ABSTRACT

Classification of Cosmological Models in Einstein's General Theory of Gravity

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In this thesis, I first review the fundamentals of Hot Big Bang cosmology, the observational cosmology and the late accelerating universe. Then I systematically study the evolution of the Friedmann-Robertson-Walker (FRW) universe with a cosmological constant Λ and a perfect fluid that has the equation of state $p = w\rho$, and classify all the solutions into various cases, where p and ρ are the pressure and energy density of the fluid, and w is a constant. In each case the main properties of the evolution of the universe are studied in detail, including the periods of deceleration or acceleration, and the existence of big bang, big crunch, and big rip singularities. Finally, I mention some future work along the direction laid down in this thesis.

Classification of Cosmological Models in Einstein's General Theory of Gravity

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DEDICATION

To my parents

CHAPTER ONE

Hot Big Bang Cosmology

Hot Big Bang Cosmology is one hypothesis for the origin of the universe: At the very beginning, the universe was very small, the size almost less than a point nucleus, and was known as the *singularity*. However, it had great heat energy, and the heat overflowed the singularity from the extremely hot moment, so that finally the explosion happened, which is called the Big Bang. This energy became the fundamental particles, and later these particles formed the substances, energy, space and time in the universe. So far the Big Bang model is one of the most convincing theories about the origin of the universe. However, the Big Bang theory is still lacking support from a large number of experiments, and we don't know the picture before the explosion.

In this chapter, some fundamental definitions and basic physical quantities will be introduced. The Hot Big Bang Cosmology contains three basic principles . Then we will summarize the main observational evidence to support this so-called standard cosmological model. Afterwards, we shall introduce the Late Acceleration of the Universe.

1.1 Relativistic Cosmology

In order to study the universe, we have to find a model to illustrate it. The easiest way is to ignore all the details like the solar system, the Milky Way, the local cluster of galaxy and so on. Then we will use a first-order ordinary differential equation called Friedmann's equation. The solutions of it are the Friedmann models and can be consider as the standard solutions of relativistic cosmology. We use this to be the basis of this part.

Relativistic Cosmology is based on three assumptions: (1) Cosmological principle; (2) Weyl's postulate; and (3) Einstein's general relativity. We will introduce these three important principles in the following sections.

1.1.1 Cosmological Principle

Cosmology is based on the cosmological principle, and it is an assumption about the large scale properties of the universe. We may illustrate it in this way: All points in space ought to experience the same physical development, correlated in time in such a way that all points at a certain distance from an observer appear to be at the same stage of development [1]. Or in the more concise way: On large spatial scales, the universe is homogeneous and isotropic. To study the cosmology, we have to believe that the place which we occupy in the universe is in no way special.

This leads to the definitions of *homogenous* and *isotropic*. Homogeneity is the statement that the universe looks the same at each point. We define that there is a cosmic time t, and t is constant in each of the spacelike slices, so each slice has no privileged points. Isotropy states that the universe looks the same in all directions. If we keep using the slices we defined, we can easily see that a manifold which has no privileged directions about a point is called isotropic and it should be spherically symmetric about that point.

Before analyzing the evolution of the universe by the cosmological principle, we have to introduce some physical terms.

We use the metric to determine distances and define the lengths of vectors. In four dimensions of space-time, the infinitesimal interval between two neighboring points x^a and $x^a + dx^a$ is defined by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1.1}$$

where $\mu, \nu = 0, 1, 2, 3,$ and $dx^0 = dt$. Eq.(1.1) is called line element, while $g_{\mu\nu}$ is called the metric or the first fundamental form.

In the *Minkowski space-time*, we denote the metric by $\eta_{\mu\nu}$, where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.2}$$

Now we can make the connection between $g_{\mu\nu}$ and $\eta_{\mu\nu}$. In special relativity, it is obvious $g_{\mu\nu} = \eta_{\mu\nu}$. But in the expanding universe, any two grid points move away from each other by the scale factor $a(t)$, i.e. if the physical distance between the two points at some earlier time t is x, and the comoving distance today is x_0, they are related by the relation:

$$x = a(t)x_0. \tag{1.3}$$

From this idea, the metric of the Minkowski space-time should be changed to

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix}, \tag{1.4}$$

which is called the *Friedmann-Robertson-Walker(FRW)* metric, where the three dimensional spatial space is flat.

Now we consider the properties of $g_{\mu\nu}$ [2]. First, we consider two three-dimensional vectors $\overrightarrow{A}$ and $\overrightarrow{B}$; each of them has three component, A^i and B^i, where $i = 1, 2, 3$. Then the dot product is

$$\overrightarrow{A} \cdot \overrightarrow{B} = \sum A^i B^i \equiv A^i B^i. \tag{1.5}$$

But in general relativity, each vector has four components, the fourth one is timelike. We use 0 to stand for time while the indices 1,2,3 stand for the spatial parts. i.e.

$$A^\mu = (A^0, A^i), \tag{1.6}$$

where $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$. Another main property of the four-dimensional vector is the relation between upper and lower indices,

$$\begin{aligned} A_\mu &= g_{\mu\nu} A^\nu \\ A^\mu &= g^{\mu\nu} A_\nu. \end{aligned} \tag{1.7}$$

In this way, the four-momentum of a massless particle must vanish, i.e.

$$P^2 \equiv P_\mu P^\mu = g_{\mu\nu} P^\mu P^\nu = 0. \tag{1.8}$$

In addition, the metric can be used to raise and lower indices on tensors with an arbitrary number of indices. i.e.

$$g^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta}. \tag{1.9}$$

Particularly when the $\alpha = \nu$, $g^{\mu\nu} = g^{\mu\alpha}$, we find that

$$g^{\nu\beta} g_{\alpha\beta} = \delta^\nu_\alpha, \tag{1.10}$$

where δ^ν_α is the Kronecker delta, and when $\nu = \alpha$, $\delta^\nu_\alpha = 1$, when $\nu \neq \alpha$, $\delta^\nu_\alpha = 0$.

We define the *Christoffel symbols of the first kind* as

$$\{ab, c\} = \frac{1}{2}(\partial_b g_{ac} + \partial_a g_{bc} - \partial_c g_{ab}), \tag{1.11}$$

and the *Christoffel symbols of the second kind* by

$$\Gamma^a_{bc} = g^{ad}\{bc, d\}. \tag{1.12}$$

Then, we find

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}). \tag{1.13}$$

We can easily find that it is symmetric, i.e.

$$\Gamma^a_{bc} = \Gamma^a_{cb}. \tag{1.14}$$

Another important tensor is the *Riemann tensor*, which is defined as :

$$
\begin{aligned}
R^a{}_{bcd} &= \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \\
R^a{}_{bcd} &= g^{ae} R_{ebcd},
\end{aligned}
\tag{1.15}
$$

The main properties of the Riemann tensor are:

$$
\begin{aligned}
R^a{}_{bcd} &= -R^a{}_{bdc} \\
R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} &= 0 \\
R_{abcd} = -R_{abdc} &= -R_{bacd} = R_{cdab} \\
R_{abcd} + R_{adbc} + R_{acdb} &= 0.
\end{aligned}
\tag{1.16}
$$

Then, the *Ricci tensor* is defined as:

$$R_{ab} = R^c{}_{acb} = g^{cd} R_{dacb}, \tag{1.17}$$

we can show that it is symmetric using Eq.(1.16). Based on this tensor, the *Ricci scalar* R is defined as :

$$R = g^{ab} R_{ab}. \tag{1.18}$$

Combining R and R_{ab}, we can define a new tensor, the *Einstein tensor*:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \tag{1.19}$$

which satisfies the *contracted Bianchi identities*

$$\nabla_b G_a{}^b \equiv 0. \tag{1.20}$$

Up to now, we have introduced all the physical quantities to be used in this thesis. In the following, we shall apply them to the FRW universe.

1.1.2 The Homogeneous and Isotropic FRW Universe

The metric given by Eq.(1.4) describes the flat FRW universe. This is a particular case in which the three-dimensional spatial space has a constant curvature.

The most general three-dimensional spatial space with a constant curvature is characterized by the line element,

$$d\sigma^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.21}$$

where k is a constant, which can be zero, positive, or negative. Without loss of generality, one can always set $k = 0, \pm 1$. Then, embedding it into the four-dimensional universe, it can be shown that the general metric in the FRW form,

$$ds^2 = dt^2 - a^2(t)(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)). \tag{1.22}$$

From the above expression, we can read off the metric g_{ab}, which can be written in several equivalent forms:

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2\sin^2\theta \end{pmatrix}, \tag{1.23}$$

or

$$\begin{aligned} g_{00} &= 1 \\ g_{11} &= -\frac{a^2(t)}{1-kr^2} \\ g_{22} &= -a^2(t)r^2 \\ g_{33} &= -a^2(t)r^2\sin^2\theta, \end{aligned} \tag{1.24}$$

from which we find

$$\begin{aligned} g^{00} &= 1 \\ g^{11} &= -\frac{1-kr^2}{a^2(t)} \\ g^{22} &= -\frac{1}{a^2(t)r^2} \\ g^{33} &= -\frac{1}{a^2(t)r^2\sin^2\theta}. \end{aligned} \tag{1.25}$$

Then, using the definition of Eq.(1.13), we find that the non-zero components of the Christoffel symbols Γ^a_{bc} are given by

$$\begin{aligned}
\Gamma^0_{11} &= -\frac{a\dot{a}}{1-kr^2} \\
\Gamma^0_{22} &= a\dot{a}r^2 \\
\Gamma^0_{33} &= a\dot{a}r^2\sin^2\theta \\
\Gamma^1_{01} &= \frac{\dot{a}}{a} \\
\Gamma^1_{11} &= \frac{kr}{1-kr^2} \\
\Gamma^1_{22} &= (-1+kr^2)r \\
\Gamma^1_{33} &= (-1+kr^2)r\sin^2\theta \\
\Gamma^2_{02} &= \frac{\dot{a}}{a} \\
\Gamma^2_{12} &= \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \\
\Gamma^3_{03} &= \frac{\dot{a}}{a} \\
\Gamma^3_{13} &= \frac{1}{r} \\
\Gamma^3_{23} &= \frac{\cos\theta}{\sin\theta},
\end{aligned} \tag{1.26}$$

where $\dot{a} = da(t)/dt$. Using the definition of the Riemann tensor R_{abcd} given by Eq.(1.15), we find that its non-zero components are given by

$$\begin{aligned}
R_{0101} &= \frac{a\ddot{a}}{1-kr^2} \\
R_{0202} &= a\ddot{a}r^2 \\
R_{0303} &= a\ddot{a}r^2\sin^2\theta \\
R_{1212} &= \frac{r^2a^2(\dot{a}^2+k)}{-1+kr^2} \\
R_{1313} &= \frac{r^2a^2(\dot{a}^2+k)\sin^2\theta}{-1+kr^2} \\
R_{2323} &= -a^2r^4(\dot{a}^2+k)\sin^2\theta,
\end{aligned} \tag{1.27}$$

where $\ddot{a} = d^2a(t)/dt^2$. Similarly, it can be shown that $R^a{}_{bcd}$ has the following non-zero components

$$\begin{aligned}
R^0{}_{101} &= \frac{a\ddot{a}}{1-kr^2} \\
R^0{}_{202} &= a\ddot{a}r^2 \\
R^0{}_{303} &= a\ddot{a}r^2\sin^2\theta \\
R^1{}_{212} &= r^2(\dot{a}^2+k) \\
R^1{}_{313} &= r^2(\dot{a}^2+k)\sin^2\theta \\
R^2{}_{323} &= r^2(\dot{a}^2+k)\sin^2\theta,
\end{aligned} \tag{1.28}$$

from which we find that the non-zero components of the Ricci tensor R_{ab} are given by

$$\begin{aligned}
R_{00} &= -\frac{3\ddot{a}}{a} \\
R_{11} &= \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2} \\
R_{22} &= a\ddot{a}r^2+2\dot{a}^2r^2+2kr^2 \\
R_{33} &= r^2\sin^2\theta(a\ddot{a}+2\dot{a}^2+2k),
\end{aligned} \tag{1.29}$$

while from Eq.(1.18) we find the Ricci scalar is given by

$$R = -\frac{6(a\ddot{a}+\dot{a}^2)+k}{a^2}. \tag{1.30}$$

Finally, the non-zero components of the Einstein tensor read

$$\begin{aligned}
G_{00} &= \frac{3(\dot{a}^2+k)}{a^2} \\
G_{11} &= \frac{2a\ddot{a}+\dot{a}^2+k}{-1+kr^2} \\
G_{22} &= -(2a\ddot{a}r^2+\dot{a}^2r^2+kr^2) \\
G_{33} &= -r^2\sin^2\theta(2a\ddot{a}+\dot{a}^2+k).
\end{aligned} \tag{1.31}$$

1.1.3 Weyl's Postulate

In order to investigate the universe, in 1923 Hermann Weyl thought about the apparent contradiction in applying a theory like General Relativity, that was set up to be generally covariant, to one particular set of circumstances, that of describing just one universe, our universe. In a universe that is expanding there seems to be a preferred coordinate system, which is comoving with the background expansion flow. Then, in the application of general relativity to a unique symmetrical system like our universe, there must be underlying phenomenologically based postulates that are formulated from local observations. Weyl reasoned that there was a privileged class of observers that is comoving with the smeared out motion of the galaxies. Then he proposed the so-called *Weyl's postulate*:

The particles of the substratum lie in space-time on a congruence of timelike geodesics diverging from a point in the finite or infinite past.

