c \geqslant 0$ and a positive Radon measure ν on $(0,\infty)$ (possibly infinite) such that*

$$\psi(r) = \dot{\psi}\left(\|r\|_p^p\right) = \psi(e) + c\|r\|_p^p + b\chi_{\mathbb{R}_\infty\setminus\{e\}}(r) + \int_0^\infty 1 - e^{-t\|r\|_p^p}\, d\nu(t)$$

Proof: $(1 \implies 2)$ By Proposition 2.30, the function $\|\cdot\|_p^p : \mathbb{R}_\infty \to \mathbb{R}_+$ is a partial morphism on $\mathbb{R}_\infty$. By Corollary 2.22, the function $\dot{\psi} : \mathbb{R}_+ \to \mathbb{R}$ defined by $\dot{\psi}\left(\|r\|_p^p\right) := \psi(r)$ is conditionally negative definite in the semi-group sense and bounded below on $\mathbb{R}_+$. By Theorem 2.29, we obtain (2).

To prove $(2 \implies 1)$, we note that the function $\|\cdot\|_p^p : \mathbb{R}_\infty \to \mathbb{R}_+$ is conditionally negative definite for all $0 < p \leqslant 2$ by Schoenberg's theorem and Proposition 2.31. Also, since the function $\chi_{\{e\}} : \mathbb{R}_\infty \to \mathbb{R}_+$ is positive definite and $\chi_{\mathbb{R}_\infty\setminus\{e\}} = 1 - \chi_{\{e\}}$, $\chi_{\mathbb{R}_\infty\setminus\{e\}}$ is conditionally negative definite on $\mathbb{R}_\infty$. Finally, since $e^{-t\|\cdot\|_p^p}$ is positive definite for all $t > 0$ by Proposition 2.30, $1 - e^{-t\|\cdot\|_p^p}$ is conditionally negative definite on $\mathbb{R}_\infty$ for all $t > 0$. Taking finite sums and limits, $\int_0^\infty 1 - e^{-t\|\cdot\|_p^p}\, d\nu(t)$ is conditionally negative definite on $\mathbb{R}_\infty$.

□

2.4.3 Case of ℓ^p Length of $\mathbb{R}^{\mathbb{N}}$ for $0 < p \leqslant 2$

Now, we focus on the case of the group $G = \mathbb{R}^{\mathbb{N}}$, the infinite direct product of countably many copies of $\mathbb{R}$ and the function $\theta = \|\cdot\|_p^p$ is the ℓ^p length of an element in $\mathbb{R}^{\mathbb{N}}$, where $0 < p \leqslant 2$. The proof is essentially similar to the case for the free real line with infinite generators. We will only state the theorems whose proofs are similar to the previous case.

Proposition 2.34. *The function $\|\cdot\|_p^p : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}_+$ is a partial morphism $\mathbb{R}^{\mathbb{N}}$.*

Now, we provide a proof that for each $t > 0$, the function $r \mapsto e^{-t\|r\|_p^p}$ is positive definite on $\mathbb{R}^{\infty}$. This is the counterpart to Proposition 2.31.

Proposition 2.35. *Let $q \in \mathbb{N} \cup \{\infty\}$, $0 < p \leqslant 2$. Then, for all $t > 0$, the function $\varphi : \mathbb{R}^q \to \mathbb{R}$ defined by $\varphi_t(r) := e^{-t\|r\|_p^p}$ is positive definite on $\mathbb{R}^q$.*

Proof: Let $t > 0$. Every $r \in \mathbb{R}^{\mathbb{N}}$ can be expressed in the form: $r = r_{d_1} + \cdots + r_{d_k}$, where r_{d_j} is the d_j-th component of $r \in \mathbb{R}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, we consider the function defined on the n-th coordinate component of $\mathbb{R}^{\mathbb{N}}$, $\phi_n : \mathbb{R}_{(n)} \to \mathbb{R}$ via $\phi_n(r_n) := e^{-t|r_j|^p}$. For each $n \in \mathbb{N}$, the function $\phi_n : \mathbb{R} \to \mathbb{R}$ is positive definite. Next, for each $r \in \mathbb{R}^{\mathbb{N}}$,

$$\varphi(r) = e^{-t\|r\|_p^p} = e^{-t\left(|r_{d_1}|^p + \cdots + |r_{d_k}|^p\right)} = e^{-t\left|r_{d_1}\right|^p} \cdots e^{-t\left|r_{d_k}\right|^p} = \phi_{d_1}(r_{d_1}) \cdots \phi_{d_k}(r_{d_k})$$

$$= (\phi_{d_1} \otimes \cdots \otimes \phi_{d_k})\,(r_{d_1} \otimes \cdots \otimes r_{d_k})$$

By [1, Corollary 3.1.13], $\phi_{d_1} \otimes \cdots \otimes \phi_{d_k}$ is positive definite on $\mathbb{R}_{(d_1)} \otimes \cdots \otimes \mathbb{R}_{(d_k)}$. It follows that φ is positive definite on $\mathbb{R}^N$. □

Corollary 2.36. *Let $0 < p \leqslant 2$. Given $\varphi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ an ℓ^p radial function, the following are equivalent.*

1. *φ is positive definite on $\mathbb{R}^{\mathbb{N}}$.*

2. *There exist a bounded, positive, Radon measure μ on $[0,\infty)$ and $b \geqslant 0$ such that*

$$\varphi(r) = \dot{\varphi}(\|r\|_p^p) = \int_0^\infty e^{-t\|r\|_p^p}\, d\mu(t) + b\chi_{\{e\}}(r).$$

Moreover, if (2) holds, then μ is uniquely determined by φ.

Corollary 2.37. *Let $0 < p \leqslant 2$. Let $\psi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be an ℓ^p-radial function. Then, the following are equivalent:*

1. *ψ is conditionally negative definite and bounded below on $\mathbb{R}^{\mathbb{N}}$.*

2. *There exist unique b, $c \geqslant 0$ and a positive Radon measure ν on $[0,\infty)$ (possibly infinite) such that*

$$\psi(r) = \dot{\psi}\left(\|r\|_p^p\right) = \psi(0) + c\|r\|_p^p + b\chi_{\mathbb{R}^{\mathbb{N}}\setminus\{e\}}(r) + \int_0^\infty 1 - e^{-t\|r\|_p^p}\, d\nu(t).$$

With these results, we provide an alternative proof of the Schoenberg-Bochner theorem [23, Theorem 13.14]

Corollary 2.38. *Let $f : [0,\infty) \to [0,\infty)$ be a function. Then, the following are equivalent.*

1. *There exists $0 < p \leqslant 2$ such that for all $d \in \mathbb{N}$, the function $\xi \mapsto f(|\xi|^p)$, $\xi \in \mathbb{R}^d$ is conditionally negative definite.*

2. *f is conditionally negative definite in the semi-group sense on $[0,\infty)$.*

3. *There exist unique b, $c \geqslant 0$ and a unique positive Radon measure ν on $[0,\infty)$ (possibly infinite) such that*

$$f(x) = \psi(0) + cx + b\chi_{(0,\infty)}(x) + \int_0^\infty 1 - e^{-tx}\, d\nu(t).$$

4. *For all* $0 < p \leqslant 2$*, the function* $g_p : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ *defined by*

$$g_p(r) := f\left(\|r\|_p^p\right) := \psi(0) + c\|r\|_p^p + b\chi_{\mathbb{R}^{\mathbb{N}}\setminus\{0\}}(r) + \int_0^\infty 1 - e^{-t\|r\|_p^p}\, d\nu(t)$$

is conditionally negative definite.

5. *For all* $0 < p \leqslant 2$ *and all* $d \in \mathbb{N}$*, the function* $g_p : \mathbb{R}^d \to \mathbb{R}$ *defined by*

$$g_p(\xi) := f\left(|\xi|_p^p\right) := \psi(0) + c|\xi|_p^p + b\chi_{\mathbb{R}^{\mathbb{N}}\setminus\{0\}}(r) + \int_0^\infty 1 - e^{-t|\xi|_p^p}\, d\nu(t)$$

is conditionally negative definite.

Proof: (1) $\implies$ (2). If $f \circ |\cdot|_p^p : \mathbb{R}^d \to \mathbb{R}$ is conditionally negative definite and bounded below for all $d \in \mathbb{N}$, then $f \circ \|\cdot\|_p^p : \mathbb{R}^{\mathbb{N}} \to [0, \infty)$ is conditionally negative definite and bounded below on $\mathbb{R}^{\mathbb{N}}$. By Lemma 2.19 and Proposition 2.30, (2) is true.

(2) $\implies$ (3) is Theorem 2.29.

(3) $\implies$ (4) This is true by Corollary 2.37, Schoenberg's theorem, Proposition 2.35 and the fact that $\chi_{\mathbb{R}^{\mathbb{N}}\setminus\{0\}}$ is conditionally negative definite on $\mathbb{R}^{\mathbb{N}}$. □

(4) $\implies$ (5). If $f \circ \|\cdot\|_p^p : \mathbb{R}^{\mathbb{N}} \to [0, \infty)$ is conditionally negative definite and bounded below on $\mathbb{R}^{\mathbb{N}}$, then $f \circ |\cdot|_p^p : \mathbb{R}^d \to \mathbb{R}$ is conditionally negative definite and bounded below for all $d \in \mathbb{N}$.

(5) $\implies$ (1) is clear.

CHAPTER THREE

Non-Commutative Khintchine Inequality

3.1 Introduction

Khintchine's inequality, which is named after Aleksandr Khinchin, plays a very important and fundamental role in classcial analysis. In the classical form, it is stated as follows, see [section A6,[6]]. For each $n \in \mathbb{N}$, let $r_n : [0,1] \to \mathbb{R}$ be defined by: $r_j(t) := \text{sgn}[\sin(2^j \pi t)]$. Then, for any $0 < p < \infty$ and any real-valued square summable sequence $(a_j)_{j=1}^{\infty}$ and $(b_j)_{j=1}^{\infty}$, there exist constants $0 < A_p < \infty$ and $0 < B_p < \infty$ such that

$$A_p \left(\sum_{j=1}^{\infty} |a_j + ib_j|^2 \right)^{\frac{1}{2}} \leqslant \left\| \sum_{j=1}^{\infty} (a_j + ib_j) r_j \right\|_{L^p([0,1])} \leqslant B_p \left(\sum_{j=1}^{\infty} |a_j + ib_j|^2 \right)^{\frac{1}{2}}.$$

The orthogonality of the Rademacher functions in the space $L^p([0,1])$ are outlined by the above inequality, even when $p \neq 2$. The earliest version of the inequality is proved by Khintchine [13], when he applied it to estimate the asymptotic behaviour of certain random walks. Later, the inequality is improved to a more general form. For example, John Edensor Littlewood [14], Raymond Paley and Antoni Zygmund [18] made significant progress in the study of Khintchine's inequality.

There is also remarkable improvement made in the search of the optimal constants A_p and B_p such that the above inequality holds. For the case where $0 < p \leqslant 2$, it is known that $B_p = 1$. Meanwhile, for $2 \leqslant p < \infty$, it was found that $A_p = 1$. Later on, Szarek [24] solved an old conjecture of Littlewood by proving $A_1 = \frac{1}{\sqrt{2}}$. For $p \geqslant 3$, Young [25] calculated the values of B_p. Finally, Haagerup [8] computed the values of the optimal constants for the remaining cases.

Next, mathematicians turn their sights to the generalization of the Khintchine type inequalities for certain types of Banach spaces. It turns out that the existence of the Khintchine type inequalities sheds some light on the geometry of the Banach spaces themselves, see [17]. One of the earliest work on non-commutative generalization of the Khintchine type inequalities with matrix-valued coefficients was pioneered by Lust-Piquard [15]. This was for the case where $1 < p < \infty$. Later, Pisier and Lust-Piquard [16] managed to prove the inequalities for the case $p = 1$. Their method of proof is inspired from classical harmonic analysis, where the classical Khintchine inequality is proved from a sequence of functions on $[0,1]$ given by $(e^{2\pi i 2^n)(\cdot)})_{n=0}^{\infty}$ via a Paley-type inequality. Using this strategy, they prove the non-commutative analogue of Khintchine inequality [Theorem II.1, [16]]. Consequently, the following non-commutative Khintchine inequality is born [Corollary II.1, [16]].