We can see that Weyl's postulate stipulates that in a fluid cosmological model, the world lines of the fluid particles, which act as the source of the gravitational field and are often taken to model galaxies, should be hypersurfaces orthogonal to a family of spatial hyperslices. This postulate says that there is only one geodesic passing through each point of spacetime. Therefore, the matter at each point possesses a unique velocity. We can easily see that the essence of Weyl's postulate is the substratum should be considered as a perfect fluid. Actually the galaxies do not follow the postulated cosmological motion exactly, and there is random directional motion of the galaxies, which is much smaller than the speed of light (one-thousandth of the speed of light). In the global cosmological perspective, this velocity is smeared out to be overall negligible and will be almost always nonrelativistic. From the observation that the general motion of the universe is expanding, we can easily find that Weyl's postulate is seen to closely show the actual situation of the real universe we live in.

Mathematically, Weyl's postulate can be expressed as that the matter in our universe can be described by a perfect fluid,

$$T_{ab} = (\rho + p)u_a u_b - p g_{ab}, \tag{1.32}$$

where ρ and p denote, respectively, the energy density and pressure of the fluid, and u_a is its 4-velocity. In the co-moving frame, it is given by $u_a = \delta^t_a$. Since in our universe there exist various matter fields such as non-relativistic matter, dark matter, radiation, dark energy, and so on, ρ and p are the sum of these individual fields

$$\rho = \sum_i \rho_i, \quad p = \sum_i p_i. \tag{1.33}$$

For non-relativistic matter, including dark matter, we have $p_M = 0$, while for radiation we have $p_R = \rho_R/3$. In general, p is a function of both ρ and T,

$$p = p(\rho, T), \tag{1.34}$$

where T is the temperature of the universe. If we consider the case where T is a constant, then the equation of state takes the form,

$$p = p(\rho). \tag{1.35}$$

1.1.4 The Friedmann Equations

The relativistic cosmology is based on the Einstein theory of gravity, which mathematically is expressed as

$$G_{ab} = \kappa T_{ab} + \Lambda g_{ab}, \tag{1.36}$$

where κ is a constant called the *gravitational coupling constant*, and in terms of the Newtonian constant G and the speed of light given by

$$\kappa = \frac{8\pi G}{c^4}.$$

In this thesis, we use the relativistic units, so that $c = 1$, which leads to $\kappa = 8\pi G$. Λ is called the cosmological constant, introduced first by Einstein in 1917 in order to construct a static universe. Since

$$\begin{aligned} T_{00} &= \rho, \\ T_{ij} &= p g_{ij}, \end{aligned} \tag{1.37}$$

inserting them, together with the expressions of G_{ab} given by Eq.(1.31), into Eq.(1.36), we find that in the present case there are only two independent equations, which can be written as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{1.38}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{1.39}$$

Combining Eqs.(1.38) and (1.39), we find that

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{1.40}$$

which represents the conservation law of the matter fields in the universe. Note that, if we introduce ρ_Λ and p_Λ via the relations,

$$\rho_\Lambda = -p_\Lambda \equiv \frac{\Lambda}{8\pi G}, \tag{1.41}$$

the Friedmann equation (1.38) and the conservation law (1.39) can be written as

$$H^2 = \frac{8\pi G}{3}\rho_{total} - \frac{k}{a^2}, \tag{1.42}$$

$$\dot{\rho}_{total} + 3H\left(\rho_{total} + p_{total}\right) = 0, \tag{1.43}$$

where

$$\rho_{total} \equiv \rho + \rho_\Lambda, \quad p_{total} \equiv p + p_\Lambda. \tag{1.44}$$

It should be noted that Eq.(1.40) or (1.43) can be obtained directly from the *conservation law*,

$$T^b_{a;b} = 0. \tag{1.45}$$

1.2 Observational Cosmology

In this part, we will introduce the observational support to the relativistic cosmology. *Observational cosmology* means one uses instruments like telescopes and cosmic ray detectors to study the structure, evolution and origin of the universe. There are three important pillars supporting the relativistic cosmology: (1) Expanding universe; (2) Big Bang nucleosynthesis (BBN); and (3) Cosmic microwave background (CMB). In the following we shall give a very brief introduction to each of them.

1.2.1 Expanding Universe

The most important observational evidence is the expansion of the universe: In 1929, Hubble observed that almost every galaxy in the universe appears to be moving away from us. By using a plot of velocity versus estimated distance for a set of many galaxies, the so-called Hubble diagram, Hubble found that they satisfy the law [4]

$$\vec{v} = H_0 \vec{r}, \tag{1.46}$$

where H_0 is a constant, usually cased the *Hubble's constant*, and $\vec{v}$ and $\vec{r}$ denote, respectively, the recession velocity of a galaxy and its distance from us. The recession velocity of an object was inferred from its *redshift*, which is basically the relativistic Doppler effect applied to light waves. Because the galaxy has both absorption and emission, if it is moving toward us, the light wave will get crowded, and the frequency will be higher. This is known as blueshift. If it is receding, the light wave length is stretched, and this is usually called the redshift. From the observation, we can see that almost all the galaxies are receding from us, so the universe is expanding. The redshift z is defined by,

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}, \tag{1.47}$$

where λ_{obs} and λ_{em} are the observed and emitted wavelengths, respectively. If a nearby object is receding at a speed v, then the redshift can be written as

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1 \simeq \frac{v}{c}. \tag{1.48}$$

Since

$$\vec{r} = a(t)\vec{x}, \tag{1.49}$$

we find that

$$1 + z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{1}{a}, \tag{1.50}$$

where $\vec{r}$ is the real distance and $\vec{x}$ is the comoving distance. Note that in writing the above expressions we assumed that the current radius of the universe is one, i.e., $a_0 = a(t_0) = 1$.

1.2.2 Big Bang Nucleosynthesis (BBN)

The Big Bang nucleosynthesis (BBN) states that during the early universe there were some light nuclei other than H-1 to have been produced. At the very beginning the nucleosythesis took place only a few minutes after the Big Bang, and soon some heavier isotope of hydrogen, such as deuterium (H-2 or D), helium isotopes He-3 and He-4, and lithium isotopes Li-6 and Li-7, were formed.

The theory of BBN gives a detailed description of the production of the deuterium, He-3, He-4, Li-7, and the mixture of these elements. In order to test these predictions, we have to reconstruct the primordial abundances, for example, observing objects very far away, and then find the conditions of the universe at a very early time. In the standard picture of BBN, all the light element abundances depend on the ratio of baryon to photon.

The precision observations of the cosmic microwave background radiation with the Wilkinson Microwave Anisotropy Probe give the independent value for the baryon to photon ratio [11]. From this observational value, we can see that for deuterium, helium-3 and helium-4, the agreement is very good; and for lithium-7, both observation and prediction give the same order of magnitude. So, the BBN extrapolates the contents and conditions of the present universe (14 billion years old) back to the times of about one second, and the results agree extremely well with observations carried out so far.

1.2.3 Cosmic Microwave Background(CMB)

Cosmic microwave background (CMB) radiation is a form of electromagnetic radiation filing the whole universe uniformly [5]. With a radio telescope, it was found that there is a faint background that is glowing almost exactly the same in all directions, and is not associated with any star, galaxy or other kind of object [6]. It glows very strongly in the microwave region of the spectrum. This phenomenon is discovered by Penzias and Wilson in 1964, and named as cosmic microwave background radiation [5].

1.2.3.1 *Properties of the microwave background radiation* This radiation bathes the Earth in all direction and is considered as the black-body radiation with the temperature, $T_0 = 2.715 \pm 0.001K$. The total energy of the radiation is determined by

$$\epsilon_{rad} \equiv \rho_{rad} c^2 = \alpha T^4. \tag{1.51}$$

With the value of $\alpha \equiv \pi^2 k_B^4 / 15\hbar^3 c^3 = 7.565 \times 10^{-16} Jm^{-3}K^{-4}$, we can easily get the present energy density of the radiation,

$$\epsilon_{rad}(t_0) = 4.17 \times 10^{-14} Jm^{-3}. \tag{1.52}$$

since the radiation obeys $p = \rho/3$, we find that the conservation law takes the form,

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0, \tag{1.53}$$

which has the solution,

$$\rho \propto \frac{1}{a^4}. \tag{1.54}$$

Then Eq.(1.51) implies that

$$T \propto \frac{1}{a}. \tag{1.55}$$

That is, the universe cools as it expands. From this relationship we can find its black-body distribution,

$$\epsilon(f)df = \frac{8\pi h}{c^3} \frac{f^3 df}{exp(hf/k_B T) - 1}, \tag{1.56}$$

while the specific intensity of a gas of photons with a blackbody spectrum is

$$I_\nu = \frac{4\pi\hbar\nu^3/c^2}{exp(2\pi\hbar\nu/k_B T) - 1}. \tag{1.57}$$

Therefore, as the universe expands, the frequency f reduces as $1/a$, but the black-body form is preserved at a lower temperature $T_{final} = T_{initial} \times a_{initial}/a_{final}$. This works for two reasons [7]: (1) The denominator is only a function of f/T and not f and T separately, so the reduction of f can be absorbed by an equivalent reduction in T; and (2) the factor f^3 in the numerator scales as inverse volume, corresponding to the evolution of the photon number density as the universe expands and cools, and the photon distribution continues to correspond to a thermal distribution with lower temperature.

1.2.3.2 *Observations of microwave background radiation* There are many experiments trying to observe the cosmic microwave background after it was observed the first time by Penzias and Wilson in 1964. These include ground, balloon and space-based receivers. Because the most challenging problems of the experiments are the receivers, telescope optics and the atmosphere, many improved microwave amplifier technologies have been designed to do the background detection [8]. In particular, COBE was the first which detected the temperature of the CMB and showed it had a black body spectrum. DASI was the first experiment to obtain the polarization signal from the CMB, while CBI made high-resolution detection and first got the E-mode polarization spectrum.

1.3 The Late Cosmic Acceleration of the Universe

In 1998, observations of Type Ia supernova [9, 10] suggested that the expansion of the universe is speeding up. This suggests

$$\ddot{a} > 0, \tag{1.58}$$

where $\ddot{a}$ is the acceleration rate of the universe. Now from Eq.(1.39),

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{1.59}$$

and the relationship between p and ρ,

$$p = w\rho, \tag{1.60}$$

we can study the nature of the accelerating universe in some detail. Since when the universe expands, the density of dark matter declines more quickly than the density of dark energy, the dark energy dominates in the later times. For example, if the volume of the universe doubles, then the density of dark matter will be halved but that of the dark energy will not change very much. Actually, we can even consider it as a constant. A particular case is the cosmological constant. As Eq.(1.58) shows, if the dark energy dominates, we have

$$\ddot{a} = \left[-\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}\right] a > 0. \tag{1.61}$$

Since $a > 0$, we find

$$-\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} > 0, \tag{1.62}$$

which means,

$$\rho + 3p < \frac{\Lambda}{4\pi G}. \tag{1.63}$$

Now we can discuss it in three different cases, $\Lambda < 0$, $\Lambda = 0$ and $\Lambda > 0$,

$$\rho + 3p \begin{cases} < \frac{\Lambda}{4\pi G}, & \Lambda < 0 \\ < 0, & \Lambda = 0 \\ < \frac{\Lambda}{4\pi G}, & \Lambda > 0. \end{cases} \tag{1.64}$$

Defining the new energy density and pressure as

$$\begin{aligned}\rho' &\equiv \rho + \frac{\Lambda}{8\pi G} \\ p' &\equiv p - \frac{\Lambda}{8\pi G} \\ p' &= w'\rho', \end{aligned} \tag{1.65}$$

we find that the acceleration equation can be re-written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho' + 3p'), \tag{1.66}$$

or

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1 + 3w')\rho'. \tag{1.67}$$

Because $\ddot{a} > 0$, we find that

$$w' < -\frac{1}{3}, \tag{1.68}$$

or

$$\frac{p - \frac{\Lambda}{8\pi G}}{\rho + \frac{\Lambda}{8\pi G}} < -\frac{1}{3}, \tag{1.69}$$

which is the basic condition for the universe to be accelerating.

CHAPTER TWO

Classification of Cosmological Models

In this chapter, we shall systematically study the evolution of the Friedmann-Robertson-Walker (FRW) universe with a cosmological constant Λ and a perfect fluid that has the equation of state $p = w\rho$, where p and ρ denote, respectively, the pressure and energy density of the fluid, and w is an arbitrary real constant. Depending on the specific values of w, Λ and the curvature k of the 3-dimensional spatial space of the universe, we classify all the solutions into various cases. In each case the main properties of the evolution are studied in detail, including the periods of deceleration and/or acceleration, and the existence of big bang, big crunch, and big rip singularities. In some particular cases, the solutions reduce to those considered in some standard textbooks, where by some typos may be corrected.