For each d, $n \in \mathbb{N}$ and $x_1, \cdots, x_d \in M_n(\mathbb{C})$, the norm $\|(x_i)_{i=1}^d\|_{\ell^1_{rc}}$ is defined by:

$$\|(x_i)_{i=1}^d\|_{\ell^1_{rc}} := \inf\left\{ \operatorname{tr}\left[\left(\sum_{i=1}^d y_i^* y_i\right)^{\frac{1}{2}} + \left(\sum_{i=1}^d z_i z_i^*\right)^{\frac{1}{2}}\right] : x_i = y_i + z_i \in M_n(\mathbb{C})\right\}$$

Then,

$$\frac{1}{1+\sqrt{2}}\|(x_i)_{i=1}^d\|_{\ell^1_{rc}} \leqslant \left\|\sum_{i=1}^d x_i \otimes e^{2\pi i 2^j(\cdot)}\right\|_{L^1([0,1];S^1)} \leqslant \|(x_i)_{i=1}^d\|_{\ell^1_{rc}}.$$

Here, we would like to recall that for $x \in M_n(\mathbb{C})$, $\|x\|_{S^1} = \operatorname{tr}[(x^*x)^{\frac{1}{2}}]$. It should be noted that in [page [16]], if the sequence of functions $(e^{2\pi i 3^n(\cdot)})_{n=1}^{\infty}$ is considered instead of $(e^{2\pi i 2^n(\cdot)})_{n=1}^{\infty}$, the constant $\frac{1}{1+\sqrt{2}}$ can be improved to $\frac{1}{2}$. By [21], one can replace the sequence of functions $(e^{2\pi i 2^n(\cdot)})_{n=1}^{\infty}$ by a sequence of either independent Rademacher random variables, independent Steinhauss random variables or independent complex Gaussian random variables to obtain the corresponding non-commutative Khintchine-type inequalities with matrix-valued coefficients, albeit with different constants.

The article seeks to generalizes the setting of the non-commutative Khintchine-type inequalities with matrix-valued coefficients to the setting of an orthogonal system with a von Neumann algebra. The existence of the Khintchine-type inequality then relies on the Z_2 property of the corresponding orthonormal system. It is worth to note that a group von Neumann algebra, the sequence of functions $(e^{2\pi i 2^n (\cdot)})_{n=1}^{\infty}$ and the sequence of independent Rademacher random variables all possess a finite Z_2 constant. Our method is largely inspired by [20] and [11]. The dual inequality above is proved more generally in the case of an orthonormal system. Later, we also obtain some partial converse result in the commutative setting where the existence of a Khintchine-type inequality implies the finiteness of the Z_2 constant.

3.2 Preliminary

3.2.1 Non-commutative L_p spaces and Orthonormal System

Suppose H is a complex Hilbert space and $B(H)$ is the algebra of all bounded linear operators on H. Equipped with the usual adjoint and involution, $B(H)$ becomes a unital C^*-algebra. Let $\mathcal{M}$ is a von Neumann algebra on H, i.e. $\mathcal{M}$ is a C^*-subalgebra of $B(H)$ which contains 1 and is σ-weakly closed. Let $\mathcal{S}_+(\mathcal{M}) = \{x \in \mathcal{M}_+ : \tau(s(x)) < \infty\}$ and $\mathcal{S}$ be the linear span of $\mathcal{S}_+$. Here $s(x)$ is the support of $x \in \mathcal{M}_+$, i.e. the least projection such that $px = x$. Let $\mathcal{M}$ is a von Neuman algebra equipped with normal semifinite faithful trace on H, then the noncommutative L_p spaces $L_p(\mathcal{M}, \tau)$ associated with $(\mathcal{M}, \tau)$ is defined to be

$$L_p(\mathcal{M}, \tau) = \overline{(\mathcal{S}, \| \cdot \|_p)}^{\| \cdot \|_p}$$

Here, $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}$. Write $L^p(\mathcal{M})$ as $L^p(\mathcal{M}, \tau)$. When $p = \infty$, set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm, i.e. $\|x\|_\infty = \|x\|$. For the rest of the article, we always assume that the von Neumann algebra $\mathcal{M}$ is finite.

Definition 3.1. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, semi-finite trace, τ on H and consider $L^2(\mathcal{M}, \tau)$. A set $W \subseteq \mathcal{M}$ is said to be an orthonormal system if for all x, $y \in W$, $\tau(x^*y) = \delta_{x,y}$.

3.2.2 Group von Neuman Algebra

Let Γ be a discrete group and λ the left regular representation $\lambda : \Gamma \to \mathfrak{B}[\ell^2(\Gamma)]$, given by $\lambda(g)\delta_h = \delta_{gh}$ where δ_g's is the canonical orthonormal basis of $\ell_2(\Gamma)$. Write $\mathfrak{L}(\Gamma)$ for its group von Neumann algebra, the weak operator closure of the linear span of $\lambda(\Gamma)$ in $\mathfrak{B}(\ell_2(\Gamma))$. Consider the standard trace $\tau(\lambda(g)) = \delta_e(g)$, where e denotes the identity of Γ. An element $f \in L^2(\widehat{\Gamma})$ can be expressed in the form:

$$f = \sum_{x \in \Gamma} c_x \lambda_x$$

with $(c_x)_{x \in \Gamma} \in \ell^2(\Gamma)$ which can be seen as the Fourier coefficients of f .

Remark 3.2. The set $\{\lambda_x \in \mathfrak{B}[\ell^2(\Gamma)] : x \in \Gamma\}$ forms an orthonormal system since $\tau(\lambda_x^* \lambda_y) = \langle \lambda_{x^{-1}y}(\delta_e), \delta_e \rangle = \delta_{x,y}$.

3.2.3 Z_2-sets

Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal, semi-finite trace τ. Let $W \subseteq \mathcal{M} \cap L^2(\mathcal{M})$ be an orthonormal system. Denote $W^* := \{x \in M \cap L^2(\mathcal{M} : x = w^*$ for some $w \in W\}$. For convenience, denote

$$N_1(W) := \sup_{x \in W} \tau(|x|^4),$$

$$N_2(W) := \inf\{\tau(|x|^2|y|^2) : x, y \in W \text{ and } x \neq y\}.$$

Denote $Z_{2,1}(W)$, $Z_{2,2}(W)$, $Z_{2,3(W)} \in \mathbb{R} \cup \{\infty\}$ by

$$Z_{2,1}(W) := \sup_{w,x\in W, w\neq x} \sum_{y,z\in W} |\tau(w^*xy^*z)|,$$

$$Z_{2,2}(W) := \sup_{x\in W} \sum_{y,z\in W, y\neq z} |\tau(|x|^2y^*z)|,$$

$$Z_{2,3}(W) := \sup_{x\in W} \sum_{y\in W, y\neq x} |\tau(|x|^2|y|^2) - N_2(W)|.$$

Definition 3.3. We say that an orthonormal system $W \subset \mathcal{M}$ satisfies the column Z_2 property if $Z_{2,1}(W)$, $Z_{2,2}(W)$, $Z_{2,3}(W) < \infty$. We say that an orthonormal system $W \subset \mathcal{M}$ satisfies the row Z_2 property if $Z_{2,1}(W^*)$, $Z_{2,2}(W^*)$, $Z_{2,3}(W^*) < \infty$. We say that an orthonormal system $W \subset \mathcal{M}$ has the Z_2 property if W satisfies the column Z_2 property and the row Z_2 property.

Remark 3.4. The definition of the Z_2 property of an orthonormal system coincides with the usual definition of the Z_2 property of a discrete group. Recall that given a subset $V \subseteq \Gamma$ a discrete group, V is said to satisfy the Z_2 property if $Z_2(V) < \infty$, where

$$Z_2(V) := \sup_{g\in\Gamma, g\neq e} \#\{(x, y) \in V \times V : x^{-1}y = g\}.$$

Suppose $\widehat{V} = \{\lambda_x : x \in V\}$ where λ_x is the left regular representation of Γ. then $\widehat{V}$ equipped with the canonical trace on $L(\widehat{\Gamma})$ forms an orthonormal system and we have $Z_2(V) = Z_{2,1}(\widehat{V}) = Z_{2,3}(\widehat{V})$.

If $Z_2(V) = 1$, then we have the characterization: $x_ix_j^{-1} = x_sx_t^{-1}$ if and only if $i = s, j = t$ or $i = j, s = t$.

Example 3.5. (i) Let $\Gamma = \mathbb{Z}$ and $W = (2^n)_{n\geqslant 1}$, then $Z_2(W) = 1$

(ii) Let $\Gamma = \mathbb{Z}^2$ and $W_k = \{(m,n) \in \mathbb{Z}^2 : m^2 + n^2 = k\}$. Then $Z_2(W_k) \leqslant 2$ for all $k \geqslant 1$.

(iii) A translation of free subset in a discrete group is also a Z_2-set with constant 1.

(iv) Let $W = (\gamma_n)_{n=1}^{\infty} \subseteq L^2(\Omega, \mathbb{P})$ be the sequence of independent, standard, complex-valued Gaussian random variables on some probability space $(\Omega, \mathbb{P})$. Recall that a complex-valued random variable is said to be Gaussian standard if its real and imaginary part are real-valued, independent Gaussian random variables on $(\Omega, \mathbb{P})$, each having mean 0 and variance $\frac{1}{2}$. Consequently, for all $n \geqslant 1$, $\int_\Omega \gamma_n \, d\mathbb{P} = 0$ and $\int_\Omega |\gamma_n|^2 \, d\mathbb{P} = 1$. It follows that $Z_{2,1}(W) = Z_{2,3}(W) = 1$ and $Z_{2,2}(W) = 0$.

Remark 3.6.

1. A subset with Z_2-constant 1 is a Sidon set in Erdős' sense[5], which means all sums of two elements are distinct.

2. Let $\Gamma = \mathbb{Z}$, the set of prime numbers is not a Z_2-set.

3.2.4 Column and Row Spaces

Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ and (x_n) is a finite sequence in $L_p(\mathcal{M}, \tau)$ for $p \in (0, \infty)$, define

$$\|(x_n)\|_{L_p(\mathcal{M};\ell_2^c)} = \|(\sum_n |x_n|^2)^{\frac{1}{2}}\|_p, \quad \|(x_n)\|_{L_p(\mathcal{M};\ell_2^r)} = \|(\sum_n |x_n^*|^2)\|_p.$$

Given $p \in (0, \infty)$, define the colunm space $L_p(\mathcal{M}; \ell_c^2) = \overline{L^p(\mathcal{M})}^{\|\cdot\|_{L_p(\mathcal{M};\ell_c^2)}}$ and the row space $L_p(\mathcal{M}; \ell_r^2) = \overline{L^p(\mathcal{M})}^{\|\cdot\|_{L_p(\mathcal{M};\ell_r^2)}}$

Let $0 < p \leqslant \infty$. We define the space $L^p(\mathcal{M}, \ell_{rc}^2)$ as follows:

1. If $0 < p < 2$,

$$L^p(\mathcal{M}; \ell_{rc}^2) = L^p(\mathcal{M}; \ell_c^2) + L^p(\mathcal{M}; \ell_r^2)$$

equipped with the intersection norm:

$$\|(x_k)\|_{L^p(\mathcal{M};\ell^2_{rc})} = \inf_{x_k = x'_k + x''_k} \{\|(x'_k)\|_{L^p(\mathcal{M};\ell^2_c)} + \|(x''_k)\|_{L^p(\mathcal{M};\ell^2_r)}\}$$

where the infimum is taken over all decompositions for which

$$\|(x'_k)\|_{L^p(\mathcal{M};\ell^2_c)} < \infty \text{ and } \|(x''_k)\|_{L^p(\mathcal{M};\ell^2_r)} < \infty.$$

2. If $p \geqslant 2$,

$$L^p(\mathcal{M};\ell^2_{rc}) = L^p(\mathcal{M},\ell^2_c) \cap L^p(\mathcal{M},\ell^2_r)$$

equipped with the intersection norm:

$$\|(x_k)\|_{L^p(\mathcal{M};\ell^2_{rc})} = \max\{\|(x_k)\|_{L^p(\mathcal{M},\ell^2_r)}; \|(x_k)\|_{L^p(\mathcal{M};\ell^2_c)}\}.$$

Denote $\|\cdot\|_{S_p^n} = \|\cdot\|_{S_p(\mathcal{B}(\ell_2^n))}$ and $\|\cdot\|_{S_p^n(\ell^2)} = \|\cdot\|_{L_p(M_n(\mathbb{C});\ell_2^{rc})}$. Let $(C_{x_i})_{i=1}^d \in [M_n(\mathbb{C})]^d$, denote the norm in the space $S_\infty^n(\ell^2_{rc})$ and by previous definition

$$\|(C_{x_i})\|_{S_\infty^n(\ell^2_{rc})} = \max\left\{\left\|\sum_{i=1}^d C_{x_i}^* C_{x_i}\right\|^{\frac{1}{2}}, \left\|\sum_{i=1}^d C_{x_i} C_{x_i}^*\right\|^{\frac{1}{2}}\right\} \text{ and}$$

$$\|(C_{x_i})\|_{S_1^n(\ell^2_{rc})} = \inf_{(C_{x_i})=(X_{x_i})+(Y_{x_i})\in M_n(\mathbb{C})^d} \left\{\left\|\sum_{i=1}^d X_{x_i}^* X_{x_i}\right\|^{\frac{1}{2}} + \left\|\sum_{i=1}^d Y_{x_i} Y_{x_i}^*\right\|^{\frac{1}{2}}\right\}.$$

3.3 Main Results and Proof

Now we are ready to state the main theorems we are going to prove.