2.1 Introduction to the FRW universe

Recent observations of supernova (SN) Ia reveal the striking discovery that our universe has lately been in its accelerated expansion phase [12, 13]. Cross checks from the cosmic microwave background radiation and large scale structure all confirm this [14, 15, 16, 17, 18]. Such an expansion was predicted neither by the standard model of particle physics nor by the standard model of cosmology, and the underlying physics still remains a complete mystery [19, 20, 21, 41]. Since the precise nature and origin of the acceleration have profound implications, understanding them is one of the major challenges of modern cosmology. As the Dark Energy Task Force (DETF) stated [23]: "*Most experts believe that nothing short of a revolution in our understanding of fundamental physics will be required to achieve a full understanding of the cosmic acceleration.*"

Within the framework of General Relativity (GR), to account for such an acceleration, it requires the introduction of either a tiny positive cosmological constant or an exotic component of matter that has a very large negative pressure and interacts with other components of matter weakly. This invisible component is usually dubbed as *dark energy*. For a perfect fluid with the equation of state, $w = p/\rho$, this implies $w < -1/3$, where p and ρ denote, respectively, the pressure and energy density of the fluid. On the other hand, a tiny positive cosmological constant is well consistent with all observations carried out so far [24, 25]. However, when we consider its physical origin, we run into other severe problems: (a) Its theoretical expectation values exceed observational limits by 120 orders of magnitude [27, 28, 29, 30]. (b) Its corresponding energy density is comparable with that of matter only recently. Otherwise, galaxies would have not been formed. Considering the fact that the energy density of matter depends on time, one has to explain why only *now* the two are in the same order. (c) Once the cosmological constant dominates the evolution of the universe, it dominates forever. An eternally accelerating universe seems inconsistent with string/M-Theory, because it is endowed with a cosmological event horizon that prevents the construction of a conventional S-matrix describing particle interaction [31, 32, 33, 34]. Other problems with an asymptotical de Sitter universe in the future were explored in [35].

In view of all the above, dramatically different models have been proposed, including quintessence [36, 37, 38], DGP branes [39, 40, 41], and the $f(R)$ models [42, 43]. For details, see [19] and references therein. However, it is fair to say that so far no convincing model has been constructed.

To introduce such a fascinating subject to students, it is always challenging both physically and mathematically. In particular, for a given model, how does one determine the evolution history of the universe without really solving the differential equations? At most times one cannot solve these equations exactly, unless some numerical methods are used. In this chapter, by providing some concrete examples, we shall show that this can be indeed done. Our method is very basic, and even undergraduate students with some knowledge of classical mechanics can easily follow it. As a matter of fact, the only knowledge required is the conservation law of total energy of a classical particle with mass m moving under a potential $V(x)$ [45],

$$\frac{1}{2}m\dot{x}^2 + V(x) = E, \tag{2.1}$$

where $\dot{x} \equiv dx/dt$ and E is the total energy of the system, which is conserved without external force. Then, taking the derivative of the above equation with respect to t, we find that

$$\ddot{x} = -\frac{dV(x)}{dx}. \tag{2.2}$$

Thus, once the potential $V(x)$ is known in terms of x, one can immediately tell if the particle is accelerating or decelerating, without integrating Eq.(2.1) explicitly. In addition, once the potential $V(x)$ is known, it is easy to determine the range of x that the motion allows. Therefore, if the problem of the evolution of the universe can be expressed in the above form, we can use the above methods of classical mechanics to study its evolution, and classify all the possible models of the universe. Another purpose of this chapter is to correct some typos appearing in some textbooks.

The rest of this chapter is organized as follows: In Sec. 2.2, we consider the Friedmann equation coupled with a cosmological constant and a perfect fluid with the equation of state $p = w\rho$ for any given curvature k. After writing it in the form of Eq.(2.1), we study the potential $V(x)$ case by case, and deduce the main properties of each model of the universe. In Sec. 2.3, we present our main conclusions.

It should be noted that classification of a (non-relativistic) matter coupled with a dark energy was considered recently in [46], and the corresponding Penrose diagrams were also presented. In this thesis, we shall use the notations and conventions defined in [47].

2.2 Classification of the FRW universe

The FRW universe is described by the metric [47],

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right), \tag{2.3}$$

in the spherically symmetric coordinates $x^a = \{t, r, \theta, \phi\}$, $(a = 0, 1, 2, 3)$, where k denotes the curvature of the three-dimensional spatial space of constant t, and can be set $k = 0, \pm 1$, without loss of generality. $a(t)$ is the expansion factor of the universe. It should be noted that Eq.(2.3) is invariant under the translation,

$$t' = t - t_s, \tag{2.4}$$

where t_s is a constant. In the following we shall use this gauge freedom to fix the origin of the timelike coordinate t. The expansion factor $a(t)$ of the universe is determined through the Einstein field equations

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab} + \Lambda g_{ab}, \tag{2.5}$$

where $\kappa[\equiv 8\pi G/c^4]$ is the Einstein coupling constant, Λ denotes the cosmological constant, and T_{ab} the energy-momentum tensor of the matter field(s) filled in the universe. For a perfect fluid, we have

$$T_{ab} = (\rho + p)\, u_a u_b - p g_{ab}, \tag{2.6}$$

where $u_a = \delta^t_a$ denotes the four-velocity of the fluid. It can be shown [47] that the Einstein field equations (2.5) for the metric (2.3) and energy-momentum tensor (2.6), have only two independent components, which can be cast in the form,

$$H^2 = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda - \frac{k}{a^2}, \tag{2.7}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{1}{3}\Lambda, \tag{2.8}$$

where $H \equiv \dot{a}/a$. Note that in writing the above equation, we have chosen units such that the speed of light is one. On the other hand, the conservation law of matter fields, $\nabla^a T_{ab} = 0$, yields

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.9}$$

It can be shown that this equation is not independent, and can be obtained from Eqs.(2.7) and (2.8).

Note that we have three unknowns, a, ρ and p, but only two independent equations. Thus, to close the system, we need to have one more equation. Usually, this is given by the equation of state of the matter field. In this chapter, we shall consider the case where

$$p = w\rho, \tag{2.10}$$

where w is an arbitrary real constant. Inserting Eq.(2.10) into Eq.(2.9), we find that it allows to integrate once and gives,

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}, \tag{2.11}$$

where ρ_0 and a_0 are the integration constants. Since ρ_0 represents the energy density when $a = a_0$, we shall assume that it is strictly positive $\rho_0 > 0$. Without loss of generality, we can always set $a_0 = 1$. Then, it can be shown that the Friedmann equation (2.7) can be cast in the form of Eq.(2.1) with $m = 1,\ E = 0,\ x(t) = a(t)$, i.e.,

$$\frac{1}{2}\dot{a}^2 + V(a) = 0, \tag{2.12}$$

where

$$V(a) = \frac{1}{2}k - \frac{1}{6}\Lambda a^2 - \frac{\mathcal{C}}{a^{1+3w}}, \tag{2.13}$$

with

$$\mathcal{C} \equiv \frac{4\pi G\rho_0}{3} > 0. \tag{2.14}$$

When $w = 0$ the problem reduces to the one treated in [47]. To study the problem further, we consider the cases $k = 0, \pm 1$ separately.

2.2.1 A. The $k = 0$ case

When $k = 0$, Eq.(2.13) reduces to

$$V(a) = -\frac{1}{6}\Lambda a^2 - \frac{\mathcal{C}}{a^{1+3w}}. \tag{2.15}$$

It is found convenient to consider the cases where $\Lambda > 0$, $\Lambda = 0$ and $\Lambda < 0$ separately.

2.2.1.1 *When $\Lambda > 0$* Eq.(2.15) can be written as

$$V(a) = -\frac{1}{6}\Lambda a^2 \left(1 + \frac{\tilde{\mathcal{C}}}{a^{1+3w}}\right), \tag{2.16}$$

where $\tilde{\mathcal{C}} \equiv 6\mathcal{C}/|\Lambda| > 0$. It is found that, depending on the value of w, the evolution of the university can be significantly different. Thus, we shall further distinguish the following sub-cases:

$$\begin{aligned} &(i)\ w > -\frac{1}{3}, \quad (ii)\ w = -\frac{1}{3}, \quad (iii)\ -1 < w < -\frac{1}{3}, \\ &(iv)\ w = -1, \quad (v)\ w < -1. \end{aligned} \tag{2.17}$$

Case A.1.1) $w > -\frac{1}{3}$: We find that $V(a)$ is strictly negative, and $V(a) \to -\infty$ for both $a = 0$ and $a \to \infty$. It also has a maximum at $a = a_m \equiv \left(3\left(1 + 3w\right)\mathcal{C}/\Lambda\right)^{1/(3(1+w))}$, for which

$$\ddot{a} = \begin{cases} < 0, & a < a_m, \\ = 0, & a = a_m, \\ > 0, & a > a_m. \end{cases} \tag{2.18}$$

Fig. 2.1 schematically shows the potential. Therefore, in this case, the evolution of the universe is dominated by matter in the early time and $a(t)$ scales like $a(t) \propto t^{2/[3(1+w)]}$, for which $\ddot{a} < 0$. As the universe expands to $a = a_m$, it reaches the turning point, after which it starts to expand acceleratingly, i.e., $\ddot{a}(a > a_m) > 0$. The universe is asymptotically de Sitter, $a(t) \propto e^{\sqrt{\Lambda/3}t}$.

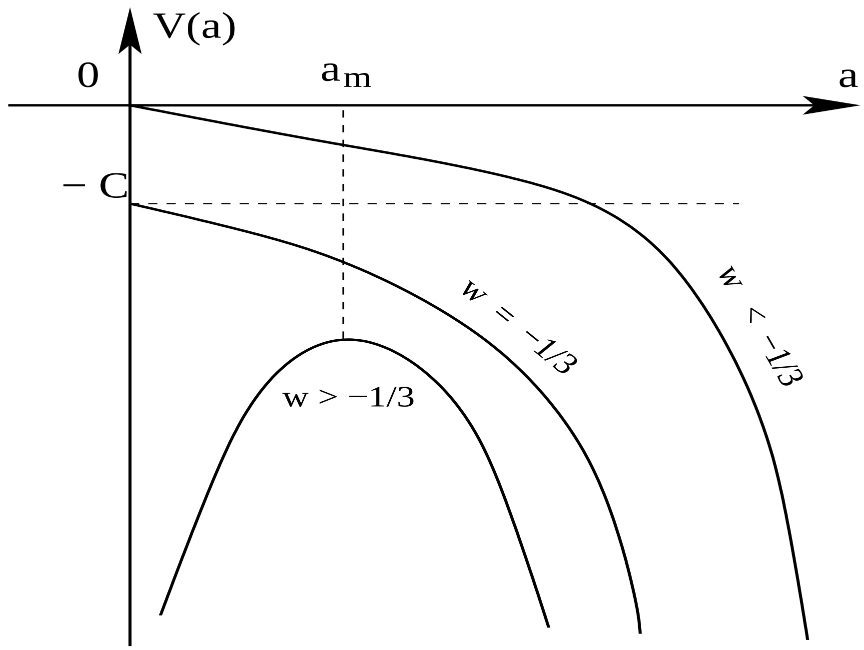

Figure 2.1: The potential given by Eq.(2.16) for $k = 0$ and $\Lambda > 0$.

Case A.1.2) $w = -\frac{1}{3}$: In this case, we find

$$\begin{aligned} V(a) &= -\frac{1}{6}\Lambda a^2 - \mathcal{C} \\ &= \begin{cases} -\mathcal{C}, & a = 0, \\ -\infty, & a \to \infty, \end{cases} \\ \ddot{a} &= -\frac{dV(a)}{da} = \frac{1}{3}\Lambda a \geq 0. \end{aligned} \tag{2.19}$$

The middle line of Fig. 2.1 is the potential for this case. Thus, in this case the universe is always accelerating except for the initial moment $a = 0$. On the other hand, from Eq.(2.11) we find that $\rho \propto a^{-2} \to \infty$ as $a \to 0$, that is, a big bang singularity still occurs at $a = 0$.

Case A.1.3) $-1 < w < -\frac{1}{3}$: In this case, we find that the potential is that given by Fig. 2.1, and

$$\ddot{a} = -\frac{dV(a)}{da} = \frac{1}{3}\Lambda a + (3|w| - 1)\,\mathcal{C}a^{3|w|-2} \geq 0, \tag{2.20}$$

where the equality holds only at $a = 0$ for $w = -2/3$. Thus, in this case the universe is always accelerating. Note that in the present case $a = 0$ still represents a big bang singularity, as can be seen from Eq.(2.11). It is also interesting to note that at $a = 0$ we have $\dot{a} = \ddot{a} = 0$, except for $w = -2/3$. Thus, when $w \neq -2/3$ the point $a = 0$ is a stationary point. However, it is not stable, and any perturbations will make the universe to expand. When $w = -2/3$ we have $\dot{a}(a = 0) = 0$ and $\ddot{a}(a = 0) = (3|w| - 1)\,\mathcal{C} > 0$. Therefore, in the latter case $a = 0$ is not stationary, and the positive force will lead the universe automatically expand.

Case A.1.4) $w = -1$: In this case, the matter acts as a vacuum energy, and the potential is given by

$$V(a) = -\frac{1}{6}\Lambda_{eff} a^2, \tag{2.21}$$

where $\Lambda_{eff} \equiv \Lambda + 6\mathcal{C}$. Therefore, in this case the universe is de Sitter, and

$$a(t) = e^{\sqrt{\Lambda_{eff}/3}(t-t_0)}. \tag{2.22}$$

Recall that the de Sitter space is free of any kind spacetime singularities at $a = 0$ as well as at $a = \infty$ [47].

Case A.1.5) $w < -1$: In this case, the behavior of the potential $V(a)$ and $a(t)$ are similar to the case $-1 < w < -1/3$, except for the fact that now $\rho \propto a^{3(|w|-1)}$ is not singular at $a = 0$, although it is at $a = \infty$, which is usually called a big rip singularity. At $a = 0$ we have $\dot{a} = \ddot{a} = 0$, that is, in this case this point also represents a unstable stationary point.