3.3.1 Statement of The Main Theorem

Theorem 3.7. *Let n, $d \in \mathbb{N}$. Let $W \subseteq \mathcal{M}$ be an orthonormal system satisfying the Z_2 property. Let x_1, ..., $x_d \in W$. Let C_{x_1}, .., $C_{x_d} \in M_n(\mathbb{C})$. Suppose $N_1(W), N_1(W^*) < \infty$, then*

$$\frac{1}{\sqrt{\max\{\alpha(W), \alpha(W^*)\}}}\|(C_{x_i})\|_{S_1^n(\ell^2_{rc})} \leqslant \left\|\sum_{i=1}^{d} C_{x_i} \otimes x_i\right\|_{L^1[M_n(\mathbb{C})\otimes\mathcal{M}]} \leqslant \|(C_{x_i})\|_{S_1^n(\ell^2_{rc})} \tag{3.1}$$

Here let $\beta(W) := N_1(W) - N_2(W) - Z_{2,1}(W) + Z_{2,2}(W) + Z_{2,3}(W)$

$$\alpha(W) := \begin{cases} N_2(W) + Z_{2,1}(W) & \text{if } \beta(W) \leqslant 0 \\ N_1(W) + Z_{2,2}(W) + Z_{2,3}(W) & \text{if } \beta(W) > 0 \end{cases}$$

Remark 3.8.

(1) Suppose Γ is a discrete group with Z_2 property and $W \subset \Gamma$ satisfies Z_2 property with the constant $Z_2(W)$. Then by the Remark 3.4, $N_1(\widehat{W}) = N_2(\widehat{W}) = 1$ and $Z_{2,1}(\widehat{W}) = Z_2(W), Z_{2,2}(\widehat{W}) = Z_{2,3}(\widehat{W}) = 0$. Then $\alpha(\widehat{W}) = \alpha(\widehat{W}^*) = 1+Z_2(W)$.

(2) Let $(\gamma_n)_{n=1}^{\infty} \subseteq L^2(\Omega, \mathbb{P})$ be the sequence of independent, standard, complex-valued Gaussian random variables on some probability space $(\Omega, \mathbb{P})$ as stated in Example 3.5(iv). Suppose $C_1, \cdots, C_d \in M_n(\mathbb{C})$, $W = \{\gamma_1, \cdots, \gamma_d\}$ Then $N_1(W) = 2, N_2(W) = Z_{2,1}(W) = 1, Z_{2,2}(W) = Z_{2,3} = 0$. Then $\beta(W) = 0$ and $\alpha(W) = 2$.

(3) Let $W = (\gamma_n)_{n=1}^{\infty} \subseteq L^2(\Omega, \mathbb{P})$ be the sequence of independent, standard, real-valued Gaussian random variables on some probability space $(\Omega, \mathbb{P})$, i.e., each

have mean 0 and variance 1. Then $N_1(W) = 3, N_2(W) = Z_{2,1}(W) = 1$ and $Z_{2,2}(W) = Z_{2,3}(W) = 0$. Hence $\beta(W) = 1$ and $\alpha(W) = 3$.

(4) Suppose $W = \{T_m(x)\}_{i=1}^d \subseteq L^2((-1,1), d\mu)$ is the subset of the Chebyshev polynomial of first kind which does not include constant function, where $d\mu = \frac{2}{\pi\sqrt{1-x^2}}dx$ and dx is the Lebesgue measure. Then $\int_{-1}^1 T_m(x)T_n(x)d\mu = \delta_{m,n}$ for $m, n \geqslant 1$. By computation, we have $N_1(W) = \frac{3}{4}$, $N_2(W) = \frac{1}{2}$, $Z_{2,1}(W) = \frac{1}{2}, Z_{2,2}(W) = \frac{1}{4}, Z_{2,3}(W) = 0$, then $\beta(W) = 0$ and $\alpha(W) = \frac{5}{8}$

Hence, we have

Corollary 3.9. *Let n, $d \in \mathbb{N}$. Suppose $W \subseteq \Gamma$ satisfies the Z_2 property. Let $x_1, \cdots, x_d \in W$. Let $C_{x_1}, \cdots, C_{x_d} \in M_n(\mathbb{C})$. Then, the following inequality holds:*

$$\frac{1}{\sqrt{Z_2(W)+1}}\|(C_{x_i})\|_{S_1^n(\ell^2_{rc})} \leqslant \|\sum_{i=1}^d C_{x_i} \otimes \lambda_{x_i}\|_{L^1(M_n(\mathbb{C})\otimes\mathcal{L}(\Gamma))} \leqslant \|(C_{x_i})\|_{S_1^n(\ell^2_{rc})}. \tag{3.2}$$

Corollary 3.10 (Haagerup,Musat[11]). *Let $(\gamma_n)_{n=1}^\infty \subseteq L^2(\Omega, \mathbb{P})$ be the sequence of independent, standard, complex-valued Gaussian random variables on some probability space $(\Omega, \mathbb{P})$. Suppose $C_1, \cdots, C_d \in M_n(\mathbb{C})$, then*

$$\frac{1}{\sqrt{2}}\|(C_i)\|_{S_1^n(\ell^2_{rc})} \leqslant \|\sum_{i=1}^d C_{x_i} \otimes \gamma_i\|_{L^1(\Omega;S_1^n)} \leqslant \|(C_i)\|_{S_1^n(\ell^2_{rc})}. \tag{3.3}$$

Corollary 3.11. *Let $(\gamma_n)_{n=1}^\infty \subseteq L^2(\Omega, \mathbb{P})$ be the sequence of independent, standard, real-valued Gaussian random variables on some probability space $(\Omega, \mathbb{P})$. Suppose $C_1, \cdots, C_d \in M_n(\mathbb{C})$, then*

$$\frac{1}{\sqrt{3}}\|(C_i)\|_{S_1^n(\ell^2_{rc})} \leqslant \|\sum_{i=1}^d C_{x_i} \otimes \gamma_i\|_{L^1(\Omega;S_1^n)} \leqslant \|(C_i)\|_{S_1^n(\ell^2_{rc})}. \tag{3.4}$$

Corollary 3.12. *Suppose $W = \{T_i(x)\}_{i=1}^d \subseteq L^2((-1,1), d\mu)$ is the subset of the Chebyshev polynomial of first kind which does not include constant function, where $d\mu = \frac{2}{\pi\sqrt{1-x^2}}dx$. Suppose $C_1, \cdots, C_d \in M_n(\mathbb{C})$, then*

$$\sqrt{\frac{8}{5}}\,\|(C_i)\|_{S_1^n(\ell_{rc}^2)} \leqslant \left\|\sum_{i=1}^d C_i \otimes T_i(\cdot)\right\|_{L^1((-1,1),d\mu;S_1^n)} \leqslant \|(C_i)\|_{S_1^n(\ell_{rc}^2)}. \tag{3.5}$$

3.4 Proof of The Main Theorem

Since $W \subseteq \mathcal{M}$ is an orthonormal system, we can extend W into an orthonormal basis B. Also, since $\mathcal{M}$ is a finite von Neumann algebra, every $f \in M_n(\mathbb{C}) \otimes \mathcal{M}$ can be expressed as $f := \sum_{y \in B} C_y \otimes y$, where the infinite sum converges in the $L^2(\mathcal{M})$ norm. For each $x \in W$, we define $\widehat{(\cdot)}(x) : M_n(\mathbb{C}) \otimes \mathcal{M} \to M_n(\mathbb{C})$ by $\hat{f}(x) = C_x$.

Lemma 3.13. *Let $f \in M_n(\mathbb{C}) \otimes \mathcal{M}$. Then, for $(x_i)_{i=1}^d \subseteq W$,*

$$\left\|\sum_{i=1}^d \left[\hat{f}(x_i)\right]^* \left[\hat{f}(x_i)\right]\right\| \leqslant \|f\|^2_{M_n(\mathbb{C})\otimes\mathcal{M}} \tag{3.6}$$

Proof. First, we consider $f \in M_n(\mathbb{C}) \otimes \mathcal{M}$ as an element of the algebraic tensor product.

$$f = \sum_{k=1}^r C_{y_k} \otimes y_k, \text{ where } y_k \in B, C_{y_k} \in M_n(\mathbb{C}) \tag{3.7}$$

$$f^*f := \left(\sum_{k=1}^r C_{y_k} \otimes y_k\right)^* \left(\sum_{k=1}^r C_{y_k} \otimes y_k\right) = \sum_{j,k=1}^r C_{y_j}^* C_{y_k} \otimes y_j^* y_k$$

Now, note that

$$\sum_{i=1}^d \left[\hat{f}(x_i)\right]^* \left[\hat{f}(x_i)\right] := \sum_{i=1}^d C_{x_i}^* C_{x_i} \leqslant \sum_{k=1}^r C_{y_k}^* C_{y_k}.$$

Let $\omega \in \mathcal{S}[M_n(\mathbb{C})]$, where $\mathcal{S}[M_n(\mathbb{C})]$ denotes the state space of $M_n(\mathbb{C})$.

$$\begin{aligned}\omega\left(\sum_{i=1}^{d}\left[\hat{f}(x_i)\right]^*\left[\hat{f}(x_i)\right]\right) &\leqslant \omega\left(\sum_{k=1}^{r} C_{y_k}^* C_{y_k}\right) \text{ since } \omega \in \mathcal{S}[M_n(\mathbb{C})] \\ &= (\omega \otimes \tau)\left(\sum_{j,k=1}^{r} C_{y_j}^* C_{y_k} \otimes y_j^* y_k\right) \\ &= (\omega \otimes \tau)(f^* f) \leqslant \|f\|_{M_n(\mathbb{C})\otimes\mathcal{M}}^2\end{aligned}$$

Taking the supremum over all $\omega \in \mathcal{S}[M_n(\mathbb{C})]$, we obtain: (3.6) for the elements with the form (3.7). After taking the completion, we obtain the above inequality for all $f \in M_n(\mathbb{C}) \otimes \mathcal{M}$. □

Lemma 3.14. *Let $W \subseteq \mathcal{M}$ be an orthonormal system which satisfies the Z_2 property. Suppose that $N_1(W), N_1(W^*) < \infty$. Given $x_1, \cdots, x_d \in W$ and $C_{x_1}, \cdots, C_{x_d} \in M_n(\mathbb{C})$, denote $f := \sum_{i=1}^{d} C_{x_i} \otimes x_i \in M_n(\mathbb{C}) \otimes \mathcal{M}$. Then we have*

$$\begin{aligned}(I_n \otimes \tau)((f^* f)^2) \leqslant & N_2(W)\left(\sum_{i=1}^{d} C_{x_i}^* C_{x_i}\right)^2 + Z_{2,1}(W)\sum_{i=1}^{d} C_{x_i}^*\left(\sum_{j=1}^{d} C_{x_j} C_{x_j}^*\right) C_{x_i} \\ &+ (N_1(W) + Z_{2,3}(W) - N_2(W) + Z_{2,2}(W) - Z_{2,1}(W))\sum_{i=1}^{d} |C_{x_i}|^4\end{aligned} \tag{3.8}$$

$$\begin{aligned}(I_n \otimes \tau)((f f^*)^2) \leqslant & N_2(W^*)\left(\sum_{i=1}^{d} C_{x_i} C_{x_i}^*\right)^2 + Z_{2,1}(W^*)\sum_{i=1}^{d} C_{x_i}\left(\sum_{j=1}^{d} C_{x_j}^* C_{x_j}\right) C_{x_i}^* \\ &+ (N_1(W^*) + Z_{2,3}(W^*) - N_2(W^*) + Z_{2,2}(W^*) - Z_{2,1}(W^*))\sum_{i=1}^{d} |C_{x_i}^*|^4\end{aligned} \tag{3.9}$$

Proof. By computation, we have

$$I_n \otimes \tau(f^*f) = \sum_{1 \leqslant i,j \leqslant d} C_i^* C_j \tau(x_i^* x_j) \tag{3.10}$$

and

$$I_n \otimes \tau((f^*f)^2) = \sum_{1 \leqslant i,j,k,l \leqslant d} C_i^* C_j C_k^* C_l \tau(x_i^* x_j x_k^* x_l). \tag{3.11}$$

Denote $R_1 = \{(i,i,i,i) \in \mathbb{N}^4 : 1 \leqslant i \leqslant d\}$, $R_2 = \{(i,i,k,k) \in \mathbb{N}^4 : 1 \leqslant i \neq k \leqslant d\}$, $U = \{(i,j,k,l) \in \mathbb{N}^4 : 1 \leqslant i \neq j \leqslant d, 1 \leqslant k,l \leqslant d\}$, and $V = \{(i,i,k,l) \in \mathbb{N}^4 : 1 \leqslant i \leqslant d, 1 \leqslant k \neq l \leqslant d\}$. If we denote $X = \{(i,j,k,l) \in \mathbb{N}^4 : 1 \leqslant i,j,k,l \leqslant d\}$. Observe that $X \backslash (U \cup V) = R_1 \cup R_2$ and $U \cap V = \varnothing, R_1 \cap R_2 = \varnothing$, then

$$I_n \otimes \tau((f^*f)^2) = \sum_{(i,j,k,l) \in R_1} + \sum_{(i,j,k,l) \in R_2} + \sum_{(i,j,k,l) \in U} + \sum_{(i,j,k,l) \in V}. \tag{3.12}$$

Next, we are going to compute each part of summation (3.12).

$$\sum_{(i,j,k,l) \in R_1} = \sum_{i=1}^{d} |C_i|^4 \tau(|x_i|^4) \leqslant N_1(W) \sum_{i=1}^{d} |C_i|^4 \tag{3.13}$$

$$\begin{aligned}
\sum_{(i,j,k,l) \in R_2} &= N_2(W) \sum_{(i,j,k,l) \in R_2} |C_i|^2 |C_j|^2 + \sum_{(i,j,k,l) \in R_2} |C_i|^2 |C_k|^2 (\tau(|x_i|^2 |x_k|^2) - N_2(W)) \\
&\leqslant N_2(W) \sum_{(i,j,k,l) \in R_2} |C_i|^2 |C_j|^2 + \frac{1}{2} \sum_{(i,j,k,l) \in R_2} (|C_i|^4 + |C_k|^4) \, |\tau(|x_i|^2 |x_k|^2) - N_2(W)| \\
&\leqslant N_2(W) \sum_{(i,j,k,l) \in R_2} |C_i|^2 |C_j|^2 + \sum_{(i,j,k,l) \in R_2} |C_i|^4 |\tau(|x_i|^2 |x_k|^2) - N_2(W)| \\
&\leqslant N_2(W) \sum_{(i,j,k,l) \in R_2} |C_i|^2 |C_j|^2 + Z_{2,3}(W) \sum_{i=1}^{d} |C_i|^4 \\
&= N_2(W) (\sum_{i=1}^{d} |C_i|^2)^2 + (Z_{2,3}(W) - N_2(W)) \sum_{i=1}^{d} |C_i|^4 \qquad (3.14)
\end{aligned}$$

For the summation of the latter two parts of the right equation in (3.12), since $U \cap V = \varnothing$ and $U \cup V$ is "symmetric" in the sense that if $(i,j,k,l) \in U \cup V$ then $(j,i,k,l), (k,l,i,j) \in U \cup V$, thus we have

$$
\begin{aligned}
\sum_{(i,j,k,l)\in U} + \sum_{(i,j,k,l)\in V} &= \sum_{(i,j,k,l)\in U\cup V} C_i^* C_j C_k^* C_l \, \tau(x_i^* x_j x_k^* x_l) \\
&\leqslant \frac{1}{2} \sum_{(i,j,k,l)\in U\cup V} (|C_k^* C_l|^2 + |C_j^* C_i|^2)\, |\tau(x_i^* x_j x_k^* x_l)| \\
&= \sum_{(i,j,k,l)\in U\cup V} |C_j^* C_i|^2 \, |\tau(x_i^* x_j x_k^* x_l)| \\
&= \sum_{(i,j,k,l)\in U} |C_j^* C_i|^2 |\tau(x_i^* x_j x_k^* x_l)| + \sum_{(i,j,k,l)\in V} |C_j^* C_i|^2 \, |\tau(x_i^* x_j x_k^* x_l)| \\
&= \sum_{1\leqslant i\neq j\leqslant d} |C_j^* C_i|^2 \sum_{1\leqslant k,l\leqslant d} |\tau(x_i^* x_j x_k^* x_l)| + \sum_{i=1}^{d} |C_i|^4 \sum_{1\leqslant k\neq l\leqslant d} |\tau(|x_i|^2 x_k^* x_l)| \\
&\leqslant Z_{2,1}(W) \sum_{1\leqslant i\neq j\leqslant d} |C_j^* C_i|^2 + Z_{2,2}(W) \sum_{1\leqslant i\leqslant d} |C_i|^4 \\
&= Z_{2,1}(W) \sum_{i=1}^{d} C_i^* \left(\sum_{j=1}^{d} C_j C_j^* \right) C_i + (Z_{2,2}(W) - Z_{2,1}(W)) \sum_{k=1}^{d} |C_k|^4.
\end{aligned} \tag{3.15}
$$

Combining (3.13), (3.14) and (3.15), we obtain the estimate to finish the proof of (3.8). The proof of (3.9) is from (3.8) by taking W by W^* and f by its adjoint f^*. □

Corollary 3.15. *Let* $W \subseteq L^2(\mathcal{M}) \cap \mathcal{M}$ *be an orthonormal system with the* Z_2 *property. Suppose that* $N_1(W), N_1(W^*) < \infty$. *Let* $C_{x_1}, \cdots, C_{x_d} \in M_n(\mathbb{C})$ *and* $f := \sum_{i=1}^{d} C_{x_i} \otimes x_i \in M_n(\mathbb{C})$. *Denote*

$$\beta(W) := N_1(W) - N_2(W) - Z_{2,1}(W) + Z_{2,2}(W) + Z_{2,3}(W),$$

$$\eta(W) := N_1(W) - N_2(W) + Z_{2,2}(W) + Z_{2,3}(W).$$

(i) If $\beta(W) \leqslant 0$, *then*

$$(I_n \otimes \tau)((f^*f)^2) \leqslant \left(N_2(W) \left\| \sum_{i=1}^{d} C_{x_i}^* C_{x_i} \right\| + Z_{2,1}(W) \left\| \sum_{i=1}^{d} C_{x_i} C_{x_i}^* \right\| \right) (I_n \otimes \tau)(f^*f) \tag{3.16}$$

(ii) If $\beta_1(W) > 0$, then

$$(I_n \otimes \tau)((f^*f)^2) \leqslant \left(N_2(W) \left\| \sum_{i=1}^{d} C_{x_i}^* C_{x_i} \right\| + \eta(W) \left\| \sum_{j=1}^{d} C_{x_j} C_{x_j}^* \right\| \right) (I_n \otimes \tau)(f^*f). \tag{3.17}$$

(iii) If $\beta(W^*) \leqslant 0$, then

$$(I_n \otimes \tau)((ff^*)^2) \leqslant \left(N_2(W^*) \left\| \sum_{i=1}^{d} C_{x_i}^* C_{x_i} \right\| + Z_{2,1}(W^*) \left\| \sum_{i=1}^{d} C_{x_i} C_{x_i}^* \right\| \right) (I_n \otimes \tau)(ff^*) \tag{3.18}$$

(iv) If $\beta(W^*) > 0$, then

$$(I_n \otimes \tau)((ff^*)^2) \leqslant \left(N_2(W^*) \left\| \sum_{i=1}^{d} C_{x_i}^* C_{x_i} \right\| + \eta(W^*) \left\| \sum_{j=1}^{d} C_{x_j} C_{x_j}^* \right\| \right) (I_n \otimes \tau)(ff^*). \tag{3.19}$$

(v) In particular, if $\left\| (C_{x_i})_{i=1}^d \right\|_{S_\infty^n(\ell^2_{rc})} = 1$, then

$$(I_n \otimes \tau)((f^*f)^2) \leqslant \alpha(W)(I_n \otimes \tau)((f^*f)) \tag{3.20}$$

$$(I_n \otimes \tau)((ff^*)^2) \leqslant \alpha(W^*)(I_n \otimes \tau)((ff^*)) \tag{3.21}$$

where

$$\alpha(W) := \begin{cases} N_2(W) + Z_{2,1}(W) & \text{if } \beta(W) \leqslant 0 \\ N_1(W) + Z_{2,3}(W) + Z_{2,2}(W) & \text{if } \beta(W) > 0 \end{cases}$$

Proof. (i) If $\beta_1(W) \leqslant 0$, then by (3.8) of Lemma 3.14, we have

$$
\begin{aligned}
(I_n\otimes\tau)((f^*f)^2) &\leqslant N_2(W)\left(\sum_{i=1}^d C_{x_i}^*C_{x_i}\right)^2 + Z_{2,1}(W)\sum_{i=1}^d C_{x_i}^*\left(\sum_{j=1}^d C_{x_j}C_{x_j}^*\right)C_{x_i}\\
&\leqslant N_2(W)\left\|\sum_{i=1}^d C_{x_i}^*C_{x_i}\right\|\left(\sum_{i=1}^d C_{x_i}^*C_{x_i}\right) + Z_{2,1}(W)\left\|\sum_{j=1}^d C_{x_j}C_{x_j}^*\right\|\sum_{i=1}^d C_{x_i}^*C_{x_i}\\
&= \left(N_2(W)\left\|\sum_{i=1}^d C_{x_i}^*C_{x_i}\right\| + Z_{2,1}(W)\left\|\sum_{i=1}^d C_{x_i}C_{x_i}^*\right\|\right)\left(\sum_{i=1}^d C_{x_i}^*C_{x_i}\right)\\
&= \left(N_2(W)\left\|\sum_{i=1}^d C_{x_i}^*C_{x_i}\right\| + Z_{2,1}(W)\left\|\sum_{i=1}^d C_{x_i}C_{x_i}^*\right\|\right)(I_n\otimes\tau)(f^*f)
\end{aligned}
$$

(ii) We have that

$$
\sum_{i=1}^d |C_i|^4 = \sum_{i=1}^d C_i^*C_iC_i^*C_i \leqslant \sup_i \|C_iC_i^*\|\left(\sum_{i=1}^d C_i^*C_i\right) \leqslant \left\|\sum_{i=1}^d C_iC_i^*\right\|\left(\sum_{i=1}^d C_i^*C_i\right) \tag{3.22}
$$

Combining the inequalities (3.16),(3.22) and (3.8), we obtain (3.17).

(iii) and (iv) follow from (i) and (ii) by taking W by W^* and f by its adjoint f^*. For the proof of (v), note that $\|(C_{x_i})_{i=1}^d\|_{S_\infty^n(\ell^2_{rc})} = 1$ means $\max\{\|\sum_{i=1}^d C_i^*C_i\|, \|\sum_{i=1}^d C_iC_i^*\|\} \leqslant 1$ by definition. Thus, combining (i),(ii),(iii),(iv) and we finish the proof. □

Lemma 3.16. *Let $W \subseteq L^2(\mathcal{M}) \cap \mathcal{M}$ be an orthornormal system satisfying the Z_2 property. $x_1, \cdots, x_d \in W$, $C_{x_1}, \cdots, C_{x_d} \in M_n(\mathbb{C})$ satisfying $\|(C_{x_i})_{i=1}^d\|_{S_\infty^n(\ell^2_{rc})} = 1$. Suppose that $N_1(W), N_1(W^*) < \infty$. Then there exists $f \in M_n(\mathbb{C})\otimes\mathcal{M}$ such that*

$$
\|f\|_{M_n(\mathbb{C})\otimes\mathcal{M}} \leqslant \frac{\sqrt{\max\{\alpha(W),\alpha(W^*)\}}}{2} \tag{3.23}
$$

and

$$
\left\|\left(C_{x_i} - \hat{f}(x_i)\right)_{i=1}^d\right\|_{S_\infty^n(\ell^2_{rc})} \leqslant \frac{1}{2} \tag{3.24}
$$

Proof. Define $g := \sum_{i=1}^d C_{x_i}\otimes\lambda_{x_i} \in M_n(\mathbb{C})\otimes\mathcal{M}$. Notice that $(\hat{g}(x_i))_{i=1}^d = (C_{x_i})_{i=1}^d$ and $\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix} \in M_2[M_n(\mathbb{C})\otimes\mathcal{M}]$ is self-adjoint. So, its spectrum is on the real line.

Let $R > 0$ and define $F_R : \mathbb{R} \to \mathbb{R}$ by:

$$F_R(t) = \begin{cases} -R & \text{if } x \leqslant -R \\ t & \text{if } -R < t < R \\ R, & \text{if } t \geqslant R \end{cases}$$

Since $F_R \in C(\mathbb{R})$ is an odd function, thus F_R can be uniformly approximated by a sequence of odd polynomials on the interval $[-R, R]$ and we can define $F_R\left(\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix}\right)$ via the functional calculus. It can be deduced that

$$\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix}^{2n+1} = \begin{bmatrix} 0 & (g^*g)^n g^* \\ (gg^*)^n g & 0 \end{bmatrix}$$

and

$$((gg^*)^n g)^* = (g^*g)^n g^*, \quad ((g^*g)^n g)^* = (gg^*)^n g$$

for each $n \in \mathbb{N}$ by induction. Given any odd polynomial $p(x) \in C(\mathbb{R})$ and by functional calculus, $p\left(\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix}\right)$ is of the form $\begin{bmatrix} 0 & q(g^*g)g^* \\ q(gg^*)g & 0 \end{bmatrix}$ for some polynomial q defined on $C^*\left(\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix}\right)$. Also, observe that $[q(gg^*)g]^* = q(g^*g)g^*$.