Finally, we note that for all the cases with $k = 0$ the corresponding Friedmann equation can be integrated explicitly, and the correspond solutions are given by

$$a(t) = \left\{\left(\frac{6\mathcal{C}}{\Lambda}\right)^{1/2} \sinh\left[(1+w)\sqrt{\frac{3\Lambda}{4}}\,(t - t_s)\right]\right\}^{\frac{2}{3(1+w)}}, \tag{2.23}$$

for $w \neq -1$, where t_s is given by

$$t_s = t_0 - \frac{1}{1+w}\sqrt{\frac{4}{3\Lambda}}\sinh^{-1}\left(\sqrt{\frac{\Lambda}{6\mathcal{C}}}\right), \tag{2.24}$$

so that $a(t = t_0) = 1$. For $w > -1$, without loss of generality, we can always use the gauge freedom of Eq.(2.4) to set $t_s = 0$, so that the big bang singularity occurs at $t = 0$. This will be the case in the rest of this chapter. When $w = -1$, the solution is that of de Sitter, given by Eq.(2.22), which is free of any kind of spacetime singularities, and the solution is valid for any t, that is, $t \in (-\infty, \infty)$. For the cases where $w < -1$, we shall not do such a translation, so that the big rip singularity happens exactly at $t = t_s$. Note that when $w < -1$, from Eq.(2.23) we find that the solution is valid only when $t \in (-\infty, t_s)$, for which we have

$$\begin{aligned} a(t) &= \left\{\sqrt{\frac{6\mathcal{C}}{\Lambda}}\sinh\left[\sqrt{\frac{3\Lambda}{4}}\left(|w|-1\right)\left(t_s - t\right)\right]\right\}^{-\frac{2}{3(|w|-1)}} \\ &= \begin{cases} \infty, & t = t_s, \\ 1, & t = t_0, \qquad (w < -1) \\ 0, & t = -\infty, \end{cases} \end{aligned} \tag{2.25}$$

Since now we have $\rho \propto a^{3(|w|-1)}$ we find that the spacetime is indeed not singular at $a(t = -\infty) = 0$, but singular at $a(t = t_s) = \infty$.

In Fig. 2.2 we summarize the main properties of the solutions for $k = 0$ and $\Lambda > 0$ with different w.

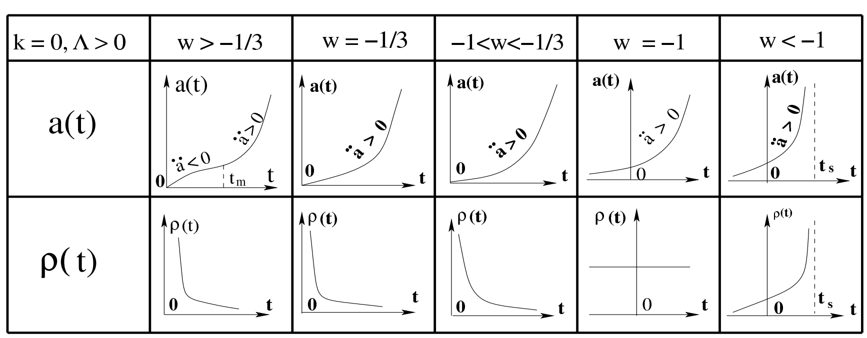

Figure 2.2: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = 0$ and $\Lambda > 0$. The spacetime has a big bang singularity at $t = 0$ for $w > -1$. It is de Sitter for $w = -1$, which is free of any kind of spacetime singularities. When $w < -1$, a big rip singularity occurs at $t = t_s$, at which we have $a\left(t_s\right) = \infty = \rho\left(t_s\right)$.

2.2.1.2 *When* $\Lambda = 0$ In this case, we have

$$\begin{aligned} V(a) &= -\frac{\mathcal{C}}{a^{1+3w}} \le 0, \\ \ddot{a} &= -(3w+1)\frac{\mathcal{C}}{a^{2+3w}} \\ &= \begin{cases} < 0, & w > -1/3, \\ = 0, & w = -1/3, \\ > 0, & w < -1/3. \end{cases} \end{aligned} \tag{2.26}$$

Fig. 2.3 shows the potential $V(a)$. Similar to the last case, now we can also integrate the Friedmann equation (2.7) to obtain the explicit solutions of $a(t)$ and $\rho(t)$,

$$\begin{aligned} a(t) &= \begin{cases} \left[3(1+w)\sqrt{\frac{\mathcal{C}}{2}}(t-t_s)\right]^{\frac{2}{3(1+w)}}, & w \neq -1, \\ e^{\sqrt{2\mathcal{C}}(t-t_0)}, & w = -1, \end{cases} \\ \rho(t) &= \begin{cases} \frac{\tilde{\rho}_0}{(t-t_s)^2}, & w \neq -1, \\ \rho_0, & w = -1, \end{cases}, \end{aligned} \tag{2.27}$$

where $\tilde{\rho}_0 \equiv 2\rho_0/[9(1+w)^2\mathcal{C}]$, and

$$t_s = t_0 - \frac{1}{1+w}\sqrt{\frac{2}{9\mathcal{C}}}. \tag{2.28}$$

When $w < -1$, Eq.(2.27) shows that in order to keep $a(t)$ real and positive we must require $t \in (-\infty, t_s)$. As $t \to -\infty$, both $a(t)$ and $\rho(t)$ vanish, while when $t \to t_s$ all of them become unbounded, that is, a big rip singularity is developed there.

In Fig. 2.4 we summarize the main properties of the solutions for $k = 0$ and $\Lambda = 0$ with different values of w.

2.2.1.3 *When* $\Lambda < 0$ In this case, we have

$$V(a) = \frac{1}{6}\left|\Lambda\right| a^2 - \frac{\mathcal{C}}{a^{1+3w}}, \tag{2.29}$$

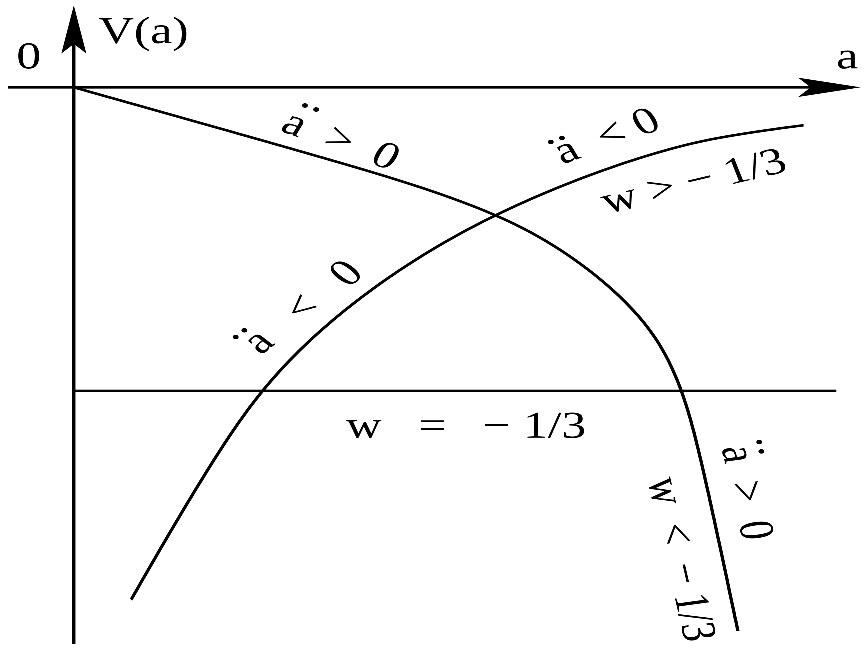

Figure 2.3: The potential given by Eq.(2.16) for $k = 0$ and $\Lambda = 0$.

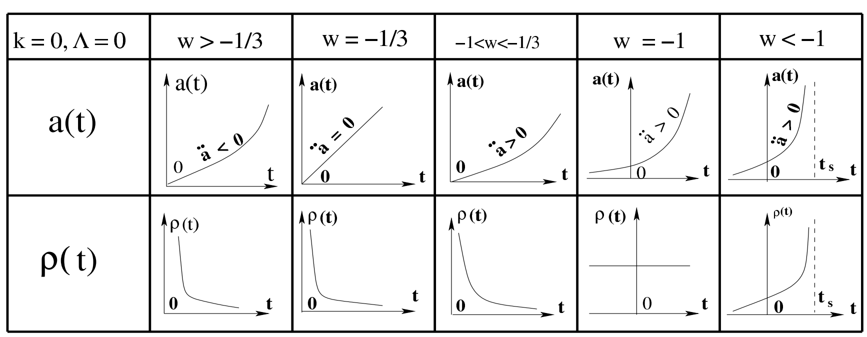

Figure 2.4: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = 0$ and $\Lambda = 0$. There is a big bang singularity at $a = 0$ for all the cases with $w > -1$. The spacetime is de Sitter for $w = -1$. When $w < -1$, a big rip singularity is developed at $t = t_s$, at which we have $a\,(t_s) = \rho\,(t_s) = \infty$.

which has been shown in Fig. 2.5 for various values of w. It can also be shown that the corresponding expansion factor and the energy density are given, respectively, by

$$
\begin{aligned}
a(t) &= \left\{\sqrt{\frac{6\mathcal{C}}{|\Lambda|}}\sin\left[\sqrt{\frac{3|\Lambda|}{4}}(1+w)\,(t-t_s)\right]\right\}^{\frac{2}{3(1+w)}}, \\
\rho(t) &= \frac{\tilde{\rho}_0}{\sin^2\left(\sqrt{\frac{3|\Lambda|}{4}}(1+w)\,(t-t_s)\right)},
\end{aligned}
\tag{2.30}
$$

for $w \neq -1$, and

$$
\begin{aligned}
a(t) &= e^{\sqrt{\Lambda_{eff}/3}(t-t_0)}, \\
\rho(t) &= \rho_0,
\end{aligned}
\tag{2.31}
$$

for $w = -1$, where $\Lambda_{eff} = |\Lambda| - 6\mathcal{C} > 0$, but now with

$$
\begin{aligned}
t_s &\equiv t_0 - \frac{\sqrt{4/(3|\Lambda|)}}{1+w}\sin^{-1}\left(\sqrt{\frac{|\Lambda|}{6\mathcal{C}}}\right), \\
\tilde{\rho}_0 &\equiv \frac{|\Lambda|\rho_0}{6\mathcal{C}}.
\end{aligned}
\tag{2.32}
$$

As we did in the previous cases, using the gauge freedom (2.4) we shall set $t_s = 0$ for $w < -1$, while keeping it as it is for $w \leq -1$.

Case A.3.1) $w > -\frac{1}{3}$: In this case, as shown in Fig. 2.5, we have $V\,(a > a_m) > 0$, so the motion of $a > a_m$ is forbidden. If the universe starts to expand from the big bang where $a = 0$, when it expands to its maximal radius a_m it will start to collapse until a big crunch singularity is developed at $t = 2t_m$, as shown in Figs. 2.5 and 2.6, where $a_m = a\,(t_m)$, and is given by

$$
a_m = \left(\frac{6\mathcal{C}}{|\Lambda|}\right)^{\frac{1}{3(1+w)}}. \tag{2.33}
$$

During the whole process, the universe is always decelerating,

$$
\ddot{a} = -\frac{dV(a)}{da} < 0, \tag{2.34}
$$

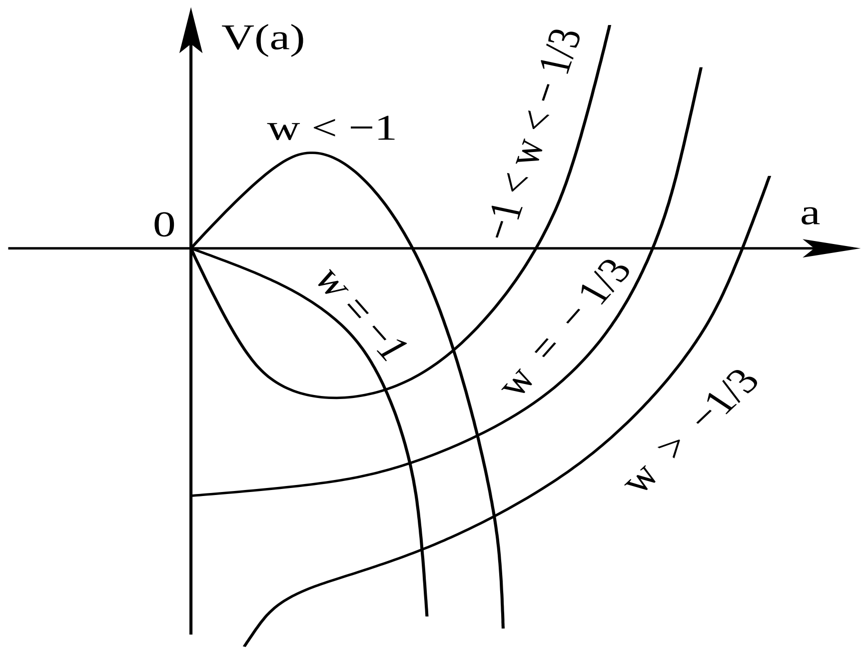

Figure 2.5: The potential given by Eq.(2.29) for $k = 0$ and $\Lambda < 0$.

as can be seen from Fig. 2.5.

Case A.3.2) $w = -\frac{1}{3}$: In this case, we find

$$
\begin{aligned}
V(a) &= \frac{1}{6}\left|\Lambda\right| a^2 - \mathcal{C} = \begin{cases} -\mathcal{C}, & a = 0, \\ \\ \infty, & a \to \infty, \end{cases} \\
\ddot{a} &= -\frac{dV(a)}{da} = -\frac{1}{3}|\Lambda| a \leq 0. \qquad (2.35)
\end{aligned}
$$

Therefore, similar to the last case, the universe expands from the big bang singularity at $a = 0$ until its maximal radius a_m, given by Eq.(2.33) with $w = -1/3$, and then starts to collapse until a big crunch singularity is formed at $t = 2t_m$.

Case A.3.3) $-1 < w < -\frac{1}{3}$: In this case, from Fig. 2.5 we can see that $V(a = 0) = 0 = V\left(a_m\right)$, and the motion is also restricted to $a \leq a_m$. However, there is a fundamental difference between this case and the last two cases: The potential $V(a)$ has a minimum at $a = a_{min}$, at which we have $dV\left(a_{min}\right)/da = 0$. The universe is initially accelerating. But, when it expands to $a = a_{min}$, it starts to decelerate until $a = a_m$, at which its expanding velocity becomes zero, and afterwards it will start to collapse, until a big crunch singularity is developed at $a = 0$, as shown in Fig. 2.6.

Case A.3.4) $w = -1$: In this case, we have

$$V(a) = -\frac{1}{6}|\Lambda|\, a^2 \left(\frac{6\mathcal{C}}{|\Lambda|} - 1\right). \tag{2.36}$$

Therefore, now there is a solution only when $\Lambda_{eff} > 0$, for which the universe is de Sitter, and

$$a(t) = e^{\sqrt{\Lambda_{eff}/3}(t-t_0)}, \tag{2.37}$$

where $\Lambda_{eff} \equiv |\Lambda| - 6\mathcal{C}$.

Case A.3.5) $w < -1$: In this case, there is a minimum a_{min} for which $V(a < a_{min}) \geq 0$. Therefore, in contrast to the previous case, now the motion of the universe is restricted to $a \geq a_{min}$. The universe starts to expand from $a = a_{min}$ until $a(t_s) = \infty$ within a finite time, whereby a big rip singularity is formed, as shown in Fig. 2.6.

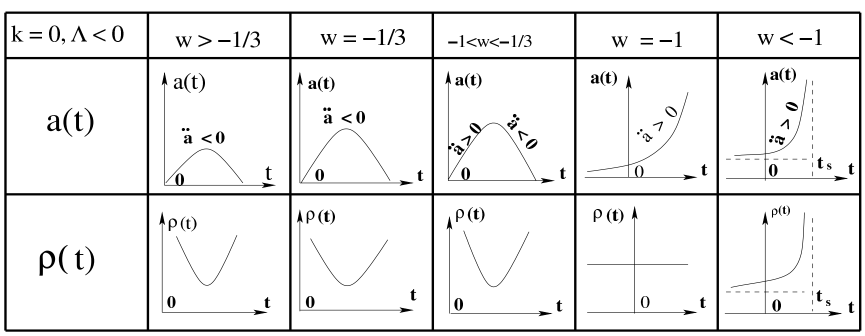

Figure 2.6: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = 0$ and $\Lambda < 0$. The spacetime is singular at $a = 0$ for all the cases with $w > -1$, (a big bang singularity). It is de Sitter for $w = -1$. When $w < -1$, a big rip singularity is developed at $t = t_s$, at which we have $a(t_s) = \rho(t_s) = \infty$.

2.2.2 B. The $k = 1$ case

In this case, the potential given by Eq.(2.13) can be written as

$$V(a) = \frac{1}{2} - \frac{1}{6}\Lambda a^2 \left[1 + \left(\frac{\tilde{\mathcal{C}}}{a}\right)^{3(1+w)}\right], \tag{2.38}$$

where $\tilde{\mathcal{C}} \equiv (12\mathcal{C}/\Lambda)^{3(1+w)}$. Following the case of $k = 0$, we also distinguish the three cases, $\Lambda > 0,\ \Lambda = 0$ and $\Lambda < 0$.

2.2.2.1 *When* $\Lambda > 0$ It is convenient to further divide it into the five sub-cases defined as in Eq.(2.17).

Case B.1.1) $w > -\frac{1}{3}$: In this case, it can be shown that for any given w and ρ_0 there always exists a critical value Λ_c and radius a_m that satisfy the conditions,

$$V\left(a_m, w, \rho_0, \Lambda_c\right) = 0, \quad V'\left(a_m, w, \rho_0, \Lambda_c\right) = 0, \tag{2.39}$$

where a prime denotes the ordinary differentiation with respect to a. It can be shown that the solutions of the above conditions are

$$\begin{aligned} a_m &= \left[3\left(1+w\right)\mathcal{C}\right]^{\frac{1}{1+3w}}, \\ \Lambda_c &= \left(\frac{1+3w}{1+w}\right)\left[3\left(1+w\right)\mathcal{C}\right]^{-\frac{2}{1+3w}}. \end{aligned} \tag{2.40}$$

As will be shown below, the solutions with $\Lambda > \Lambda_c$ have quite different properties from the ones with $\Lambda < \Lambda_c$. Therefore, in the following we shall further distinguish the three different cases, $\Lambda > \Lambda_c,\ \Lambda = \Lambda_c$ and $\Lambda < \Lambda_c$.

Case B.1.1.1) $w > -\frac{1}{3},\ \Lambda > \Lambda_c$: In this case, the potential $V(a)$ is always negative for any given a, as shown in Fig. 2.7. Therefore, the corresponding solutions have no turning point. If the universe initially starts to expand from a big bang singularity at $a = 0$, it will expand forever, as shown by Fig. 2.8. However, the potential has a maximum at $a = a_m$, for which we have

$$\ddot{a} = \begin{cases} < 0, & a < a_m, \\ = 0, & a = a_m, \\ > 0, & a > a_m, \end{cases} \tag{2.41}$$

that is, the universe is initially decelerating. Once it expands to $a = a_m$, it starts to expand acceleratingly.

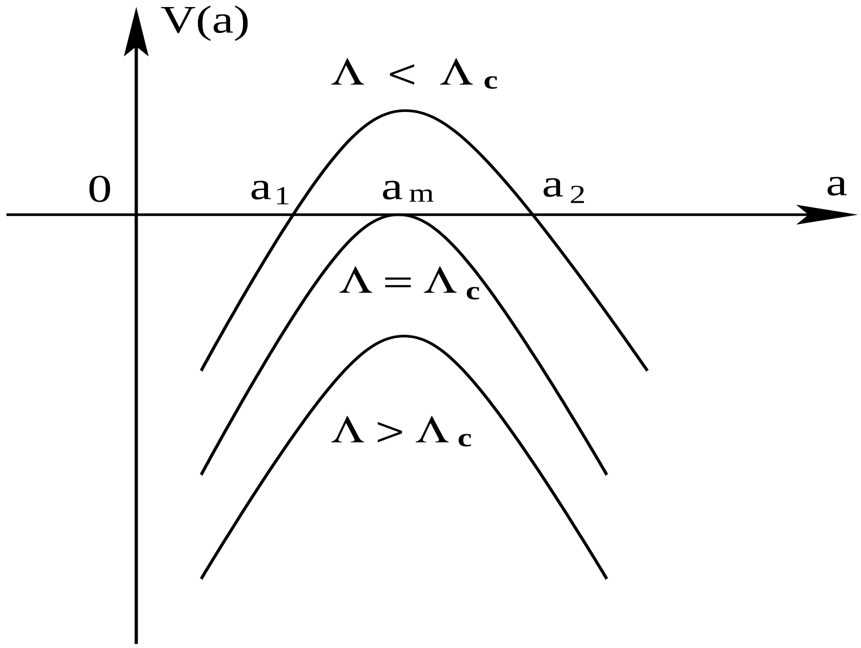

Figure 2.7: The potential given by Eq.(2.38) for $k = 1$, $w > -\frac{1}{3}$ and $\Lambda > 0$, where $\Lambda_c = \Lambda_c\,(w, \rho_0)$.

Case B.1.1.2) $w > -\frac{1}{3}$, $\Lambda = \Lambda_c$: In this case, there exists a static point a_m, at which we have $V\,(a_m) = V'\,(a_m) = 0$, as one can see from Fig. 2.7. Therefore, if the universe starts to expand from the big bang at $a = 0$, it will expand until $a = a_m$. The universe is decelerating during this period. Since at the point $a = a_m$, we have $\dot{a} = 0 = \ddot{a}$, the universe will become static once it reaches this point. However, it is not stable, and with small perturbations, the universe will either collapse until a singularity is developed at $a = 0$ or expand forever with $\ddot{a} > 0$. If the universe initially at $a = a_i > a_m$, from Fig. 2.7 we can see that it will expand forever. Since $V'(a)$ is always negative, so the universe in this region is always accelerating.

Case B.1.1.3) $w > -\frac{1}{3}$, $0 < \Lambda < \Lambda_c$: In this case, $V(a) = 0$ has two real and positive roots, a_1 and a_2, as shown in Fig. 2.7. Without loss of generality, we assume that $a_2 > a_1$. Since $V(a) > 0$ for $a \in (a_1, a_2)$, the motion is forbidden in this region. Similar to the last case, depending on the initial conditions, the universe can have quite different evolutions. In particular, if it starts to expand from the big bang singularity at $a = 0$, it will expand until $a = a_1$, at which we have $\dot{a} = 0$ and $\ddot{a} < 0$. Since $\ddot{a} < 0$ at this point, the universe will start to collapse afterwards, until a big crunch singularity is developed at $a = 0$. If the universe starts to expand at $a_i \geq a_2$, it will expand forever. In the latter case, the universe is always accelerating, as can be seen from Fig. 2.7.

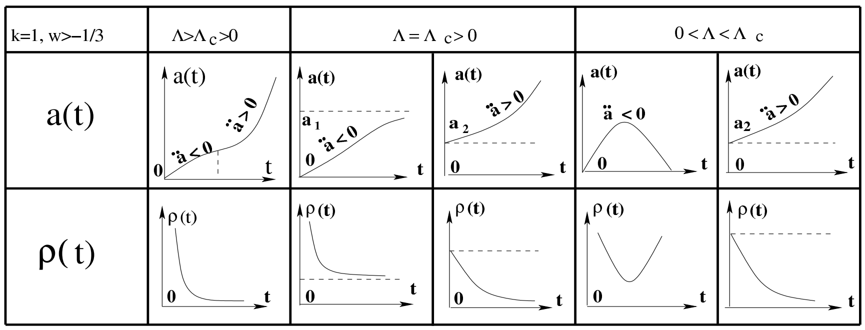

Figure 2.8: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = 1$, $w > -\frac{1}{3}$ and $\Lambda > 0$, where Λ_c is given by Eq.(2.40). A big bang singularity occurs at $a = 0$ in all cases with $\Lambda \geq \Lambda_c$. In the first sub-case of $0 < \Lambda < \Lambda_c$, both big bang and big crunch singularities occur, while in the second sub-case the spacetime is free of any kind of spacetime singularities.

Case B.1.2) $w = -\frac{1}{3}$: In this case, we have

$$V(a) = \frac{1}{2}(1 - \mathcal{C}) - \frac{1}{6}\Lambda a^2. \tag{2.42}$$

Thus, depending on the value of $\mathcal{C}(\rho_0)$, the motion of the universe will be different. In particular, when $\mathcal{C}(\rho_0) < 1/2$, there exists a minimal a_{min}, for which $V(a < a_{min}) > 0$, that is, the motion in the region $0 < a < a_{min}$ is forbidden, as shown in Fig. 2.9. When $\mathcal{C}(\rho_0) \geq 1/2$, the universe can start to expand from the big bang singularity at $a = 0$. In all the cases we have $V'(a) < 0$, so that the universe is always accelerating [cf. Fig. 2.10].

Case B.1.3) $-1 < w < -\frac{1}{3}$: In this case, we find that $V'(a)$ is strictly negative for any $a \geq 0$ with $V(0) = 1/2$. Therefore, similar to the case $w = -1/3$ and $\mathcal{C} < 1/2$, there exists a minimal a_{min}, for which $V(a < a_{min}) > 0$, and the motion in the region $0 < a < a_{min}$ is forbidden, as shown in Fig. 2.9. Thus, in

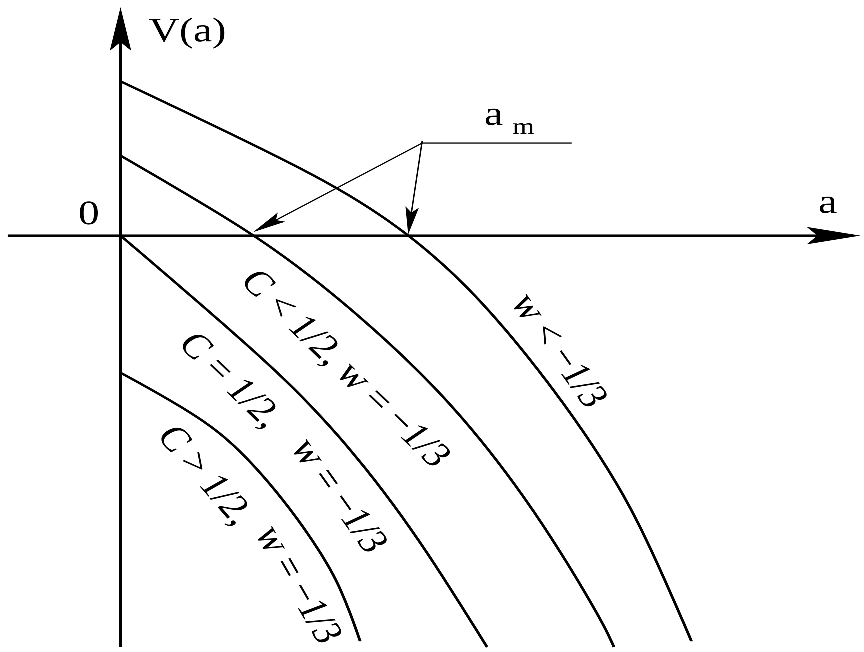

Figure 2.9: The potential given by Eq.(2.42) for $k = 1$, $w \leq -\frac{1}{3}$ and $\Lambda > 0$.

the present case the universe starts to expand from a radius $a_i \geq a_{min}$ until $a = \infty$ without turning point. Again, because now $V'(a) < 0$ for any $a \geq a_{min}$, the universe is always accelerating. However, there is no any kind of singularities to be developed, either big bang, big crunch, or big rip, as shown by Fig. 2.10.