So, there exists a sequence $\left(\begin{bmatrix} 0 & q_n(g^*g)g^* \\ q_n(gg^*)g & 0 \end{bmatrix}\right)_{n=1}^{\infty} \subseteq M_2[M_n(\mathbb{C}) \otimes \mathcal{M}]$ and $\begin{bmatrix} c & d \\ f & e \end{bmatrix} \in M_2[M_n(\mathbb{C}) \otimes \mathcal{M}]$ such that

$$\left\| \begin{bmatrix} 0 & q_n(g^*g)g^* \\ q_n(gg^*)g & 0 \end{bmatrix} - \begin{bmatrix} c & d \\ f & e \end{bmatrix} \right\|_{M_2[M_n(\mathbb{C})\otimes\mathcal{M}]}$$
$$= \left\| \begin{bmatrix} -c & q_n(g^*g)g^* - d \\ q_n(gg^*)g - f & -e \end{bmatrix} \right\|_{M_2[M_n(\mathbb{C})\otimes\mathcal{M}]}$$

which tends to 0 as $n \to \infty$. So, $c = 0$, $e = 0$ and $d = f^*$ and $F_R\left(\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & f^* \\ f & 0 \end{bmatrix}$. Thus,

$$\|f\|_{M_n(\mathbb{C})\otimes\mathcal{M}} = \left\| \begin{bmatrix} 0 & f^* \\ f & 0 \end{bmatrix} \right\|_{M_2[M_n(\mathbb{C})\otimes\mathcal{M}]} = \left\| F_R\left(\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix}\right) \right\|_{M_2[M_n(\mathbb{C})\otimes\mathcal{M}]} \leqslant R. \quad (3.25)$$

Next, define $G_R(t) = t - F_R(t)$ for all $t \in \mathbb{R}$. Using the functional calculus,

$$G_R\left(\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix} - F_R\left(\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & g^* - f^* \\ g - f & 0 \end{bmatrix}$$

and

$$\left[G_R\left(\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix}\right)\right]^2 = \begin{bmatrix} 0 & g^* - f^* \\ g - f & 0 \end{bmatrix}^2 = \begin{bmatrix} (g^* - f^*)(g - f) & 0 \\ 0 & (g - f)(g^* - f^*) \end{bmatrix}.$$

Because
$$\left[G_R\left(\begin{bmatrix} 0 & g^* \\ g & 0 \end{bmatrix}\right)\right] = \begin{bmatrix} 0 & g^* - f^* \\ g - f & 0 \end{bmatrix}$$
is self-adjoint, thus

$$\left|G_R\left(\begin{bmatrix}0 & g^*\\ g & 0\end{bmatrix}\right)\right|^2 = \left[G_R\left(\begin{bmatrix}0 & g^*\\ g & 0\end{bmatrix}\right)\right]^2.$$

Since $|G_R(t)| \leqslant \frac{t^2}{4R}$ for all $t \in \mathbb{R}$,

$$\left[G_R\left(\begin{bmatrix}0 & g^*\\ g & 0\end{bmatrix}\right)\right]^2 \leqslant \frac{1}{16R^2}\begin{bmatrix}0 & g^*\\ g & 0\end{bmatrix}^4 = \frac{1}{16R^2}\begin{bmatrix}(g^*g)^2 & 0\\ 0 & (gg^*)^2\end{bmatrix}$$

that is,

$$\begin{bmatrix}\frac{1}{16R^2}(g^*g)^2 - (g-f)^*(g-f) & 0\\ 0 & \frac{1}{16R^2}(gg^*)^2 - (g-f)(g-f)^*\end{bmatrix} \geqslant 0.$$

Consequently, we obtain

$$(g-f)^*(g-f) \leqslant \frac{1}{16R^2}(g^*g)^2, \quad (g-f)(g-f)^* \leqslant \frac{1}{16R^2}(gg^*)^2. \qquad (3.26)$$

Therefore, there exists an element $f \in M_n(\mathbb{C}) \otimes \mathcal{M}$ satisfying the inequality (3.25) and (3.26). Hence, according to corollary 3.15, we have

$$\begin{aligned}\sum_{i=1}^{d}\left[C_{x_i} - \hat{f}(x_i)\right]^*\left[C_{x_i} - \hat{f}(x_i)\right] &\leqslant \sum_{x\in G}\left[C_x - \hat{f}(x)\right]^*\left[C_x - \hat{f}(x)\right]\\ &= (I_n \otimes \tau)[(g-f)^*(g-f)]\\ &\leqslant \frac{1}{16R^2}(I_n \otimes \tau)\left[(g^*g)^2\right] \text{ as } I_n \otimes \tau \text{ is positive-preserving}\\ &\leqslant \frac{\alpha(W)}{16R^2}(I_n \otimes \tau)(g^*g) \quad \text{(by Corollary 3.15}(v))\\ &= \frac{\alpha(W)}{16R^2}\sum_{i=1}^{d} C_{x_i}^* C_{x_i}\end{aligned}$$

Hence, we have

$$\left\|\sum_{i=1}^{d}\left(C_{x_i}-\hat{f}(x)\right)^*\left(C_{x_i}-\hat{f}(x)\right)\right\| \leqslant \frac{\alpha(W)}{16R^2}\left\|\sum_{i=1}^{d}C_{x_i}^*C_{x_i}\right\| = \frac{\alpha(W)}{16R^2}$$

Similarly,

$$\left\|\sum_{i=1}^{d}\left(C_{x_i}-\hat{f}(x)\right)\left(C_{x_i}-\hat{f}(x)\right)^*\right\| \leqslant \frac{\alpha(W^*)}{16R^2}.$$

Setting $R = \frac{\sqrt{\max\{\alpha(W),\alpha(W^*)\}}}{2}$, by inequality (3.25) we have

$$\|f\| \leqslant R = \frac{\sqrt{\max\{\alpha(W),\alpha(W^*)\}}}{2} \tag{3.27}$$

and the inequality (3.24). □

Now, let $(C_{x_i})_{i=1}^d \in [M_n(\mathbb{C})]^d$. If $\|(C_{x_i})_{i=1}^d\|_{S_\infty^n(\ell^2_{rc})} = 1$, by Lemma 3.16, there exists $f \in M_n(\mathbb{C}) \otimes \mathcal{M}$ satisfying (3.23) and (3.24). By homogeneity, we can deduce that for any $(C_{x_i})_{i=1}^d$ of the above form, there exists $f \in M_n(\mathbb{C}) \otimes \mathcal{M}$ such that

$$\|f\|_{M_n(\mathbb{C})\otimes\mathcal{M}} \leqslant \frac{\sqrt{\max\{\alpha(W),\alpha(W^*)\}}}{2}\|(C_{x_i})_{i=1}^d\|_{S_\infty^n(\ell^2_{rc})} \tag{3.28}$$

$$\left\|\left(C_{x_i}-\hat{f}(x_i)\right)_{i=1}^d\right\|_{S_\infty^n(\ell^2_{rc})} \leqslant \frac{1}{2}\|(C_{x_i})_{i=1}^d\|_{S_\infty^n(\ell^2_{rc})} \tag{3.29}$$

Lemma 3.17. *Let $W \subseteq \cap\mathcal{M}$ be an orthonormal system with the Z_2 property, $(x_i)_{i=1}^d \subset W$ and $(C_{x_i})_{i=1}^d \in [M_n(\mathbb{C})]^d$. Then there exists $h \in M_n(\mathbb{C}) \otimes \mathcal{M}$ such that $\hat{h}(x_i) = C_{x_i}, i \in \{1, \cdots, d\}$ and*

$$\|h\|_{M_n(\mathbb{C})\otimes\mathcal{M}} \leqslant \sqrt{\max\{\alpha(W),\alpha(W^*)\}}\|(C_{x_i})_{i=1}^d\|_{S_\infty^n(\ell^2_{rc})} \tag{3.30}$$

Proof. Denote $S_0 = (C_{x_i})_{i=1}^d \in M_n(\mathbb{C})^d$ and define the map $\phi : M_n(\mathbb{C})^d \to M_n(\mathbb{C})\otimes\mathcal{M}$ by $\phi(S_0) = f$, where f is defined via the functional calculus approach in lemma 3.16 that satisfies

$$\|\phi(S_0)\|_{M_n(\mathbb{C})\otimes\mathcal{M}} = \|f\|_{M_n(\mathbb{C})\otimes\mathfrak{L}(\Gamma)} \leqslant \frac{\sqrt{\max\{\alpha(W),\alpha(W^*)\}}}{2}\|S_0\|_{S_\infty^n(\ell^2_{rc})}$$

and

$$\left\| \left(C_{x_i} - \widehat{\phi((S_0)}(x_i) \right)_{i=1}^d \right\|_{S_\infty^n(\ell_{rc}^2)} \leqslant \frac{1}{2} \|S_0\|_{S_\infty^n(\ell_{rc}^2)}.$$

Define $S_1 = (C_{1,x_i})_{i=1}^d \in M_n(\mathbb{C})^d$ by $C_{1,x_i} = C_{x_i} - \widehat{\phi(S_0)}(x_i)$ for $i = 1, \cdots, d$. By induction on $k = 0, 1, 2, \cdots$, we can define $S_{k+1} = (C_{k+1,x_i})_{i=1}^d \in M_n(\mathbb{C})^d$ by

$$S_{k+1} = \left(C_{k,x_i} - \widehat{\phi(S_k)}(x_i) \right)_{i=1}^d = S_k - \left(\widehat{\phi(S_k)}(x_i) \right)_{i=1}^d.$$

Similarly, by lemma 3.16 or by inequality (3.29),

$$\|S_{k+1}\|_{S^\infty(\ell_{rc}^2)} = \left\| \left(C_{k,x_i} - \widehat{\phi(S_k)}(x_i) \right)_{i=1}^d \right\|_{S_\infty^n(\ell_{rc}^2)} \leqslant \frac{1}{2} \|S_k\|_{S_\infty^n(\ell_{rc}^2)}.$$

and

$$\|\phi(S_k)\|_{M_n(\mathbb{C})\otimes\mathcal{M}} \leqslant \frac{\sqrt{\max\{\alpha(W), \alpha(W^*)\}}}{2} \|S_k\|_{S_\infty^n(\ell_{rc}^2)} \tag{3.31}$$

Thus, for all $k \geqslant 1$, we have

$$\|S_{k+1}\|_{S_\infty^n(\ell_{rc}^2)} \leqslant \frac{1}{2^{k+1}} \|S_0\|_{S_\infty^n(\ell_{rc}^2)} \tag{3.32}$$

By reversing the induction process, we also obtain

$$\begin{aligned} S_0 &= (\widehat{\phi(S_0)}(x_i))_{i=1}^d + S_1 = (\widehat{\phi(S_0)}(x_i))_{i=1}^d + (\widehat{\phi(S_1)}(x_i))_{i=1}^d + S_2 = \cdots \\ &= \sum_{j=0}^k \left(\widehat{\phi(S_j)}(x_i) \right)_{i=1}^d + S_{k+1} \end{aligned}$$

By (3.32) $S_k \to 0$ as $k \to \infty$. Hence,

$$S_0 = \left(\sum_{j=0}^\infty \widehat{\phi(S_j)}(x_i) \right)_{i=1}^d \tag{3.33}$$

Note that

$$\sum_{j=0}^{\infty} \|\phi(S_j)\|_{M_n(\mathbb{C})\otimes\mathcal{M}} \leqslant \frac{\sqrt{\max\{\alpha(W),\alpha(W^*)\}}}{2} \sum_{j=0}^{\infty} \|S_k\|_{S_\infty^n(\ell_{rc}^2)}$$
$$\leqslant \frac{\sqrt{\max\{\alpha(W),\alpha(W^*)\}}}{2} \sum_{j=0}^{\infty} \frac{1}{2^j} \|S_0\|_{S_\infty^n(\ell_{rc}^2)} \leqslant \sqrt{\max\{\alpha(W),\alpha(W^*)\}} \, \|S_0\|_{S_\infty^n(\ell_{rc}^2)}$$

Thus, $\sum_{j=0}^{\infty} \phi(S_j)$ converges in the norm topology and

$$\|\sum_{j=0}^{\infty} \phi(S_j)\| \leqslant \sqrt{\max\{\alpha(W),\alpha(W^*)\}} \, \|S_0\|_{S_\infty^n(\ell_{rc}^2)} \,. \tag{3.34}$$

Therefore, by the equation (3.33) and continuity of Fourier transform

$$(C_{x_i})_{i=1}^d = S_0 = \left(\widehat{\sum_{j=0}^{\infty} \phi(S_j)}(x_i) \right)_{i=1}^d . \tag{3.35}$$

Let $h := \sum_{j=1}^{\infty} \phi\left[(C_{j,x_i})_{i=1}^d\right]$. By (3.34) and (3.35), h is the function that satisfies the requirements in the lemma. □

Recall that the Fourier transform $\hat{f}$ of $f = \sum_{x\in\Gamma} C_x \otimes x$ mapping from $M_n(\mathbb{C}) \otimes \mathcal{M}$ to $S_\infty^n(\ell_{rc}^2)$ is bounded with norm 1 by lemma 3.13. Thus, its adjoint (which we call it as the Fourier inverse transform) is well-defined with norm 1 mapping from $(S_\infty^n(\ell_{rc}^2))^*$ to $[M_n(\mathbb{C}) \otimes \mathcal{M}]^*$. Fix $(x_i)_{i=1}^d \subset \Gamma$ and suppose $(C_{x_i})_{i=1}^d \in S_1^n(\ell_{rc}^2)$, then the Fourier inverse transform of $(C_{x_i})_{i=1}^d$ (denoted by $\overbrace{(C_{x_i})_{i=1}^d}$) is

$$\overbrace{(C_{x_i})_{i=1}^d} = \sum_{i=1}^{d} C_{x_i} \otimes x$$

Thus by duality, lemma 3.13 and lemma 3.17, we proved the main theorem.