Case B.1.4) $w = -1$: In this case, the potential is a simple parabola,

$$V(a) = \frac{1}{2} - \frac{1}{6}\left(\Lambda + 6\mathcal{C}\right) a^2, \tag{2.43}$$

schematically shown by the top curve in Fig. 2.9. As a result, the motion is similar to the last case, except for the fact that now $\rho = \rho_0$.

Case B.1.5) $w < -1$: In this case, we also have $V'(a) < 0$ and there exists a finite radius, a_{min}, such that when $a < a_{min}$ we have $V(a) > 0$, and when $a \geq a_{min}$ we have $V(a) \leq 0$. The only difference is that in the present case there is a big rip singularity that happens at $a = \infty$, as now we have $\rho \propto a^{3(|w|-1)}$, as shown in Fig. 2.10.

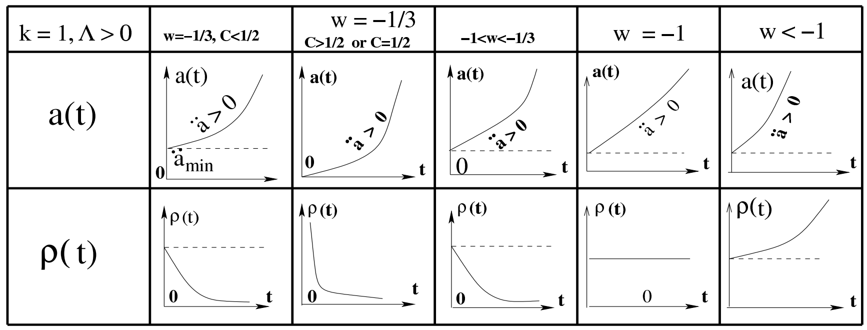

Figure 2.10: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = 1,\ w \leq -\frac{1}{3}$ and $\Lambda > 0$. A big bang singularity occurs only in the case $w = -1/3$ and $\mathcal{C} \geq 1/2$. In the case $w < -1$, a big rip singularity occurs at $a = \infty$.

2.2.2.2 *When* $\Lambda = 0$ In this case, we find that

$$V(a) = \frac{1}{2} - \frac{\mathcal{C}}{a^{1+3w}}. \tag{2.44}$$

Fig. 2.11 shows the potential for various values of w, from which we can see that when $w > -1/3$, the motion of the universe is restricted to $a \leq a_m$, where a_m is the solution of $V(a) = 0$. The universe starts to expand at the big bang singularity $a = 0$ until the turning point $a = a_m$. Afterwards, it will start to collapse until a big crunch singularity is developed at $a = 0$, as shown in Fig. 2.12.

When $w = -1/3$, there is motion only for $\mathcal{C} > 1/2$, for which the universe expands linearly from a big bang singularity at $a = 0$ with $\ddot{a} = 0$.

When $-1 \leq w < -1/3$, the motion is possible only for $a > a_{min}$, as shown in Figs. 2.11 and 2.12. The universe starts to expand from the initial point $a_i \geq a_{min}$ with $\ddot{a} > 0$. No turning point exists, so the universe will expand forever. During the whole process, the matter density remains finite, so no singularity exists in this case.

When $w < -1$, it can be shown that the motion for $a < a_{min}$ is also forbidden. As a result, no big bang singularity exists in the present case. But, a big rip singularity will be developed as $a \to \infty$, as shown by Fig. 2.12. In the whole process, we

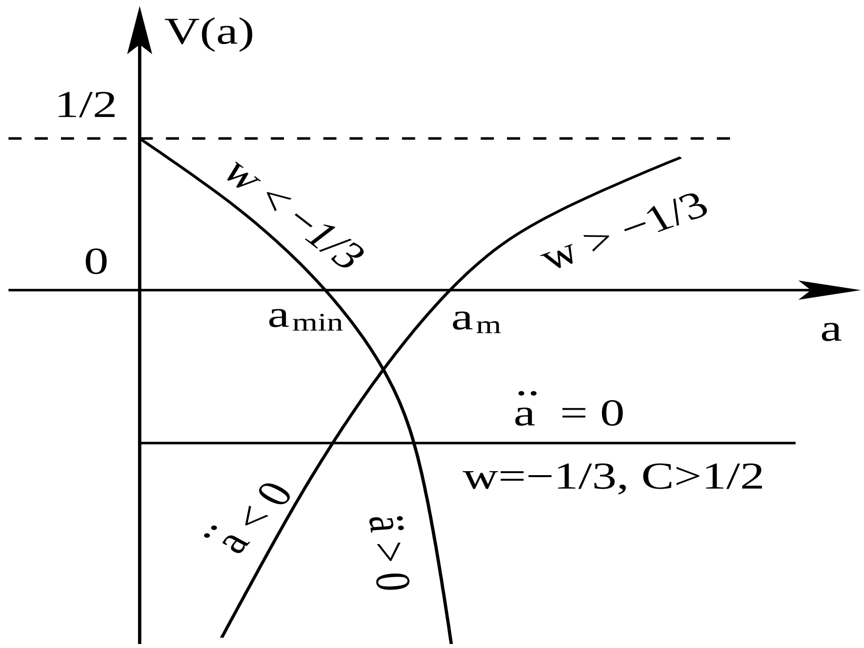

Figure 2.11: The potential given by Eq.(2.44) for $k = 1$ and $\Lambda = 0$.

have $\ddot{a} > 0$.

2.2.2.3 *When* $\Lambda < 0$ In this case, we have

$$V(a) = \frac{1}{2} + \frac{1}{6}|\Lambda| a^2 - \frac{\mathcal{C}}{a^{1+3w}}, \tag{2.45}$$

which has the properties as shown by Fig. 2.13. In particular, when $w > -1/3$, we find that the universe starts to expand from a big bang singularity at $a = 0$ until a maximal radius a_m where $V(a_m) = 0$. Afterwards, the universe starts to collapse, and finally a big crunch is developed at $t = 2t_m$ where t_m is determined by $a_m = a(t_m)$. In the whole process, the universe is decelerating, as shown by Fig. 2.14.

When $w = -1/3$, the potential is non-negative for $\mathcal{C} \leq 1/2$, so the motion is forbidden. When $\mathcal{C} > 1/2$ we have $V(a) < 0$ for $a < a_m$, where a_m is the root of $V(a) = 0$, as shown in Fig. 2.13. Thus, the motion now is possible in the region $a < a_m$, for which the universe starts to expand from a big bang singularity at

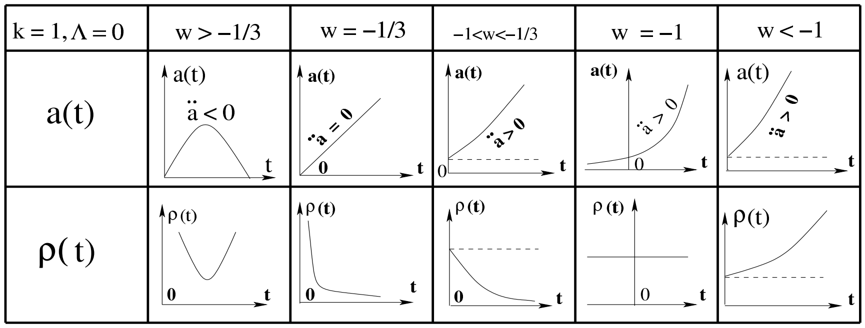

Figure 2.12: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = 1$ and $\Lambda = 0$. There are both big bang and big crunch singularities in the case $w > -1/3$, while only a big bang singularity occurs in the case $w = -1/3$. There is no singularity in the cases with $-1 \leq w < -1/3$. A big rip singularity occurs at $a = \infty$ for $w < -1$.

$a = 0$. Once it reaches its maximum at a_m, it starts to collapse until a big crunch is developed at $t = 2t_m$.

When $-1 < w < -1/3$, similar to the last case, the potential is negative only for $a < a_m$, as shown in Fig. 2.13. In particular, a big bang (crunch) singularity happens at $t = 0$ ($t = 2t_m$). The difference is that now there exists a time t_{max} so that for $0 < t < t_{max}$ or $2t_m - t_{max} < t < 2t_m$ the universe is accelerating, while during the time $t_m - t_{max} < t < 2t_m - t_{max}$ it is decelerating, where t_{max} is the root $V'(t_{max}) = 0$.

When $w = -1$, the potential is non-positive only for $\mathcal{C} > |\Lambda|/6$ and $a \geq a_{min}$, where a_{min} is the root of $V(a) = 0$, as shown in Fig. 2.13. Therefore, in this case the universe starts to expand always at an initial radius $a_i \geq a_{min}$. The universe will expand forever with $\ddot{a} > 0$. However, the spacetime is not singular even when $a = \infty$.

When $w < -1$, the potential is non-positive only for $a \geq a_{min}$, where again a_{min} is the root of $V(a) = 0$, as shown in Fig. 2.13. The evolution of the universe in this case is similar to the last one, except for that now a big rip singularity will be developed at $a = \infty$.

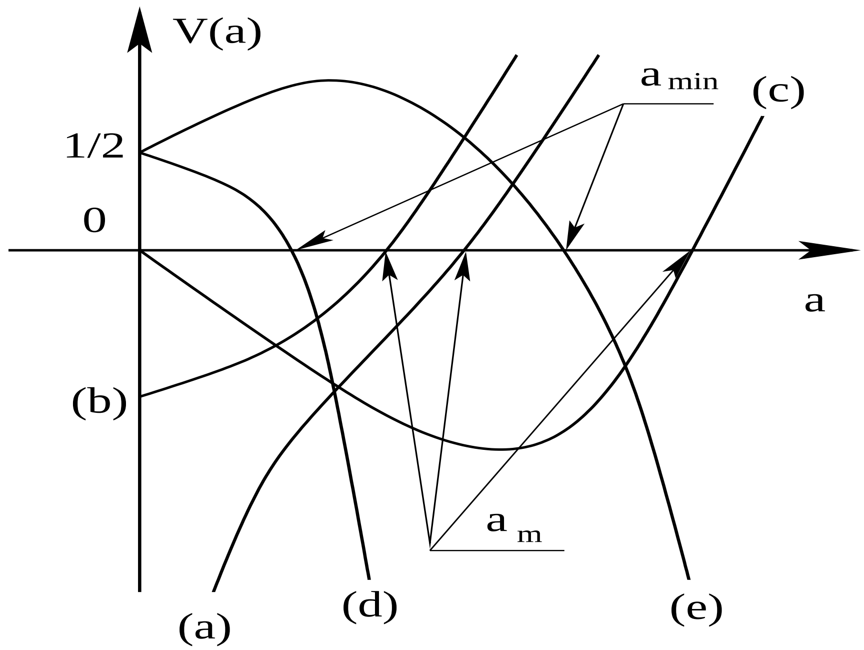

Figure 2.13: The potential given by Eq.(2.45) for $k = 1$ and $\Lambda = 0$: (a) for $w > -1/3$; (b) for $w = -1/3$ and $\mathcal{C} > 1/2$; (c) for $-1 < w < -1/3$; (d) for $w = -1$ and $\mathcal{C} > |\Lambda|/6$; and (e) for $w < -1$.

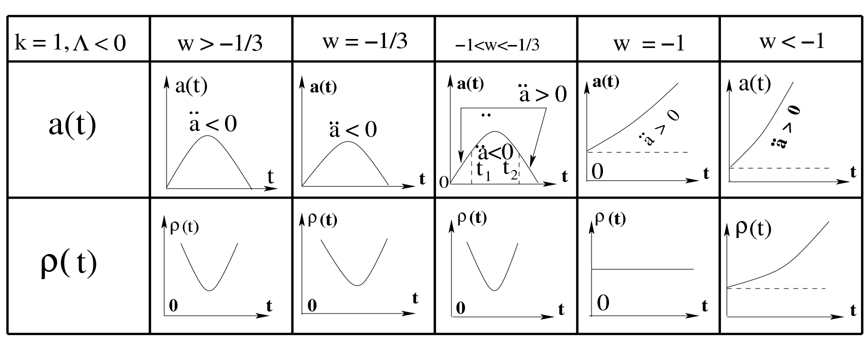

Figure 2.14: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = 1$ and $\Lambda < 0$. There are both big bang and big crunch singularities in the case with $w > -1$, while only a big bang singularity occurs in the case $w = -1$, and no singularities for the case $w = -1$, while there is a big rip singularity at $a = \infty$ for $w < -1$.

2.2.3 C. The $k = -1$ case

When $k = -1$, the potential is given by

$$V(a) = -\frac{1}{2} - \frac{1}{6}\Lambda a^2 - \frac{\mathcal{C}}{a^{1+3w}}. \tag{2.46}$$

To study the motion of the universe in this case, it is also convenient to distinguish the three cases $\Lambda > 0$, $\Lambda = 0$ and $\Lambda < 0$, and in each case there are five sub-cases with different choices of w.

2.2.3.1 *When $\Lambda > 0$* In this case, we find that $V(a) \to -\infty$ as $a \to \infty$, and

$$V(a) = \begin{cases} -1/2, & w < -1/3, \\ -(1/2 + \mathcal{C}), & w = -1/3, \\ -\infty, & w > -1/3, \end{cases} \tag{2.47}$$

when $a \to 0$, as shown by Fig. 2.15. Thus, when $w < -1/3$, the potential has a maximum at a_m where $V(a_m) = 0$. The universe starts to expand from a big bang at $a = 0$. Initially, it is decelerating, $\ddot{a} < 0$. However, when it expands to a_m, it turns to expand at an accelerating rate, $\ddot{a} > 0$, as shown in Fig. 2.16. When $-1 < w \leq -1/3$, the universe expands from a big bang at $a = 0$ until $a = \infty$, and there is no turning point. It expands with $\ddot{a} > 0$ in the whole process. The case of $w = -1$ is similar to the case of $-1 < w \leq -1/3$, except that the spacetime is not singular either at $a = 0$ or at $a = \infty$, as shown in Fig.2.16. When $w < -1$, one can see that the universe starts to expand from $a = 0$ with $\ddot{a} > 0$ for any given a. There is no big bang singular at $a = 0$, but there is a big rip singularity at $a = \infty$.

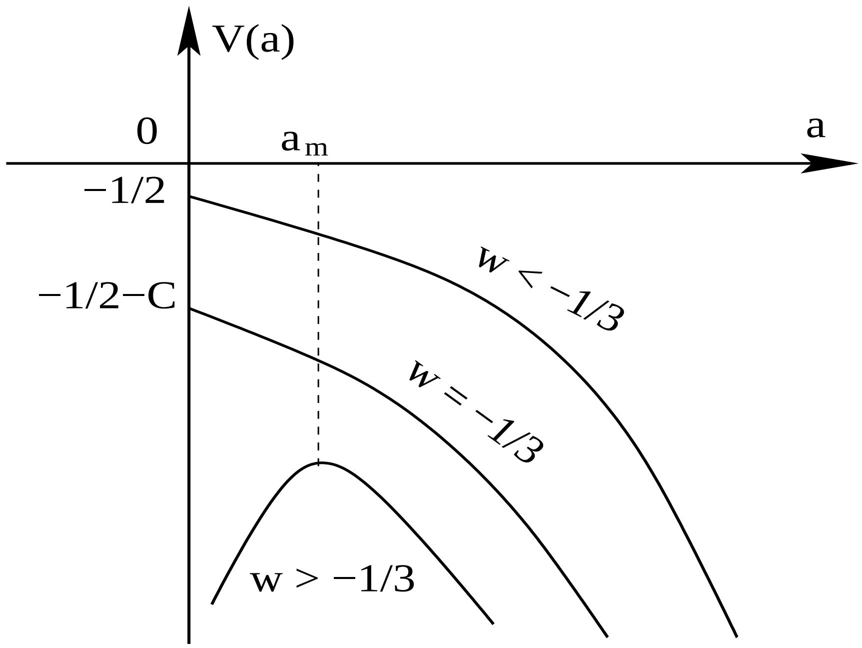

Figure 2.15: The potential given by Eq.(2.46) for $k = -1$ and $\Lambda > 0$.

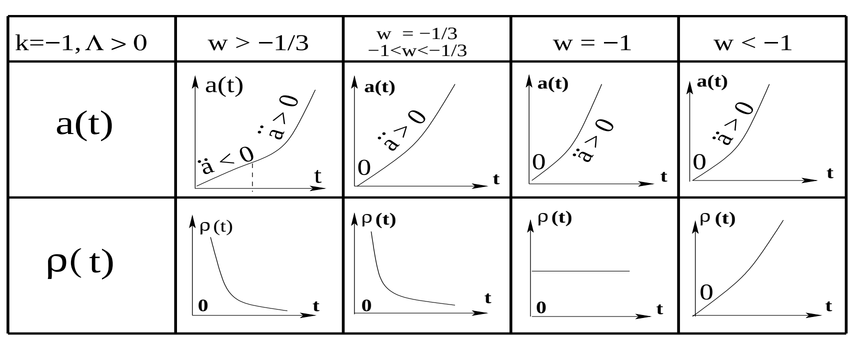

Figure 2.16: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = -1$ and $\Lambda > 0$. There are a big bang singularity for $w > -1$, no singularity for $w = -1$, and a big rip singularity at $a = \infty$ for $w < -1$.

2.2.3.2 *When* $\Lambda = 0$ In this case, we have

$$
\begin{aligned}
V(a) &= -\frac{1}{2} - \frac{\mathcal{C}}{a^{1+3w}} < 0, \\
\ddot{a} &= -\frac{(3w+1)\mathcal{C}}{a^{2+3w}} = \begin{cases} < 0, & w > -1/3, \\ 0, & w = -1/3, \\ > 0, & w < -1/3, \end{cases}
\end{aligned}
\tag{2.48}
$$

when $a \in [0, \infty)$, as shown by Fig. 2.17. We also have

$$
\rho(a) == \begin{cases} \infty, & w > -1/3, \\ \rho_0, & w = -1/3, \\ 0, & w < -1/3, \end{cases}
\tag{2.49}
$$

as $a \to 0$, and

$$
\rho(a) == \begin{cases} 0, & w > -1/3, \\ \rho_0, & w = -1/3, \\ \infty, & w < -1/3, \end{cases}
\tag{2.50}
$$

as $a \to \infty$. Fig. 2.18 shows the motion of the universe for each given w.

2.2.3.3 *When* $\Lambda < 0$ In this case, we have

$$
V(a) = -\frac{1}{2} + \frac{1}{6}|\Lambda|a^2 - \frac{\mathcal{C}}{a^{1+3w}}.
\tag{2.51}
$$

Depending on the vales Λ, $\mathcal{C}$ and w, the potential will have quite different properties. In the following we shall study all of them case by case.

Case C.3.1) $w > -1/3$: In this case, the potential is shown schematically in Fig. 2.19, from which we can see that it is non-positive only for $a \leq a_m$, where a_m is the positive root of $V(a) = 0$. Clearly, in this case there is a big bang singularity at $a = 0$, from which the universe starts to expand until $a = a_m$. Afterwards, it will

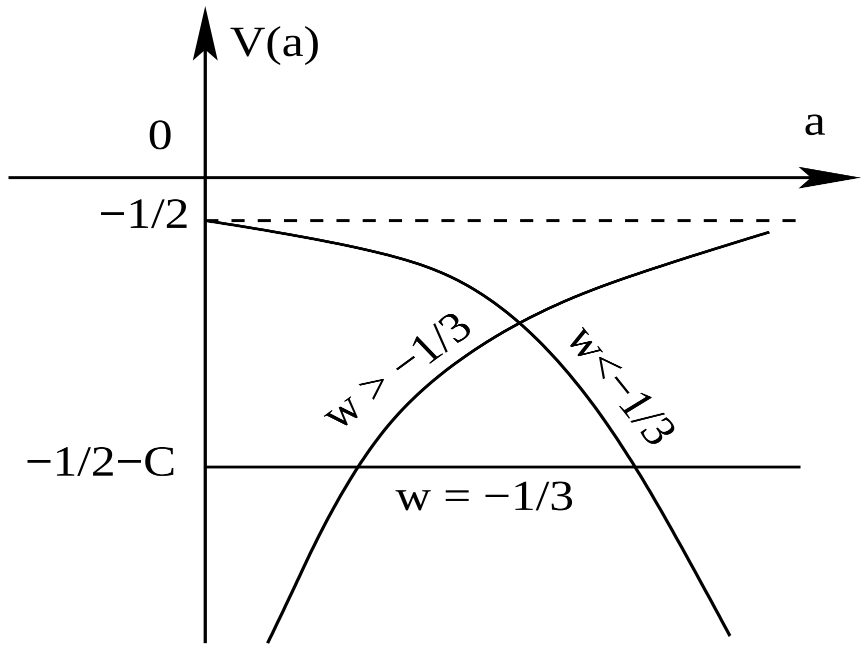

Figure 2.17: The potential given by Eq.(2.48) for $k = -1$ and $\Lambda = 0$.

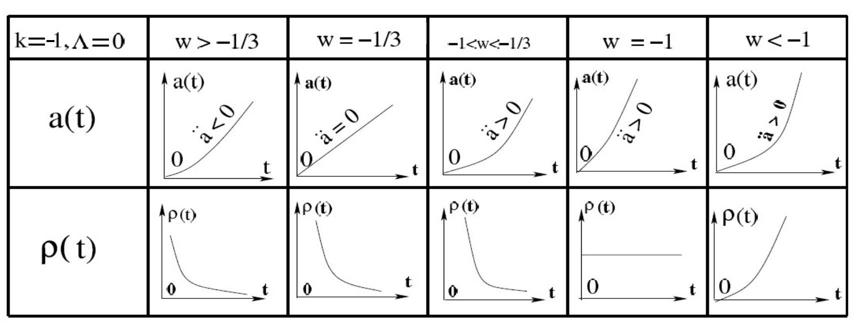

Figure 2.18: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = -1$ and $\Lambda = 0$. There are a big bang singularity for $w > -1$, no singularity for $w = -1$, and a big rip singularity at $a = \infty$ for $w < -1$.

collapse so that finally a big crunch singularity is developed at $t = 2t_m$, at which we have $a\,(2t_m) = 0$, as shown by Fig. 2.20.

Case C.3.2) $w = -1/3$: The potential in this case is similar to the last case, except for the fact that now $V(0) = -1/2 - \mathcal{C}$, as shown in Fig. 2.19. The motion of the universe is qualitatively the same as that in the last case, as shown by Fig. 2.20.

Case C.3.3) $-1 < w < -1/3$: In this case the potential has a minimum at $a = a_{min}$, as shown in Fig. 2.19, for which we find that $\ddot{a} < 0$ for $a < a_{min}$, and $\ddot{a} > 0$ for $a > a_{min}$, as shown by Fig. 2.20.

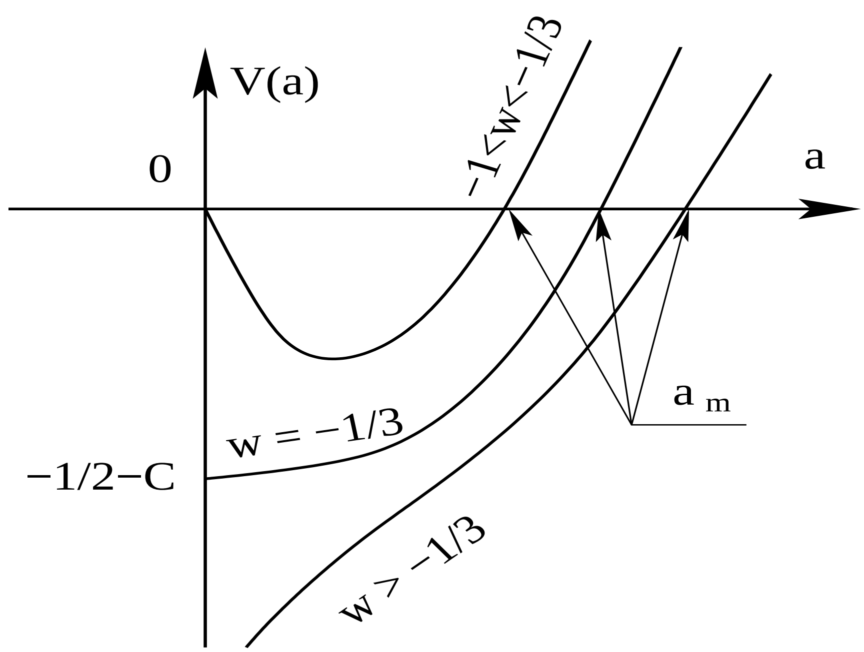

Figure 2.19: The potential given by Eq.(2.51) for $k = -1$, $\Lambda < 0$ and $w > -1$.

Case C.3.4) $w = -1$: In this case, depending on the ratio $6\mathcal{C}/|\Lambda|$, there are three distinguished sub-cases. When $6\mathcal{C}/|\Lambda| < 1$, the potential is non-positive only when $a \leq a_m$ where $a_m \equiv [3/(|\Lambda| - 6\mathcal{C})]^{1/2}$, as shown by Fig. 2.21. Then, the universe starts to expand from $a = 0$ until $a = a_m$. Afterwards, it will start to collapse until $a = 0$ again. But in the whole process, no spacetime singularity is

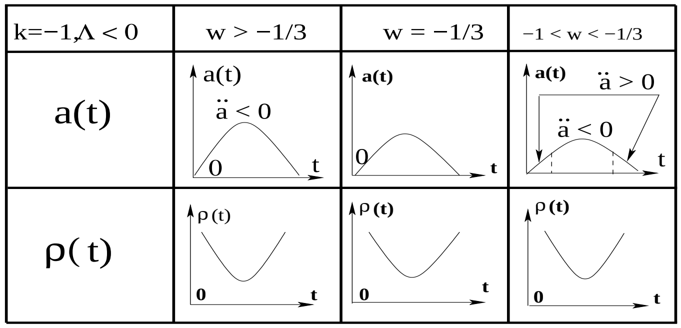

Figure 2.20: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = -1,\ \Lambda < 0$ and $w > -1$. There are both big bang and big crunch singularities for all the cases with $w > -1$.

developed. So, the universe is oscillating between $a = 0$ and $a = a_m$, as shown in Fig. 2.22.