Remark 3.18. Pisier and Ricard obtained similar results without estimating the constants. Assume that $Z_2(W) = 1$ in Corollary 3.9, the constant in lower Kintchine's inequality will be $\frac{1}{\sqrt{2}}$. Conversely, we may ask if lower Kintchine's inequality (3.2) holds with the constant $\frac{1}{\sqrt{2}}$, can we deduce that $Z_2(W) = 1$? The answer is no. For if we consider the result of Szarek who proved the best constant of lower Khintchine inequality is $\frac{1}{\sqrt{2}}$ in the case of Rademacher variables which could be seen as a subset $R \subset \{-1,1\}^{\mathbb{N}}$ and $Z_2(R) = 2$. However, We do not know the answer for the case of operator valued coefficients.

3.4.1 Some Converse Results

If we consider the integer group $\mathbb{Z}$, then we have the following partial converse to the Khintchine's inequality in Corollary 3.9.

Theorem 3.19. *Let $W \subseteq \mathbb{Z}$. Suppose that the following inequality*

$$\frac{1}{\sqrt{2}} \left\|(C_i)_{i=1}^d\right\|_{S^1([M_n(\mathbb{C})]^d)} \leqslant \left\|\sum_{l=1}^d C_i \otimes e^{2\pi i k_l(\cdot)}\right\|_{L^1(\mathbb{T};S_1^n)} \tag{3.36}$$

holds for all $d \in \mathbb{N}$, for all C_1, ..., $C_d \in M_n(\mathbb{C}), n \in \mathbb{N}$ and for all k_1, ..., $k_d \in W$ distinct, Then, $Z_2(W) \leqslant 6$.

Before we prove theorem 3.19, we need the following lemma by Haagerup and Itoh [9]. A sketch of proof and two examples are provided as well for completion.

Lemma 3.20 (Haagerup-Itoh,1995). *For any integer $n \geqslant 1$, there exist a Hilbert space H of dimension $d = \binom{2n+1}{n}$ and $2n+1$ partial isometries $a_1, \ldots, a_{2n+1} \in \mathcal{B}(H) = M_d(\mathbb{C})$ such that*

1) *$Tr(a_i^* a_i) = \binom{2n}{n}$ for $1 \leqslant i \leqslant 2n+1$, where Tr denotes the trace on $M_d(\mathbb{C})$.*

2) *$\sum_{i=1}^{2n+1} a_i^* a_i = \sum_{i=1}^{2n+1} a_i a_i^* = (n+1)I_d$, where I_d denotes the identity of the Hilbert space.*

3) *For any $(g_k)_{k=1}^{2n+1} \subset \mathbb{C}$ with $\sum_{k=1}^{2n+1} |g_k|^2 = 1$, the operator $b = \sum_{k=1}^{2n+1} g_k a_k$ is a partial isometry and $Tr(b^* b) = \binom{2n}{n}$.*

Proof. We will use the notation in [4]. Suppose $\mathfrak{h}$ is a Hilbert space of dimension $2n+1$ with an orthonormal basis $B = \{e_1, \cdots, e_{2n+1}\}$ and the inner product $(\cdot,\cdot)$ antilinear in the first slot and linear in the second slot. Denote the n-fold tensor product of $\mathfrak{h}$ itself by $\mathfrak{h}^n = \mathfrak{h}\otimes\mathfrak{h}\otimes\cdots\otimes\mathfrak{h}$ and the Fock space by $\mathfrak{F}(\mathfrak{h}) = \oplus_{n\geqslant 0}\mathfrak{h}^n$, where $\mathfrak{h}^0 = \mathbb{C}$. Denote the operator P_- on Fock space $\mathfrak{F}(\mathfrak{h})$ by

$$P_-(f_1\otimes f_2\otimes\cdots\otimes f_n) = \frac{1}{n!}\sum_{\pi}\epsilon_\pi f_{\pi_1}\otimes f_{\pi_2}\otimes\cdots\otimes f_{\pi_n}.$$

for all $f_1,\cdots,f_n\in\mathfrak{h}$, where the sum is over all all permutations π and ϵ_π is 1 if π is even and -1 if π is odd. Extension by linearity and the Fermi-Fock space is defined by $\mathfrak{F}_-(\mathfrak{h}) = P_-\mathfrak{F}(\mathfrak{h})$. Also denote number operator N on $\mathfrak{F}_-(\mathfrak{h})$ by

$$D(N) = \{\psi = \{\psi^{(n)}\}_{n\geqslant 0} : \psi^{(n)}\in\mathfrak{F}_-(\mathfrak{h}^n), \sum_{n\geqslant 0} n^2\|\psi^{(n)}\|^2 < \infty\}$$

and

$$N\psi = \{n\psi^{(n)}\}_{n\geqslant 0}$$

for each $\psi\in D(N)$.

Suppose $f, h_1,\cdots,h_n\in\mathfrak{h}$, the annihilation operator $a_-(f):\mathfrak{F}_-(\mathfrak{h}^n)\to\mathfrak{F}_-(\mathfrak{h}^{n-1})$ and creation operator $a_-^*(f):\mathfrak{F}_-(\mathfrak{h}^n)\to\mathfrak{F}_-(\mathfrak{h}^{n+1})$ by initially setting $a_-(f)\psi^{(0)} = 0$, $a_-^*(f)\psi^{(0)} = \psi^{(0)}f$ if $\psi^{(0)}\in\mathbb{C}$ and

$$a_-(f)(h_1\otimes\cdots\otimes h_n) := \frac{1}{\sqrt{n}}\sum_{k=1}^{n}(-1)^{k-1}(f,h_k)P_-(h_1\otimes\cdots\otimes\widehat{h}_K\otimes\cdots\otimes h_n),$$

$$a_-^*(f)(h_1\otimes\cdots\otimes h_n) := \sqrt{n+1}P_-(f\otimes h_1\otimes\cdots\otimes h_n).$$

where $h_1\otimes\cdots\otimes\widehat{h}_k\otimes\cdots\otimes h_n$ means variable h_k is omitted in the tensor product. Suppose $g, f\in\mathfrak{h}$, from the definition of annihilation and creation operators, we have

$$a_-^*(g)a_-(g)P_-(h_1 \otimes \cdots \otimes h_n) = \sum_{k=1}^{n} (f, h_k) P_-((h_1 \otimes \cdots \otimes h_n)_k) \tag{3.37}$$

where $(h_1 \otimes \cdots \otimes h_n)_k$ means the h_k is replaced by g in the tensor product $h_1 \otimes \cdots \otimes h_n$.

There are following relations by computation similarly:

- (Adjoint relation) $(a_-^*(f)\phi, \psi) = (\phi, a_-(f)\psi)$, for $\phi, \psi \in D(N^{\frac{1}{2}})$
- (CAR relations)

$$\{a_-(f), a_-(g)\} = 0 = \{a_-^*(f), a_-^*(g)\} \tag{3.38}$$

$$\{a_-(f), a_-(g)\} = (f, g)\mathbb{1}. \tag{3.39}$$

where the notation $\{A, B\} = AB + BA$ is used.

In particular, choose $g = f = e_k$, and denote $a_k = a_-(e_k)$ for $k = 1, \cdots, 2n+1$. By (3.37) and CAR relation, we have

$$(a_k^* a_k)^2 = a_k^* \{a_k, a_k^*\} a_k = \|e_k\|^2 a_k^* a_k = a_k^* a_k, \quad k = 1, \cdots, 2n+1.$$

Hence, a_k is a partial isometry for $k = 1, \cdots, 2n+1$ if we restrict a_k to the subspace $\mathfrak{F}_-(\mathfrak{h}^{n+1})$. Also, suppose that $1 \leqslant i_1 < \cdots < i_{n+1} \leqslant 2n+1$. Then, by (3.37),

$$a_k^* a_k \left(P_-(e_{i_1} \otimes \cdots \otimes e_{i_{n+1}})\right) = \begin{cases} P_-(e_{i_1} \otimes \cdots \otimes e_{i_{n+1}}) & \text{if } e_k \in \{e_1, \cdots, e_{i_{n+1}}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.40}$$

Consider $a_k^* a_k : \mathfrak{F}_-(\mathfrak{h}^{n+1}) \to \mathfrak{F}_-(\mathfrak{h}^{n+1})$ as a d by d matrix, where $d = \binom{2n+1}{n+1}$ is the dimension of $\mathfrak{F}_-(\mathfrak{h}^{n+1})$ with the basis $\{P_-(e_{i_1} \otimes \cdots \otimes e_{i_{n+1}}) : 1 \leqslant i_1 < \cdots < i_{n+1} \leqslant 2n+1\}$. Thus, by (3.40), we know that there are $\binom{2n+1-1}{n+1-1}$ entries equal to 1 on the diagonal of $a_k^* a_k$, with the other entries being 0. Thus $Tr(a_k^* a_k) = \binom{2n}{n}$, for

$k = 1, \cdots, 2n+1$. Again, by (3.40), we know that $\sum_{k=1}^{2n+1} a_k^* a_k = (n+1)I_d$ restricting to $\mathfrak{F}_-(\mathfrak{h}^{n+1})$.

On the other hand, if we restrict a_k to the subspace $\mathfrak{F}_-(\mathfrak{h}^n)$, we will have $\sum_{k=1}^{2n+1} a_k^* a_k = nI_d$ since $a_k a_k^* : \mathfrak{F}_-(\mathfrak{h}^n) \to \mathfrak{F}_-(\mathfrak{h}^n)$ is also a d by d matrix. Note that $\binom{2n+1}{n+1} = \binom{2n+1}{n}$. Then, using the CAR relations (3.39), $\sum_{k=1}^{2n+1} a_k a_k^* = (2n+1-n)I_d = (n+1)I_d$. Therefore, we prove 1) and 2) of lemma.

For 3) of lemma, $b = \sum_{k=1}^{2n+1} g_k a_k = \sum_{k=1}^{2n+1} g_k a_-(e_k) = a_-(\sum_{k=1}^{2n+1} g_k e_k)$. Hence, $(b^*b)^2 = \| \sum_{k=1}^{2n+1} g_k e_k \|^2 b^*b = b^*b$ since $\| \sum_{k=1}^{2n+1} g_k e_k \| = \sum_{k=1}^{2n+1} |g_k|^2 = 1$. Thus b is also a partial isometry. Besides, using the relation that $Tr(a_k^* a_j) = \delta_{kj}$ obtained from [9], we have

$$Tr(b^*b) = Tr((\sum_{k=1}^{2n+1} \bar{g}_k a_k^*)(\sum_{j=1}^{2n+1} g_j a_j)) = \sum_k \sum_j \bar{g}_k g_j Tr(a_k^* a_j) = \sum_{k=1}^{2n+1} |g_k|^2 \binom{2n}{n} = \binom{2n}{n}.$$

. Therefore, 3) of lemma is proved. □

Remark 3.21. Let $n = 2$ in the Lemma 3.20, we can obtain the 5 partial isometries

$$a_1 = E_{6,1} + E_{5,2} + E_{4,3} + E_{3,4} + E_{2,5} + E_{1,6},$$

$$a_2 = -E_{9,1} - E_{8,2} - E_{7,3} + E_{3,7} + E_{2,8} + E_{1,9},$$

$$a_3 = -E_{8,4} - E_{7,5} - E_{5,7} - E_{4,8} + E_{1,10} + E_{10,1},$$

$$a_4 = E_{10,2} + E_{9,4} + E_{6,7} - E_{7,6} - E_{4,9} - E_{2,10},$$

$$a_5 = E_{10,3} + E_{9,5} + E_{8,6} + E_{6,8} + E5,9 + E_{3,10}.$$

satisfying the 1),2) and 3) of Lemma 3.20 where $E_{i,j}$ is a 10×10 matrix with entry equals 1 at position (i,j) and 0 at other places.

Lemma 3.22. *Let $W \subseteq \mathbb{Z}$. Suppose that for all $l \in \mathbb{N}$, for all $C_1, \ldots, C_l \in M_n(\mathbb{C}), n \in \mathbb{Z}$ and for all $k_1, \ldots, k_l \in W$ distinct, the following inequality holds:*

$$\frac{1}{\sqrt{2}}\left\|(C_i)_{i=1}^l\right\|_{S_1^n(\ell_{rc}^2)} \leqslant \left\|\sum_{i=1}^l C_i \otimes e^{2\pi i k_i(\cdot)}\right\|_{L^1(\mathbb{T};S_1^n)}.$$

Then

1) *there do not exist* $k_1, k_2, ..., k_{10} \in W$ *distinct such that* $k_2 - k_1 = k_4 - k_3 = ... = k_{10} - k_9$

2) *there does not exist distinct* $k_1, k_2, ..., k_9 \in W$ *such that* (k_1, k_2, k_3), (k_4, k_5, k_6) *and* (k_7, k_8, k_9) *are arithmetic progressions of length 3 with same common difference.*

3) W *can only contain an arithmetic sequence of at most length 4.*

Proof. 1) Suppose that there exist distinct $k_1, ..., k_{10} \in W$ and $A_1, ..., A_{10} \in M_n(\mathbb{C})$ such that $k_2 - k_1 = k_4 - k_3 = ... = k_{10} - k_9$. By Lemma 3.20, choose $n = 2$, there exist partial isometries $B_1, ..., B_5 \in M_{10}(\mathbb{C})$ such that

- $Tr\,(B_i^* B_i) = 6$ for all $1 \leqslant i \leqslant 5$.
- $\sum_{i=1}^5 B_i^* B_i = \sum_{i=1}^5 B_i B_i^* = 3I_{10}$. Consequently, $\|(B_i)_{i=1}^5\|_{S_1^{10}(\ell_{rc}^2)} = 10\sqrt{3}$
- Let $\beta_1, ..., \beta_5 \in \mathbb{C}$ where $\sum_{i=1}^5 |\beta_i|^2 = 1$, the operator $B = \sum_{i=1}^5 \beta_i B_i \in M_{10}(\mathbb{C})$ is a partial isometry with $Tr(B^*B) = 6$.

Now, consider the element $(A_i)_{i=1}^{10}$ with $A_{2i-1} = A_{2i} = B_i$ for $i = 1, \ldots, 5$. Then by definition and duality,

$$\begin{aligned}\|(A_i)_{i=1}^{10}\|_{S_1^{10}(\ell_{rc}^2)} &= \sup\left\{\left|\mathrm{Tr}\left(\sum_{i=1}^{10} A_i X_i\right)\right| : \|(X_i)_{i=1}^{10}\|_{S_\infty^{10}(\ell_{rc}^2)} \leqslant 1\right\} \quad (\text{ by duality})\\ &\geqslant \left|\mathrm{Tr}\left[\sum_{i=1}^{10} A_i\left(\frac{1}{\sqrt{6}}A_i^*\right)\right]\right| \quad (\text{ since } \sum_{i=1}^{10} A_i^* A_i = \sum_{i=1}^{10} A_i A_i^* = 6I)\end{aligned}$$

$$= \frac{1}{\sqrt{6}}\ \mathrm{Tr}\left(\sum_{i=1}^{10} A_i A_i^*\right) = \frac{1}{\sqrt{6}}\ \mathrm{Tr}(6I_{10}) = \frac{60}{\sqrt{6}} = 10\sqrt{6}$$

Here, Tr is the unnormalized trace on $M_{10}(\mathbb{C})$. On the other hand, we have

$$\begin{aligned}
&\left\|(A_i)_{i=1}^{10}\right\|_{S_1^{10}(\ell_{rc}^2)} \\
&= \inf\left\{\mathrm{Tr}\left[\left(\sum_{i=1}^{10} Y_i^* Y_i\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{10} Z_i Z_i^*\right)^{\frac{1}{2}}\right] : A_i = Y_i + Z_i \in M_{10}(\mathbb{C}), 1 \leqslant i \leqslant 10\right\} \\
&\leqslant \mathrm{Tr}\left[\left(\sum_{i=1}^{10} A_i^* A_i\right)^{\frac{1}{2}}\right] = \mathrm{Tr}\left[\left(2\sum_{i=1}^{5} B_i^* B_i\right)^{\frac{1}{2}}\right] = \mathrm{Tr}\left(\sqrt{6} I_{10}\right) = 10\sqrt{6}
\end{aligned}$$

We now consider $f : \mathbb{T} \to S_1^{10}$ defined by

$$f(t) := \sum_{m=1}^{10} A_m e^{2\pi i k_m t} = \sum_{m=1}^{5} B_m (e^{2\pi i k_{2m-1} t} + e^{2\pi i k_{2m} t})$$

Then we will compute the norm $\|f\|_{L^1(\mathbb{T};S_1^{10})}$ and compare it with $\frac{1}{\sqrt{2}}\left\|(A_i)_{i=1}^{10}\right\|_{S_1^{10}(\ell_{rc}^2)}$

$$\begin{aligned}
\|f\|_{L^1(\mathbb{T};S_1^{10})} &= \int_{\mathbb{T}} \mathrm{Tr}(|f(t)|) dt = \int_0^1 \left|\mathrm{Tr}\left[\sum_{m=1}^{5} B_m (e^{2\pi i k_{2m-1} t} + e^{2\pi i k_{2m} t})\right]\right| dt \\
&= \int_0^1 \mathrm{Tr}\left(\left|\sum_{m=1}^{5} B_m e^{2\pi i k_{2m-1} t}(1 + e^{2\pi i (k_{2m} - k_{2m-1}) t})\right|\right) dt \\
&= \int_0^1 \left|1 + e^{2\pi i (k_2 - k_1) t}\right| \mathrm{Tr}\left(\left|\sum_{m=1}^{5} B_m e^{2\pi i k_{2m-1} t}\right|\right) dt \\
&= \int_0^1 \sqrt{5}|1 + e^{2\pi i (k_2 - k_1) t}| \left|\mathrm{Tr}\left(\frac{1}{\sqrt{5}}\left|\sum_{m=1}^{5} B_m e^{2\pi i k_{2m-1} t}\right|\right)\right| dt \\
&= \int_0^1 6\sqrt{5}|1 + e^{2\pi i (k_2 - k_1) t}|\ dt \quad \text{(by Lemma 3.20)} \\
&= \frac{24\sqrt{5}}{\pi}
\end{aligned}$$

Hence $\frac{1}{\sqrt{2}}\left\|(A_i)_{i=1}^{10}\right\|_{S_1^{10}(\ell_{rc}^2)} > \|f\|_{L^1(\mathbb{T};S_1^{10})}$, which is a contradiction.

2) Suppose that there exist $k_1, ..., k_9 \in W$ such that $k_2 - k_1 = k_3 - k_2 = k_5 - k_4 = k_6 - k_5 = k_8 - k_7 = k_9 - k_8$. By Lemma 3.20, choose $n = 1$, there exist partial isometries $B_1, B_2, B_3 \in M_3(\mathbb{C})$ such that

- $Tr\,(B_i^* B_i) = 2$ for all $1 \leqslant i \leqslant 3$.
- $\sum_{i=1}^3 B_i^* B_i = \sum_{i=1}^3 B_i B_i^* = 2I_3$. Consequently, $\|(B_i)_{i=1}^3\|_{S_1^3(\ell_{rc}^2)} = 3\sqrt{2}$
- Let $\beta_1, \beta_2, \beta_3 \in \mathbb{C}$ where $\sum_{i=1}^3 |\beta_i|^2 = 1$, the operator $B = \sum_{i=1}^3 \beta_i B_i \in M_3(\mathbb{C})$ is a partial isometry with $Tr(B^* B) = 2$.

Now, consider the element $(C_i)_{i=1}^9$ with $C_{3i-2} = C_{3i} = B_i$ and $C_{3i-1} = 2B_i$ for $i = 1, 2, 3$. Similarly, $\|(C_i)_{i=1}^9\|_{S_1^3(\ell_{rc}^2)} = 6\sqrt{3}$ and $\left\|\sum_{i=1}^9 C_i \otimes e^{2\pi i k_l(\cdot)}\right\|_{L^1(\mathbb{T};S_1^n)} = 4\sqrt{3} > \frac{1}{\sqrt{2}}\|(C_i)_{i=1}^9\|_{S_1^3(\ell_{rc}^2)}$, which is a contradiction.

3) Suppose that W contains an arithmetic sequence of length 5 with common difference m, i.e., there exist $k_1, ..., k_5 \in W$ distinct such that $m = k_2 - k_1 = k_3 - k_2 = k_4 - k_3 = k_5 - k_4$. Consider the element $(C_k)_{k=1}^5 = (1, 2, 3, 2, 1) \in \mathbb{C}^5$. Then

$$\left\|(C_i)_{i=1}^5\right\|_{S_1^1(\ell_{rc}^2)} = \sqrt{1^2 + 2^2 + 3^2 + 2^2 + 1^2} = \sqrt{19}$$

On the other hand,

$$\|\sum_{i=1}^5 C_i \otimes e^{2\pi i k_i(\cdot)}\|_{L^1(\mathbb{T};S_1^1)} = 3.$$

Thus $\frac{1}{\sqrt{2}}\|(C_i)_{i=1}^5\|_{S_1^1(\ell_{rc}^2)} > \|\sum_{i=1}^5 C_i \otimes e^{2\pi i k_i(\cdot)}\|_{L^1(\mathbb{T};S_1^1)}$, which is a contradiction.

□

Now we come to the proof theorem 3.19.

Proof. Suppose that $Z_2(W) \geqslant 7$. Then there exist $k_1, ..., k_{14} \in W$ (it is possible that $k_s = k_{s+1}$ for some s) such that $k_2 - k_1 = \cdots = k_{14} - k_{13} = m \in \mathbb{N}$, which we consider them as seven distinct intervals of length m on the real line, denoted as $[k_{2i-1}, k_{2i}], i =$

$1, \cdots, 7$. For convenience, assume that $k_{2i} < k_{2j}$ for $i < j$ and we denote the distance between interval $[k_{2i-1}, k_{2i}]$ and $[k_{2i+1}, k_{2i+2}]$ by $d_i = |k_{2i+1} - k_{2i}|, i = 1, \cdots, 7$. The Figure 3.1 shows one possible case in the assumption. In particular, $d_i = 0$ means $k_{2i} = k_{2i+1}$.

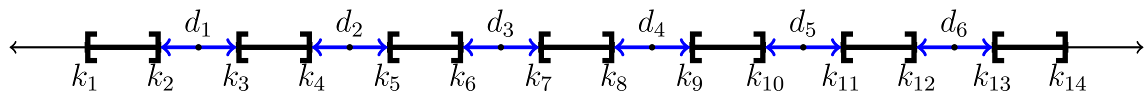

Figure 3.1. The case when $d_i \neq 0, i = 1, \cdots, 6$

By 3) of Lemma 3.22, $\#\{d_i : d_i = 0\} \geqslant 3$; By 1) of Lemma 3.22, $\#\{d_i : d_i = 0\} < 5$. Thus $\#\{d_i : d_i = 0\} = 3$ or 4.

- If $\#\{d_i : d_i = 0\} = 4$, this means that there are only 2 elements in $(d_i)_{i=1}^6$ that is not 0. On the other hand, by 2) of Lemma 3.22, there cannot be three consecutive elements be zero in $(d_i)_{i=1}^6$. Thus, it will lead us to find that discussing the cases when $d_3 \neq 0$ is sufficient by the symmetry.
  - If $d_4 \neq 0$, then $d_1 = d_2 = d_5 = d_6 = 0$ which is impossible, for there will be five distinct intervals of same length: $(k_1, k_2), (k_5, k_6), (k_7, k_8), (k_9, k_{10})$ and (k_{13}, k_{14}). This contradicts 1) of Lemma 3.22. See Figure 3.2. Similarly, the case when $d_6 = 0$ is also impossible.

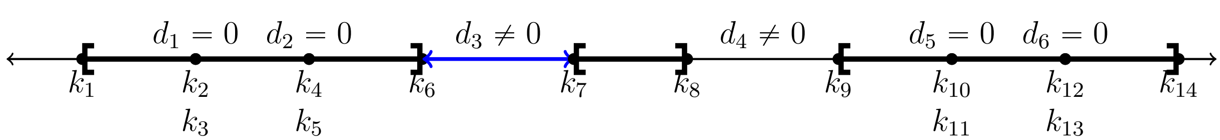

Figure 3.2. The case when $d_3 \neq 0, d_4 \neq 0$ and all else are 0

– If $d_5 \neq 0$, then this contradicts the 2) of Lemma 3.22, for there will be three distinct arithmetic progressions of length 3 with common difference: $(k_1, k_2, k_4), (k_7, k_8, k_9)$ and (k_9, k_{10}, k_{12}). See Figure 3.3.

- If $\#\{d_i : d_i = 0\} = 3$, the cases are discussed similarly as above by Lemma 3.22.