When $6\mathcal{C}/|\Lambda| = 1$, we find that $V(a) = -1/2$, and the universe expands linearly starting from $a = 0$. There is no turning point, and no spacetime singularity, as shown by Figs. 2.21 and 2.22.

When $6\mathcal{C}/|\Lambda| > 1$, we find that $V(a) < -1/2$ for any given a. Then, starting from $a = 0$, the universe expands always acceleratingly ($\ddot{a} > 0$) until $a = \infty$, as shown by Figs. 2.21 and 2.22. No spacetime singularity is developed during the whole process.

Case C.3.5) $w < -1$: In this case, it can be shown that for any given ρ_0 there always exists a critical value Λ_c so that $V(a) = 0$ has two root positive roots when $|\Lambda| > |\Lambda_c|$; one positive root when $|\Lambda| = |\Lambda_c|$; and no positive root when $|\Lambda| < |\Lambda_c|$, as shown by Fig.2.19, where Λ_c is the solution of the equations $V\left(a_m, \Lambda_c\right) = 0$, and $V'\left(a_m, \Lambda_c\right) = 0$. It can be shown that it is given by,

$$\Lambda_c = \left(\frac{3|w| - 1}{|w| - 1}\right) \left[3\left(|w| - 1\right) \mathcal{C}\right]^{\frac{2}{3|w|-1}} . \tag{2.52}$$

Case C.3.5.1) $|\Lambda| > |\Lambda_c|$: In this case, the potential is positive in the region $a_1 < a < a_2$, where a_1 and a_2 are the two positive roots of $V(a) = 0$ with $a_2 > a_1$. Therefore, the motion of the universe now is restricted to the regions $0 \leq a \leq a_1$

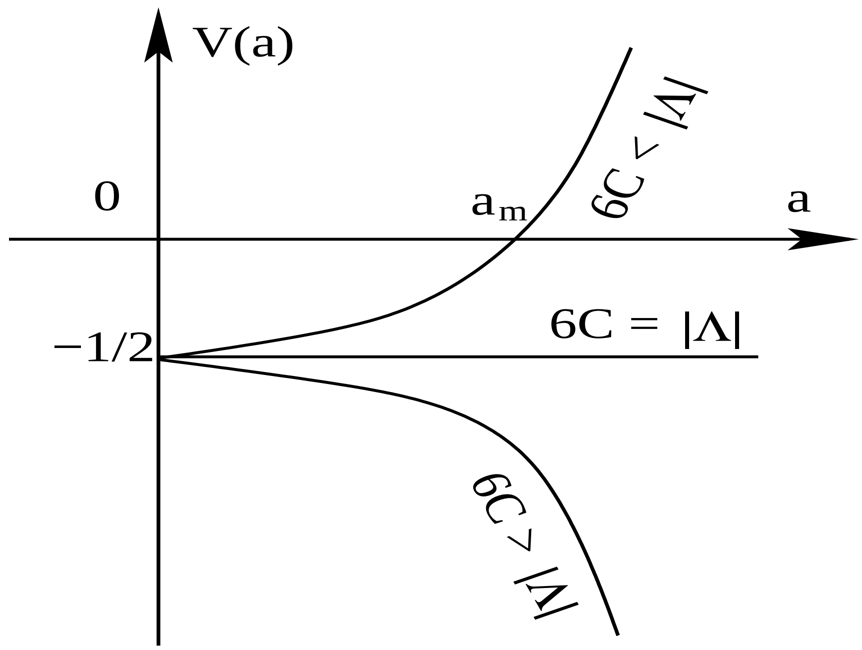

Figure 2.21: The potential given by Eq.(2.51) for $k = -1$, $\Lambda < 0$ and $w = -1$.

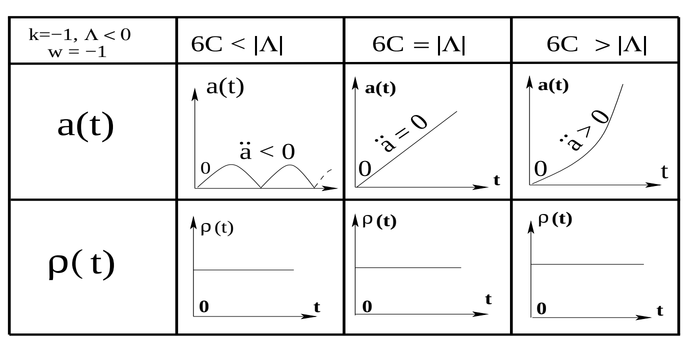

Figure 2.22: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = -1$, $\Lambda < 0$ and $w = -1$. The spacetime is not singular in any of these cases.

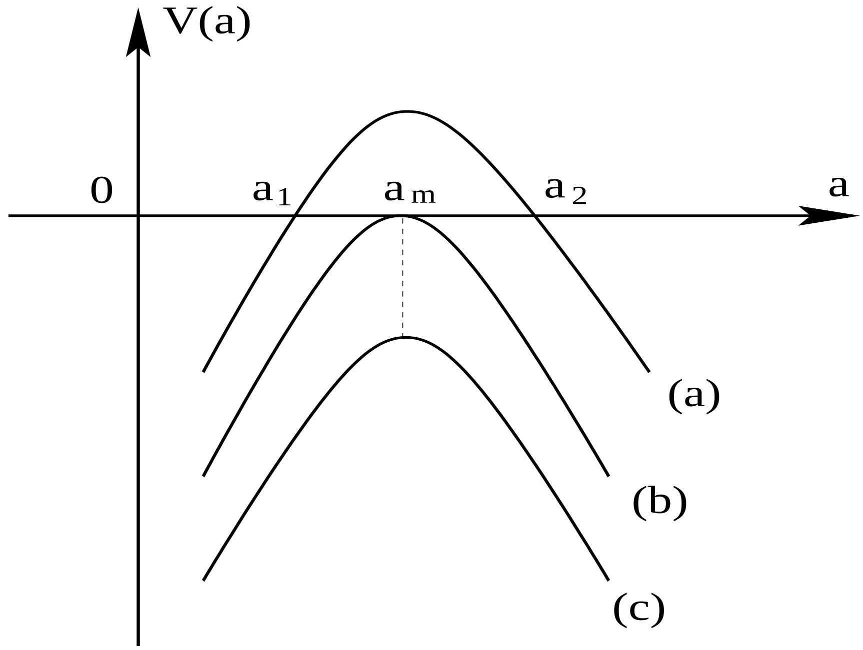

Figure 2.23: The potential given by Eq.(2.51) for $k = -1$, $w < -1$ and $\Lambda < 0$, where Λ_c is given by Eq.(2.52), where (a) is for $|\Lambda| > |\Lambda_c|$; (b) is for $|\Lambda| = |\Lambda_c|$; and (c) is for $|\Lambda| < |\Lambda_c|$.

and $a \geq a_2$, depending on its initial condition. If the universe starts to expand at $a = 0$, it will expand until its maximal radius $a = a_1$, and then collapse until $a = 0$. In the whole process, we have $\ddot{a} < 0$. Since at $a = 0$ the spacetime is not singular, so the universe will start to expand again. This process will be repeating endlessly, as shown in Fig. 2.24. However, if it starts to expand at a radius $a_i \geq a_2$, the universe will expand forever and never stops, as now $\ddot{a} > 0$ for any given $a \geq a_2$. A big rip singularity will be finally developed at $a = \infty$, since now we have $\rho \to \infty$, as $a \to \infty$.

Case C.3.5.2) $|\Lambda| = |\Lambda_c|$: In this case, there exists a static point a_m, at which we have $V(a_m) = V'(a_m) = 0$, as one can see from Fig. 2.23, where $a_m = [3(|w| - 1)\mathcal{C}]^{-1/(3|w|-1)}$. Therefore, if the universe starts to expand from $a = 0$, it will expand until $a = a_m$ with $\ddot{a} < 0$. Since at the point $a = a_m$, we have $\dot{a} = 0 = \ddot{a}$, the universe will become static at this point. However, it is not stable, and with small perturbations, the universe will either collapse until $a = 0$ or expand forever with $\ddot{a} > 0$. It should be noted that the spacetime is not singular at $a = 0$. So, if it collapses, it will start to expand again when the point $a = 0$ is reached. If the universe initially at $a = a_i > a_m$, from Fig. 2.23 we can see that it will expand forever. Since $V'(a)$ is always negative, so the universe in this case is always accelerating. A big rip singularity will be also finally developed at $a = \infty$.

Case C.3.5.3) $|\Lambda| < |\Lambda_c|$: In this case, the potential $V(a)$ is always negative for any given a, as shown in Fig. 2.23. Therefore, the corresponding solutions have no turning point. If the universe initially starts to expand from $a = 0$, it will expand forever. However, the potential has a maximum at $a = a_m$, for which we have

$$\ddot{a} = \begin{cases} < 0, & a < a_m, \\ = 0, & a = a_m, \\ > 0, & a > a_m, \end{cases} \tag{2.53}$$

that is, the universe is initially decelerating. Once it expands to $a = a_m$, it starts to expand at an accelerating rate. Similar to the last two cases, a big rip singularity will also develop at $a = \infty$.

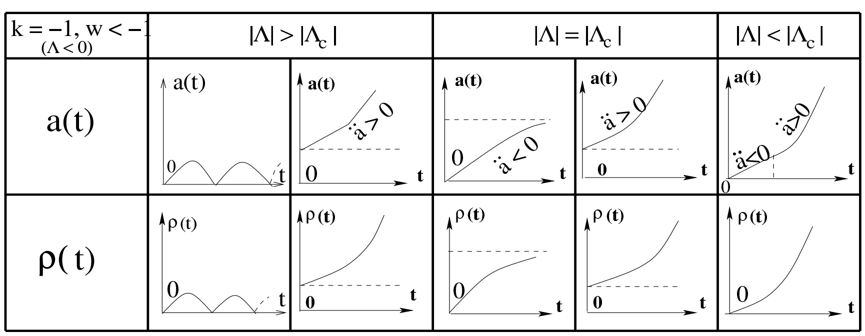

Figure 2.24: The expansion factor $a(t)$, the acceleration $\ddot{a}(t)$, and the energy density $\rho(t)$ for $k = -1$, $w < -1$ and $\Lambda < 0$, where Λ_c is given by Eq.(2.52). There are big rip singularities in all the cases, except for the first sub-cases of $|\Lambda| > |\Lambda_c|$ and $|\Lambda| = |\Lambda_c|$.

CHAPTER THREE

Summary and Future work

In this thesis, we have systematically studied the solutions of the Friedmann-Robertson-Walker (FRW) universe with a cosmological constant and a perfect fluid that has the equation of state $p = w\rho$, where p and ρ denote, respectively, the pressure and energy density of the fluid, and w is an arbitrary real constant. Writing the motion of the universe in the form,

$$\frac{1}{2}\dot{a}^2 + V\left(a, k, \rho_0, w, \Lambda\right) = 0. \tag{3.1}$$

We have been able to classify all the solutions according to the different values of $k,\ \rho_0,\ w$ and Λ, by simply using the knowledge of one-dimensional motion in classical mechanics [45], where $k[= 0, \pm 1]$ denote the curvature of the FRW space, ρ_0 the energy density of the matter field when $a = 1$, and Λ the cosmological constant. All these solutions are classified and presented in Figs. 2.2, 2.4, 2.6, 2.8, 2.10, 2.12, 2.14, 2.16, 2.18, 2.20, 2.22, and 2.24. Some particular cases were already discussed in various standard textbooks, whereby some typos may be corrected.

The method used in this thesis is simply the conservation law of kinetic and potential energies in classical mechanics, which can be easily followed to do the analysis and applied to the studies of other cosmological models of the universe. In particular, it can be applied to any model in which the motion of the universe can be cast in the form of Eq.(3.1) [19], including brane worlds in string/M theory [48, 49] and asymmetric branes [50, 51, 52, 53].

To see this clearly, in the following we briefly review the $f(R)$ and DGP models.

f(R) model: f(R) gravity is a kind of modified gravity theory. It is an alternative to Einstein's general relativity and can explain the accelerated expansion of the universe without dark energy. In f(R) gravity the action of the theory is given by

$$S[g] = \int \frac{1}{2\kappa} f(R)\sqrt{-g}d^4x, \tag{3.2}$$

where $f(R)$ is a function of the Ricci curvature. Then, it can be shown that the generalized Friedmann equations can be cast in the form,

$$\begin{aligned} 3FH^2 &= \rho_m + \rho_{rad} + \frac{1}{2}(FR - f) - 3H\dot{F} \\ -2F\dot{H} &= \rho_m + \frac{4}{3}\rho_{rad} + \ddot{F} - H\dot{F}, \end{aligned} \tag{3.3}$$

where $F(R) = \frac{\partial f(R)}{\partial R}$.

DGP model: DGP model is a model proposed by Dvali, Gabadadze and Porrati[54]. The corresponding Friedmann equations are given by

$$\begin{aligned} H^2 - \frac{\sigma}{r_c}\sqrt{H^2 + \frac{K}{a^2}} &= \frac{\mu^2}{3}\Sigma\rho_i - \frac{K}{a^2} \\ \dot{\rho} + 3H(\rho + p) &= 0. \end{aligned} \tag{3.4}$$

Clearly, in both theories, we can write the problems in the form of Eq.(3.1). Once this is done, one can simply follow what we have done in this thesis to study their cosmology.

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