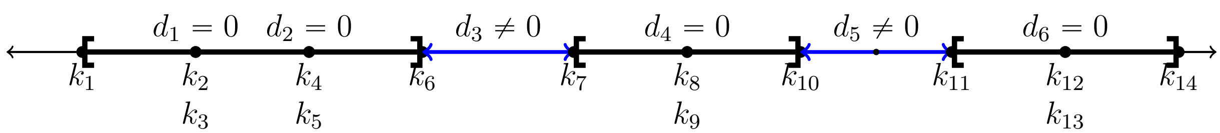

Figure 3.3. The case when $d_3 \neq 0, d_5 \neq 0$ and all else are 0
.

Therefore, by discussing all possible cases and we finish our proof by contradiction.

□

Remark 3.23. It is not known whether the Z_2 constant in Theorem 3.19 can be reduced to 2 or not.

For some abelian groups, we also have some results if the best constant of the lower Khintchine inequalities is greater than certain numbers.

Theorem 3.24. *(1) Suppose* $G := \mathbb{Z}/k_1\mathbb{Z} \times ... \times \mathbb{Z}/k_m\mathbb{Z} \times (\prod_{\alpha\in\Lambda} \mathbb{T})$. *Let* $W \subseteq \widehat{G}$. *Suppose that there exists* $K > \frac{2}{\pi}$ *such that for all* $n \in \mathbb{N}$, *for all* $d \in \mathbb{N}$, *for all* $C_1, ..., C_d \in M_n(\mathbb{C})$ *and for all* $x^1, ..., x^d \in W$,

$$K \left\|(C_k)_{k=1}^d\right\|_{S_1^n(\ell^2_{rc})} \leqslant \left\| \sum_{k=1}^d C_k \otimes \gamma_{x^k} \right\|_{L^1(\widehat{G};S_1)}.$$

Then, $Z_2(W) < \infty$.

(2) *Suppose* $G := \prod_{\alpha\in\Lambda} \mathbb{Z}/2\mathbb{Z}$. *Let* $W \subseteq \widehat{G}$. *Suppose that there exists* $K > \frac{1}{2}$ *such that for all* $n \in \mathbb{N}$, *for all* $d \in \mathbb{N}$, *for all* B_1, ..., $B_d \in M_n(\mathbb{C})$ *and for all* x_1, ..., $x_d \in W$,

$$K \left\|(C_k)_{k=1}^d\right\|_{S_1^n(\ell^2_{rc})} \leqslant \left\|\sum_{k=1}^d C_k \otimes \gamma_{x^k}\right\|_{L^1(\widehat{G};S_1)}.$$

Then, $Z_2(W) < \infty$.

Lemma 3.25. *Suppose* $G := \mathbb{Z}/k_1\mathbb{Z} \times ... \times \mathbb{Z}/k_m\mathbb{Z} \times (\prod_{\alpha\in\Lambda} \mathbb{T})$. *Let* $W \subseteq \widehat{G}$ *such that* $Z_2(W) = \infty$. *Then, for every* $d \in \mathbb{N}$ *sufficiently large, there exist* $x^1, ..., x^d, y^1, ..., y^d \in \widehat{G}$ *such that:*

a) $y^1 - x^1 = y^2 - x^2 = ... = y^d - x^d \neq 0 \in \widehat{G}$.

b) there exists $\beta \in \Lambda$ *such that* $(y^1 - x^1)_\beta \neq 0 \in \mathbb{Z}$.

c) $Ran[\gamma_{y^1 - x^1}] = \mathbb{T}$., *where* γ_x *is the group homomorphism* $\gamma_x : G \to \mathbb{T}$ *associated to* $x \in \widehat{G}$.

Proof. Note that $\widehat{G} = \mathbb{Z}/k_1\mathbb{Z} \times ... \times \mathbb{Z}/k_m\mathbb{Z} \times (\bigoplus_{\alpha\in\Lambda} \mathbb{Z})$. So, each $x \in \widehat{G}$ is of the form $x = (x_1, ..., x_m, (x_\alpha)_{\alpha\in\Lambda}) \in \widehat{G}$, where $x_1 \in \mathbb{Z}/k_1\mathbb{Z}$, ..., $x_m \in \mathbb{Z}/k_m\mathbb{Z}$ and there is only finitely many $\alpha \in \Lambda$ such that $x_\alpha \neq 0$. Then, the character γ_x associated to $x \in \widehat{G}$ is of the form

$$\gamma_x = (\gamma_{x_1}, ...\gamma_{x_m}, \gamma_{(x_\alpha)_{\alpha\in\Lambda}}) : \mathbb{Z}/k_1\mathbb{Z} \times ... \times \mathbb{Z}/k_m\mathbb{Z} \times \prod_{\alpha\in\Lambda} \mathbb{T} \to \mathbb{T}$$
$$\gamma_x(j_1, ..., j_m, (z_\alpha)_{\alpha\in\Lambda}) = \prod_{r=1}^m \gamma_{x_r}(j_r) \prod_{x_\alpha\neq 0} \gamma_{x_\alpha}(z_\alpha) = \prod_{r=1}^m e^{\frac{2\pi i j_r x_r}{k_r}} \prod_{x_\alpha\neq 0} z_\alpha^{x_\alpha}.$$

Suppose that $Z_2(W) = \infty$. Note that the only elements in $\widehat{G}$ that have finite order are of the form $x = (x_1, ..., x_m, (0_\alpha)_{\alpha_\Lambda})$. Since there are only finitely many factors of the form $\mathbb{Z}/k_i\mathbb{Z}$, there can only be finitely many elements in $\widehat{G}$ that have a finite order.

Then, for $d \in \mathbb{N}$ sufficiently large (for example, $d > \prod_{j=1}^{m} k_j$), there exist distinct x^1, ..., x^d, y^1, ..., $y^d \in W$ such that:

1) $y^1 - x^1 = y^2 - x^2 = ... = y^d - x^d \neq 0 \in \widehat{G}$.

2) there exists $\beta \in \Lambda$ such that $(y^1 - x^1)_\beta \neq 0 \in \mathbb{Z}$.

Note that (1) follows from the fact that $Z_2(W) = \infty$. For simplicity, denote $w := y^1 - x^1$.

Next, we have that for each $g = (j_1, ..., j_m, (z_\alpha)_{\alpha\in\Lambda}) \in G$,

$$\gamma_w(g) = e^{\frac{2\pi i j_1 w_1}{k_1}} \cdot ... \cdot e^{\frac{2\pi i j_m w_m}{k_m}} \cdot z_{\alpha_1}^{w_{\alpha_1}} \cdot ... \cdot z_\beta^{w_\beta} \cdot ... \cdot z_{\alpha_\ell}^{w_{\alpha_\ell}}, \text{ where } \alpha_1, ..., \beta, ..., \alpha_\ell \neq 0 \in \mathbb{Z}.$$

In particular, we consider the element $g_z = [0, ..., 0, (z_\beta)_{\alpha\in\Lambda}]$, where $(z_\beta)_{\alpha\in\Lambda}$ takes the value $z \in \mathbb{T}$ at the β-th position and zero elsewhere. So, $\gamma_w(g_z) = z_\beta^{w_\beta}$. Since $w_\beta \neq 0$, $\mathrm{Ran}(\gamma_w) = \mathbb{T}$. Thus the lemma is proved. □

Nowe we come to the proof of Theorem 3.24.

Proof. (1) Suppose $G := \mathbb{Z}/k_1\mathbb{Z} \times ... \times \mathbb{Z}/k_m\mathbb{Z} \times (\prod_{\alpha\in\Lambda} \mathbb{T})$. For each $l \in \mathbb{N}$ sufficiently large, there exist distinct x^1, ..., x^l, y^1, ..., $y^l \in \widehat{G}$ satisfies the conclusions of Lemma 3.25.

So, $\sigma(\gamma_{y^1-x^1}) = \mathbb{T}$. By Gelfand's theorem, $C^*(\gamma_{y^1-x^1}, 1) = C(\mathbb{T})$. Moreover, the canonical trace on $L^\infty(G)$, τ_G coincides with the canonical trace on $\mathfrak{L}(\gamma_{y^1-x^1}) = L^\infty(\mathbb{T})$. In other words, for $f \in \mathfrak{L}(\gamma_{y^1-x^1})$, $\int_G f \, d\mu(g) = \int_{\mathbb{T}} f \, d\lambda(z)$.

We show that there exist $d \in \mathbb{N}$, x^1, ..., x^l, $y^1, ..., y^l \in W$ and C_1, ..., $C_{2l} \in M_d(\mathbb{C})$ such that the lower Khintchine inequality fails to hold. By Lemma 3.20, Choose $n = m$ and denote $d = \binom{2m+1}{m}$ and $l = 2m + 1$, there exist partial isometries B_1, ..., $B_{2m+1} \in M_d(\mathbb{C})$ such that

- $Tr(B_i^*B_i) = \binom{2m}{m}$ for $1 \leqslant i \leqslant 2m+1$, where Tr denotes the trace on $M_d(\mathbb{C})$.
- $\sum_{i=1}^{2m+1} B_i^*B_i = \sum_{i=1}^{2m+1} B_iB_i^* = (m+1)I_d$, where I_d denotes the identity of $M_d(\mathbb{C})$.
- For any $(\beta_k)_{k=1}^{2m+1} \subset \mathbb{C}$ with $\sum_{i=1}^{2m+1} |\beta_k|^2 = 1$, the operator $b = \sum_{i=1}^{2m+1} \beta_k B_k$ is a partial isometry and $Tr(b^*b) = \binom{2m}{m}$.

Now, consider the element $(C_i)_{i=1}^{2l}$ with $C_{2i-1} = C_{2i} = B_i$ for $i = 1, \dots, l$. Then by definition and duality and follow the calculations in Lemma 3.22, we have

$$A_m = \|(C_i)_{i=1}^{2l}\|_{S_1^d(\ell_{rc}^2)} = d\sqrt{2(m+1)}$$

and

$$B_m = \| \sum_{k=1}^{l} (C_{2k} \otimes \gamma_{x_k} + C_{2k+1} \otimes \gamma_{y_k}) \|_{L^1(\hat{G};S_1^d)} = \frac{4d(m+1)}{\pi\sqrt{2m+1}}$$

We show that there exists m such that $KA_m > B_m$, i.e., $K > \frac{B_m}{A_m} = \frac{\sqrt{8(m+1)}}{\pi\sqrt{2m+1}}$. Since $\lim_{m\to\infty} \frac{\sqrt{8(m+1)}}{\pi\sqrt{2m+1}} = \frac{2}{\pi}$ and $K > \frac{2}{\pi}$, there exists m big enough such that $KA_m > B_m$, which is a contradiction.

(2) Suppose $G := \prod_{\alpha\in\Lambda} \mathbb{Z}/2\mathbb{Z}$. First, note that $\widehat{G} = \bigoplus_{\alpha\in\Lambda} \mathbb{Z}/2\mathbb{Z}$. For each $(x_\alpha)_{\alpha\in\Lambda} \in \bigoplus_{\alpha\in\Lambda} \mathbb{Z}/2\mathbb{Z}$, $x_\alpha \neq 0$ for only a finite number of α. Suppose that $Z_2(W) = \infty$. Then, for each $d \in \mathbb{N}$, there exist $x_1, ..., x_l,\ y_1, ..., y_l \in W$ such that

(a) all the x_i's are distinct.

(b) $y_1 - x_1 = y_2 - x_2 = ... = y_l - x_l \neq 0$

Now, let $(a_\alpha)_\Lambda \in G$, where a_α takes on the value of either 0 or 1 at each $\alpha \in \Lambda$. The group homomorphism associated with the element $y_1 - x_1$, $\gamma_{y_1-x_1} \in \widehat{G}$ is given by $\gamma_{y_1-x_1} : \prod_{\alpha\in\Lambda} \mathbb{Z}/2\mathbb{Z} \to \mathbb{T}$, where $\gamma[(a_\alpha)_{\alpha\in\Lambda}] = (-1)^{\sum_{a_\alpha\neq 0, \alpha\in\Lambda} a_\alpha}$.

Similarly, by Lemma 3.20, Choose $n = m$ and denote $d = \binom{2m+1}{m}$ and $l = 2m+1$, and as in (1) we still consider the element $(C_i)_{i=1}^{2l}$ with $C_{2i-1} = C_{2i} = B_i$ for $i = 1, \dots, l$. Following the calculations above, we have

$$A_m = \|(C_i)_{i=1}^{2l}\|_{S_1^d(\ell_{rc}^2)} = d\sqrt{2(m+1)}$$

and

$$B_m = \| \sum_{k=1}^{l} (C_{2k} \otimes \gamma_{x_k} + C_{2k+1} \otimes \gamma_{y_k}) \|_{L^1(\hat{G};S_1^d)} = \frac{d(m+1)}{\sqrt{2m+1}}$$

We show that there exists m such that $KA_m > B_m$, i.e., $K > \frac{B_m}{A_m} = \frac{\sqrt{(m+1)}}{\sqrt{2(2m+1)}}$. Since $\lim_{m\to\infty} \frac{\sqrt{(m+1)}}{\sqrt{2(2m+1)}} = \frac{1}{2}$ and $K > \frac{1}{2}$, there exists m big enough such that $KA_m > B_m$, which is a contradiction.

□

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